# GRAVITATIONAL LENSING

## 14 - LENS MODELS: ISOTHERMAL SPHERE

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# THE SINGULAR ISOTHERMAL SPHERE

*The Singular Isothermal Sphere is a simple model to describe the distribution of matter in galaxies and clusters. It can be derived assuming that the matter content of the lens behaves like an ideal gas confined by a spherically symmetric gravitational potential. If the gas is isothermal and in hydrostatic equilibrium, its density profile is*

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

*velocity dispersion of the gas particles*

*The profile has some good properties:*

- *simple*
- *Reproduces the flat rotation curves of galaxies*

*The profile is "unphysical"*

- *singularity near the center*
- *mass is infinite*

# THE SINGULAR ISOTHERMAL SPHERE

*For lensing purposes, we are interested in the projection of this profile:*

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \qquad\qquad r^2 = \xi^2 + z^2$$

$$\Sigma(\xi) = 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{dz}{\xi^2 + z^2} = \frac{\sigma_v^2}{\pi G \xi} \arctan \frac{z}{\xi}\Bigg|_0^\infty = \frac{\sigma_v^2}{2G\xi}$$

*Using angular units:* $\Sigma(\theta) = \dfrac{\sigma_v^2}{2GD_L\theta}$

*Now we can compute the convergence:*

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{cr}} \qquad\qquad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

$$\Rightarrow \kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{cr}} = \frac{\sigma_v^2}{2GD_L\theta}\frac{4\pi G}{c^2}\frac{D_L D_{LS}}{D_S} = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}\frac{1}{2\theta}$$

# THE SINGULAR ISOTHERMAL SPHERE

$$\kappa(\theta) = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}\frac{1}{2\theta}$$

*If we set* $\theta_0 = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}$ *and* $x = \frac{\theta}{\theta_0}$, *then* $\kappa(x) = \frac{1}{2x}$

*Remember that, for a power-law lens,* $\kappa(x) = \frac{m'(x)}{2x} = \frac{3-n}{2}x^{1-n}$

*Thus, the SIS lens is a power-law lens with n=2, and* $\theta_0$ *is the Einstein radius!*

$$\theta_{E,SIS} = \theta_0 = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}$$

*Note that* $\kappa(\theta_E) = 1/2$ *for a SIS lens.*

# THE SINGULAR ISOTHERMAL SPHERE

*We can use the usual formulas to derive all the relevant lens quantities. For example:*

$$m(x) = x$$

*We can compute the deflection angle as*

$$\alpha(x) = \frac{m(x)}{x} = \frac{|x|}{x}$$

*This has an easy form, so that we can solve the lens equation analytically. Indeed, the lens equation is:* $y = x - \frac{|x|}{x}$ *,which can be split into*

$y = x - 1$ *for* $x>0$*, and* $y = x + 1$ *for* $x<0$

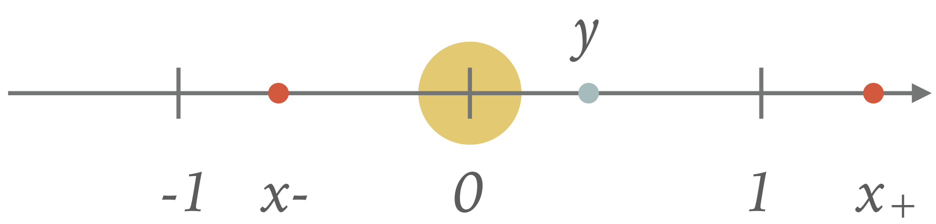
*Two solutions exist only if* $|y|<1$*!*

# THE SINGULAR ISOTHERMAL SPHERE

$y = x - 1$ *for* $x > 0$*, and* $y = x + 1$ *for* $x < 0$

*The solutions of the lens equation are:*

$x_+ = y + 1$ $\qquad$ $x_- = y - 1$

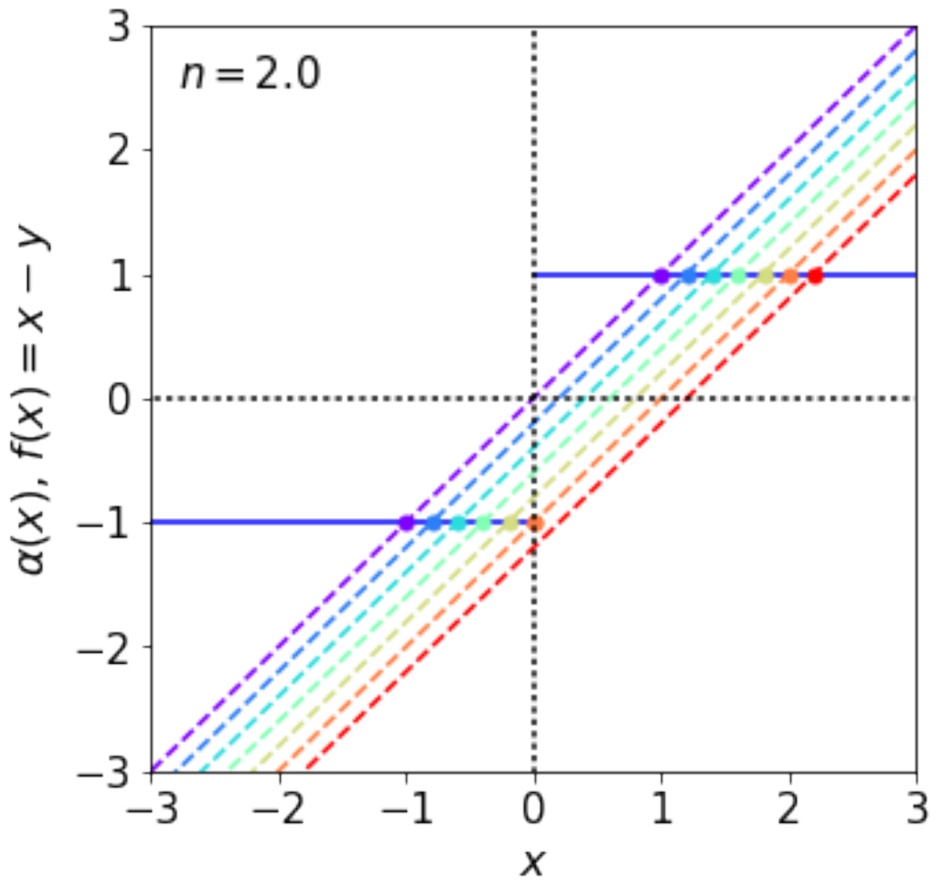



*Going back to the angular units:* $x = \dfrac{\theta}{\theta_E}$ $\quad y = \dfrac{\beta}{\theta_E}$

