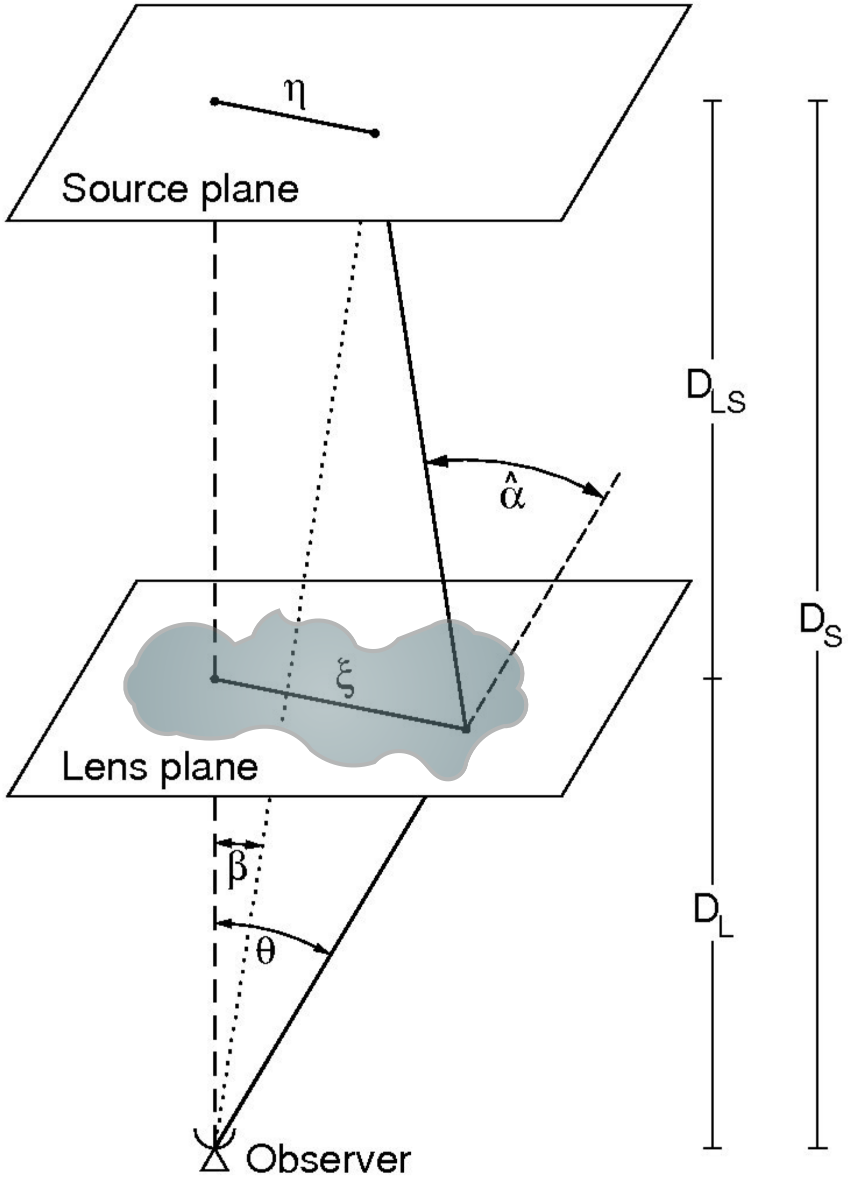
GRAVITATIONAL LENSING

5- THE LENSING POTENTIAL & SHEAR

R. Benton Metcalf

2022-2023

LENS EQUATION



Remember that:

1) *positions on the lens and source planes are defined by vectors*
2) *the deflection angle itself is a vector*

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S}$$

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\hat{\vec{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

THE LENSING POTENTIAL

$$\psi(\theta) = \left(\frac{2}{c^2} \frac{D_{ls}}{D_s D_l} \right) \int_{-\infty}^{\infty} dz \ \phi_N(D_l \theta, z)$$

$$\boldsymbol{\nabla}_\theta \psi(\theta) = \boldsymbol{\alpha}(\theta)$$

$$\nabla^2_\theta \psi(\theta) = 2\kappa(\theta)$$

dimensionless surface density

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{\mathrm{crit}}}$$

critical surface density

$$\Sigma_{\mathrm{crit}} = \frac{c^2}{4\pi G} \frac{D_s}{D_{ls} D_l}$$

THE LENSING POTENTIAL

Green's functions relating the surface density to the potential and deflection.

$$\psi(\theta) = \frac{1}{\pi} \int d^2\theta' \ \kappa(\boldsymbol{\theta}') \ \ln\left(|\boldsymbol{\theta} - \boldsymbol{\theta}'|\right)$$

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{\rm crit}}$$

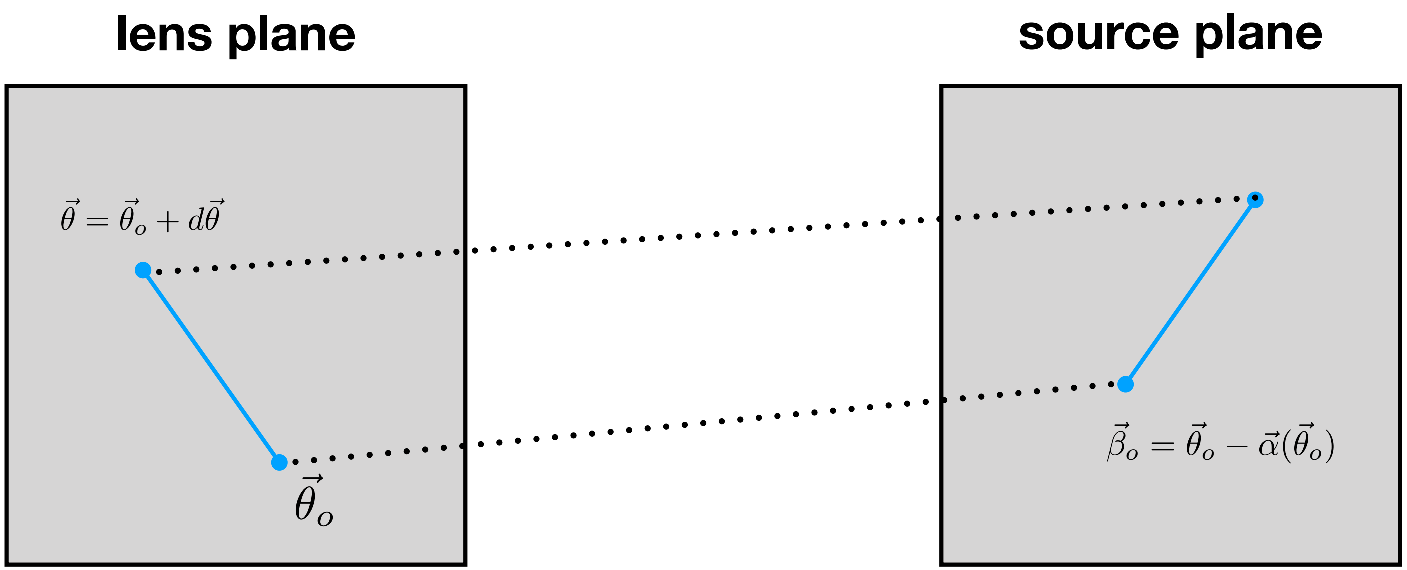
$$\boldsymbol{\alpha}(\theta) = \frac{1}{\pi} \int d^2\theta' \ \kappa(\boldsymbol{\theta}') \ \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}')}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

$$\Sigma_{\rm crit} = \frac{c^2}{4\pi G} \frac{D_s}{D_{ls} D_l}$$

LENS MAPPING

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

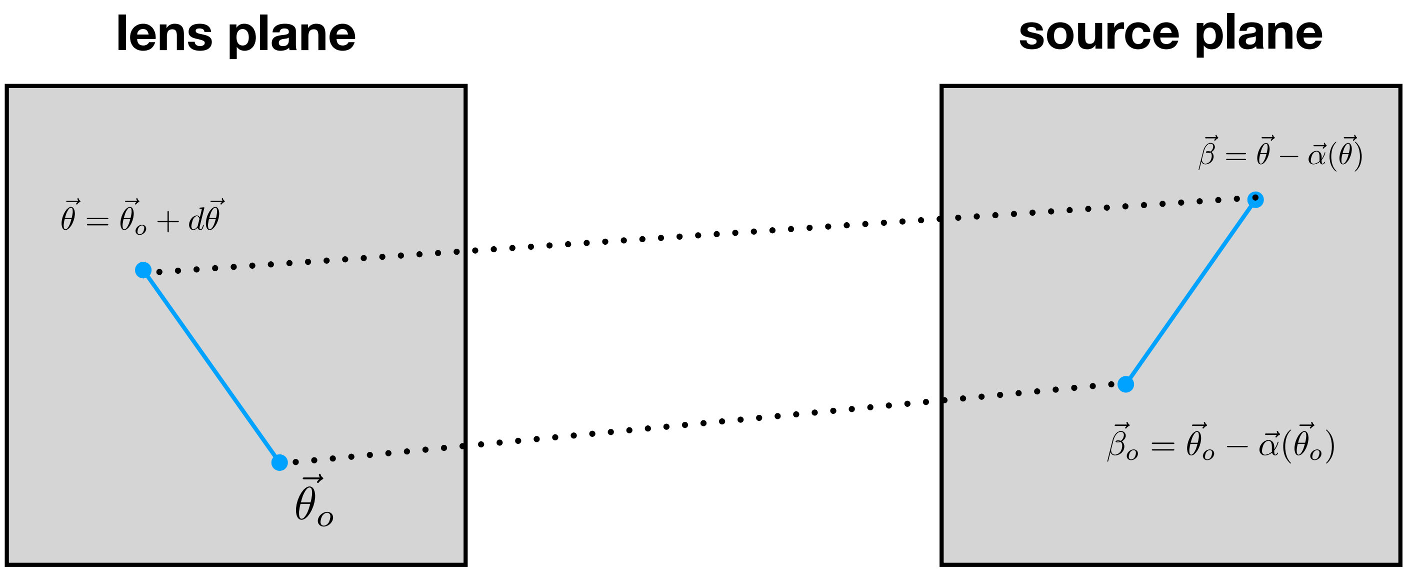
$$\vec{\alpha}(\vec{\theta}) = \int d^2\theta' \; \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \frac{\Sigma(\vec{\theta}')}{\Sigma_{crit}}$$



