THE ALMOST ALTERNATING DIAGRAMS OF THE TRIVIAL KNOT

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Abstract. Bankwitz characterized an alternating diagram representing the trivial knot. A non-alternating diagram is called almost alternating if one crossing change makes the diagram alternating. We characterize an almost alternaing diagram representing the trivial knot. As a corollary we determine an unknotting number one alternating knot with a property that the unknotting operation can be done on its alternating diagram.

1. Introduction

Our concern in this paper is to decide if a given link diagram on S^2 represents a trivial link in S^3. This basic problem of Knot Theory has been worked in three directions with respect to the properties which we require the diagram to have: closed braid position; positivity; and alternation. We pursue the third direction. For the first direction see [BM] and for the second direction see [Crm] and [St].

A link diagram is *trivial* if the diagram has no crossings. Obviously a trivial link diagram represents a trivial link. A portion of a non-trivial link diagram depicted at the left of Figure 1 is called a *nugatory crossing*. Such a local kink may be eliminated for our purpose. Therefore we consider only *reduced* link diagrams, i.e. link diagrams with no nugatory crossings.

Let L be a link diagram on S^2 and let $\hat{L}$ be the link projection obtained from L by changing each crossing to a double point. If there is a simple closed curve C on $S^2 - \hat{L}$ such that each component of $S^2 - C$ contains a component of $\hat{L}$, then we call L *disconnected* and C a *separating curve for* L. Otherwise we call L *connected*.

A non-trivial link diagram is *alternating* if overcrossings and undercrossings alternate while running along the diagram. We know that a reduced alternating link diagram never represents a trivial link.

Theorem 1. (Crowell [Crw], Murasugi [Mu]) *A splittable link never admits a connected alternating diagram.*

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Theorem 2. (Bankwitz [Ba]) *The trivial knot never admits a reduced alternating diagram.*

FIGURE 1. Nugatory crossings and a trivial clasp

We consider the problem for a link diagram which is alternating except one crossing. Such a link diagram is called almost alternating and first studied by C.Adams et al. [Ad1]. A link diagram is *almost alternating* if the diagram is neither trivial nor alternating, but one crossing change makes the diagram alternating. A crossing of an almost alternating link diagram is called a *dealternator* if the crossing change at the crossing makes the diagram alternating. In [Ad1, Ad3], the decision problem for an almost alternating link diagram is asked. M.Hirasawa gave a solution for special almost alternating link diagrams in [Hi].

If an almost alternating link diagram has a trivial clasp (the right of Figure 1), then we obtain either a trivial link diagram or an alternating link diagram with fewer crossings from the diagram by the Reidemeister move of type II. Thus we may assume our diagram is *strongly reduced*, i.e. a reduced diagram with no trivial clasps.

Let L be a non-trivial link diagram on S^2 and let $\hat{L}$ be the link projection obtained from L by changing each crossing to a double point. If there is a simple closed curve C on S^2 intersecting $\hat{L}$ transversely in just two points such that $\hat{L}$ is not a trivial arc in each component of $S^2 - C$, then we call L *non-prime* and C a *decomposing curve for* L. Otherwise we call L *prime*. Note that a prime link diagram is connected.

We call a portion of a link diagram depicted in Figure 2 a *flyped tongue*, where the shadowed disks indicate alternating 2-tangles. Then the author showed the following in [Ts].

Theorem 3. ([Ts])

(1) *A splittable link with* n*-components* ($n \geq 3$) *never admits a connected almost alternating diagram.*
(2) *A prime, strongly reduced almost alternating diagram of a splittable link with* 2*-components has a flyped tongue.*

The following is the main theorem of this paper.

Theorem 4. *A strongly reduced almost alternating diagram of the trivial knot has a flyped tongue.*

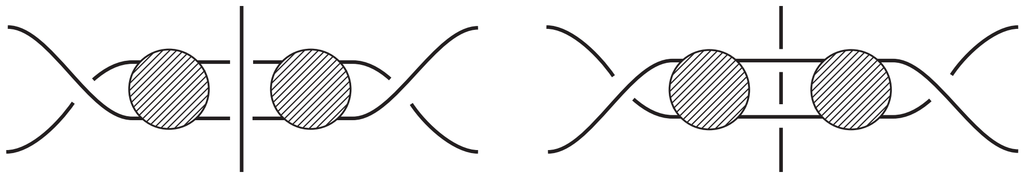

FIGURE 2. Flyped tongues

1.1. **The almost alternating diagrams of the trivial knot.** Theorem 4 yields a simple finite algorithm to see if a given reduced almost alternating knot diagram represents the trivial knot without increasing the number of crossings of diagrams in the process. In fact Adams et.al. in [Ad2] conjectured that we have a calculs to reduce a given almost alternating diagram of the trivial knot consisting of three kinds of local moves on link diagrams: a *flype move* defined by Figure 3; an *untongue move* defined by Figure 4; and an *untwirl move* defined by Figure 5, where we allow the move obtained by changing all the crossings in or taking the mirror image of each figure. Note that each move does not change the link type which a diagram represents. The last two moves are introduced in [Ad2]. We show their conjecture is true. A similar algorithm for a reduced almost alternating link diagram with more than one component is obtained in [Ts].

Let K be a reduced almost alternating knot diagram. If K is not strongly reduced, then apply the Reidemeister move of type II to K to have another diagram K', which is trivial or alternating. In the first case, we can see that K is not reduced, which contradicts the assumption. Consider the second case. Since K' has at most two nugatory crossings, K' represents the trivial knot if and only if K' is a coiled diagram for a non-zero integer m from Theorem 2 (Figure 6). Next consider the case when K is strongly reduced. If K has no flyped tongues, then K represents a non-trivial knot from Theorem 4. Otherwise, we obtain another almost alternating diagram K'' which has fewer crossings than K by an untongue move or an untwirl move after sufficient flype moves. It is easy to see that K'' is reduced, since K is strongly reduced. Then we go back to the beginning and continue the process.

Going over the above process assuming that K represents the trivial knot, we obain the following. Here note that if K is not strongly reduced, then K is a diagram in Figure 7, which we denote by $\mathcal{C}_m$.

Theorem 5. *Let K be a reduced almost alternating diagram of the trivial knot. Then there are a non-zero integer m and a sequence of reduced almost alternating diagrams*

$$K = K_1 \to \cdots \to K_p = C_m$$

such that K_{i+1} is obtained from K_i by a flype move, an untongue move or an untwirl move.

□

Therefore we can obtain all the almost alternating diagrams of the trivial knot. Here we defince a *tongue move* and a *twirl move* as the converse of an untongue move and an untwirl move, respectively.

Corollary 1.1. *A reduced almost alternating diagram of the trivial knot is obtained from C_m for a non-zero interger m by tongue moves, twirl moves and flype moves.*

□

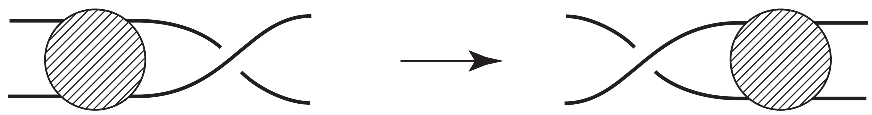

Figure 3. A flype move

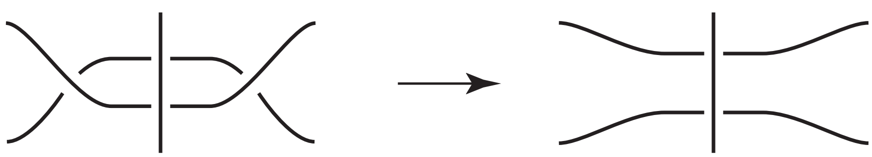

Figure 4. An untongue move

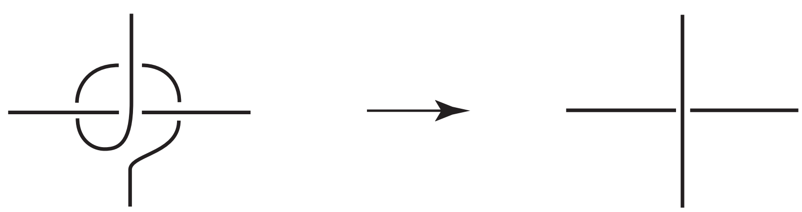

Figure 5. An untwirl move

Figure 6. Coiled diagrams

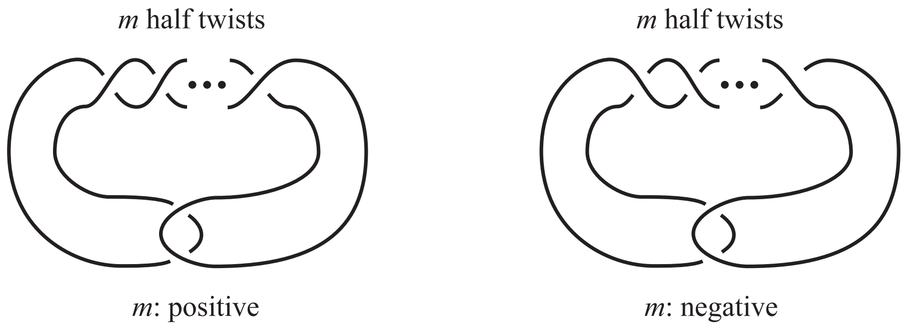

FIGURE 7. C_m

1.2. **Alternating knots with unknotting number one.** In [Ko] P.Kohn made a conjecture, which says that a link with unknotting number one has a minimal diagram which has a crossing such that the crossing change at the crossing makes the link trivial. This conjecture was shown to be true for large algebraic alternating knots by C.Gordon and J.Luecke in [GL]. We remark here that we can obtain all the alternating knots with unknotting number one satisfying the conjecture from Corollary 1.1, since we obtain an alternating knot with unknotting number one from a reduced almost alternating diagram of the trivial knot by the crossing change at the dealternator.

1.3. **Organization of the paper.** Theorem 1 and Theorem 2 were proved algebraically using the Alexander polynomial of a link in [Crw] and [Mu] and using the determinant of a knot in [Ba], respectively. After those, geometric proofs were given in [Me] and in [MT1] using the "crossing-ball" technique invented by W.Menasco. Namely he embed a link in a "branched" sphere S to realize a diagram as a geometrical object. We succeed his technique to prove Theorem 4 and review it in Section 2. In Section 3, we introduce key concepts which play important roles in this paper: special position for a spanning surface of a link; short arcs; and short bridges. We show that if a given spanning surface is in special position, then the boundary of a neighborhood of it is in standard position (Proposition 3.1). In [Ts] to prove Theorem 3 the author in fact showed that if a prime, strongly reduced almost alternating link diagram on S admits a sphere in its complement which is in standard position, then the diagram admits a flyped tongue (Theorem 6). Therefore we are done if our almost alternating diagram on S of the trivial knot admits a spanning disk in special position. In Section 4, for a spanning surface E of a link which is given as a connected, reduced almost alternating diagram on S, we show that E is in special position if and only if E has no short arcs. In Section 5, we show that if a spanning disk E of the trivial knot which is given as a strongly reduced almost alternating diagram on S has a short arc, then we can cut E along the short arc or short bridges to have a connected, strongly reduced almost alternating diagram on S of the trivial 2-component link with spanning disks in special position. Then we study the intersection diagram of the spanning disks and S to show that the given diagram has a flyped tongue in Section 6.

