GRAVITATIONAL LENSING

13 - LENS MODELS: POWERLAW LENSES

R. Benton Metcalf

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AXIALLY SYMMETRIC POWER-LAW LENSING

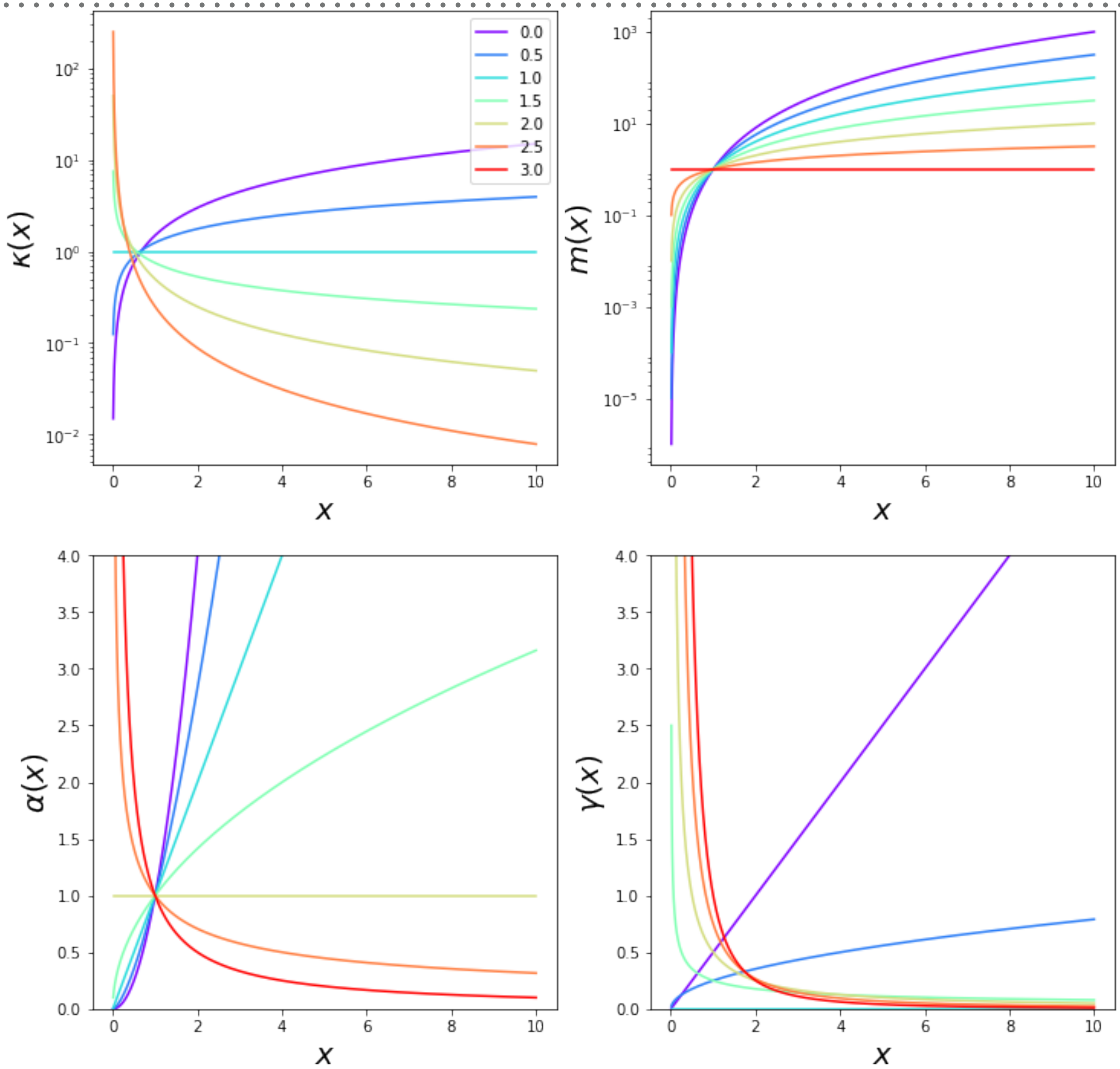
$$m(x) = x^{3-n}$$

$$\alpha(x) = \frac{m(x)}{x} = x^{2-n}$$

$$\kappa(x) = \frac{m'(x)}{2x} = \frac{3-n}{2}\ x^{1-n}$$

$$\gamma(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = x^{1-n} - \frac{3-n}{2}\ x^{1-n} = \frac{n-1}{2} x^{1-n}$$

POWER-LAW LENS



POWER-LAW LENS

- $n < 1$ convergence increases with x. This makes them not suitable for gravitationally bound objects, i.e. galaxies & clusters
- $n = 1$ convergence constant
- $1 < n < 2$ convergence decreasing function of x. The deflection is zero at the centre: $\alpha(x = 0) = 0$
- $n = 2$ deflection is constant. $\alpha(x) = \text{const.}$
- $2 < n < 3$ deflection diverges at origin. Lens potential diverges at origin.
- $n = 3$ corresponds to a point mass. $\alpha(x) = 1/x$
- $n > 3$ are unphysical because they produce 2D mass profiles, *M(r)*, that decrease with radius.

POWER-LAW LENS: CRITICAL LINES AND CAUSTICS

$$\lambda_t = 1 - \frac{m(x)}{x^2} = 0 = 1 - x^{1-n} \Rightarrow x_{crit,t} = 1$$

The tangential critical line has equation $x=1$ for any value of the slope parameter n.

This tells us that the reference angular scale used to define the dimensionless dimensions was the Einstein radius.

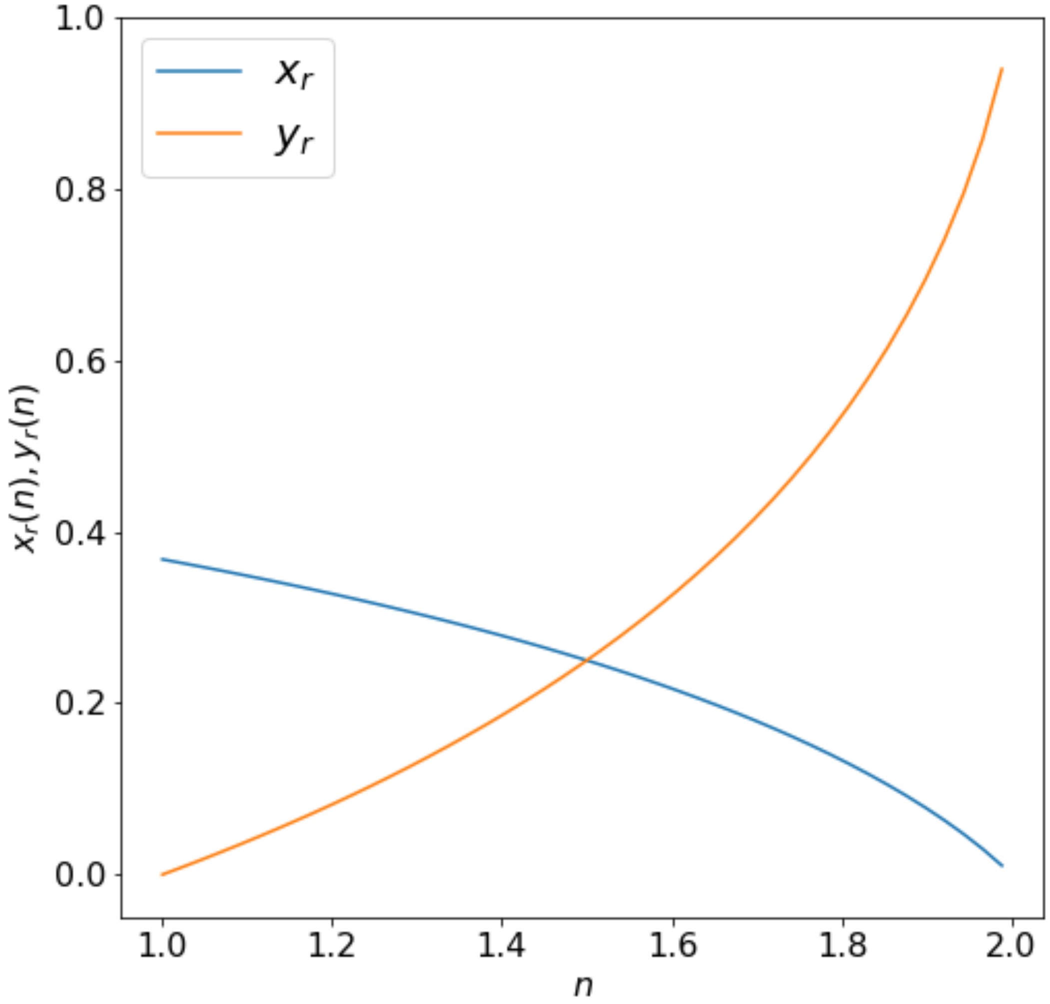
The caustic is the point $y=0$

$$y_{cau} = x_{crit} - \alpha(x_{crit}) \ ; \ \alpha(x) = x^{2-n} \ \Rightarrow y_{cau,t} = 0 \ \ \forall\, n$$

Actually, this is true for any axially symmetric lens!

$$\lambda_t = 1 - \frac{m(x)}{x^2} = 0 \ \Rightarrow y_{cau,t} = x_{crit,t}\left[1 - \frac{m(x_{crit,t})}{x^2_{crit,t}}\right] = 0$$

