GRAVITATIONAL LENSING

11 - MICROLENSING IV : MULTIPLY IMAGED QUASARS

R. Benton Metcalf

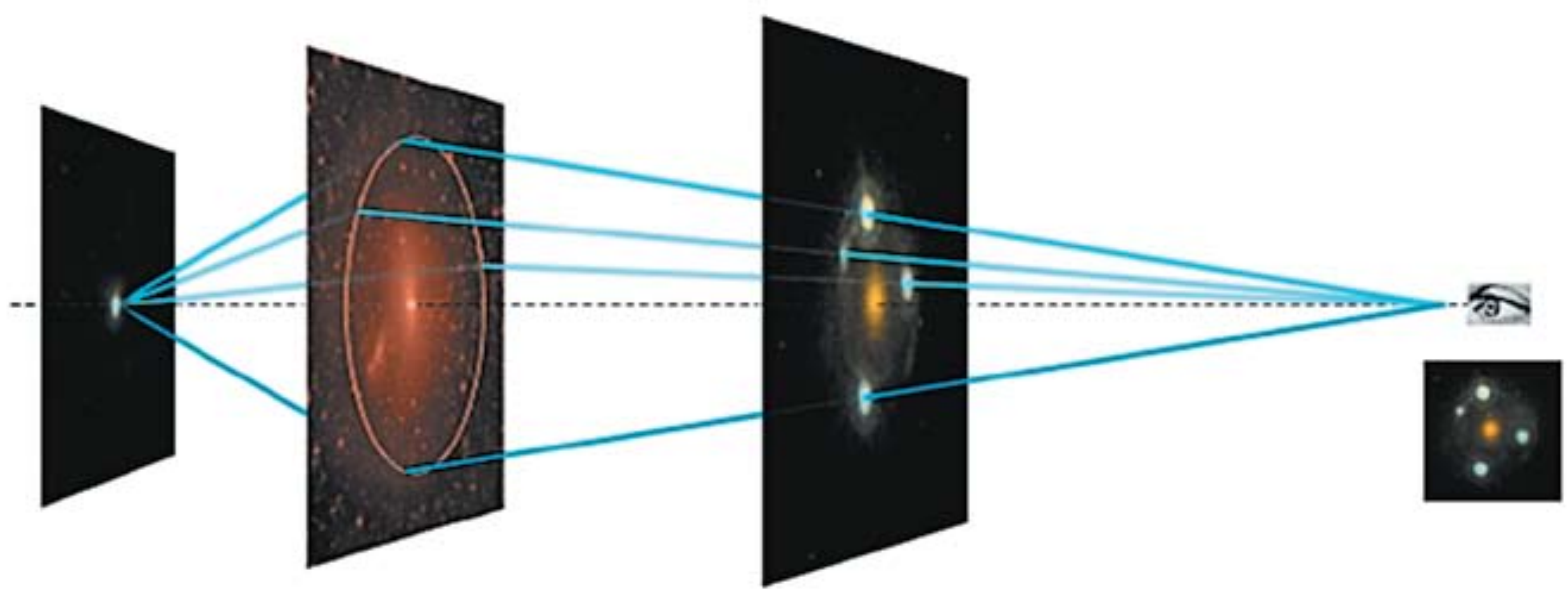
2022-2023

MICROLENSING OF MULTIPLY IMAGED QSOs

Distant quasars can be gravitationally lensed into multiple (two or four) resolved images.

The light path for each of these images may be microlensed by stars in the lens galaxy.

This can be detected by independent variations in the light curve of the images once the time-delay between them have been subtracted.



COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o}\int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

critical density

$$\Omega_m = \frac{\rho}{\rho_{\rm crit}} \qquad \rho_{\rm crit} = \frac{3H_o^2}{8\pi G}$$

H_o Hubble parameter

comoving angular size distance

$$D_{CA}(z) = \begin{cases} R_{\rm curv}\sin\left(\frac{\chi}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda < 1 \\ \chi & \Omega_m + \Omega_\Lambda = 1 \\ R_{\rm curv}\sinh\left(\frac{\chi}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o} \int_0^{z_s} \frac{dz}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)(1+z)^2}}$$

critical density

$$\Omega_m = \frac{\rho}{\rho_{\rm crit}} \qquad \rho_{\rm crit} = \frac{3H_o^2}{8\pi G}$$

H_o Hubble parameter

comoving angular size distance

$$D_{CA}(z_1, z_2) = \begin{cases} R_{\rm curv} \sin\left(\frac{\chi_1 - \chi_2}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda < 1 \\ \chi_1 - \chi_2 & \Omega_m + \Omega_\Lambda = 1 \\ R_{\rm curv} \sinh\left(\frac{\chi_1 - \chi_2}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o \sqrt{1 - \Omega_m - \Omega_\Lambda}}$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o}\int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

comoving angular size distance

$$D_{CA}(z_1, z_2) = \begin{cases} R_{\mathrm{curv}}\sin\left(\frac{\chi_1-\chi_2}{R_{\mathrm{curv}}}\right) & \Omega_m+\Omega_\Lambda < 1 \\ \chi_1-\chi_2 & \Omega_m+\Omega_\Lambda = 1 \\ R_{\mathrm{curv}}\sinh\left(\frac{\chi_1-\chi_2}{R_{\mathrm{curv}}}\right) & \Omega_m+\Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\mathrm{curv}} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

(proper) angular size distance

$$D_A(z, z_o) = \frac{D_{CA}(z, z_o)}{(1+z)}$$

luminosity distance

$$D_L(z) = (1+z)D_{CA}(z)$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o}\int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

angular size distance

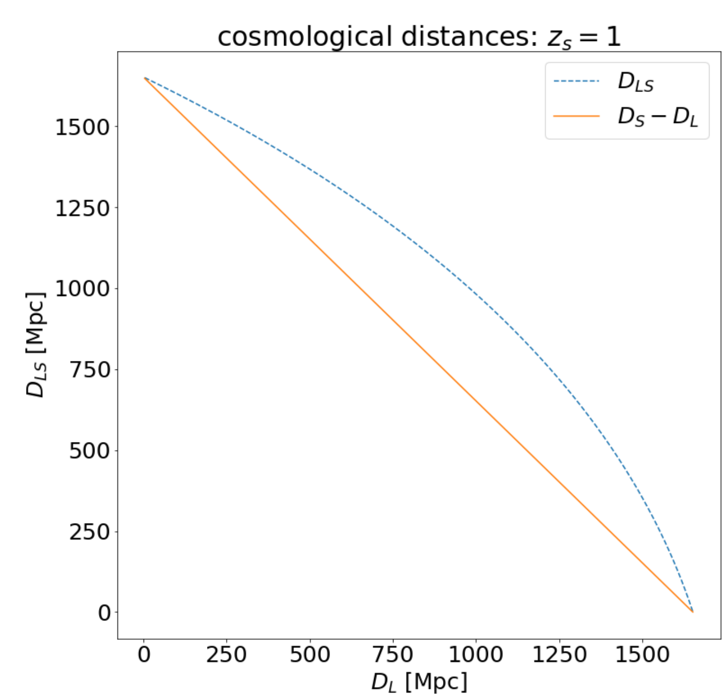
$$D_A(z_1,z_2) = \frac{1}{1+z_1}\begin{cases} R_{\rm curv}\sin\left(\frac{\chi_1-\chi_2}{R_{\rm curv}}\right) & \Omega_m+\Omega_\Lambda<1 \\ \chi_1-\chi_2 & \Omega_m+\Omega_\Lambda=1 \\ R_{\rm curv}\sinh\left(\frac{\chi_1-\chi_2}{R_{\rm curv}}\right) & \Omega_m+\Omega_\Lambda>1 \end{cases}$$

$z_1 > z_2$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

$$D_{ls} \neq D_s - D_l$$



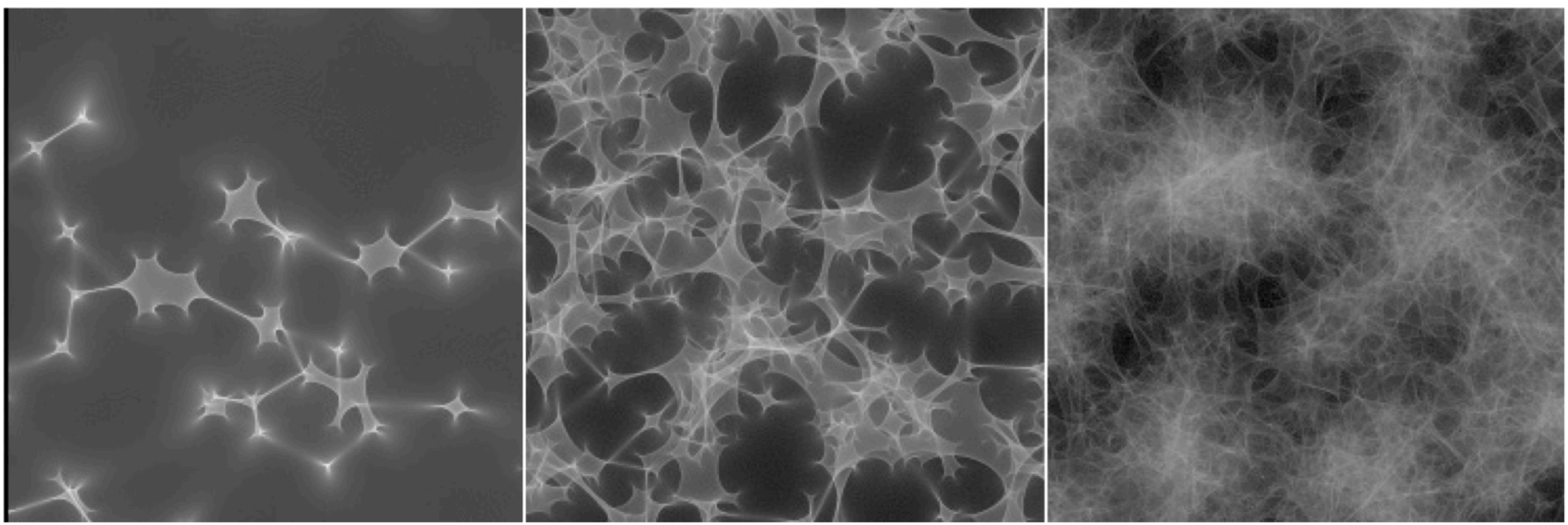
MICROLENSING OF MULTIPLY IMAGED QSOs

The optical depth in this case can be much higher than for stars in our galaxy, even one or greater, making it probably that the source is being microlensed at any given time.