LENS MAPPING

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

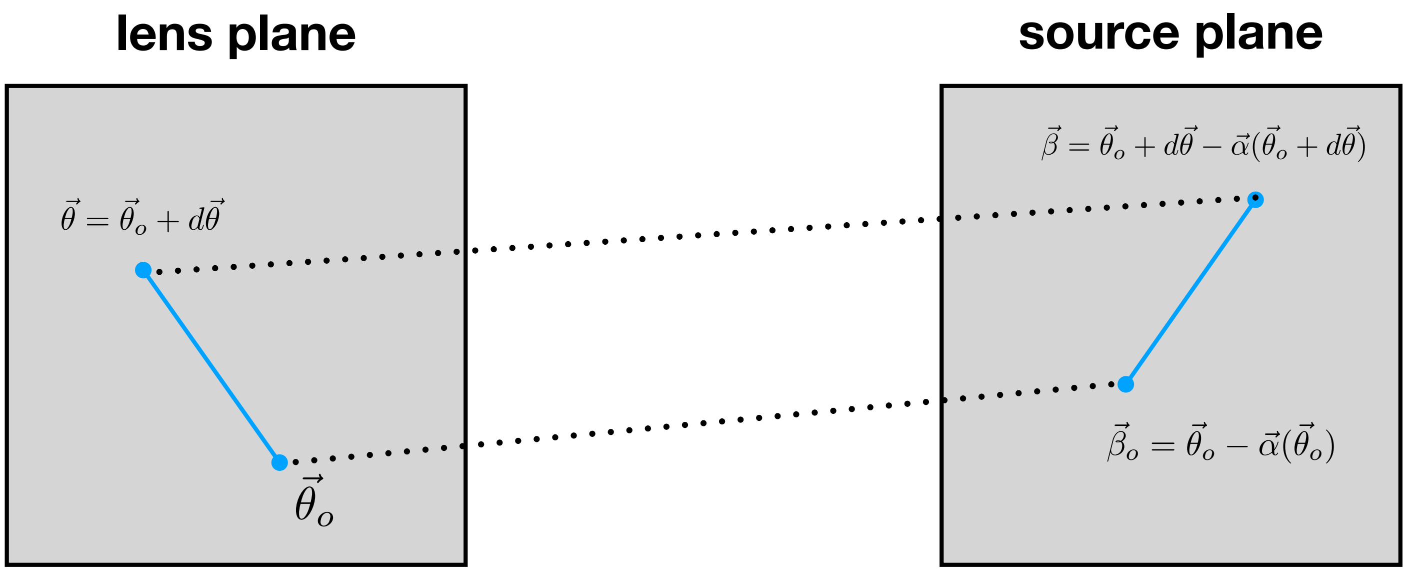
$$\vec{\alpha}(\vec{\theta}) = \int d^2\theta' \, \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \frac{\Sigma(\vec{\theta}')}{\Sigma_{crit}}$$



LENS MAPPING

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

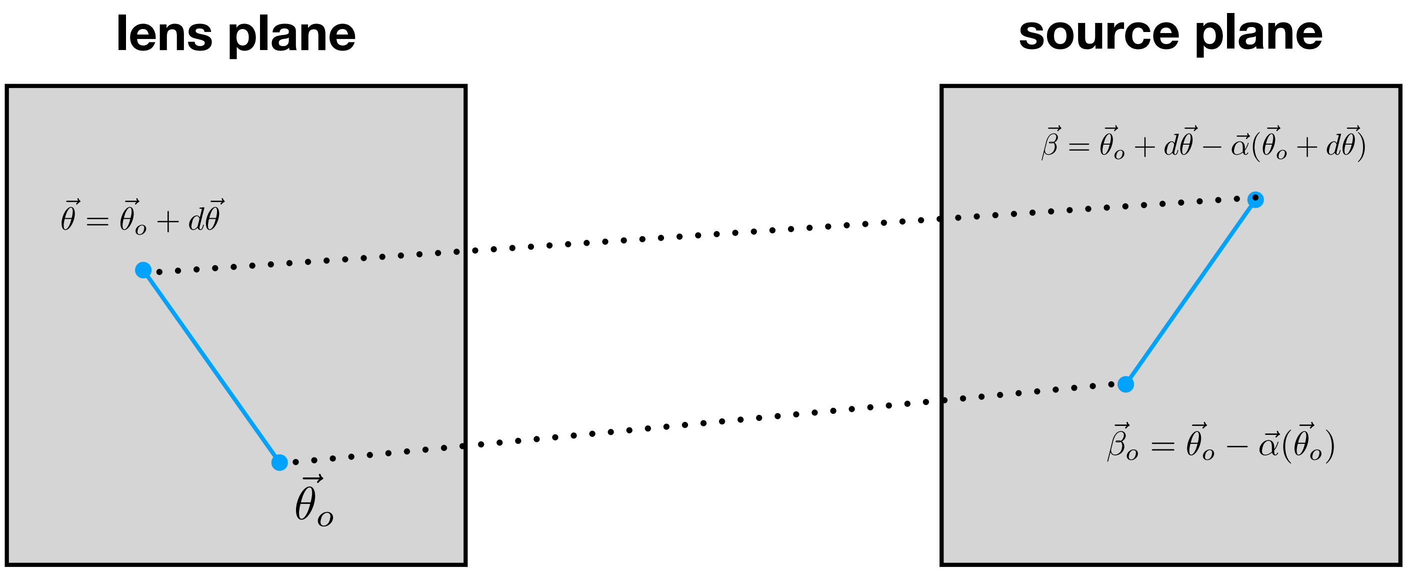
$$\vec{\alpha}(\vec{\theta}) = \int d^2\theta' \, \frac{\vec{\theta} - \vec{\theta'}}{|\vec{\theta} - \vec{\theta'}|^2} \frac{\Sigma(\vec{\theta'})}{\Sigma_{crit}}$$



$$d\vec{\beta} = \vec{\beta} - \vec{\beta}_o = d\vec{\theta} - \left(\vec{\alpha}(\vec{\theta}_o + d\vec{\theta}) - \vec{\alpha}(\vec{\theta}_o)\right)$$

LENS MAPPING

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \qquad \vec{\alpha}(\vec{\theta}) = \int d^2\theta' \, \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \frac{\Sigma(\vec{\theta}')}{\Sigma_{crit}}$$



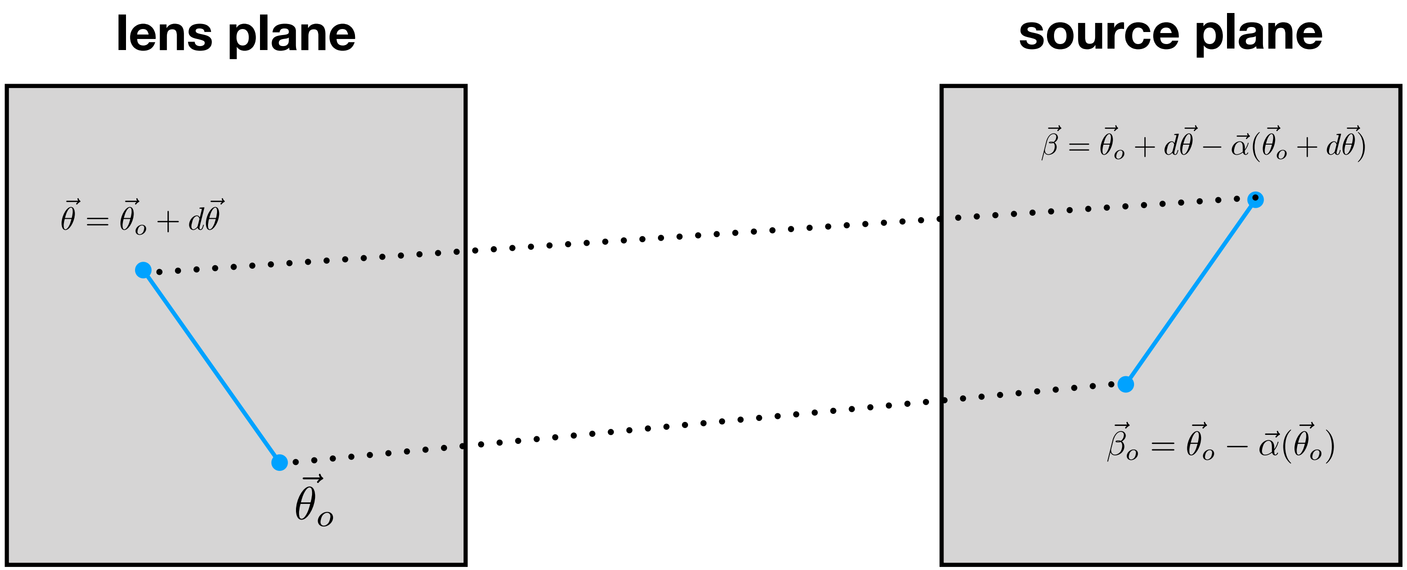
$$d\vec{\beta} = \vec{\beta} - \vec{\beta}_o = d\vec{\theta} - \left(\vec{\alpha}(\vec{\theta}_o + d\vec{\theta}) - \vec{\alpha}(\vec{\theta}_o)\right)$$

$$= d\vec{\theta} - d\vec{\theta} \cdot \nabla_\theta \vec{\alpha}(\vec{\theta}_o)$$

LENS MAPPING

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \int d^2\theta' \, \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \frac{\Sigma(\vec{\theta}')}{\Sigma_{crit}}$$



$$\begin{aligned} d\vec{\beta} = \vec{\beta} - \vec{\beta}_o &= d\vec{\theta} - \left(\vec{\alpha}(\vec{\theta}_o + d\vec{\theta}) - \vec{\alpha}(\vec{\theta}_o)\right) \\ &= d\vec{\theta} - d\vec{\theta} \cdot \nabla_\theta \vec{\alpha}(\vec{\theta}_o) \\ &= \boldsymbol{A}\, d\vec{\theta} \end{aligned}$$