RADIAL CRITICAL LINE



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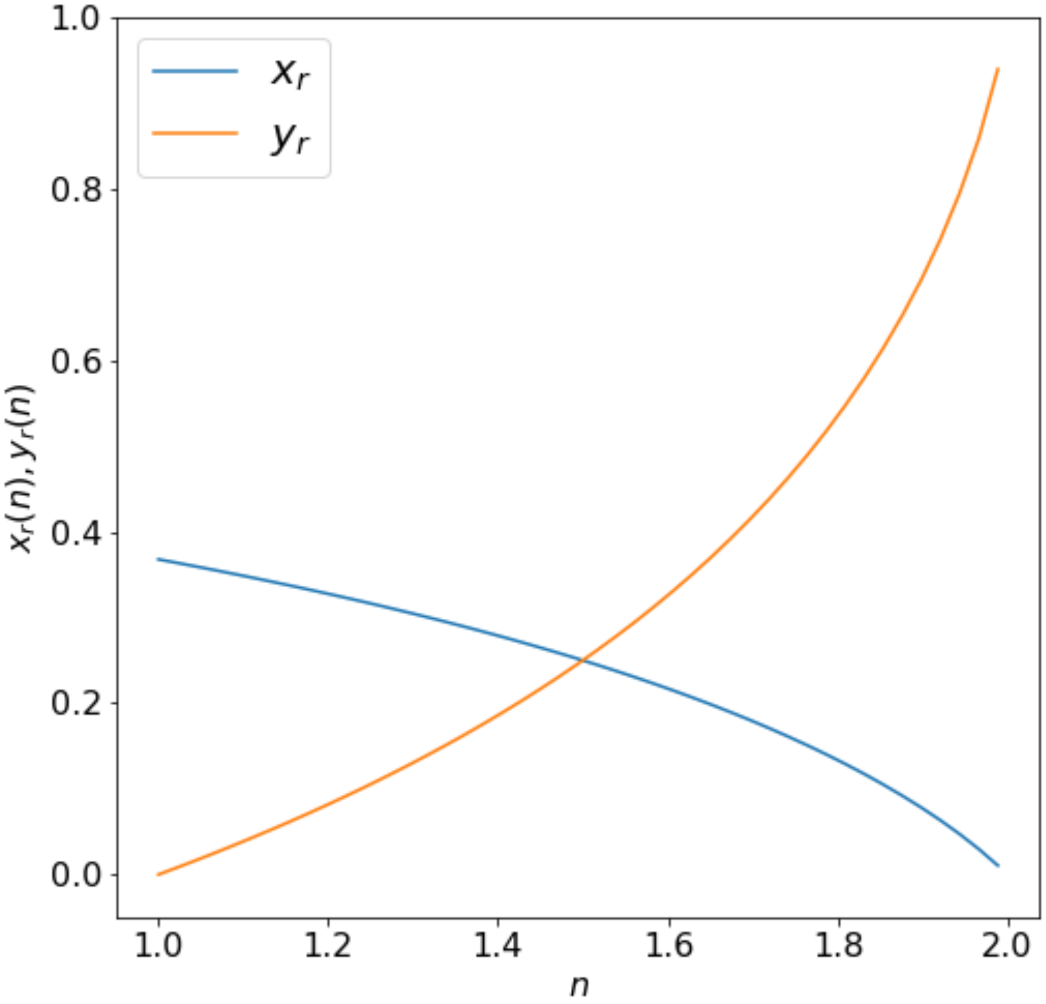


Large n, small radial critical line

AGAIN ON THE EXISTENCE OF THE RADIAL CRITICAL LINE

Another way to write the radial eigenvalue (see past lesson) is:

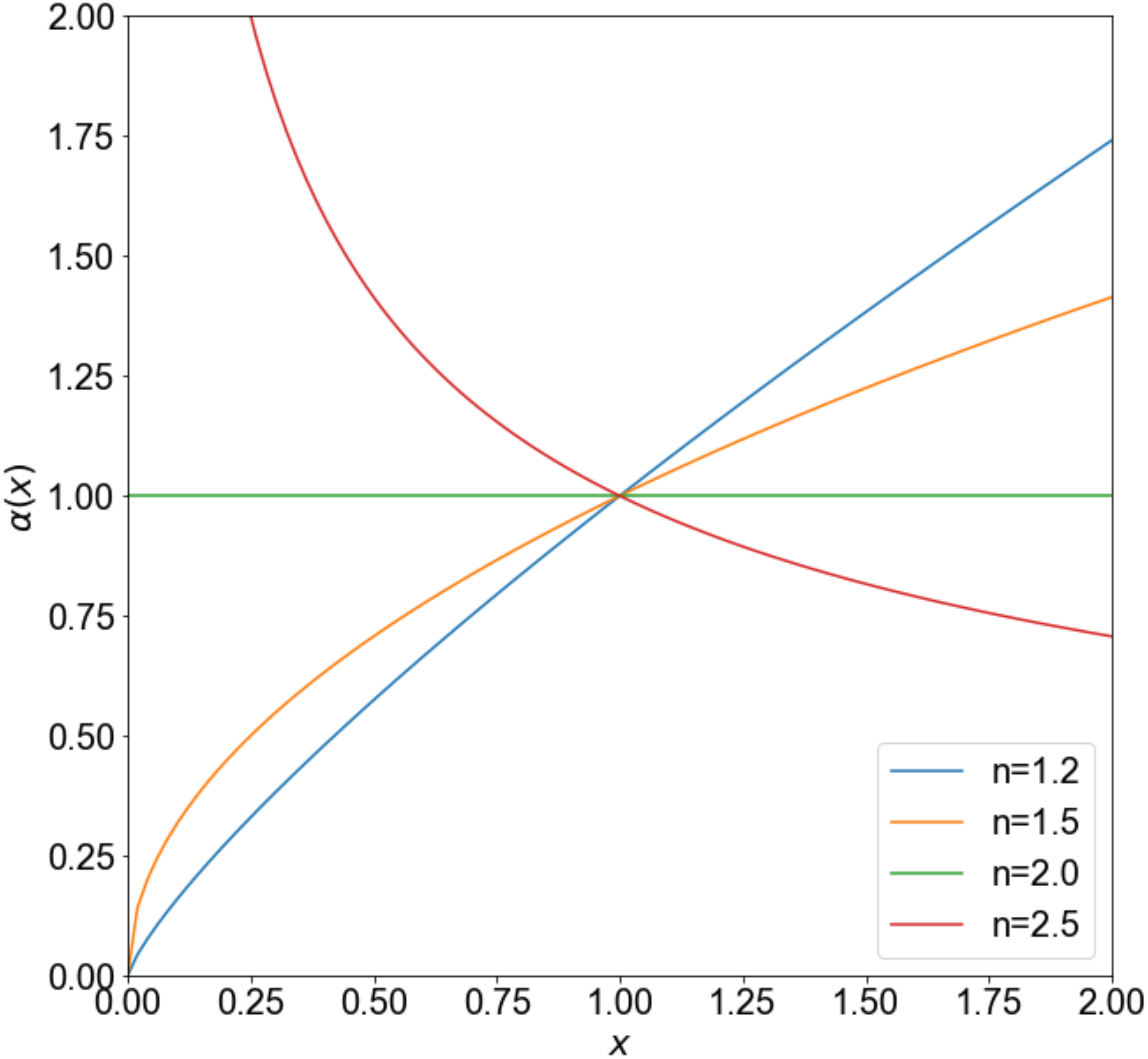
$$\lambda_r(x) = 1 - \alpha'(x)$$

This shows us that the radial critical line exists only if at some distance from the lens center

$\alpha'(x) = 1,$

i.e. the curve $\alpha(x)$ is tangent to some line

$f(x) = x + q$



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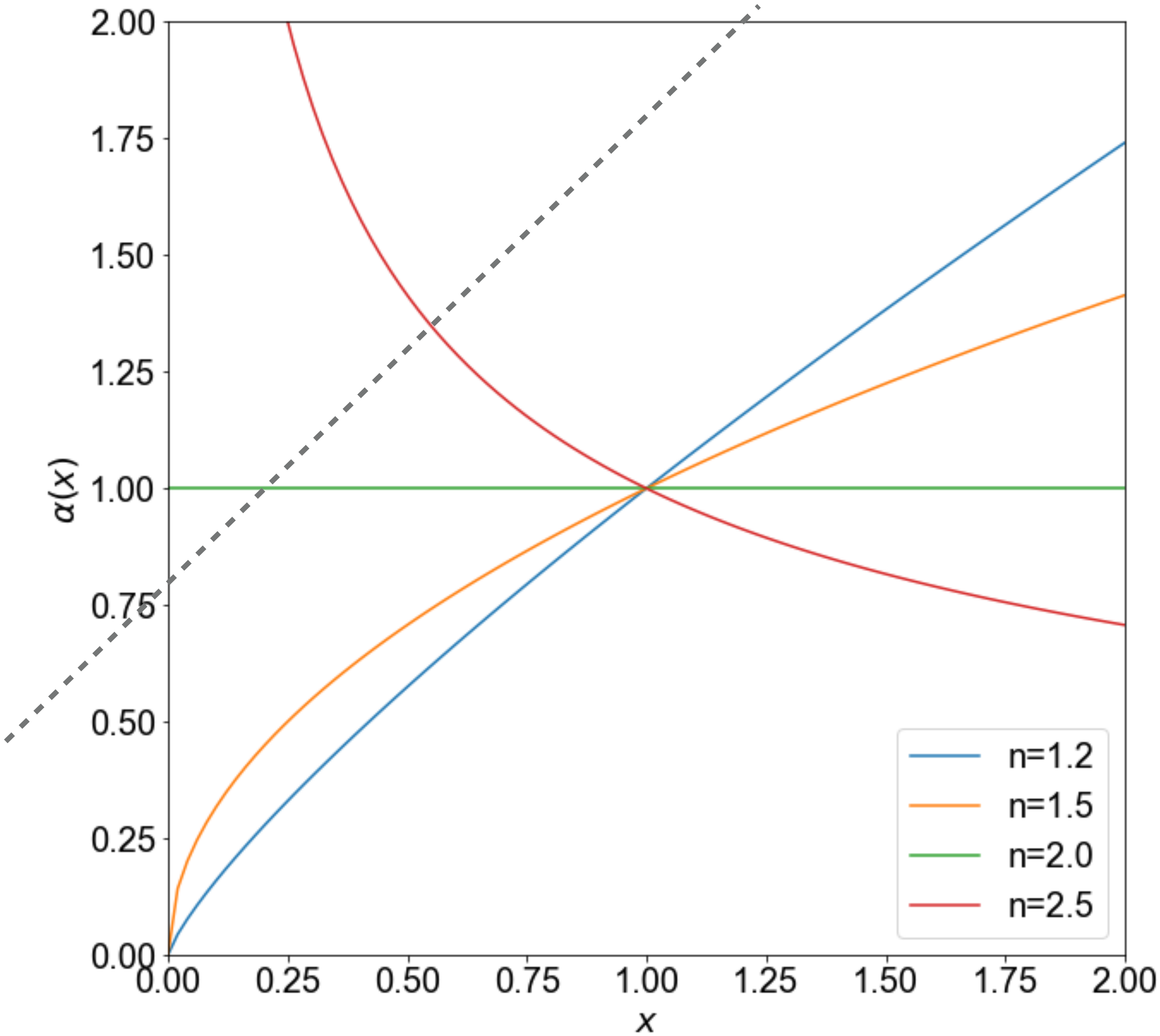