$$\tau = \frac{\kappa_*}{1 - \kappa_{\rm macro}}$$

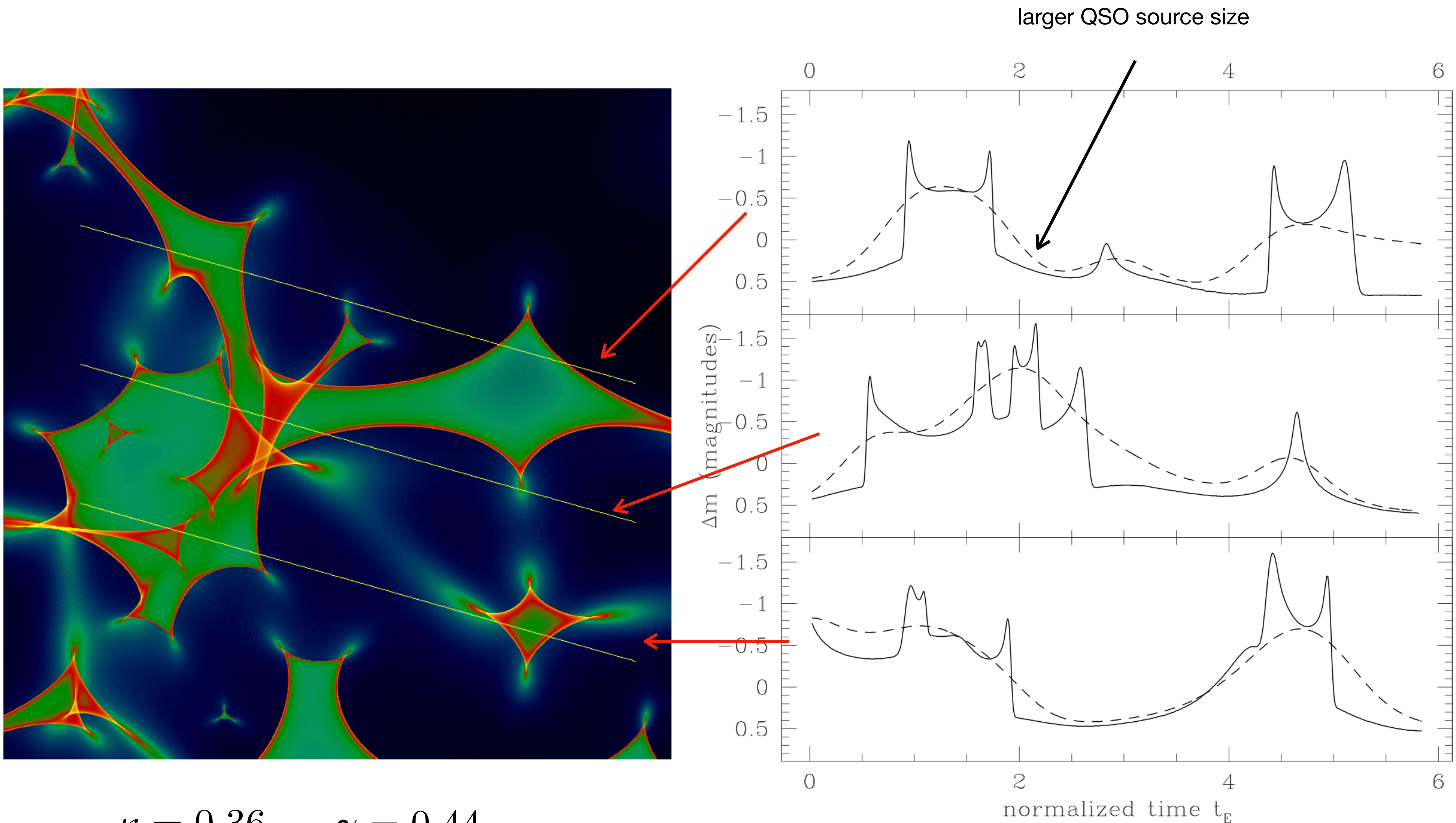
The nature of the caustics depends strongly on the density of stars.

