Bob Morris is a retired metallurgist/materials scientist who has been dabbling in Free Flight modeling since 2005 with an interest in old timer, Nostalgia and small gliders. He's a member of the Brooklyn Skyscrapers and regular attendee at the Geneseo, New York Great Grape Gathering.

A wound rubber motor is a complicated looking thing but I think I can offer a relatively simple model that explains a lot. This approach, previously published in the April 2011 and July 2014 issues of *Free Flight Quarterly* (ref. 1), uses the topological variables *Twist* **Tw** and *Writhe* **Wr**, which apply to deformable ribbons like the short strip of Tan Super Sport shown in the three photos in Figure 1. *Twist* is the number of rotations about the long axis of the strip and *Writhe* is the number of times the axis appears to cross over itself in an average side view. **Tw** and **Wr** are related by a simple relationship called the Calengareanu Invariant (ref. 2):

$$\mathbf{Tw} + \mathbf{Wr} = \textbf{Linking Number} \quad \text{eq.(1)}$$

In a rubber motor, Linking Number corresponds to the total number of turns **N**; writhe turns **Wr** are the kinks or knots. The Linking Number **N** in the left photo is 0 and in the center and right photos **N** = 1. Stretching a wound motor will cause writhe turns to transform to twist and visa-versa, as illustrated in the center and right photos. In general, an elastic strip will tend to distribute applied turns between twist and writhe in a way that minimizes total elastic energy (ref. 3). However, in a wound rubber motor this mechanical equilibration may be impeded by friction.

The simplified geometrical model of a rubber motor, shown in Figure 2, is a single cylindrical strand. **R** and **L** are the radius and length, respectively, of the un-stretched motor. The maximum extension ratio is **x** so that the length of the fully extended motor is **x L**. The radius of the fully extended motor is **r**. Assuming the rubber is incompressible, the volume **V** is independent of stretching and twisting etc.:

$$V = \pi R^2 L = \pi r^2 x L = \pi r^2 l = \pi r'^2 l' \quad \text{eq. (2)}$$

which can be rearranged to give:

$$r'^2 = R^2 L / l' \quad \text{eq. (3)}$$

In the April 2011 *FFQ* article, max turns N_{max} was calculated by imagining that the fully stretched motor was coiled around itself spool-wise so as to fit in the original **R** by **L** cylinder, as shown in the upper left corner of Figure 2. Dividing the small stretched cross-sectional area **a** into the rectangular area **R L**, or dividing the stretched length **x L** by the average circumference of a coil **π R** both yield:

$$N_{max} = x L / \pi R \quad \text{eq. (4)}$$

Albert Einstein supposedly said "Make everything as simple as possible but not simpler." This result for a pure writhe motor is too simple. It gives the correct dependence on motor cross-section but underestimates breaking