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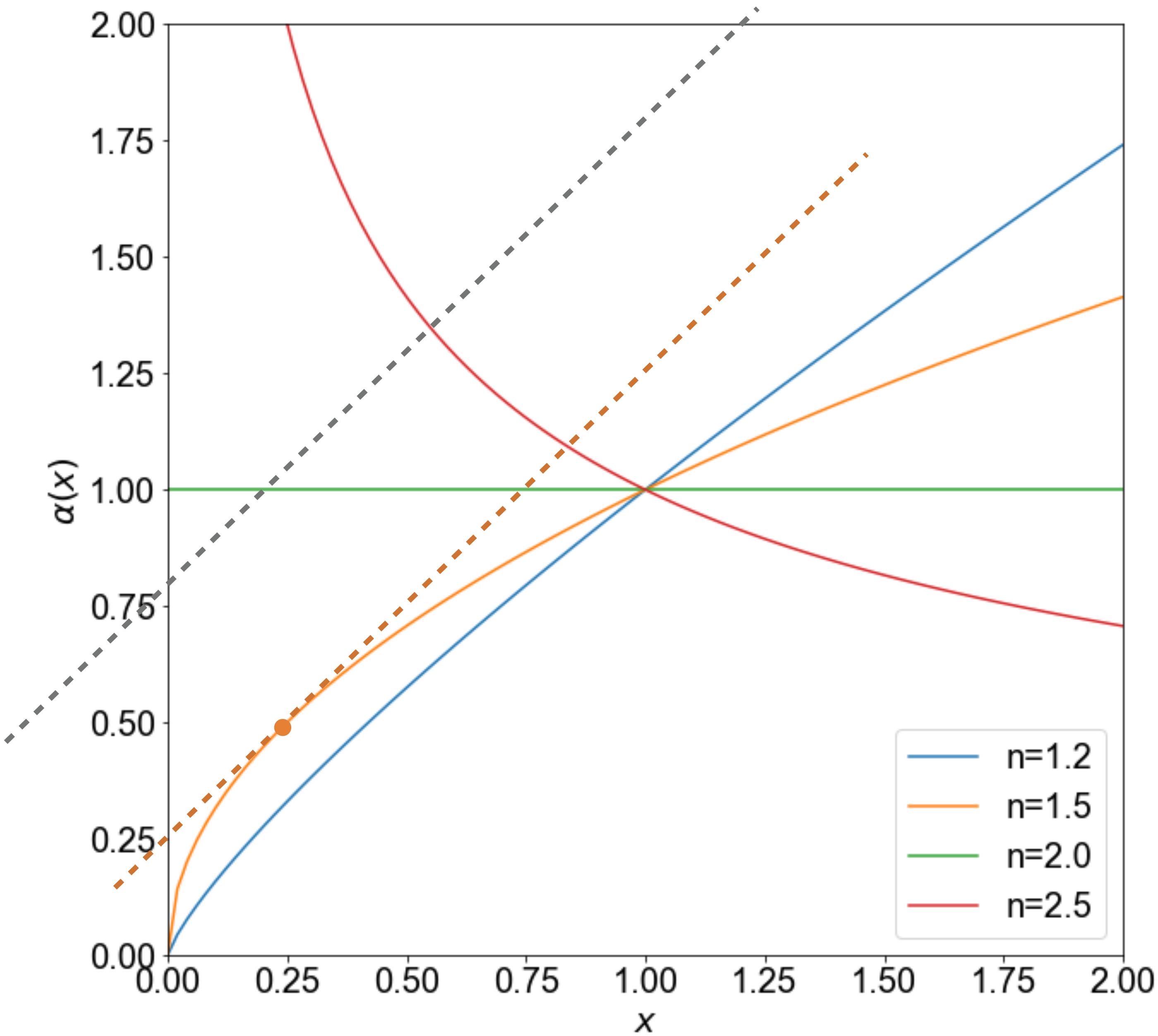

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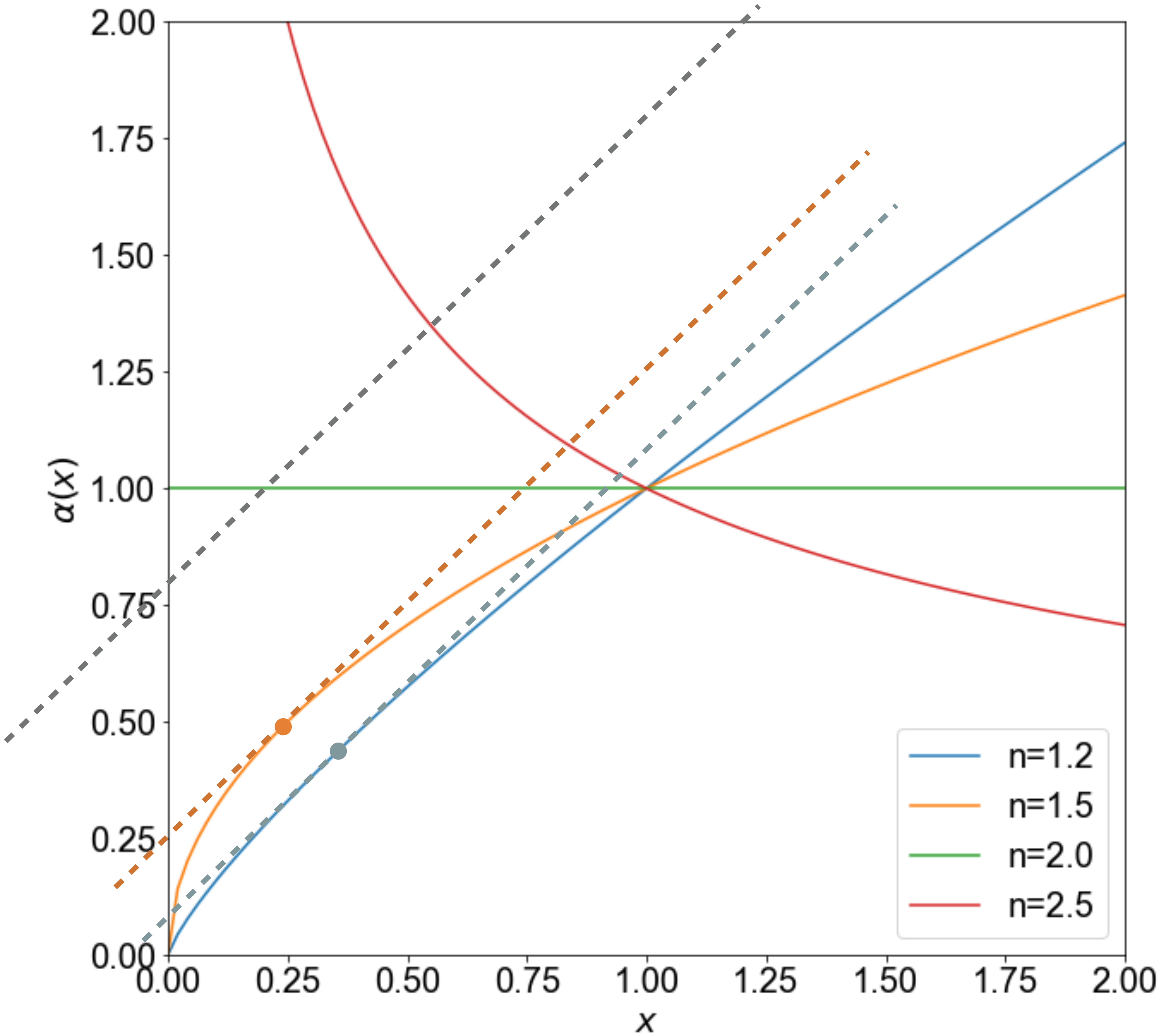


HOW MANY IMAGES DOES A POWER-LAW LENS PRODUCE?

$$y = x - \alpha(x) \Rightarrow x^{2-n} - x - y = 0$$

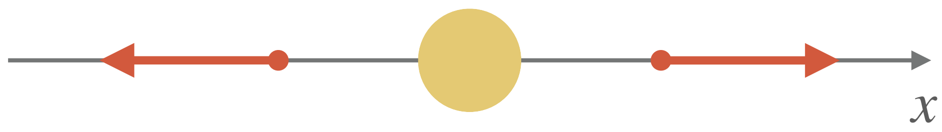
Apart from some special cases, solving analytically the lens equation is impossible.

However, we can determine graphically how many images a give source produce, by building the so called "image diagram"!