$\kappa = 0.2$ $\kappa = 0.5$ $\kappa = 0.8$



J. Wambsganss

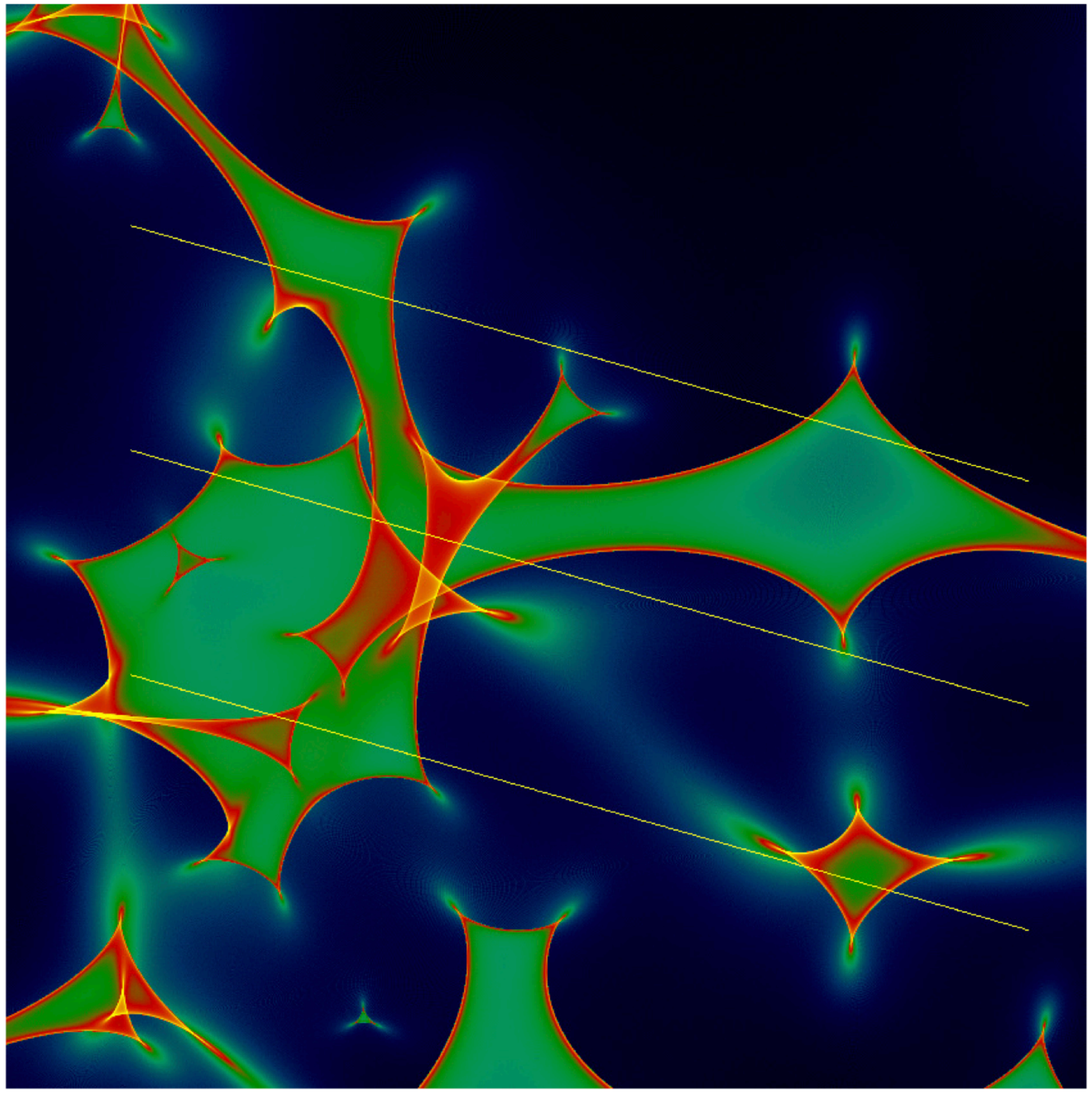
MICROLENSING OF MULTIPLY IMAGED QSOs

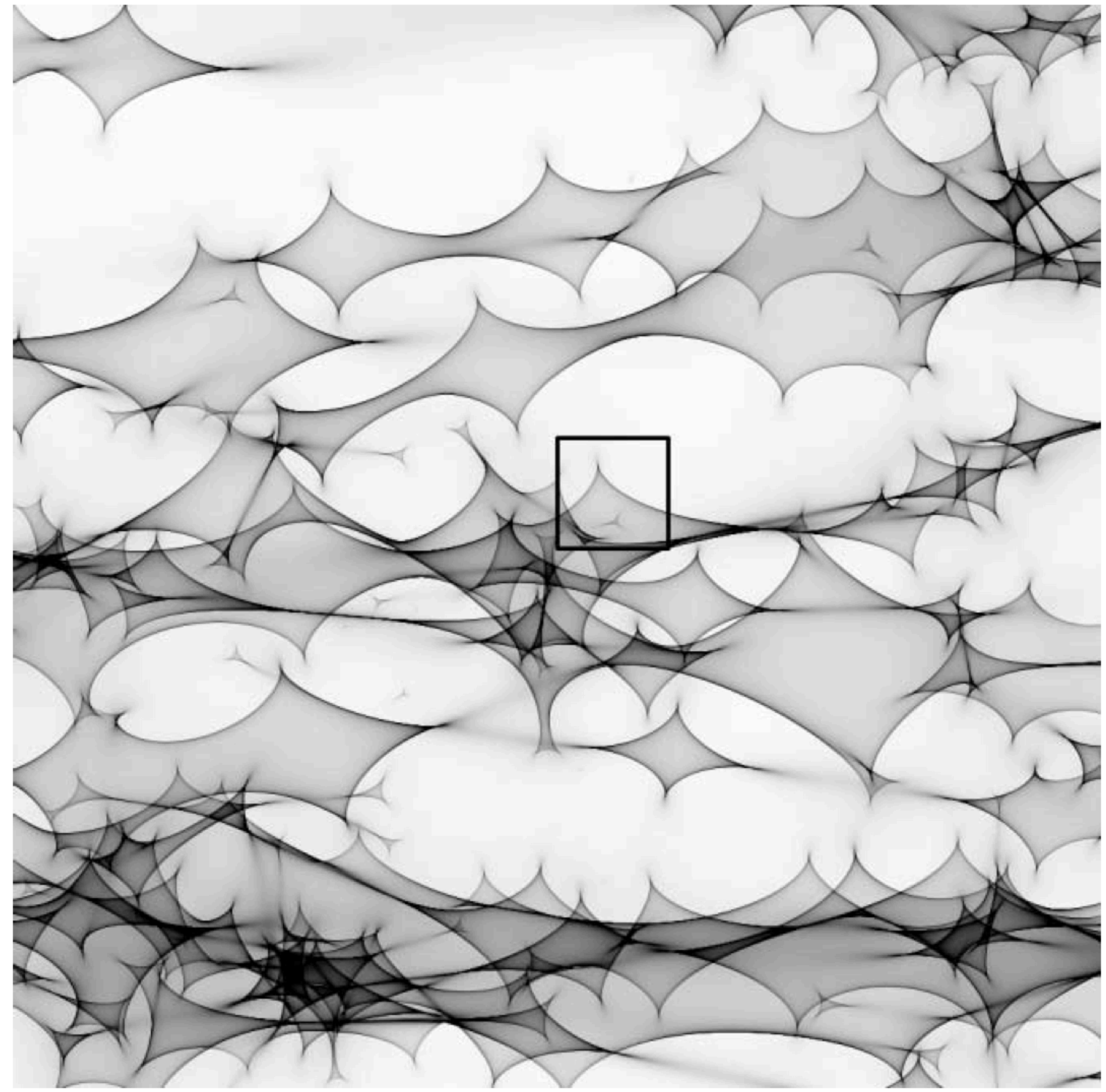


$\kappa = 0.36 \qquad \gamma = 0.44$

J. Wambsganss

MICROLENSING OF MULTIPLY IMAGED QSOs

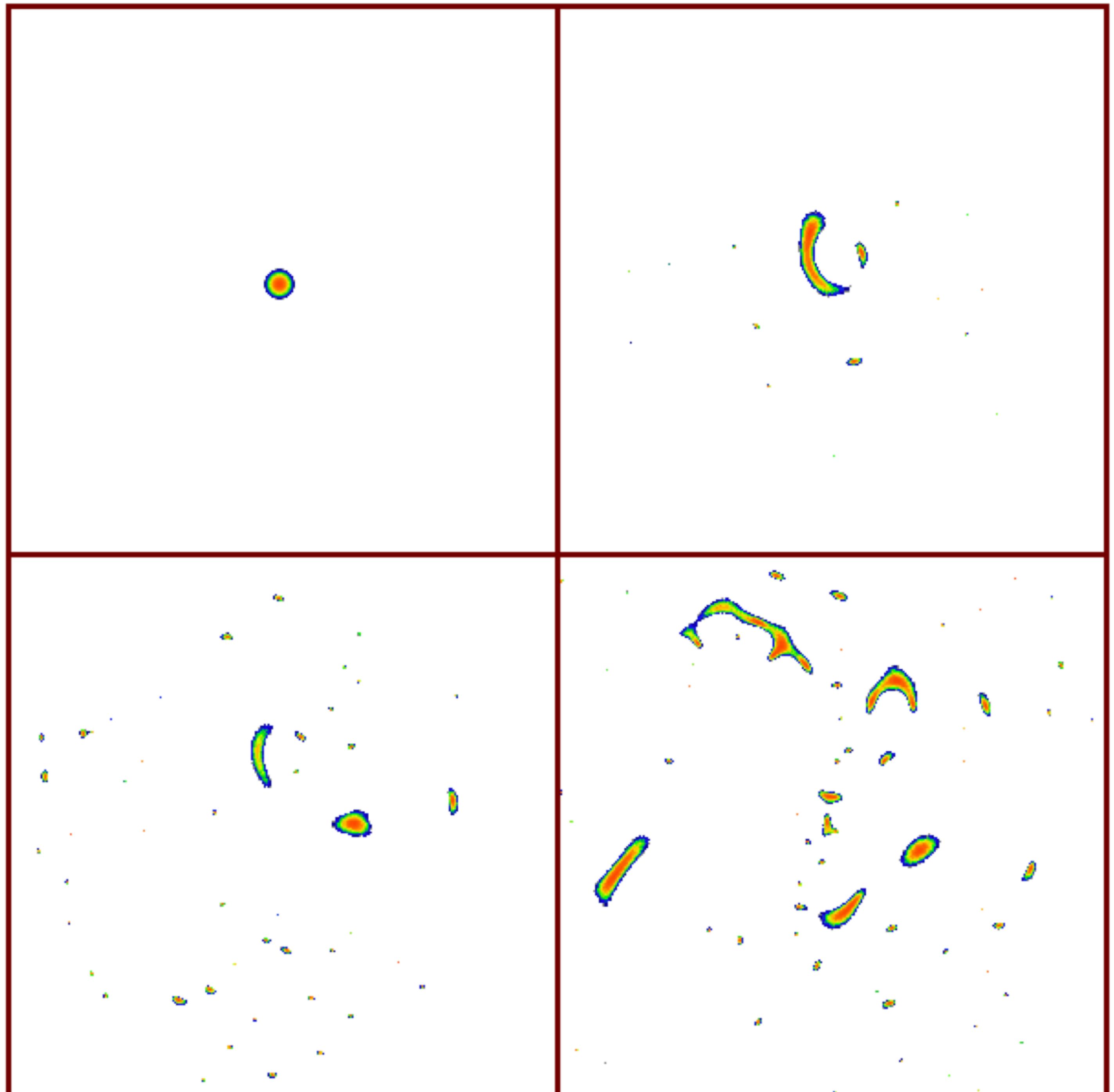




Microlensing map of QSO 2237+0305A image

MICROLENSING OF MULTIPLY IMAGED QSOs

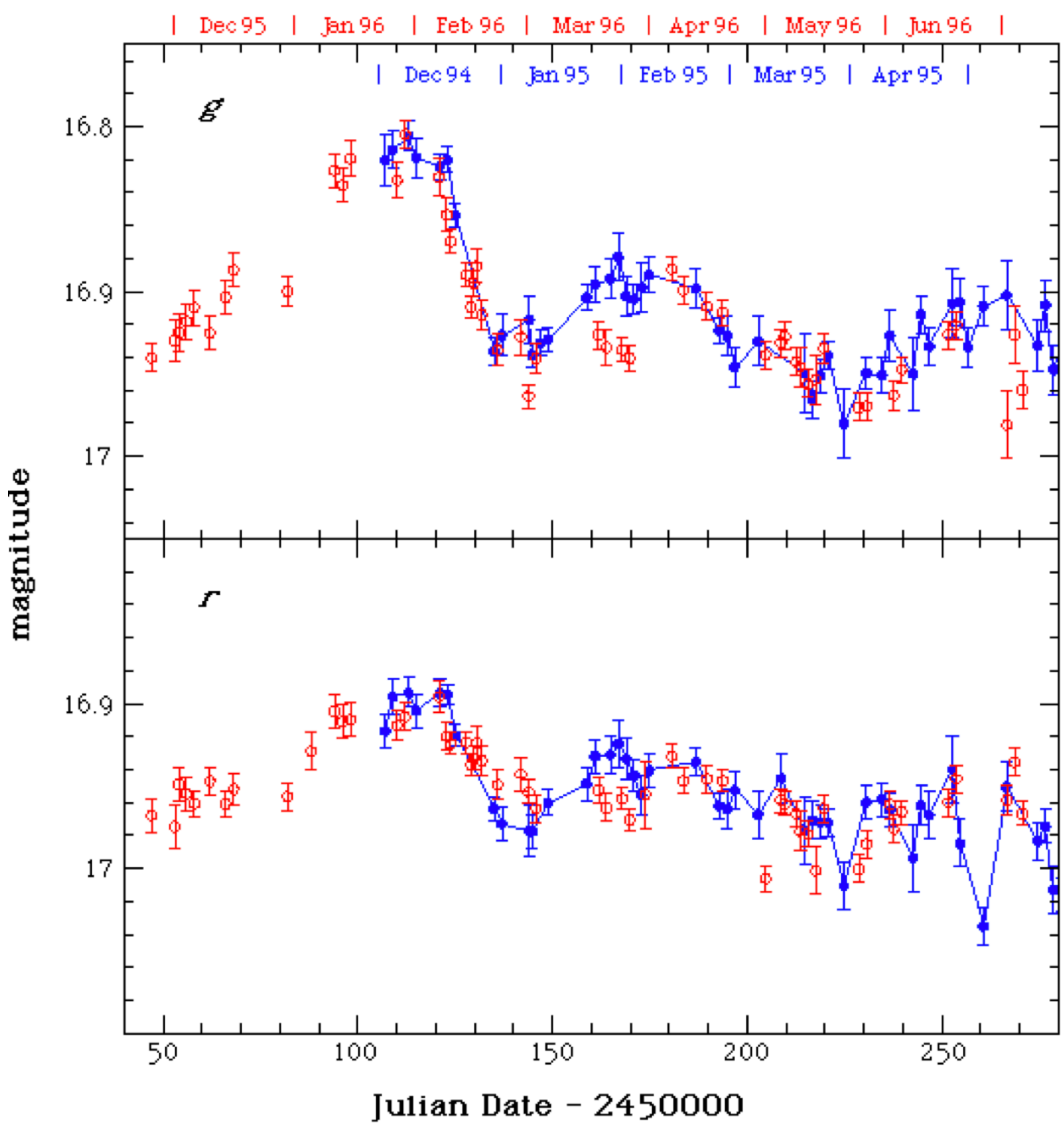
micro-images of the accession disk of the quasar

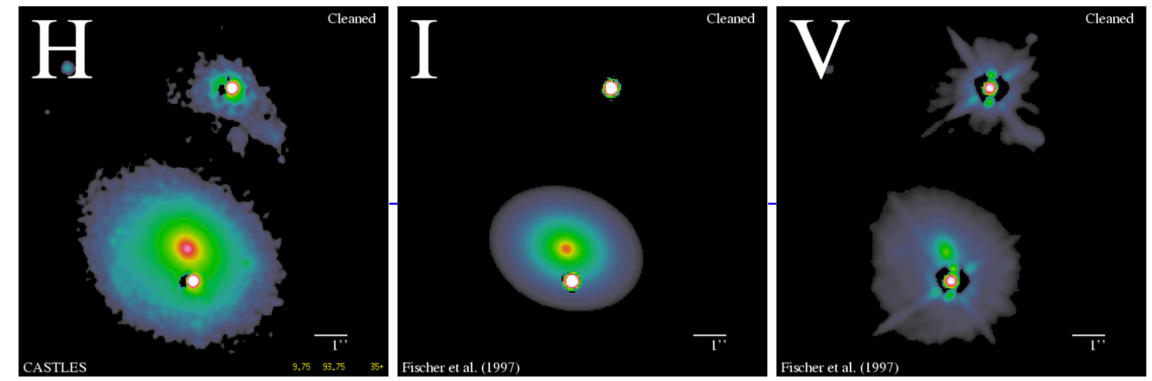


J. Wambsganss

MICROLENSING OF MULTIPLY IMAGED QSOs

Kundrić et al.





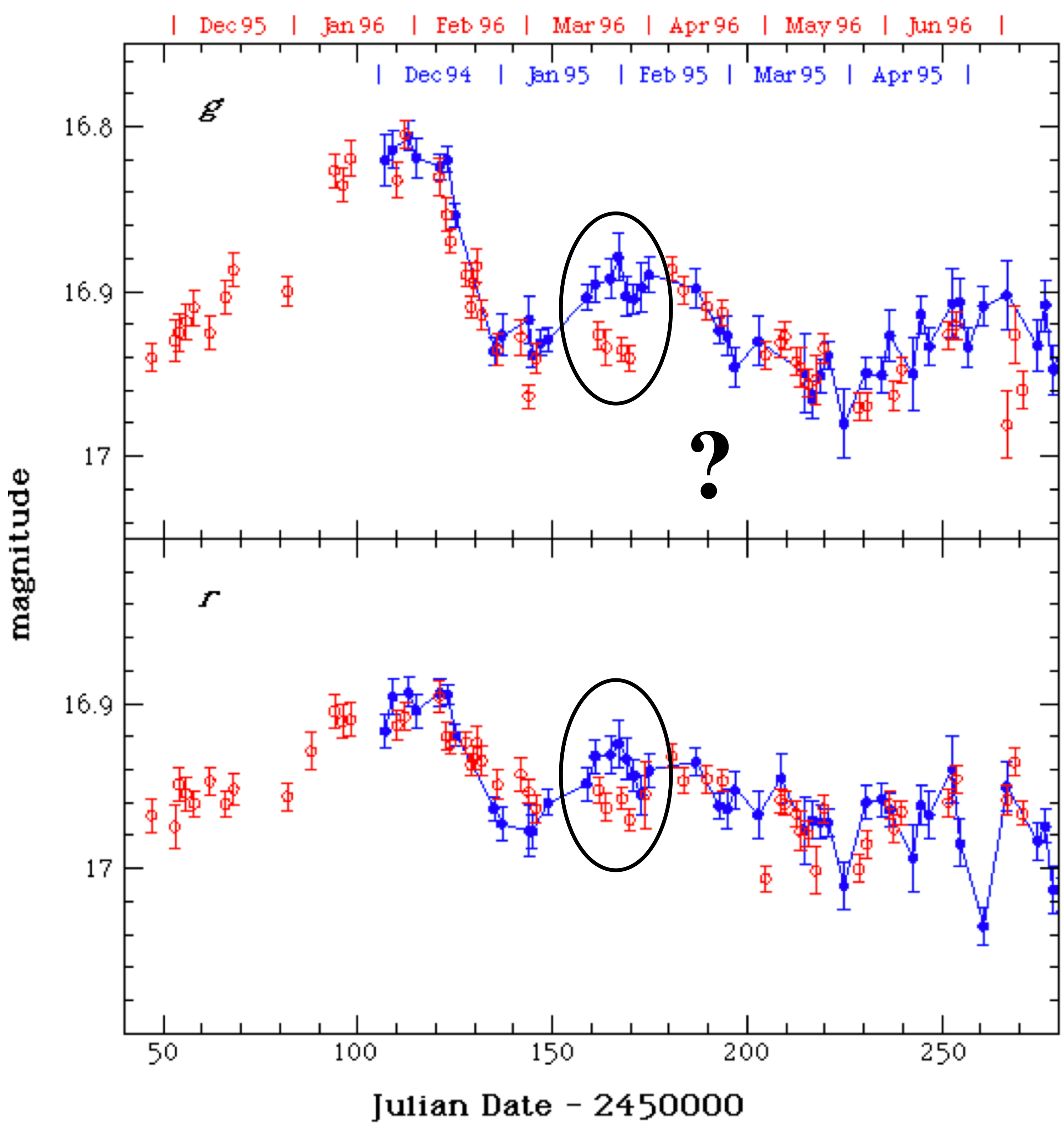
Q0957+561
images A (blue) and B (red)

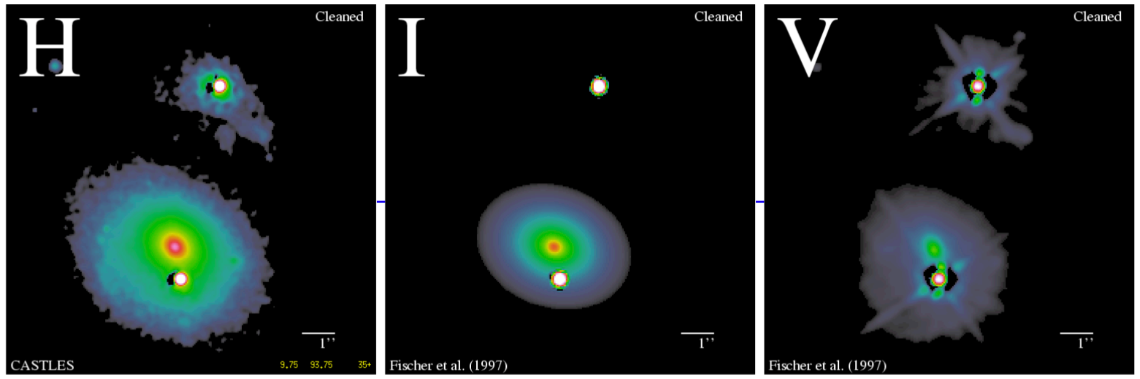
First the time-delay must be measured.

