GRAVITATIONAL LENSING

15 - LENS MODELS: ELLIPTICAL LENSES

R. Benton Metcalf

2022-2023

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

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Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

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Elliptical contours with their major axis along the ξ_2 axis

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$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

Ensures that area of ellipse is equal to the area of the circle of radius ξ

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Let's derive the convergence in dimensionless units:

SINGULAR ISOTHERMAL ELLIPSOID

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}} \frac{\xi_0}{\xi_0}$$

Let's derive the convergence in dimensionless units:

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_\mathrm{L} D_\mathrm{LS}}{D_\mathrm{S}}$$

$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

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In polar coordinates:

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2}$$

$$\kappa(x, \varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

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With the ansatz $\Psi(x,\varphi) := x\tilde{\Psi}(\varphi)$

$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

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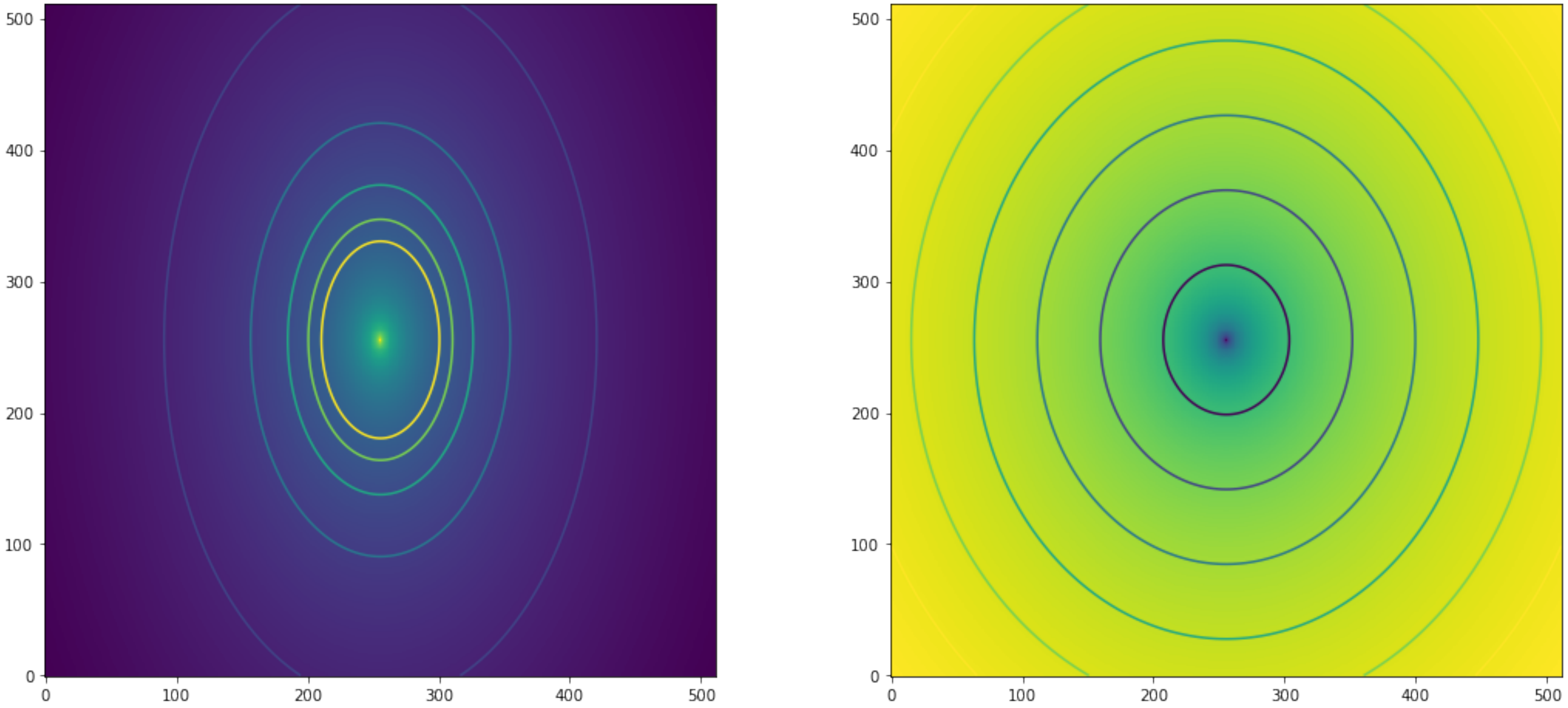
Solved with Green's method (Kormann et al. 1994):

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

CONVERGENCE AND POTENTIAL $f = 0.7$



SINGULAR ISOTHERMAL ELLIPSOID

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Let's compute the deflection angle:

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$$\begin{aligned}\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)\\ \alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\end{aligned}$$

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Analogy with the SIS: the deflection angle does not depend on x!

SINGULAR ISOTHERMAL ELLIPSOID

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The component of the shear:

SINGULAR ISOTHERMAL ELLIPSOID

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The component of the shear:

$$\begin{aligned}\gamma_1 &= \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial x_1}-\frac{\partial\alpha_2}{\partial x_2}\right)\\ \gamma_2 &= \frac{\partial\alpha_1}{\partial x_2}\end{aligned}$$

$$\begin{aligned}\gamma_1 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi\\ \gamma_2 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\alpha_1(\vec{x}) = \frac{\sqrt{f}}{f'}\text{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

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$$\gamma_1 = \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_2}{\partial x_2}\right)$$

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$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$

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Similarly to the SIS: $\gamma = \kappa$

SINGULAR ISOTHERMAL ELLIPSOID

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$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

We have now the ingredients to compute the lensing Jacobian matrix

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-2\kappa\sin^2\varphi & \kappa\sin 2\varphi \\ \kappa\sin 2\varphi & 1-2\kappa\cos^2\varphi \end{bmatrix}$$

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whose eigenvalues are:

$$\begin{aligned}\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,.\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

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As the SIS, the SIE does not have a radial critical line!

SINGULAR ISOTHERMAL ELLIPSOID

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The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

SINGULAR ISOTHERMAL ELLIPSOID

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The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

$$\kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)} \quad \Rightarrow \quad \vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}\left[\cos\varphi, \sin\varphi\right]$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)} [\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)} \cos\varphi - \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f} \cos\varphi\right) \\
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There is no radial caustic, but there is the cut, which can be computed as

