GRAVITATIONAL LENSING

GRAVITATIONAL LENSING OF THE CMB

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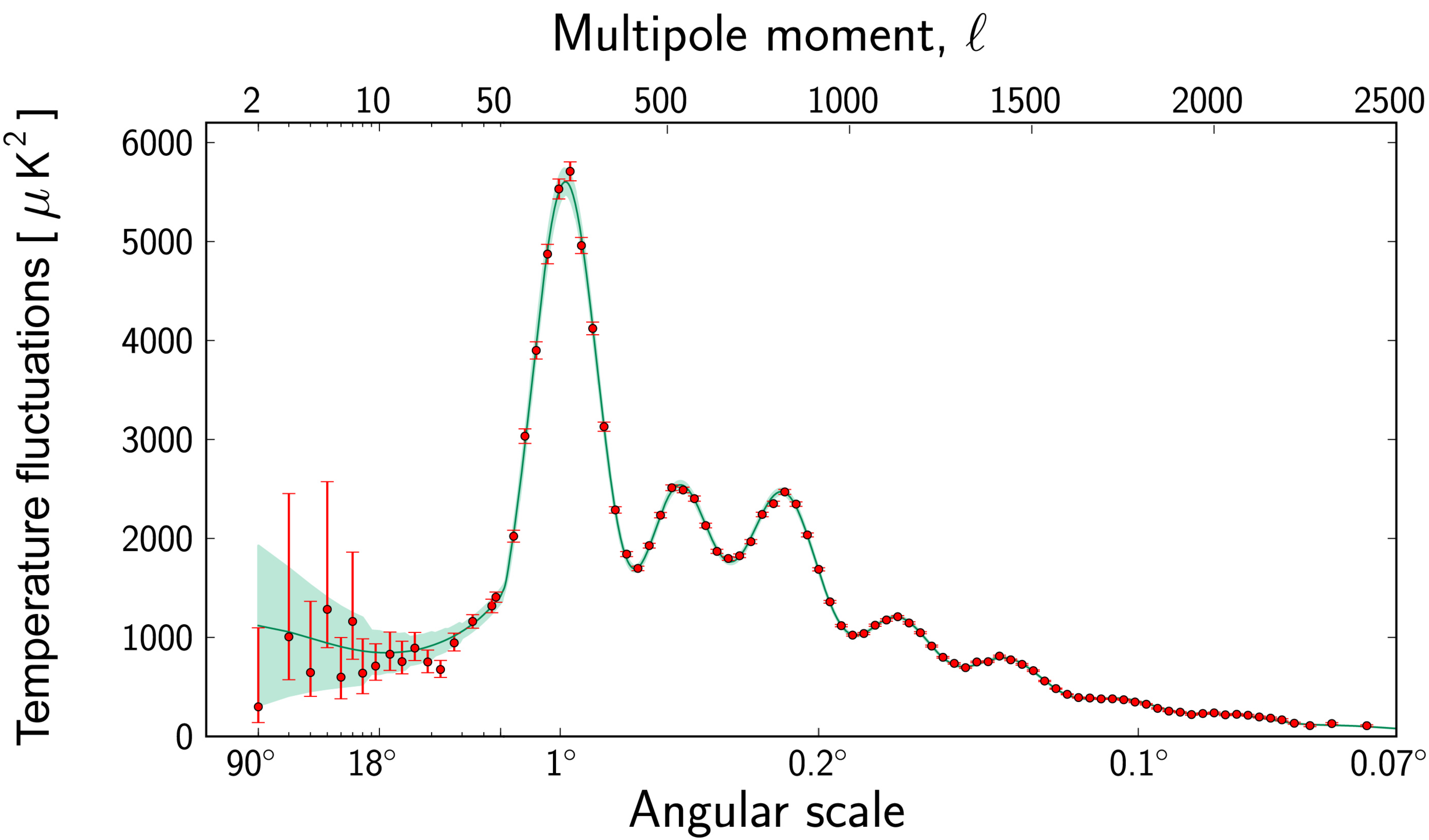
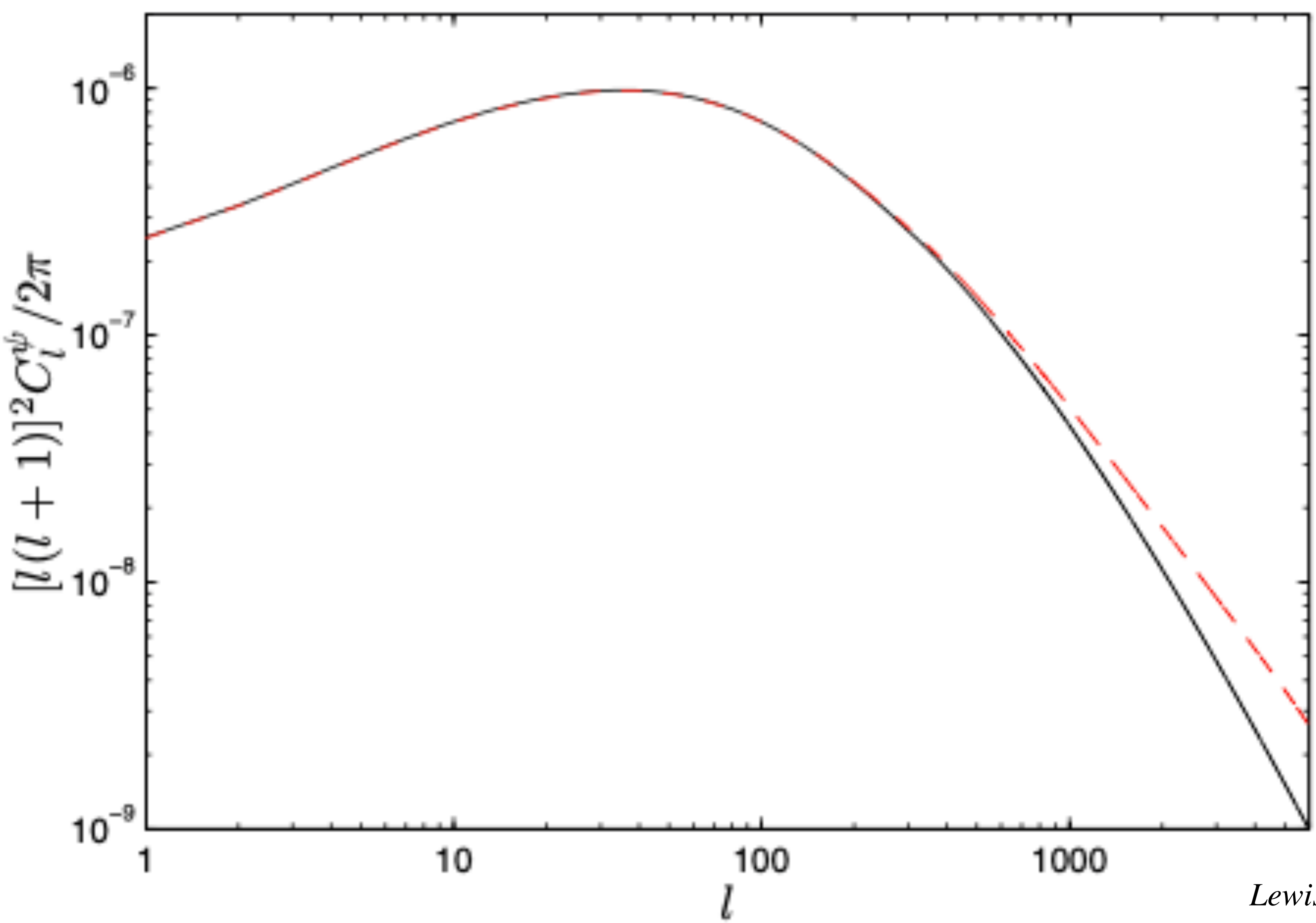
LENSING OF THE CMB