Look for clear features in the light curves of different images that are repeated ofter a delay.

MICROLENSING OF MULTIPLY IMAGED QSOs

Kundrić et al.



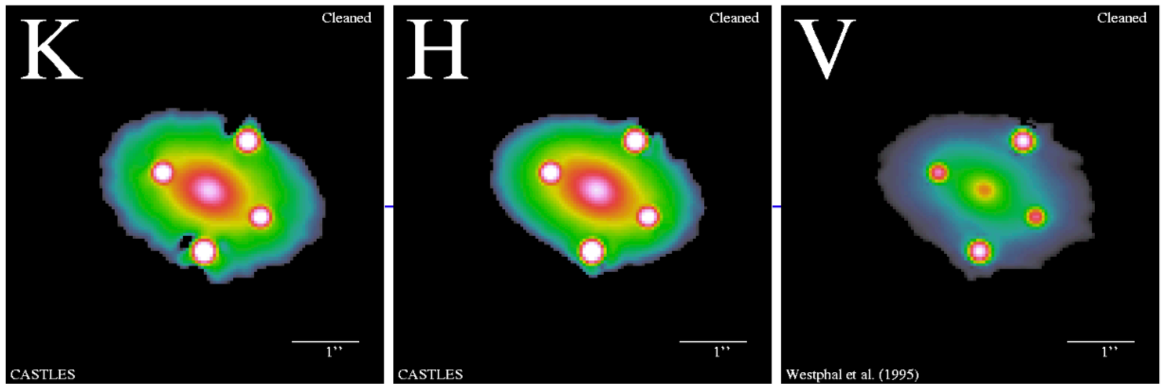


Q0957+561
images A (blue) and B (red)

First the time-delay must be measured.

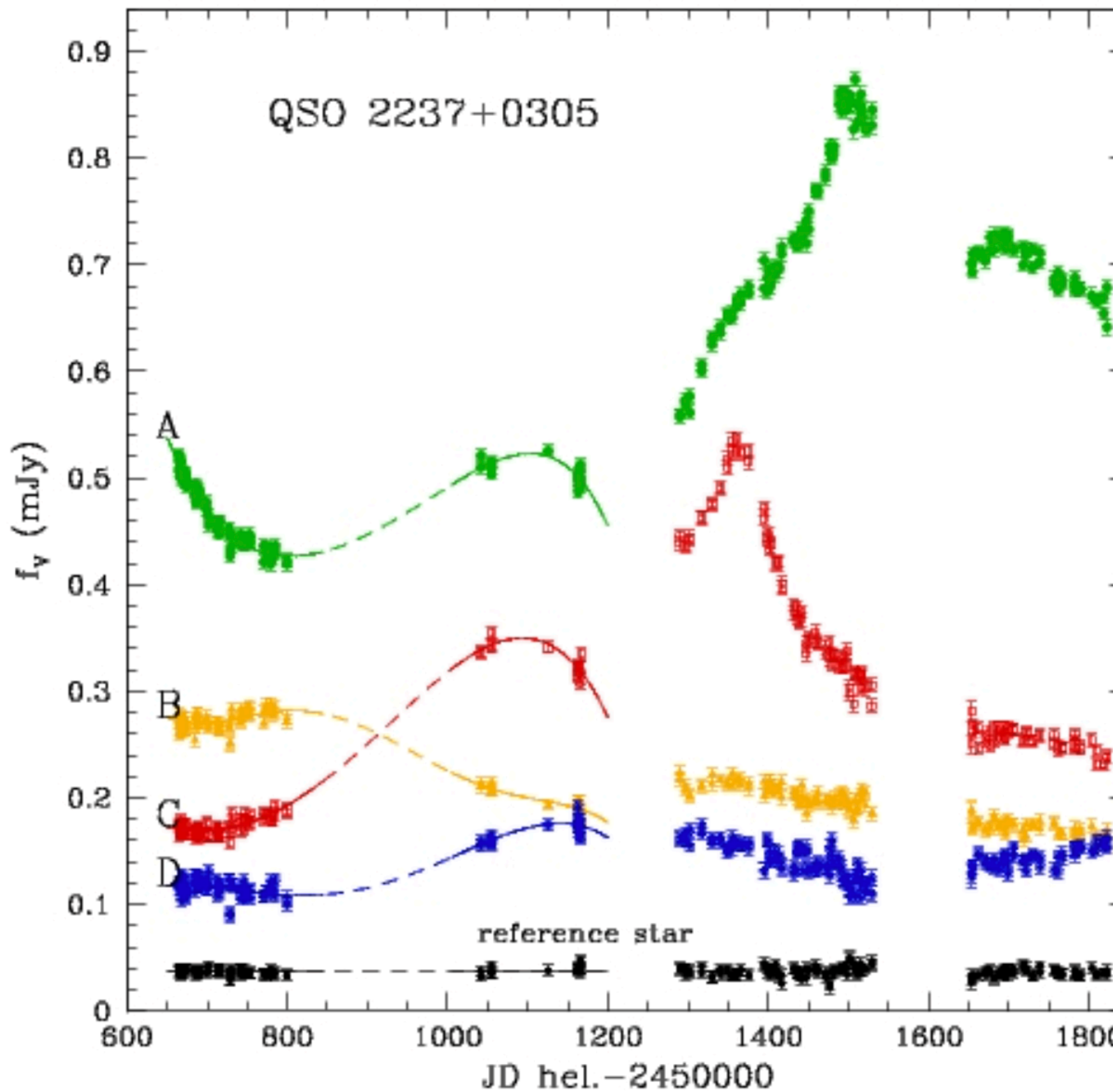
Look for clear features in the light curves of different images that are repeated ofter a delay.

MICROLENSING OF MULTIPLY IMAGED QSOs



Example : 2237+0305

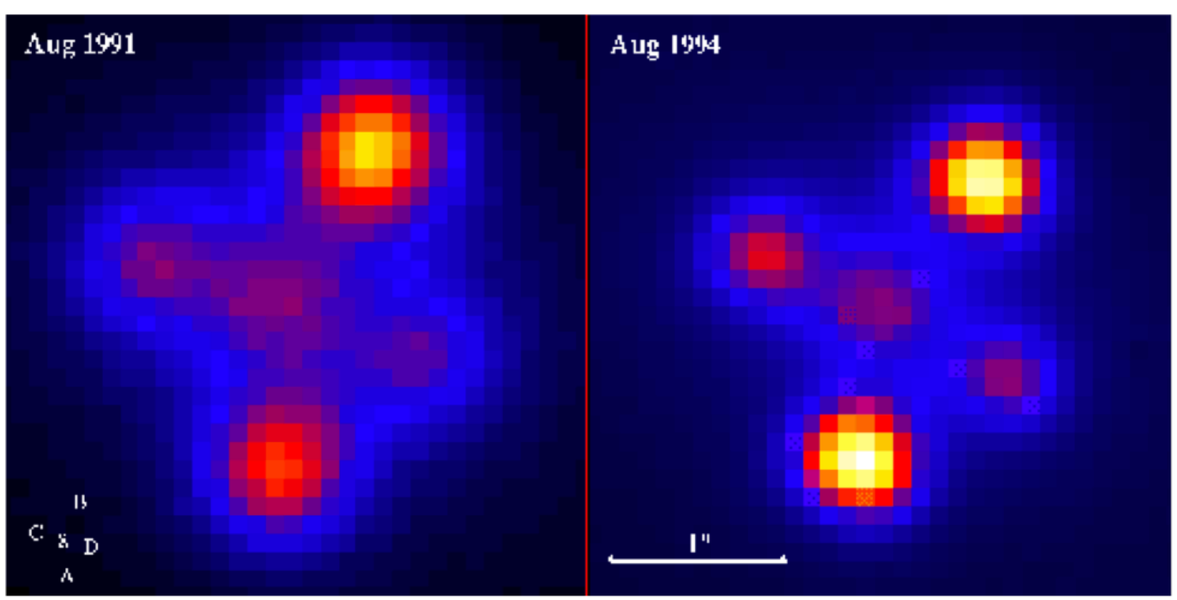
OGLE-II data



Particularly susceptible to micro lensing because the low lens redshift results in the images being close to the centre of the galaxy where the star density is high.

$$z_l = 0.04$$
$$z_s = 1.69$$

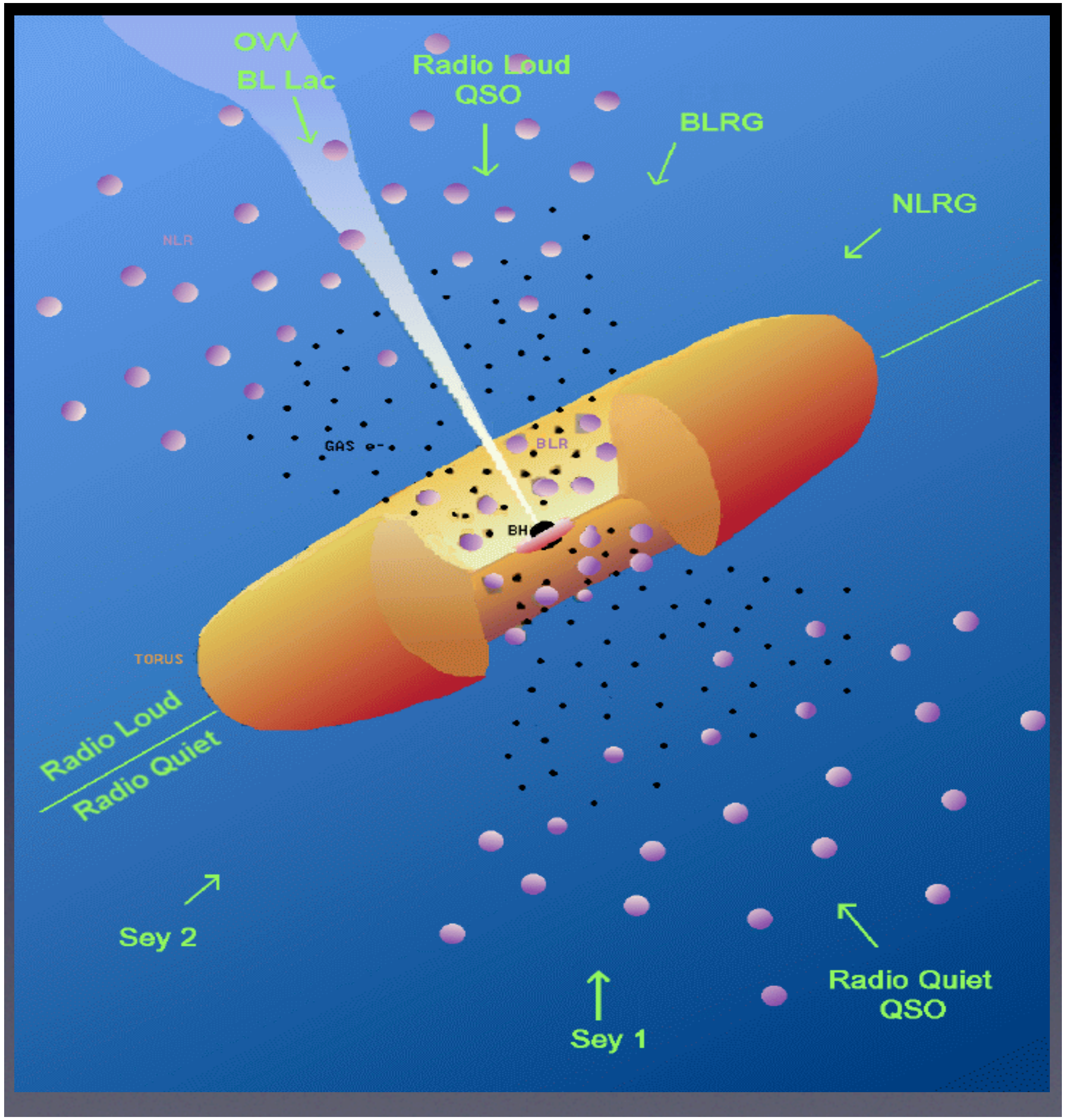
C image is clearly dimmer in 1991 then in 1994.