2. Preliminary

In this section we bliefly review concepts introduced by Menasco with some additional or modified notations. For details, see [Me], [MT1] etc.

Let $\widetilde{S}$ be a 2-sphere in $S^3 = \mathbb{R}^3 \cup \infty$. Denote by $\widetilde{B}^-$ the 3-ball which $\widetilde{S}$ bounds in $\mathbb{R}^3$ and by $\widetilde{B}^+$ $S^3 - \mathrm{int}\widetilde{B}^-$. Take m halls out from $\widetilde{S}$ and denote the result by $\widetilde{S}_m$. To each hall, put a 2-sphere θ_i with an equater ε_i specified so that the equater is on the hall. We call each θ_i a *bubble* and the 3-ball which a bubble bounds in $\mathbb{R}^3$ a *crossing-ball*, denoted by Θ_i. We call the disk $\theta_i \cap \widetilde{B}^\pm$ an *upper/lower* hemisphere and denote it by $\theta_i^\pm$. A *bubbled sphere* S_m is a union of $\widetilde{S}_m$ and the m bubbles. We denote the 2-sphere $\widetilde{S}_m \cup (\cup \theta_i^\pm)$ by $S_m^\pm$ and the 3-ball which $S_m^\pm$ bounds in $\widetilde{B}^\pm$ by $B_m^\pm$. A link L in S^3 is called a *link diagram on* S_m if L is on S_m, meets a bubble θ_i in a pair of two arcs $a_i^+ b_i^+$ on θ_i^+ and $a_i^- b_i^-$ on θ_i^-, and meets the equater transversely so that a_i^+, a_i^-, b_i^+ and b_i^- are positioned on ε_i in this order. Note that $L \cap S_m^+$ on S_m^+ is a link diagram in a usual sense. We call a diagram on S_m simply a diagram unless any confusion is expected. We say that a link diagram L on S_m has a specific property, e.g. alternation, if $L \cap S_m^+$ on S_m^+ has the property. Then we also say that L is in alternating *position*. We assume that m is sufficiently large and omit m from now on.

Let L be an n-component link diagram $L_1 \cup \cdots \cup L_n$ on S. We call the intersection $L \cap \theta_i$ a *crossing* if it is not empty. A *segment* λ_j is a component of $L \cap (S^+ \cap S^-)$, and a *positive/negative long segment* $\Lambda_k^\pm$ is a component of $L \cap S^\pm$. We say that $\Lambda_k^\pm$ *runs through* a bubble θ_i if $\Lambda_k^\pm$ contains the arc $a_i^\pm b_i^\pm$ of $L \cap \theta_i$, and that $\Lambda_k^\pm$ is *p-/n-adjacent to* θ_i if an end of $\Lambda_k^\pm$ is on θ_i. The *length* of a long segment Λ_k is the number of segments which Λ_k has. A segment λ_j is *p-/n-adjacent* to θ_i if the positive/negative long segment containing λ_j is p-/n-adjacent to θ_i. If λ_j is p-adjacent and n-adjacent to bubbles, then it is called *alternating*. Otherwise λ_j is called *non-alternating*. A bubble θ_i is *p-/n-adjacent* to another bubble θ_l if there is a segment which has its ends on θ_i and θ_l and is p-/n-adjacent to θ_l. A crossing x is *p-/n-adjacent* to another crossing y if the bubble at x is p-/n-adjacent to the bubble at y. A *region* R_k is the closure of a component of $(S^+ \cap S^-) - L$ and its degree, denoted by $\mathrm{deg}R_k$, is the number of segments on its boundary. Let N_j be a sufficiently small tubular neighborhood of L_j such that $\partial N_j \cap \Theta_i$ is a pair of a saddle-shaped disk in B^+ and a saddle-shaped disk in B^-.

2.1. **Standard position for a closed surface in a link complement.**

Let L be a link diagram on S and let F be a closed surface in $S^3 - L$. Then we may isotop F so that F satisfies the following conditions, and then we say that F is in *basic position*.

- (Fb1) F intersects $S^\pm$ transversely in a pairwise disjoint collection of simple closed curves;
- (Fb2) F does not intersect N_j for any j; and
- (Fb3) F intersects each crossing-ball Θ_i in a collection of saddle-shaped disks in $\Theta_i - \cup N_j$ (Figure 8).

Definition. Let L be a link diagram on S and let F be a closed surface in $S^3 - L$ which is in basic position with $F \cap S^{\pm} \neq \emptyset$. Let C be a curve of $F \cap S^{\pm}$. We say that C is *standard* if C satisfies the following conditions and that F is in *standard position* if any curve of $F \cap S^{\pm}$ is standard.

(Ft1) C bounds a disk in $F \cap B^{\pm}$;
(Ft2) C meets at least one bubble; and
(Ft3) C meets a bubble in an arc.

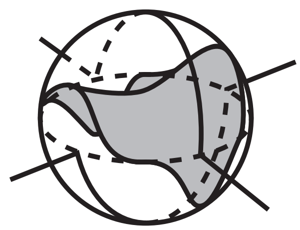

FIGURE 8. A saddle-intersection in a crossing-ball

2.2. **Standard position for a spanning surface of a link.** Let E be a spanning surface of a link diagram L on S. We may isotop E so that E satisfies the following conditions, and then we say that E is in *basic position*.

(Eb1) E intersects $S^{\pm}$ transversely in a pairwise disjoint collection of simple closed curves;
(Eb2) E intersects N_j in an annulus M_j so that $M_j \cap \Theta_i = L_j \cap \theta_i$ and $\partial M_j \cap \partial N_j$ proceeds along ∂N_j monotonely with respect to the longitudinal coordinate of ∂N_j; and
(Eb3) $E - K$ intersects each crossing-ball Θ_i in a collection of saddle-shaped disks in $\Theta_i - \cup N_j$.

We call a component of $(E - K) \cap \theta_i^{\pm}$ a *positive/negative saddle-arc*, and call a component of $\partial E \cap S^{\pm}$ a *positive/negative boundary-arc*. Each end of a boundary-arc is called a *junction*. Note that each alternating and non-alternating segment has odd and even number of junctions, respectively. The closure of a component of the intersection of E and the interior of a region is an *inside arc* if its both ends are on bubbles; an *outside arc* if one end is on a bubble and the other is a junction; and an *isolated arc* if both ends are junctions. Let C be a curve of $L \cap S^{\pm}$. We say that C *runs through the center* (resp. *through a side*) *of* $\theta_i^{\pm}$ if C meets θ_i in $L \cap \theta_i^{\pm}$ (resp. in a saddle-arc on $\theta_i^{\pm}$). We say that C *runs through* (resp. *touches*) a segment λ_i if C meets λ_i in a boundary arc in the interior of λ_i whose end points belong to outside or isolated arcs in different regions (resp. in a same region).

Definition. Let E be a spanning surface in basic position of a link diagram L on S and C be a curve of $E \cap S^{\pm}$. We say that C is *standard* if C satisfies the following conditions and that E is in *standard position* if any curve of $E \cap S^{\pm}$ is standard.

(Et1) C bounds a disk in $E \cap B^{\pm}$;

(Et2) C meets a bubble or a segment;
(Et3) C meets a bubble in an arc.
(Et4) C never runs through a side of an upper/lower hemisphere with meeting a segment which is adjacent to the bubble; and
(Et5) C never touches a segment.

2.3. **Band moves along bridges of a spanning surface of a link.** Let E be a spanning surface in basic position of a link diagram L on S. Assume that there is a disk Δ_η in $B^\pm$ such that:

(Br1) $\Delta_\eta \cap E = \eta$ is an arc in $\partial\Delta_\eta$;
(Br2) $\Delta_\eta \cap S^\pm = \zeta$ is an arc in $\partial\Delta_\eta$;
(Br3) $\eta \cup \zeta = \partial\Delta_\eta$ and $\eta \cap \zeta = \partial\eta = \partial\zeta$; and
(Br4) ζ is in a region R.

We call η a *bridge of E*. We say that η is *trivial* if there is a disk Δ' in $E \cap B^\pm$ such that $\partial\Delta' = \eta \cup \zeta'$ with ζ' in R. A *band move along a bridge η* is an isotopy performed by sliding η across Δ_η and past ζ (see Figure 9).

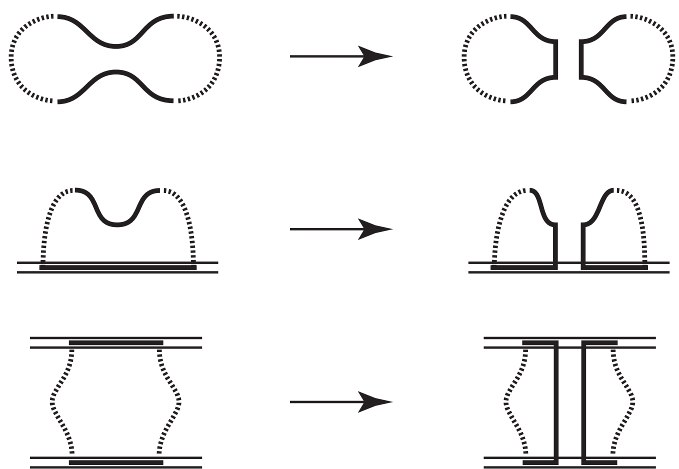

FIGURE 9. Three kinds of bridges and the band moves along the bridges

2.4. **The complexity of a spanning surface of a link.** Let E be a spanning surface in basic position of a link diagram L on S. In general E is not in standard position. However E can be isotoped into standard position if E is incompressible. Here we define the *complexity* of E as the ordered pair (t, u), where t is the number of saddle-intersections of $E \cap \cup\Theta_i$ and u is the total number of components of $E \cap S^\pm$.

Proposition 2.1. ([MT2] **Proposition 2.2, 2.3**) *If E is incompressible and has a minimal complexity, then E is in standard position.*

3. A spanning surface of a link diagram on S

Let E be an incompressible spanning surface in standard position of a link diagram L on S.

3.1. **Special position for E.** Let C be a curve of $E \cap S^{\pm}$. We say that C is *special* if C satisfies the following conditions, and that E is in *special position* if any curve of $E \cap S^{\pm}$ is special.

- (Ep1) C never runs through the center of an upper/lower hemisphere with meeting a segment which is adjacent to the bubble;
- (Ep2) C shares at most 1 junction with an alternating segment; and
- (Ep3) C shares no junctions with a non-alternating segment.

Then we have the following.

Proposition 3.1. *If E is in special position, then the boundary of a neighborhood of E is in standard position.*

Proof. Let M' be a product neighborhood $E \times [1, -1]$ of E which is sufficiently small compare to the tubular neighborhood $\cup N_j$ of L. Take a neighborhood M of E as the union of M' and a neighborhood of $\cup N_j$. Clearly we have that the boundary ∂M of M is in basic position and that $\partial M \cap S^{\pm} \neq \emptyset$. We show only that the positive curves are standard, since we can similarly show that the negative curves are standard.