LENSING OF THE CMB

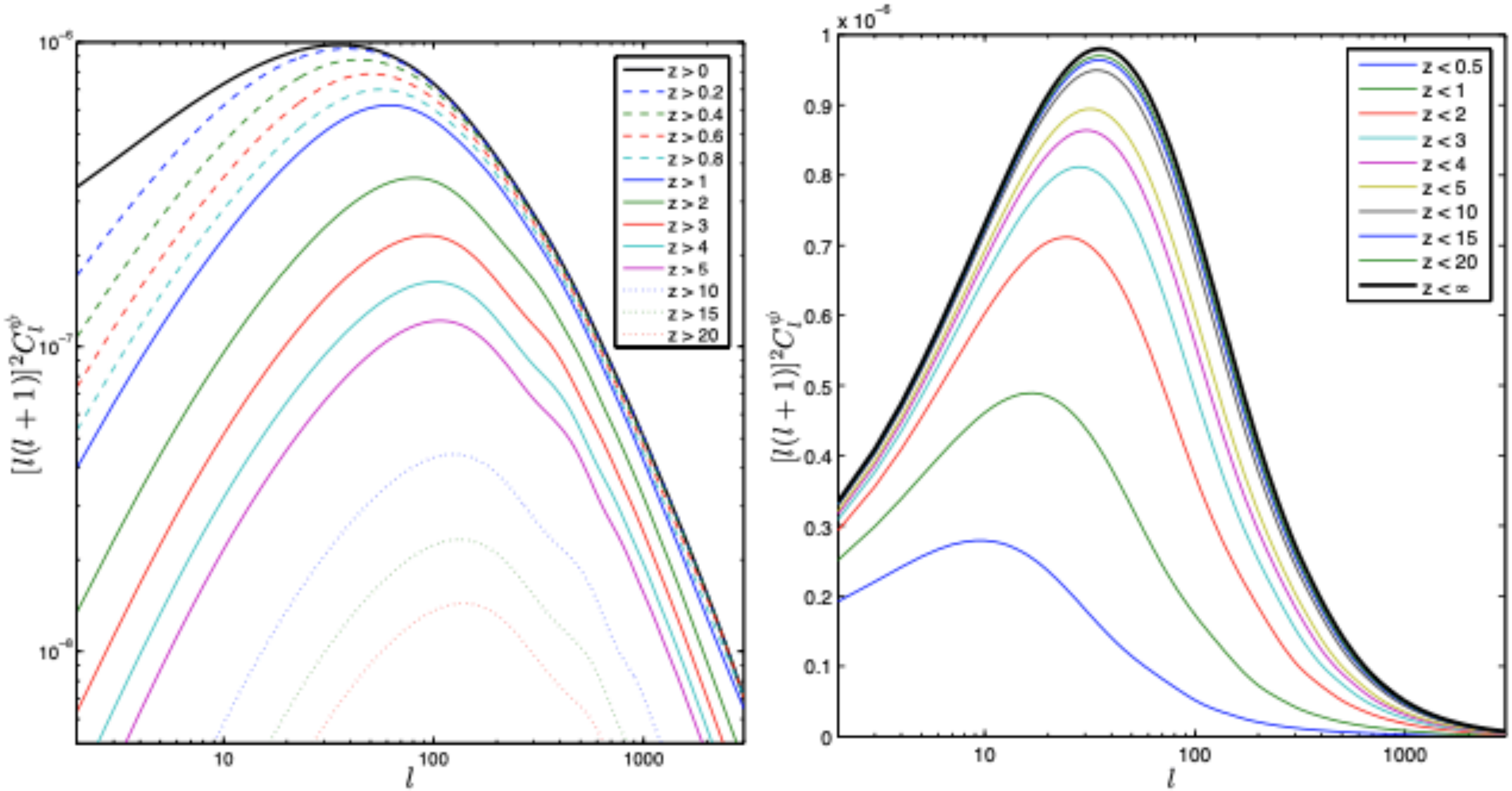
power spectrum of lensing deflection



Lewis & Challinor, 2006

LENSING OF THE CMB

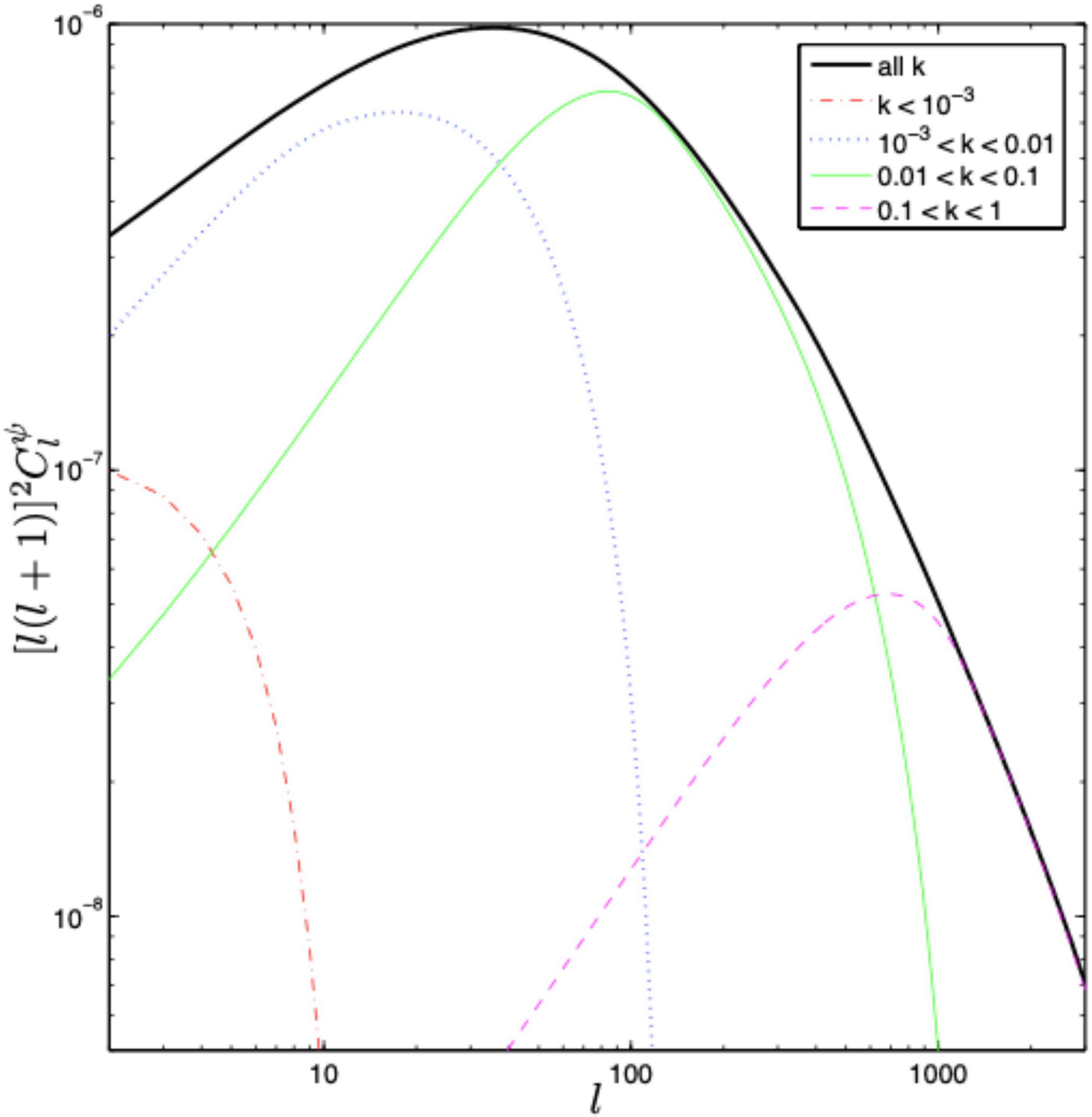
Cumulative contributions to the lensing potential from structure in different redshift ranges



Lewis & Challinor, 2006

LENSING OF THE CMB

Contribution to the lensing potential from structure of different scales



LENSING OF THE CMB

Correlation of CMB temperature taking lensing into account

$$\begin{aligned}
\xi^{ob}(|\boldsymbol{\theta} - \boldsymbol{\theta}'|) &= \left\langle T^{ob}(\boldsymbol{\theta}) T^{ob}(\boldsymbol{\theta}') \right\rangle \\
&= \langle T(\boldsymbol{\theta} + \boldsymbol{\alpha}) T(\boldsymbol{\theta}' + \boldsymbol{\alpha}') \rangle \\
&= \int \frac{d^2\ell}{(2\pi)^2} C_\ell^T \, e^{i\boldsymbol{\ell}\cdot(\boldsymbol{\theta}-\boldsymbol{\theta}')} \left\langle e^{i\boldsymbol{\ell}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')} \right\rangle \\
&= \int \frac{d^2\ell}{(2\pi)^2} C_\ell^T \, e^{i\boldsymbol{\ell}\cdot(\boldsymbol{\theta}-\boldsymbol{\theta}')} e^{-\frac{1}{2}\left\langle \left[\boldsymbol{\ell}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')\right]^2 \right\rangle}
\end{aligned}$$

Correlation of CMB temperature taking lensing into account

$$\left\langle \left[\boldsymbol{\ell} \cdot (\boldsymbol{\alpha} - \boldsymbol{\alpha}')\right]^2 \right\rangle = \ell^2 \left[C_0(0) - C_0(\theta) + \cos(2\phi) C_2(\theta)\right]$$
$$= \ell^2 \left[\sigma^2(\theta) + \cos(2\phi) C_2(\theta)\right]$$

$$C_0(\theta) = \langle \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}' \rangle$$
$$= \int \frac{d\ell}{2\pi} \ell^3 C_\ell^\psi J_0(\ell\theta)$$