MICROLENSING OF MULTIPLY IMAGED QSOs

Spectroscopic microlensing

Classical model for the emission regions of a QSO



MICROLENSING OF MULTIPLY IMAGED QSOs

Spectroscopic microlensing

The different emission regions of a QSO produce emission lines in the spectra of each line.

Each emission region has a distribution of velocities causing the line profile.

Since the magnification is position dependent it will affect the line profile in a different way in each image that is micro lensed.

Hutsemékers et al. Constraints on Quasar BLRs from Microlensing

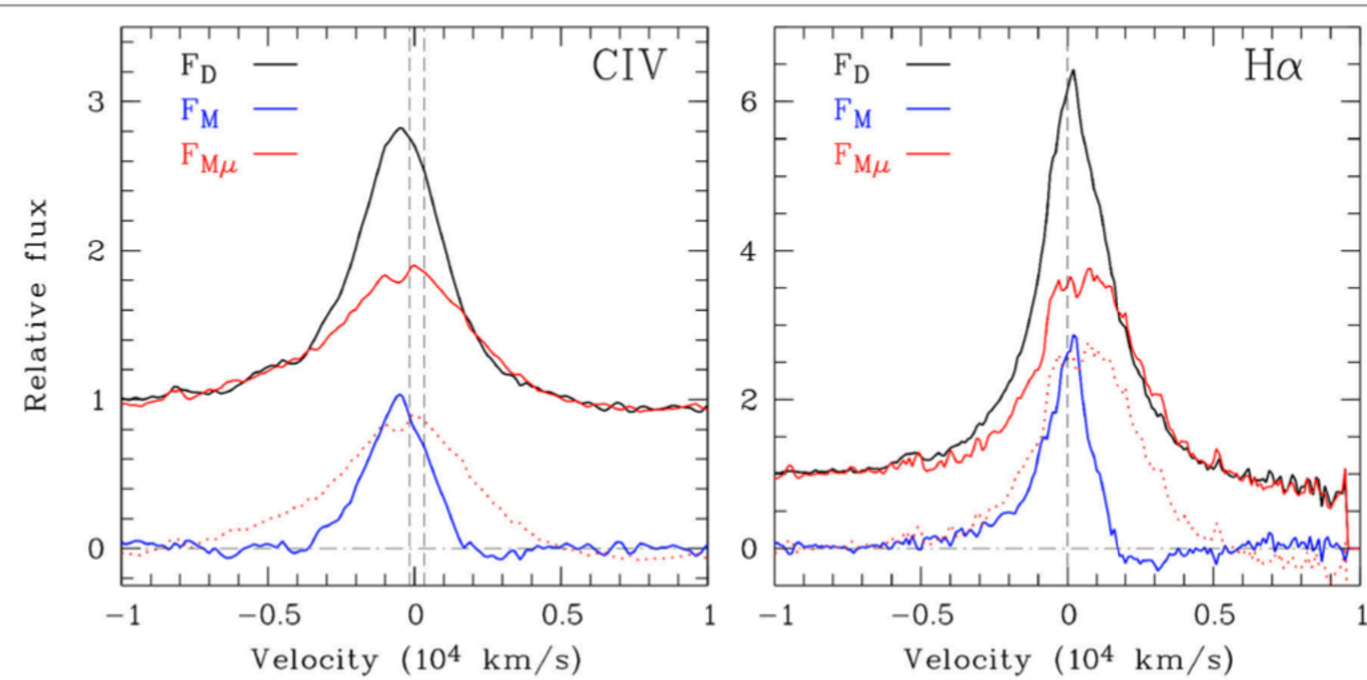


FIGURE 2 | Microlensing in the CIV and Hα lines of the quasar Q2237+030. F_D is the observed spectrum of image D which is not affected by microlensing. The line profile can be decomposed in a part affected by microlensing, $F_{M\mu}$, and another part not affected by microlensing, F_M, with $F_D = F_M + F_{M\mu}$. The dotted line profile represents the continuum-subtracted $F_{M\mu}$.

Here the emission line is decomposed into a microlensed part and an unlensed part.

MICROLENSING OF MULTIPLY IMAGED QSOs

Spectroscopic microlensing

The different emission regions of a QSO produce emission lines in the spectra of each line.

Each emission region has a distribution of velocities causing the line profile.

Since the magnification is position dependent it will affect the line profile in a different way in each image that is micro lensed.

Hutsemékers et al. Constraints on Quasar BLRs from Microlensing

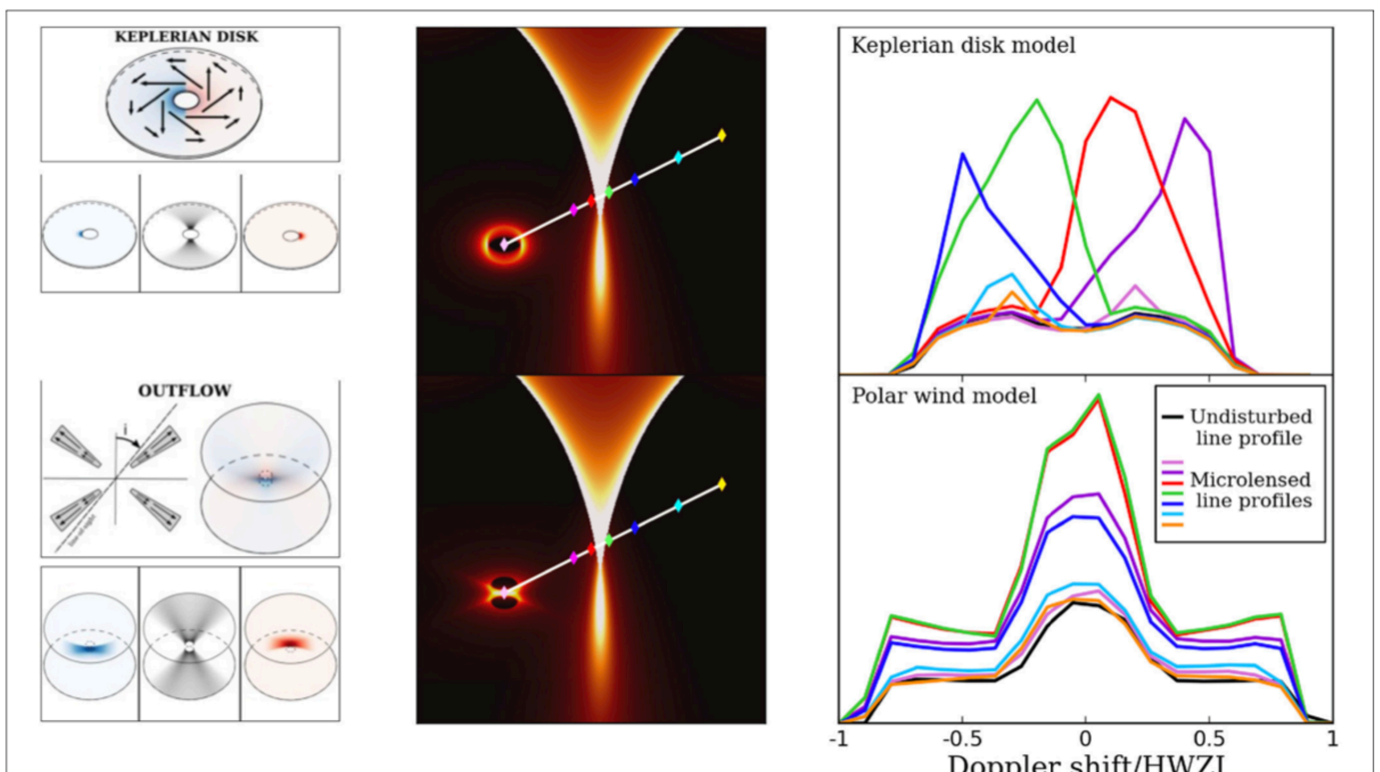


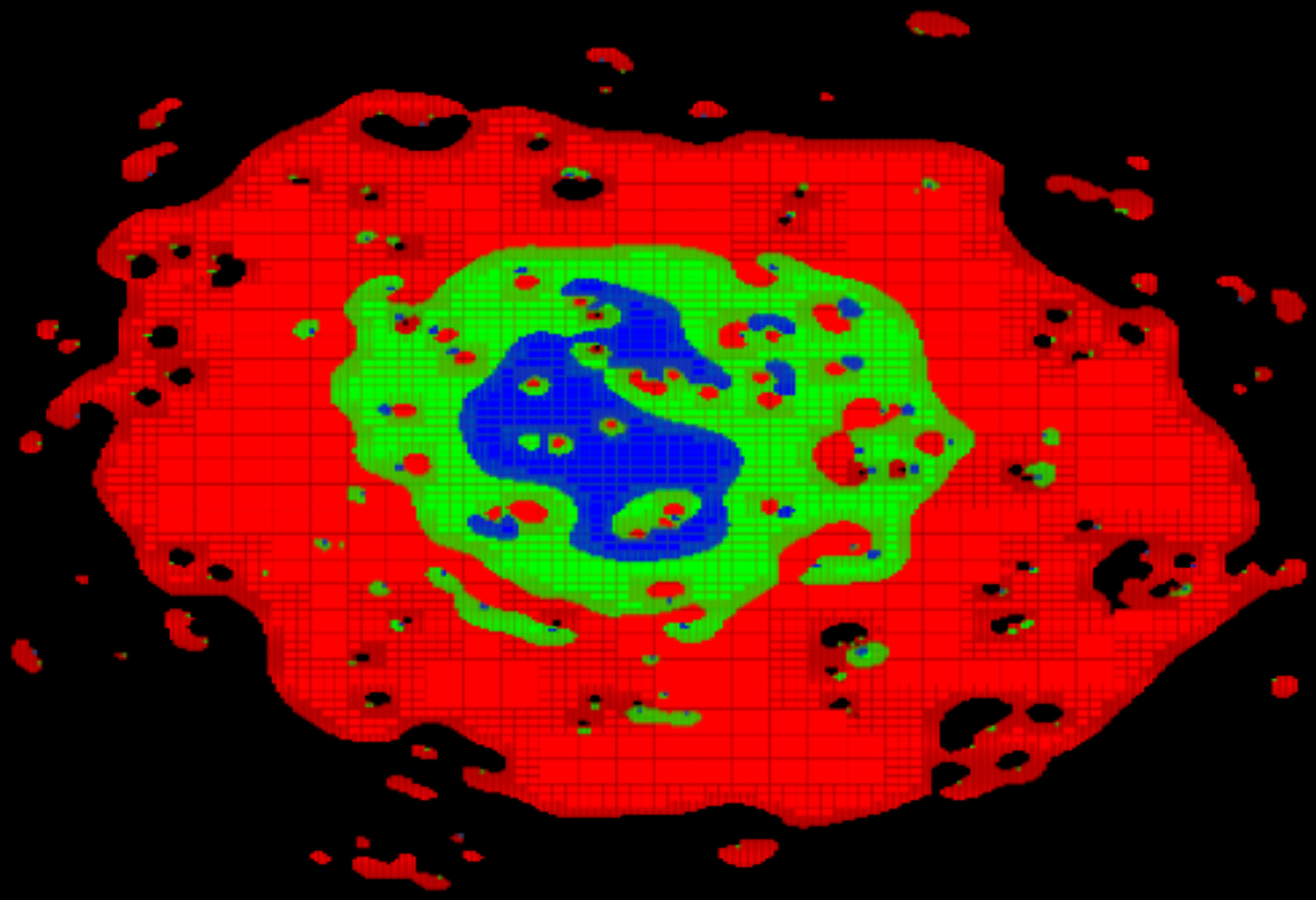
FIGURE 3 | Microlensing of different BELR structures and the resulting line profile distortions. The left panel sketches two models: a rotating Keplerian disk and a biconical outflow, both seen at intermediate inclination. The location of the approaching and receding gas is indicated in blue and red, respectively. The middle panel shows the BELR superimposed on a typical caustic. At the different positions indicated by colored losanges, different subregions of the BELR are magnified by the caustic. The right panel illustrates the line profiles corresponding to the different positions of the BELR onto the caustic pattern. Microlensing of the Keplerian disk is characterized by asymmetric red/blue line profile distortions while microlensing of the biconical outflow is characterized by symmetric wings/core distortions.

