GRAVITATIONAL LENSING

12 - LENS MODELS: AXIALLY SYMMETRIC LENSES

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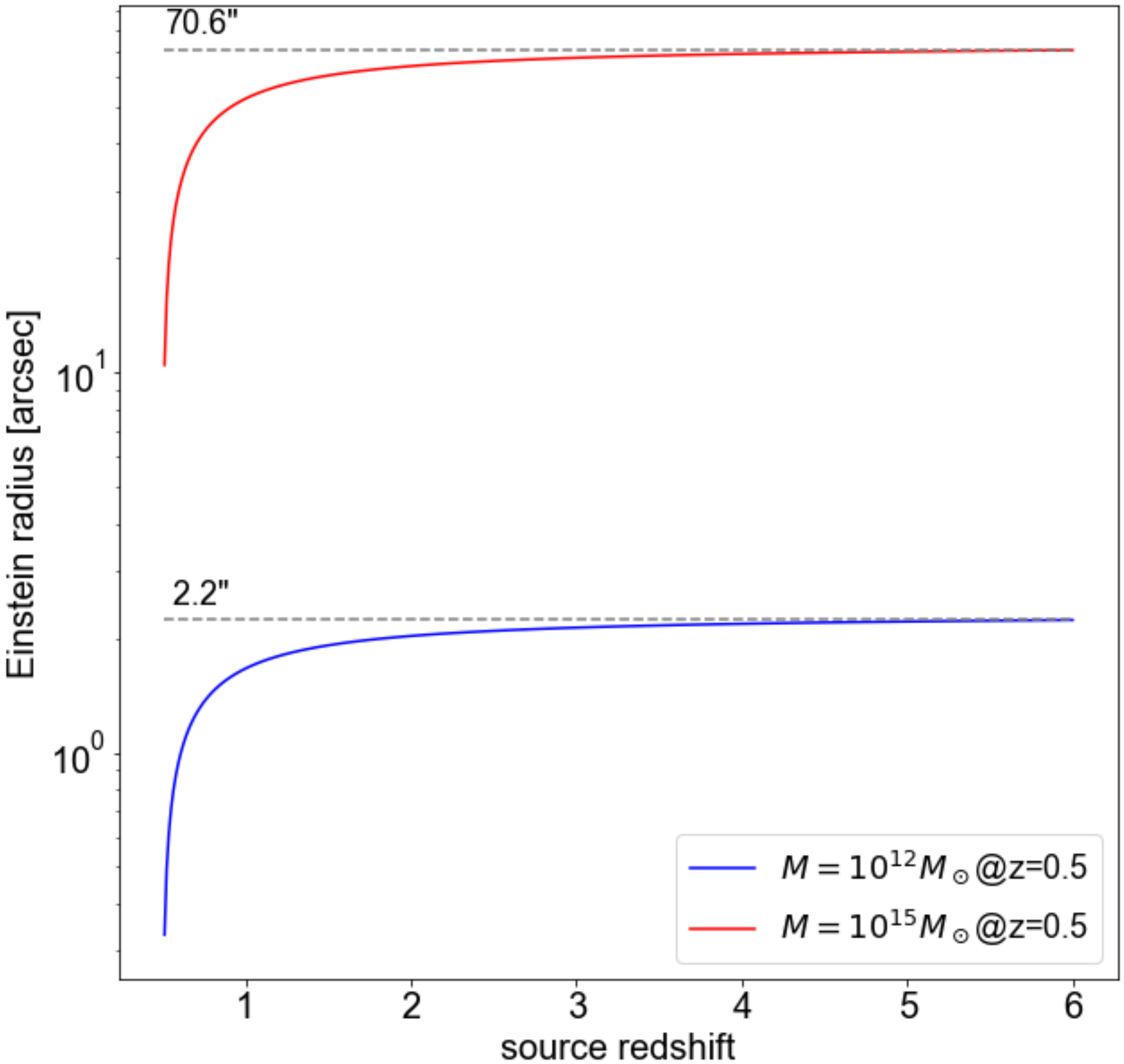
LENSING BY GALAXIES AND GALAXY CLUSTERS

