2.2.1.2.2 REYNOLDS AVERAGED NAVIER-STOKES EQUATIONS

By hand of a time-averaging of the NS equations and the continuity equation for incompressible fluids, the basic equations for the averaged turbulent flow will be derived in the following. The flow field can then be described only with help of the mean values.

In order to be able to take a time-average, the momentary value is decomposed into the parts mean value and fluctuating value. This is shown graphically in Figure 2-17.

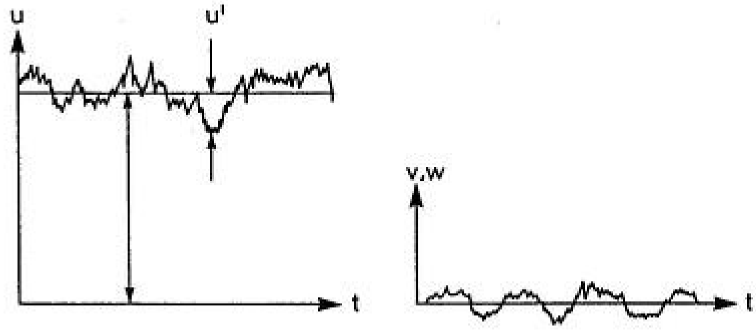

Figure 2-17: Turbulent velocity fluctuation in pipe flow as a function of time, taken from *[Fredsøe, 1990].*

The momentary velocity components is u, the time-averaged value is named $\bar{u}$ and the fluctuating velocity has the letter u'. With help of this definition the decomposition can mathematically be written as:

$$u=\bar{u}+u', \; v=\bar{v}+v', \; w=\bar{w}+w', \; p=\bar{p}+p' \qquad \text{Eq. 2-135}$$

Analogously for the density and the temperature:

$$\rho=\bar{\rho}+\rho', \; T=\bar{T}+T', \qquad \text{Eq. 2-136}$$

which will however be considered constant in the following.

The chosen averaging method takes the mean values at a fix place in space and averaged over a time span that is large enough for the mean values to be independent of it.

$$\bar{u}=\frac{1}{\Delta t}\int_{t_0}^{t_0+t_1} u\,dt \qquad \text{Eq. 2-137}$$

The time-averaged values of the fluctuating values are defined to be zero:

$$\overline{u'}=0, \; \overline{v'}=0, \; \overline{w'}=0, \; \overline{p'}=0 \qquad \text{Eq. 2-138}$$

Firstly the continuity equation is averaged. If we substitute the expressions for the velocities from Eq. 2-135 into the continuity equation (see Eq. 2-131) we get:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial v'}{\partial y}+\frac{\partial \bar{w}}{\partial z}+\frac{\partial w'}{\partial z}=0 \qquad \text{Eq. 2-139}$$

The time-average of the last equation is written as:

$$\overline{\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial v'}{\partial y}+\frac{\partial \bar{w}}{\partial z}+\frac{\partial w'}{\partial z}}=0 \qquad \text{Eq. 2-140}$$

Before we look at the transformation and reduction of Eq. 2-140, a summary of rules for time-averaging shall be given:

$$\overline{\frac{\partial \bar{u}}{\partial x}}=\frac{1}{\Delta t}\int_{t_0}^{t_o+t_1}\frac{\partial u}{\partial x}dt=\frac{\partial}{\partial x}\frac{1}{\Delta t}\int_{t_0}^{t_o+t_1}u\,dt=\frac{\partial \bar{u}}{\partial x}$$

$$\overline{\frac{\partial u'}{\partial x}}=\frac{1}{\Delta t}\int_{t_0}^{t_o+t_1}\frac{\partial u'}{\partial x}dt=\frac{\partial}{\partial x}\frac{1}{\Delta t}\int_{t_0}^{t_o+t_1}u'dt=0 \qquad \text{Eq. 2-141}$$

$$\bar{\bar{f}}=\bar{f}\,,\quad \overline{f+g}=\bar{f}+\bar{g}\,,\quad \overline{\bar{f}\cdot g}=\bar{f}\cdot\bar{g}\,,\quad \overline{\frac{\partial f}{\partial s}}=\frac{\partial \bar{f}}{\partial s}\,,\ \overline{\int f\,ds}=\int \bar{f}\,ds \qquad \text{Eq. 2-142}$$

$$\textit{but}\quad \overline{f\cdot g}\neq \bar{f}\cdot\bar{g}$$

The averaged derivatives of the fluctuations are also zero according to these rules, so that the time-averaged continuity equation is:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0 \qquad \text{Eq. 2-143}$$

Now the NS equations will be time-averaged. The averaging will be exemplified for the x-component. Beforehand a small transformation of the advection term from Eq. 2-131:

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}&=\frac{\partial(u^2)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}-u\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{=0}\\ &=\frac{\partial(u^2)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\end{aligned} \qquad \text{Eq. 2-144}$$