Different space-velocity distributions will result in different types of distortions got the lines.

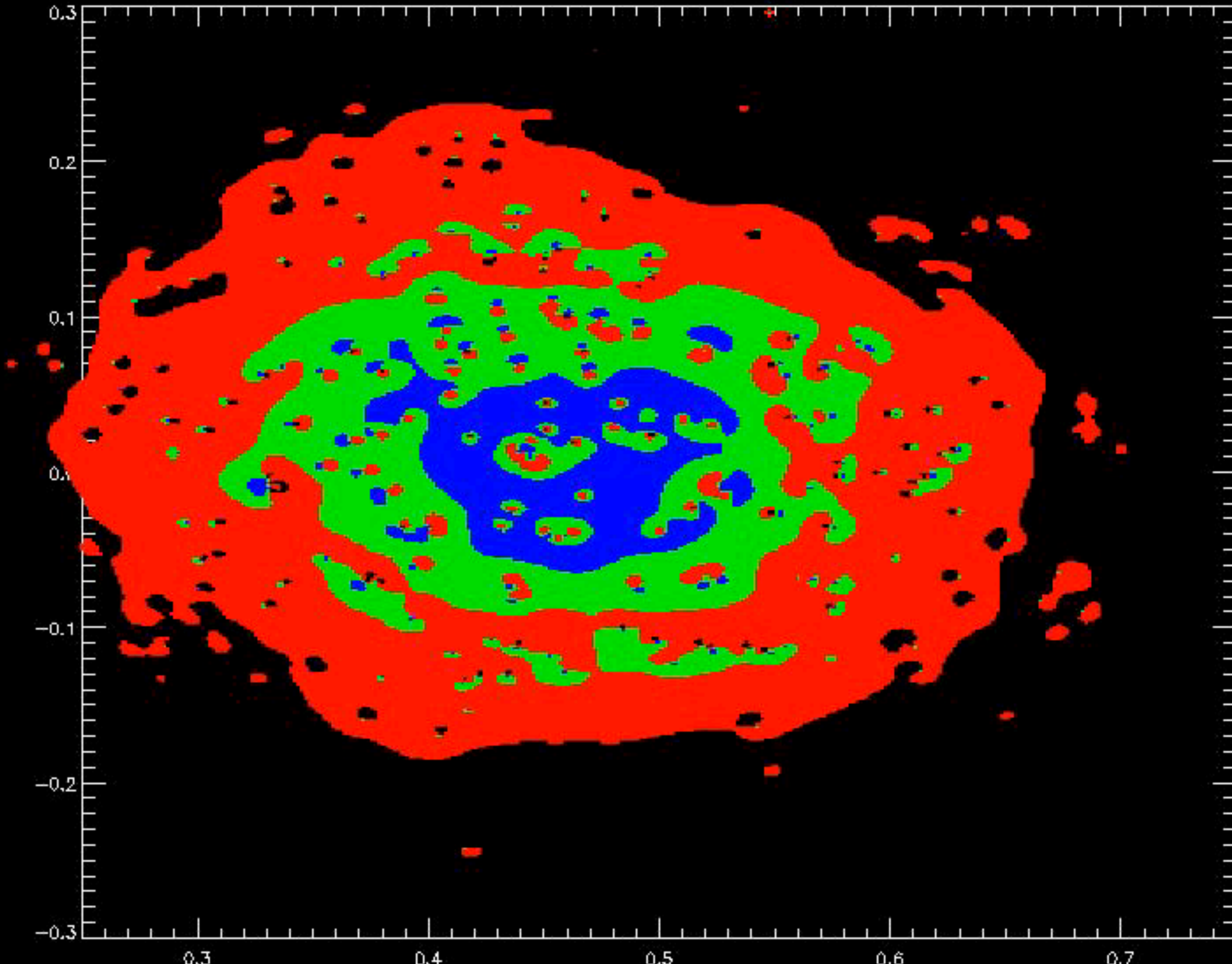
An outflow will generally produce a symmetric distortion.

A orbital geometry will generally produce an asymmetric distortion.

QSO Microlensing

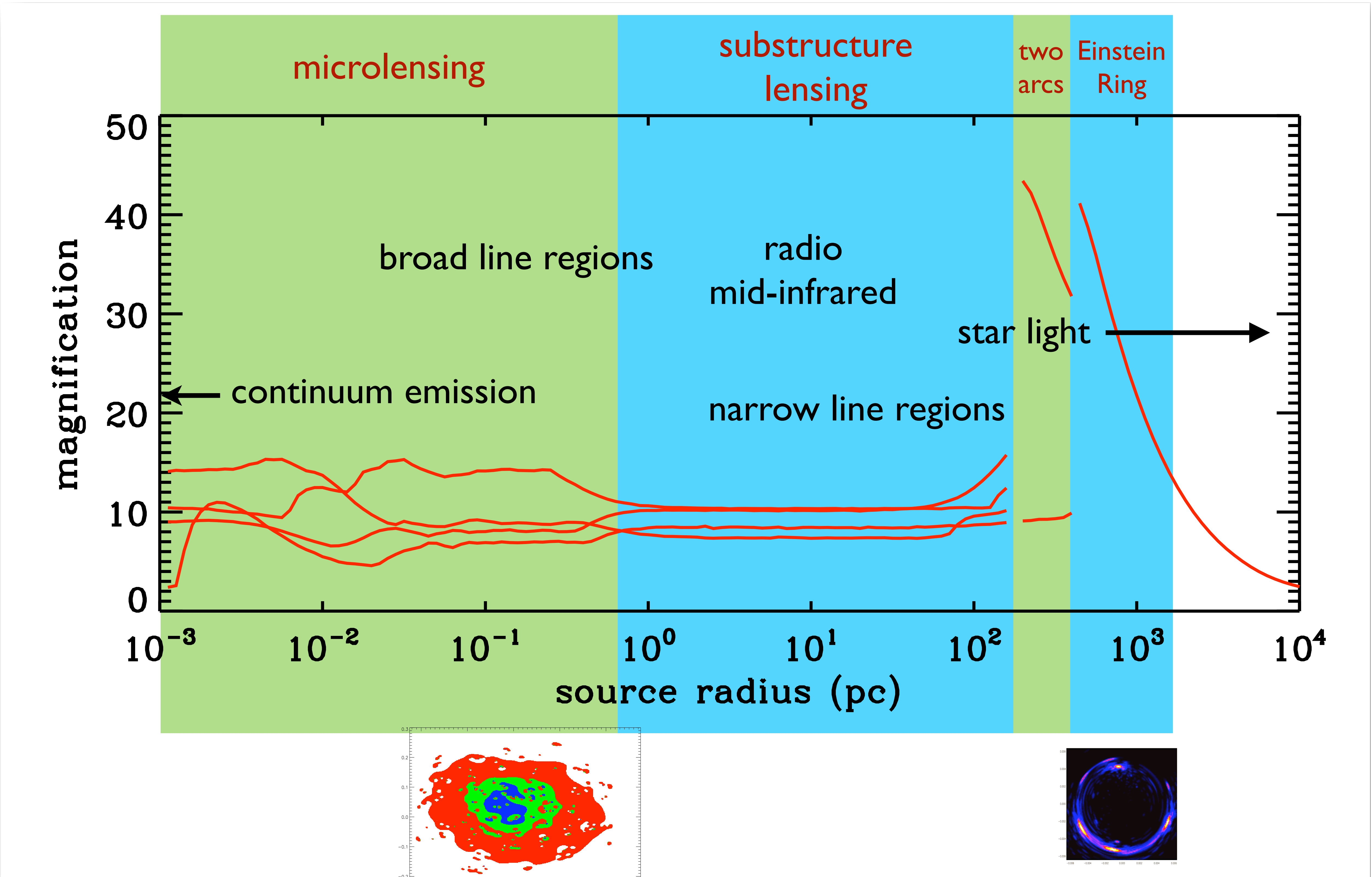


QSO Microlensing



MICROLENSING OF MULTIPLY IMAGED QSOs

Lensing as a function of source size.