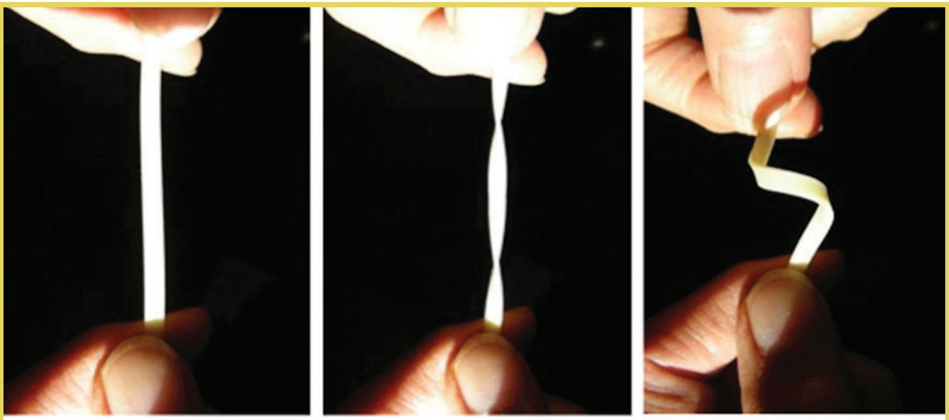

Figure 1 Twist - Writhe transformation in a rubber strip

PHOTOGRAPHY: CONTRIBUTED

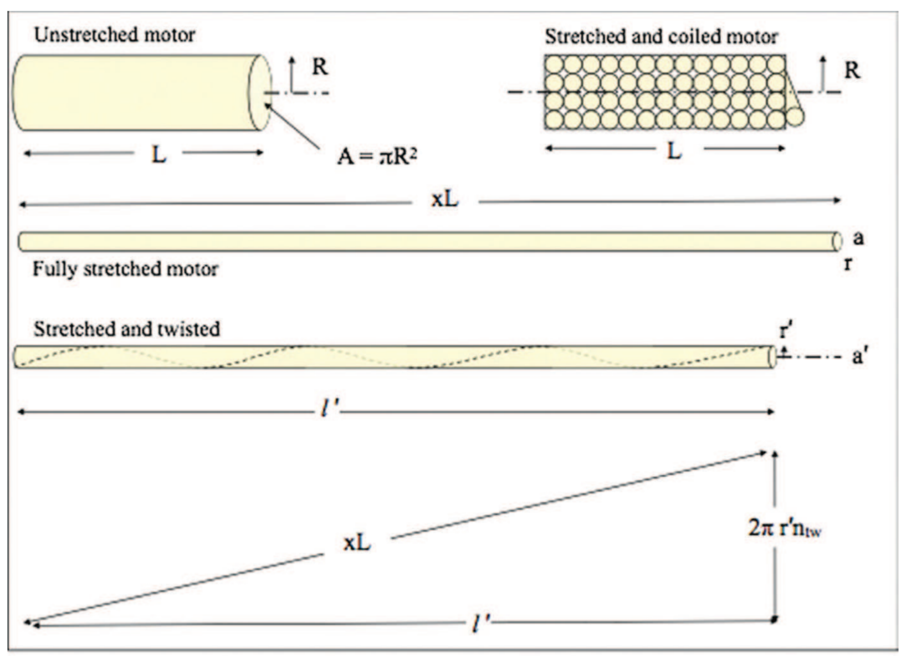

Figure 2 Schematic of simplified rubber motor model

turns by a factor of two or so.

To create a more realistic model one needs to twist the fully-extended motor prior to coiling, but twisting at constant length would increase the tension in the outer fibers. When the stretched **r-by-l** cylinder is twisted by n_t turns, a straight line on the surface of the motor becomes a helix, as shown in Figure 2. To approximate constant tension, the length of the helical line is held constant at **xL**. As twisting progresses, **r** increases to **r′** while **l** shrinks to **l′** to maintain constant volume, which decreases the number of writhe turns to:

$$n_{wr} = R\,L / a' = l' / \pi R \qquad \text{eq. (5)}$$

Unwrapping the helix gives a triangle with height $2\pi r' n_{tw}$, base **l′**, and hypotenuse **x L**, so:

$$(n_{tw}\, 2\pi r')^2 + l'^2 = (x\,L)^2 \qquad \text{eq. (6)}$$

$$n_{tw}^{\,2}\, 4\pi^2 r'^2 + l'^2 = x^2 L^2$$

substituting for **r′** from equation (3):

$$n_{tw}^{\,2}\, 4\pi^2 R^2 L / l' = x^2 L^2 - l'^2$$

re-arranging and solving for n_{tw} gives:

$$n_{tw} = [(x^2 L^2 l' - l'^3) / 4\pi^2 R^2 L]^{1/2} \qquad \text{eq. (7)}$$

$$N_{total} = n_{tw} + n_{wr} = l' / \pi R + [(x^2 L^2 l' - l'^3) / 4\pi^2 R^2 L]^{1/2} \qquad \text{eq.(8)}$$