$$\theta_- = \beta - \theta_E$$

$$\theta_+ = \beta + \theta_E$$

# IMAGE DIAGRAM (SIS)

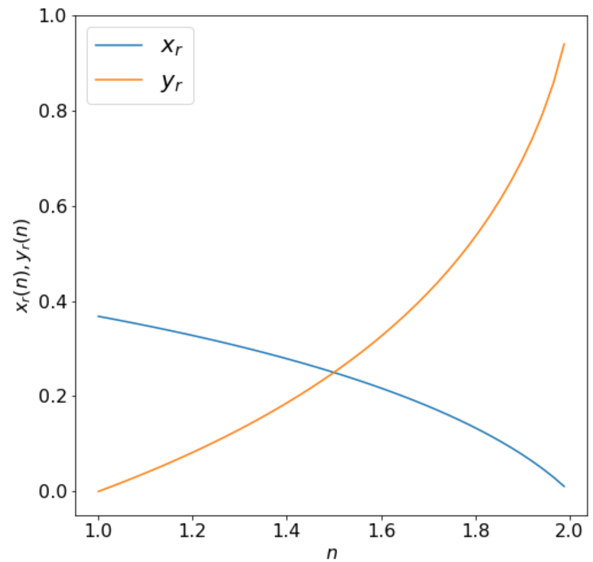



$$\frac{d\alpha}{dx} = 0 \;\; \forall \, x$$

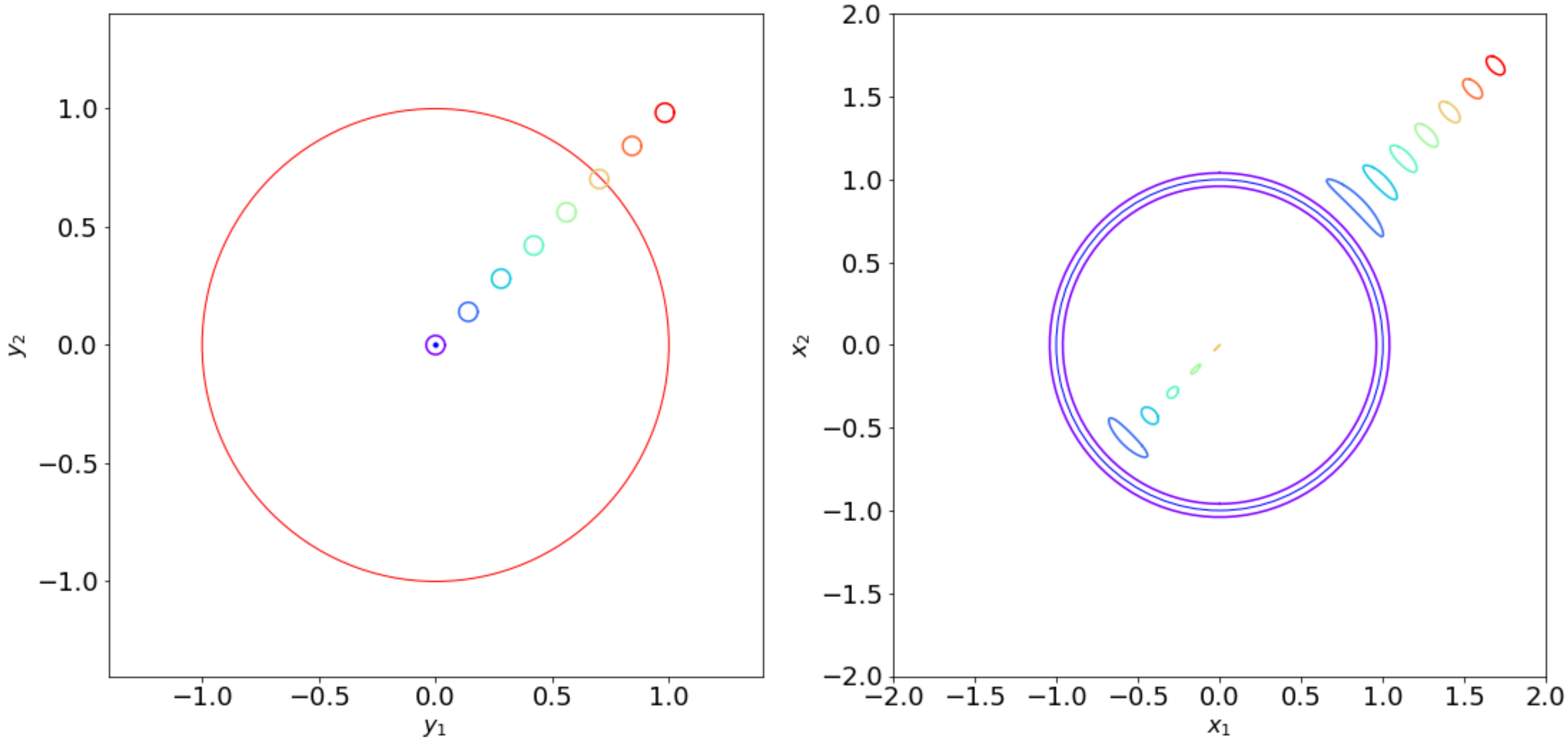
*There is no radial critical line!*

*However, there is a line that almost plays the role of the caustic (at least to determine the image multiplicity…)*

*A limiting case of radial caustic for* $n \to 2$



# THE SINGULAR ISOTHERMAL SPHERE



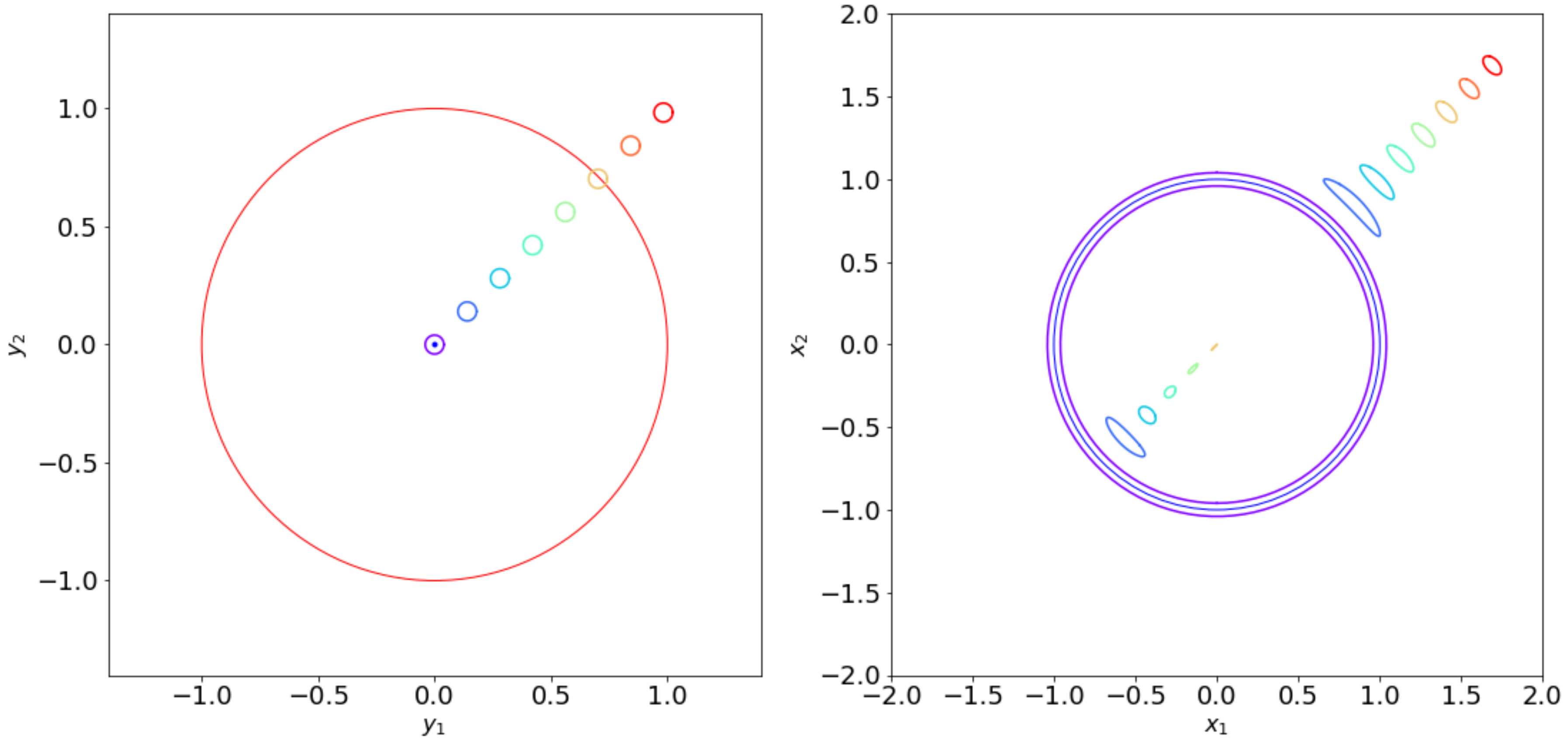
# THE SINGULAR ISOTHERMAL SPHERE

$$\frac{d\alpha}{dx} = 0 \ \ \forall \, x$$

*This implies that the radial eigenvalue of the Jacobian matrix is always* $\lambda_r = 1$

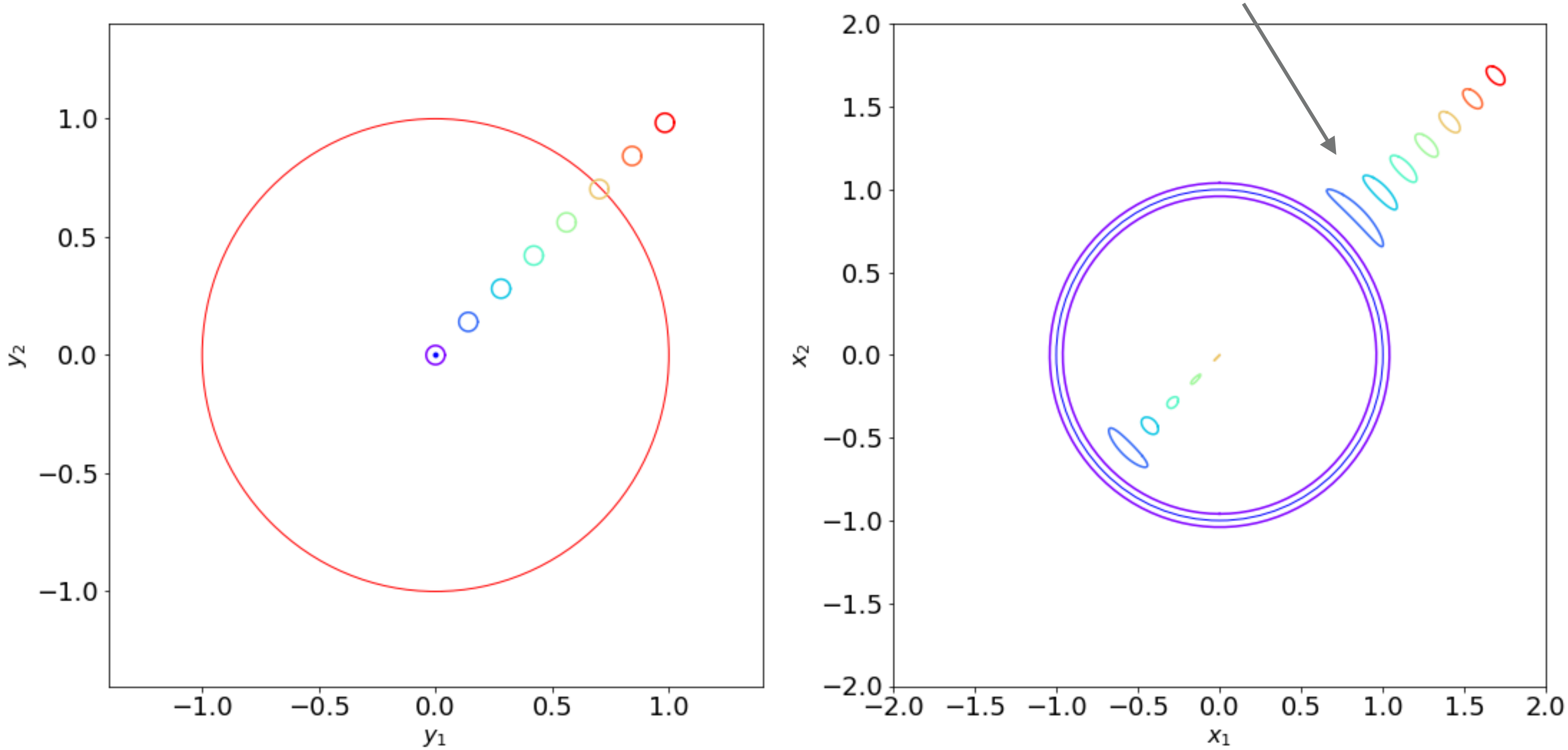
*Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.*