It is easy to see that $M \cap S^+$ is a neighborhood of the union of positive curves and positive long segments. Note that a positive long segment of length p meets exactly one positive curve if $p \geq 2$ and no positive curves if $p = 1$ from conditions (Ep2) and (Ep3). Therefore we have that

$$\partial M \cap S^+ = \{C_1', C_1'', \cdots, C_m', C_m'', C_{m+1}, \cdots, C_{m+q}\},$$

where $E \cap S^+ = \{C_1, \cdots, C_m\}$ and $C_i' \cup C_i''$ is the boundary of a neighborhood M_i of the union of C_i and the positive long segments which C_i meets, and C_{m+k} is the boundary of a neighborhood of a positive long segment Λ_k of length 1.

Then it is clear that C_{m+k} is standard and that C_i' and C_i'' satisfy condition (Ft1) from the construction. Note that C_i meets a bubble from conditions (Et2), (Ep2) and (Ep3). Therefore C_i' and C_i'' satisfy condition (Ft2). Assume that C_i' or C_i'', say C_i' does not satisfy condition (Ft3). Note that the pair of the curves of $\partial M \cap S^+$ which is closest to the center of an upper hemisphere θ_k^+ is the boundary of M_l such that C_l runs through the center of θ_k^+. Thus C_i' runs through one side of a bubble twice. This implies that C_i does not satisfy condition (Et3), (Et4) or (Ep1), which is a contradiction (see Figure 10). □

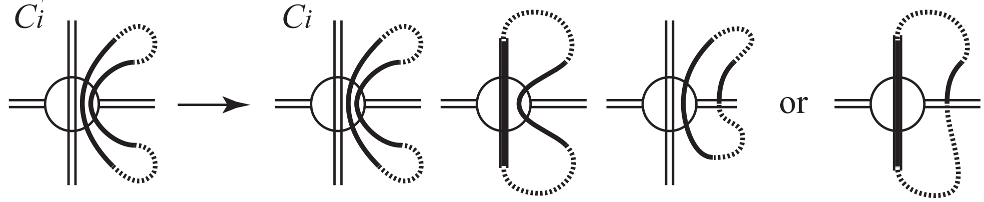

FIGURE 10

3.2. **short arcs of E.** A *short arc of* E is an isolated arc ξ whose ends are on distinct segments which are adjacent to a common crossing. Depending upon how the positive curve containing ξ meets the segments, we have four types of short arcs as in Figure 11, where taking the mirror images do not change their types. The *cut surgery along a short arc* ξ is the operation of replacing E with $E^\xi = E - \xi \times (-1, 1)$, where we isotop E^ξ so that $L^\xi = \partial E^\xi$ be a link diagram on S.

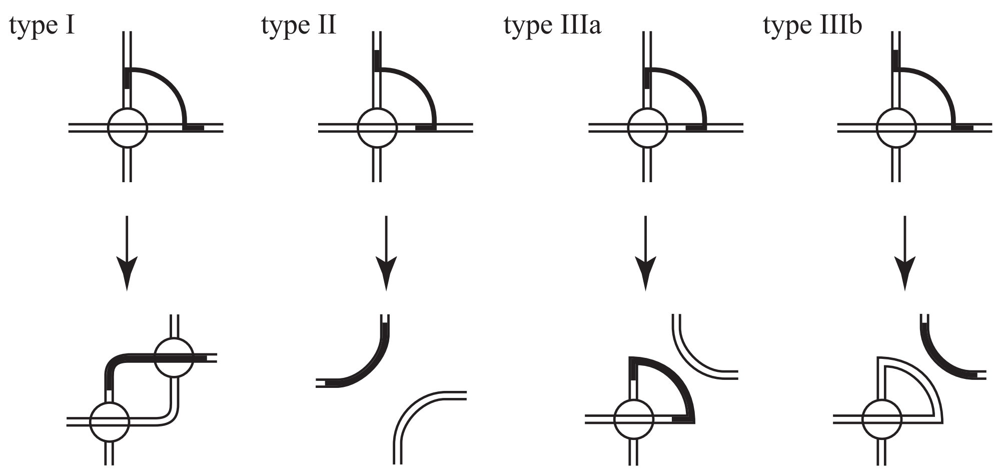

FIGURE 11

3.3. **short bridges of E.** If η is a non-trivial bridge with its ends on distinct segments which are adjacent to a common crossing x, then we call η a *short bridge of* E. The *cut surgery along a short bridge* η is the operation of replacing E with $E^\eta = (E - \eta \times (-1, 1)) \cup \Delta_{\eta \times \{-1\}} \cup \Delta_{\eta \times \{1\}}$ (see Figure 12).

Lemma 3.1. *If a crossing x admits a short arc or a short bridge, then θ_x has no saddle-intersections.*

Proof. Assume otherwise. Then, there exists a curve which does not satisfies condition (Et3) or (Et4). □

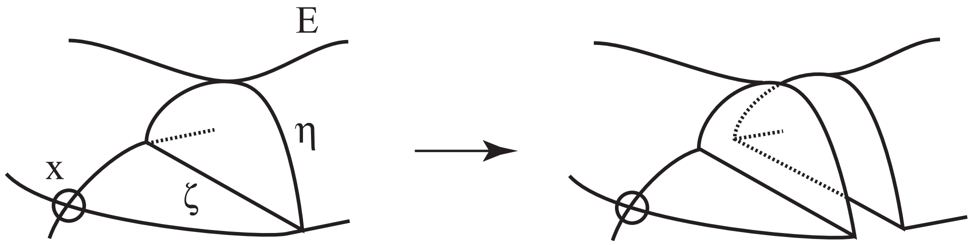

FIGURE 12

3.4. **short cuts of E.** A *short cut* μ of E is a short arc of type III or a short bridge. The *cut surgery along a short cut* μ is the operation of replacing E with $E^\mu = (E - \mu \times (-1,1)) \cup \Delta_{\mu \times \{-1\}} \cup \Delta_{\mu \times \{1\}}$ and we let $L^\mu = \partial E^\mu$. Note that this is equivalent to the cut surgery along a short arc (resp. a short bridge) if μ is a short arc (resp. a short bridge).

If a curve does not satisfies condition (Ep1) at a bubble θ_x, then we say that the curve *has a neck* (at crossing x). Then we have the following.

Lemma 3.2. *If a curve has a neck at a crossing x, then the curve admits a short cut on a region with x, and a short arc of type* II *or a non-trivial bridge on another region with x.*

□

Here we define two types of curves each of which consists of two short arcs and two boundary arcs: a curve of type $\Gamma_1^\pm$ is a curve with one neck which admits a short arc of type II and a short arc of type III; and a curve of type $\Gamma_2^\pm$ is a curve with two necks around a non-trivial clasp each of which admits a short arc of type II (see Figure 13 for curves of type Γ_1^+ and Γ_2^+).

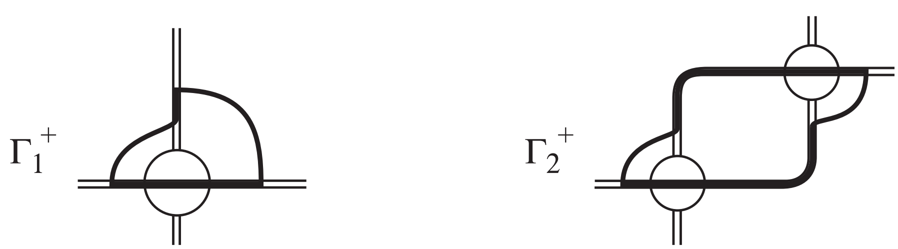

FIGURE 13

Assume that E admits two short cuts μ and μ'. We say that μ and μ' are *equivalent* if they have the ends on same segments. If μ and μ' are disjoint and are not equivalent, then let Ω be the subdisk of E bounded by μ and μ'. Assuming that $\mu \times \{1\}$ and $\mu' \times \{-1\}$ belong to Ω, define $(\Omega^\mu)^{\mu'}$ as $(\Omega^\mu)^{\mu'} = (\Omega - \mu \times (0,1) - \mu' \times (-1,0)) \cup \Delta_{\mu \times \{1\}} \cup \Delta_{\mu' \times \{-1\}}$. We say that μ and μ' are *parallel* if each intersection curve of $(\Omega^\mu)^{\mu'} \cap S^\pm$ has type Γ_1 or Γ_2.

4. A spanning surface of an almost alternating link diagram on S

In this section we study an incompressible spanning surface E of a connected, reduced almost alternating link diagram L on S. Here we assume that L is not the diagram of Figure 14. Thus L has only one dealternator from the following proposition, and we denote the dealternator by δ. We denote the bubble at a crossing x by θ_x. We call a curve of $E \cap S^{\pm}$ *anchored* if it runs through $\theta_\delta^{\pm}$, and otherwise we call the curve *floating*.

Proposition 4.1. *A connected, reduced almost alternating link diagram with more than one dealternator is the diagram depicted in Figure* 14.

Proof. Let α be one of the dealternators of the link diagram. Then α is adjacent to four crossings. Since the crossing change at another dealternator β makes the link diagram alternating and the diagram is reduced, each of the four crossings is β. $\square$

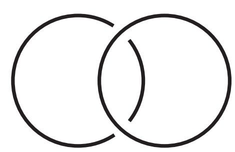

Figure 14

Lemma 4.1. *Assume that E is in basic position. Let C be an innermost curve of $E \cap S^{\pm}$ which is standard and floating. If C admits only trivial bridges, then C has type $\Gamma_1^{\pm}$.*

Proof. We prove only the case when C is positive, since the other case can be shown similarly. Since C is standard, C meets a bubble or a segment. Moreover since C is innermost, C bounds a disk D on S^+ whose interior contains no positive curves. Thus the interior of D does not contain the center of an upper hemisphere, since otherwise D contains the curve running through the center of the upper hemisphere. If C meets bubbles in succession, then D contains the center of one of the two upper hemispheres, since C is floating. Thus C meets a segment λ. Assume that λ is n-adjacent to a bubble θ and that C runs through the center of θ^+. Then C meets a segment λ' which is p-adjacent to θ, i.e. C has a neck, since C is innermost and floating. Thus from Lemma 3.2, C has type Γ_1^+, since C admits only trivial bridges. Next assume that C runs through λ. Then λ is p-adjacent to a bubble θ which is not the dealternator, and C runs through the center of θ^+, since C is innermost and floating. Thus C has type Γ_1^+ as shown above. $\square$

Lemma 4.2. *Assume that E is in standard position. If E either has a floating curve or admits a non-trivial bridge, then E has a short arc of type* II *or* III.

Proof. We show only the case when E has a positive floating curve or a non-trivial positive bridge, since other cases can be shown similarly.

First consider the case when E has a positive floating curve. Take an innermost one C. If C admits only trivial bridges, then C has type Γ_1^+ from Lemma 4.1, and thus we are done. If C admits a non-trivial bridge η, then operate the band move on C along η to have two positive curves C' and C''. Here we have the following.