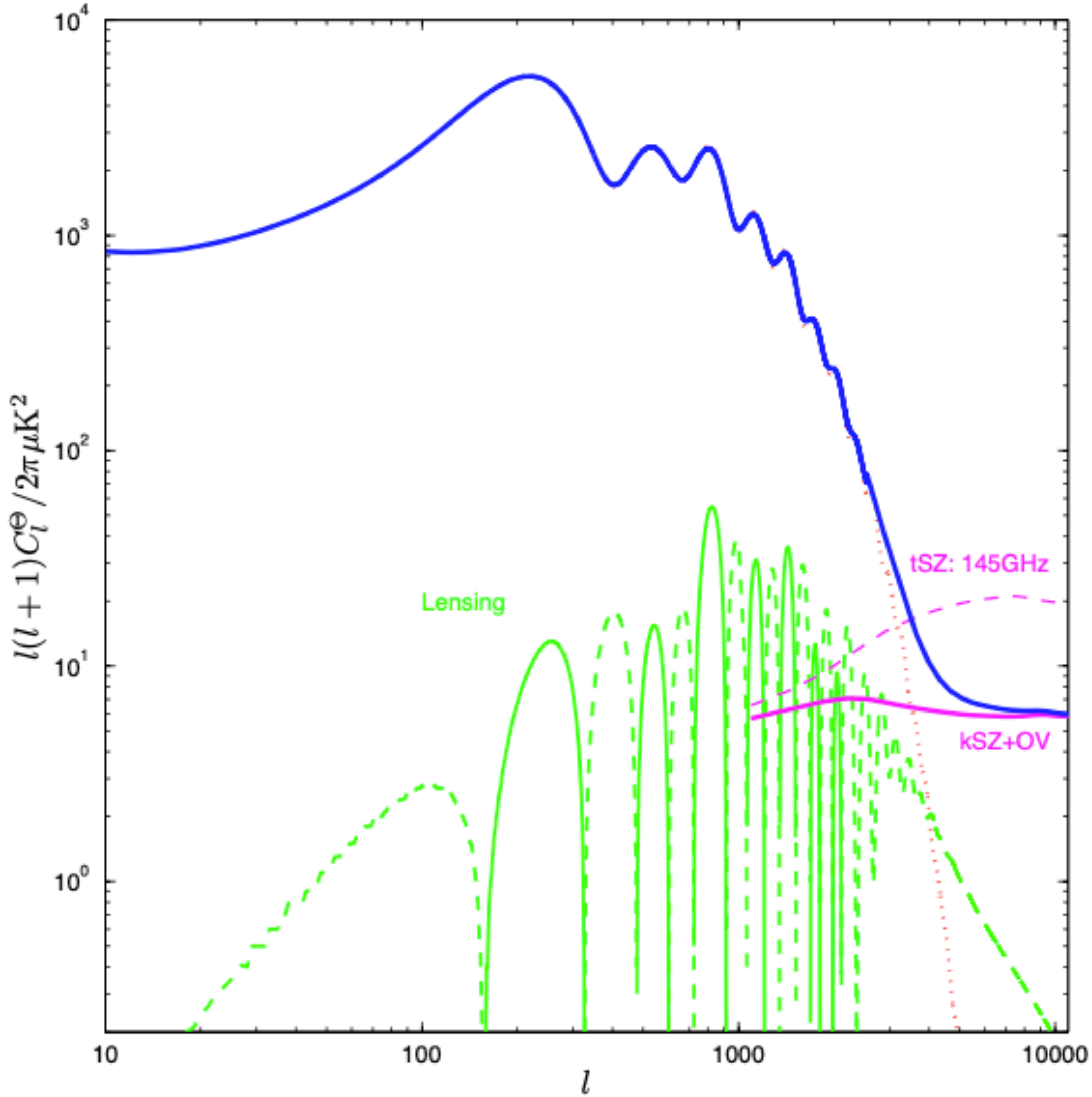
$$C_2(\theta) = \int \frac{d\ell}{2\pi} \ell^3 C_\ell^\psi J_2(\ell\theta)$$

LENSING OF THE CMB

Correlation of CMB temperature taking lensing into account

$$
\begin{aligned}
\xi^{ob}(|\boldsymbol{\theta}-\boldsymbol{\theta}'|) &= \int \frac{d^2\ell}{(2\pi)^2} C_\ell^T e^{i\boldsymbol{\ell}\cdot(\boldsymbol{\theta}-\boldsymbol{\theta}')} e^{-\frac{1}{2}\left\langle\left[\boldsymbol{\ell}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')\right]^2\right\rangle} \\
&= \int \frac{d^2\ell}{(2\pi)^2} C_\ell^T e^{-i\ell\cos(\phi)} e^{-\frac{\ell^2}{2}\left[\sigma^2(\theta)+\cos(2\phi)C_2(\theta)\right]} \\
&= \int \frac{d^2\ell}{(2\pi)^2} C_\ell^T e^{-i\ell\cos(\phi)} e^{-\frac{\ell^2}{2}\sigma^2(\theta)} \left[1-\frac{\ell^2}{2}\cos(2\phi)C_2(\theta)+\ldots\right] \\
&= \int \frac{d\ell}{(2\pi)} \ell C_\ell^T e^{-\frac{\ell^2}{2}\sigma^2(\theta)} \left[J_0(\theta\ell)+\frac{\ell^2}{2}J_2(\theta\ell)C_2(\theta)+\ldots\right]
\end{aligned}
$$

LENSING OF THE CMB



Lensed polarisation correlation functions

$$P(\boldsymbol{x}) = Q + iU$$ *Stock's parameters Q and U*

$$P_r(\boldsymbol{x}) = e^{-i2\phi_r} P(\boldsymbol{x})$$ *angle between **x**-**x**' and the x-axis used in P(x)*

three correlation functions

$$\begin{aligned}
\xi_+(r) &\equiv \langle P_r^*(\mathbf{x}) P_r(\mathbf{x}')\rangle = \langle P^*(\mathbf{x}) P(\mathbf{x}')\rangle \\
\xi_-(r) &\equiv \langle P_r(\mathbf{x})\, P_r(\mathbf{x}')\rangle = \langle e^{-4i\phi_r} P(\mathbf{x}) P(\mathbf{x}')\rangle \\
\xi_X(r) &\equiv \langle P_r(\mathbf{x})\, \Theta(\mathbf{x}')\rangle = \langle e^{-2i\phi_r} P(\mathbf{x}) \Theta(\mathbf{x}')\rangle.
\end{aligned}$$

in terms of Stock's parameters

$$\begin{aligned}
\xi_+(r) &= \langle Q_r(\mathbf{x}) Q_r(\mathbf{x}') + U_r(\mathbf{x}) U_r(\mathbf{x}')\rangle \\
\xi_-(r) &= \langle Q_r(\mathbf{x}) Q_r(\mathbf{x}') - U_r(\mathbf{x}) U_r(\mathbf{x}')\rangle + \cancel{i\langle Q_r(\mathbf{x}) U_r(\mathbf{x}') + U_r(\mathbf{x}) Q_r(\mathbf{x}')\rangle} \\
\xi_X(r) &= \langle Q_r(\mathbf{x}) \Theta(\mathbf{x}')\rangle + \cancel{i\langle U_r(\mathbf{x}) \Theta(\mathbf{x}')\rangle}.
\end{aligned}$$

must be zero by symmetry

Lensed polarisation correlation functions

$$\xi_+(\theta) = \frac{1}{2\pi} \int d\ell\ell \;\left(C_\ell^E + C_\ell^B\right) e^{-\frac{\ell^2}{2}\sigma^2(\theta)} \left[J_0(\theta\ell) + \frac{\ell^2}{2} J_2(\theta\ell) C_2(\theta) + \dots\right]$$

$$\xi_-(\theta) = \frac{1}{2\pi} \int d\ell\ell \;\left(C_\ell^E - C_\ell^B\right) e^{-\frac{\ell^2}{2}\sigma^2(\theta)} \left[J_4(\theta\ell) + \frac{\ell^2}{4} \left[J_2(\theta\ell) + J_6(\theta\ell)\right] C_2(\theta) + \dots\right]$$

$$\xi_\times(\theta) = \frac{1}{2\pi} \int d\ell\ell \; C_\ell^{TE} e^{-\frac{\ell^2}{2}\sigma^2(\theta)} \left[J_2(\theta\ell) + \frac{\ell^2}{4} \left[J_0(\theta\ell) + J_4(\theta\ell)\right] C_2(\theta) + \dots\right]$$