# THE SINGULAR ISOTHERMAL SPHERE



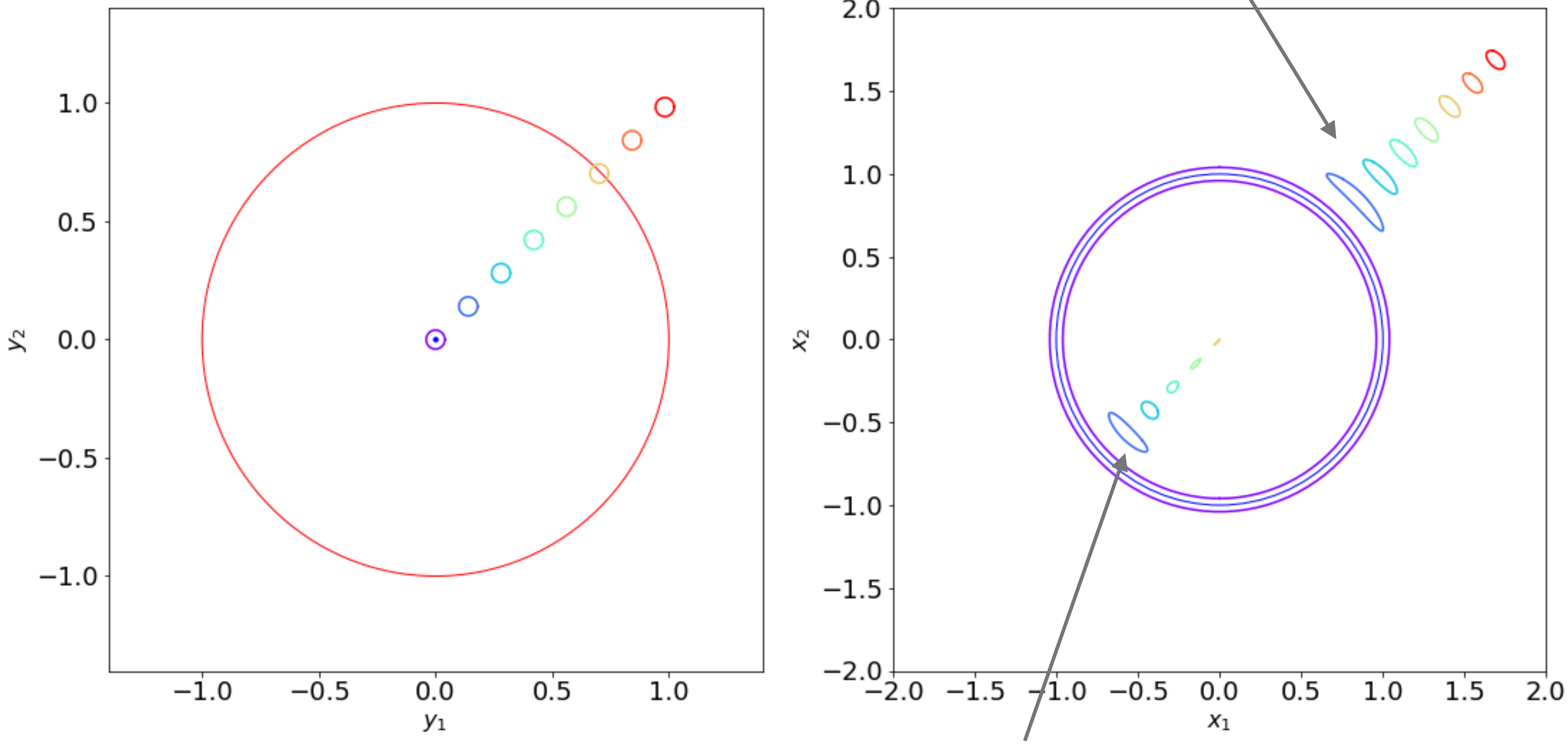
# THE SINGULAR ISOTHERMAL SPHERE

*Minimum of TD surface*



# THE SINGULAR ISOTHERMAL SPHERE

*Minimum of TD surface*




*Saddle point of TD surface*

# THE SINGULAR ISOTHERMAL SPHERE

*The shear and the convergence profiles are identical!*

$$\gamma(x) = \bar{\kappa}(x) - \kappa(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = \frac{1}{x} - \frac{1}{2x} = \frac{1}{2x} = \kappa(x)$$

*Note that both $\kappa(x)$ and $\gamma(x)$ are defined to be positive! If $x$ can assume negative values, we may write*

$$\kappa(x) = \gamma(x) = \frac{1}{2|x|}$$

$$\gamma_1 = \frac{1}{2|x|}\cos 2\phi$$

$$\gamma_2 = \frac{1}{2|x|}\sin 2\phi$$

# THE SINGULAR ISOTHERMAL SPHERE

*We talked about the radial magnification already… what about the tangential one?*

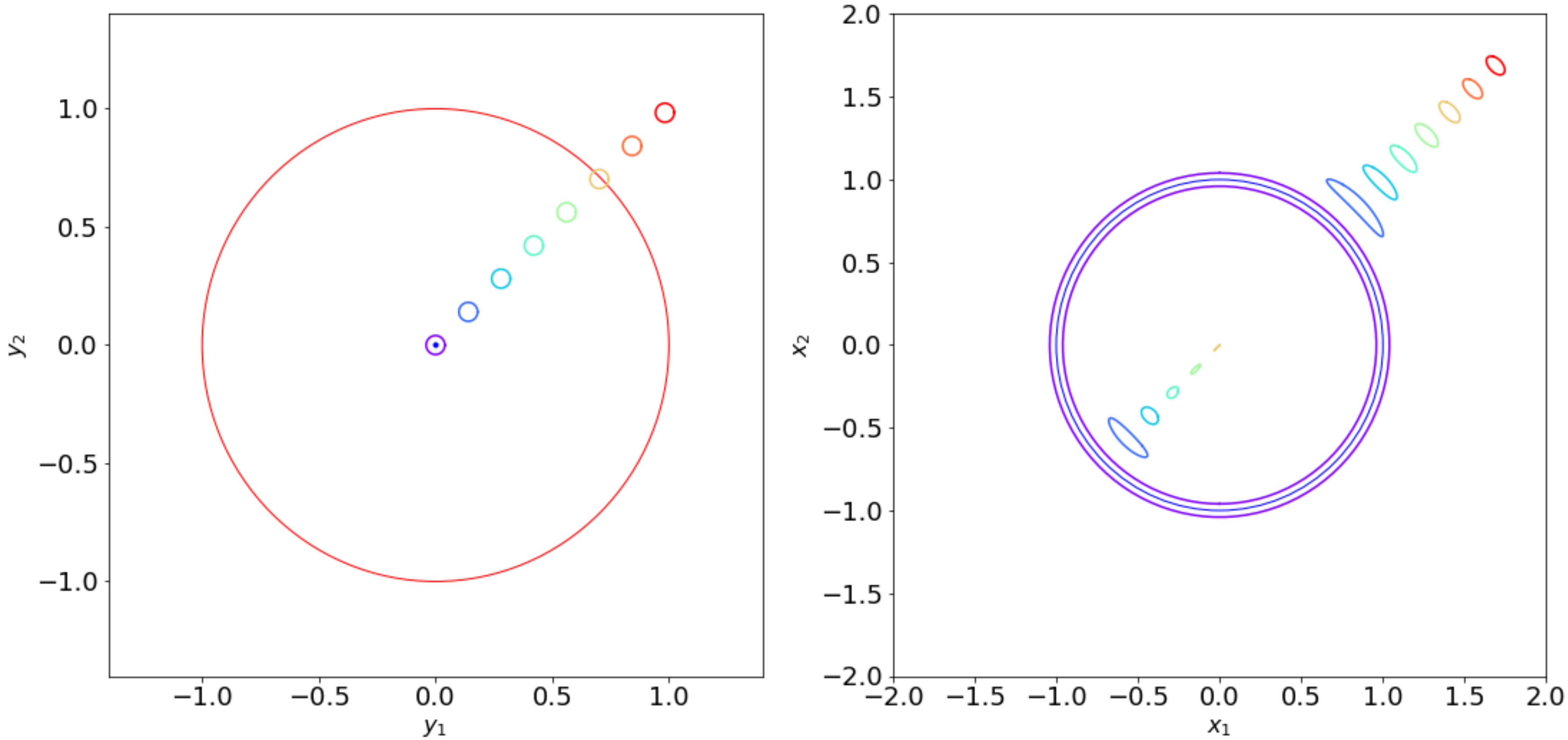
$$\mu(x) = [1 - \kappa(x) - \gamma(x)]^{-1} = \left[1 - \frac{1}{|x|}\right]^{-1} = \frac{|x|}{|x| - 1}$$