LENS MAPPING : ROTATION OF SHEAR

$$R^T \Gamma R$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

LENS MAPPING : ROTATION OF SHEAR

$$R^T \Gamma R$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 \cos(\phi) + \gamma_2 \sin(\phi) & -\gamma_1 \sin(\phi) + \gamma_2 \cos(\phi) \\ \gamma_2 \cos(\phi) - \gamma_1 \sin(\phi) & -\gamma_2 \sin(\phi) - \gamma_1 \cos(\phi) \end{pmatrix}$$

LENS MAPPING : ROTATION OF SHEAR

$$R^T \Gamma R$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 \cos(\phi) + \gamma_2 \sin(\phi) & -\gamma_1 \sin(\phi) + \gamma_2 \cos(\phi) \\ \gamma_2 \cos(\phi) - \gamma_1 \sin(\phi) & -\gamma_2 \sin(\phi) - \gamma_1 \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2 \sin(\phi)\cos(\phi) & \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1 \sin(\phi)\cos(\phi) \\ \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1 \sin(\phi)\cos(\phi) & -\gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2 \sin(\phi)\cos(\phi) \end{pmatrix}$$

LENS MAPPING : ROTATION OF SHEAR

$$R^T \Gamma R$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1\cos(\phi) + \gamma_2\sin(\phi) & -\gamma_1\sin(\phi) + \gamma_2\cos(\phi) \\ \gamma_2\cos(\phi) - \gamma_1\sin(\phi) & -\gamma_2\sin(\phi) - \gamma_1\cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2\sin(\phi)\cos(\phi) & \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1\sin(\phi)\cos(\phi) \\ \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1\sin(\phi)\cos(\phi) & -\gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2\sin(\phi)\cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1\cos(2\phi) + \gamma_2\sin(2\phi) & \gamma_2\cos(2\phi) - \gamma_1\sin(2\phi) \\ \gamma_2\cos(2\phi) - \gamma_1\sin(2\phi) & -(\gamma_1\cos(2\phi) + \gamma_2\sin(2\phi)) \end{pmatrix}$$

LENS MAPPING : ROTATION OF SHEAR

$$R^T \Gamma R$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 \cos(\phi) + \gamma_2 \sin(\phi) & -\gamma_1 \sin(\phi) + \gamma_2 \cos(\phi) \\ \gamma_2 \cos(\phi) - \gamma_1 \sin(\phi) & -\gamma_2 \sin(\phi) - \gamma_1 \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2 \sin(\phi)\cos(\phi) & \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1 \sin(\phi)\cos(\phi) \\ \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1 \sin(\phi)\cos(\phi) & -\gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2 \sin(\phi)\cos(\phi) \end{pmatrix}$$

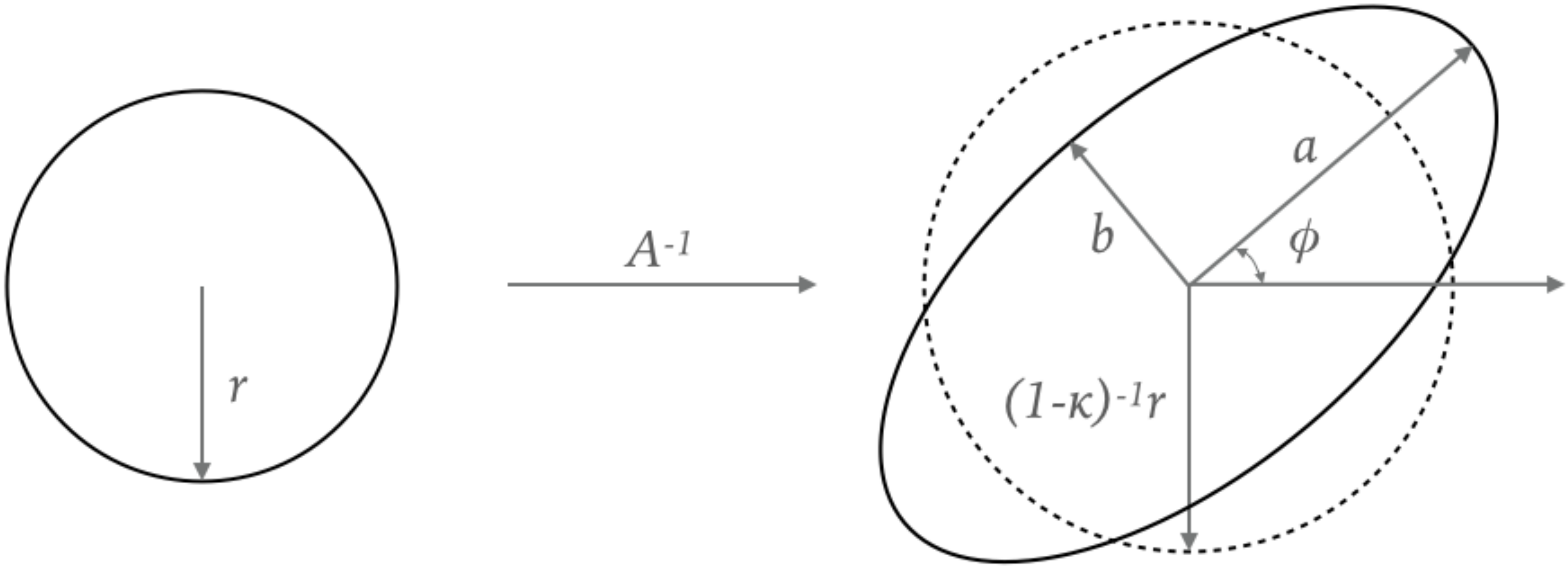
$$\begin{pmatrix} \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi) & \gamma_2 \cos(2\phi) - \gamma_1 \sin(2\phi) \\ \gamma_2 \cos(2\phi) - \gamma_1 \sin(2\phi) & -(\gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi)) \end{pmatrix}$$

$$\boxed{\begin{aligned} \gamma_1' &= \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi) \\ \gamma_2' &= \gamma_2 \cos(2\phi) - \gamma_1 \sin(2\phi) \end{aligned}}$$

Not a vector!

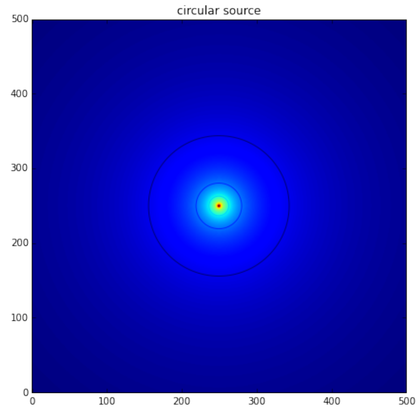
spin-2 object

LENS MAPPING



ON THE SPIN-2 NATURE OF SHEAR

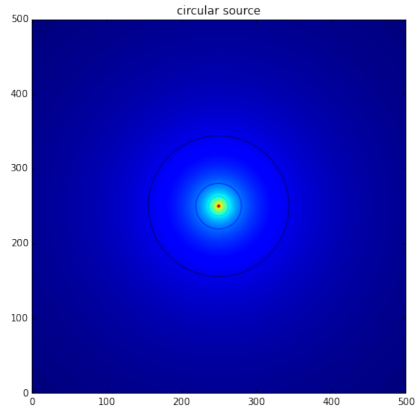
Consider a circular source



ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

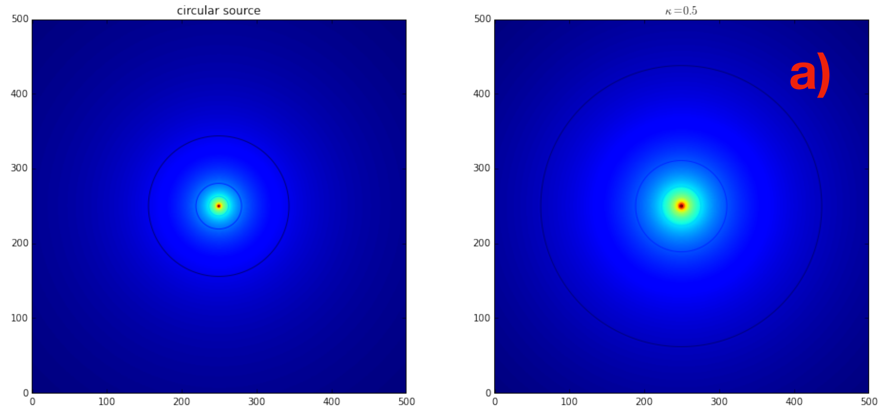
How is it distorted if we apply a pure convergence transformation?



ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

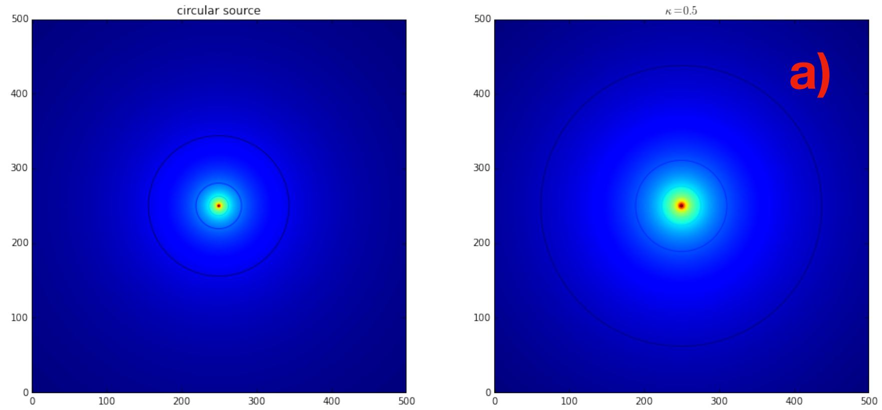
a) **How is it distorted if we apply a pure convergence transformation?**



ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

a) **How is it distorted if we apply a pure convergence transformation?**



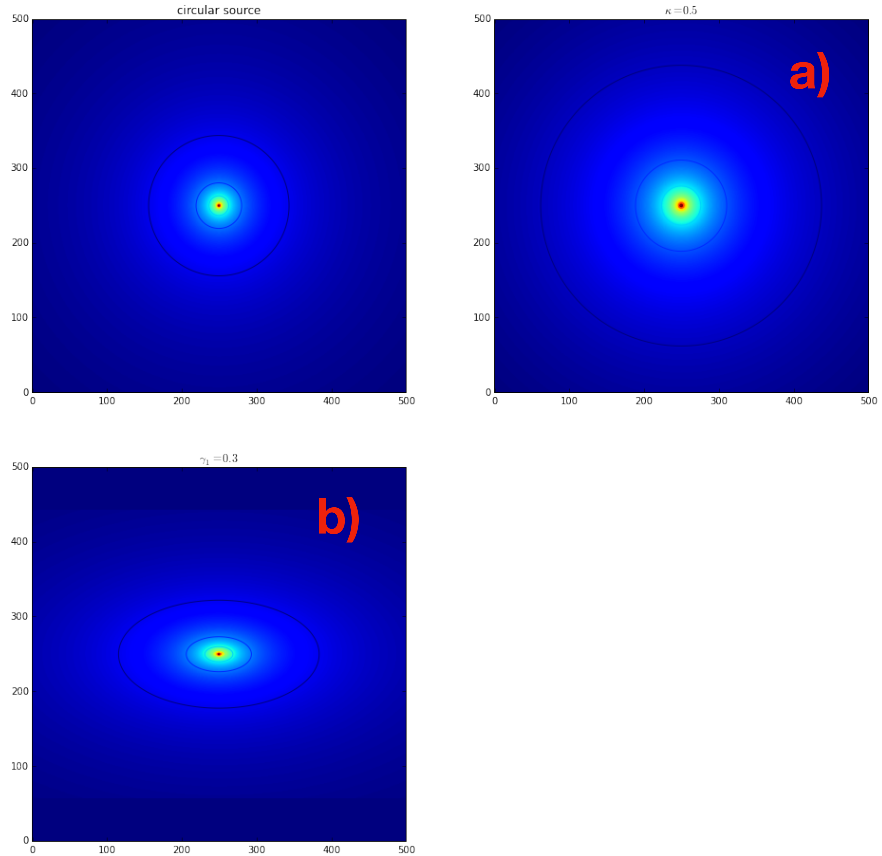
➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$



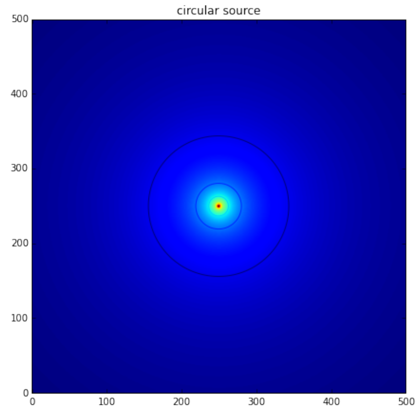
ON THE SPIN-2 NATURE OF SHEAR

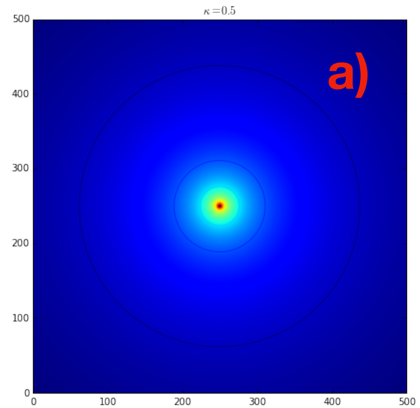
Consider a circular source

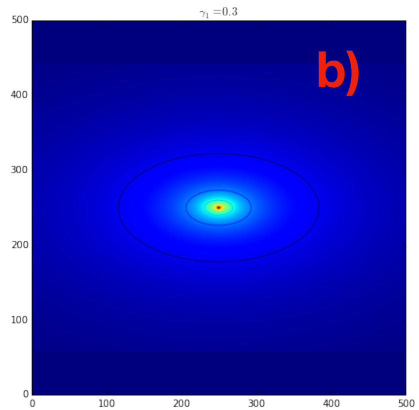
a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

➤ $\gamma_2 = 0 \quad \gamma_1 < 0$







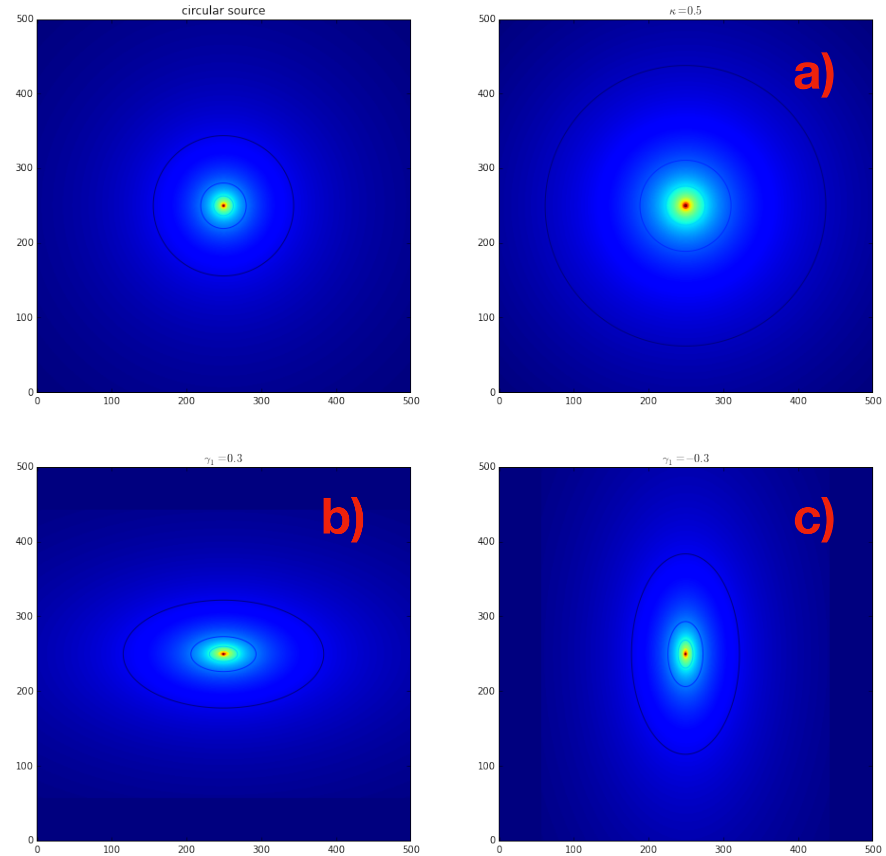
ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