$$\vec{y}_c = \lim_{x\to 0}\vec{y}(x,\varphi) = -\vec{\alpha}(\varphi)$$

SINGULAR ISOTHERMAL ELLIPSOID

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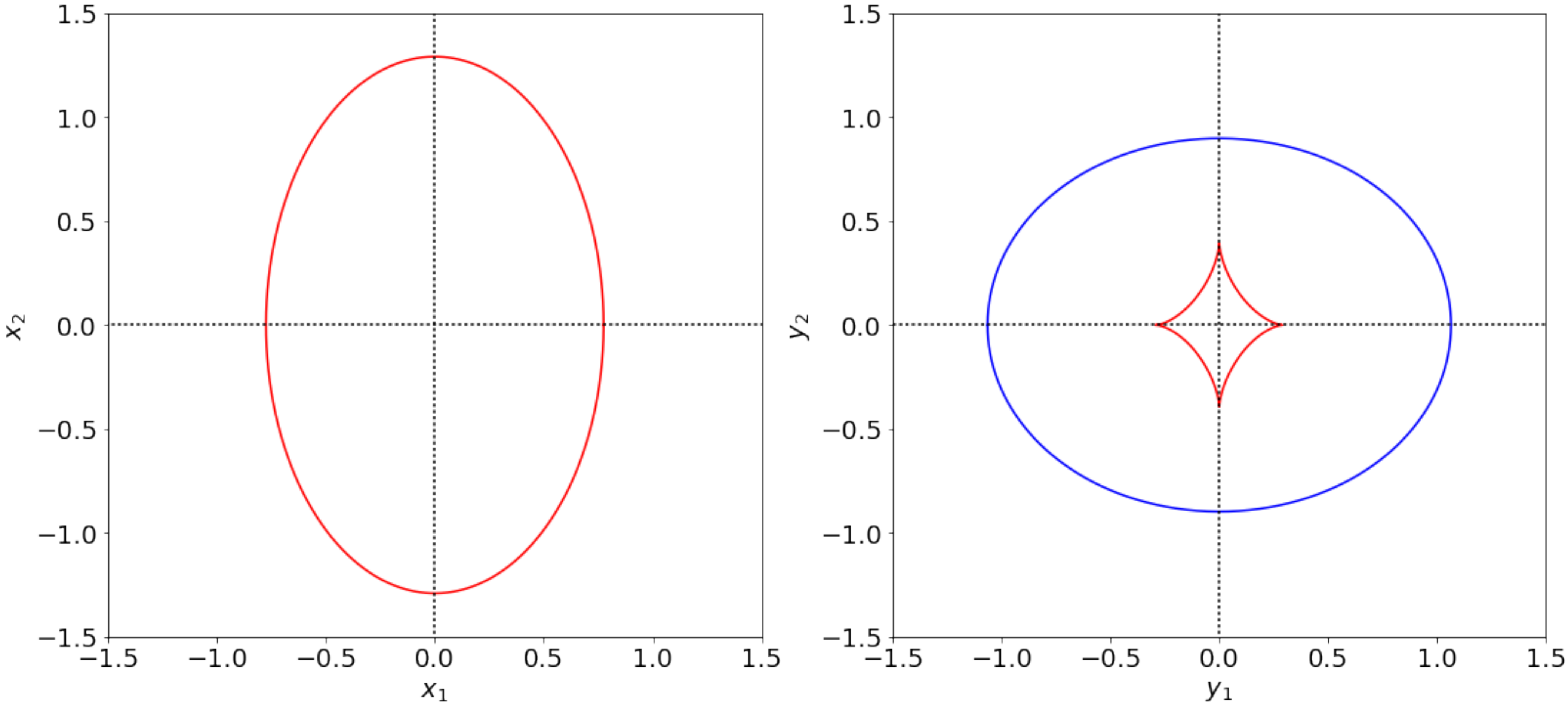
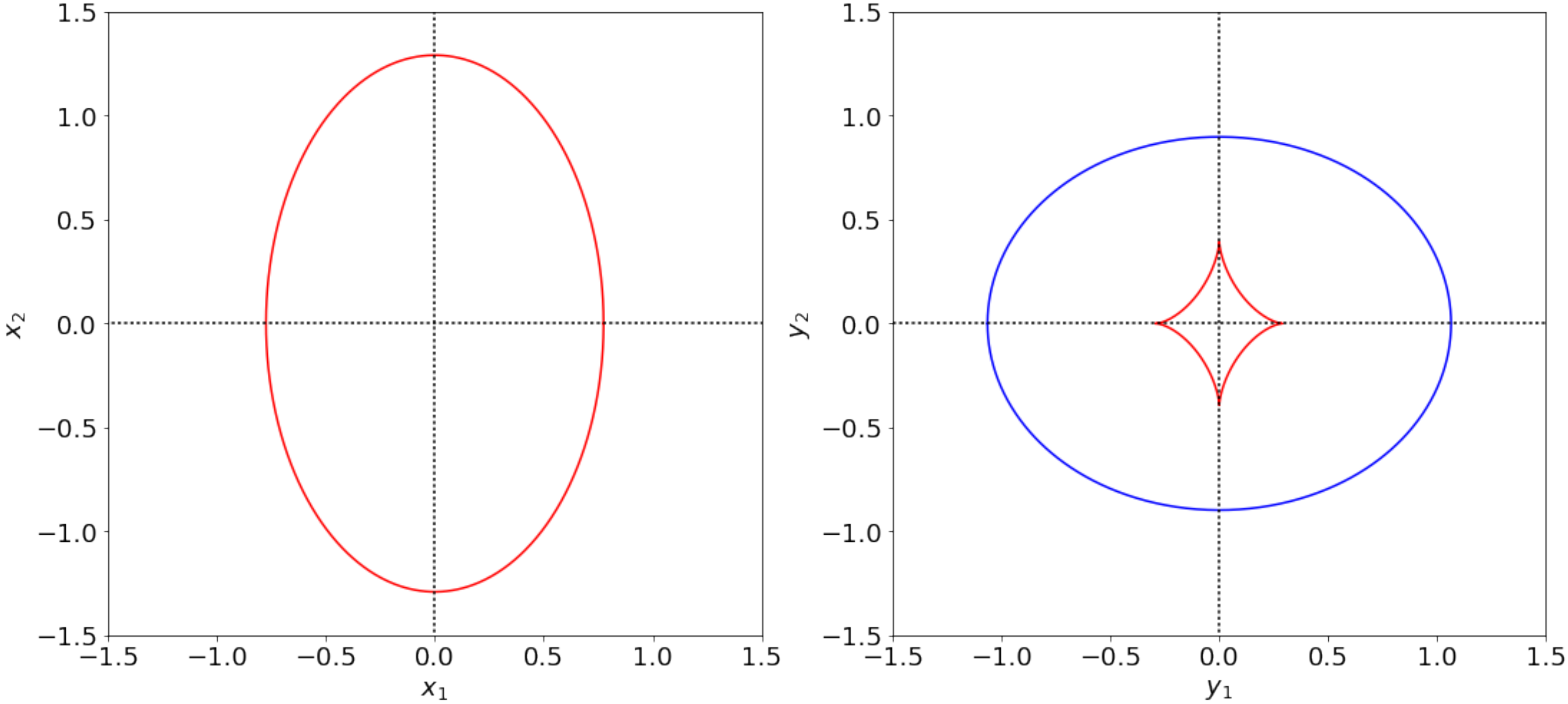
CRITICAL LINE, CUT, CAUSTIC ($f = 0.7$)



HOW MANY IMAGES?



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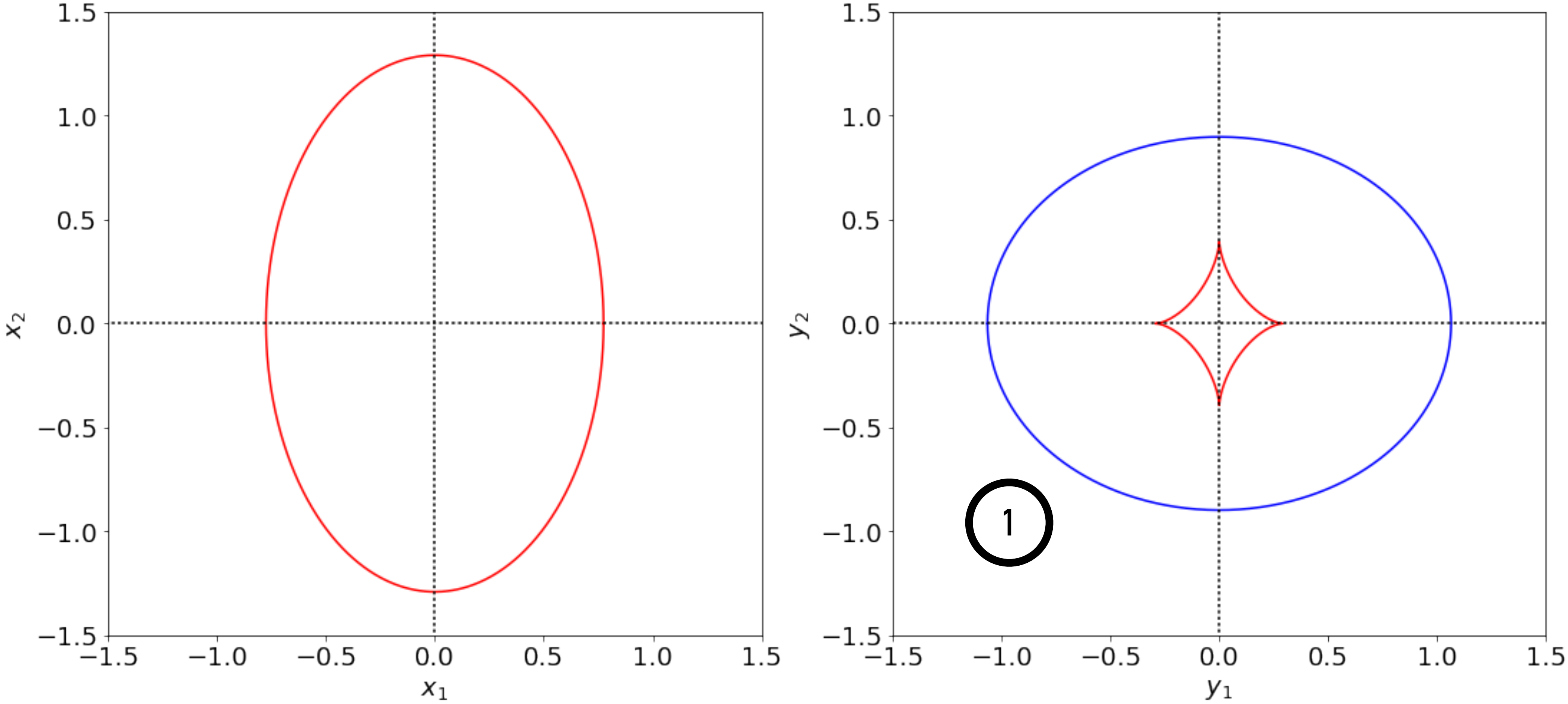