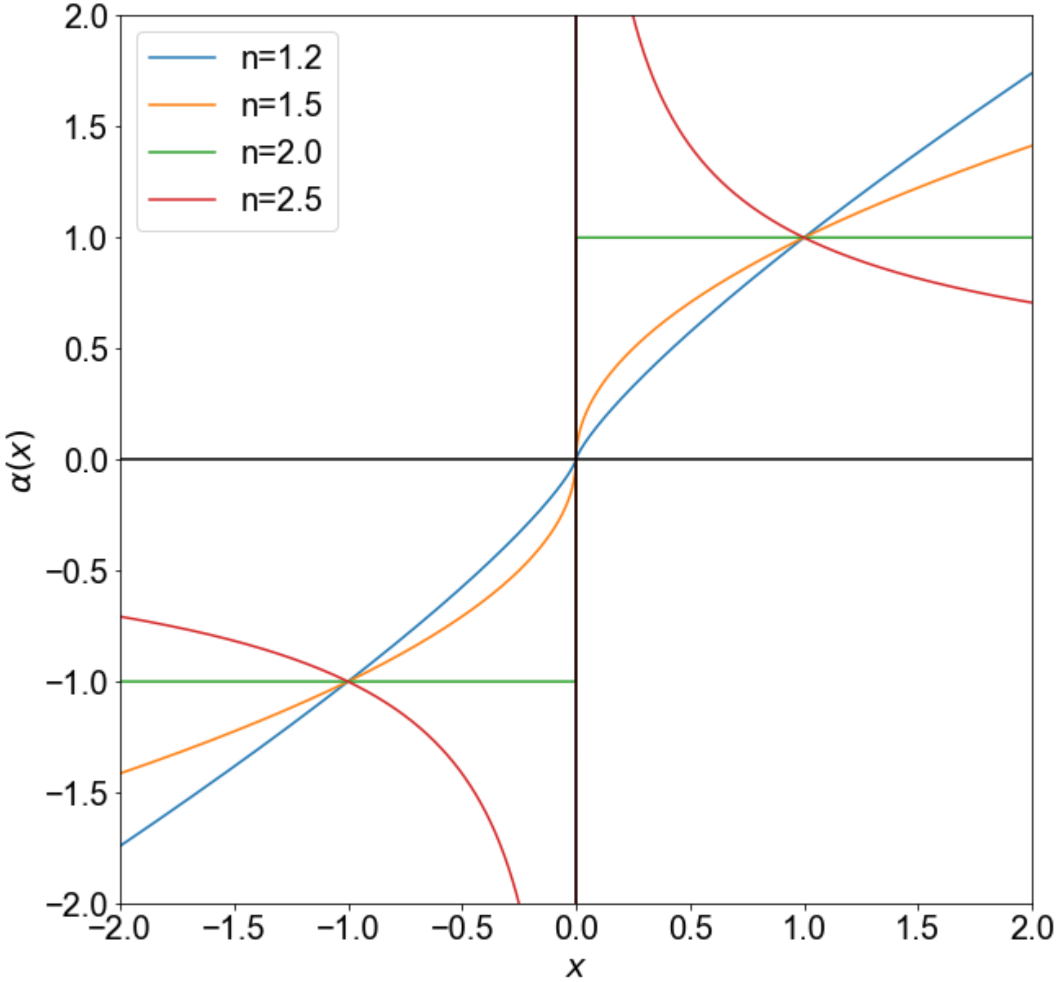
Method:

- Rewrite the lens equation as $\alpha(x) = x - y$
- It become obvious that, for a given source at y, the solutions of the lens equations are the points x where the line $f(x) = x - y$ intercepts the curve $\alpha(x)$

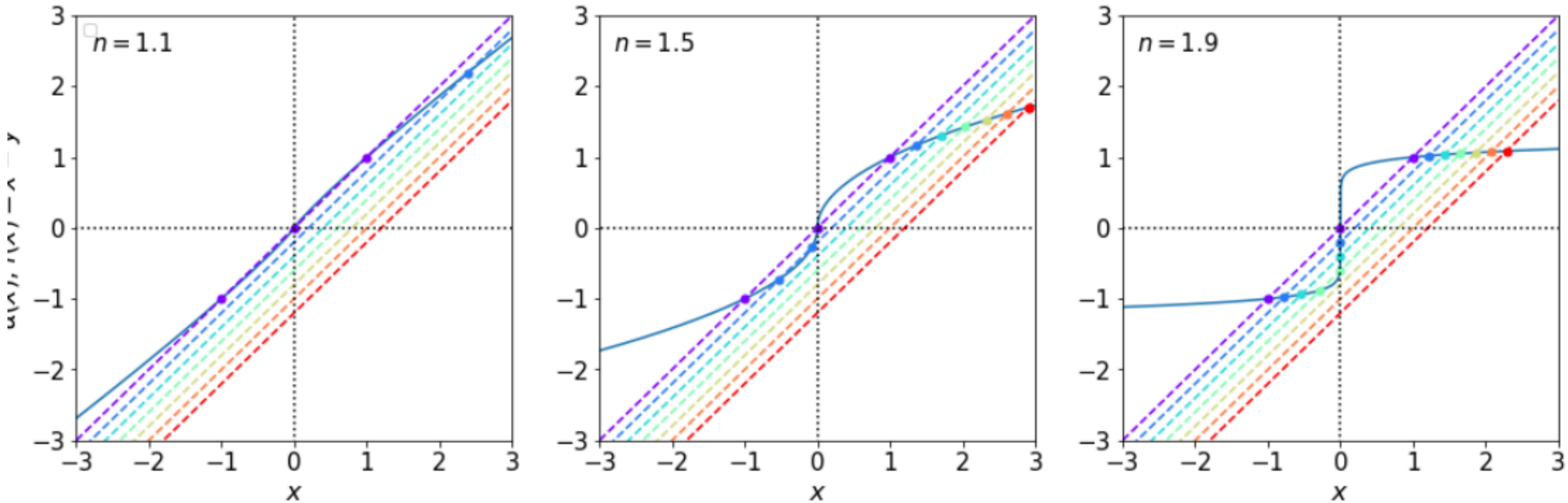
THE DOMAIN OVER WHICH $\alpha(x)$ IS DEFINED EXTENDS TO THE NEGATIVE x AXIS



$$\vec{\alpha}(x) = \alpha(x)\vec{e}_x$$



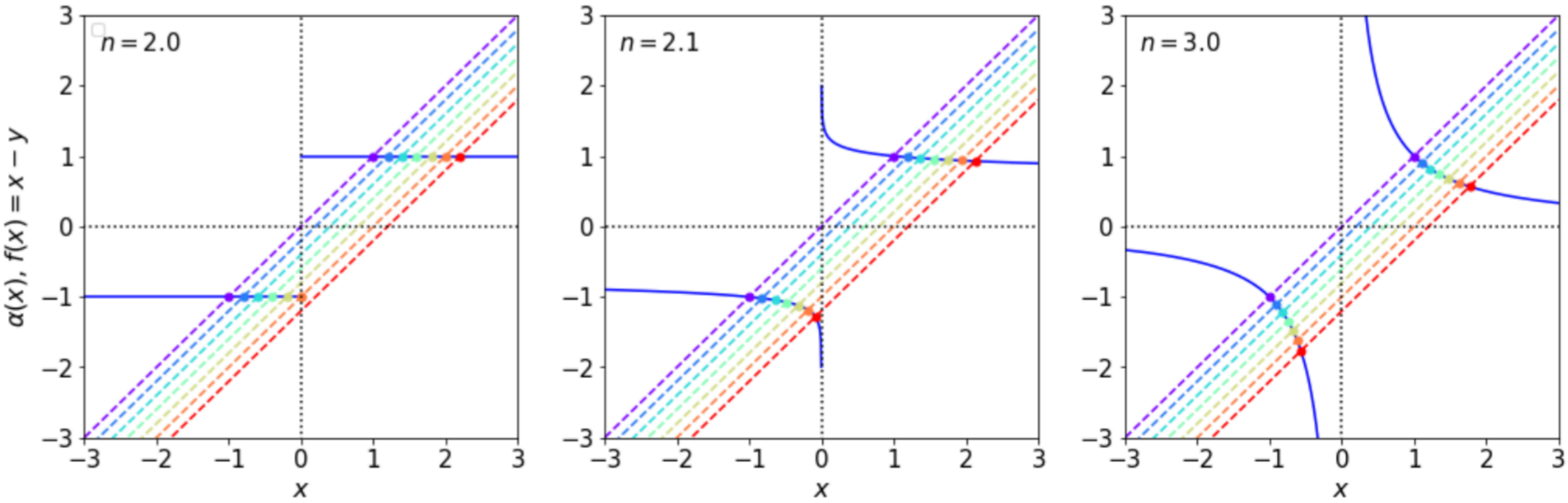
POWER-LAW LENSES : IMAGES



For $1 < n < 2$ there are 3 or 1 images.

3 images when the source is within the radial caustic.