c) ➤ $\gamma_2 = 0 \quad \gamma_1 < 0$



ON THE SPIN-2 NATURE OF SHEAR

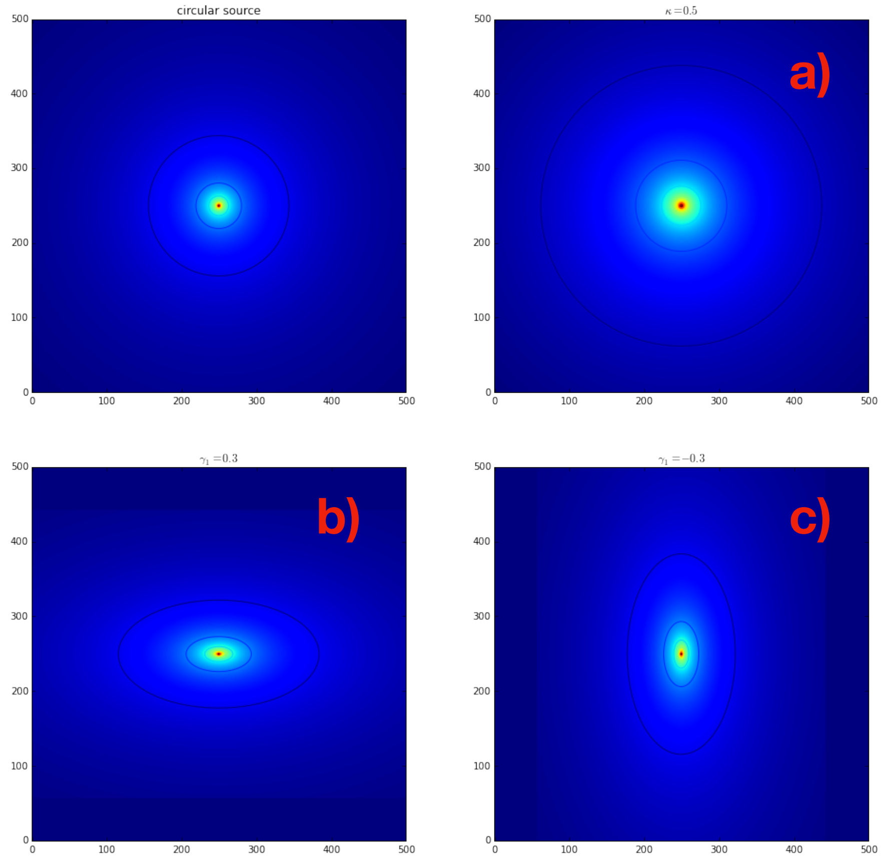
Consider a circular source

a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

c) ➤ $\gamma_2 = 0 \quad \gamma_1 < 0$

➤ $\gamma_2 > 0 \quad \gamma_1 = 0$



ON THE SPIN-2 NATURE OF SHEAR

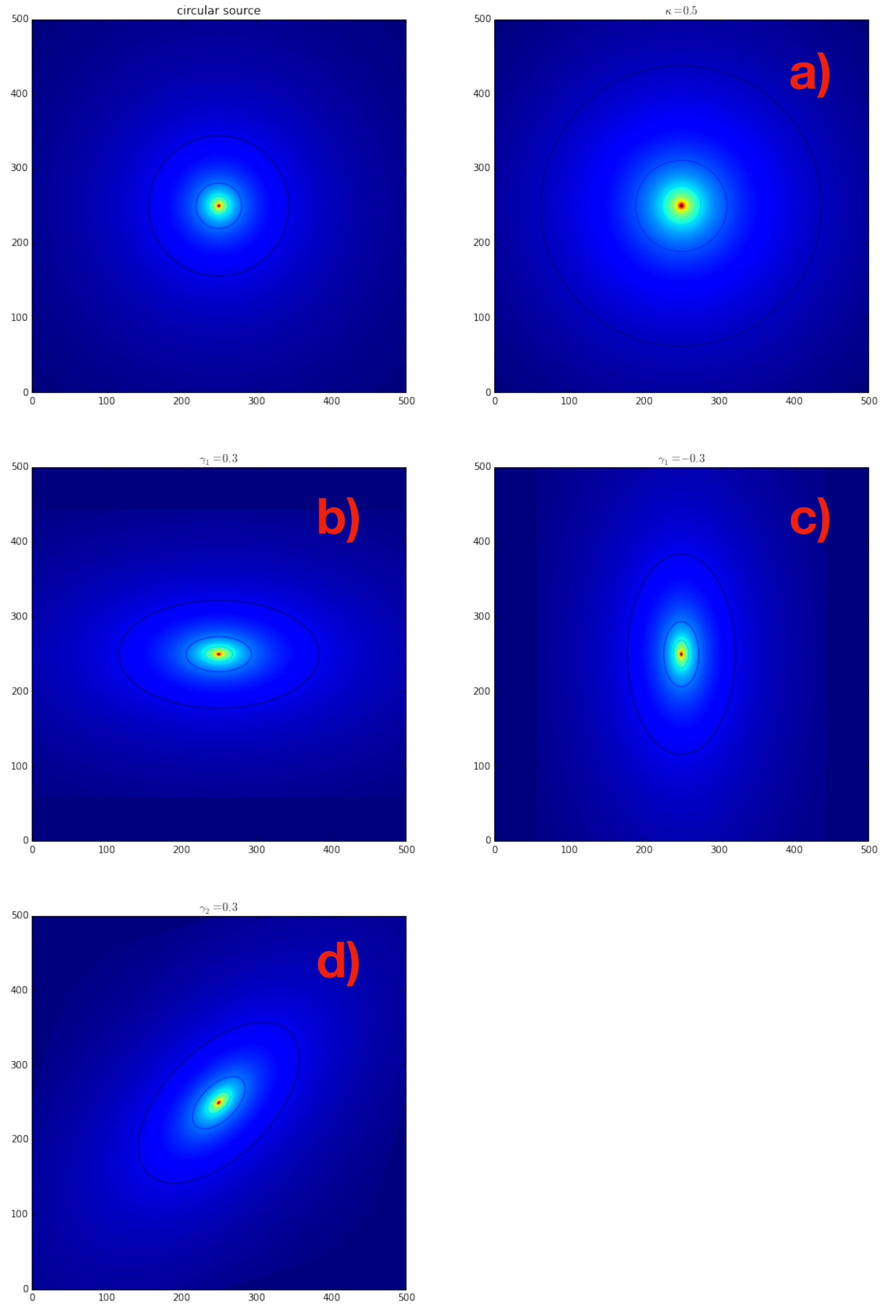
Consider a circular source

a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

c) ➤ $\gamma_2 = 0 \quad \gamma_1 < 0$

d) ➤ $\gamma_2 > 0 \quad \gamma_1 = 0$



ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

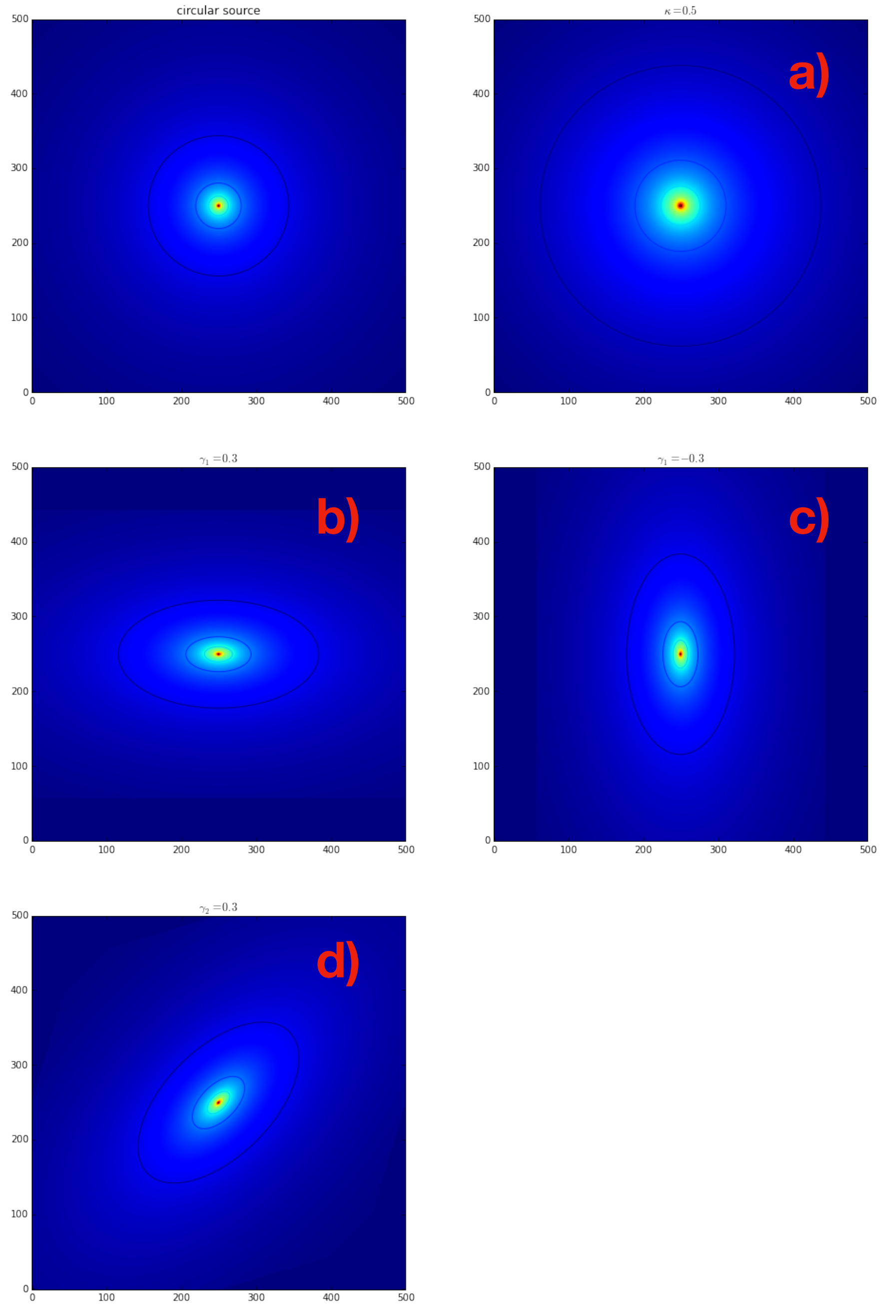
a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

c) ➤ $\gamma_2 = 0 \quad \gamma_1 < 0$

d) ➤ $\gamma_2 > 0 \quad \gamma_1 = 0$

➤ $\gamma_2 < 0 \quad \gamma_1 = 0$



ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

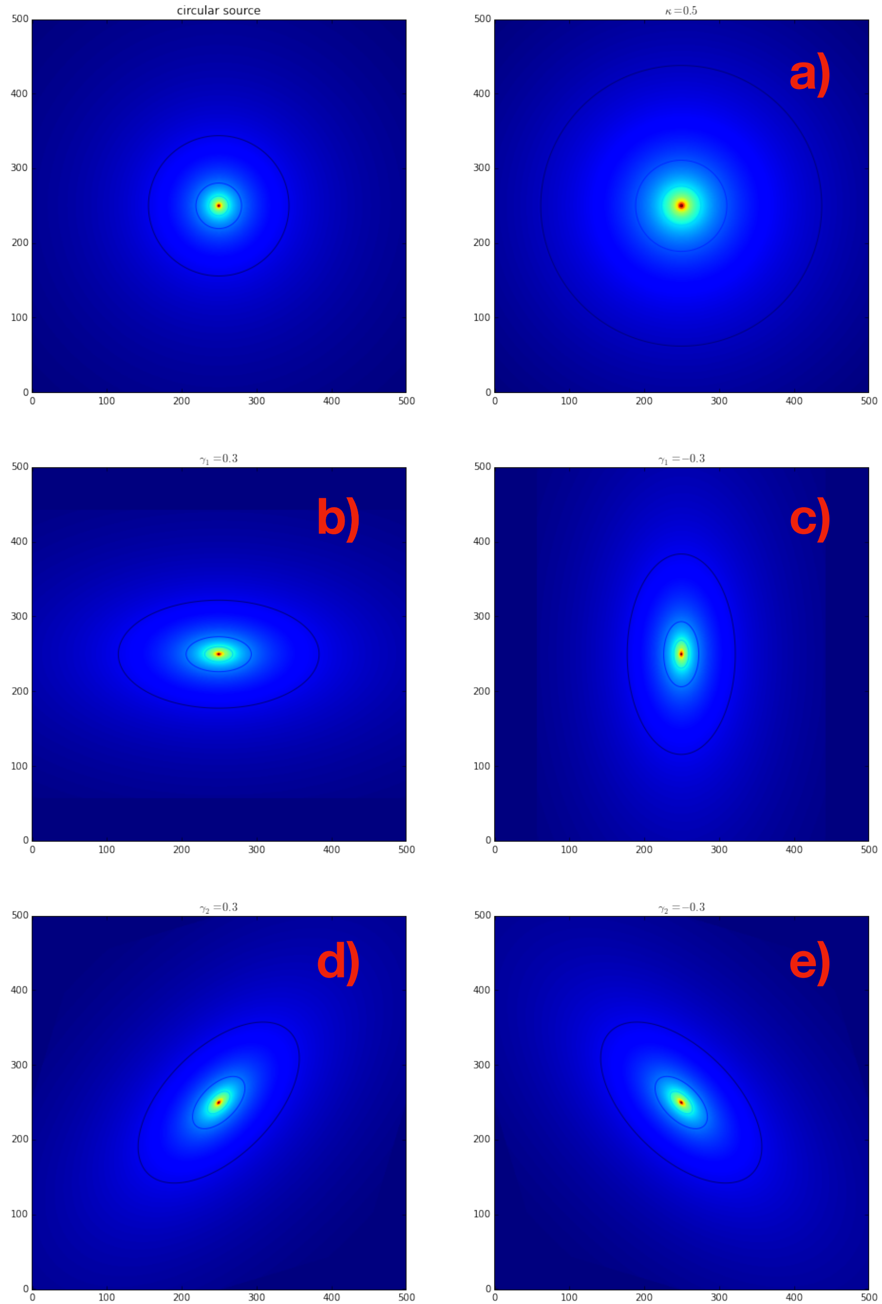
a) **How is it distorted if we apply a pure convergence transformation?**