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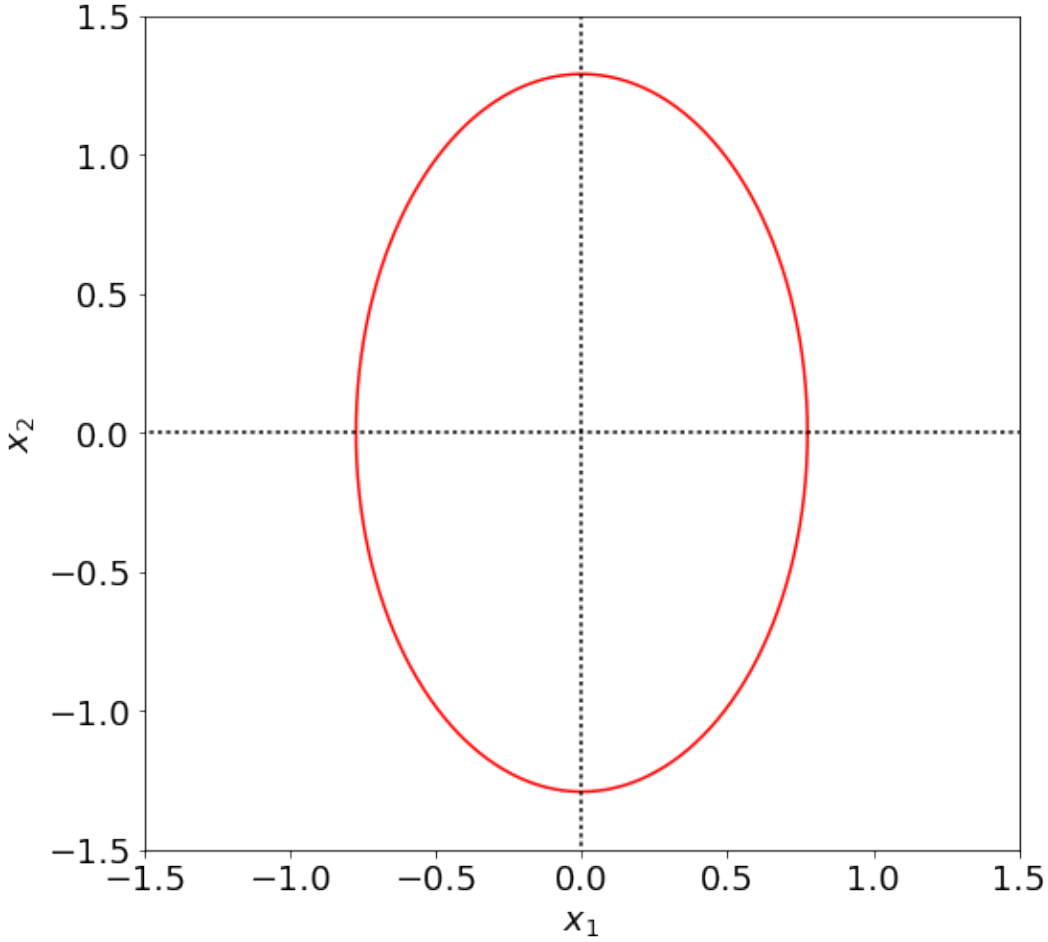
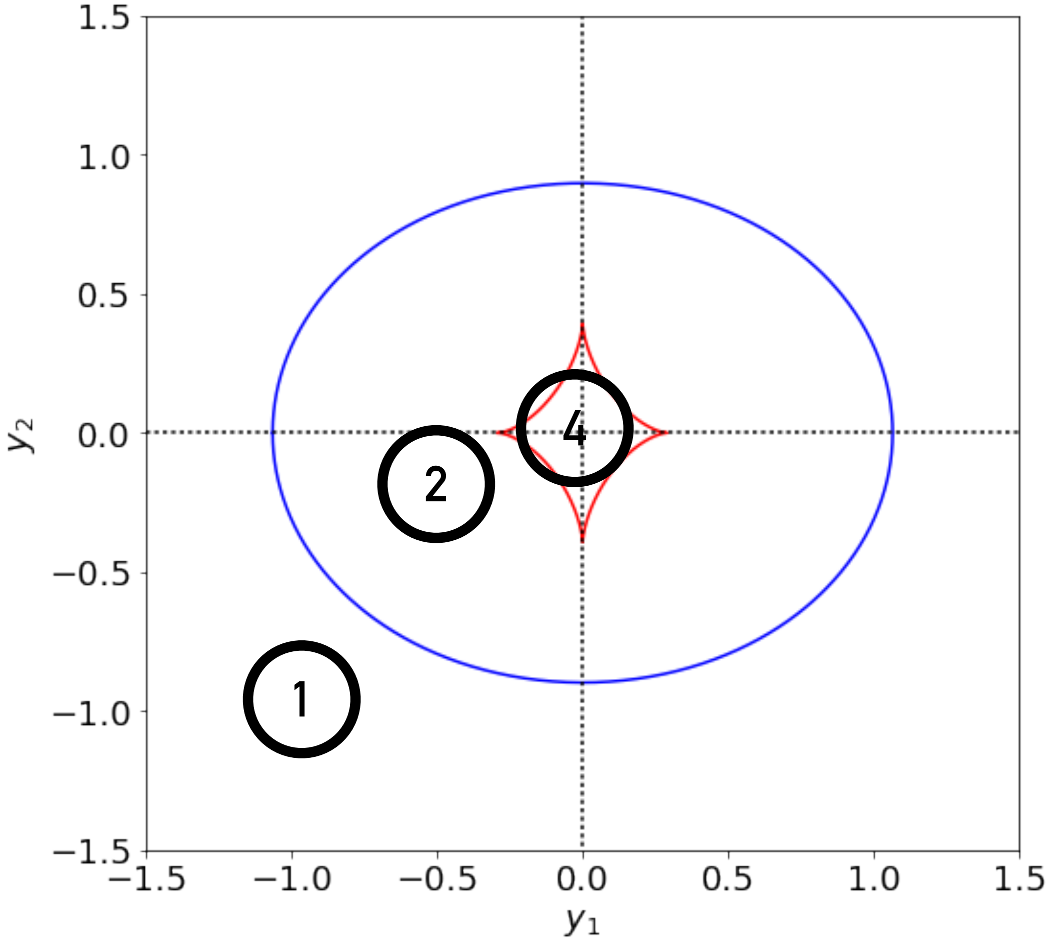
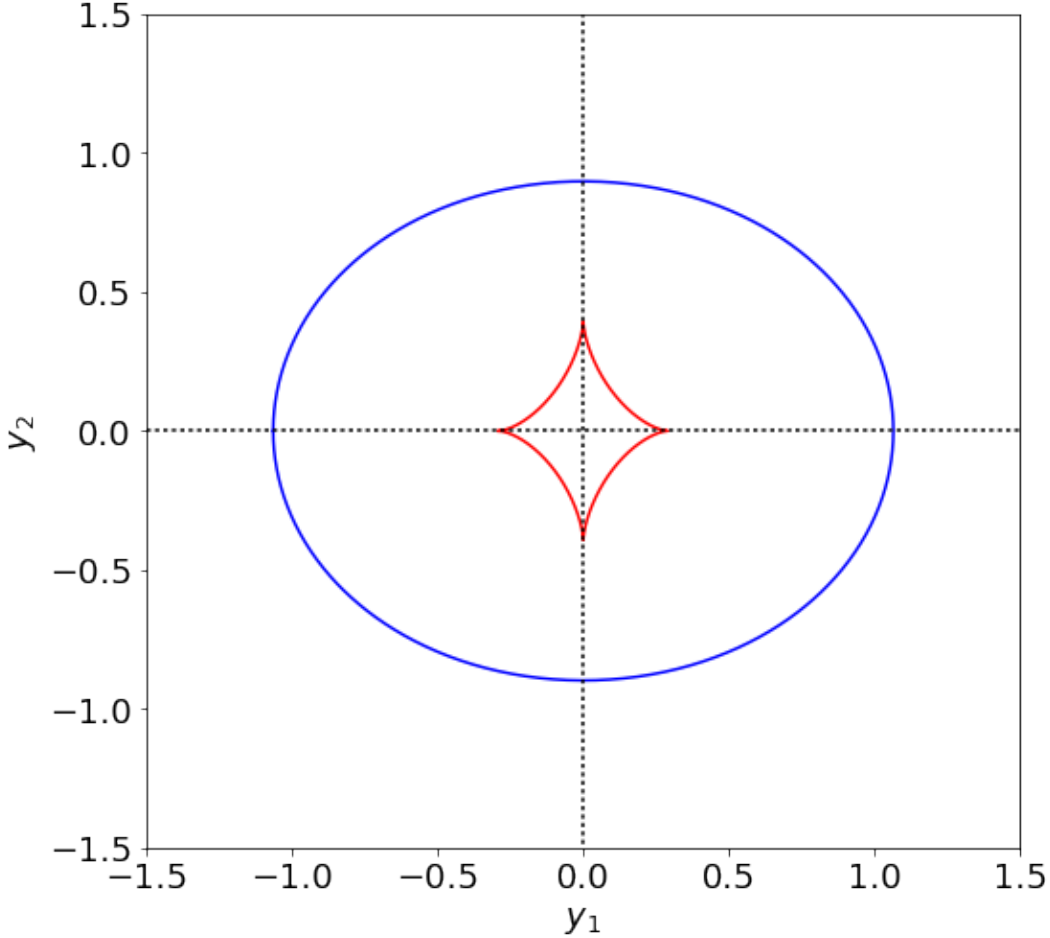