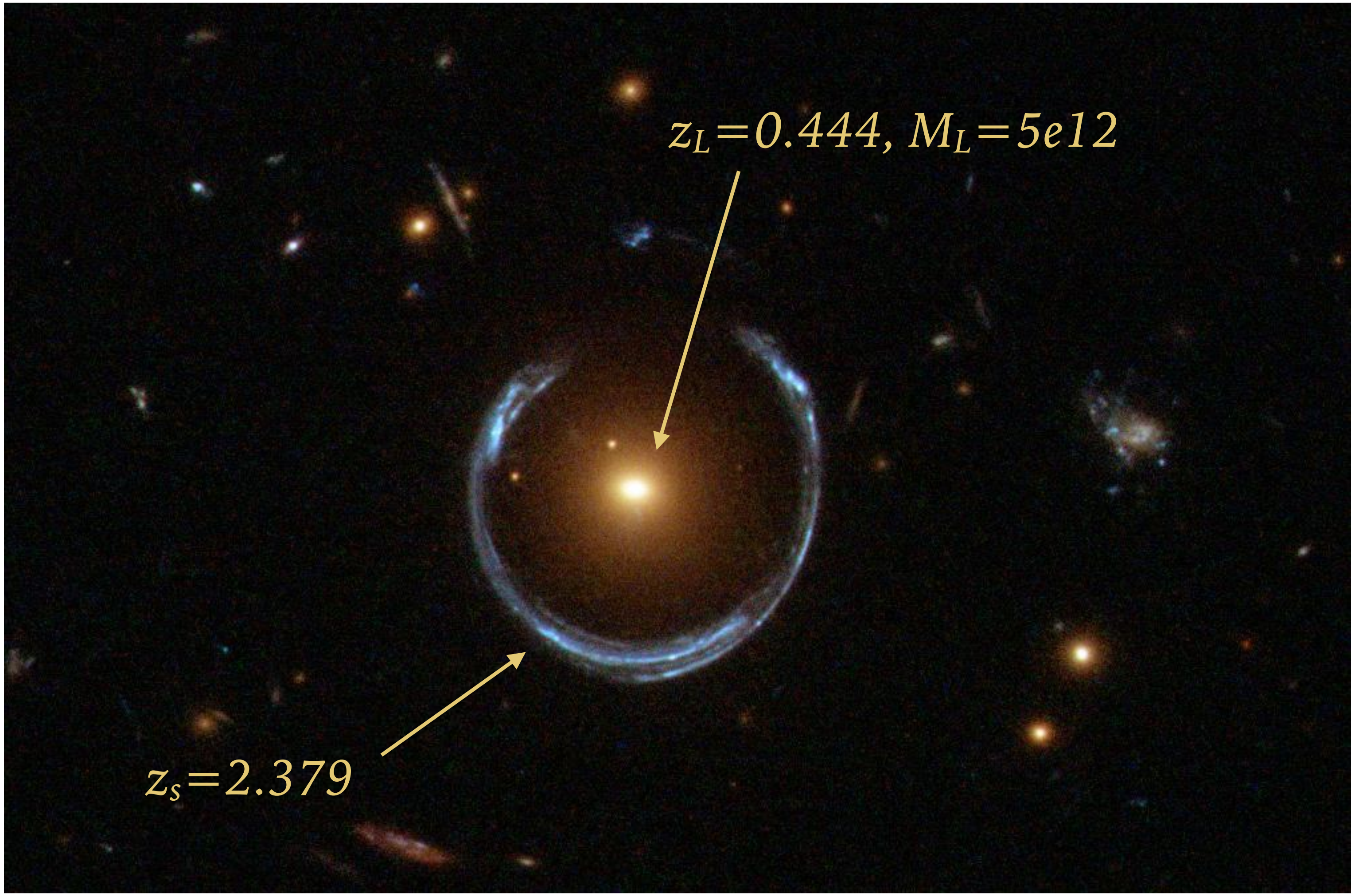
- Much more massive than stars!
- At cosmological distances
- Extended!
- Complex (more parameters)