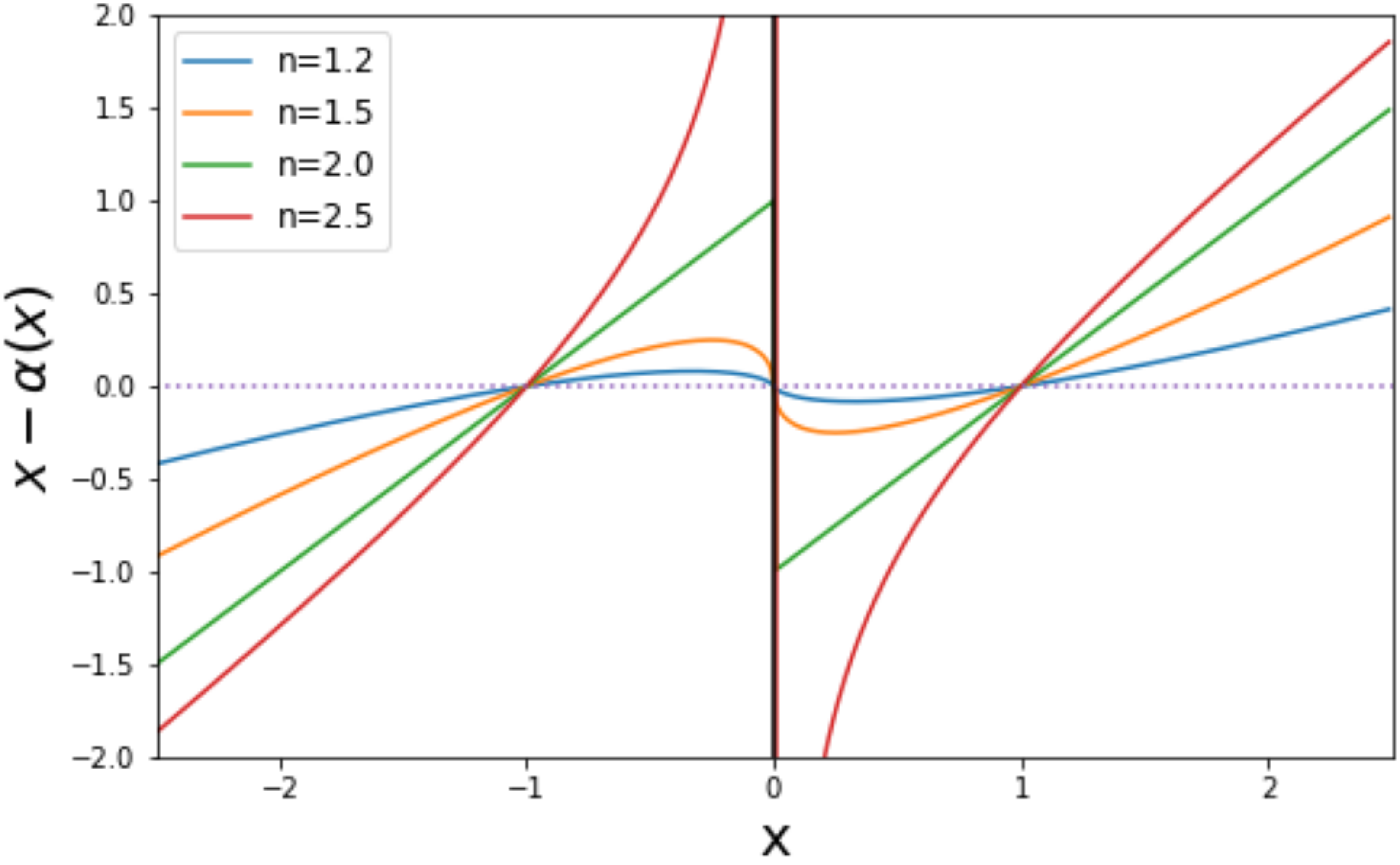
POWER-LAW LENSES : IMAGES



For $2 < n < 3$ there are always 2 images.

For n = 3 there are always 2 when $y < 1$ and 1 image when $y > 1$.

POWER-LAW LENSES : IMAGES



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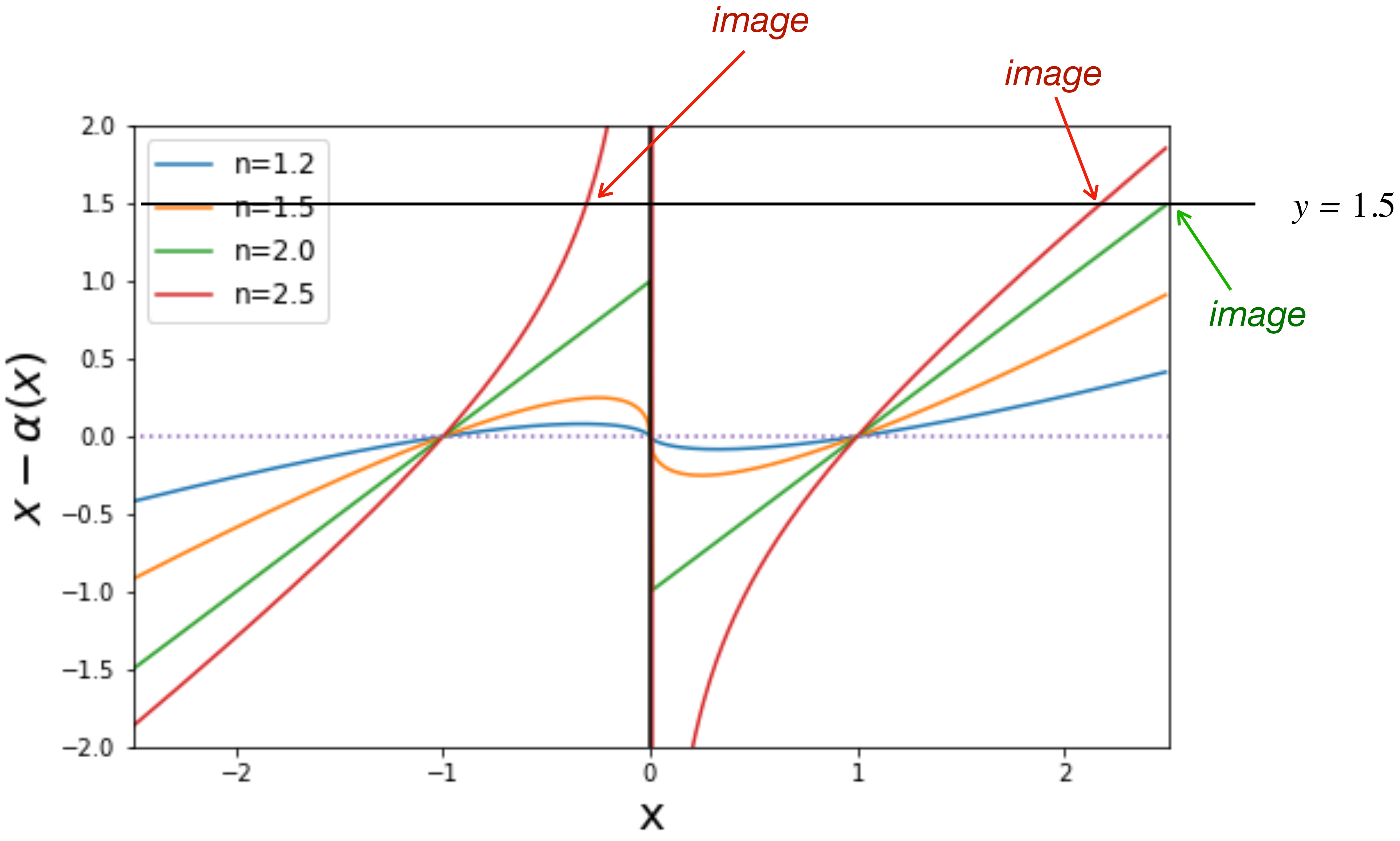

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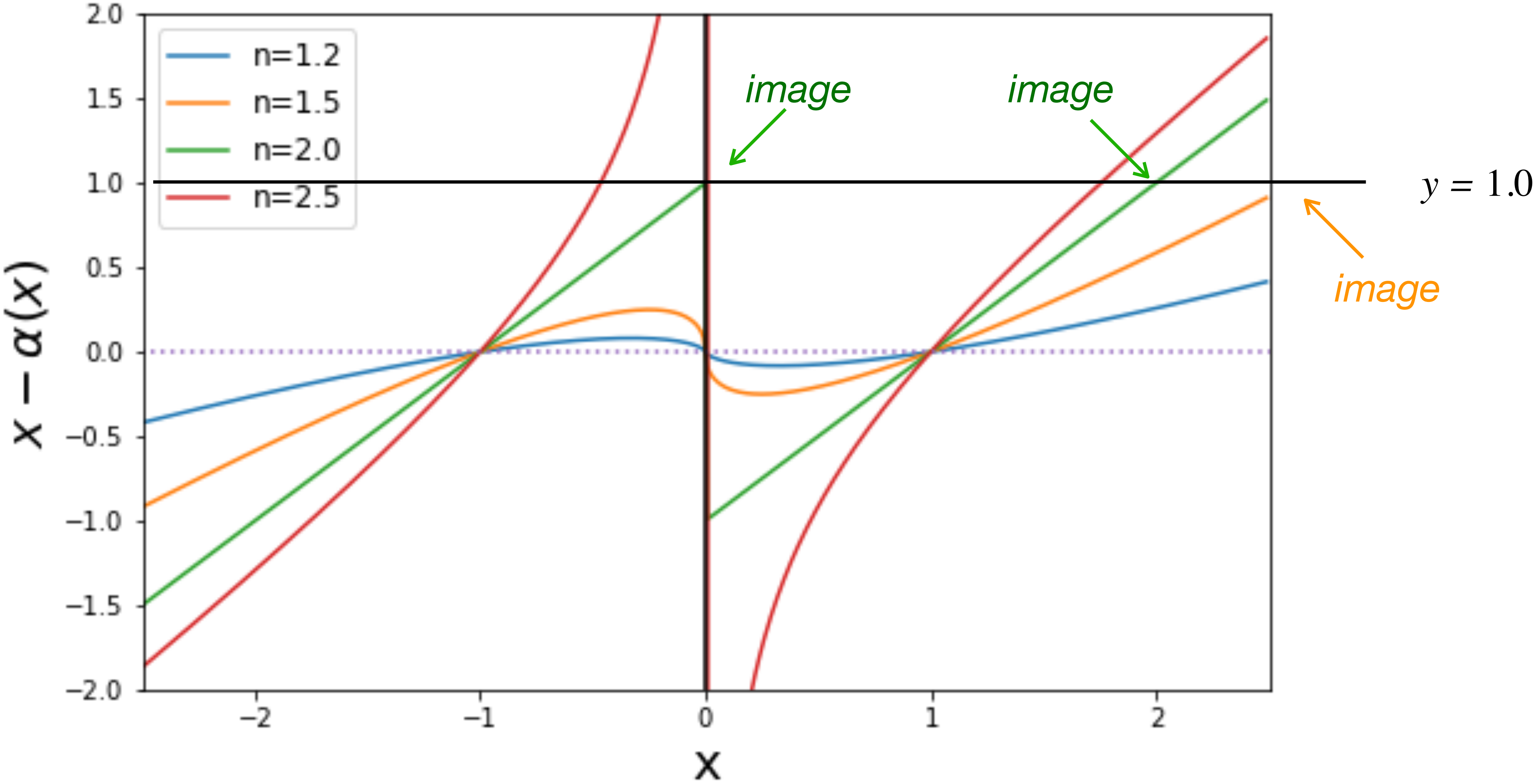