Claim 1. *We may take η so that C' admits only trivial bridges and that C' and C'' are standard.*

Proof. Let D be the subdisk of E which C spans in B^+. Then C' spans a disk $D' = \Delta_{\eta\times\{1\}} \cup \widetilde{D}'$, where $\widetilde{D}'$ is the component of $D - \eta \times (-1, 1)$ which contains $\eta \times \{1\}$. Here we may assume that D admits only trivial bridges in $\widetilde{D}'$. Therefore if C' admits a non-trivial bridge η', then η' intersects $\eta \times \{1\}$ in one point x. Now let α_η and β_η be the ends of $\eta \times \{1\}$, and let $\alpha_{\eta'}$ (resp. $\beta_{\eta'}$) be the end of η' which is on the boundary of $\widetilde{D}'$ (resp. of $\Delta_{\eta\times\{1\}}$). Let $\eta_{\alpha_\eta x}$ (resp. $\eta_{x\alpha_{\eta'}}$) be the subarc of $\eta \times \{1\}$ (resp. of η') whose ends are x and α_η (resp. $\alpha_{\eta'}$). Then $\eta_{\alpha_\eta x} \cup \eta_{x\alpha_{\eta'}}$ is a non-trivial bridge of D in $\widetilde{D}'$, which is a contradiction.

Since η is a bridge which is non-trivial, it is clear that C' and C'' satisfy conditions (Et1-4). We can also see that C' and C'' satisfy condition (Et5) by taking η so that D admits only trivial bridges in $\widetilde{D}'$ as above. □

From Claim 1 and Lemma 4.1, C' has type Γ_1^+. Therefore C has a short arc of type II or IIIa, since C is obtained by connecting C' with C'' along a subarc of only one of the two short arcs.

Next consider the case when E has no floating curves but admits a non-trivial bridge. Take a curve C which admits a non-trivial bridge. Then we can obtain from C an innermost curve which is standard, floating, and admits only trivial bridges by the band move along a non-trivial bridge. Hence we know that C has a short arc of type II or IIIa as above. □

Proposition 4.2. *Assume that E is in standard position. Then E is in special position if and only if E has no short arcs.*

Proof. Note that if E either has a floating curve or admits a non-trivial bridge, then E has a short arc from Lemma 4.2. Assume that $E \cap S^{\pm}$ has a curve C which is not special. If C has a neck, then C admits a non-trivial bridge or a short arc from Lemma 3.2.

Claim 2. *If C shares more than* 1 *junction with an alternating segment, then E admits a short arc.*

Proof. Assume that C meets an alternating segment λ in two junctions a and b. Here we consider the case when C is positive. The case when C is negative can be shown similarly, and thus we omit it. We see that a curve of $E \cap S^{\pm}$ can share at most 2 junctions with a segment, which are the ends of a boundary arc considering the orientation of the curve and the segment, since the curve is a simple closed curve and E is in standard position. Thus C runs through λ. Let θ_1 be the bubble which λ is n-adjacent to. Take the positive curve C_1 which runs through the center of θ_1^+ and let c be the junction of C_1 and λ. Here we may assume that b and c are neighboring junctions. Moreover then we may assume that C neighbors C_1, i.e. there is no positive curves on S^+ between C and C_1, since otherwise E has a floating curve. Let R be the region which contains the outside or isolated arc of C with b as an end of it. If R has degree 2, then obviously C has a short arc in R. If R has degree no less than 3, then take the bubble θ_2 of R which is p-adjacent to θ_1. Note that θ_1 does not have saddle-intersections, since C (resp. C_1) does not run through a side of θ_1^+ from condition (Et4) (resp. (Et3)) and C neighbors C_1. Moreover we may assume that θ_2 is not at the dealternator, since otherwise either C or C_1 is floating. Therefore $\lambda_{\theta_1\theta_2}$ is alternating, and thus has a junction. Let C_2 be the positive curve with the closest junction to θ_1 on $\lambda_{\theta_1\theta_2}$. Since θ_1 has no saddle-intersections and C neighbors C_1, we have that $C_2 = C$ or that $C_2 = C_1$. In the former case C_2 admits a non-trivial bridge or has a short arc, and in the latter case C_2 has a neck. □

Assume that C is positive and that C meets a non-alternating segment λ. The case when C is negative can be shown similarly, and thus we omit the proof. If λ is p-adjacent to the dealternator, then C runs through λ. Then C does not run through a side of θ_δ^+ from condition (Et4). Thus C either is floating or has a neck. Next if λ is n-adjacent to the dealternator, then there is a negative curve which runs through λ, and thus we can be done similarly.

Conversely assume that E has a short arc. Then take a crossing x which admits a short arc and take the closest short arc ξ to x. Let $C_\xi^\pm$ be the positive/negative curve which contains ξ and let λ be the end segment of ξ which is p-adjacent to x. If λ is non-alternating, then neither C_ξ^+ nor C_ξ^- satisfies condition (Ep3). Next assume that λ is alternating. If ξ has type I or IIIa, then C_ξ^+ has two junctions with λ, since C_ξ^+ does not run through the center of θ_x^-. Thus C_ξ^+ does not satisfies condition (Ep2). If ξ has type II or IIIb, then C_ξ^- does not satisfies condition (Ep1) or (Ep2). □

5. A spanning disk of the trivial knot in almost alternating position I

Let K be the trivial knot in strongly reduced almost alternating position. Let E be a spanning disk for K in basic position with minimal complexity. Thus K has only one dealternator from Proposition 4.1, and E is in standard position from Proposition 2.1. Moreover K is prime from the following proposition.

Proposition 5.1. *A connected, reduced almost alternating diagram of a trivial link is prime.*

Proof. Let L be a non-prime, connected, reduced almost alternating link diagram. From Proposition 4.1, we have that L has only one dealternator. Thus L can be decomposed into a connected alternating diagram L' and a connected almost alternating diagram L'' such that L' is reduced. Then L' does not represent a trivial link from Theorem 1 or from Theorem 2. This implies that L does not represent a trivial link (see [BZ] Corollary 7.5 (b)). $\square$

Lemma 5.1. *The dealternator of K does not admit a short cut.*

Proof. Assume that the dealternator admits a short cut μ. Then the dealternator of K^μ is a nugatory crossing. Since K is prime, K^μ is connected. Thus we obtain a connected alternating diagram of the trivial 2-component link from K^μ by the Reidemeister move of type I. This contradicts to Theorem 1. $\square$

Proposition 5.2. *Non-equivalent disjoint short cuts of E are parallel.*

Proof. Assume that E admits non-equivalent disjoint short cuts μ and μ'. Let $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{E}_3$ be subdisks of E such that $\widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}_3 = E$, $\widetilde{E}_1 \cap \widetilde{E}_2 = \mu$ and $\widetilde{E}_2 \cap \widetilde{E}_3 = \mu'$. Then we obtain disks E_1, E_2 and E_3 by cut surgeries along μ and μ' such that:

$E_1 = (\widetilde{E}_1 - \mu \times (-1, 0)) \cup \Delta_{\mu \times \{-1\}}$;
$E_2 = (\widetilde{E}_2 - \mu \times (0, 1) - \mu' \times (-1, 0)) \cup \Delta_{\mu \times \{1\}} \cup \Delta_{\mu' \times \{-1\}}$; and
$E_3 = (\widetilde{E}_3 - \mu' \times (0, 1)) \cup \Delta_{\mu' \times \{1\}}$.

Let $K_i = \partial E_i$ $(i = 1, 2, 3)$. Note that K^μ and $K^{\mu'}$ are connected, since K is prime. Thus $(K^\mu)^{\mu'}$ is a disconnected almost altnernating diagram of the trivial 3-component link consisting of two connected components of an almost alternating diagram $K_1 \cup K_3$, and a trivial or alternating diagram K_2 from Lemma 5.1, Theorem 3 (1) and Theorem 1. Since μ and μ' do not have the ends on same segments, K_2 has a crossing and thus K_2 is a coiled diagram from Theorem 2.

Let $x_1, \cdots, x_k$ be the crossings of K_2, where x_i and x_{i+1} belong to a common region R_i of degree 2, and x_1 and x_k admits μ and μ' in E, respectively. We claim here that each θ_{x_i} has no saddle-intersections. From Lemma 3.1, we know that neither θ_{x_1} nor θ_{x_k} has saddle-intersections. Thus assume that k is no less than 3 and that θ_{x_2} has a saddle-intersection. Then there is a positive curve which runs through a side of $\theta^+_{x_2}$ and goes into R_1. Then the curve runs through a side of $\theta^+_{x_1}$, since R_1 has degree 2 and E is in standard position. However this contradicts that θ_{x_1} has no saddle-intersections. Now the claim holds by an induction. Therefore the curves of $(E_1 \cup E_3) \cap S^\pm$ are away from K_2. Then E_2 is coiled, i.e. each curve of $E_2 \cap S^\pm$ has type $\Gamma^\pm_1$ or $\Gamma^\pm_2$, since E has a minimal complexity. Hence μ and μ' are parallel. $\square$

Corollary 5.1. *A curve of $E \cap S^\pm$ does not admit non-equivalent disjoint short cuts.*

$\square$

Corollary 5.2. *An innermost floating curve C of $E \cap S^{\pm}$ has type $\Gamma_1^{\pm}$ or $\Gamma_2^{\pm}$.*

Proof. If C admits only trivial bridges, then C has type $\Gamma_1^{\pm}$ from Lemma 4.1. If C admits a non-trivial bridge, then sufficiently many band moves on bridges split C into a set of curves each of which has type $\Gamma_1^{\pm}$. Since C does not admit non-equivalent disjoint short cuts from Corollary 5.1, the set consists of two curves and the bridge of C is a short cut. Therefore C has type $\Gamma_2^{\pm}$, since K is strongly reduced. $\square$

Lemma 5.2. *A crossing of K does not admit two short arcs in a same region.*

Proof. Assume that a crossing admits a short arc ξ of type I or II. Then the two surgered segments of K^{ξ} belong to different components of the trivial 2-component link, implying the conclusion.

Next assume that a crossing admits two short arcs ξ and ξ' in a region. Here we assume that there are no short arcs on the region between ξ and ξ'. Thus both of ξ and ξ' have the same type of IIIa or IIIb from the above. We consider only the former case, since the latter case can be shown similarly.

Let x be a crossing which admits short arcs $\xi = a_1a_2$ and $\xi' = b_1b_2$ of type IIIa in a region R, where a_i and b_i are junctions on a segment λ_i $(i = 1, 2)$ and ξ' is closer to x than ξ on R. Let $\eta = c_1c_2$ be a positive trivial bridge $\xi' \times (-1)$ at x such that c_i is on $C_{\xi'} \cap \lambda_i$, where $C_{\xi'}$ is the positive curve containing ξ'. Let D be the subdisk of E bounded by ξ and η. Take a disk D' in B^+ such that $\partial D = \eta \cup \lambda_{c_1a_1} \cup \xi \cup \lambda_{a_2c_2}$ and $D' \cap E = D' \cap D$, where $\lambda_{c_ia_i}$ (resp. $\lambda_{a_ic_i}$) is the subsegment of λ_i with a_i and c_i as its ends. Then replace D with D' to have another spanning disk for K, which is clearly in basic position with a fewer complexity than that of E. This contradicts the minimality of E. $\square$

Lemma 5.3. *The bubble at the dealternator of K has a saddle-intersection.*

Proof. Assume otherwise and let $C_{\delta}^{\pm}$ be the curve which runs through the center of $\theta_{\delta}^{\pm}$. Then C_{δ}^{+} and C_{δ}^{-} are the only curves which run through θ_{δ}. Let $D_i^{\pm}$ be a disk bounded by $C_{\delta}^{\pm}$ on $S^{\pm}$ such that $D_1^{\pm} \cap D_2^{\pm} = C_{\delta}^{\pm}$ and $D_1^{\pm} \cup D_2^{\pm} = S^{\pm}$ $(i = 1, 2)$. We claim here that the interior of $D_1^{\pm}$ or of $D_2^{\pm}$ contains no positive/negative curves. Assume otherwise and let $C_i^{\pm}$ be an innermost curve in the interior of $D_i^{\pm}$. Since any curve other than $C_{\delta}^{\pm}$ is floating, $C_i^{\pm}$ admits a short cut $\eta_i^{\pm}$ from Corollary 5.2. However then $\eta_1^{\pm}$ and $\eta_2^{\pm}$ are not parallel, since we have $C_{\delta}^{\pm}$ on $S^{\pm}$ between $\eta_1^{\pm}$ and $\eta_2^{\pm}$. This contradicts Proposition 5.2.