*Which allows to calculate the magnification of the two images as*

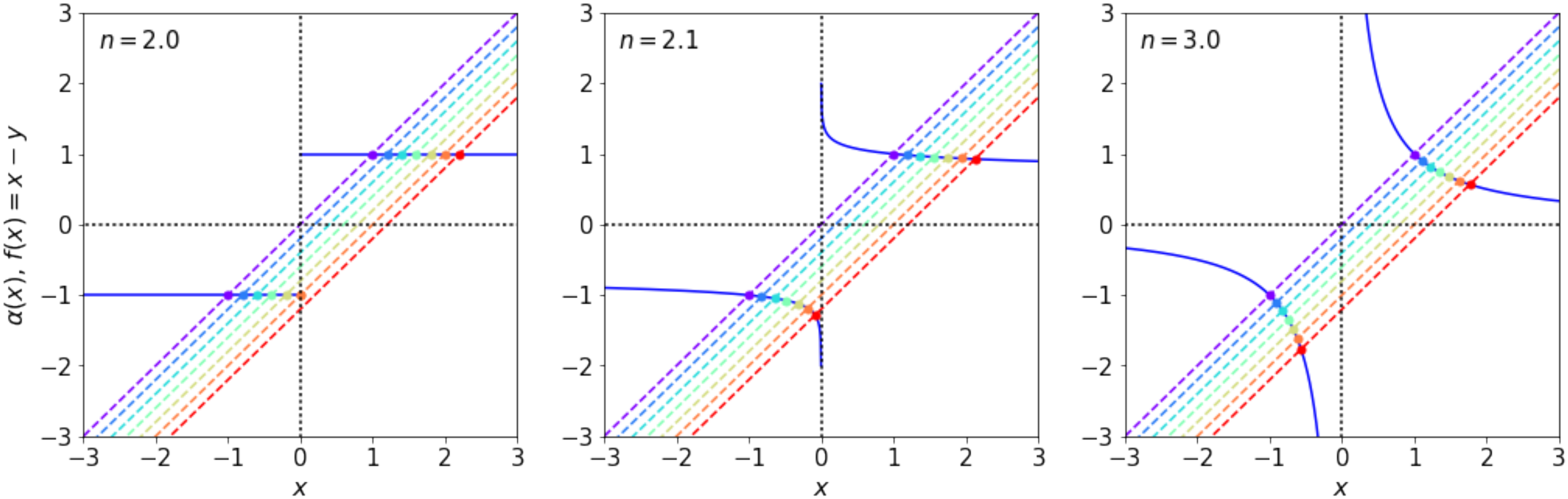
$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y} \quad \textit{and} \quad \mu_- = \frac{|y-1|}{|y-1|-1} = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

*Thus, the largest the projected distance of lens and source, the closest to unity is the magnification of the image $x_+$. On the contrary, the image at $x_-$ has zero magnification when the source is located on the cut.*

# THE SINGULAR ISOTHERMAL SPHERE



# IMAGE DIAGRAM (N>=2)




*PL lenses with n>2 always have 2 images, because the time delay surface is not continuously deformable.*

*In addition, the images are radially de-magnified!*

# SOFTENED PROFILES: THE NON-SINGULAR ISOTHERMAL SPHERE

*The profiles considered so far have surface density profiles with a singularity at $x=0$. We consider another class of lenses which have a flat core.*

*Given the simplicity of the model, we investigate the effects of the core by modifying the SIS lens:*

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\frac{1}{\sqrt{\xi^2+\xi_c^2}} = \frac{\Sigma_0}{\sqrt{1+\xi^2/\xi_c^2}}$$

$$\Sigma_0 = \frac{\sigma_v^2}{2G\xi_c}$$

# NON SINGULAR ISOTHERMAL SPHERE (NIS)

*Choosing* $$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{\xi^2 + \xi_c^2}} = \frac{\Sigma_0}{\sqrt{1 + \xi^2/\xi_c^2}}$$

$$\kappa(x) = \frac{1}{2\sqrt{x^2 + x_c^2}}$$

# NON SINGULAR ISOTHERMAL SPHERE (NIS)

*The mass profile is computed as follows*

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2 + x_c^2} - x_c$$

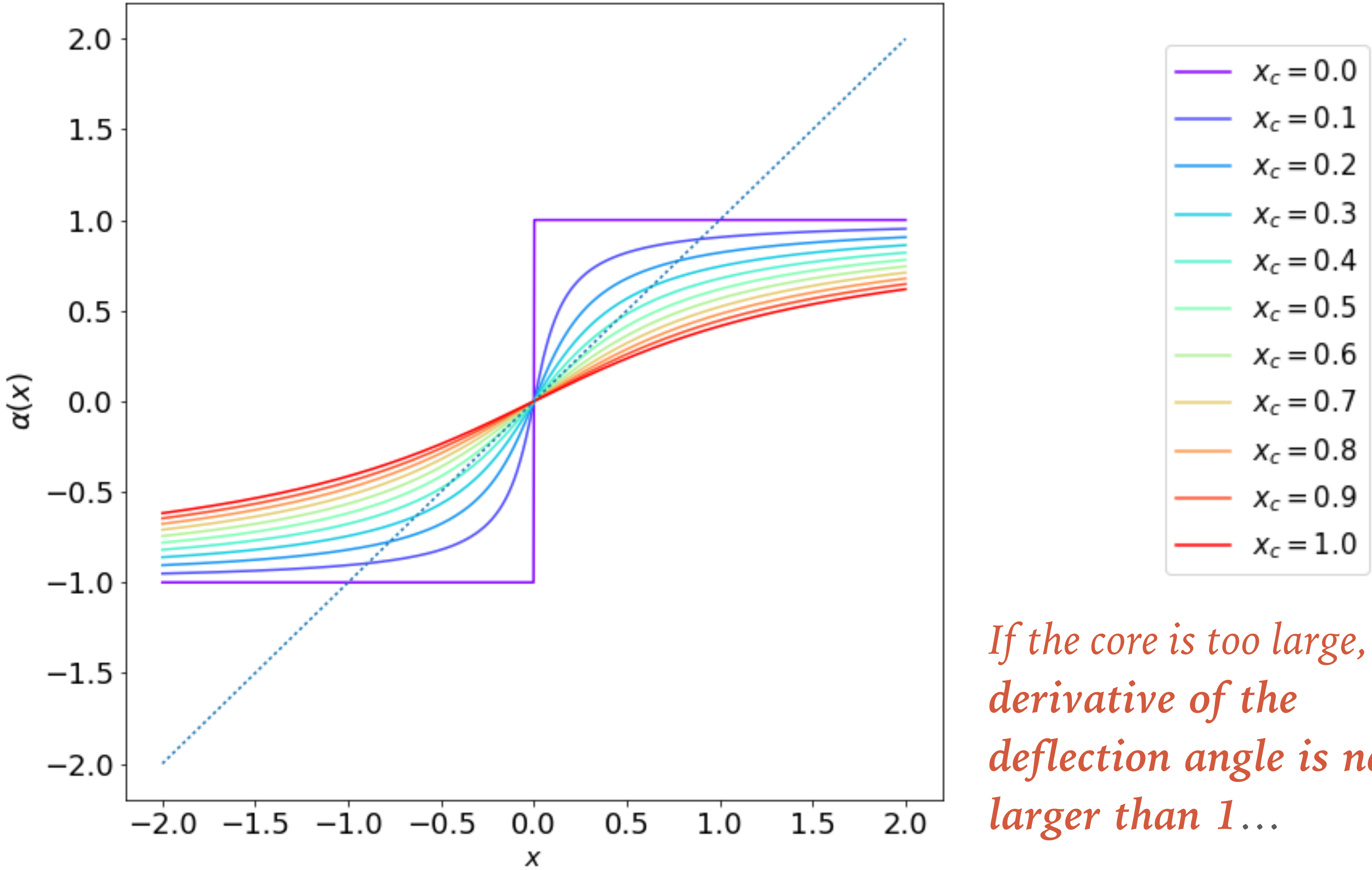
*The deflection angle is*

$$\alpha(x) = \frac{m(x)}{x} = \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

*The shear is*

$$\gamma(x) = \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2} - \frac{1}{2\sqrt{x^2 + x_c^2}}$$

# NON SINGULAR ISOTHERMAL SPHERE (NIS)




*If the core is too large,* ***the derivative of the deflection angle is never larger than 1…***

***No radial critical line!***

# NON SINGULAR ISOTHERMAL SPHERE (NIS)

*The radius of the radial critical line can be found by solving the equation:*

$$\left(1-\frac{d\alpha(x)}{dx}\right)=1+\frac{m(x)}{x^2}-2\kappa(x)=0$$

$$1+\frac{\sqrt{x^2+x_c^2}-x_c}{x^2}-\frac{1}{\sqrt{x^2+x_c^2}}=0$$

$$x_r^2=\frac{1}{2}\left(2x_c-x_c^2-x_c\sqrt{x_c^2+4x_c}\right)$$

$$x_r^2\geq 0 \text{ for } x_c\leq 1/2.$$

# NON SINGULAR ISOTHERMAL SPHERE (NIS)

*We can search for the tangential critical line:*

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2+x_c^2} - x_c \qquad\qquad m(x)/x^2 = 1$$