Note: computed using

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}$$

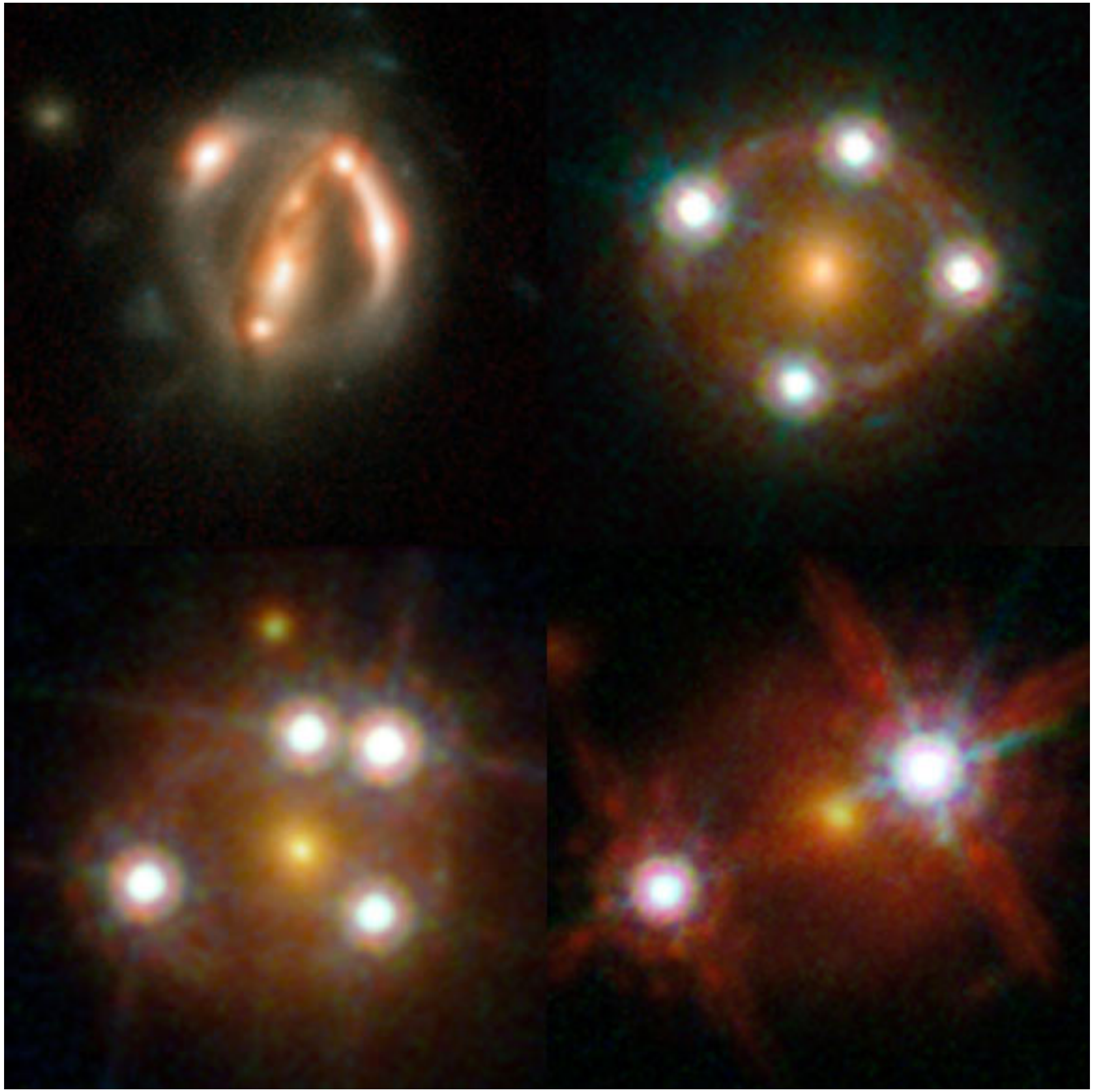


EXTENDED LENSES

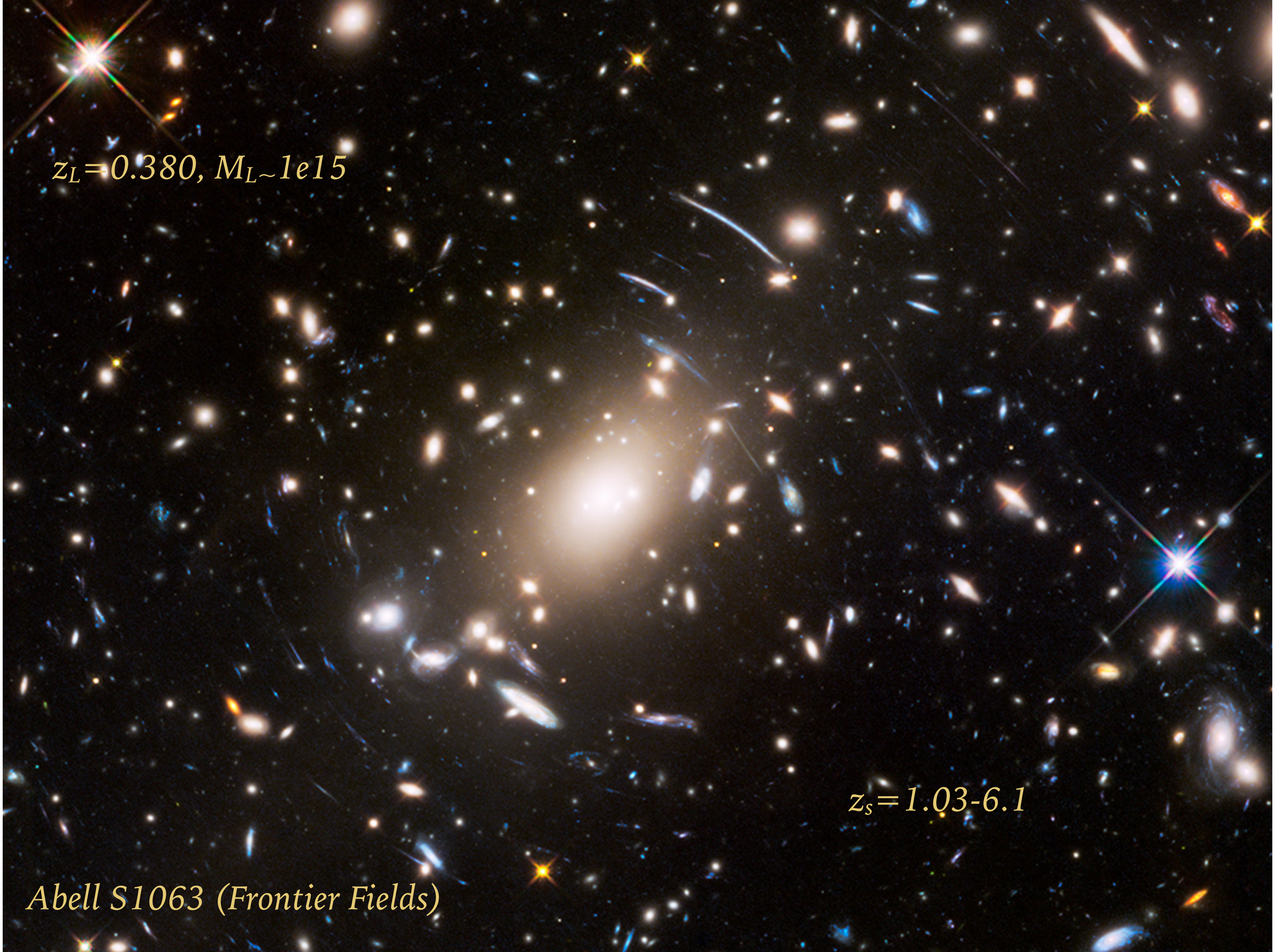


Cosmic horseshoe (Belokurov et al. 2007)