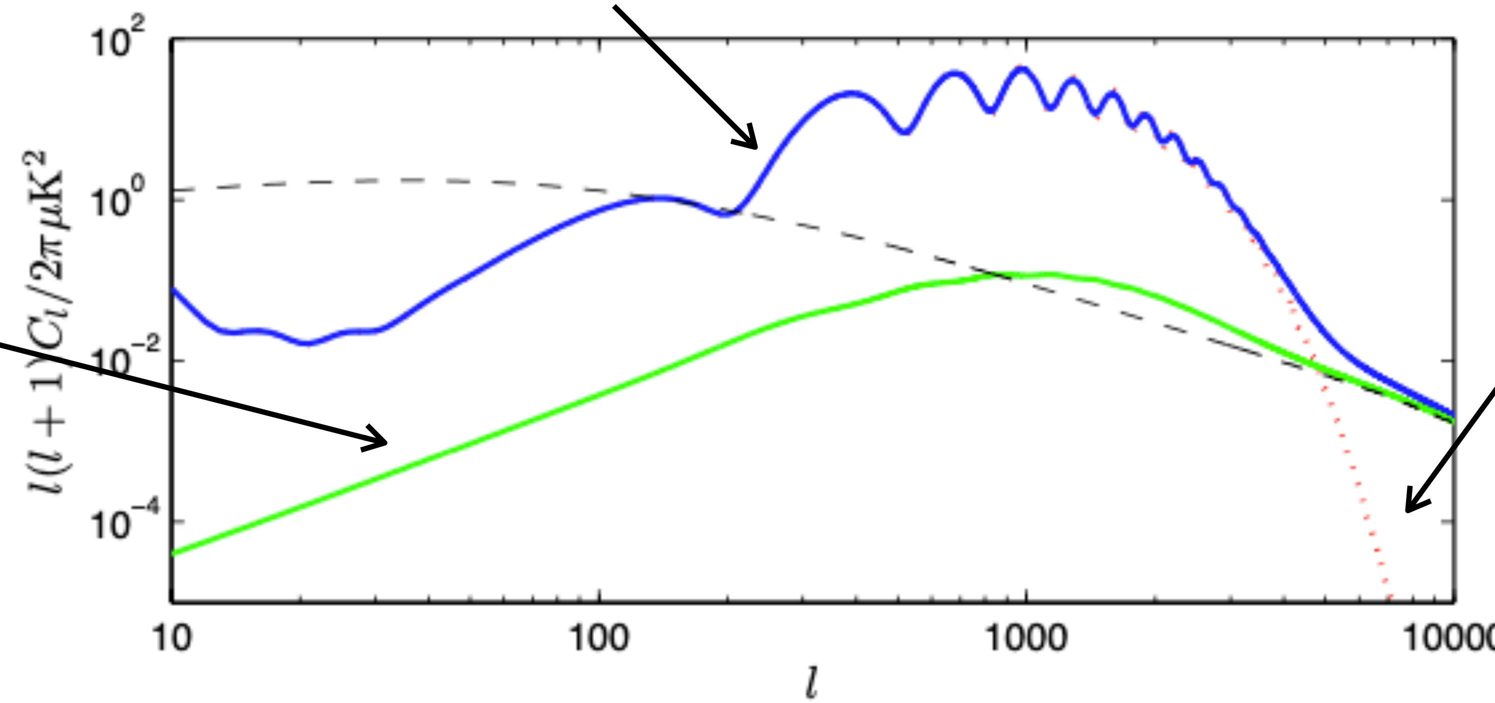
LENSING OF THE CMB

CMB polarisation power spectra

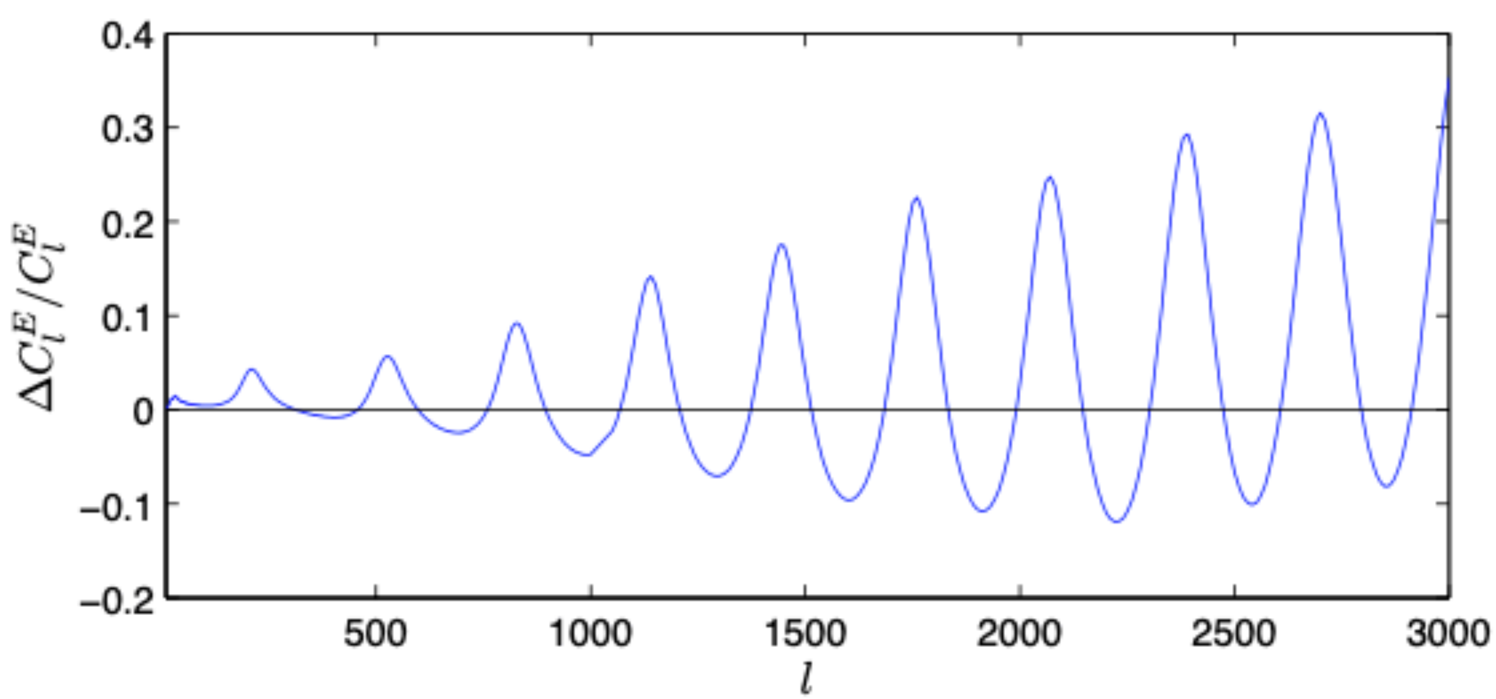
Lensed E-mode power spectrum

unlensed E-mode power spectrum

Lensed B-mode power spectrum



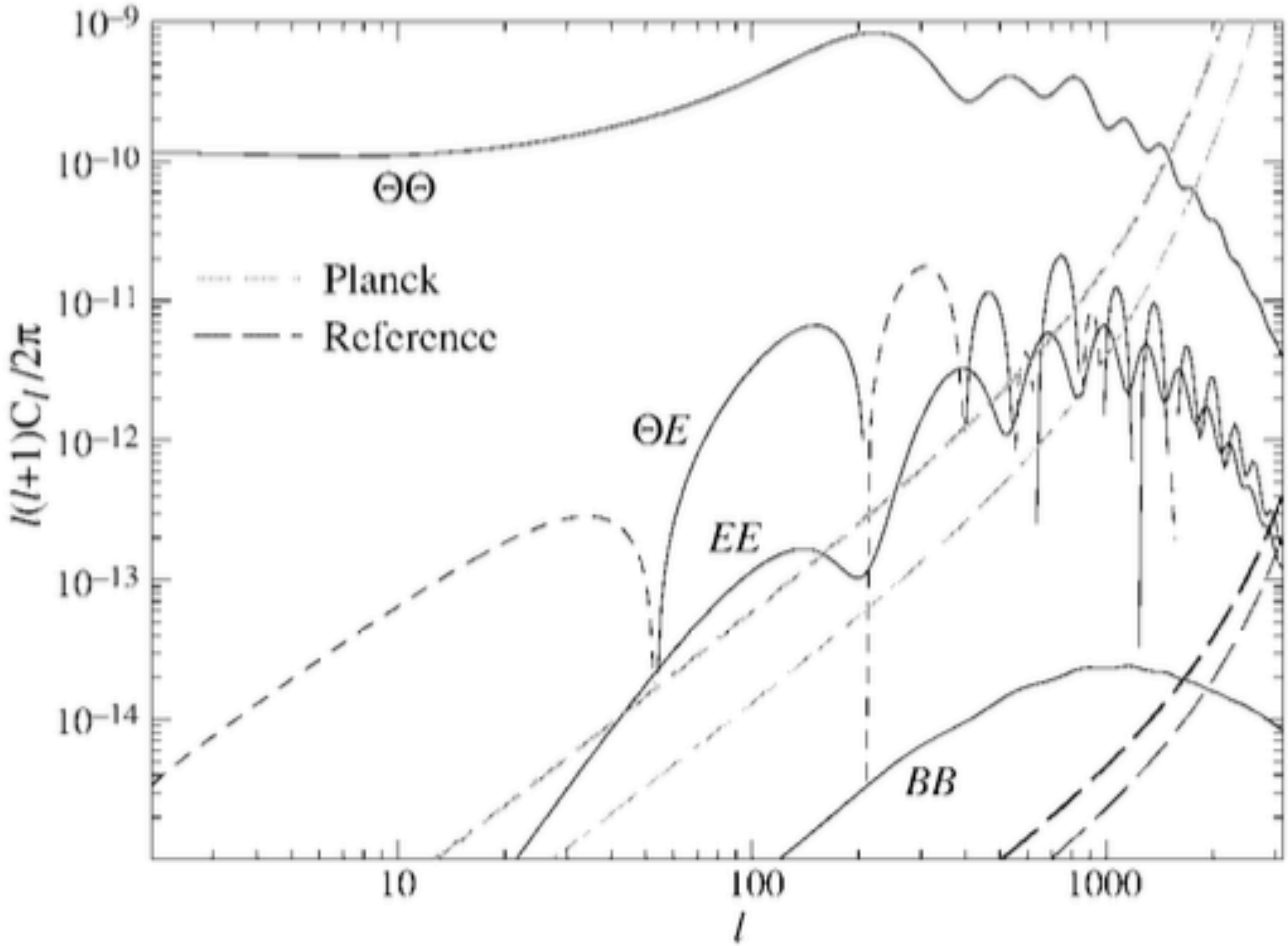
Fractional change in E-mode power spectrum caused by lensing