GRAVITATIONAL LENSING

12 - LENS MODELS: AXIALLY SYMMETRIC LENSES

R. Benton Metcalf

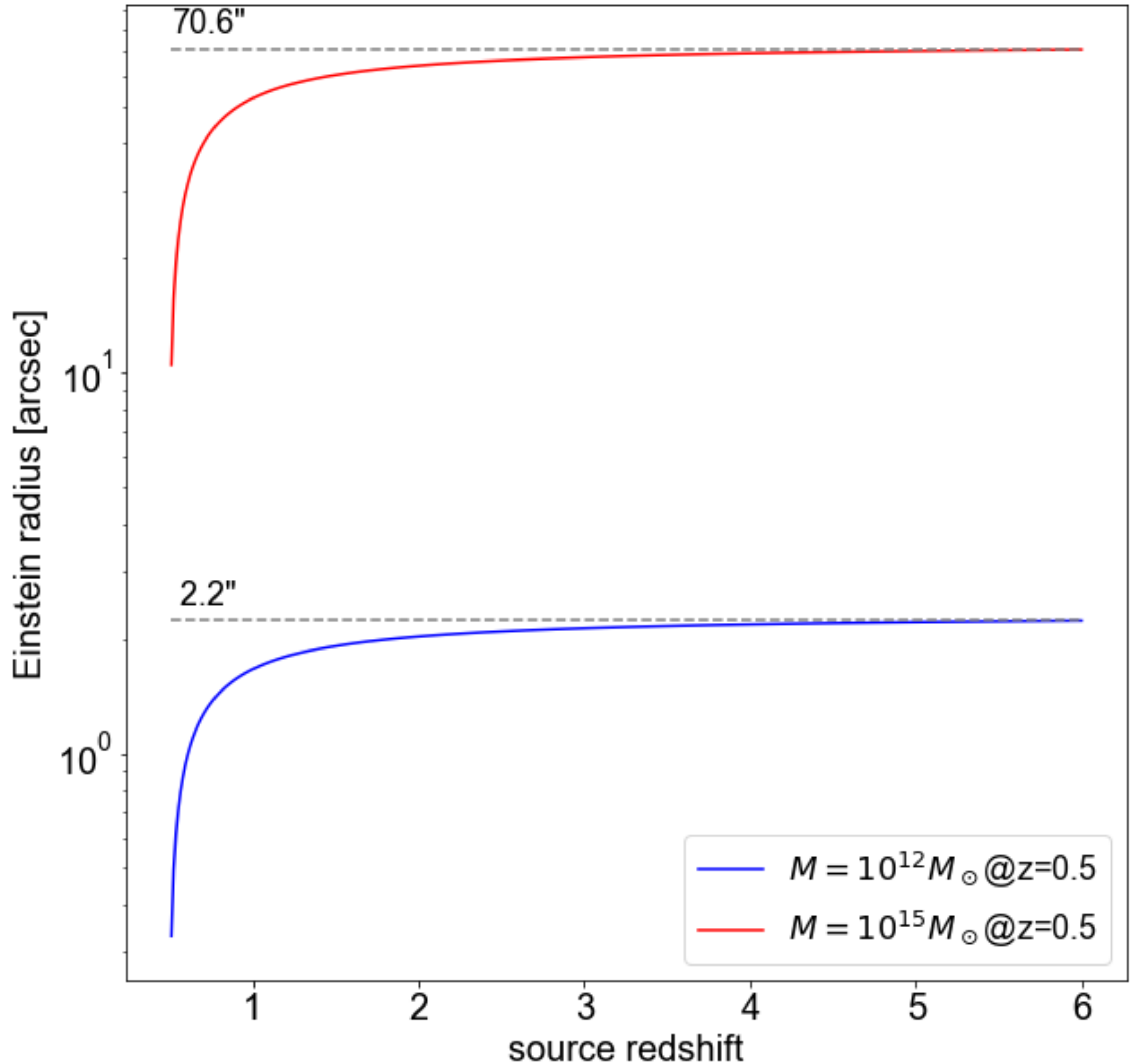
2022-2023

LENSING BY GALAXIES AND GALAXY CLUSTERS

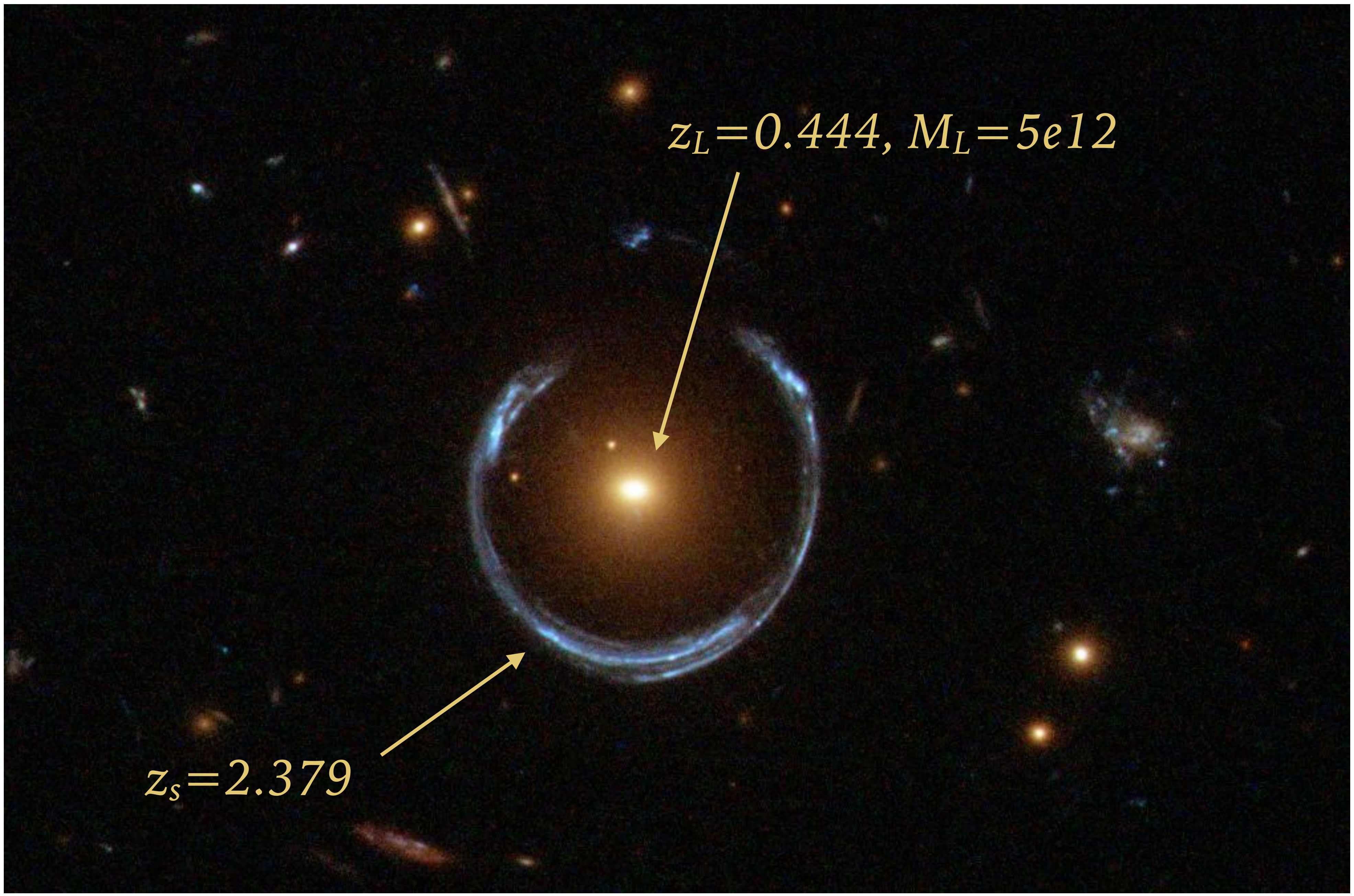
- Much more massive than stars!
- At cosmological distances
- Extended!
- Complex (more parameters)