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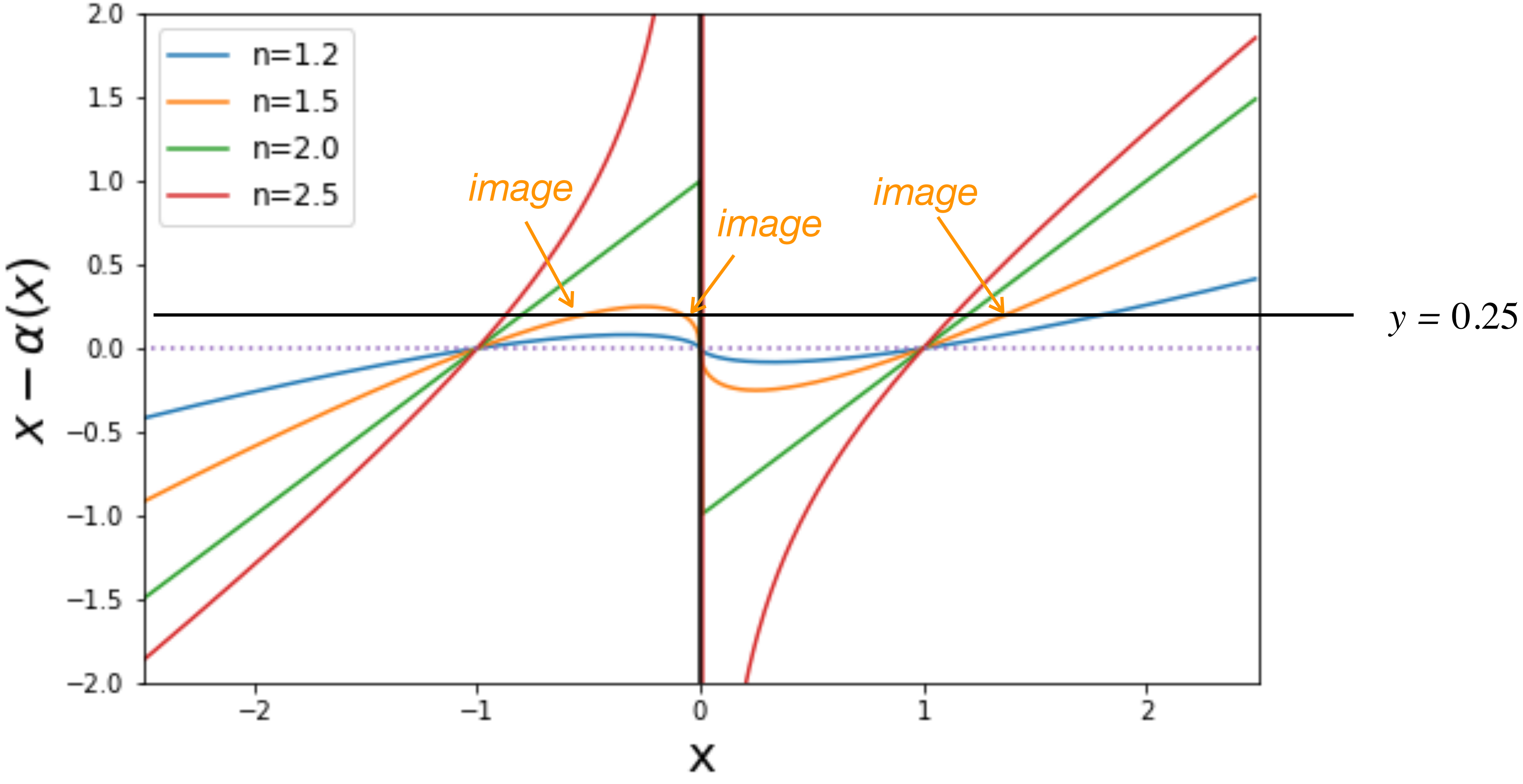
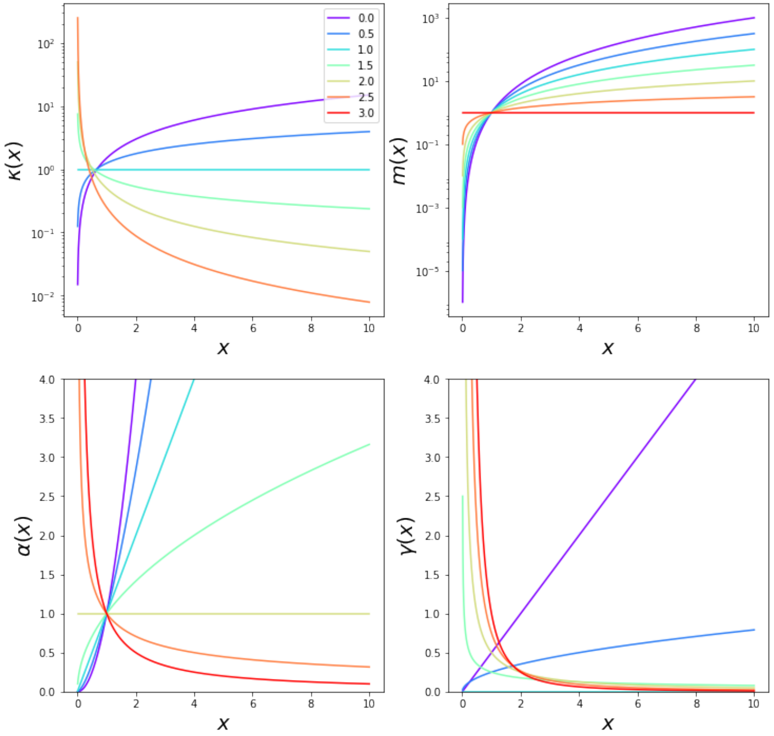
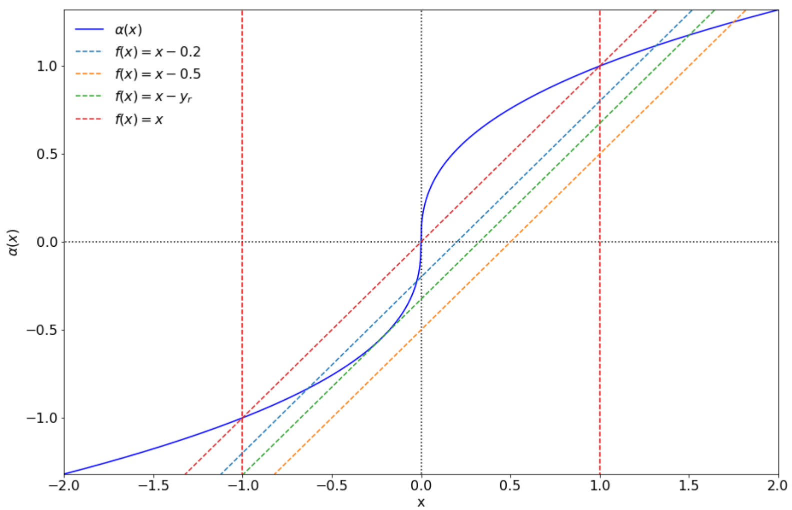


RECAP: POWER-LAW LENS (N<2)



$$m(x) = x^{3-n} \qquad \kappa(x) = \frac{m'(x)}{2x} = \frac{3-n}{2} x^{1-n}$$

$$\alpha(x) = \frac{m(x)}{x} = x^{2-n}$$



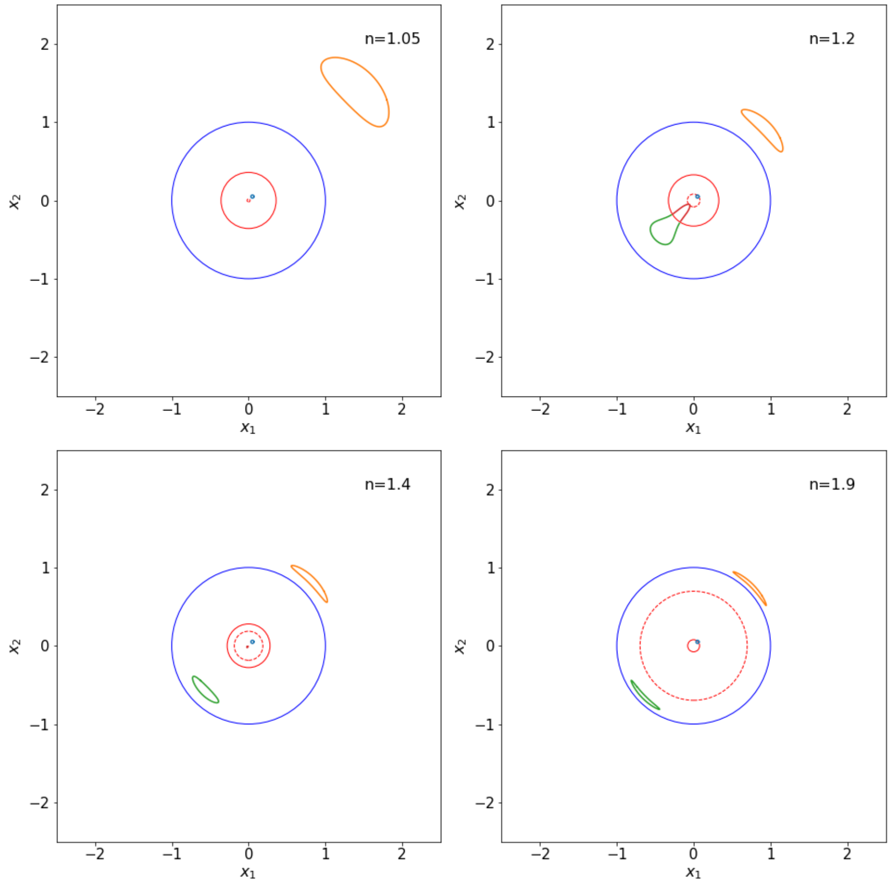
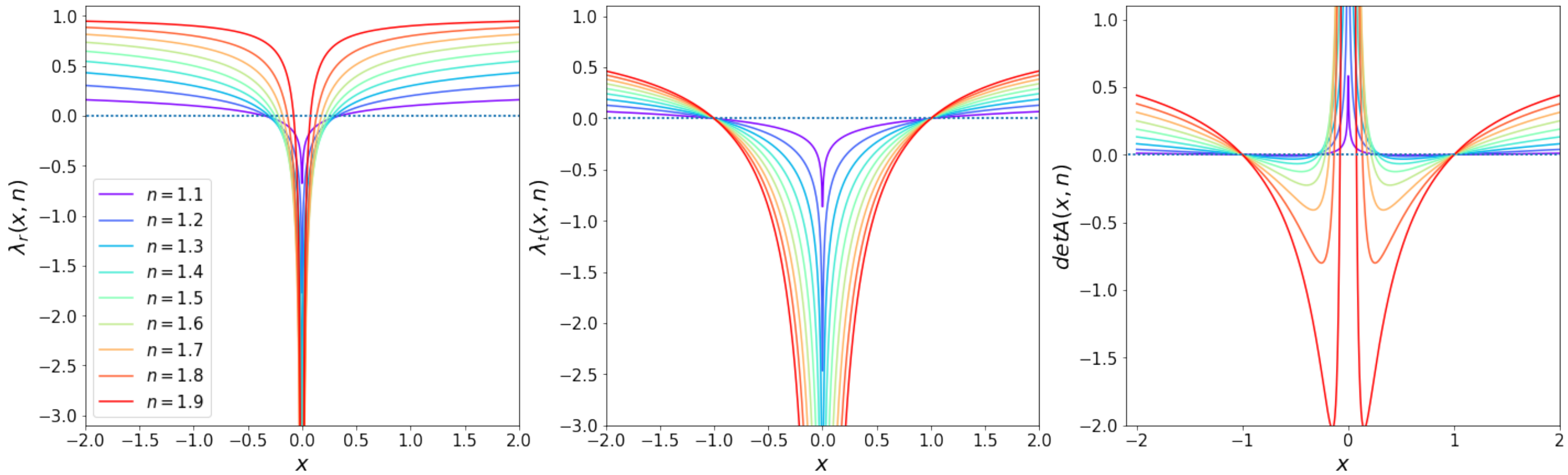
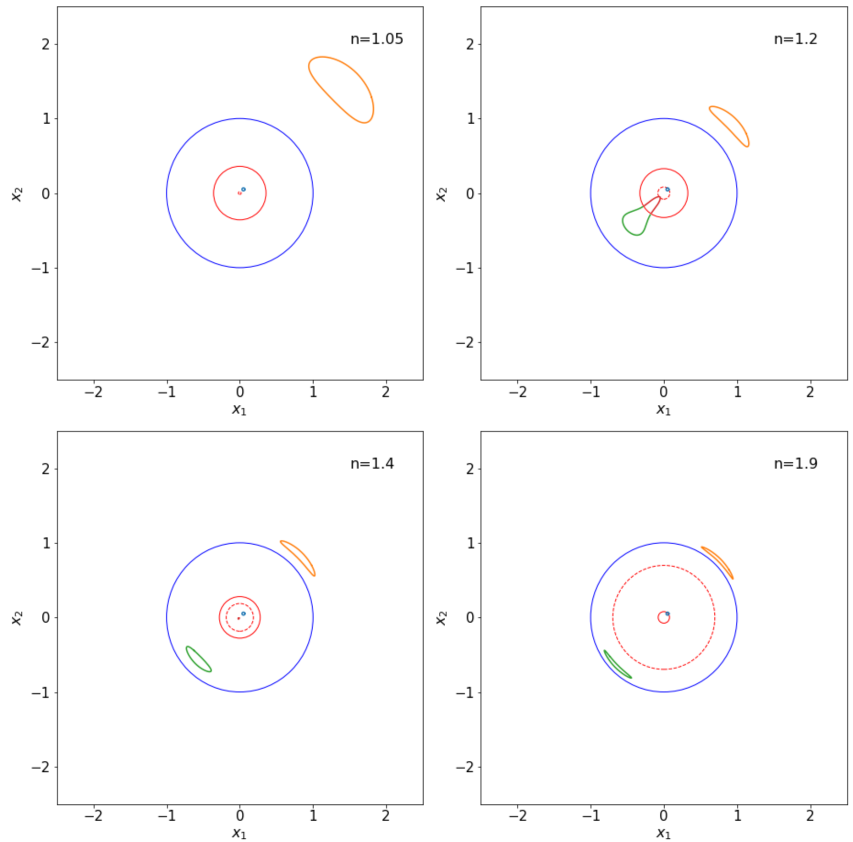