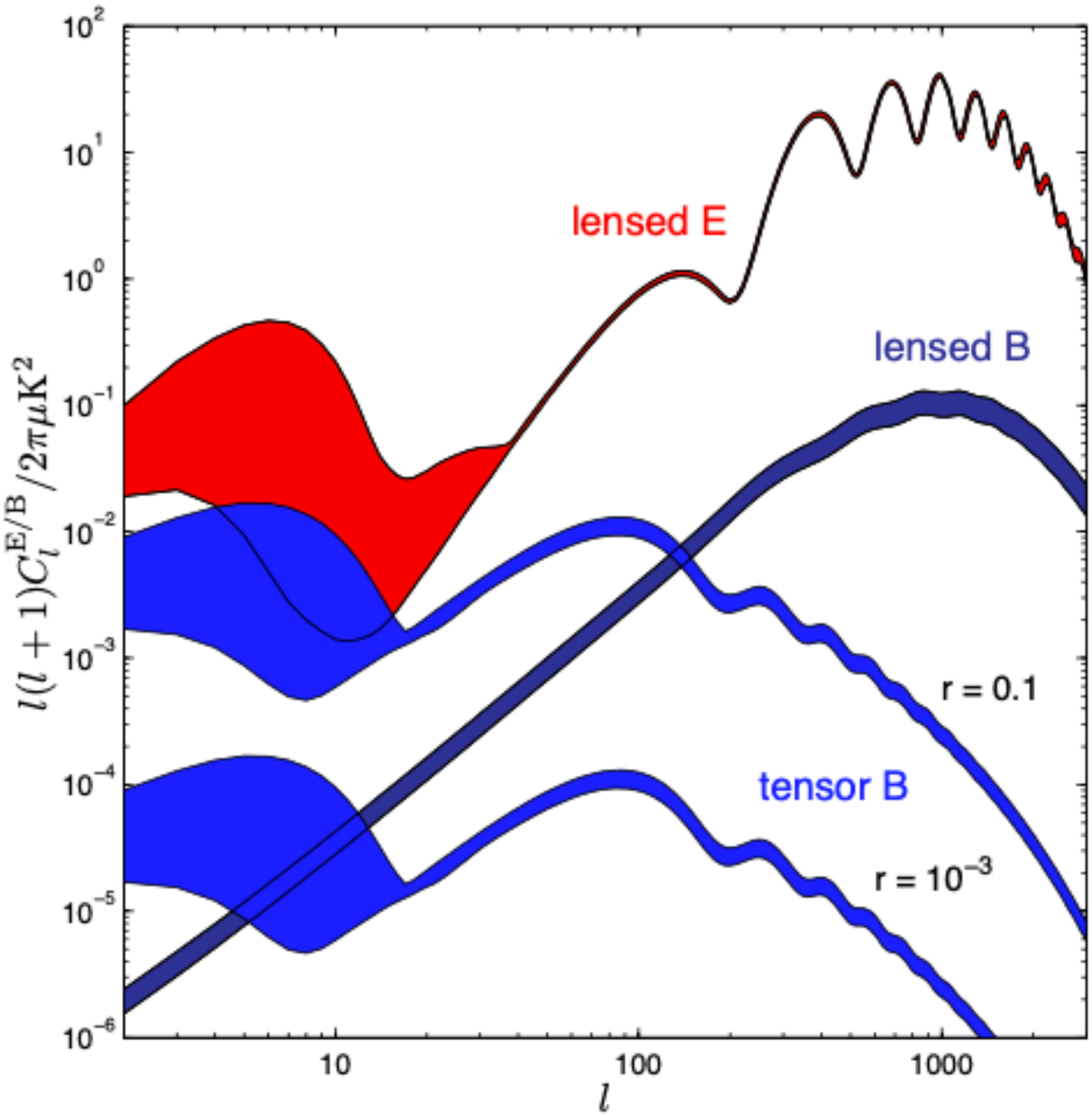
LENSING OF THE CMB

power spectra and cross-power spectra between temperature, E polarisation and E polarisation

$$\Theta = T$$



LENSING OF THE CMB



Potential reconstruction

To lowest order the lensed temperature is

$$T^{ob}(\boldsymbol{\theta}) = T(\boldsymbol{\theta} + \boldsymbol{\alpha})$$
$$\simeq T(\boldsymbol{\theta}) + \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} T(\boldsymbol{\theta}) + \ldots$$

Fourier transforming the gradients gives

$$\boldsymbol{\nabla}\psi(\boldsymbol{\theta}) = -i \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\tilde{\psi}_{\boldsymbol{\ell}}\; e^{-i\boldsymbol{\ell}\cdot\boldsymbol{\theta}} \qquad \boldsymbol{\nabla} T(\boldsymbol{\theta}) = -i \int \frac{d^2\ell'}{(2\pi)^2}\, \boldsymbol{\ell}'\tilde{T}_{\boldsymbol{\ell}'}\; e^{-i\boldsymbol{\ell}'\cdot\boldsymbol{\theta}}$$

$$T^{ob}(\boldsymbol{\theta}) \simeq T(\boldsymbol{\theta}) - \int \frac{d^2\ell}{(2\pi)^2} \int \frac{d^2\ell'}{(2\pi)^2}\, \boldsymbol{\ell}\cdot\boldsymbol{\ell}'\; \tilde{\psi}_{\boldsymbol{\ell}}\tilde{T}_{\boldsymbol{\ell}'} e^{-i\boldsymbol{\theta}\cdot(\boldsymbol{\ell}'+\boldsymbol{\ell})} + \ldots$$

Potential reconstruction

$$T^{ob}(\boldsymbol{\theta}) \simeq T(\boldsymbol{\theta}) - \int \frac{d^2\ell}{(2\pi)^2} \int \frac{d^2\ell'}{(2\pi)^2}\, \boldsymbol{\ell}\cdot\boldsymbol{\ell}'\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}'} e^{-i\boldsymbol{\theta}\cdot(\boldsymbol{\ell}'+\boldsymbol{\ell})} + \ldots$$

Now Fourier transforming both sides

$$\begin{aligned}\tilde{T}^{ob}_{\boldsymbol{\ell}_1} &= \tilde{T}_{\boldsymbol{\ell}_1} - \int \frac{d^2\ell}{(2\pi)^2} \int \frac{d^2\ell'}{(2\pi)^2}\, \boldsymbol{\ell}\cdot\boldsymbol{\ell}'\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}'} \int d^2\theta e^{-i\boldsymbol{\theta}\cdot(\boldsymbol{\ell}'+\boldsymbol{\ell}-\boldsymbol{\ell}_1)} + \ldots \\ &= \tilde{T}_{\boldsymbol{\ell}_1} - \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1-\boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}} + \ldots\end{aligned}$$

this is a delta function

Potential reconstruction

$$
\begin{aligned}
\tilde{T}^{ob}_{\boldsymbol{\ell}_1} &= \tilde{T}_{\boldsymbol{\ell}_1} - \int \frac{d^2\ell}{(2\pi)^2} \int \frac{d^2\ell'}{(2\pi)^2}\, \boldsymbol{\ell}\cdot\boldsymbol{\ell}'\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}'} \int d^2\theta e^{-i\boldsymbol{\theta}\cdot(\boldsymbol{\ell}'+\boldsymbol{\ell}-\boldsymbol{\ell}_1)} + \dots \\
&= \tilde{T}_{\boldsymbol{\ell}_1} - \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1 - \boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}} + \dots
\end{aligned}
$$

for a different l value this is

$$
\begin{aligned}
\tilde{T}^{ob*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} &= \tilde{T}^{*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} - \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1 - \boldsymbol{L} - \boldsymbol{\ell})\ \tilde{\psi}^{*}_{\boldsymbol{\ell}} \tilde{T}^{*}_{\boldsymbol{\ell}_1-\boldsymbol{L}-\boldsymbol{\ell}} + \dots \\
&= \tilde{T}^{*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1 - \boldsymbol{L} + \boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}^{*}_{\boldsymbol{\ell}_1-\boldsymbol{L}+\boldsymbol{\ell}} + \dots
\end{aligned}
$$