CRITICAL LINE, CUT, CAUSTIC



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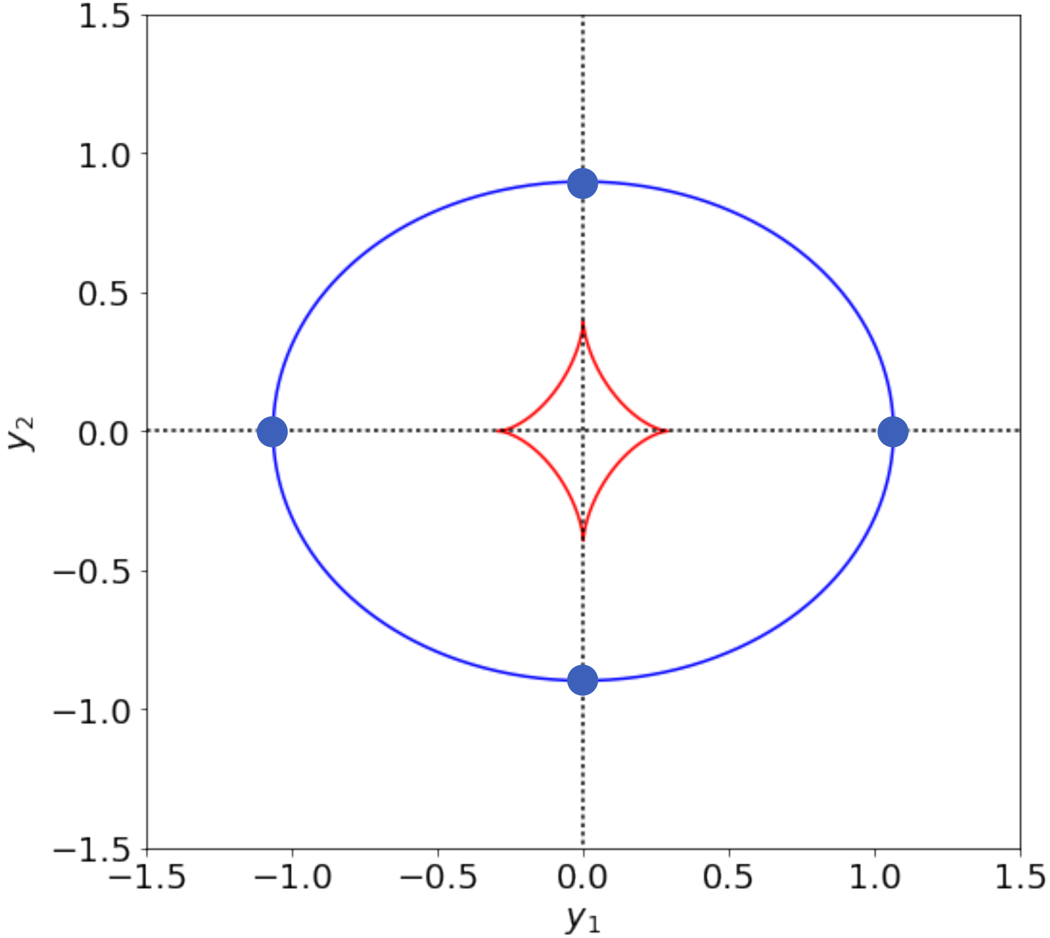
$$f' = \sqrt{1-f^2}$$

CRITICAL LINE, CUT, CAUSTIC



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$$
\begin{aligned}
s_{1,\pm,c} &= [y_{c,1}(\varphi = 0, \pi), 0]\,, \\
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\end{aligned}
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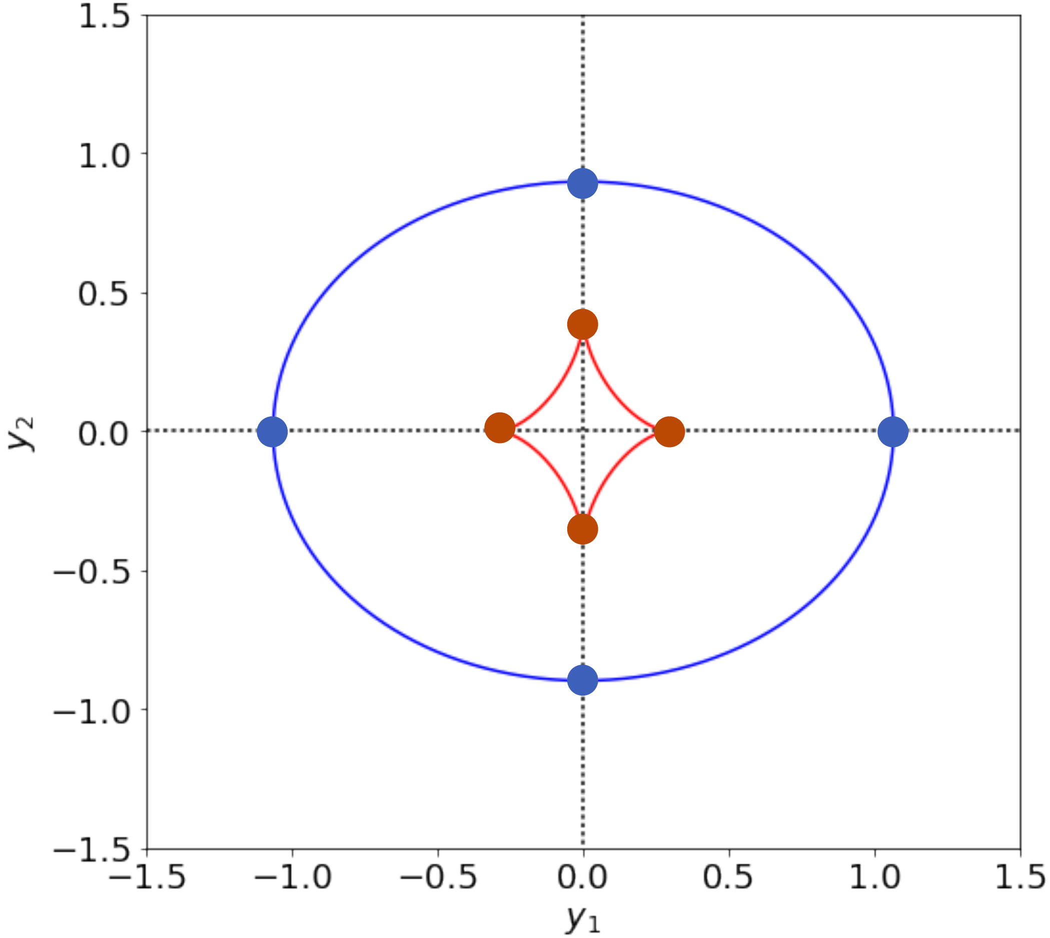
CRITICAL LINE, CUT, CAUSTIC



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$$s_{1,\pm,t} = [y_{t,1}(\varphi = 0, \pi), 0]\ ,$$