IMAGE PARITY





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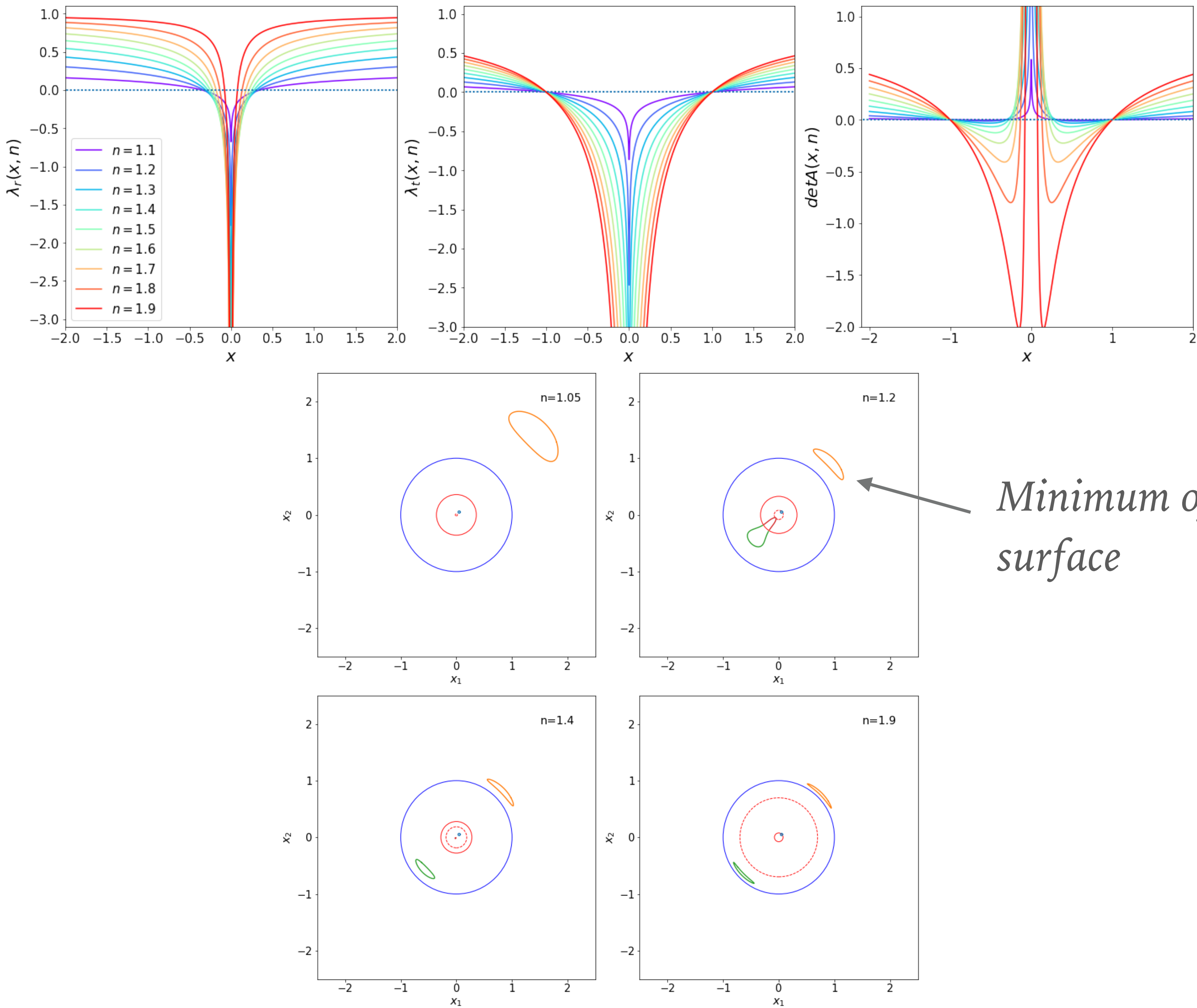
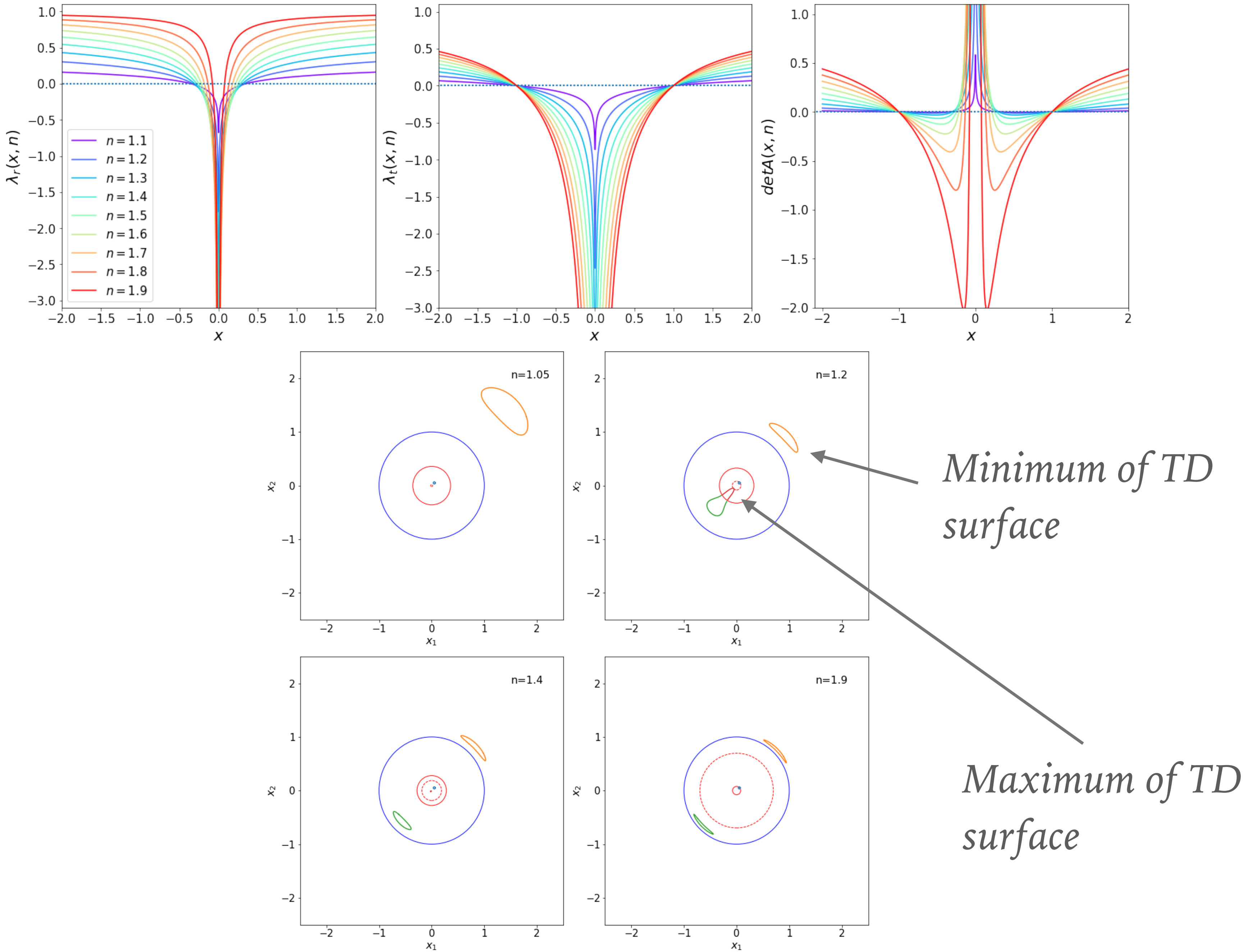