b) ➤ $\gamma_2 = 0 \quad \gamma_1 > 0$

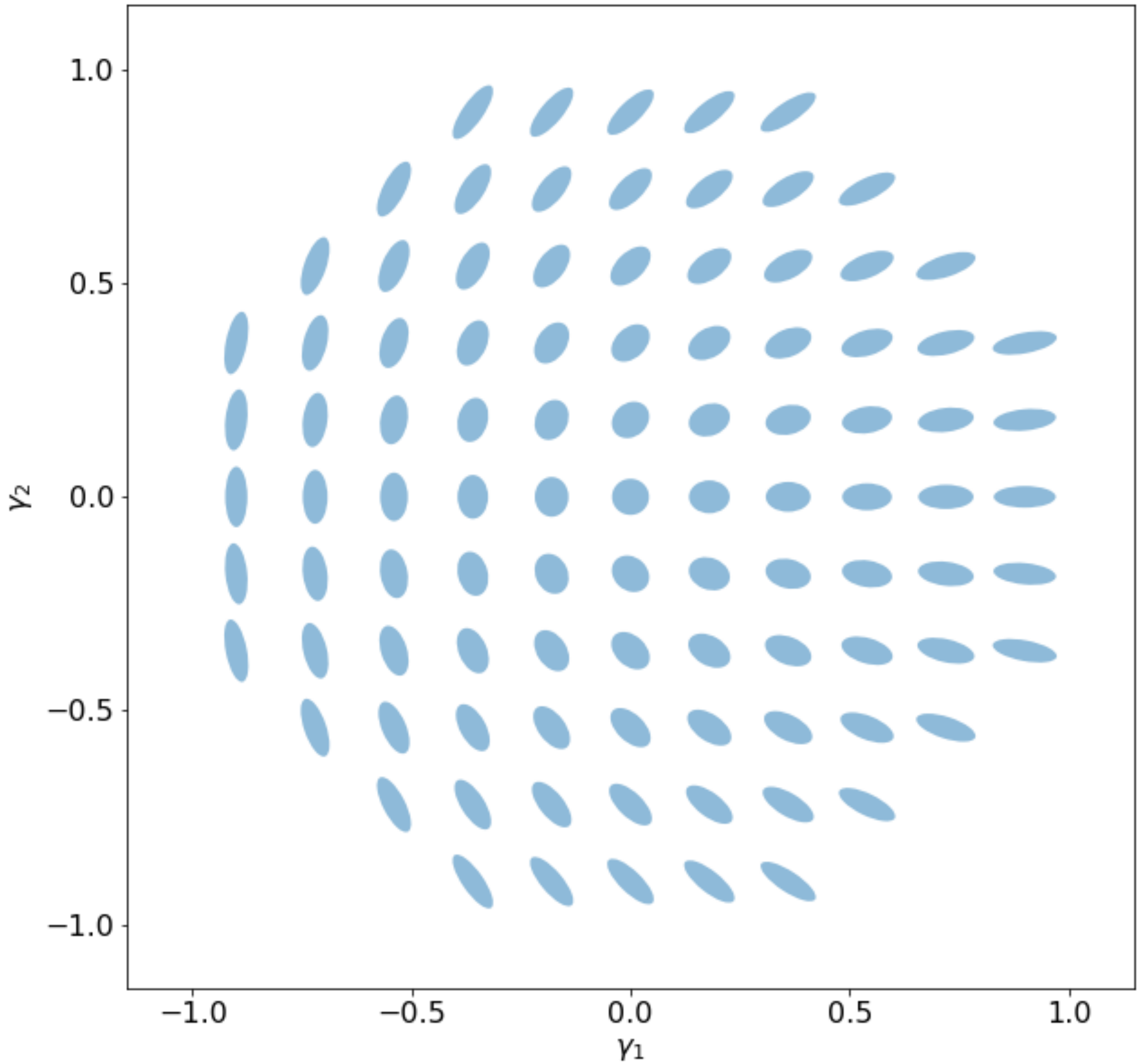
c) ➤ $\gamma_2 = 0 \quad \gamma_1 < 0$

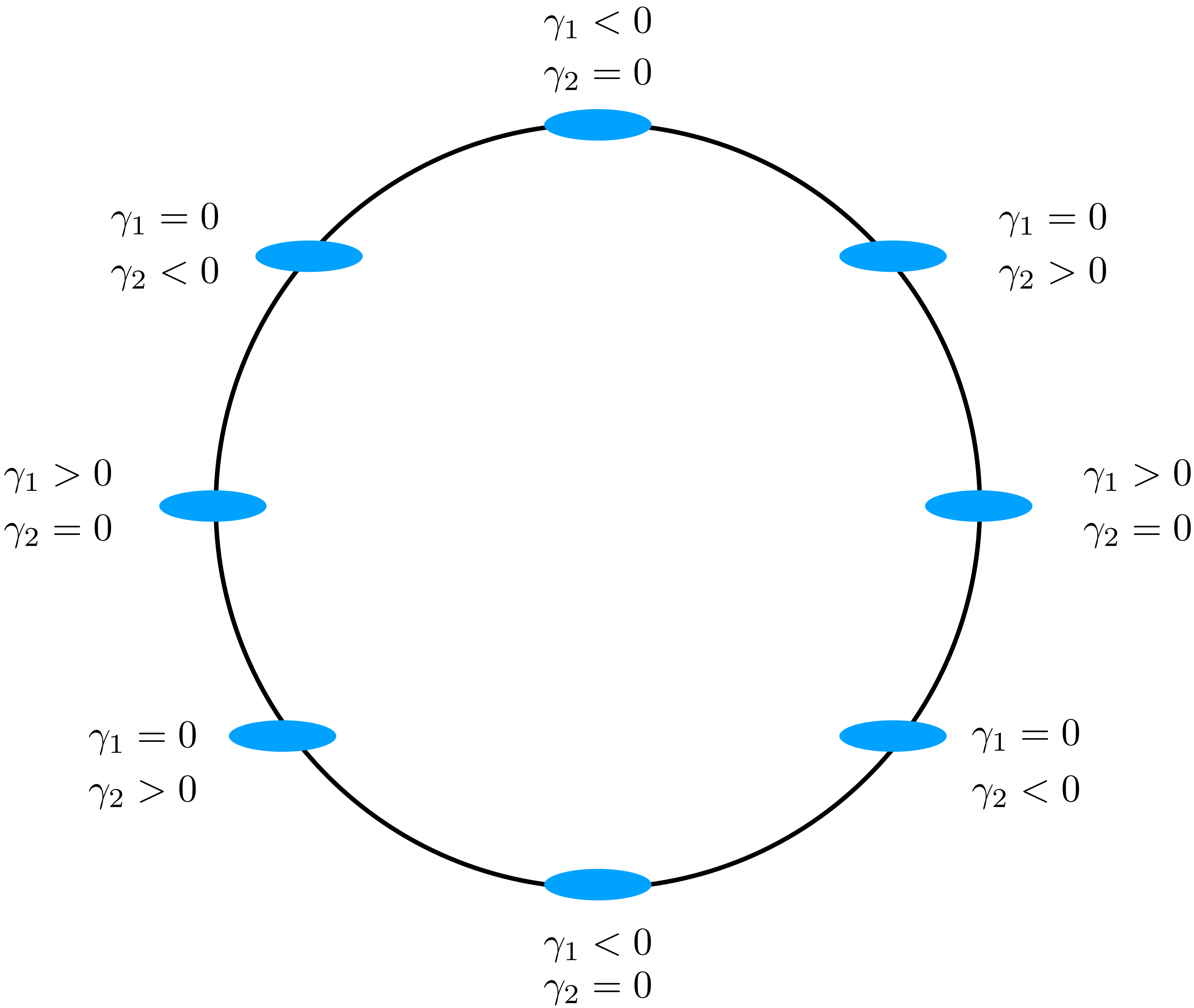
d) ➤ $\gamma_2 > 0 \quad \gamma_1 = 0$

e) ➤ $\gamma_2 < 0 \quad \gamma_1 = 0$



SHEAR DISTORTIONS




$\gamma_1 < 0$
$\gamma_2 = 0$
$\gamma_1 = 0$
$\gamma_2 < 0$
$\gamma_1 = 0$
$\gamma_2 > 0$
$\gamma_1 > 0$
$\gamma_2 = 0$
$\gamma_1 > 0$
$\gamma_2 = 0$
$\gamma_1 = 0$
$\gamma_2 > 0$
$\gamma_1 = 0$
$\gamma_2 < 0$
$\gamma_1 < 0$
$\gamma_2 = 0$

DEPENDENCE ON REDSHIFT

We have seen that the lensing potential, the deflection angle, the convergence, the shear… depend on a combination of distances.

For example:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_L D_S}\frac{2}{c^2}\int_0^\infty \Phi(D_L\vec{\theta})dz$$

$$\vec{\alpha}(\vec{\theta}) = \vec{\nabla}\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\alpha}(\vec{\theta})$$

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{cr}} = \frac{1}{2}\Delta_\theta\hat{\Psi}(\vec{\theta}) \qquad \gamma_1 = \frac{1}{2}(\hat{\Psi}_{11} - \hat{\Psi}_{22}) \qquad \gamma_2 = \hat{\Psi}_{12} = \hat{\Psi}_{21} \qquad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

The distance ratio $D_{LS}D_L/D_S$ is called “lensing distance”.