Potential reconstruction

$$\tilde{T}^{ob}_{\boldsymbol{\ell}_1} = \tilde{T}_{\boldsymbol{\ell}_1} - \int \frac{d^2\ell}{(2\pi)^2} \int \frac{d^2\ell'}{(2\pi)^2}\, \boldsymbol{\ell}\cdot\boldsymbol{\ell}'\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}'} \int d^2\theta e^{-i\boldsymbol{\theta}\cdot(\boldsymbol{\ell}'+\boldsymbol{\ell}-\boldsymbol{\ell}_1)} + \dots$$

$$= \tilde{T}_{\boldsymbol{\ell}_1} - \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1-\boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}} + \dots$$

$$\tilde{T}^{ob*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} = \tilde{T}^{*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} - \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1-\boldsymbol{L}-\boldsymbol{\ell})\ \tilde{\psi}^*_{\boldsymbol{\ell}} \tilde{T}^*_{\boldsymbol{\ell}_1-\boldsymbol{L}-\boldsymbol{\ell}} + \dots$$

$$= \tilde{T}^{*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1-\boldsymbol{L}+\boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \tilde{T}^*_{\boldsymbol{\ell}_1-\boldsymbol{L}+\boldsymbol{\ell}} + \dots$$

multiplying them together and averaging while keeping only terms linear in ψ *gives*

$$\langle \tilde{T}^{ob}_{\boldsymbol{\ell}_1} \tilde{T}^{ob*}_{\boldsymbol{\ell}_1-\boldsymbol{L}} \rangle = \langle \tilde{T}_{\boldsymbol{\ell}_1} \tilde{T}^*_{\boldsymbol{\ell}_1-\boldsymbol{L}} \rangle - \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1-\boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \langle \tilde{T}_{\boldsymbol{\ell}_1-\boldsymbol{\ell}} \tilde{T}^*_{\boldsymbol{\ell}_1-\boldsymbol{L}} \rangle$$

$$+ \int \frac{d^2\ell}{(2\pi)^2}\, \boldsymbol{\ell}\cdot(\boldsymbol{\ell}_1-\boldsymbol{L}+\boldsymbol{\ell})\ \tilde{\psi}_{\boldsymbol{\ell}} \langle \tilde{T}_{\boldsymbol{\ell}_1} \tilde{T}^*_{\boldsymbol{\ell}_1-\boldsymbol{L}+\boldsymbol{\ell}} \rangle + \dots$$

$$= (2\pi)^2 \delta(\boldsymbol{L}) C_{\ell_1} - \boldsymbol{L}\cdot(\boldsymbol{\ell}_1-\boldsymbol{L})\ \tilde{\psi}_{\boldsymbol{L}} C_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}_1\ \tilde{\psi}_{\boldsymbol{\ell}} C_{\ell_1} + \dots$$

$$= (2\pi)^2 \delta(\boldsymbol{L}) C_{\ell_1} + \left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell}_1)\ C_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}_1\ C_{\ell_1}\right] \tilde{\psi}_{\boldsymbol{\ell}} + \dots$$

Potential reconstruction

Now we seek a linear estimator for the potential of the form $$\hat{\psi}(\boldsymbol{L}) = \int d^2\ell \; A(\boldsymbol{L},\boldsymbol{\ell})\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}$$

The average estimator is $$\langle\hat{\psi}(\boldsymbol{L})\rangle = \int d^2\ell \; A(\boldsymbol{L},\boldsymbol{\ell})\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\rangle$$

this is a constraint $$= \psi(\boldsymbol{L})\int d^2\ell \; A(\boldsymbol{L},\boldsymbol{\ell})\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell}_1)\; C_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}_1\; C_{\ell_1}\right]$$
$$= \psi(\boldsymbol{L})$$

We want this to be equal to the potential so we want the integral to be 1.

The variance of this estimator is given by

$$\langle|\hat{\psi}(\boldsymbol{L})|^2\rangle = \int d^2\ell \int d^2\ell' \; A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\tilde{T}^{ob*}_{\boldsymbol{\ell}'}\tilde{T}^{ob}_{\boldsymbol{\ell}'-\boldsymbol{L}}\rangle$$
$$= \int d^2\ell \int d^2\ell' \; A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')\left[\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\tilde{T}^{ob*}_{\boldsymbol{\ell}'}\tilde{T}^{ob}_{\boldsymbol{\ell}'-\boldsymbol{L}}\rangle\right]$$

This is a property of a Gaussian distribution

$$= \int d^2\ell \int d^2\ell' \; A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')2(2\pi)^2\delta(\boldsymbol{\ell}'-\boldsymbol{l})(2\pi)^2\delta(\boldsymbol{\ell}'-\boldsymbol{l})C^{tot}_{\ell}C^{tot}_{\ell'-L}$$

$$= 2|A(\boldsymbol{L},\boldsymbol{\ell})|^2 C^{tot}_{\ell}C^{tot}_{\ell-L}$$

Potential reconstruction

$$\langle\hat{\psi}(\boldsymbol{L})\rangle = \int d^2\ell\ A(\boldsymbol{L},\boldsymbol{\ell})\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\rangle$$
$$= \psi(\boldsymbol{L})\int d^2\ell\ A(\boldsymbol{L},\boldsymbol{\ell})\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell}_1)\ C_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}_1\ C_{\ell_1}\right]$$

$$\langle|\hat{\psi}(\boldsymbol{L})|^2\rangle = \int d^2\ell\int d^2\ell'\ A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\tilde{T}^{ob*}_{\boldsymbol{\ell}'}\tilde{T}^{ob}_{\boldsymbol{\ell}'-\boldsymbol{L}}\rangle$$
$$= \int d^2\ell\int d^2\ell'\ A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')\left[\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\tilde{T}^{ob*}_{\boldsymbol{\ell}'}\tilde{T}^{ob}_{\boldsymbol{\ell}'-\boldsymbol{L}}\rangle\right]$$
$$= \int d^2\ell\int d^2\ell'\ A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')2(2\pi)^2\delta(\boldsymbol{\ell}'-\boldsymbol{l})(2\pi)^2\delta(\boldsymbol{\ell}'-\boldsymbol{l})C^{tot}_{\ell}C^{tot}_{\ell'-L}$$
$$= 2|A(\boldsymbol{L},\boldsymbol{\ell})|^2 C^{tot}_{\ell}C^{tot}_{\ell-L}$$

Now we can minimise the noise subject to the constraint of the average using a Lagrangian multiplier

$$\frac{\delta}{\delta A}\left[\langle|\hat{\psi}(\boldsymbol{L})|^2 + \lambda(\langle\hat{\psi}(\boldsymbol{L})\rangle - \psi(\boldsymbol{L}))\right] = 0$$

$$4A(\boldsymbol{L},\boldsymbol{\ell})C^{tot}_{\ell}C^{tot}_{\ell-L} + \lambda\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\ C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\ C_{\ell}\right] = 0$$

Solving for A gives

$$A(\boldsymbol{L},\boldsymbol{\ell}) = -\lambda\frac{\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\ C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\ C_{\ell}\right]}{4C^{tot}_{\ell}C^{tot}_{\ell-L}}$$

LENSING OF THE CMB

Potential reconstruction

$$\langle\hat{\psi}(\boldsymbol{L})\rangle = \int d^2\ell \; A(\boldsymbol{L},\boldsymbol{\ell})\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\rangle$$
$$= \psi(\boldsymbol{L})\int d^2\ell \; A(\boldsymbol{L},\boldsymbol{\ell})\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell}_1)\; C_{\boldsymbol{\ell}_1-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}_1 \; C_{\ell_1}\right]$$

$$\langle|\hat{\psi}(\boldsymbol{L})|^2\rangle = \int d^2\ell \int d^2\ell' \; A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\tilde{T}^{ob*}_{\boldsymbol{\ell}'}\tilde{T}^{ob}_{\boldsymbol{\ell}'-\boldsymbol{L}}\rangle$$
$$= \int d^2\ell \int d^2\ell' \; A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')\left[\langle\tilde{T}^{ob}_{\boldsymbol{\ell}}\tilde{T}^{ob*}_{\boldsymbol{\ell}-\boldsymbol{L}}\tilde{T}^{ob*}_{\boldsymbol{\ell}'}\tilde{T}^{ob}_{\boldsymbol{\ell}'-\boldsymbol{L}}\rangle\right]$$
$$= \int d^2\ell \int d^2\ell' \; A(\boldsymbol{L},\boldsymbol{\ell})A^*(\boldsymbol{L},\boldsymbol{\ell}')2(2\pi)^2\delta(\boldsymbol{\ell}'-\boldsymbol{l})(2\pi)^2\delta(\boldsymbol{\ell}'-\boldsymbol{l})C^{tot}_{\ell}C^{tot}_{\ell'-L}$$
$$= 2|A(\boldsymbol{L},\boldsymbol{\ell})|^2 C^{tot}_{\ell}C^{tot}_{\ell-L}$$

Now we can minimise the noise subject to the constraint of the average using a Lagrangian multiplier

$$\frac{\delta}{\delta A}\left[\langle|\hat{\psi}(\boldsymbol{L})|^2 + \lambda(\langle\hat{\psi}(\boldsymbol{L})\rangle - \psi(\boldsymbol{L}))\right] = 0$$

$$4A(\boldsymbol{L},\boldsymbol{\ell})C^{tot}_{\ell}C^{tot}_{\ell-L} + \lambda\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\; C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\; C_{\ell}\right] = 0$$

Solving for A gives

$$A(\boldsymbol{L},\boldsymbol{\ell}) = -\lambda\frac{\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\; C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\; C_{\ell}\right]}{4C^{tot}_{\ell}C^{tot}_{\ell-L}}$$

Potential reconstruction

$$4A(\boldsymbol{L},\boldsymbol{\ell})C_\ell^{tot}C_{\ell-L}^{tot} + \lambda\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\; C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\; C_\ell\right] = 0$$

Solving for A gives

$$A(\boldsymbol{L},\boldsymbol{\ell}) = -\lambda\frac{\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\; C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\; C_\ell\right]}{4C_\ell^{tot}C_{\ell-L}^{tot}}$$

Minimum variance linear estimator for the lensing potential.