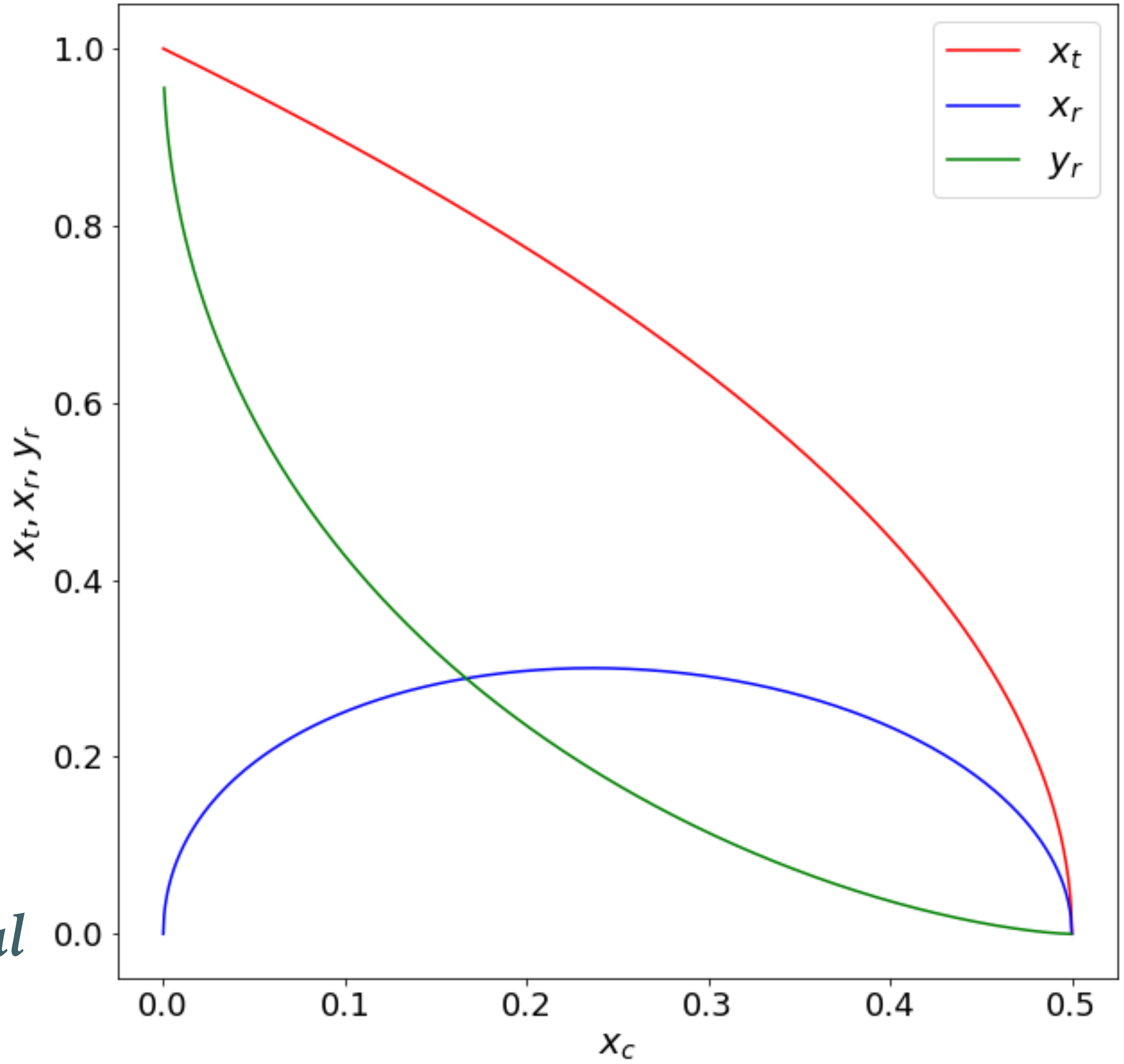
$$\sqrt{x^2+x_c^2} - x_c = x^2$$

$$x^2(x^2+2x_c-1) = 0$$

$$x_t = \sqrt{1-2x_c}$$

*The tangential critical line also exists only if $x_c$<1/2!*

# NON SINGULAR ISOTHERMAL SPHERE (NIS)



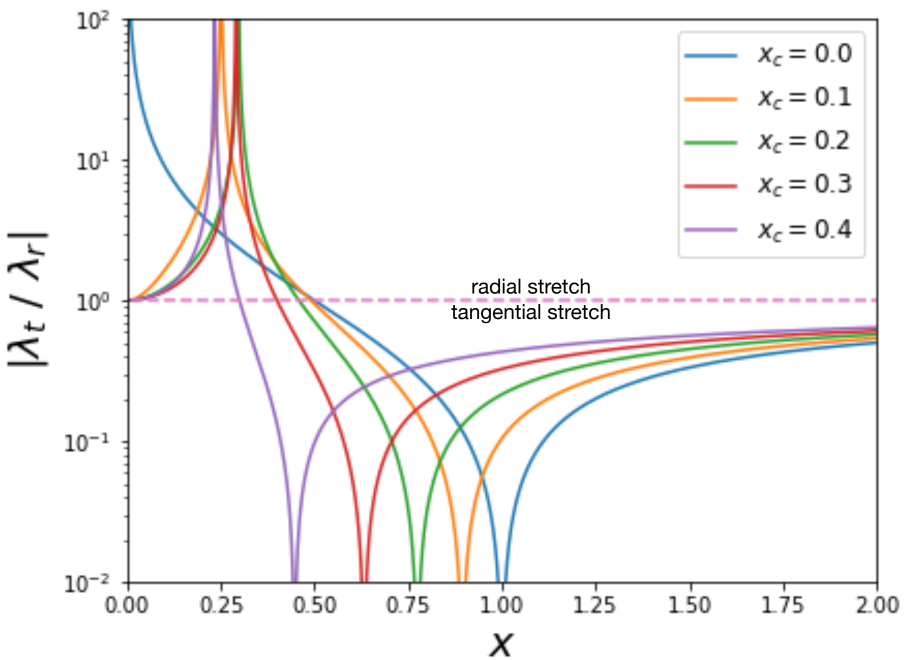

*Small core, small radial critical line, large tangential critical line*

# NON SINGULAR ISOTHERMAL SPHERE (NIS)

*Distortion of infinitesimal images*

*radial eigenvalue of A* $$\lambda_r = 1 - \alpha'(x) = 1 + \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2} - \frac{1}{\sqrt{x^2 + x_c^2}}$$

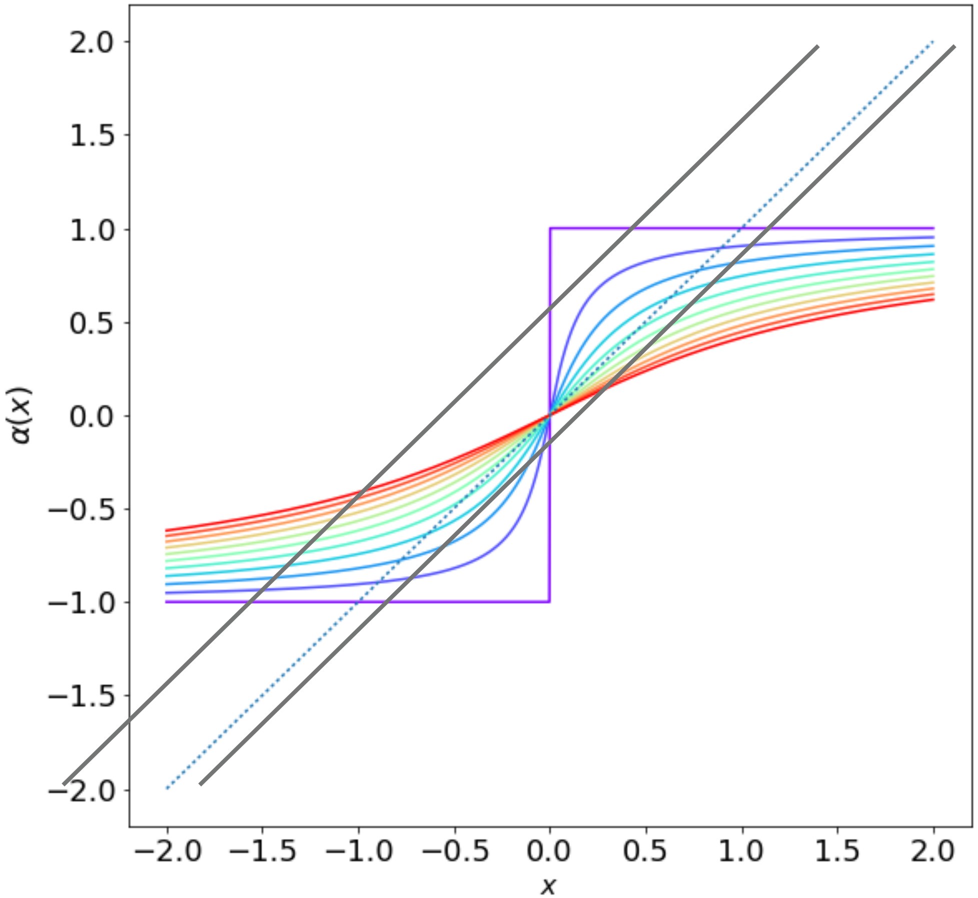
*tangential eigenvalue of A* $$\lambda_t = 1 - \frac{\alpha(x)}{x} = 1 - \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2}$$




$$\left|\frac{\lambda_t}{\lambda_r}\right| < 1$$ *image is tangentially stretched*

$$\left|\frac{\lambda_t}{\lambda_r}\right| > 1$$ *image is radially stretched*

# NON SINGULAR ISOTHERMAL SPHERE (NIS)




*As you can see, this has implications also for the existence of multiple images…*