Therefore $C_{\delta}^{\pm}$ bounds a disk $D^{\pm}$ on $S^{\pm}$ whose interior contains no positive/negative curves. Let x be the crossing which is p-adjacent to the dealternator so that $\lambda_{x\delta}$ meets D^{+}. If C_{δ}^{+} meets $\lambda_{x\delta}$, then the dealternator admits a short cut from Lemma 3.2. This contradicts Lemma 5.1. Thus $\lambda_{x\delta}$ is contained in the interior of D^{+}. Since the interior of D^{+} contains no positive curves, $\lambda_{x\delta}$ has no junctions and then C_δ^- runs through the center of θ_x^-. Thus θ_x has no saddle-intersections, since C_δ^- bounds D^-. Hence C_δ^+ runs through the center of θ_x^+. Then we can take a disk Δ in B^+ such that $\partial\Delta = \alpha \cup \beta \cup \lambda_{x\delta} \cup \gamma$, where $\Delta \cap E = \alpha$, $\Delta \cap S^+ = \beta \cup \lambda_{x\delta} \cup \gamma$ and β (resp. γ) is on θ_x^+ (resp. θ_δ^+). Thus both of $\eta_1 = \alpha \times \{-1\}$ and $\eta_2 = \alpha \times \{1\}$ are positive bridges. Then $(K^{\eta_1})^{\eta_2}$ is a connected almost alternating diagram of the trivial 3-component link, since K is strongly reduced. However this contradicts Theorem 3 (1). □

Lemma 5.4. *The dealternator of K does not admit a short arc.*

Proof. Let x be a crossing which admits a short arc. Then θ_x does not have a saddle-intersection from Lemma 3.1. Thus x is not the dealternator from Lemma 5.3. □

5.1. **A spanning disk with a short arc.**

Proposition 5.3. *If E has a short arc ξ of type* I*, then K^ξ is the trivial* 2*-component link in prime, strongly reduced almost alternating position.*

Proof. Let x be a crossing which admits ξ and take a look at the diagram for K^ξ in Figure 11. Since x is not the dealternator from Lemma 5.4, we can see that K^ξ is almost alternating and strongly reduced. Moreover since K^ξ is clearly connected, K^ξ is prime from Proposition 5.1. □

Lemma 5.5. *If E has a short arc of type* I*, then E has no other short arcs.*

Proof. If E has a short arc ξ of type I and another short arc ξ', then $(K^\xi)^{\xi'}$ is a connected almost alternating diagram of the trivial 3-component link from Lemma 5.4 and Proposition 5.3. This contradicts Theorem 3 (1). □

Proposition 5.4. *E has a short arc of type* II *or* III *if and only if E admits non-equivalent disjoint short cuts.*

Proof. If E admits a short cut, then E has a short arc of type II or III from Lemma 4.2, since a short cut is a short arc of type III or a short bridge.

Next assume that E has a crossing x which admits a short arc ξ of type II. Let $C_\xi^\pm$ be the positive/negative curve containing ξ and let λ_i^+ (resp. λ_i^-) be the segment which is p-adjacent (resp. n-adjacent) to x $(i = 1, 2)$, where λ_1^+ and λ_1^- are the end segments of ξ. Since $C_\xi^\pm$ runs through the center of $\theta_x^\pm$ from Lemma 5.2, $C_\xi^\pm$ has a short cut $\mu^\pm$ whose ends are on $\lambda_1^\pm$ and $\lambda_2^\mp$. Then μ^+ and μ^- are non-equivalent and disjoint, since μ^+ and μ^- have different end segments.

If E has a floating curve, then an innermost floating curve has type Γ_1 or Γ_2 from Corollary 5.2, and thus a short arc of type II. Therefore we are done from the above. We complete the proof by showing the following claim.

Claim 3. *If E has a short arc of type* III*, then E has a short arc of type* II.

Proof. We show only the case when E has a short arc of type IIIa, since the other case can be shown similarly. In addition we may assume that E has no floating curves from the above. Let x be a crossing which admits a short arc ξ of type IIIa and let C_ξ be the positive curve containing ξ. Then C_ξ runs through the center of θ_x^+ from Lemma 5.2. Let y and z be the crossings which are p-adjacent to x with segment λ_{yx} containing an end of ξ.

Assume that y is the dealternator. Then C_ξ runs through the center of θ_y^+ from condition (Et4). However then, C_ξ has a neck and thus has a short cut from Lemma 3.2, contradicting Lemma 5.1.

Assume that z is the dealternator. Consider the case when C_ξ runs through the center of θ_z^+. From Lemma 5.3, θ_z has a saddle-intersection. Then a positive curve running through θ_z^+ on the side of λ_{zx} runs through θ_x^+ on the side of λ_{zx}. However this contradicts Lemma 3.1. Next consider the case when C_ξ runs through a side of θ_z^+. If there is a positive curve running through a side of θ_z^+ closer to λ_{zx} than C_ξ, then we obtain a contradiction to Lemma 3.1 as above. Otherwise C_ξ bounds a disc D on S^+ such that the center of θ_z^+ is in the interior of D and $S^+ - D$ has no positive anchored curves. Then the curve running through the center of θ_y^+ is in $S^+ - D$, and thus floating. This contradicts the assumption.

Now assume that neither y nor z is the dealternator. Thus both of λ_{yx} and λ_{zx} are alternating segments. Let a_1, a_2 and a_3 be consecutive junctions on λ_{yx} such that $\lambda_{a_1a_2}$ is $\lambda_{yx} \cap C_\xi$ with a_2 an end of ξ. Then the positive curve C_a which runs through a_3 is not C_ξ but neighbors C_ξ, i.e. there are no positive curves on S^+ between C_a and C_ξ, since E has no floating curves. Next let C_b be the positive curve which runs through the closest junction b to θ_x on λ_{zx}. Then C_b is not C_ξ, since otherwise $C_\xi = C_b$ admits non-equivalent disjoint short cuts at x, contradicting Corollary 5.1. Thus C_b neighbors C_ξ, since no curves run through a side of θ_x and E has no floating curves. Therefore C_ξ has two neighbors in a same component of $S - C_\xi$, and thus we have that $C_a = C_b$. However this is impossible, since the segment running through θ_y^+ and the segment running through θ_z^+ belong to different components in K^ξ. □

□

5.2. **A spanning disk with non-equivalent disjoint short cuts.** Let X be a set of mutually non-equivalent disjoint short cuts such that any short cut of E either intersects or is equivalent to an element of X. From Proposition 5.2, there is a pair of short cuts, say η_l and η_r, which bounds a subdisk Ω of E containing all the elements of X. We define the *extract surgery on E* as the operation of getting rid of $(\Omega^{\eta_l})^{\eta_r}$ from $(E^{\eta_l})^{\eta_r}$, and denote the result by E^* and ∂E^* by K^*.

Proposition 5.5. *The extract surgery on E is well-defined.*

Proof. Assume that there is a short cut η' which is not equivalent to any element of X. Then there is a crossing x which admits a short cut η of X which intersects η'. Let C be the curve of $E \cap S^\pm$ admitting both of η and η'. If η is neither η_l nor η_r, then C has type $\Gamma_1^\pm$ or $\Gamma_2^\pm$ from Proposition 5.2. However this is a contradiction, since a curve with type $\Gamma_1^\pm$ or $\Gamma_2^\pm$ does not admit non-equivalent short cuts. Thus assume that η is η_l. Then C has a neck and a short arc of type II at x from Proposition 5.2. Thus η' has a common end segment with η. Let a and b (resp. a' and b') be the ends of η (resp. η'), where a and a' are on a same segment. Then we can obtain non-equivalent disjoint short cuts, one of which has ends a and b' and the other has ends a' and b by smoothing the intersection of η and η'. However this contradicts Corollary 5.1. $\square$

Take an arc $\widetilde{\psi}$ on Ω which connects η_l and η_r. Let ψ be a projection of $\widetilde{\psi}$ on $S^+ \cap S^-$ and call ψ a *band-trace* for E.

Proposition 5.6. *Let L be a connected, reduced almost alternating diagram of the trivial 2-component link. If L is not strongly reduced, then L is the diagram of Figure 14.*

Proof. Apply the Reidemeister move of type II to L to have another diagram L'. Then L' is alternating or trivial. In the former case L' is disconnected from Theorem 1. Since L is prime from Proposition 5.1, each component of L' is reduced. This contradicts Theorem 2. In the latter case we have the conclusion. $\square$

Proposition 5.7. *Assume that E admits non-equivalent disjoint short cuts. Then K^* is the trivial 2-component link in prime, strongly reduced almost alternating position and E^* is in special position.*

Proof. Take a look at E and recall notations in the definition of the extract surgery. Since K is prime, there uniquely exists a region which contains the two end segments of η_i $(i = l, r)$. We denote the region by R_i. Define crossings x_i, y_i and z_i of $(E-\Omega)\cap S$ if $\deg R_i = 2$, and of ∂R_i if $\deg R_i \geq 3$ as follows: x_i be the crossing which admits η_i; and y_i (resp. z_i) be the crossing which is p-adjacent (resp. n-adjacent) to x_i. Let C_{η_i} be the curve which admits η_i, where we take the one which is not contained in Ω if η_i is a short arc.

Since K is almost alternating and prime, we have that $x_r \neq y_l$, z_l and equivalently that $x_l \neq y_r$, z_r. Next assume that $y_l = z_r = \delta$ or that $y_r = z_l = \delta$. Take a look at the region R whose boundary crossings are δ, x_l, x_r and other crossings of Ω. From Lemma 5.3, there is a positive curve C which runs through a side of θ^+_δ and goes into R. However then C meets neither $\lambda_{\delta x_l}$ nor $\lambda_{\delta x_r}$ from condition (Et4), and C does not meet Ω from the proof of Proposition 5.2. Hence the dealternator is not adjacent to both of x_l and x_r, and thus we may assume that x_r is not adjacent to the dealternator. In addition, we may assume that C_{η_l} is positive, since the other case can be shown similarly.