COMPARISON WITH EXPERIMENT

Now we can plug some numbers into a spreadsheet and see how this works for a 12 inch long Coupe motor made from 10 strands of 1/8" wide, 0.043" thick rubber strip with maximum extension ratio **x** = 10.5 (ref. 4). The volume of rubber is 0.645 cubic inches which implies a weight of 9.83 grams (ref. 5). The motor is initially stretched to 126 inches, 10.5 times the un-stretched length. With no twist applied, the stretched cross-section is 0.0048 square inches and the calculated maximum number of writhe turns, **R L / a** is 329.6. The spreadsheet then decreases the length in steps and calculates the corresponding number of twist turns using equation (7), and the new cross-section. The number of writhe turns, and total turns

$$N_{total} = n_{tw} + n_{wr}$$

follow from equation (5). The results are shown as a graph in Figure 3.

Starting at the right side of Figure 3, N_{total} initially increases with increasing twist (decreasing length), reaches a maximum at a reduced length of about 96 inches, then decreases with further twist. The calculated N_{max} of 532 turns, comprises 281 twist turns and 251 writhe turns, or 1.12 twists per writhe, not quite equal partitioning as had been postulated in *FFQ* No. 39, but not that far off. The $N_{total} = n_t + n_w$ maximum is very broad and a fourth order polynomial gives a good fit to the calculated N_{total} vs **l′** curve.

Calculated max turns N_{max} values for a range of motor sizes are shown in Figure 4 and all are a little below experimental values. The curve fit to the calculated values closely follows the expected **1/√area** dependence on motor cross-section (number of strands). The calculated twist-to-writhe ratio at maximum turns ranges between 1.12 and 1.25 with an average of 1.19.

DISCUSSION

This model relies on several simplifying assumptions, the most important of which is that the stress and strain are constant and directed along the axis of the rubber strands.

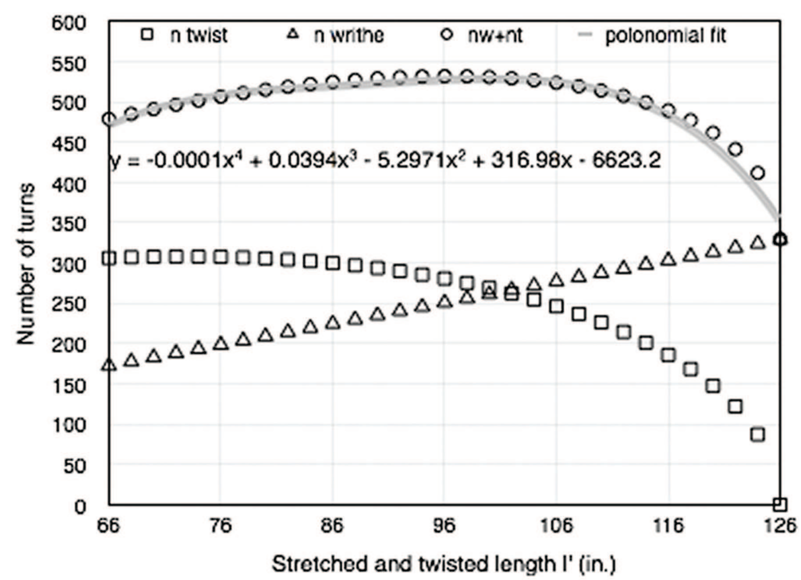

Figure 3 Calculated twist and writhe turns for a 10 strand Coupe motor for x = 10.5