Both the shear and the convergence, being second derivatives of the lensing potential, scale as the lensing distance

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o}\int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

critical density

$$\Omega_m = \frac{\rho}{\rho_{\rm crit}} \qquad \rho_{\rm crit} = \frac{3H_o^2}{8\pi G}$$

H_o Hubble parameter

comoving angular size distance

$$D_{CA}(z) = \begin{cases} R_{\rm curv}\sin\left(\frac{\chi}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda < 1 \\ \chi & \Omega_m + \Omega_\Lambda = 1 \\ R_{\rm curv}\sinh\left(\frac{\chi}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o} \int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

critical density

$$\Omega_m = \frac{\rho}{\rho_{\rm crit}} \qquad \rho_{\rm crit} = \frac{3H_o^2}{8\pi G}$$

H_o Hubble parameter

comoving angular size distance

$$D_{CA}(z_1, z_2) = \begin{cases} R_{\rm curv} \sin\left(\frac{\chi_1 - \chi_2}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda < 1 \\ \chi_1 - \chi_2 & \Omega_m + \Omega_\Lambda = 1 \\ R_{\rm curv} \sinh\left(\frac{\chi_1 - \chi_2}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o} \int_0^{z_s} \frac{dz}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda + (1-\Omega_m - \Omega_\Lambda)(1+z)^2}}$$

comoving angular size distance

$$D_{CA}(z_1, z_2) = \begin{cases} R_{\rm curv} \sin\left(\frac{\chi_1 - \chi_2}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda < 1 \\ \chi_1 - \chi_2 & \Omega_m + \Omega_\Lambda = 1 \\ R_{\rm curv} \sinh\left(\frac{\chi_1 - \chi_2}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o \sqrt{1 - \Omega_m - \Omega_\Lambda}}$$

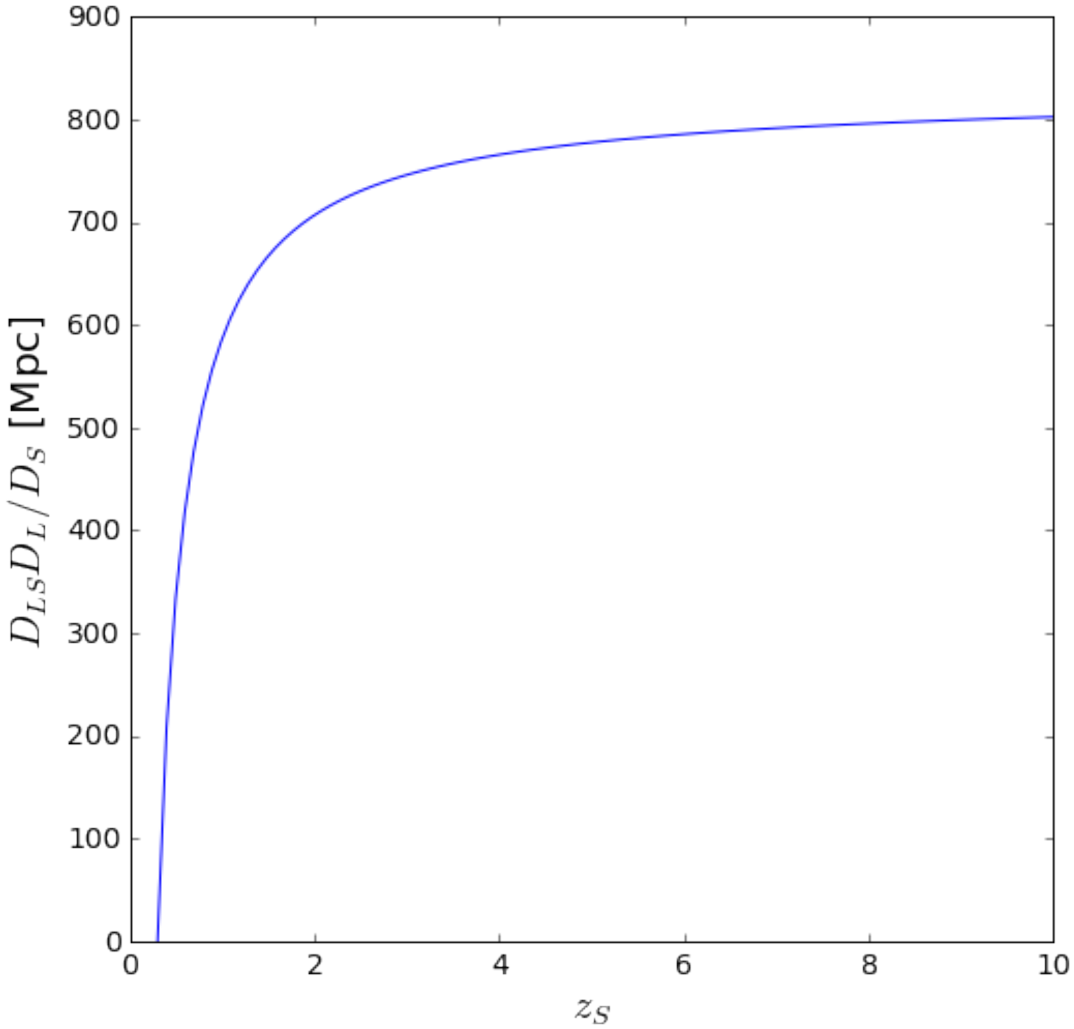
(proper) angular size distance

$$D_A(z, z_o) = \frac{D_{CA}(z, z_o)}{(1+z)}$$

luminosity distance

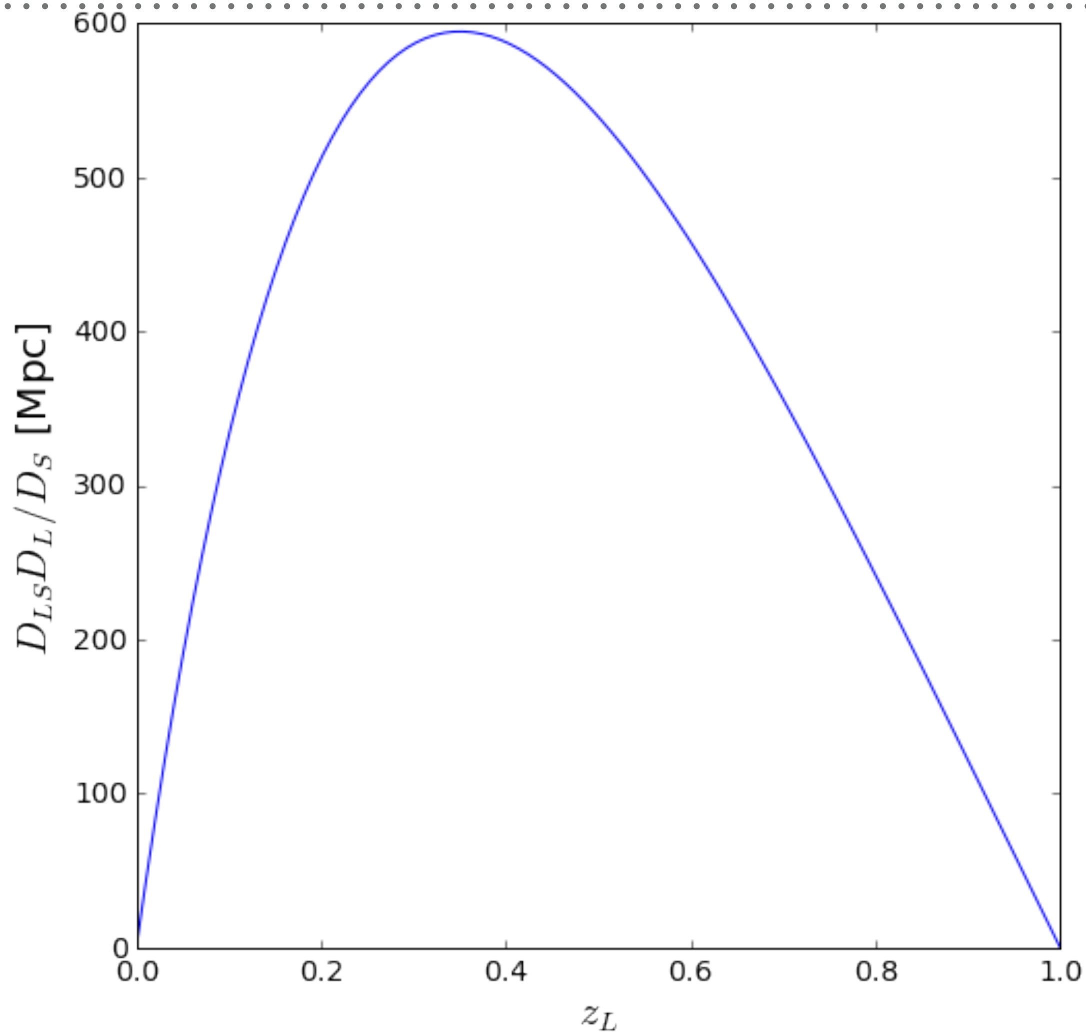
$$D_L(z) = (1+z) D_{CA}(z)$$

HOW DOES THE LENSING DISTANCE SCALE WITH SOURCE REDSHIFT?