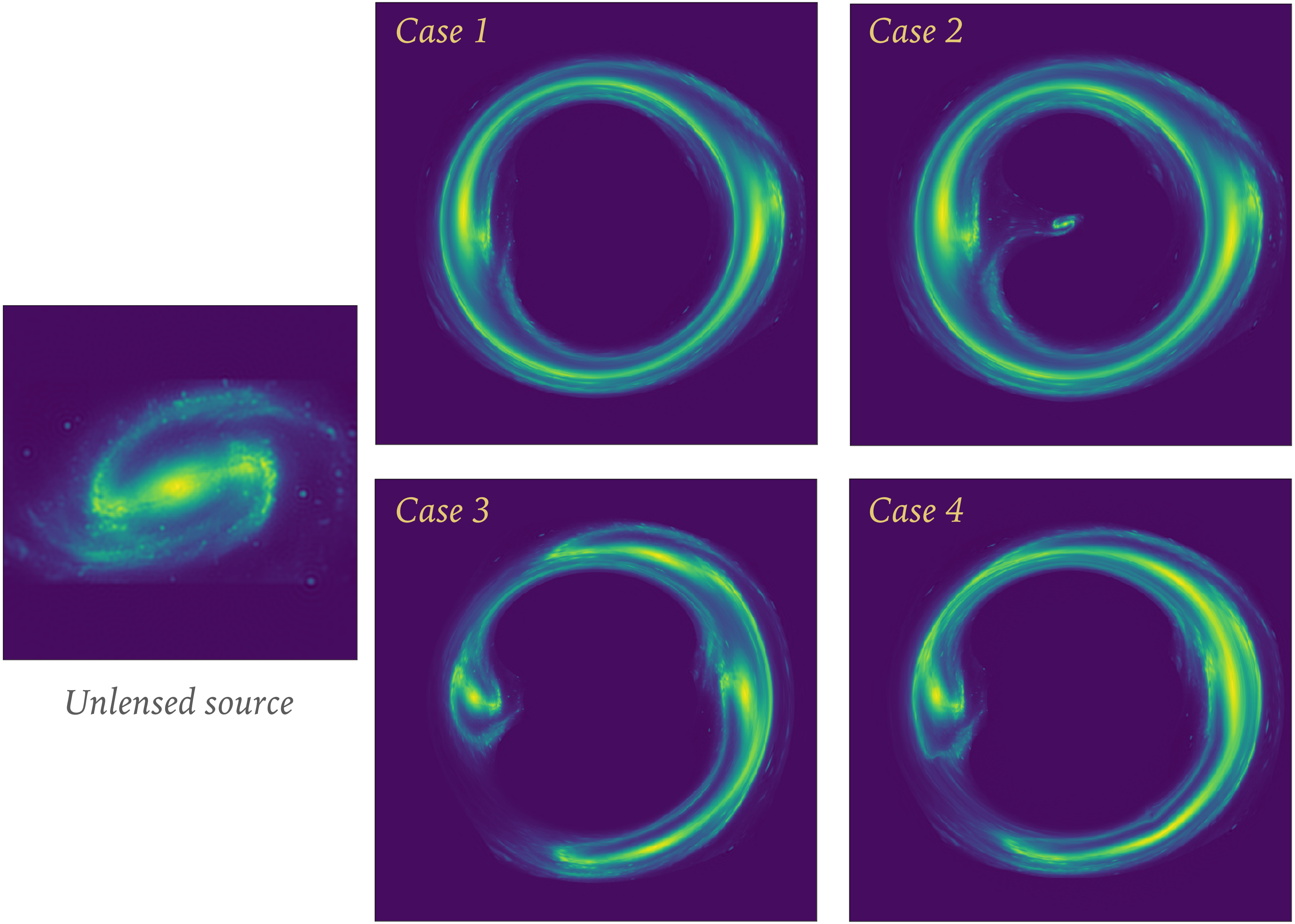
EXTENDED LENSES

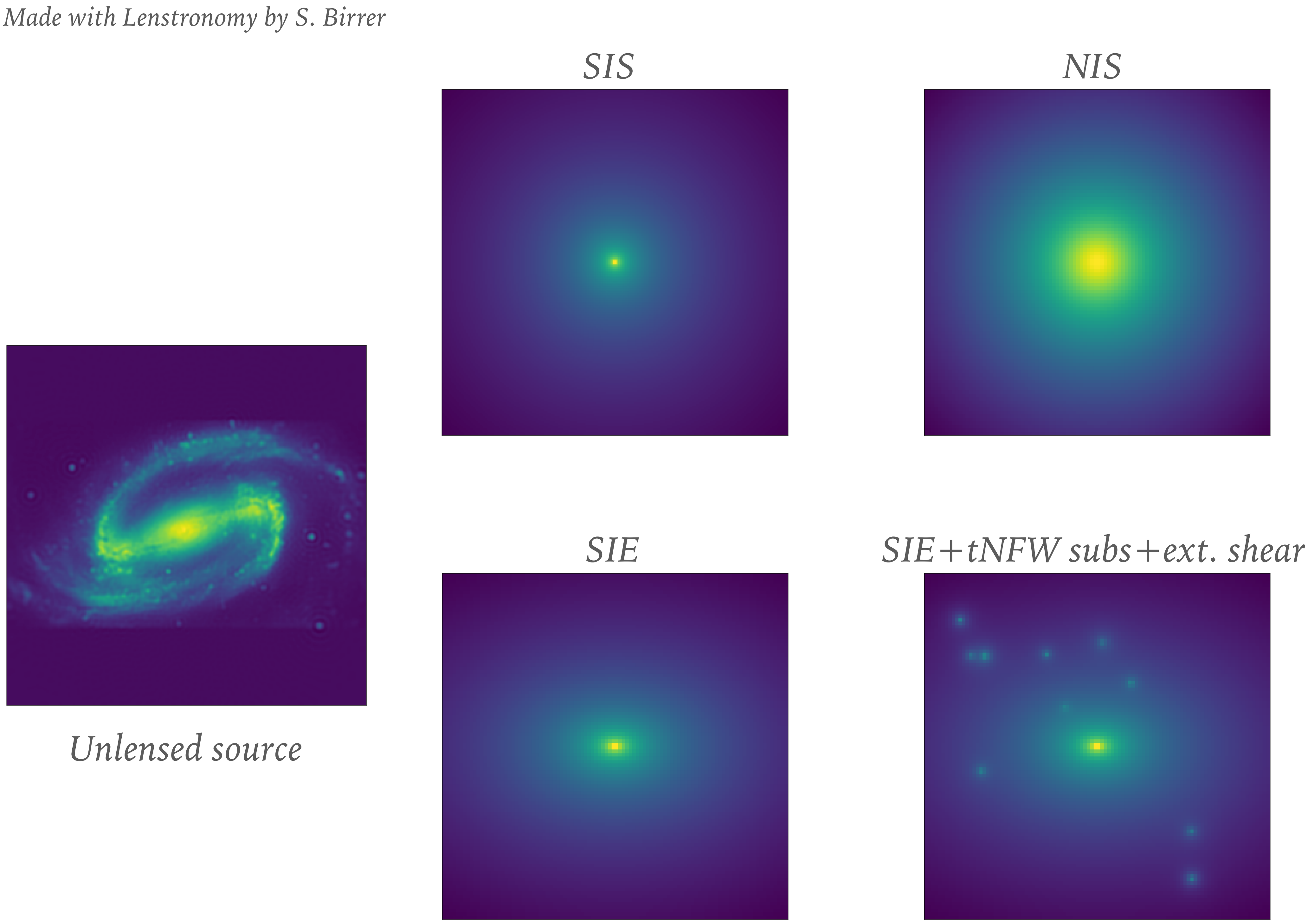


Suyu et al. (H0LiCOW team)


$z_L = 0.380$, $M_L \sim 1e15$
$z_s = 1.03$-6.1
Abell S1063 (Frontier Fields)

Made with Lenstronomy by S. Birrer



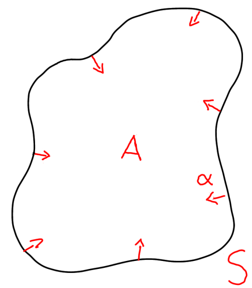

Made with Lenstronomy by S. Birrer
SIS
NIS
SIE
SIE+tNFW subs+ext. shear
Unlensed source

SHEAR IN POLAR COORDINATES

Gauss' Law

Relates deflection along a closed curve to the convergence within the curve.