Here we define C^+_l and C^-_l. Let C^+_l be C_{η_l}. If R_l has degree 2 (resp. has degree no less than 3 and $\lambda_{x_l z_l}$ has a junction), then let C^-_l be the curve of $(E-\Omega)\cap S^-$ sharing a junction with C^+_l on the segment facing R_l which is p-adjacent (resp. n-adjacent) to x_l; and if R_l has degree no less than 3 and $\lambda_{x_l z_l}$ has no junctions, then let C^-_l be the curve which runs through θ_{z_l} the closest side to $\lambda_{x_l z_l}$, where such a curve exists from Lemma 5.3, since now z_l is the dealternator.

Next define C_r^+ and C_r^- as follows; if C_{η_r} is a positive curve, then let C_r^+ be C_{η_r} and let C_r^- be the curve of $(E-\Omega)\cap S^-$ sharing a junction with C_r^+ on the segment facing R_r which is n-adjacent (resp. p-adjacent) to x_r if R_r has degree no less than 3 (resp. degree 2); and if C_{η_r} is a negative curve, then let C_r^- be C_{η_r} and let C_r^+ be the curve of $(E-\Omega)\cap S^+$ sharing a junction with C_r^- on the segment facing R_r which is p-adjacent (resp. n-adjacent) to x_r if R_r has degree no less than 3 (resp. degree 2).

Claim 4. *E^* has no floating curves.*

Proof. It is sufficient to show that E does not have a second-innermost floating curve, since every innermost floating curve of E belongs to $\Omega\cap S^{\pm}$. If $E\cap S^{\pm}$ has no innermost floating curves, then we are done. Thus assume otherwise. From the proof of Proposition 5.2, no curves of $(\overline{E-\Omega})\cap S^{\pm}$ run through the bubble at a crossing of $\Omega\cap S$. Therefore if E has a second-innermost floating curve, then it is $C_l^{\pm}=C_r^{\pm}$. However this is impossible, since: $C_l^+\cap(E-\Omega)$ and $C_r^+\cap(E-\Omega)$ belong to different components of $E-\Omega$; $C_l^-\cap(E-\Omega)$ and $C_r^-\cap(E-\Omega)$ belong to different components of $E-\Omega$ if R_l has degree 2 or if R_l has degree no less than 3 and $\lambda_{x_l y_l}$ has a junction; and $C_l^-=C_r^-$ runs through the dealternator if R_l has degree no less than 3 and $\lambda_{x_l y_l}$ has no junctions. □

Here assume that K^* is reduced and E^* is in standard position. Consider E^* with a band-trace ψ. Here note that K^* is a connected almost alternating diagram of the trivial 2-component link. Assume that K^* is not strongly reduced. Then K^* is the diagram of Figure 14 from Proposition 5.6. Since each of the four regions of S^+ with K^* is a trivial clasp and ψ is in one of the four regions, K is not strongly reduced, either. Thus reducedness of K^* implies strongly reducedness of K^*. Next if K^* has a short arc ξ, then E^* has a short cut η from the assumption, Lemma 5.5 and Proposition 5.4, since the extract surgery does not creat new non-boundary arcs for E^*. Therefore $(K^*)^{\eta}$ is a connected almost alternating diagram of the trivial 3-component link from Lemma 5.1, since K^* is prime from Proposition 5.1. This contradicts Theoerm 3 (1). Thus E^* is in special position from Proposition 4.2. Therefore it is sufficient to show the following two claims.

Claim 5. *K^* is reduced.*

Proof. It is sufficient to show that each of R_l and R_r has degree no less than 3. Take a look at $E\cap S^{\pm}$. Since the proof is similar to the proof of Claim 3, we omit the detail in the following.

Assume that R_l has degree 2. Then y_l is not the dealternator, since otherwise C_r^+ is floating, implying a contradiction to Claim 4. Thus $\lambda_{x_l y_l}$ is alternating, and we have a positive curve which has the closest junction to x_l on segment $\lambda_{x_l y_l}$. Then we obtain a contradiction by considering the curve with C_l^+ and C_r^+.

Assume that R_r has degree 2 and C_{η_r} is positive. Then we have a positive curve which has the closest junction to x_r on segment $\lambda_{x_r y_r}$, since y_r is not the dealternator. Then we obtain a contradiction by considering the curve with C_l^+ and C_r^+.

Now assume that R_l has degree no less than 3, and that R_r has degree 2 and C_{η_r} is negative. Note that C_r^+ runs through the center of $\theta_{x_r}^+$. Let C be the negative curve which shares a junction with C_r^+ on segment $\lambda_{x_r z_r}$. Assume that $\lambda_{x_l z_l}$ has no junctions. Since no curves of $(E - \Omega) \cap S^+$ run through a crossing of $\Omega \cap S$, $C_r^- = C_l^-$ and thus C_r^- runs through $\theta_{z_l}^- = \theta_\delta^-$ on the closest side to $\lambda_{z_l x_l}$. Thus C is floating, implying a contradiction to Claim 4. If $\lambda_{x_l z_l}$ has a junction, then we obtain a contradiction as above by considering the curves C_l^-, C_r^- and C. □

Claim 6. *E^* is in standard position.*

Proof. Take a look at $E \cap S^{\pm}$. First we show that y_l is not the dealternator. Assume otherwise. Note that C_l^+ is anchored from Claim 4. If C_l^+ runs through a side of $\theta_{y_l}^+$, then C_l^+ does not satisfies condition (Et4). If C_l^+ runs through the center of $\theta_{y_l}^+$, then C_l^+ admits a neck at y_l, and thus y_l admits a short cut from Lemma 3.2. This contradicts Lemma 5.1.

Since C_{η_l} and C_{η_r} are the only curves which are changed by the extract surgery, it is sufficient to show that these two curves are standard after the surgery. This is done by showing that C_{η_l} runs through the center of $\theta_{y_l}^+$ in E and that C_{η_r} runs through the center of $\theta_{y_r}^+$ (resp. $\theta_{z_r}^-$) if C_{η_r} is positive (resp. negative). If any of these claims does not hold, we can show that y_l, y_r or z_r admits a short cut from Lemma 5.5 by following the proof of Claim 2. However this is a contradiction. □

□

6. A spanning disk of the trivial knot in almost alternating position II

Let K be the trivial knot in strongly reduced almost alternating position and let E be a spanning disk for K in basic position with minimal complexity. We assume here that E has a short arc ξ. Then after a proper surgery, K^* is the trivial 2-component link in prime, strongly reduced almost alternating position from results in Section 5, where we denote K^ξ by K^* if ξ has type I, since it causes no contradiction from Lemma 5.5. Therefore K^* has a flyped tongue from Theorem 3 (2). We show in this section that we can operate the inverse of the surgery on E^* without harming the flyped tongue of K^*.

Define the *left* and *right* side of a non-alternating segment $\lambda_{\delta q}$ by running along $\lambda_{\delta q}$ from crossing q to the dealternator δ. Denote the region which faces $\lambda_{\delta q}$ from the left (resp. right) side by O_l^q (resp. O_r^q). Denote the region sharing with O_i^q a segment ($\neq \lambda_{\delta q}$) which is adjacent to δ (resp. q) by P_i^q (resp. Q_i^q) ($i = l,\ r$). We say that $\lambda_{\delta q}$ has a *flype-component* on the left (resp. right) side or a flype-component with a *flype-crossing* x if P_l^q and Q_l^q

(resp. P_r^q and Q_r^q) share a common crossing x. If $\lambda_{\delta q}$ has flype-components on both sides, K^* has a flyped tongue. Then we call the pair of O_l^q and O_r^q the *core* of a flyped tongue.

Take a look at a flype-component of K^* with a flype-crossing x. We may denote O_i^q, P_i^q and Q_i^q by O_x^q, P_x^q and Q_x^q ($i = l$ or r). We call the 2-tangle W_x^q with δ, q and x as its ends, the *flype-tangle of* $\lambda_{\delta q}$ with x. We say that W_x^q is *trivial* if W_x^q consists of two segments $\lambda_{x\delta}$ and λ_{xq}. In the following we omit q of regions unless we need to emphasize.

6.1. **A spanning disk with a short arc of type I.**

Proposition 6.1. *If E has a short arc ξ of type* I*, then K has a flyped tongue.*

Proof. From Proposition 5.3, K^* is the trivial 2-component link in prime, strongly reduced almost alternating position. Thus K^* has a non-alternating segment $\lambda_{\delta q}$ which has a flype-component with crossing x_l (resp. x_r) on the left (resp. right) side from Theorem 3 (2). Consider the inverse operation of the cut surgery along ξ paying attention only on K^*, which can be regarded as an operation of smoothing one of the two crossings of a non-trivial clasp Σ of K^*. If none of δ, q, x_l and x_r belongs to Σ, then we see that K also admits the flype-components and thus we are done.

Since the dealternator cannot belong to Σ, it is sufficient to consider the cases when q or x_l belongs to Σ. Assume that q belongs to Σ. Since K^* is strongly reduced, each region facing $\lambda_{\delta q}$ has degree no less than 3. Thus we may assume that Q_l is Σ. Then flype-tangle W_{x_l} is trivial, and thus K is not strongly reduced no matter which crossing of q and x_l we smooth.

Next assume that x_l belongs to Σ but q does not. Since δ does not belong to Σ, one of the two regions ($\neq P_l$, Q_l) which has x is Σ. In either case, P_l and Q_l share the other crossing y of Σ. Then y is another flype-crossing of $\lambda_{\delta q}$ on the left side. Therefore K has a flype-component of $\lambda_{\delta q}$ with x_l if we smooth y, or with y if we smooth x_l. □

6.2. **A spanning disk with non-equivalent disjoint short cuts.**

Next we consider the case when E has a short arc of type II or III, i.e. admits non-equivalent disjoint short cuts from Proposition 5.4. Then K^* is the trivial 2-component link in prime, strongly reduced almost alternating position and E^* is in special position from Proposition 5.7. We consider only the case when K^* has a flyped tongue as the left of Figure 2, since the other case can be shown similarly. Since E^* is in special position, E^* admits neither a floating curve nor a non-trivial bridge from Lemma 4.2 and Proposition 4.2. Thus first we have the following.

Lemma 6.1. *A curve of $E^* \cap S^{\pm}$ meets a region in an arc.*

Proof. Otherwise E^* admits a non-trivial bridge. □

Since E^* does not have a floating curve, curves with a same sign are concentric on $S^\pm$. Let C_δ, C_1 and C_2 be curves of $E \cap S^\pm$, where C_δ is the curve running through the center of $\theta^\pm_\delta$. Then we say that $C_\delta > C_1 > C_2$ if C_1 bounds a disk in a component of $S^\pm - C_\delta$ which contains C_2. If C_1 and C_2 are in different components, we say that $C_1 > C_\delta > C_2$ or $C_2 > C_\delta > C_1$.

In the following we denote by C_x the positive curve which runs through the center of θ^+_x of a crossing x. From Lemma 6.1, we know a curve precisely if we are given which crossings and how the curve runs through, since K^* is prime and E^* is in special position. Thus we may denote a curve only by giving crossings with order which the curve runs through, where we denote a crossing by itself (resp. itself with a bar on top) if the curve runs through a side (resp. the center) of the upper hemisphere of the crossing. We denote an arc of a positive curve similarly, e.g. an inside arc by γ_{xy}; an outside arc by $\gamma_{\bar{x}y}$; and an isolated arc by $\gamma_{\bar{x}\bar{y}}$.