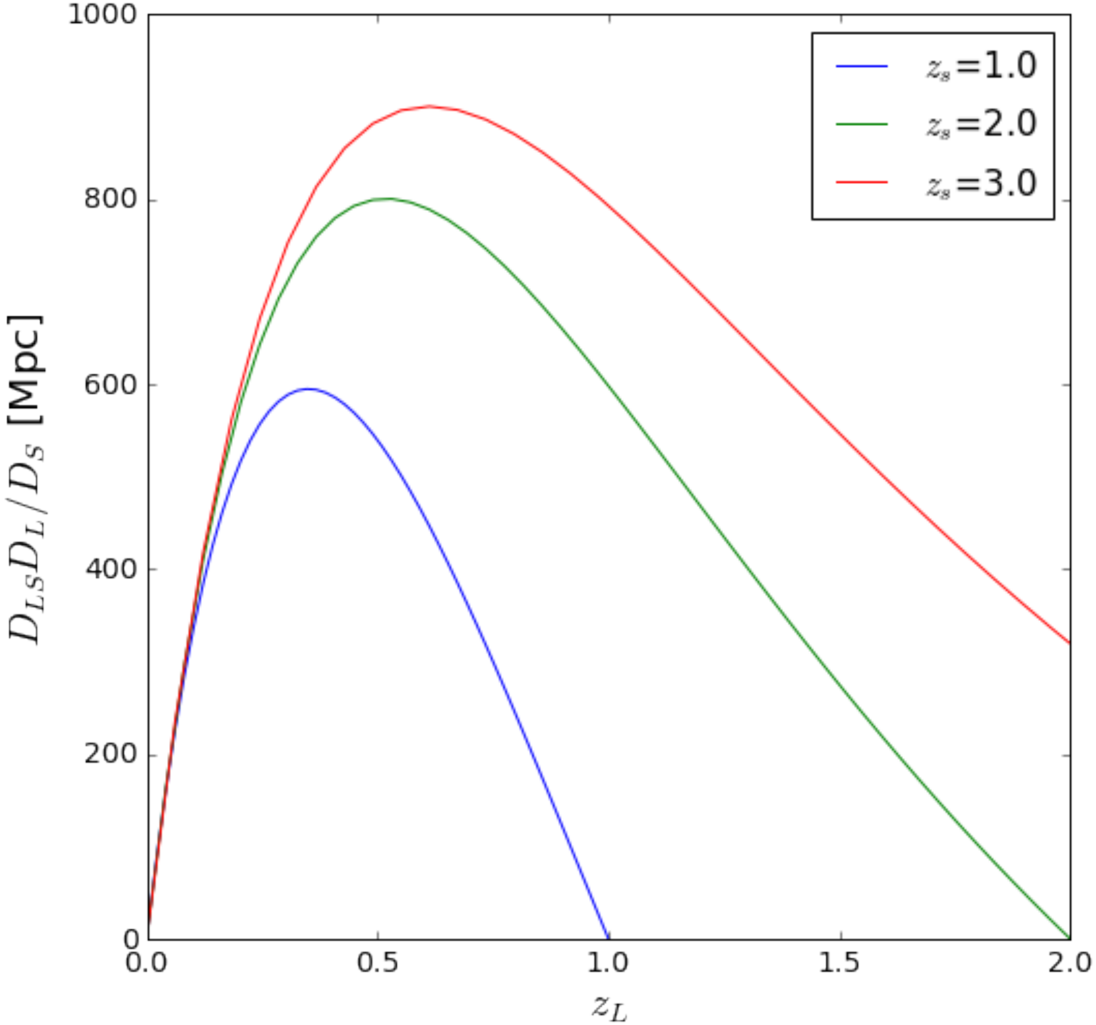
Note that if the lensing distance grows, the critical surface density decreases, the convergence and the shear grow!

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



The lensing distance peaks at ~half way between the source and the observer, meaning that there is an optimal distance where the lens produces its largest effects.

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



Of course, the peak moves to larger distances as the distance to the source increases.

CONSERVATION OF SURFACE BRIGHTNESS

*The source **surface brightness** is*

$$I_\nu = \frac{dE}{dt dA d\Omega d\nu}$$

In phase space, the radiation emitted is characterized by the density

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p}$$

In absence of photon creations or absorptions, f is conserved (Liouville theorem)

$$dN = \frac{dE}{h\nu} = \frac{dE}{cp}$$

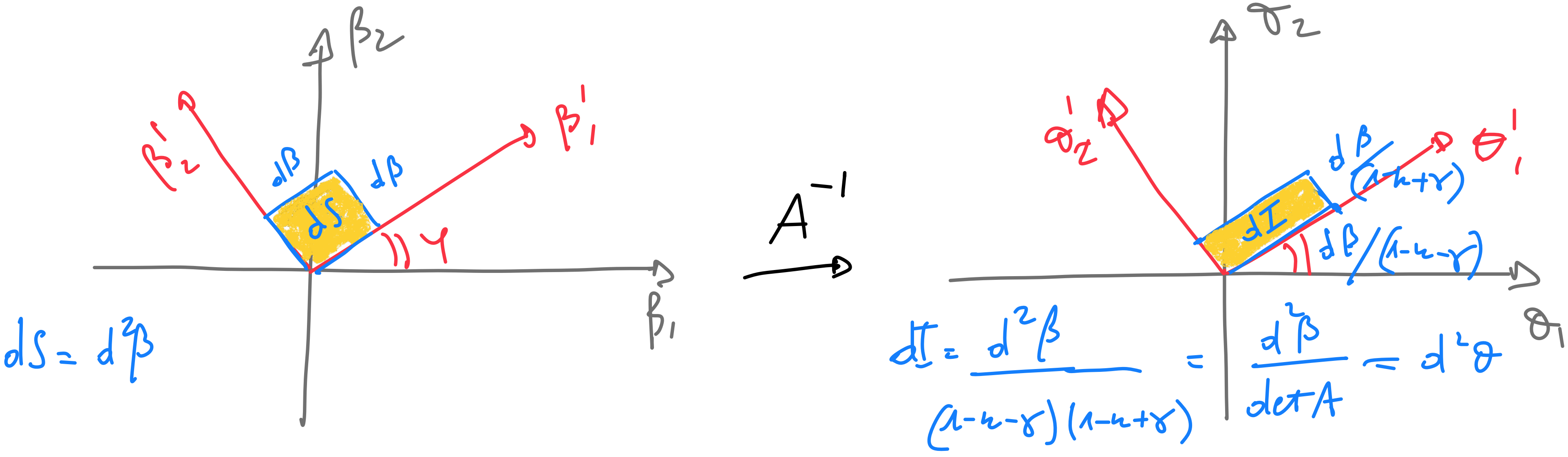
$$d^3x = c dt dA$$

$$d^3\vec{p} = p^2 dp d\Omega$$

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p} = \frac{dE}{hcp^3 dA dt d\nu d\Omega} = \frac{I_\nu}{hcp^3}$$

Since GL does not involve creation or absorption of photons, neither it changes the photon momenta (achromatic!), surface brightness is conserved!

MAGNIFICATION

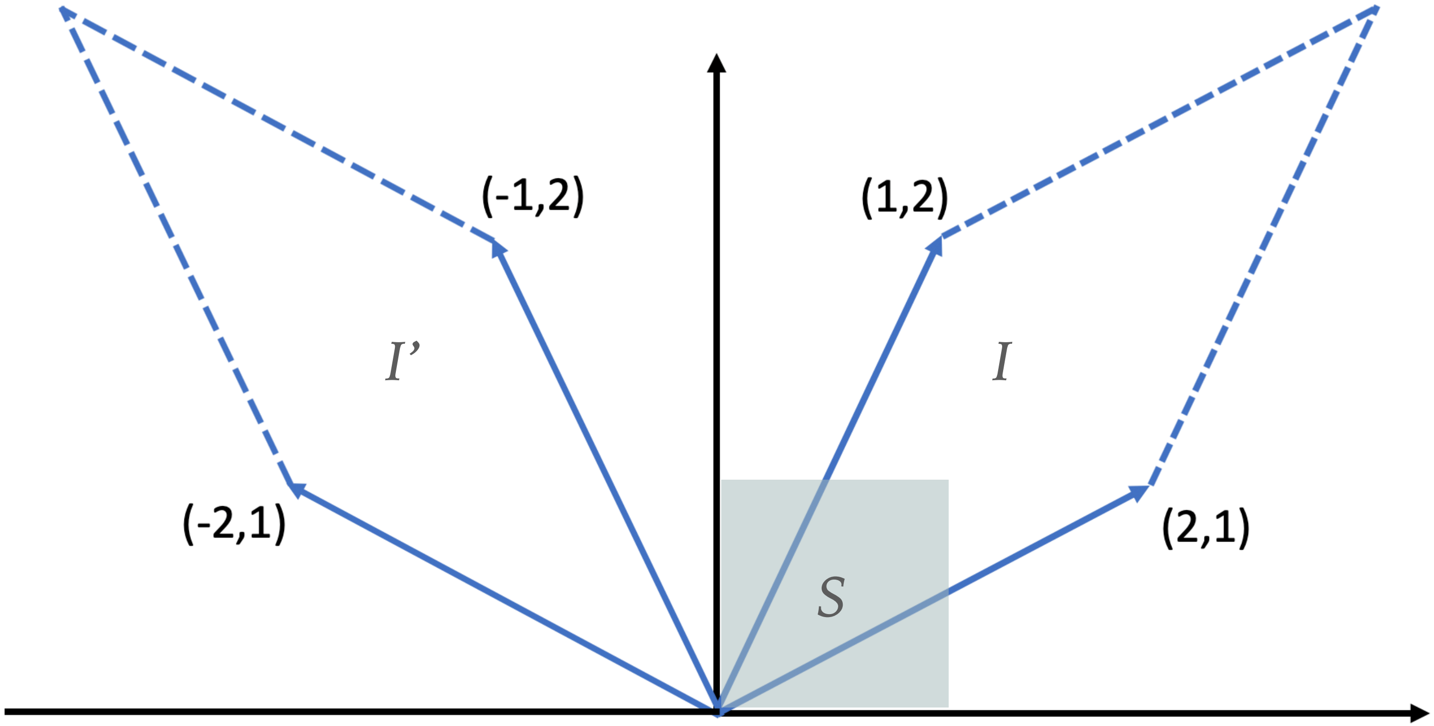


$$\mu(\overrightarrow{\theta}) = \frac{dI}{dS} = \frac{d^2\theta}{d^2\beta} = \det A^{-1}(\overrightarrow{\theta})$$

$$F_\nu = \int_I I_\nu(\overrightarrow{\theta})d^2\theta = \int_S I_\nu^S[\overrightarrow{\beta}(\overrightarrow{\theta})]\mu[\overrightarrow{\beta}(\overrightarrow{\theta})]d^2\beta$$

Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

POSITIVE AND NEGATIVE MAGNIFICATION



$$I' = \det\begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix} = -3 \qquad I = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$$