Applying the constraint gives the Lagrangian multiplier and the final properly normalised estimator is

$$\boxed{\begin{aligned}\hat{\psi}(\boldsymbol{L}) &= N_L^{-1}\int d^2\ell\; \frac{\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\; C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\; C_\ell\right]}{C_\ell^{tot}C_{\ell-L}^{tot}}\tilde{T}_{\boldsymbol{\ell}}^{ob}\tilde{T}_{\boldsymbol{\ell}-\boldsymbol{L}}^{ob*} \\ N_L &= \int d^2\ell\; \frac{\left[\boldsymbol{L}\cdot(\boldsymbol{L}-\boldsymbol{\ell})\; C_{\boldsymbol{\ell}-\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\ell}\; C_\ell\right]^2}{C_\ell^{tot}C_{\ell-L}^{tot}}\end{aligned}}$$

Potential reconstruction

Lensing will cause a correlation between modes (Fourier or spherical harmonic) that would not exist for a pure Gaussian field.

Using this property we can, to first order, reconstruct the lensing potential.

$$\hat{\psi}(\boldsymbol{L}) = N^{-1} \int d^2\ell \ \frac{\boldsymbol{L} \cdot \boldsymbol{\ell}}{C_\ell^{tot} C_{|L-\ell|}^{tot}} \tilde{T}_{\boldsymbol{\ell}} \tilde{T}_{\boldsymbol{\ell}-\boldsymbol{L}}$$

LENSING OF THE CMB

Estimators can also be constructed from the polarisation power spectra in a very similar way.

The lack of primordial B-modes on the same scales makes using the polarisation improve the signal-to-noise significantly.

$$\boldsymbol{L} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$$

$$\hat{\psi}(\boldsymbol{L}) = \frac{A(L)}{L} \int \frac{d^2\ell}{(2\pi)^2} x(\boldsymbol{\ell}_1) x'(\boldsymbol{\ell}_2) F_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2)$$

$$F_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = \frac{f_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2)}{2C^{xx}_{\ell_1} C^{xx}_{\ell_1}} \qquad \alpha = TT, EE, BB$$

$$F_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = \frac{f_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2)}{C^{xx}_{\ell_1} C^{x'x'}_{\ell_1}} \qquad \alpha = TB, EB$$

$$A(L) = L^2 \left[\int \frac{d^2\ell}{(2\pi)^2} f_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) F_\alpha(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \right]^{-1}$$

Hu & Okamoto 2002

LENSING OF THE CMB

Estimators can also be constructed from the polarisation power spectra as well.

MINIMUM VARIANCE FILTERS

α	$f_\alpha(\boldsymbol{l}_1, \boldsymbol{l}_2)$
$\Theta\Theta$........	$\tilde{C}_{l_1}^{\Theta\Theta}(\boldsymbol{L}\cdot\boldsymbol{l}_1) + \tilde{C}_{l_2}^{\Theta\Theta}(\boldsymbol{L}\cdot\boldsymbol{l}_2)$
ΘE........	$\tilde{C}_{l_1}^{\Theta E}\cos\varphi_{\boldsymbol{l}_1\boldsymbol{l}_2}(\boldsymbol{L}\cdot\boldsymbol{l}_1) + \tilde{C}_{l_2}^{\Theta E}(\boldsymbol{L}\cdot\boldsymbol{l}_2)$
ΘB........	$\tilde{C}_{l_1}^{\Theta E}\sin 2\varphi_{\boldsymbol{l}_1\boldsymbol{l}_2}(\boldsymbol{L}\cdot\boldsymbol{l}_1)$
EE........	$[\tilde{C}_{l_1}^{EE}(\boldsymbol{L}\cdot\boldsymbol{l}_1) + \tilde{C}_{l_2}^{EE}(\boldsymbol{L}\cdot\boldsymbol{l}_2)]\cos 2\varphi_{\boldsymbol{l}_1\boldsymbol{l}_2}$
EB........	$[\tilde{C}_{l_1}^{EE}(\boldsymbol{L}\cdot\boldsymbol{l}_1) - \tilde{C}_{l_2}^{BB}(\boldsymbol{L}\cdot\boldsymbol{l}_2)]\sin 2\varphi_{\boldsymbol{l}_1\boldsymbol{l}_2}$
BB........	$[\tilde{C}_{l_1}^{BB}(\boldsymbol{L}\cdot\boldsymbol{l}_1) + \tilde{C}_{l_2}^{BB}(\boldsymbol{L}\cdot\boldsymbol{l}_2)]\cos 2\varphi_{\boldsymbol{l}_1\boldsymbol{l}_2}$

$$\langle x(\boldsymbol{\ell})x'(\boldsymbol{\ell}')\rangle_{CMB} = f_\alpha(\boldsymbol{\ell},\boldsymbol{\ell}')\tilde{\psi}(\boldsymbol{\ell}-\boldsymbol{\ell}')$$