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CRITICAL LINE, CUT, CAUSTIC

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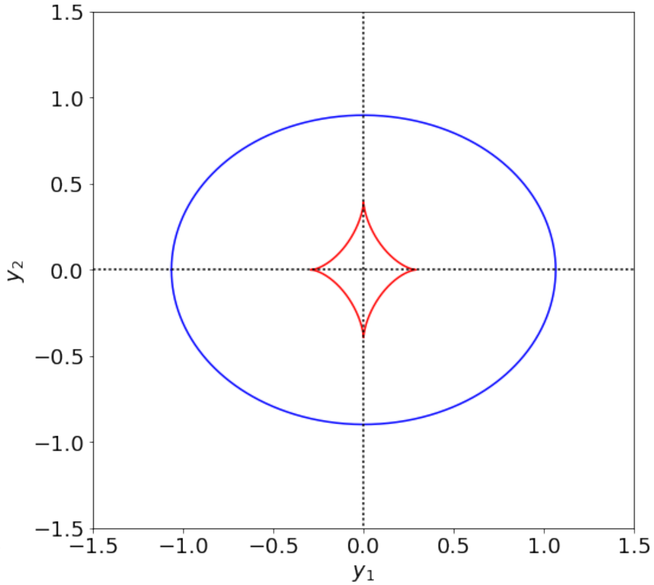
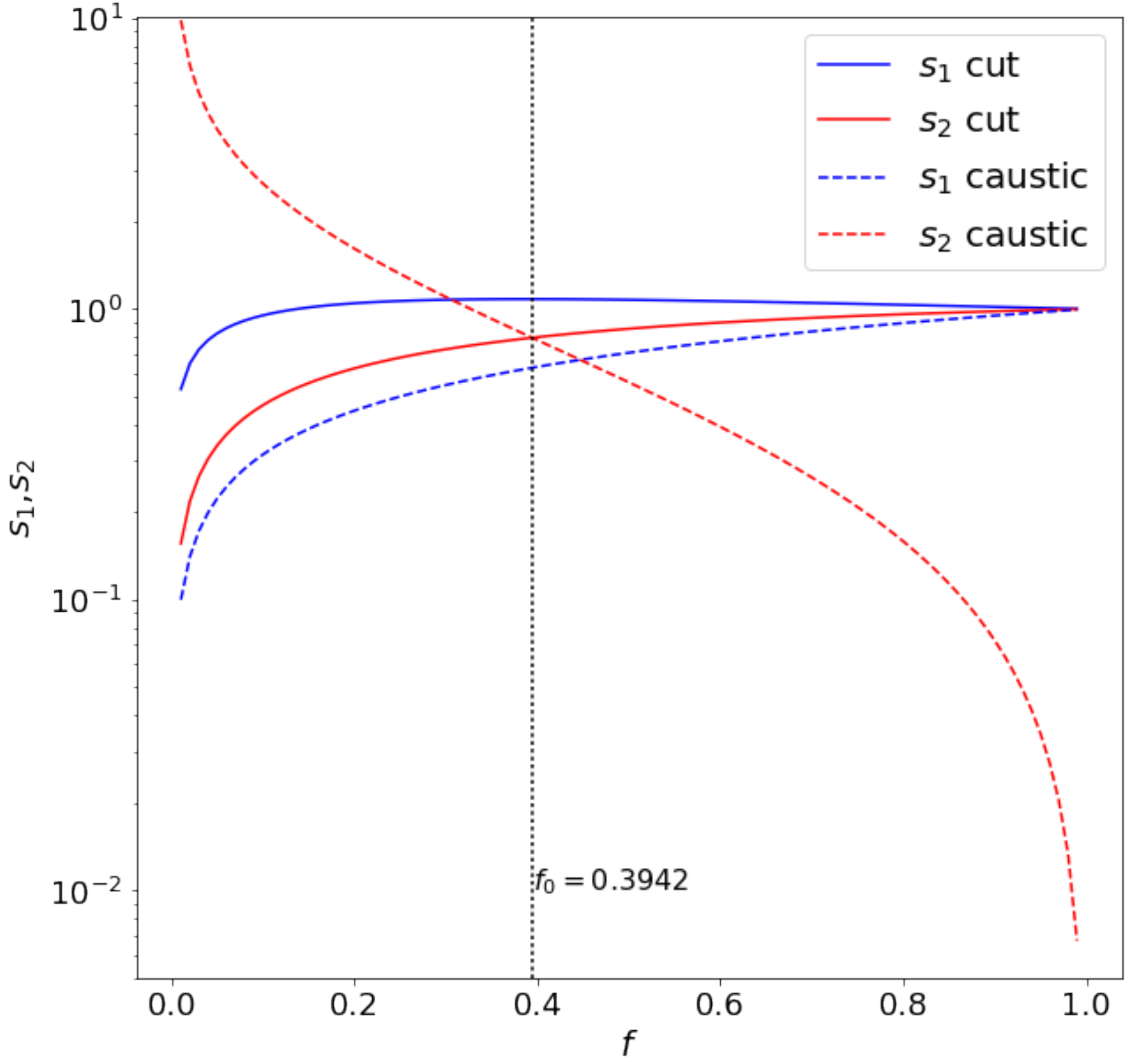
We can easily see that

$$s_{1,c} > s_{1,t}$$

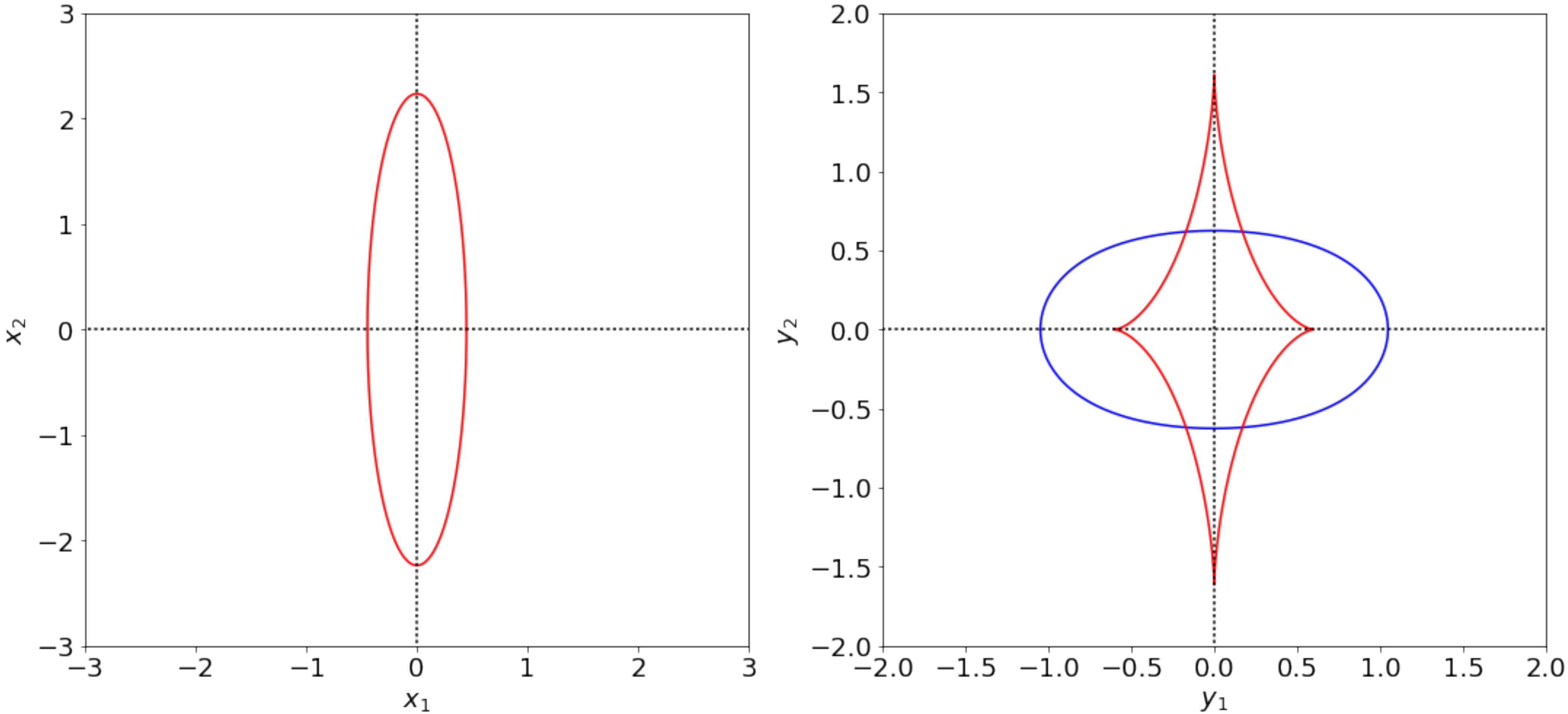
independent on f.



CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC



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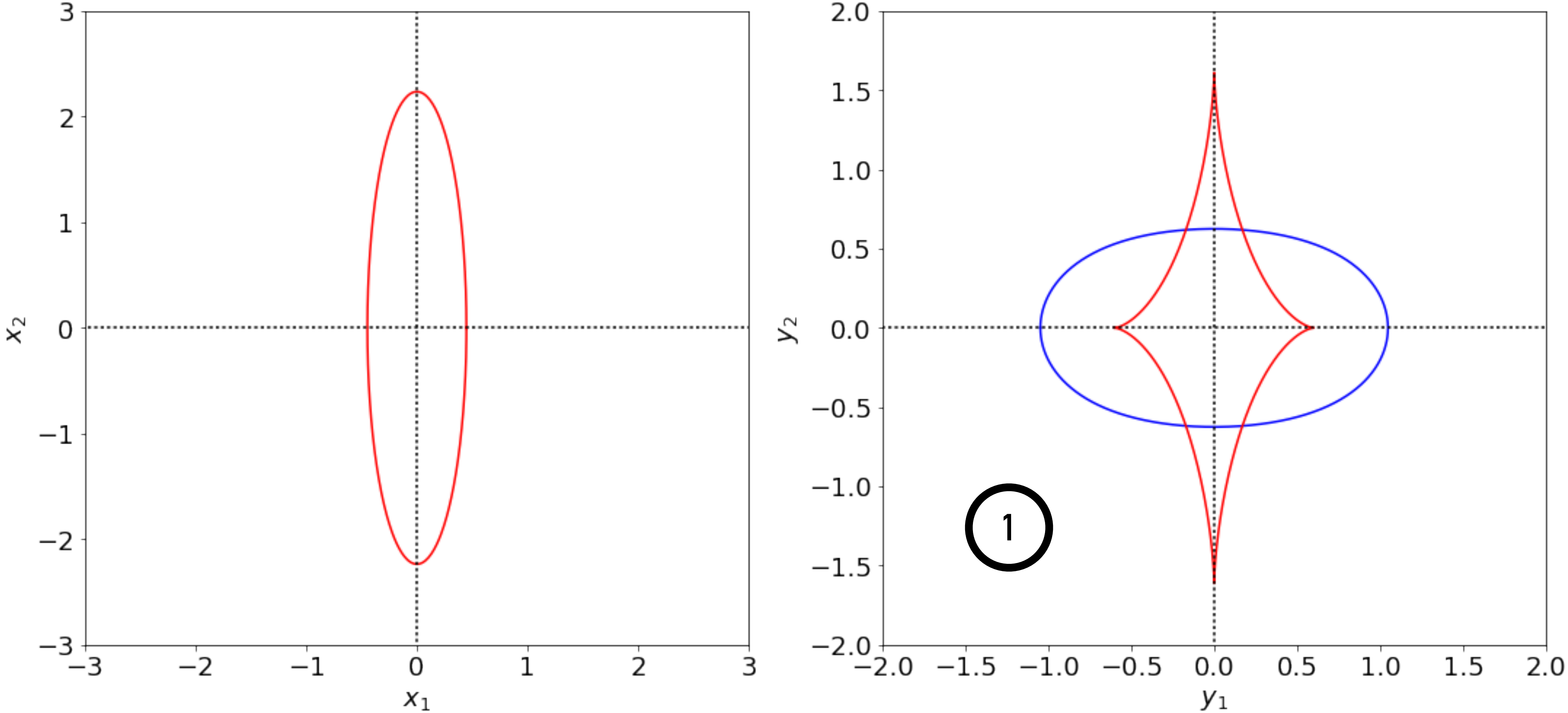
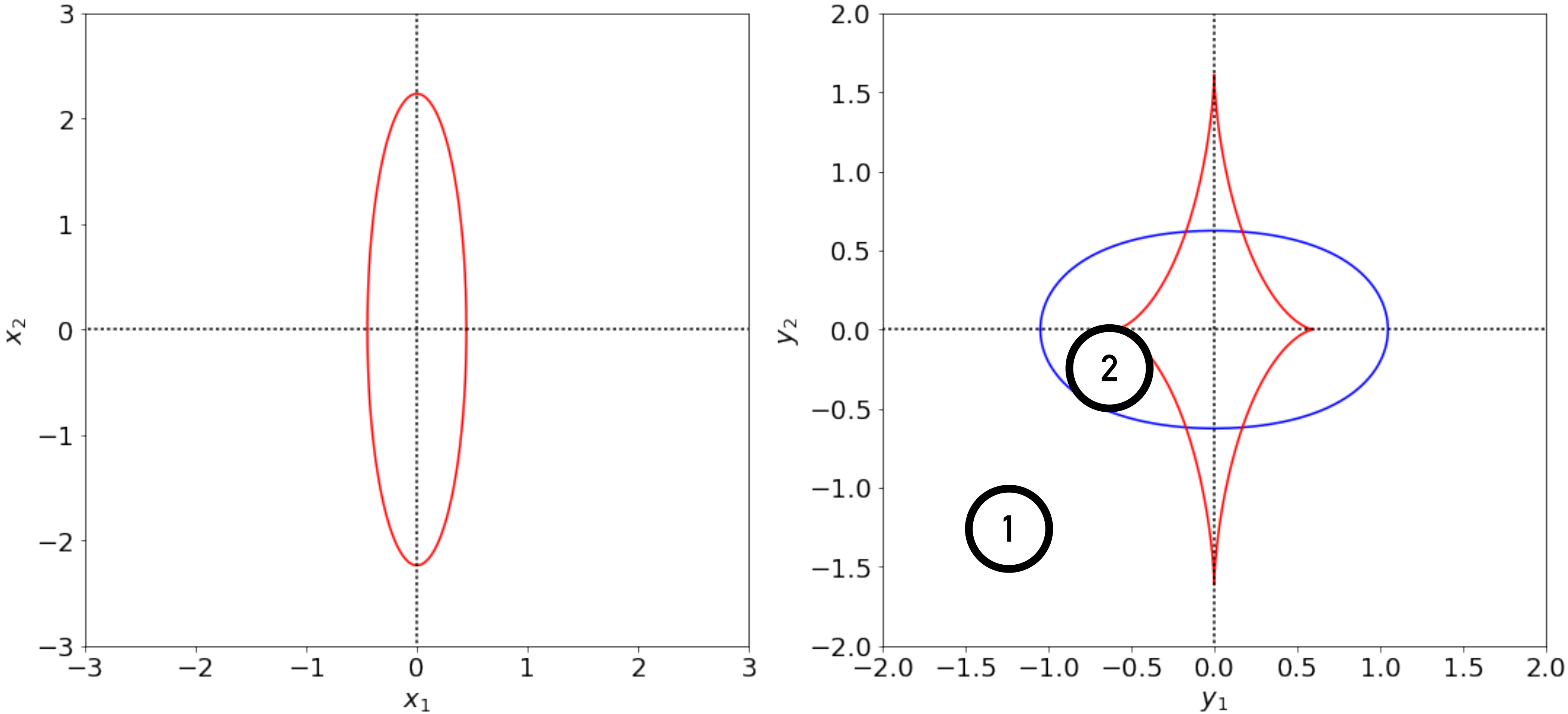
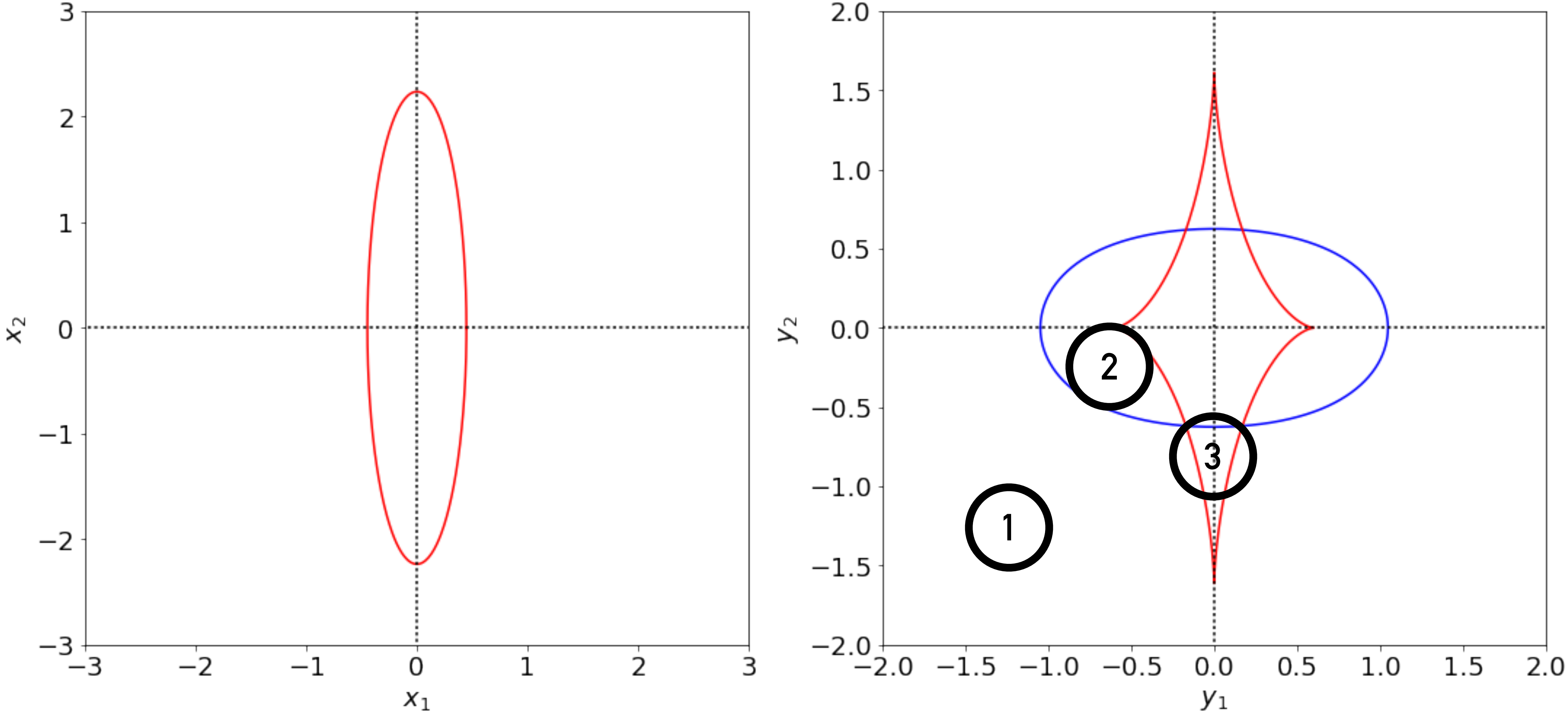


CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC



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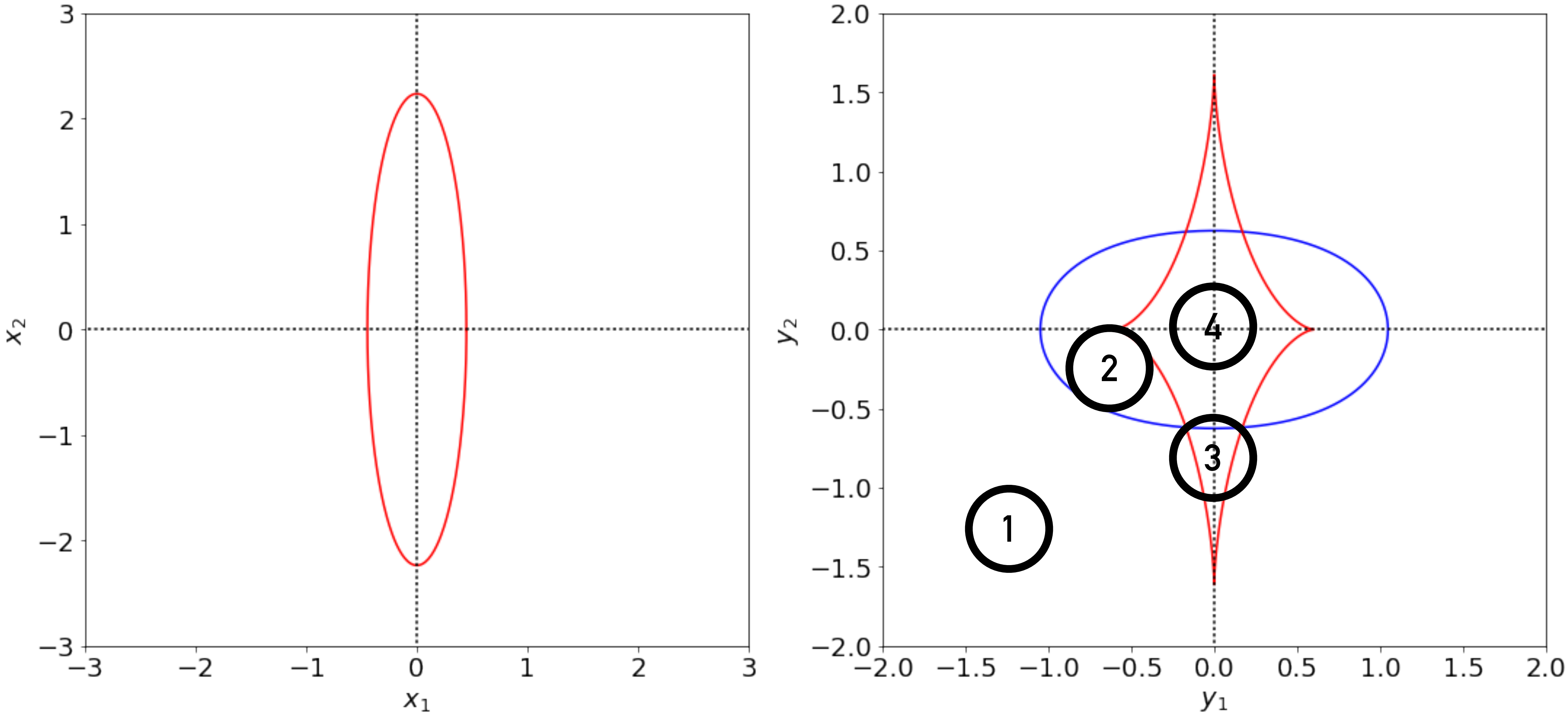


NON-SINGULAR-ISOTHERMAL-ELLIPSOID

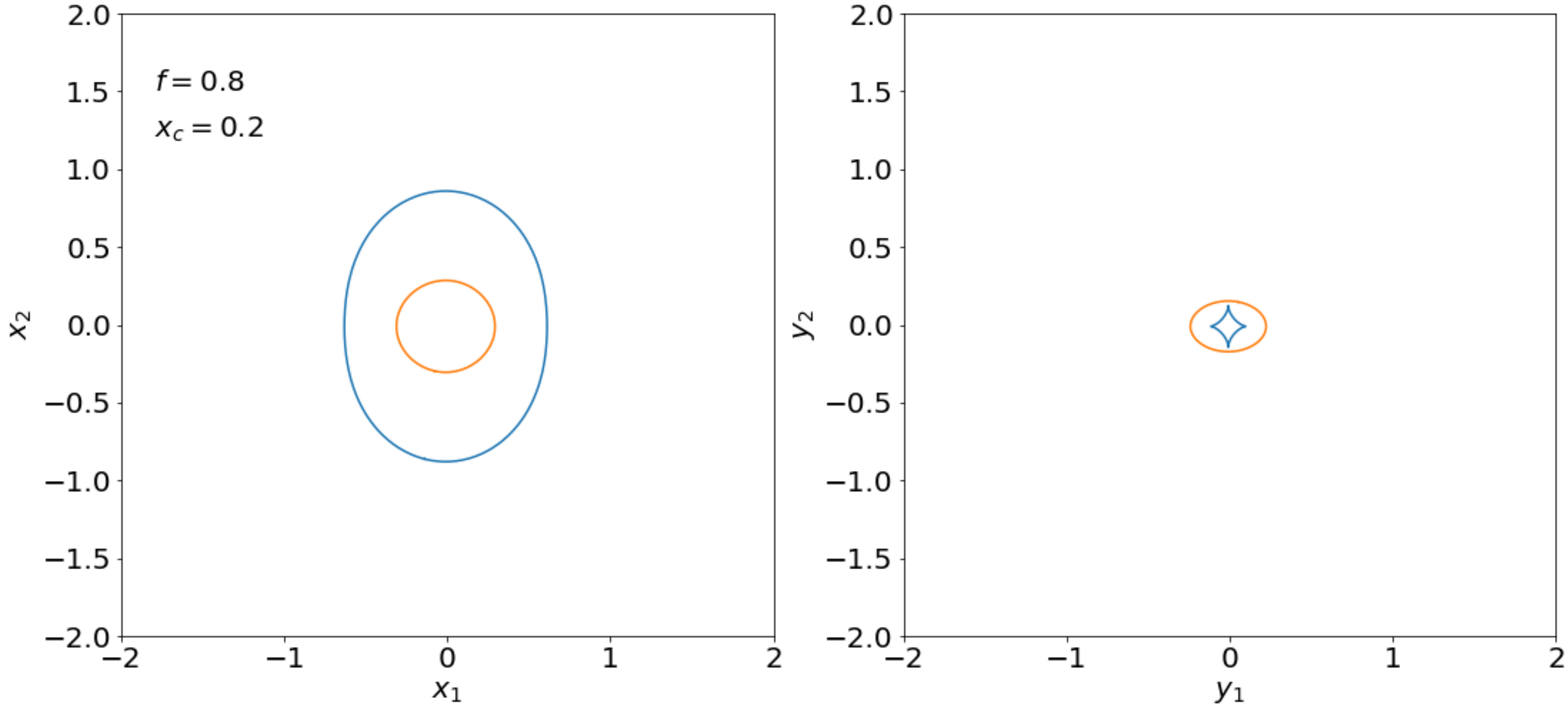
The SIE can be turned into a non-singular model by adding a core:

$$\Sigma(\vec{\xi}) = \frac{\sigma^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2\xi_2^2 + \xi_c^2}}$$

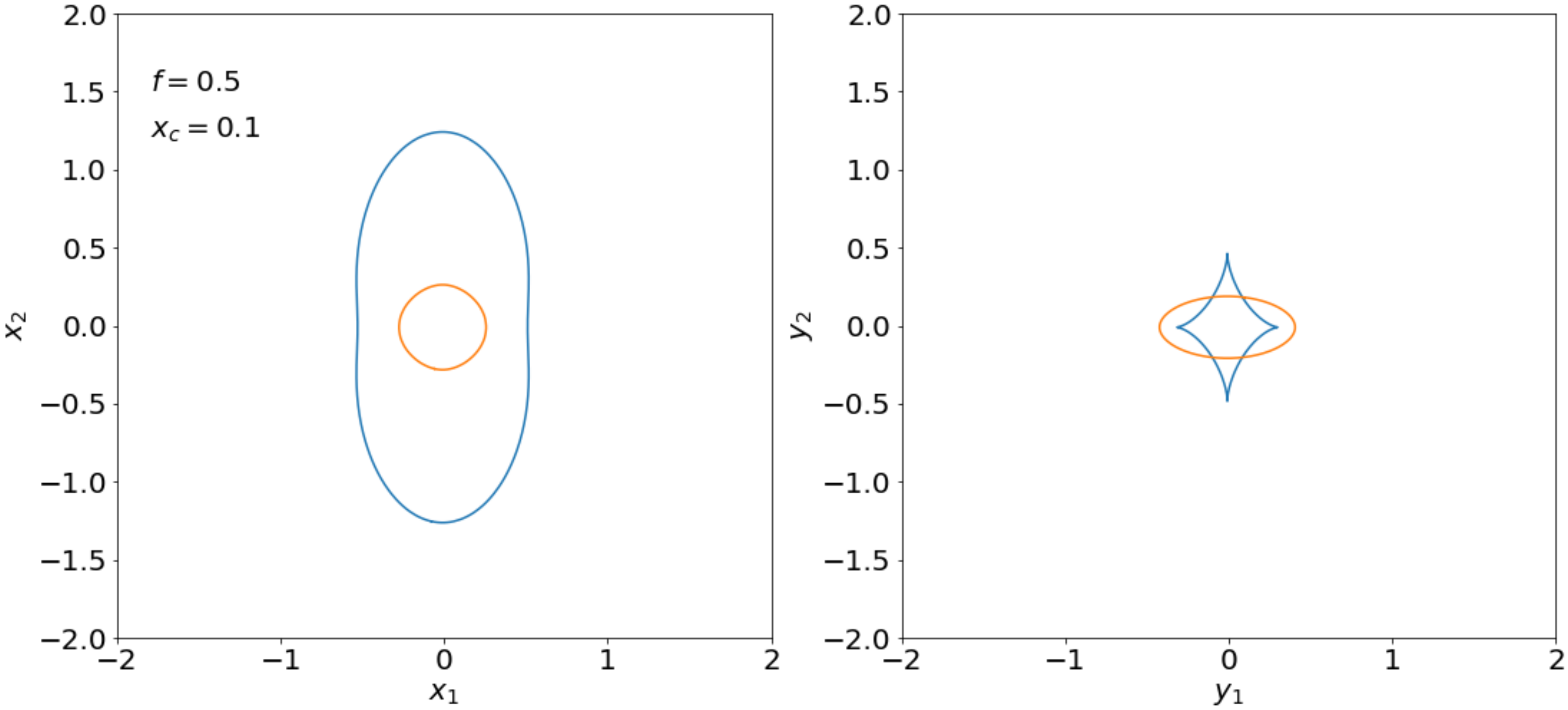
$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2 + x_c^2}} .$$