Lemma 6.2. *Let R be a region which has the dealternator δ. Let y_1 be the crossing of R which is p-adjacent to δ and let y_i be the crossing of R which is n-adjacent to y_{i-1} $(i = 2, 3, \cdots, n)$ so that y_n is n-adjacent to δ. Then we have that C_{y_j} contains an outside arc $\gamma_{\bar{y}_j\delta}$ in R and that $C_{y_n} = C_\delta > C_{y_k} > C_{y_l}$ if $k > l$ $(j, k, l = 1, \cdots, n-1)$.*

Proof. From Lemma 6.1, $C_{y_n} = C_\delta$ meets R only along segment $\lambda_{\delta y_n}$. Thus each C_{y_j} runs through θ^+_δ on the side of $\lambda_{\delta y_1}$ $(j = 1, \cdots, n-1)$. We have the conclusion from Lemma 6.1. □

Denote the number of crossings of a flype-tangle W_x by $\deg W_x$, and the number of crossings of W_x which belong to a region R by $\deg W_x|_R$. Since K^* is reduced, W_x is trivial if and only if $\deg W_x = 0$. Let p be the crossing of P_x which is n-adjacent to δ. Denote by U_x the region ($\neq P_x$, Q_x) which has x but has neither a crossing of W_x nor $\lambda_{\delta q}$. Let u (resp. v) be the crossing of U_x which is n-adjacent (resp. p-adjacent) to x.

Lemma 6.3. *Let x be a flype-crossing of $\lambda_{\delta q}$ of K^*. Then we have the followings.*

(1) *If $\deg W_x = 0$, then $C_x = \bar{x}\delta$.*
(2) *If $\deg W_x \neq 0$ and $\deg P_x \geq \deg W_x|_{P_x} + 3$, then $C_x = \bar{x}q\delta$.*

Proof. (1) Since K^* is strongly reduced, we have that $p \neq x$. Thus applying Lemma 6.2 to P_x and O_x, we have the conclusion. (2) Since $\deg P_x \geq \deg W_x|_{P_x} + 3$, we have that $p \neq x$, and thus C_x contains an outside arc $\gamma_{\bar{x}\delta}$ in P_x from Lemma 6.2. Since W_x is not trivial, W_x has a crossing x_1 which is p-adjacent to q. Let x_i be the crossing of W_x which is p-adjacent to x_{i-1} and belongs to Q_x $(i = 2, \cdots, n-1)$ so that $x_n = x$. From Lemma 6.2, C_{x_1} contains an outside arc $\gamma_{\bar{x}_1\delta}$ in O_x. If $C_\delta > C_{x_2} > C_{x_1}$, then we have that $C_{x_2} = \bar{x}_2 q\delta\cdots$. If $C_{x_2} > C_\delta > C_{x_1}$, then C_δ goes into W_x and out from W_x either to O_x, to P_x or to Q_x. Either case contradicts Lemma 6.1. Then inductively we obtain that $C_{x_n} = C_x = \bar{x}q\delta\cdots$. Therefore we can conclude that $C_{x_n} = C_x = \bar{x}q\delta$. □

Lemma 6.4. *Let x be a flype-crossing of $\lambda_{\delta q}$ of K^* and assume that* $\deg P_x \geq \deg W_x|_{P_x} + 4$*. If* $\deg Q_x = \deg W_x|_{Q_x} + 2$ *or* $\deg U_x \geq 3$*, then* $C_u = \bar{u}xq\delta$*.*

Proof. Since $\deg P_x \geq \deg W_x|_{P_x} + 4$, we have that $p \neq x, u$. From Lemma 6.3, we have that $C_x = \bar{x}\delta$ if $\deg W_x = 0$ and that $C_x = \bar{x}q\delta$ if $\deg W_x \neq 0$. Moreover we have that $C_\delta > C_u > C_x$ from Lemma 6.2. Thus we have that $C_u = \bar{u}\delta q \cdots$. If $\deg Q_x = \deg W_x|_{Q_x} + 2$, then we have that $v = q$ and thus C_u runs through a side of θ_x^+. Hence we have the conclusion. Next assume that $\deg Q_x \geq \deg W_x|_{Q_x} + 3$ and $\deg U_x \geq 3$. Then we have that $v \neq u, q$. If $C_\delta > C_u > C_v$, then $C_v = u\delta q \cdots$, since $C_u = \bar{u}\delta q \cdots$. However then C_v admits a non-trivial bridge in Q_x or in U_x, which contradicts Lemma 6.1. If $C_\delta > C_v > C_u$ or $C_v > C_\delta > C_u$, then C_u runs through a side of θ_x^+. Hence we have the conclusion. □

Lemma 6.5. *Let x be a flype-crossing of $\lambda_{\delta q}$ of K^*. If W_x is not trivial and does not have a flype-crossing for $\lambda_{\delta q}$, then θ_x has a saddle-intersection.*

Proof. Since W_x is not trivial, there is a crossing x_1 in W_x which is p-adjacent to x. Then we have that $C_\delta > C_x > C_{x_1}$ and that C_{x_1} contains $\gamma_{\bar{x}_1\delta}$ from Lemma 6.2. Moreover since W_x has no flype-crossings for $\lambda_{\delta q}$, there exists a crossing x_2 in W_x which is p-adjacent to x_1. Then we have that $C_\delta > C_x > C_{x_2} > C_{x_1}$ from Lemma 6.1. Therefore C_{x_2} runs through a side of θ_x^+. Thus we are done. □

Proposition 6.2. *If E admits non-equivalent disjoint short cuts, then K has a flyped tongue.*

Proof. From Proposition 5.7, K^* is the trivial 2-component link in prime, strongly reduced almost alternating position. Thus K^* has a flyped tongue from Theorem 3 (2). Consider K^* with a band-trace ψ of E. Note that ψ is properly embedded in $R - (E - K)$ for a region R. Since the crossings of K^* are preserved by the inverse operation of the extract surgery, we use the same notations in K for the crossings of K^*.

Claim 7. *Let x be a flype-crossing of $\lambda_{\delta q}$ of K^*. If $E^* \cap S^+$ has inside arcs $\gamma_{\delta q}$ in O_x and γ_{xq} in Q_x, then K admits a flype-component of $\lambda_{\delta q}$ with x.*

Proof. From Lemma 6.1, the negative curve containing arc $\gamma_{\delta q}$ is δq. Then ψ does not have an end on $\lambda_{\delta q}$, and thus K has $\lambda_{\delta q}$. Also from Lemma 6.1, the curve which shares a saddle-intersection at θ_x with the curve containing γ_{xq} contains an inside arc $\gamma_{x\delta}$ in P_x. Since ψ does not meet non-boundary arcs, x faces δ (resp. q) through $\gamma_{x\delta}$ (resp. γ_{xq}) in S with K. □

Claim 8. *If $\lambda_{\delta q}$ admits two flype-crossings on one side in K^*, then $\lambda_{\delta q}$ admits a flype-component on the side in K.*

Proof. Let x and y be flype-crossings of $\lambda_{\delta q}$ such that W_y has x. We only consider the case when W_x is trivial, since the other case can be shown similarly. We have that $C_x = \bar{x}\delta$ and $C_y = \bar{y}q\delta$ from Lemma 6.3. Let D_x (resp. D_y) be the disc spanned by C_x (resp. C_y) in $S^+ - C_\delta$. Then we can take arcs α_x in $(P_x \cap D_x) - E$ (resp. β_x in $(O_x \cap D_x) - E$) with ends on θ_x and θ_δ (resp. θ_q), and α_y in $(P_x \cap (D_y - D_x)) - E$ (resp. β_y in $(Q_x - D_y) - E$) with ends on θ_y and θ_δ (resp. θ_q). Note that ψ is properly embedded in

$R-(E-K)$ for a region R. Thus y faces δ (resp. q) through α_y (resp. β_y) in K if ψ is in $D_x-(E-K)$ or in $(D_y\cap Q_x)-(E-K)$, and x faces δ (resp. q) through α_x (resp. β_x) in K otherwise. $\square$

Claim 9. *Let x be a flype-crossing of $\lambda_{\delta q}$ of K^*. If W_x is not trivial and C_δ does not run through a side of θ_x^+, then $\lambda_{\delta q}$ admits a flype-component in K on the same side as x.*

Proof. If W_x has a flype-crossing for $\lambda_{\delta q}$, we are done from Claim 8. Thus assume otherwise. Then there are positive curves C ($\neq C_\delta$) and C' ($\neq C_\delta$) which share a saddle-intersection at θ_x such that $C_\delta > C > C_x > C'$ from Lemma 6.5 and Lemma 6.2. Therefore C has γ_{xq} in Q_x and $\gamma_{q\delta}$ in O_x. Hence we are done by Claim 7. $\square$

Claim 10. *Let x be a flype-crossing of $\lambda_{\delta q}$ of K^*. If $\deg P_x \geq \deg W_x|_{P_x}+4$, then K admits a flype-component of $\lambda_{\delta q}$ on the same side with x.*

Proof. If $\deg U_x = 2$, then $\lambda_{\delta q}$ admits a flype-component with x or with $u=v$ from Claim 8. If $\deg U_x \geq 3$, then we have that $C_u = \bar{u}xq\delta$ from Lemma 6.4. Thus we are done by Claim 7. $\square$

Now take a look at a flyped tongue of K^*. Let x_l (resp. x_r) be a flype-crossing of $\lambda_{\delta q}$ of K^* on the left (resp. right) side. We divide the case with respect to the degrees of P_l, P_r W_l and W_r. First we have that $\deg P_i \neq 2$, since K^* is strongly reduced ($i=l$ or r). Second if $\deg P_l \geq \deg W_l|_{P_l}+4$ and $\deg P_r \geq \deg W_r|_{P_r}+4$, then K admits a flyped tongue from Claim 10. Third consider the case when $\deg W_l = \deg W_r = 0$. Then we may assume that $\deg P_l = 3$, since we are done from Claim 10 if $\deg P_l \geq 4$ and $\deg P_r \geq 4$. Thus we obtain an alternating diagram $\bar{K}^*$ of the trivial 2-component link by an untongue move and the Reidemeister move of type II (see Figure 15). Then $\bar{K}^*$ is disconnected from Theorem 1. Thus K^* is a diagram of Figure 16 from Theorem 2, since K^* is prime. Since ψ is in a region and connects different components of K^*, K has a flyped tongue. Fourth consider the case when $\deg P_i = 3$ with $\deg W_i = 0$ for $i=l$ or r. We are done, since this case is equivalent to the third case. Fifth consider the case when $\deg P_l = \deg W_l|_{P_l}+3$ and $\deg P_r = \deg W_r|_{P_r}+3$. Here we may additionally assume that $\deg W_l \neq 0$ and $\deg W_r \neq 0$ from the above. Thus it is sufficient to show that K has a flype-component on the left side under the assumption that C_δ runs through a side of $\theta_{x_l}^+$ from Claim 9. Then note that $\deg Q_l \geq \deg W_l|_{Q_l}+3$, since otherwise C_δ is not standard. Thus there is a crossing w ($\neq \delta$) which is n-adjacent to q. Then we have that $C_\delta > C_w > C_{x_r}$, since $C_{x_r} = \bar{x}_r q\delta$ from Lemma 6.3 and thus $C_w = \bar{w}q\delta p\cdots$. Therefore the curve sharing a saddle-intersection at θ_p with C_w is $C = px_lq\delta$, since $C_{x_l} = \bar{x}_l q\delta$ from Lemma 6.3. Hence we are done from Claim 7.