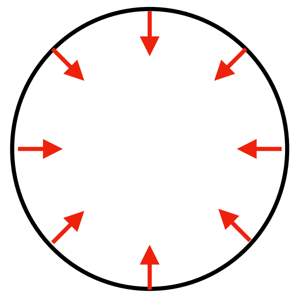
$$\int_C ds\ \boldsymbol{n} \cdot \boldsymbol{\nabla}\psi = \int d^2x\ \nabla^2\psi$$



$$\int_C ds\ \boldsymbol{\alpha} \cdot \boldsymbol{n} = 2 \int d^2x\ \kappa(\boldsymbol{x}) = 2A\langle\kappa\rangle$$

For the special case of a circular curve

$$\boxed{\overline{\alpha_r} = r\ \langle\kappa\rangle = \frac{1}{\pi\Sigma_{crit}}\frac{M(r)}{r}}$$



POLAR COORDINATES

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r} \frac{\partial}{\partial \phi}$$

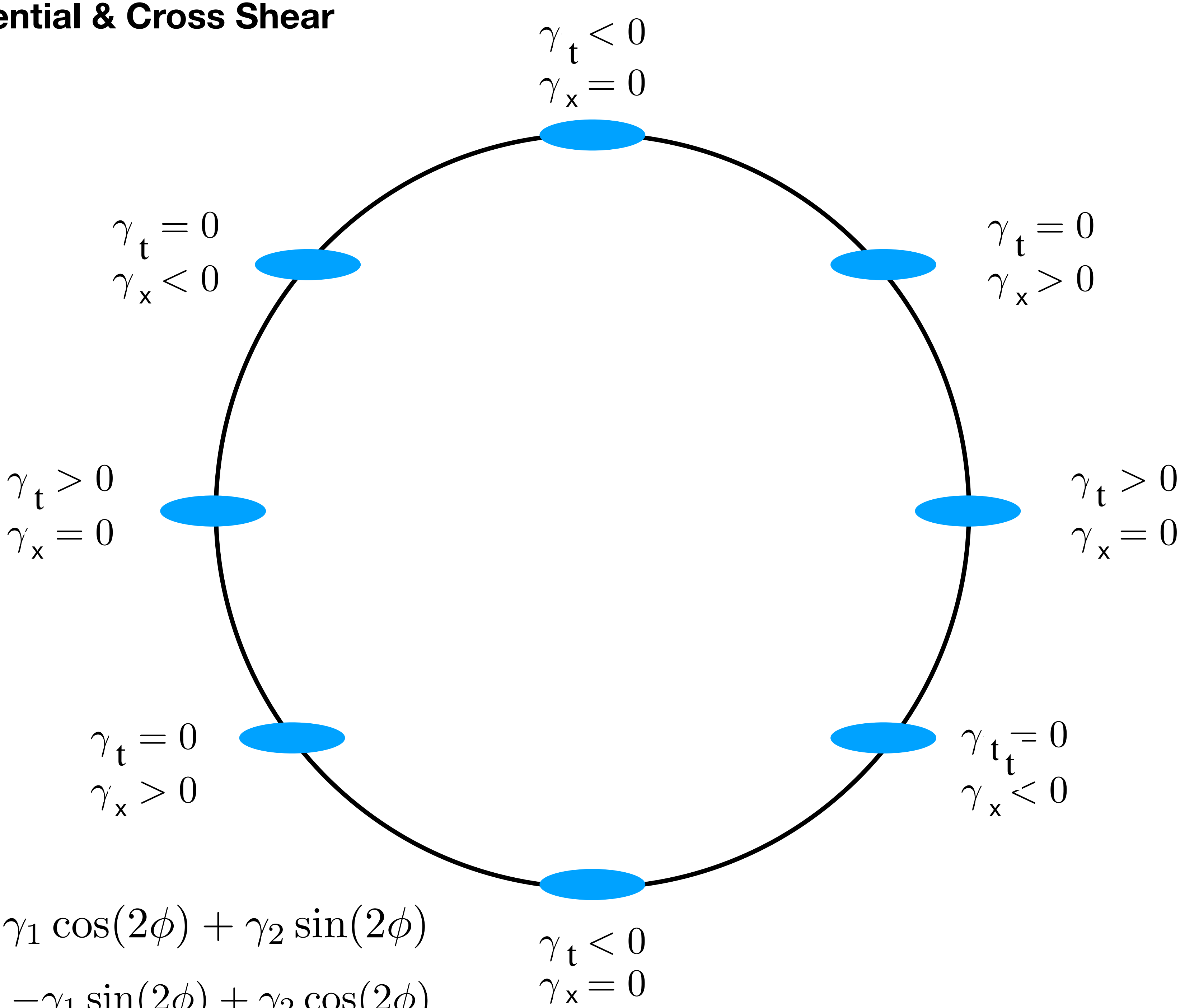
$$\begin{aligned} \hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial x} &= \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}} \frac{\partial}{\partial r} - \frac{\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\phi}}}{r} \frac{\partial}{\partial \phi} \\ &= \cos(\phi) \frac{\partial}{\partial r} - \frac{\sin(\phi)}{r} \frac{\partial}{\partial \phi} \end{aligned}$$