Hu & Okamoto 2002

LENSING OF THE CMB

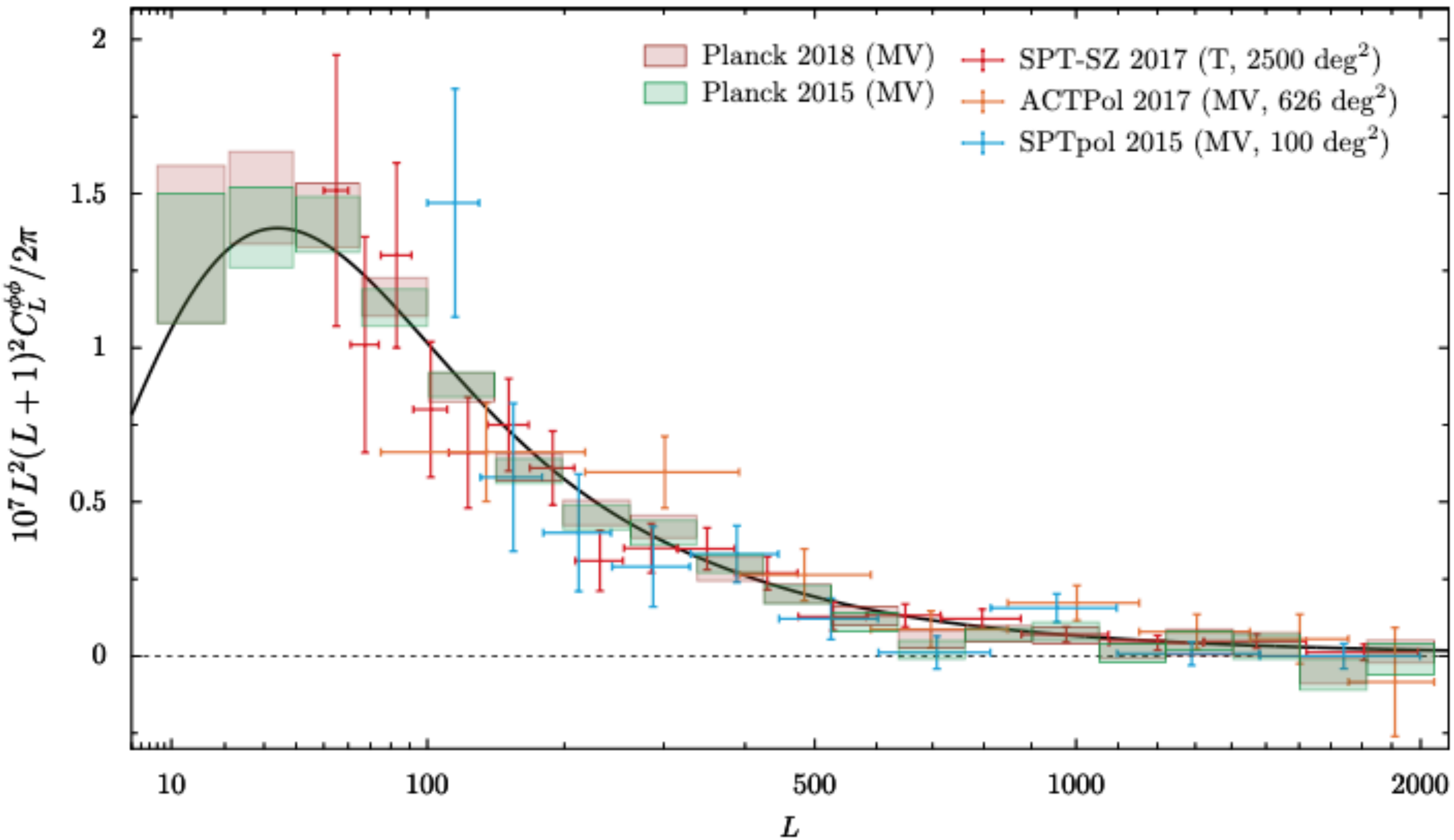
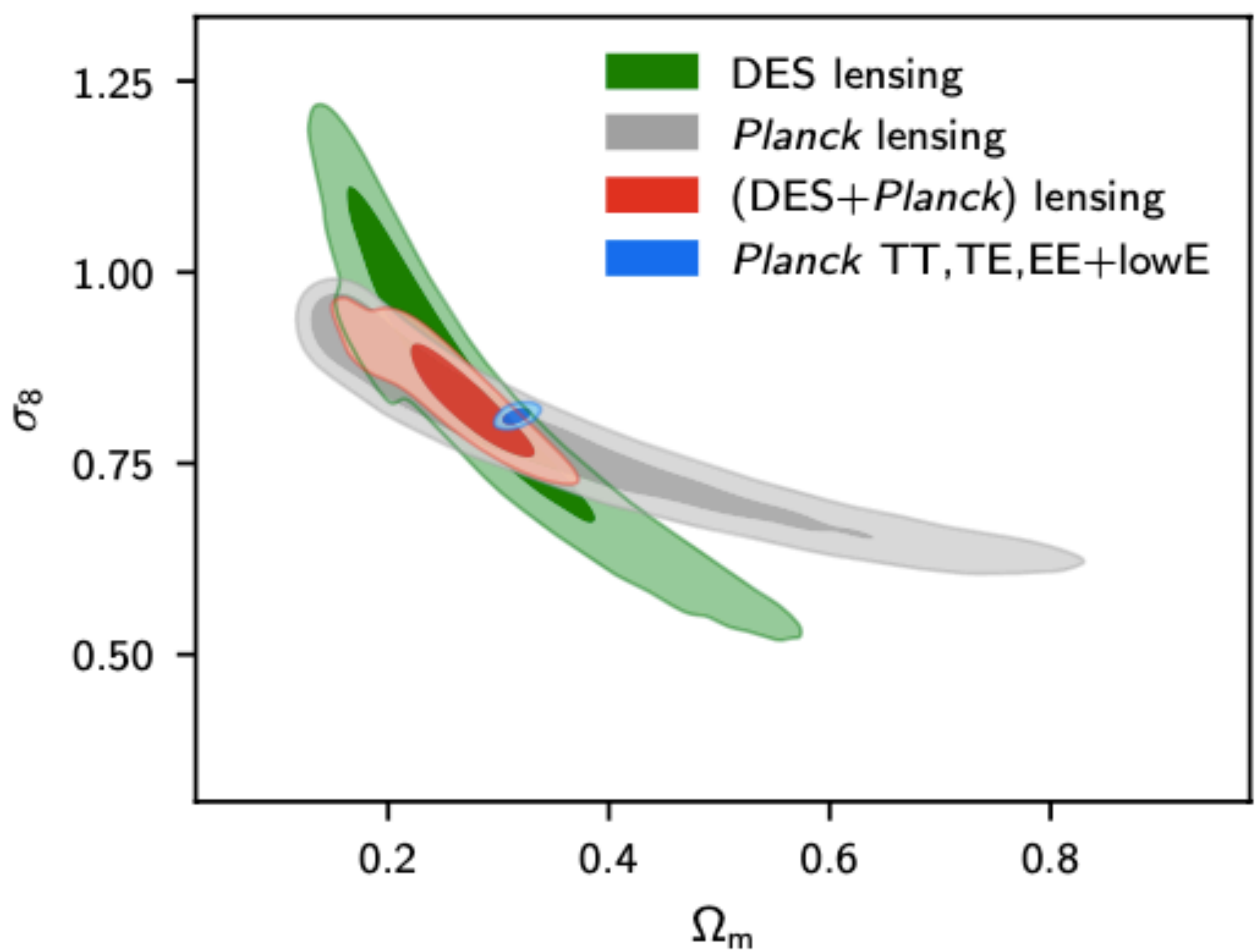
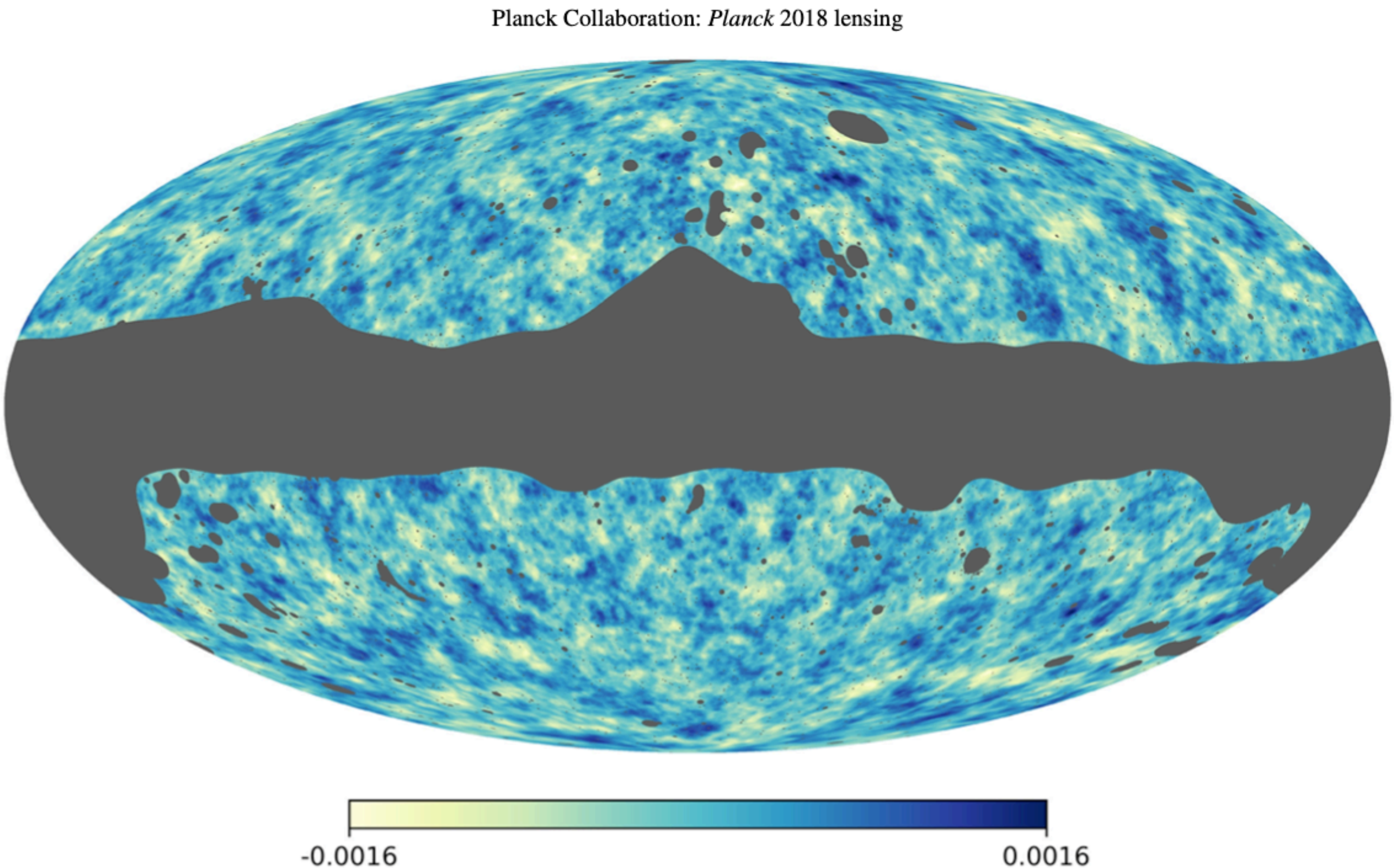


Fig. 5. *Planck* 2018 lensing power-spectrum band powers (pink boxes) over the aggressive multipole range. The 2015 analysis band powers (green) were calculated assuming a slightly different fiducial model and have not been (linearly) corrected to the 2018 model. Also shown are recent measurements by the ACTPol (Sherwin et al. 2017), SPTpol (Story et al. 2015), and SPT-SZ (Simard et al. 2017) collaborations. The SPT-SZ measurement is not completely independent, since the SPT-SZ reconstruction also uses temperature data from *Planck*, but with subdominant weight over the smaller sky area used. The black line shows the lensing potential power spectrum for the ΛCDM best-fit parameters to the *Planck* 2018 likelihoods (*Planck* TT,TE,EE+lowE, which excludes the lensing reconstruction).

LENSING OF THE CMB



LENSING OF THE CMB



LENSING OF THE CMB