Note that it is impossible to have that $\deg P_l = \deg W_l|_{P_l}+2$ and $\deg P_r = \deg W_r|_{P_r}+2$, since δ cannot be n-adjacent to both x_l and x_r. Therefore we are left with the following cases from the symmetry:

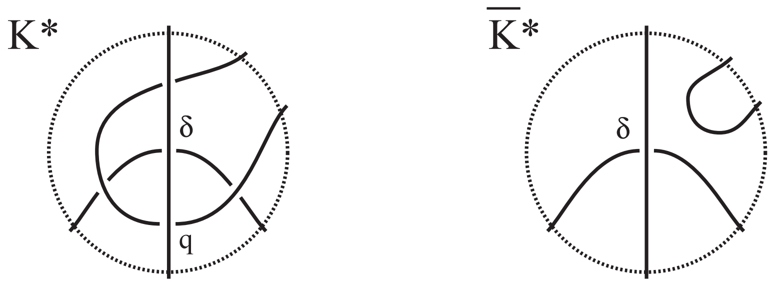

FIGURE 15

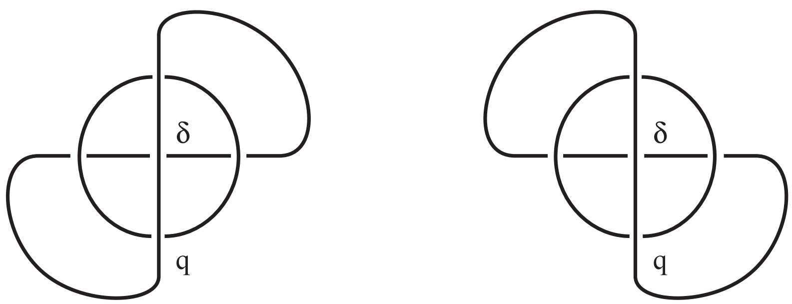

FIGURE 16

$\deg P_l = \deg W_l|_{P_l} + 3$ and $\deg P_r \geq \deg W_r|_{P_r} + 4$ ($\deg W_l \neq 0$);
$\deg P_l = \deg W_l|_{P_l} + 2$ and $\deg P_r \geq \deg W_r|_{P_r} + 4$ ($\deg W_l \neq 0$); and
$\deg P_l = \deg W_l|_{P_l} + 2$ and $\deg P_r = \deg W_r|_{P_r} + 3$ ($\deg W_l \neq 0$ and $\deg W_r \neq 0$).

In the first case, let K^* be as the left of Figure 17 with regions A_i, B_j, I_k ($i, j = l, r$ and $k = 1, \cdots, 4$) and let x be the crossing of I_2 which is n-adjacent to p. We may assume that W_l has no flype-crossings for $\lambda_{\delta q}$, since otherwise we are done by Claim 8 and Claim 10.

Claim 11. *If* $\deg I_1 = \deg W_l|_{I_1} + 2$*,* $\deg I_2 = 2$ *or* $\deg I_3 \geq 3$*, then* K *has a flyped tongue.*

Proof. From Claim 10, K has a flype-component of $\lambda_{\delta q}$ on the right side. If $\deg I_1 = \deg W_l|_{I_1} + 2$, then C_δ does not run through a side of $\theta^+_{x_l}$, and thus we are done by Claim 9. If $\deg I_2 = 2$, then we are done by Claim 8. Assume that $\deg I_1 \geq \deg W_l|_{I_1} + 3$, $\deg I_2 \geq 3$ and $\deg I_3 \geq 3$. Then I_3 has a crossing x' ($\neq x$) which is p-adjacent to p. Note that $x \neq x_l$ and $x' \neq x_r$. From Claim 9 we may assume that C_δ runs through a side of $\theta^+_{x_l}$, and thus $C_\delta = \bar{q}\bar{\delta}\bar{p}x_l$. Then we have that $C_\delta > C_x > C_{x'}$ from Lemma 6.1, since $C_{x'} = \bar{x}'\delta q \cdots$ from Lemma 6.2. Therefore we have that $C_x = \bar{x}p\delta q \cdots$, and in fact C_x runs through a side of θ^+_p. Hence the positive curve sharing a saddle-intersection with C_x at θ_p is $px_lq\delta$, and thus we are done by Claim 7. □

From Claim 11 we may assume that $\deg I_1 \geq \deg W_l|_{I_1} + 3$, $\deg I_2 \geq 3$ and $\deg I_3 = 2$. Then we have three subcases: $\deg W_r \neq 0$; $\deg W_r = 0$ and $\deg I_4 = 2$; and $\deg W_r = 0$ and $\deg I_4 \neq 2$. In the first subcase, apply flype moves and the untongue move on K^* to have another almost alternating diagram $\bar{K}^*$ of the trivial 2-component link with fewer crossings than K^*. From given conditions and assumptions, we have that W_l is not trivial and has no flype-crossings for $\lambda_{\delta q}$, and that W_r is not trivial. Let W_1 (resp. W_2) be the flyped W_l (resp. W_r). Since neither W_l nor W_r is trivial, we have that neither W_1 nor W_2 is trivial. Thus we can see that $\bar{K}^*$ is connected and reduced, and that $\bar{K}^*$ has no less than 4 crossings. Therefore $\bar{K}^*$ is prime from Proposition 5.1 and strongly reduced from Proposition 5.6. Hence $\bar{K}^*$ has a flyped tongue from Theorem 3 (2). Let A_i, B_j and H_k be mutually distinct regions of S with $\bar{K}^*$ as the right of Figure 17 ($i, j = 1, 2$, $k = 1, 2, 3$). Let q' be the crossing of B_2 which is n-adjacent to the dealternator δ of $\bar{K}^*$. Since W_l has no flype-crossings for $\lambda_{\delta q}$, W_1 has no flype-crossings for $\lambda_{\delta q'}$, i.e. B_1 and H_1 do not share a crossing. Now each pair of regions (A_i, B_j) $(i, j = 1, 2)$ can be the core of a flyped tongue of $\bar{K}^*$. If (A_1, B_1) is the core of a flyped tongue of $\bar{K}^*$, then B_2 shares a crossing with the region sharing with A_1 a segment which is n-adjacent to the crossing of W_1 which is p-adjacent to the dealternator of $\bar{K}^*$. Then the only possibility is H_1, contradicting the condition that W_1 has no flype-crossings. We can be done similarly for (A_1, B_2) and for (A_2, B_2). If (A_2, B_1) is the core of a flyped tongue of $\bar{K}^*$, then B_2 shares a crossing with H_2. However then we have that $H_3 = B_2$, contradicting that W_2 is not trivial. In the second subcase, the only possibility that K does not have a flype-component of $\lambda_{\delta q}$ with x_l is when band-trace ψ connects segment $\lambda_{q x_r}$ and the segment $(\neq \lambda_{x_l p})$ which is n-adjacent to x_l. However then, K has flype-components of $\lambda_{\delta x_r}$ with q and with x_l. Now consider the third subcase. If $\deg A_r \neq 4$, i.e. x is not n-adjacent to x_r, then we obtain a contradiction by applying flype moves and the untongue move on K^* as the first subcase. If $\deg A_r = 4$, then there is a crossing y $(\neq p, x_l, x_r)$ which is n-adjacent to x. We have that $C_{x_r} = \bar{x}_r q \delta$ from Lemma 6.3, and we may assume that $C_\delta = \bar{p}\bar{\delta}\bar{q} x_l$ from Claim 9. Therefore we have that $C_y = \bar{y} x \delta q \cdots$ or that $C_y = \bar{y} x p \delta q \cdots$. In either case there is a positive curve $p \delta q x_l$, since I_3 has degree 2. Hence we are done from Claim 7.

In the last two cases, we can show that W_l has a flype-crossing for $\lambda_{\delta q}$ by analyzing the diagram after applying flype moves and the untongue move on K^* as the previous case. However then each case is equivalent to a case which is done before.

$\square$

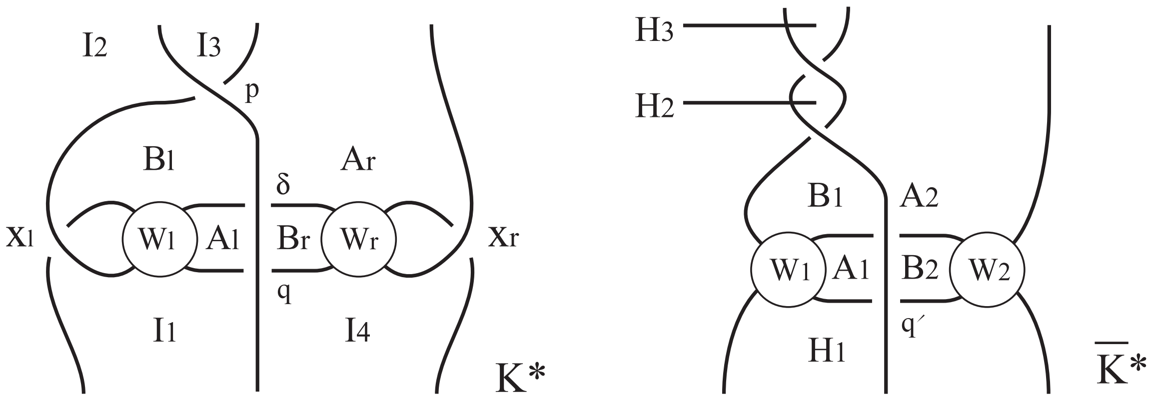

FIGURE 17

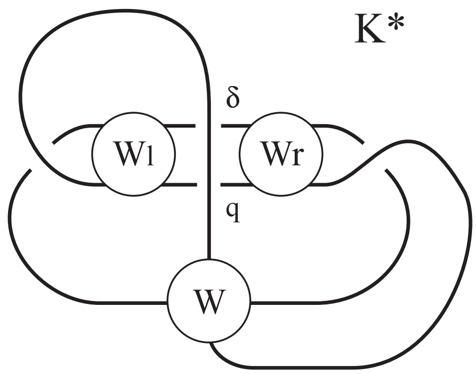

FIGURE 18

7. PROOF OF THEOREM 4

To prove Theorem 3 we actually showed the following in [Ts].

Theorem 6. ([Ts]) *Let L be a prime, strongly reduced almost alternating link diagram on S. Assume that L admits a sphere in $S^3 - L$ which is in standard position. Then L admits a flyped tongue.*

Proof of Theorem 4. Let K be the trivial knot in strongly reduced almost alternating position and let E be a spanning disk for K in basic position with minimal complexity. Then K is prime from Proposition 5.1 and E is standard position from Proposition 2.1. Therefore if E has no short arcs, then E is in special position from Proposition 4.2. Then we can take a neighborhood of E whose boundary is in standard position from Proposition 3.1. Thus K has a flyped tongue from Theorem 6. Next assume that E has a short arc. If E has a short arc of type I, then K has a flyped tongue from Proposition 6.1. If E has a short arc of type II or III, then K has a flyped tongue from Proposition 5.4 and Proposition 6.2.

□

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