$$\begin{aligned} \hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial y} &= \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{r}} \frac{\partial}{\partial r} - \frac{\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\phi}}}{r} \frac{\partial}{\partial \phi} \\ &= \sin(\phi) \frac{\partial}{\partial r} + \frac{\cos(\phi)}{r} \frac{\partial}{\partial \phi} \end{aligned}$$

POLAR COORDINATES

$$
\begin{aligned}
\nabla^2 &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} \\
&= \left(\cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\right)\left(\cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\right) \\
&+ \left(\sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\right)\left(\sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\right) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}
\end{aligned}
$$

Tangential & Cross Shear

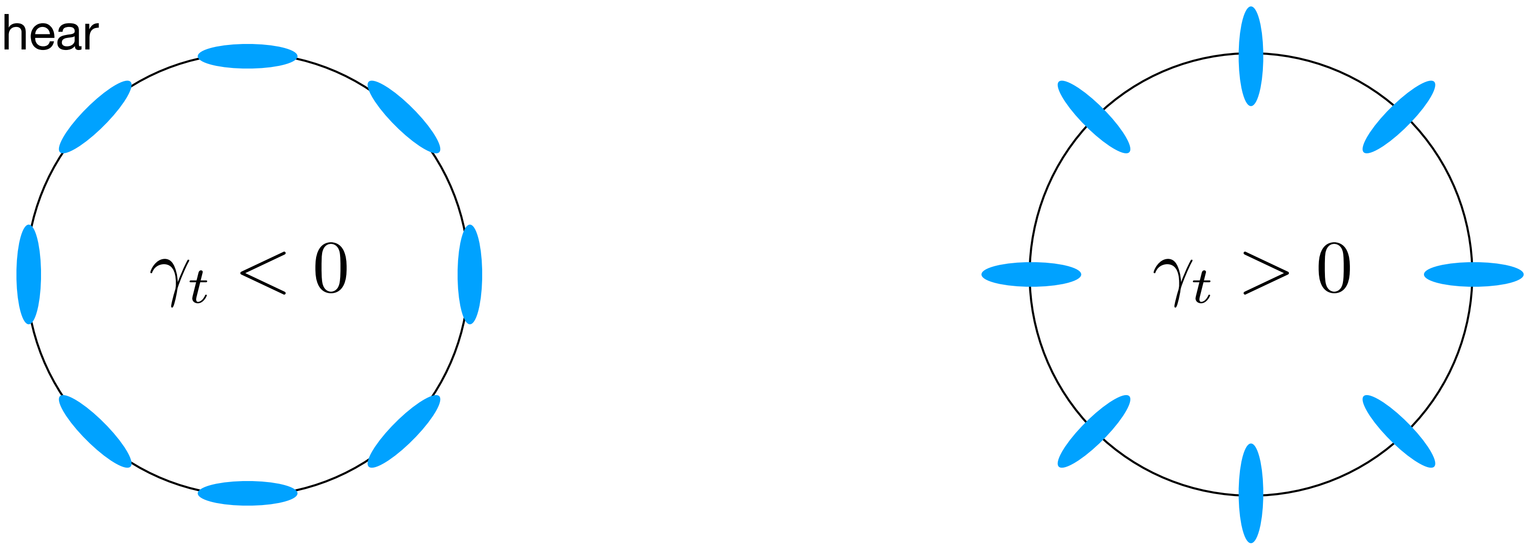


$$\gamma_t = \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi)$$

$$\gamma_\times = -\gamma_1 \sin(2\phi) + \gamma_2 \cos(2\phi)$$

SHEAR IN POLAR COORDINATES

tangential shear



cross shear

SHEAR IN POLAR COORDINATES

$$\gamma_2 = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi = \left[\cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\right]\left[\sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\right]\psi$$