In this case, the analytical treatment of the lens is much more complicated. We limit the discussion to the topology of the critical lines and caustics and infer information about the image multiplicities…

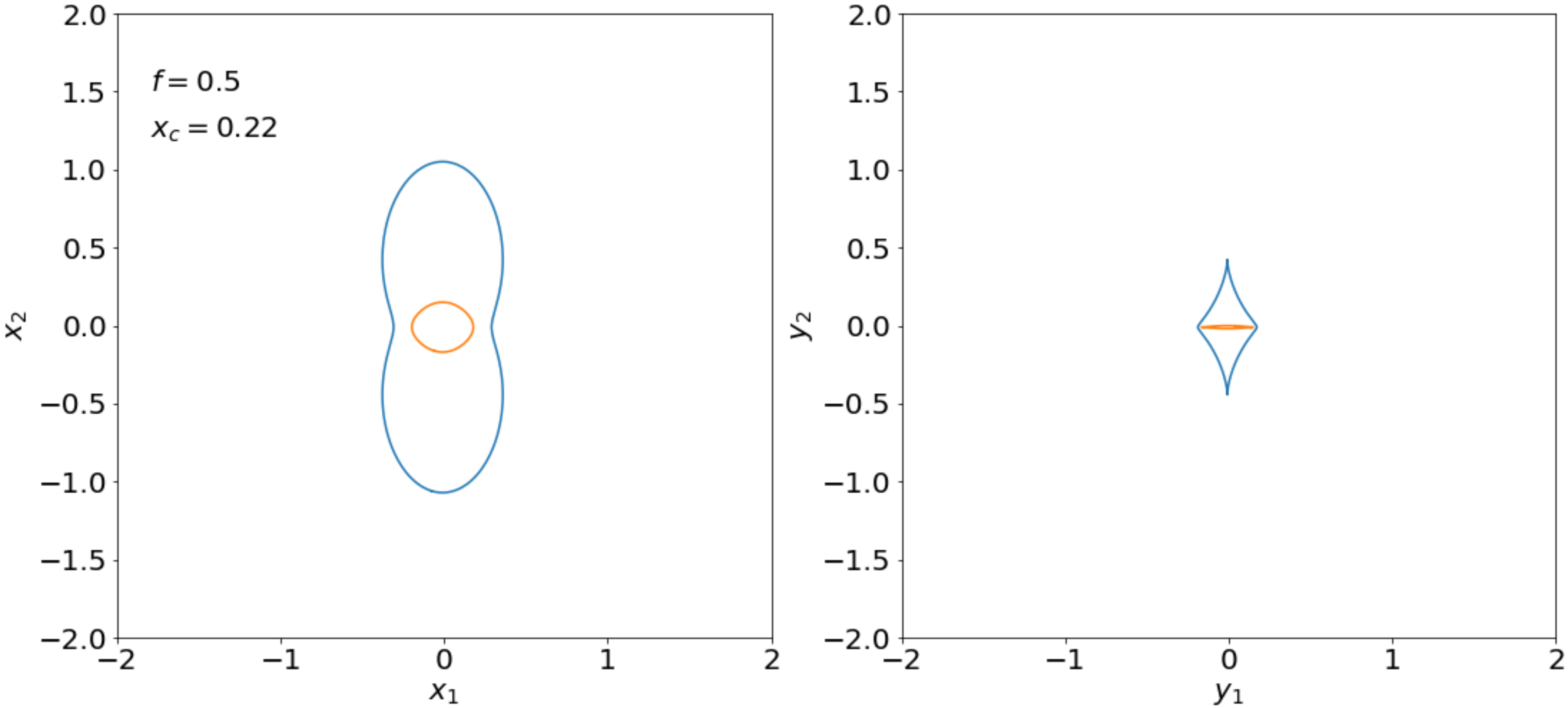
SMALL CORE RADIUS AND ELLIPTICITY



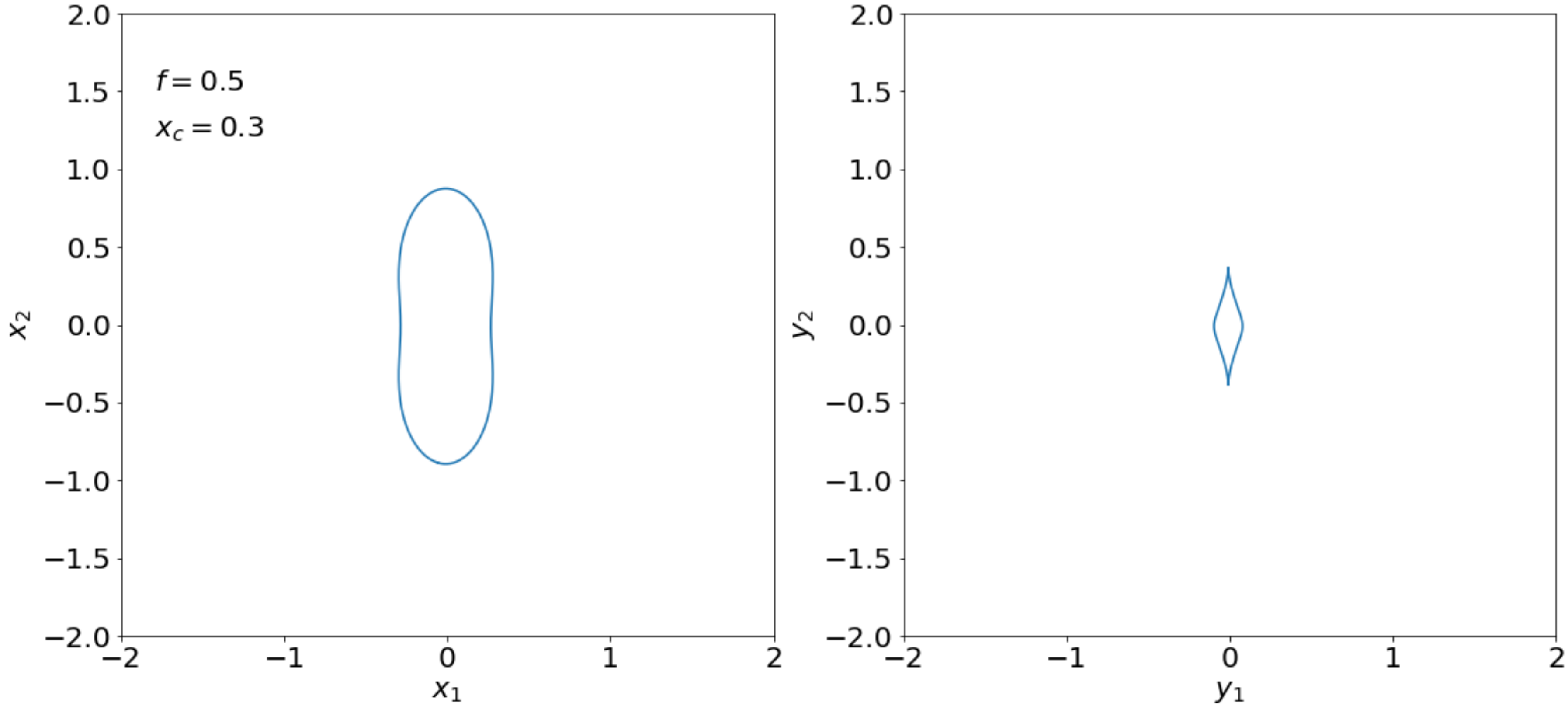
NAKED CUSP



INCREASING THE CORE SIZE…



NO RADIAL CRITICAL LINE AND CAUSTIC



CAUSTIC TOPOLOGIES (SEE KORMANN, BARTELMANN & SCHNEIDER, 1994)