# NON SINGULAR ISOTHERMAL SPHERE (NIS)

*The lens equation can be reduced to the form:*

$$y = x - \frac{m(x)}{x} = x - \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

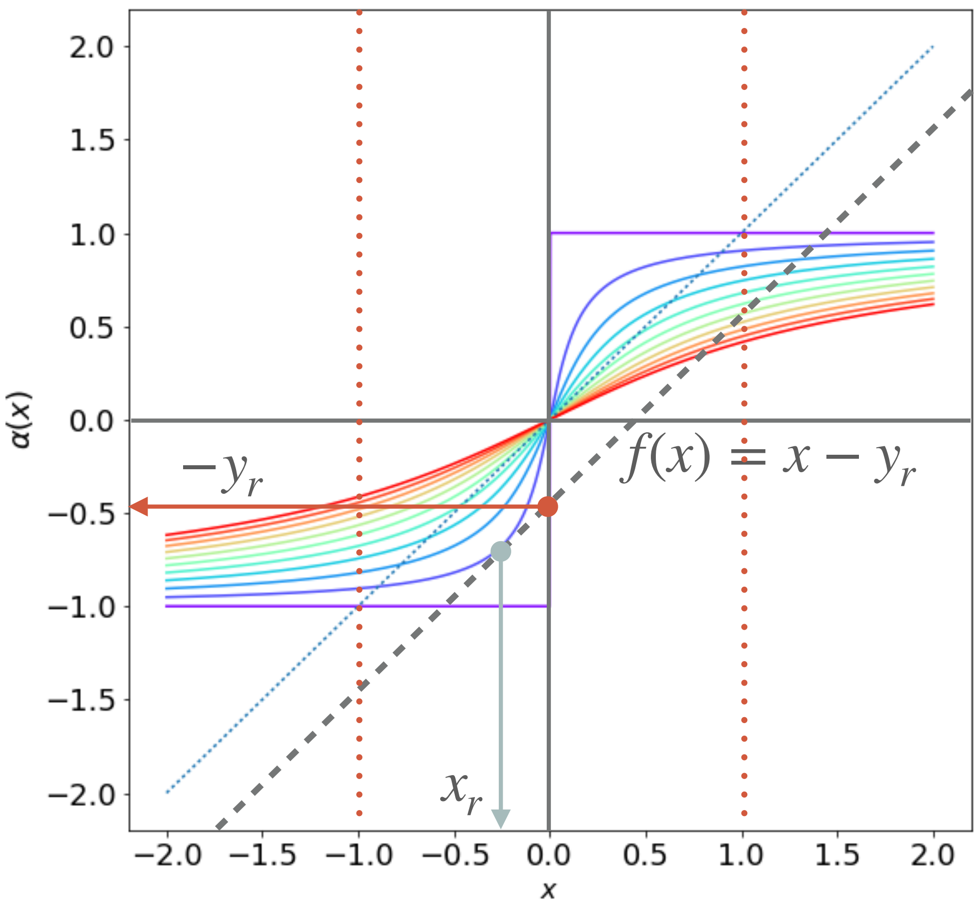
$$x^3 - 2yx^2 + (y^2 + 2x_c - 1)x - 2yx_c = 0\,.$$

*There are up to three solutions, but, again the existence of multiple images depends on y and $x_c$…*

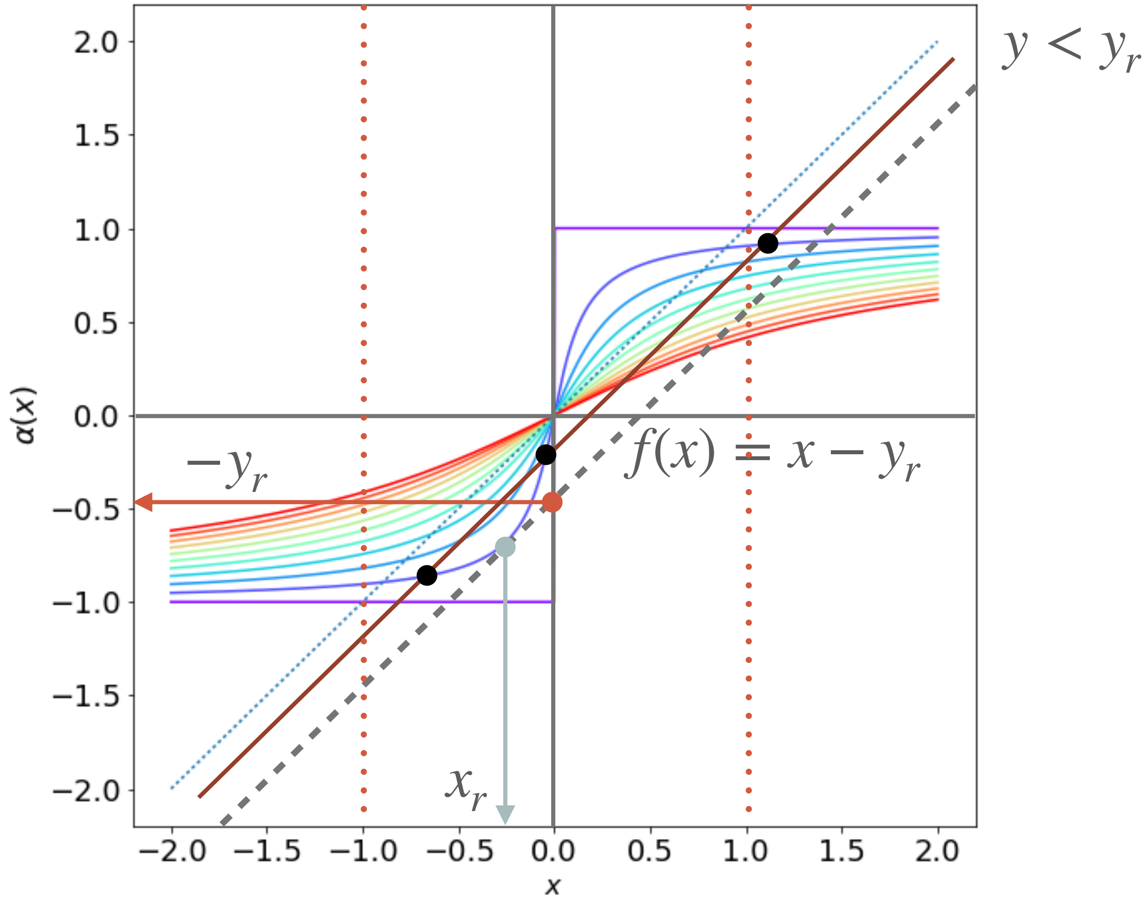
*In particular on whether:*

- *the radial caustic exists*
- *the source is inside or outside the radial caustic*

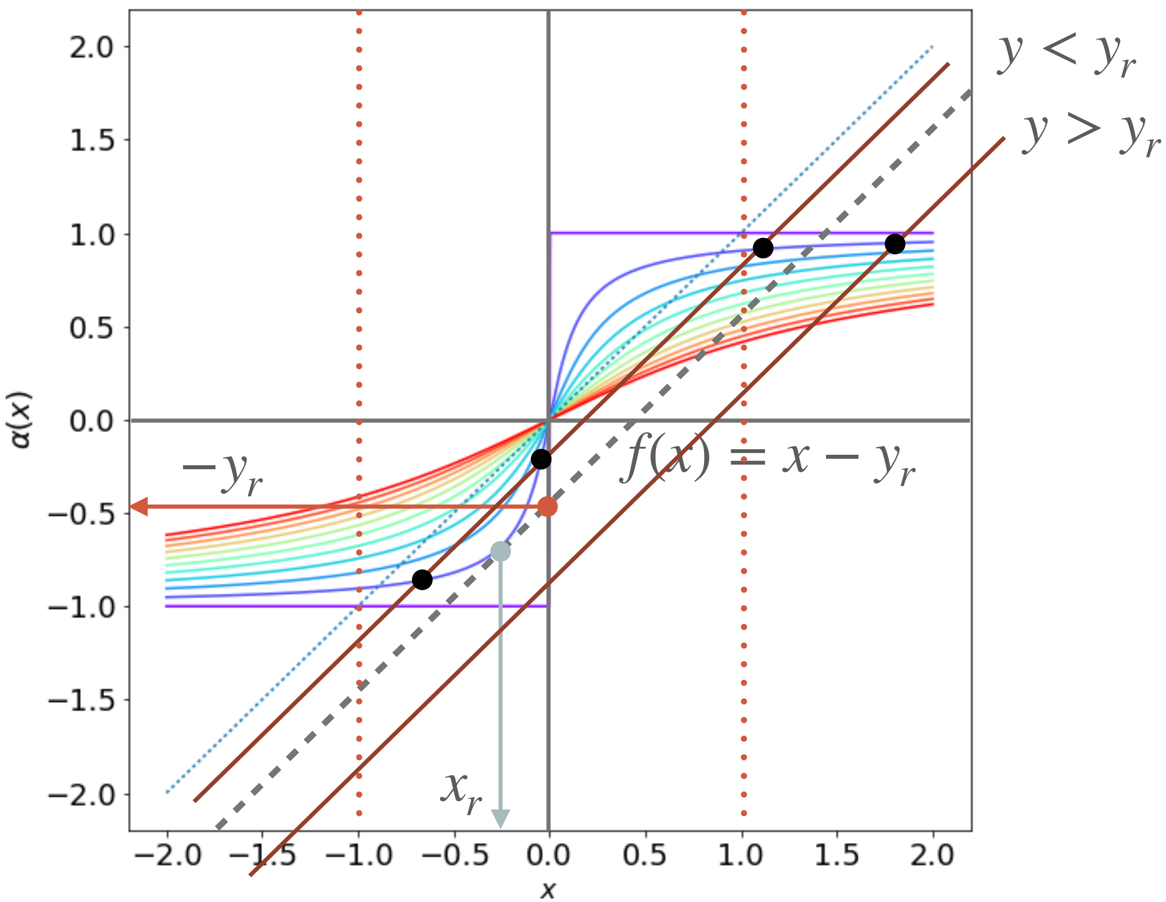
# NON SINGULAR ISOTHERMAL SPHERE (NIS)