$$= \left[\cos(\phi)\sin(\phi)\frac{\partial^2}{\partial r^2} + \cos^2(\phi)\left(-\frac{1}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \phi}\right) - \frac{\sin(\phi)}{r}\left(\cos(\phi)\frac{\partial}{\partial r} + \sin(\phi)\frac{\partial^2}{\partial r \partial \phi}\right) - \frac{\sin(\phi)}{r^2}\left(-\sin(\phi)\frac{\partial}{\partial \phi} + \cos(\phi)\frac{\partial^2}{\partial \phi^2}\right)\right]\psi$$

$$= \cos(\phi)\sin(\phi)\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi + \left(\cos^2(\phi) - \sin^2(\phi)\right)\left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} - \frac{1}{r^2}\frac{\partial}{\partial \phi}\right)\psi$$

$$= \frac{1}{2}\sin(2\phi)\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi + \cos(2\phi)\left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} - \frac{1}{r^2}\frac{\partial}{\partial \phi}\right)\psi$$

$$= \gamma_t \sin(2\phi) + \gamma_\times \cos(2\phi)$$

tangential shear operator

$$\Delta_t = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) = \frac{\partial^2}{\partial r^2} - \frac{1}{2}\nabla^2$$

cross shear operator

$$\Delta_\times = \left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} - \frac{1}{r^2}\frac{\partial}{\partial \phi}\right)$$

TANGENTIAL SHEAR THEOREM

$$\int_C ds\, \boldsymbol{\alpha} \cdot \boldsymbol{n} = 2 \int d^2x\ \kappa(\boldsymbol{x})$$

Gauss' Law

$$\int_0^{2\pi} d\phi\ r\, \alpha_r = 2 \int_0^{2\pi} d\phi \int_0^r dr' r' \kappa(r', \phi)$$

on a circle

$$\frac{\partial}{\partial r} \int_0^{2\pi} d\phi\ r\, \alpha_r = \frac{\partial}{\partial r} 2 \int_0^{2\pi} d\phi \int_0^r dr'\ r'\, \kappa(r', \phi)$$

taking the derivative of both sides

$$\int_0^{2\pi} d\phi\ \left(\alpha_r + r \frac{\partial \alpha_r}{\partial r} \right) = 2 \int_0^{2\pi} d\phi\ r\, \kappa(r, \phi)$$

$$\int_0^{2\pi} d\phi\ r \frac{\partial \alpha_r}{\partial r} = 2 \int_0^{2\pi} d\phi\ r\, \kappa(r, \phi) - \int_0^{2\pi} d\phi\ \alpha_r$$

rearrange it

$$\begin{aligned} \int_C ds\ \frac{\partial \alpha_r}{\partial r} &= 4\pi r\, \overline{\kappa}(r) - \int_0^{2\pi} d\phi\ \alpha_r \\ &= 4\pi r\, \overline{\kappa} - \frac{1}{r} \int_C ds\ \alpha_r \\ &= 4\pi r\, \overline{\kappa} - \frac{2}{r} \int_V d^2x\ \kappa(\boldsymbol{x}) \\ &= 4\pi r\, \overline{\kappa} - 2\pi r \langle \kappa \rangle \end{aligned}$$

Gauss' Law again

TANGENTIAL SHEAR THEOREM

$$
\begin{aligned}
\int_C ds\ \gamma_t = 2\pi r \overline{\gamma}_t &= \int_C ds\ \Delta_t \psi \\
&= \int_C ds\ \left[\frac{\partial^2}{\partial r^2} - \frac{1}{2}\nabla^2\right]\psi \\
&= \int_C ds\ \left[\frac{\partial \alpha_r}{\partial r} - \kappa\right] \\
&= 4\pi r\, \overline{\kappa} - 2\pi r \langle \kappa \rangle - \int_C ds\ \kappa \\
&= 2\pi r\, \overline{\kappa} - 2\pi r \langle \kappa \rangle
\end{aligned}
$$

using previous result

$$
\boxed{\overline{\gamma}_t = \overline{\kappa} - \langle \kappa \rangle}
$$

for any circle

SUMMARY OF AXIALLY SYMMETRIC LENSING

deflection angle : $$\alpha(r) = \frac{m(r)}{r}$$

$$m(r) = 2\int_0^r dr'\ r'\kappa(r') = \frac{M(r)}{\pi\Sigma_{\mathrm{crit}}}$$

convergence : $$\kappa(r) = \frac{m'(r)}{2r}$$

shear : $$\gamma(r) = \frac{m(r)}{r^2} - \frac{m'(r)}{2r}$$

$$\gamma_2 = \gamma(r)\sin(2\phi)$$

$$\gamma_1 = \gamma(r)\cos(2\phi)$$

inverse magnification : $$det\boldsymbol{A} = \frac{1}{\mu} = \left[1 - \frac{m(r)}{r^2}\right]\left[1 + \frac{m(r)}{r^2} - \frac{m'(r)}{r}\right]$$

tangential critical curve : $$\frac{m(r)}{r^2} = 1$$

radial critical curve : $$\frac{m'(r)}{r} - \frac{m(r)}{r^2} = 1$$