Minimum of TD surface

IMAGE PARITY



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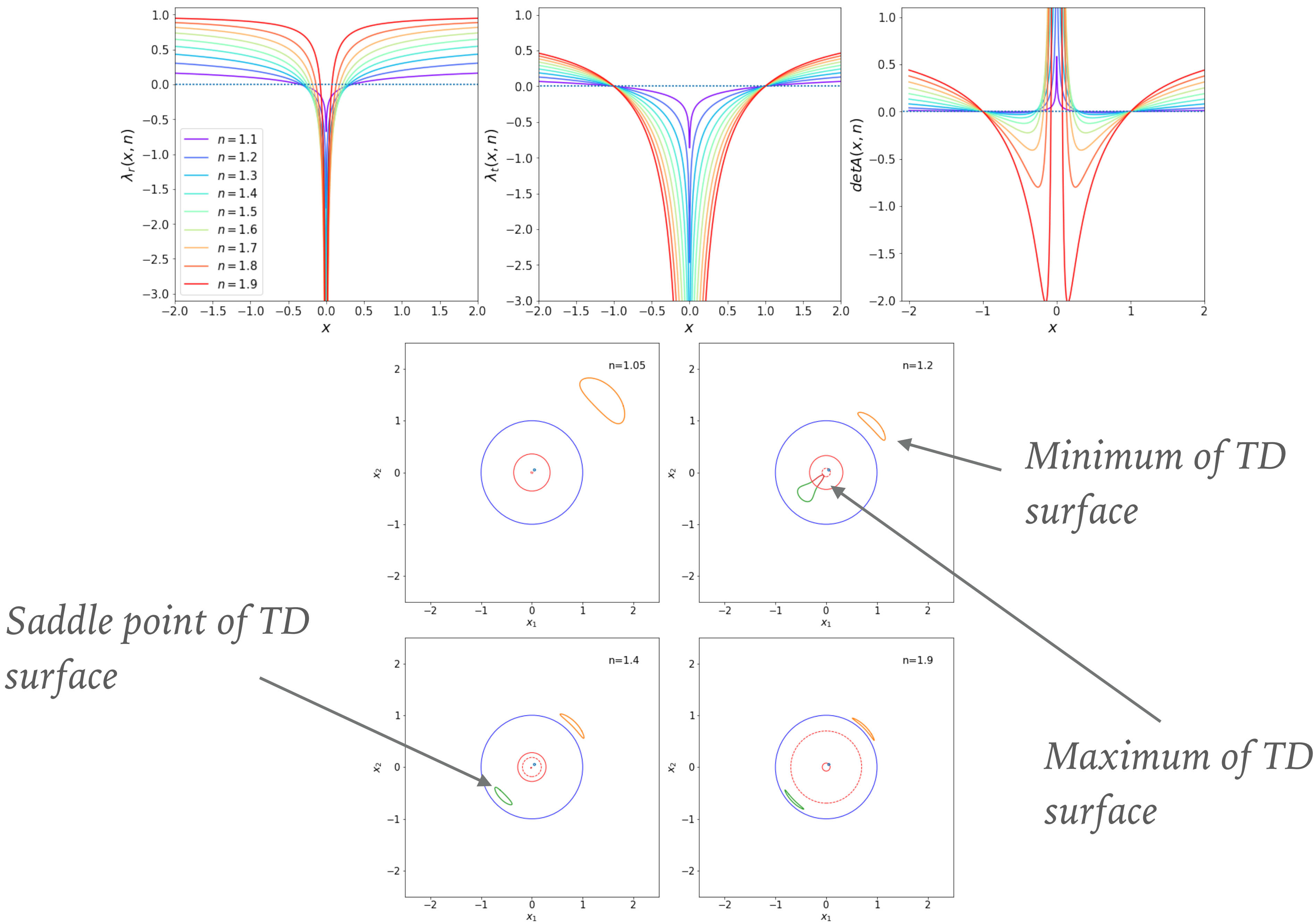


POWER-LAW LENSES : IMAGE DISTORTION

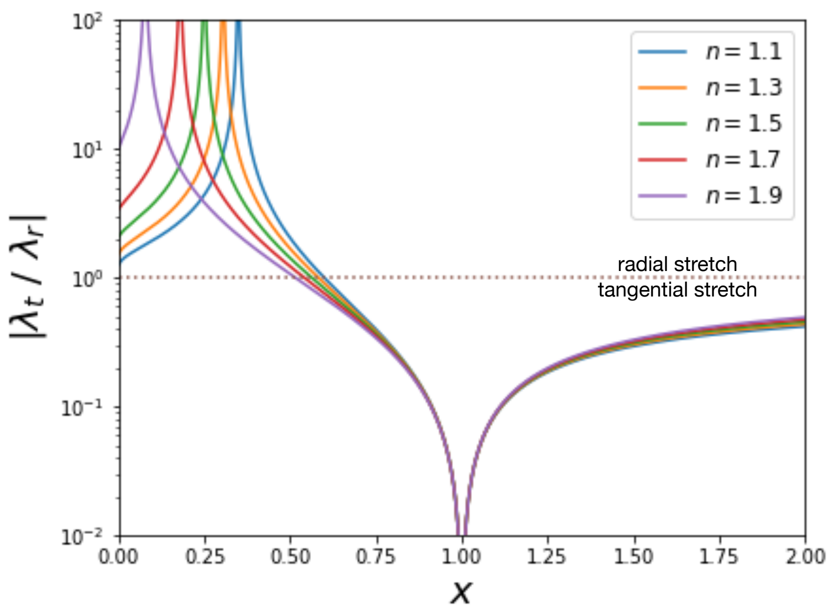
Distortion of infinitesimal images

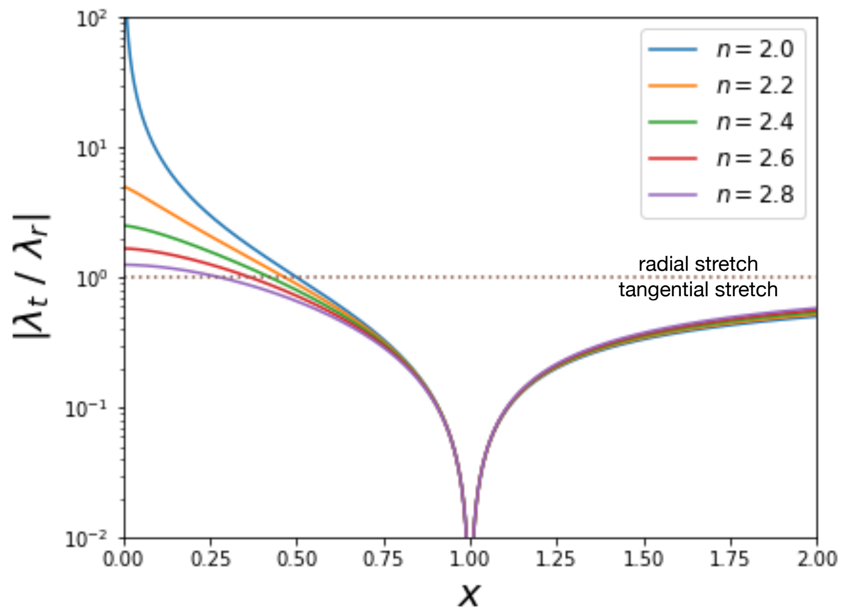
radial eigenvalue of A $\lambda_r = 1 - \alpha'(x)$

tangential eigenvalue of A $\lambda_t = 1 - \dfrac{\alpha(x)}{x}$

$\left|\dfrac{\lambda_t}{\lambda_r}\right| < 1$ *image is tangentially stretched*

$\left|\dfrac{\lambda_t}{\lambda_r}\right| > 1$ *image is radially stretched*





POWER-LAW LENSES : TIME-DELAY

$$t(\boldsymbol{x}) = \frac{(1+z_l)}{c}\frac{D_l D_s}{D_{ls}}\left(\frac{\xi_o}{D_l}\right)^2\left[\frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2 - \Psi(\boldsymbol{x})\right]$$

ξ_o *- units in which x,y are measured*

$$t(\boldsymbol{x}) = \frac{(1+z_l)}{c}\frac{D_l D_s}{D_{ls}}\left(\frac{\xi_o}{D_l}\right)^2\left[\frac{1}{2}|\boldsymbol{\alpha}(\boldsymbol{x})|^2 - \Psi(\boldsymbol{x})\right]$$

$$= \frac{(1+z_l)D_{\Delta t}}{c}\,\tau(\boldsymbol{x})$$

for a power-law lens

$$\Psi(\boldsymbol{x}) = \frac{x^{3-n}}{3-n} \qquad \alpha(x) = x^{2-n}$$

relative time-delay

$$\tau_{iA} = \tau_i - \tau_A = \left(\frac{\xi_o}{D_l}\right)^2\left[\frac{1}{2}\left(|x_i|^{2(2-n)} - |x_A|^{2(2-n)}\right) - \frac{1}{3-n}\left(|x_i|^{3-n} - |x_A|^{3-n}\right)\right]$$

Images that have the same separation will have a time delay that depends on the slope of the lens, n.

Generally, steeper mass profiles will have longer time-delays for the same image separation.