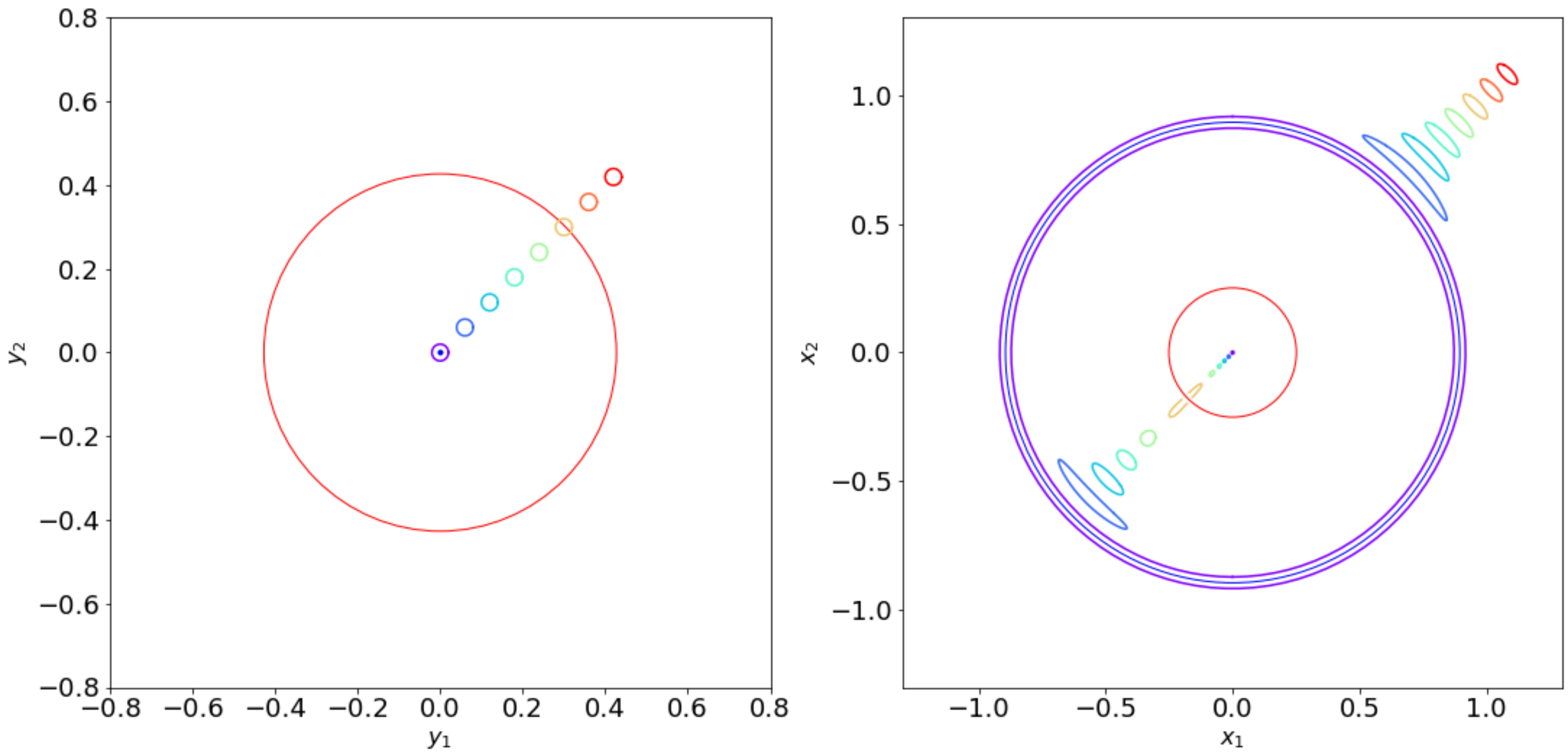
# NON SINGULAR ISOTHERMAL SPHERE (NIS)



# NON SINGULAR ISOTHERMAL SPHERE (NIS)



# NON SINGULAR ISOTHERMAL SPHERE (NIS)




*Three images if the source is inside the radial caustic; One image otherwise.*

*Parity: changes at each critical line (remember: maxima, minima, saddle points of TDS)*

# OTHER INTERESTING PROFILES: NFW DENSITY PROFILE

*Numerical simulations in the framework of the Cold-Dark-Matter model show that halos develop a sort of "universal" density profile of the kind*

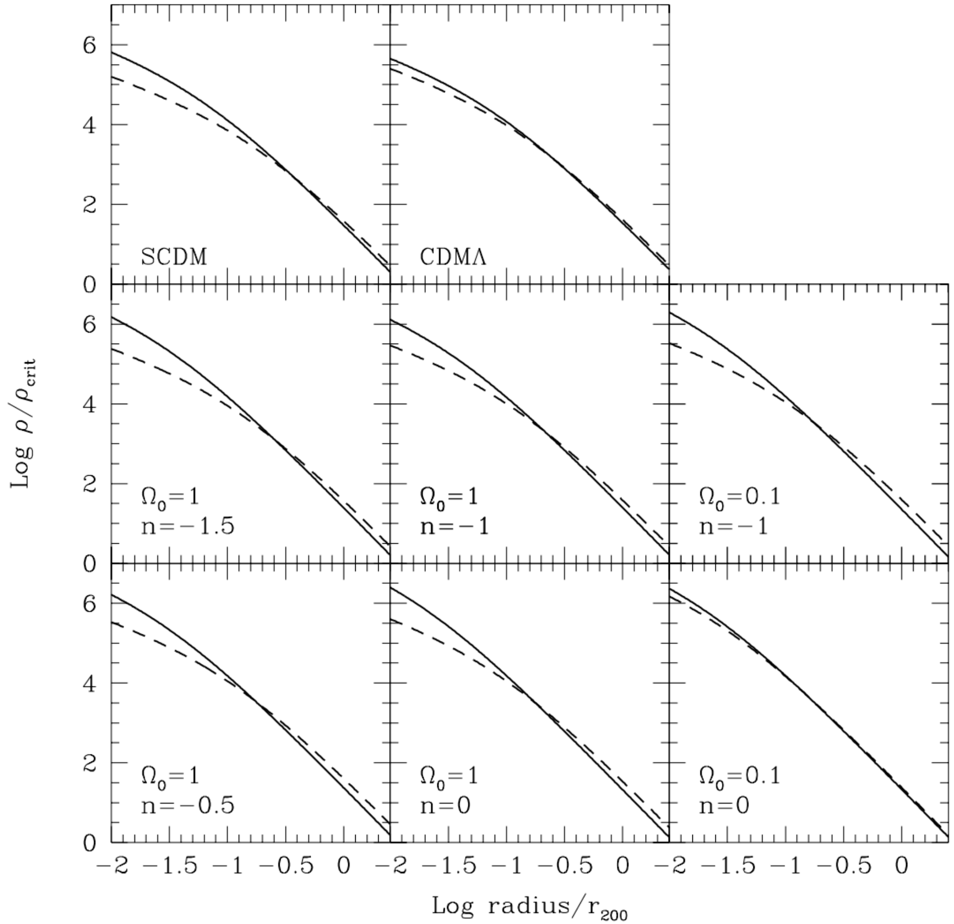
$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2}$$

*The profile depends on two parameters ($\rho_s, r_s$). They can be expressed in terms of the mass and of the concentration of the halo:*

$$r_{200} = 1.63 \times 10^{-2} \left(\frac{M_{200}}{h^{-1}M_\odot}\right)^{1/3} \left[\frac{\Omega_0}{\Omega(z)}\right]^{-1/3} (1+z)^{-1}h^{-1}$$

$$\rho_s = \frac{200}{3}\rho_{cr}\frac{c^3}{[\ln(1+c) - c/(1+c)]} \qquad c = \frac{r_{200}}{r_s}$$