The expressions for the decomposition of the velocities from Eq. 2-135 are now substituted into the transformed Navier-Stokes equation (see Eq. 2-131) and a time-average is done:

$$\overline{\rho\left\{\frac{\partial(\overline{u}+u')}{\partial t}+\frac{\partial(\overline{u}+u')^2}{\partial x}+\frac{\partial(\overline{u}+u')(\overline{v}+v')}{\partial y}+\frac{\partial(\overline{u}+u')(\overline{w}+w')}{\partial z}\right\}}$$

$$=\overline{\underbrace{F_x}_{\substack{\text{is not subject to}\\ \text{turbulent}\\ \text{fluctuation}}}-\frac{\partial(\overline{p}+p')}{\partial x}+\mu\left(\frac{\partial^2(\overline{u}+u')}{\partial x^2}+\frac{\partial^2(\overline{u}+u')}{\partial y^2}+\frac{\partial^2(\overline{u}+u')}{\partial z^2}\right)} \qquad \text{Eq. 2-145}$$

Application of the rules from Eq. 2-141 and Eq. 2-142 shows that among others the terms $\frac{\partial\left(\overline{\overline{u}_i u_j'}\right)}{\partial x_j}, \frac{\partial \overline{u_i'}}{\partial t}, \frac{\partial \overline{u_i'}}{\partial x_j}$ from the equation above can be reduced and the equation can be transformed to:

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{uu}}{\partial x}+\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{uv}}{\partial y}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}+\frac{\partial\overline{uw}}{\partial z}\right)$$

$$=F_x-\frac{\partial\overline{p}}{\partial x}+\mu\underbrace{\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}\right)}_{\Delta\overline{u}} \qquad \text{Eq. 2-146}$$

Further small transformations, for example a repeated application of the product rule and the continuity equation to the advection term, lead to a form of the time-averaged NS equations for all three directions as:

$$\rho\left(\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}\right)=F_x-\frac{\partial\overline{p}}{\partial x}+\mu\Delta\overline{u}-\rho\left(\frac{\partial\overline{u'u'}}{\partial x}+\frac{\partial\overline{u'v'}}{\partial y}+\frac{\partial\overline{u'w'}}{\partial z}\right)$$

$$\rho\left(\frac{\partial\overline{v}}{\partial t}+\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}+\overline{w}\frac{\partial\overline{v}}{\partial z}\right)=F_y-\frac{\partial\overline{p}}{\partial y}+\mu\Delta\overline{v}-\rho\left(\frac{\partial\overline{u'v'}}{\partial x}+\frac{\partial\overline{v'v'}}{\partial y}+\frac{\partial\overline{v'w'}}{\partial z}\right) \qquad \text{Eq. 2-147}$$

$$\rho\left(\frac{\partial\overline{w}}{\partial t}+\overline{u}\frac{\partial\overline{w}}{\partial x}+\overline{v}\frac{\partial\overline{w}}{\partial y}+\overline{w}\frac{\partial\overline{w}}{\partial z}\right)=F_z-\frac{\partial\overline{p}}{\partial z}+\mu\Delta\overline{w}-\rho\left(\frac{\partial\overline{u'w'}}{\partial x}+\frac{\partial\overline{v'w'}}{\partial y}+\frac{\partial\overline{w'w'}}{\partial z}\right)$$

Or in tensor form:

$$\rho\frac{D\overline{u_i}}{Dt}=F_i-\frac{\partial\overline{p}}{\partial x_i}+\mu\,\Delta\overline{u_i}-\rho\underbrace{\left(\frac{\partial\overline{u_i'u_j'}}{\partial x_j}\right)}_{\text{Re}\,ynolds-stress} \qquad \text{Eq. 2-148}$$

From now on the time-averaged fields will not be overlined anymore. So for example *u* stands for the time-averaged velocity component in direction of the x-axis.

We pay attention to the last two terms of the right side of Eq. 2-148:

$$\mu \Delta u_i - \rho \left(\frac{\partial \overline{u_i' u_j'}}{\partial x_j} \right)$$

$$= \mu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \right) - \rho \frac{\partial}{\partial x_j} \left(\overline{u_i' u_j'} \right) \qquad \text{Eq. 2-149}$$

$$= \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} - \rho \overline{u_i' u_j'} \right)$$

The expression in the brackets above corresponds to the total shear stress:

$$\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} - \rho \overline{u_i' u_j'} \qquad \text{Eq. 2-150}$$

If we compare to the Navier-Stokes equations Eq. 2-131, it is conspicuous that besides the viscous part an additional term has been added to the total shear stress. This term results from the time-average and is generally the dominant part of the total shear stress. Since the term only appears due to the Reynolds average, it is called Reynolds stress or apparent turbulent shear stress. As stated in the introduction to the RANS approach, to lead to the closure of the equation system, an approximation for the Reynolds stresses has to be done, which sets in relation the apparent shear stresses with the velocity field of the average flow.

With the approach of the **eddy viscosity principle after Boussinesq** 1877, the general time-averaged NS equations, also called Reynolds equations, can thus be written in tensor form as:

$$\rho \left(\frac{Du_i}{Dt} \right) = F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \tau_{ij}$$

Eq. 2-151

$$with \quad \tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \rho \left(\nu_T \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \, \delta_{ij} \right)$$