Another key assumption is that the volume is constant or, equivalently, that Poisson's ratio for the rubber is exactly 0.5. This is an excellent approximation for rubber up to strains of 300% or so, but at higher extensions Holt and McPherson (ref. 6) showed that the volume decreases somewhat, probably as a result of crystallization of the rubber molecules. Extrapolating Holt and McPherson's results to an extension ratio of 10 gives a volume reduction of about 3% which would translate to 3% smaller cross-sectional area implying 3% more writhe turns. An extension ratio **x** of 10.5 has been assumed based on F1B/G flyer Tom Vaccaro's rubber tests. In a typical test the rubber doesn't quite break, so the model predictions herein could be interpreted as "almost breaking turns"

whereas the experimental values in Figure 3 are actual breaking turns. Agreement with these experiments on Tan Super Sport is pretty good given the simplicity of the model. The remaining discrepancy might be further reduced by using the actual fracture strain value for **x** and applying the Holt and McPherson volume change correction. It appears that this model predicts the twist-to-writhe ratio that produces the minimum strain value for a given number of turns. To maximize motor life it is important to minimize exposure to high strain values. By reducing the value of **x**, the model could be used to devise a winding sequence to maximize turns while not exceeding say 80% of the breaking strain. This might reliably provide five or six runs on a given motor while delivering respectable performance. This model may help refine the rule-of-thumb “stretch, wind half of the expected number of turns then come in gradually while continuing to wind” by adding a little more twist before coming in. This would probably result in little if any altitude advantage but standardizing the winding process and twist-to-writhe ratio might reduce the odds of motor breakage near max turns.

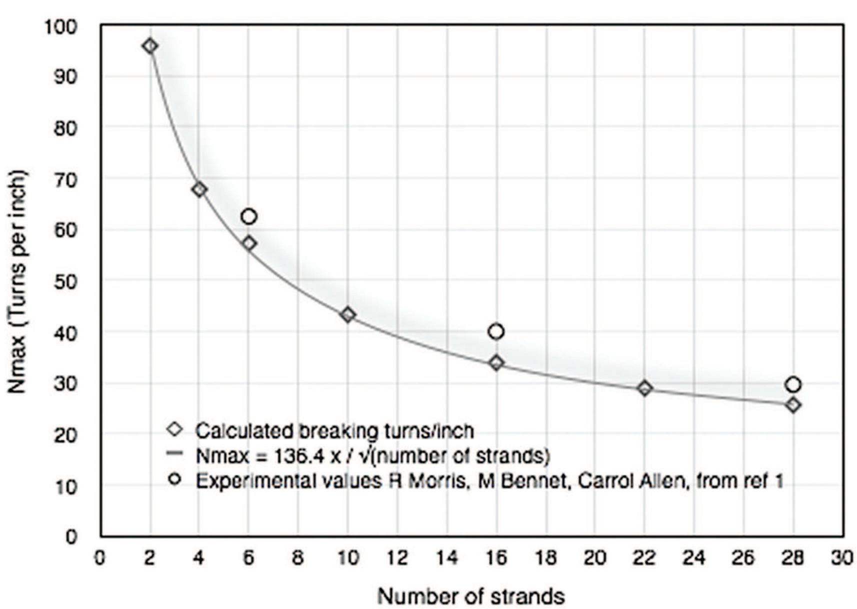

Figure 4 Summary of breaking turns vs number of strands of 1/8" for x = 10.5

An interesting feature of the twist-plus-writhe model is the possibility that the twist-to-writhe ratio may vary along the length of a wound motor depending on the stretching and winding sequence and friction, which could affect the final fore-and-aft distribution of rubber weight for motors longer than the hook-to-pin distance, a problem discussed by Peter Hall in (ref. 7).

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4) Tom Vaccaro (private communication) measured total extension values for a box of 2012 Tan Super Sport ranging between 10.1 and 10.7. The test does not break the samples.

5) Carrol Allen (private communication) measured Tan Super Sport specific gravity of 0.93 using immersion in ethanol-water mixtures which he probably then drank.

6) W. Holt and A.T. McPherson, Change of volume of rubber on stretching: effects of time, elongation and temperature, Res. Paper RP936,/ *Journal of Research of the National Bureau of Standards/*, 17, 1936.

7) P. Hall, A Knotty Problem, 2013 BMFA *Free-Flight Forum*.