*Navarro, Frenk & White, 1997*



# LENSING PROPERTIES OF THE NFW PROFILE

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2-1} f(x) \,, \qquad x = \xi / r_s$$

with

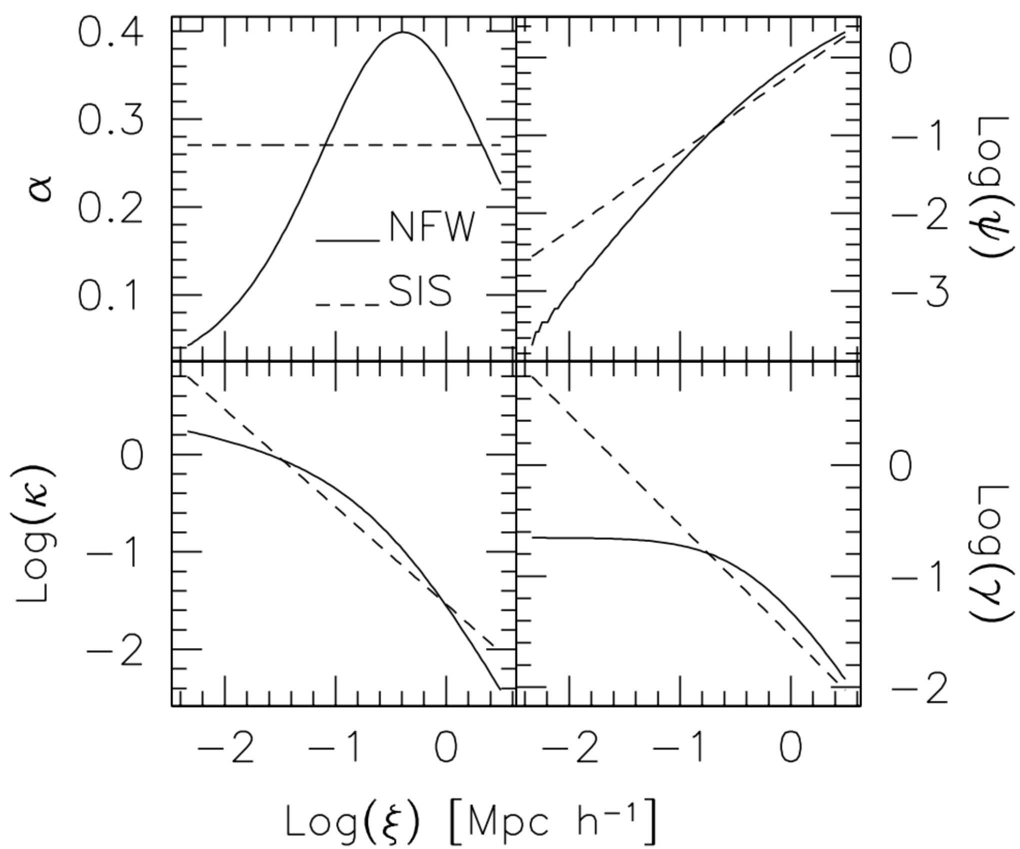
$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \arctan\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

$$\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2-1} \qquad \kappa_s \equiv \rho_s r_s \Sigma_{\rm cr}^{-1}$$

$$m(x) = 2\int_0^x \kappa(x')x'dx' = 4k_s h(x)$$

$$h(x) = \ln\frac{x}{2} + \begin{cases} \frac{2}{\sqrt{x^2-1}} \arctan\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 1 & (x = 1) \end{cases}$$

$$\alpha(x) = \frac{4\kappa_s}{x} h(x)$$



# LENSING PROPERTIES OF THE NFW PROFILE

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2-1} f(x) \,, \qquad x = \xi / r_s$$

with

$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \operatorname{arctan} \sqrt{\frac{x-1}{x+1}} & (x > 1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

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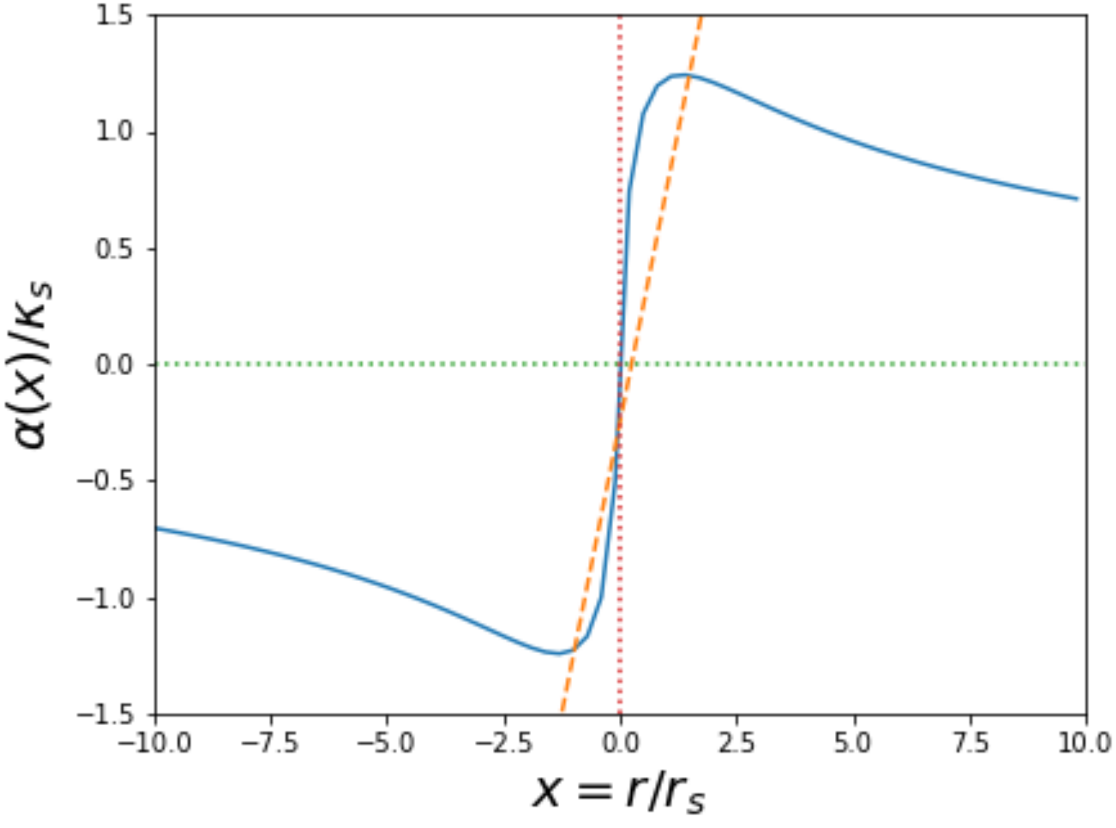
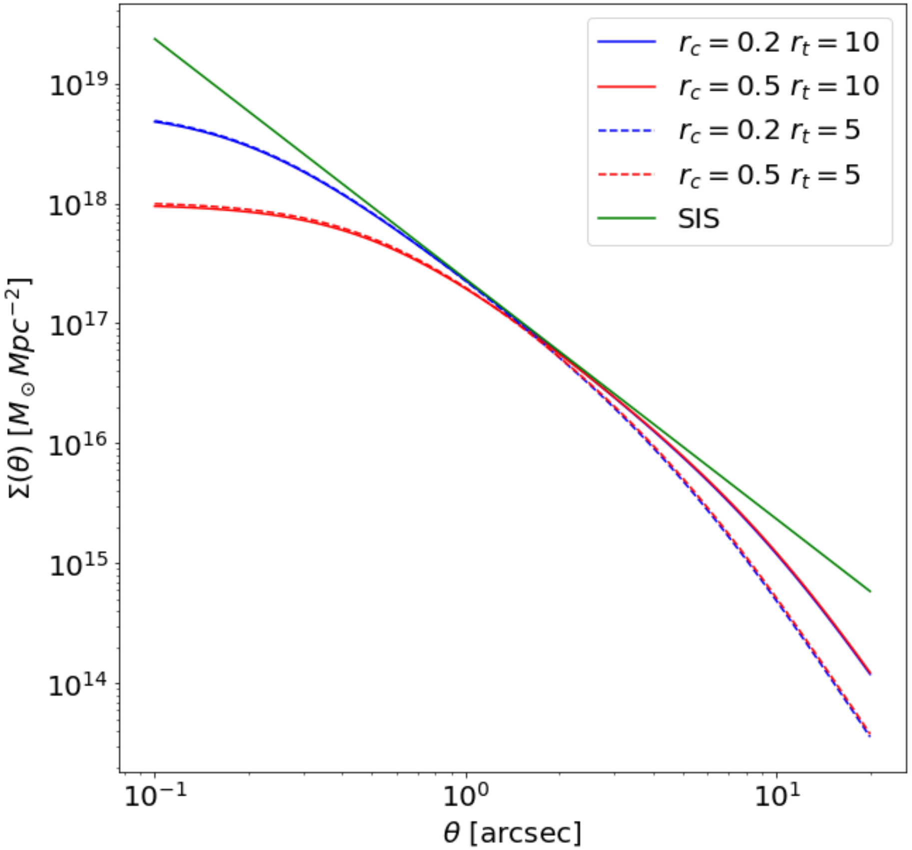
$$\alpha(x) = \frac{4\kappa_s}{x} h(x)$$

*One or three images*



# PIEMD (OR DPIE) PROFILE




*The PIEMD model is a particular mass profile which includes both a core and a truncation radius*

$$\rho(r) = \frac{\rho_0}{\left(1 + \frac{r^2}{r_c^2}\right)\left(1 + \frac{r^2}{r_t^2}\right)}$$

$$\rho_0 = \frac{\sigma_0^2}{2\pi G}\frac{r_c + r_t}{r_c^2 r_t} \qquad r_t > r_c$$

*The surface density is*

$$\Sigma(\xi) = \frac{\sigma_0^2}{2G}\frac{r_t}{r_t - r_c}\left(\frac{1}{\sqrt{\xi^2 + r_c^2}} - \frac{1}{\sqrt{\xi^2 + r_t^2}}\right)$$

*Thus, it is equivalent to the difference of two NIS models (see last lesson).*