Note: computed using

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}$$

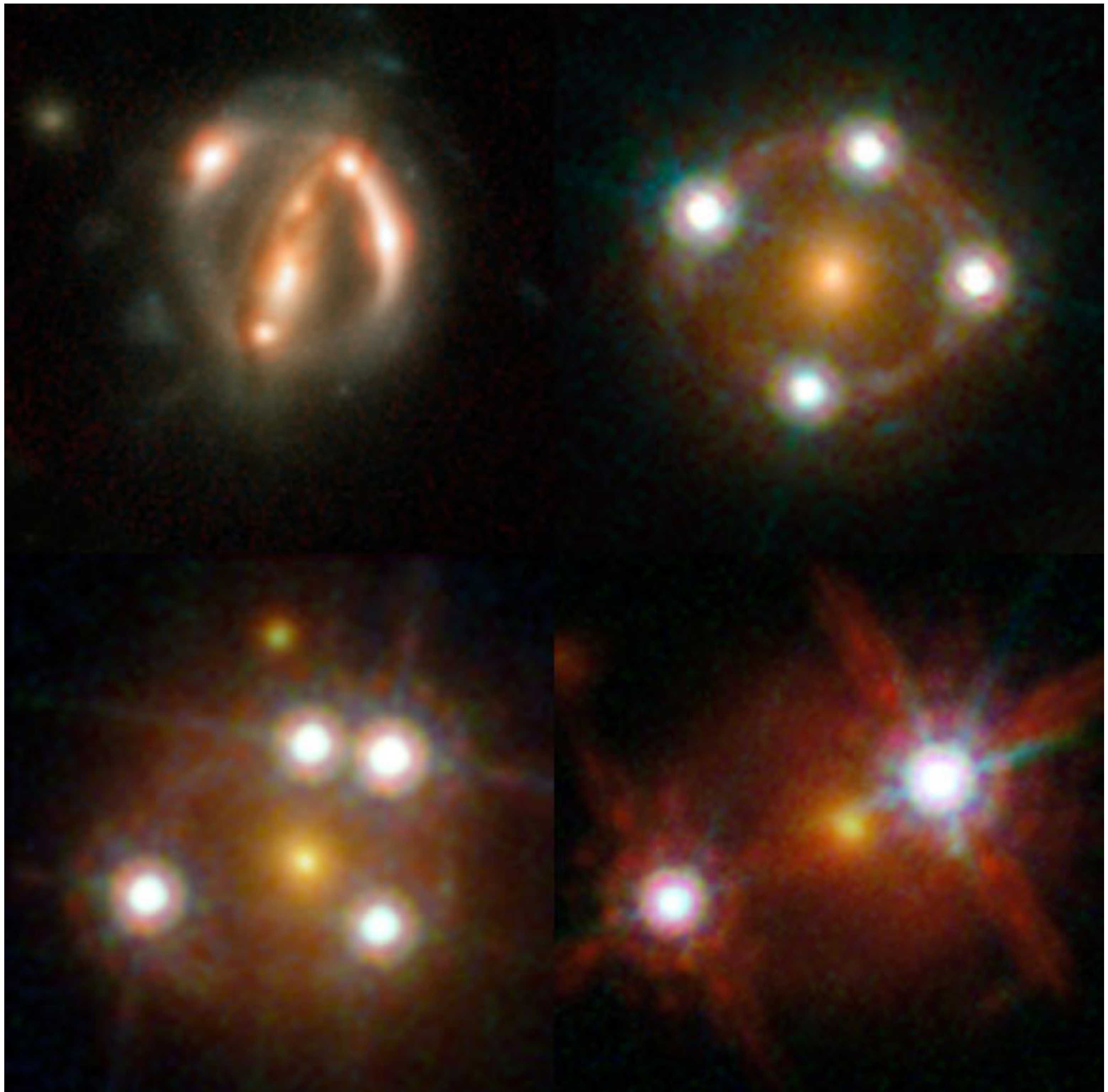


EXTENDED LENSES

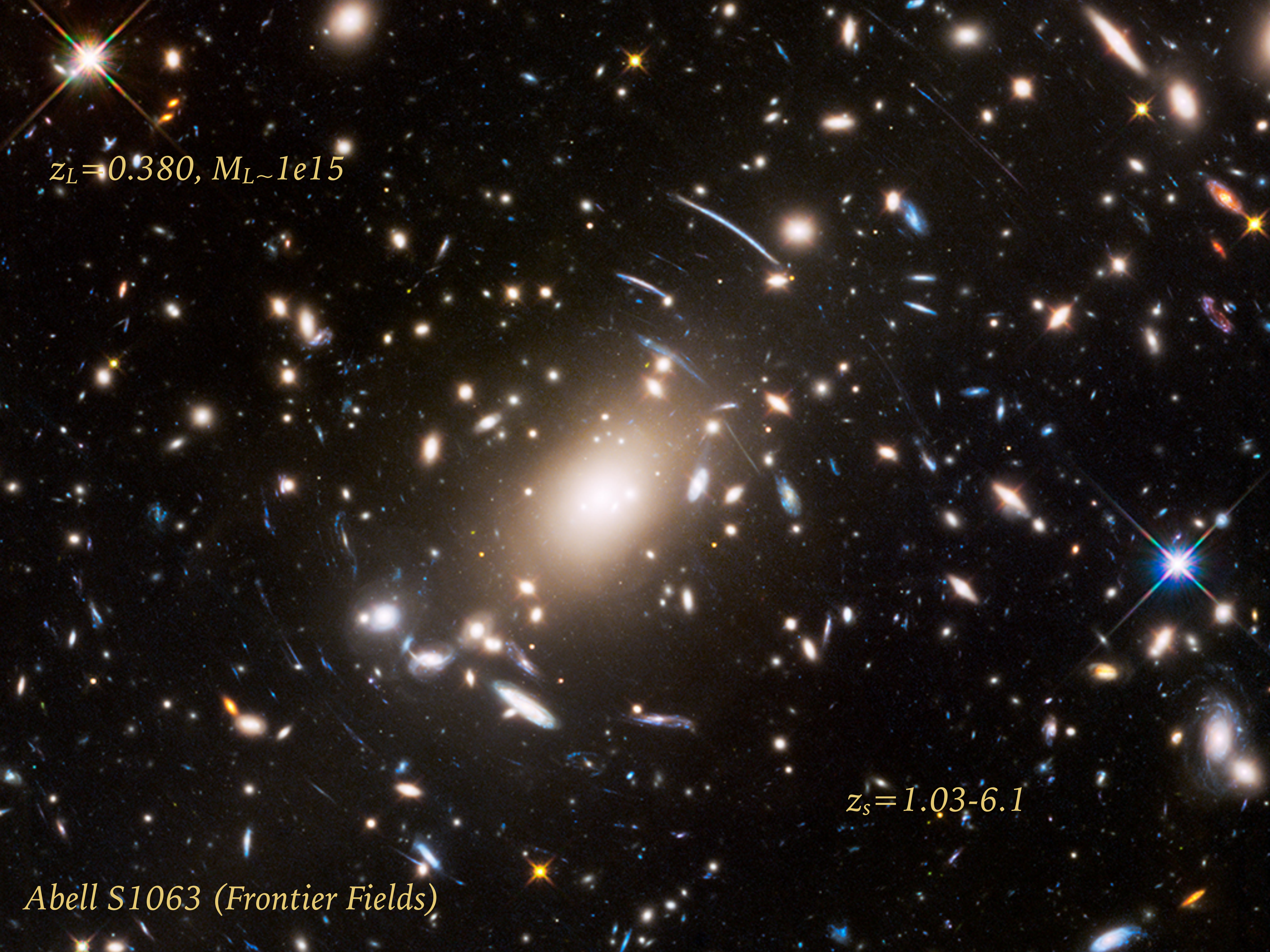


Cosmic horseshoe (Belokurov et al. 2007)

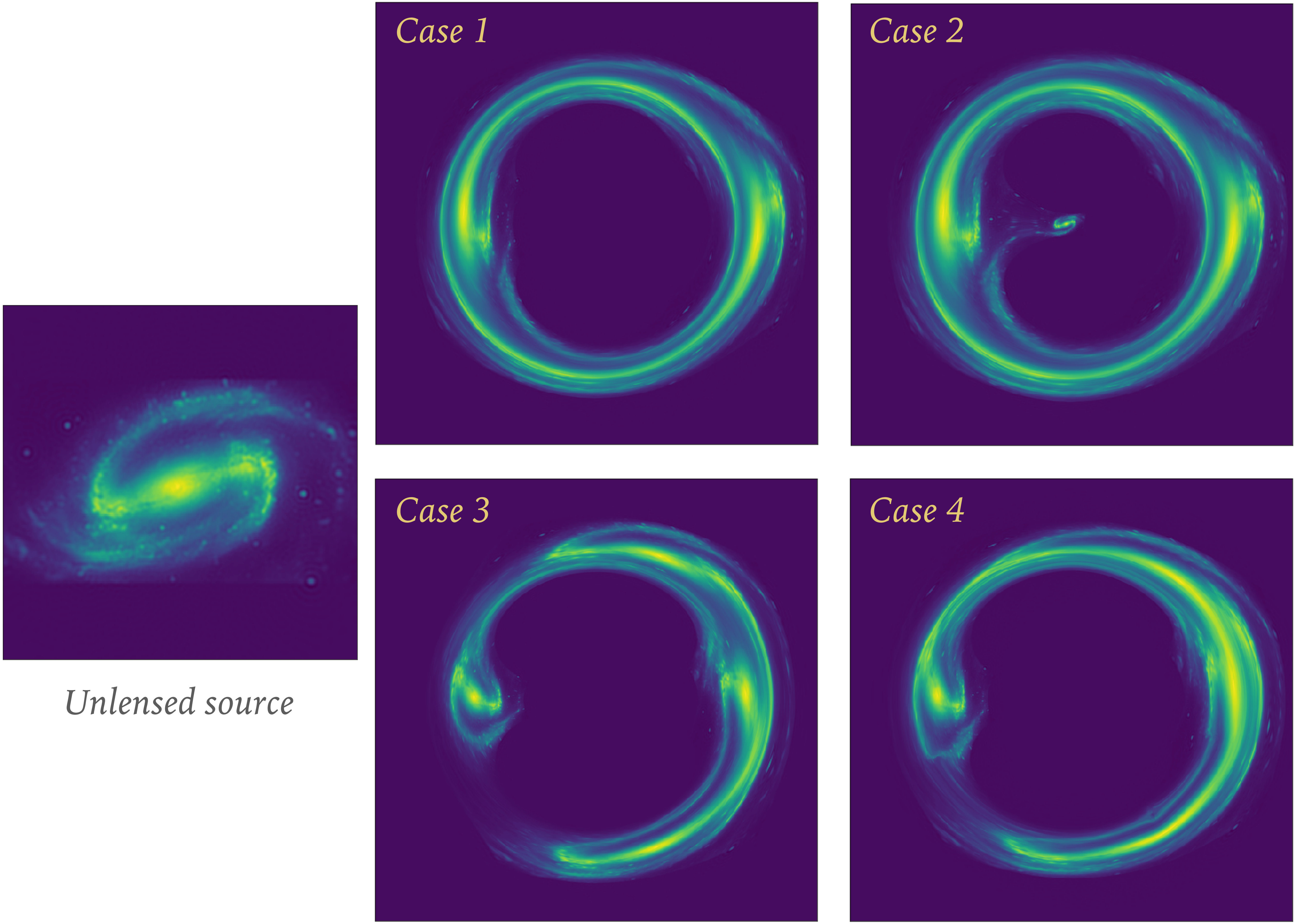
EXTENDED LENSES

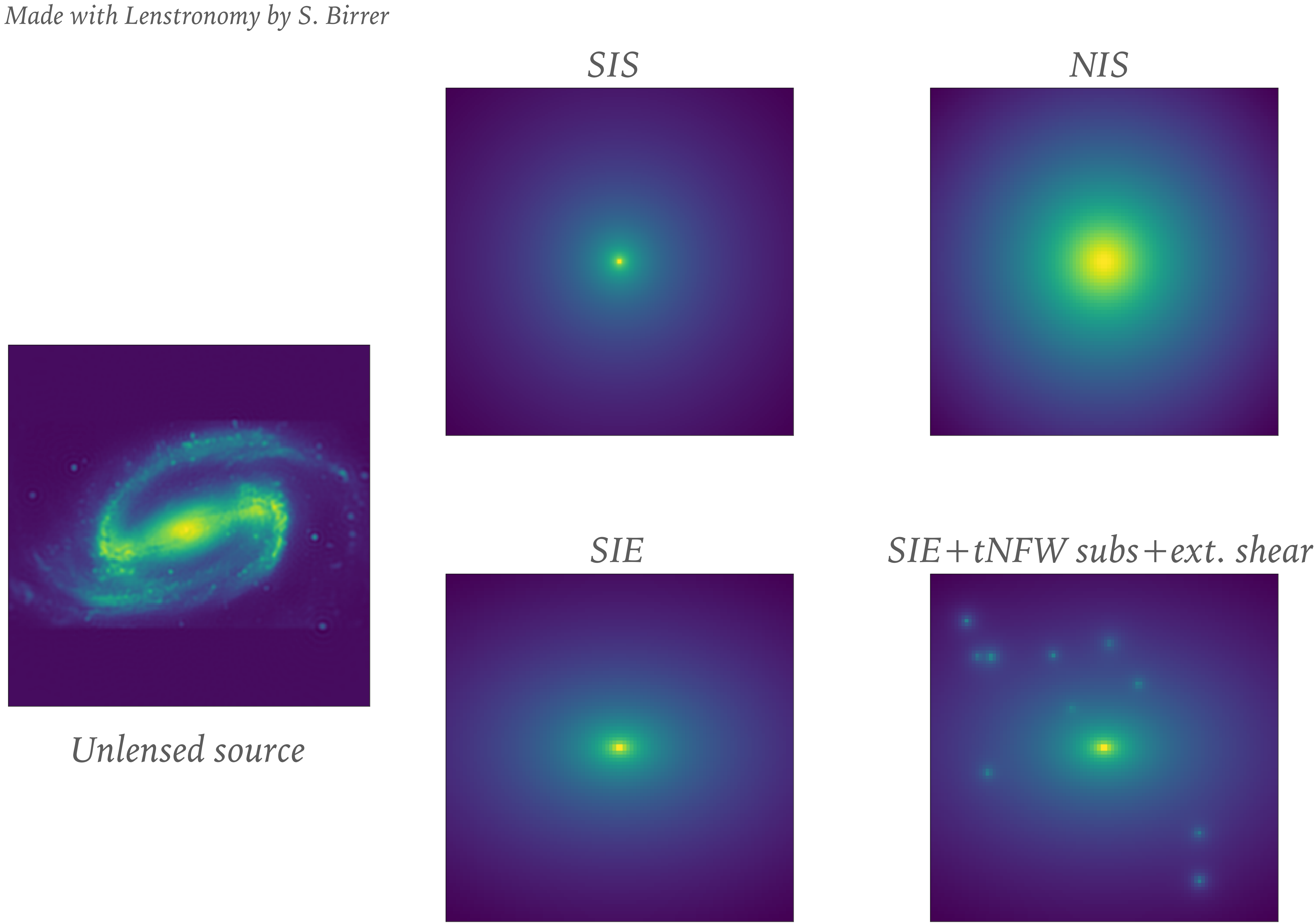


Suyu et al. (H0LiCOW team)


$z_L = 0.380$, $M_L \sim 1e15$
$z_s = 1.03\text{-}6.1$
Abell S1063 (Frontier Fields)

Made with Lenstronomy by S. Birrer



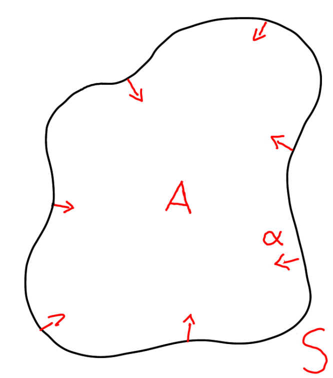

Made with Lenstronomy by S. Birrer
SIS
NIS
SIE
SIE+tNFW subs+ext. shear
Unlensed source

SHEAR IN POLAR COORDINATES

Gauss' Law

Relates deflection along a closed curve to the convergence within the curve.

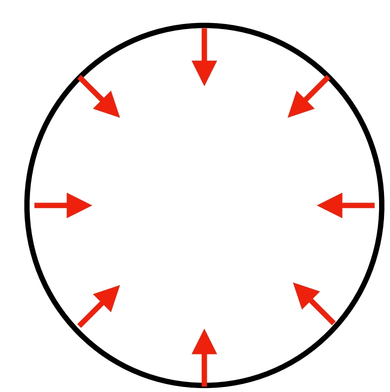
$$\int_C ds\ \boldsymbol{n}\cdot\boldsymbol{\nabla}\psi = \int d^2x\ \nabla^2\psi$$



$$\int_C ds\ \boldsymbol{\alpha}\cdot\boldsymbol{n} = 2\int d^2x\ \kappa(\boldsymbol{x}) = 2A\langle\kappa\rangle$$

For the special case of a circular curve

$$\boxed{\overline{\alpha_r} = r\ \langle\kappa\rangle = \frac{1}{\pi\Sigma_{crit}}\frac{M(r)}{r}}$$



POLAR COORDINATES

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi}$$

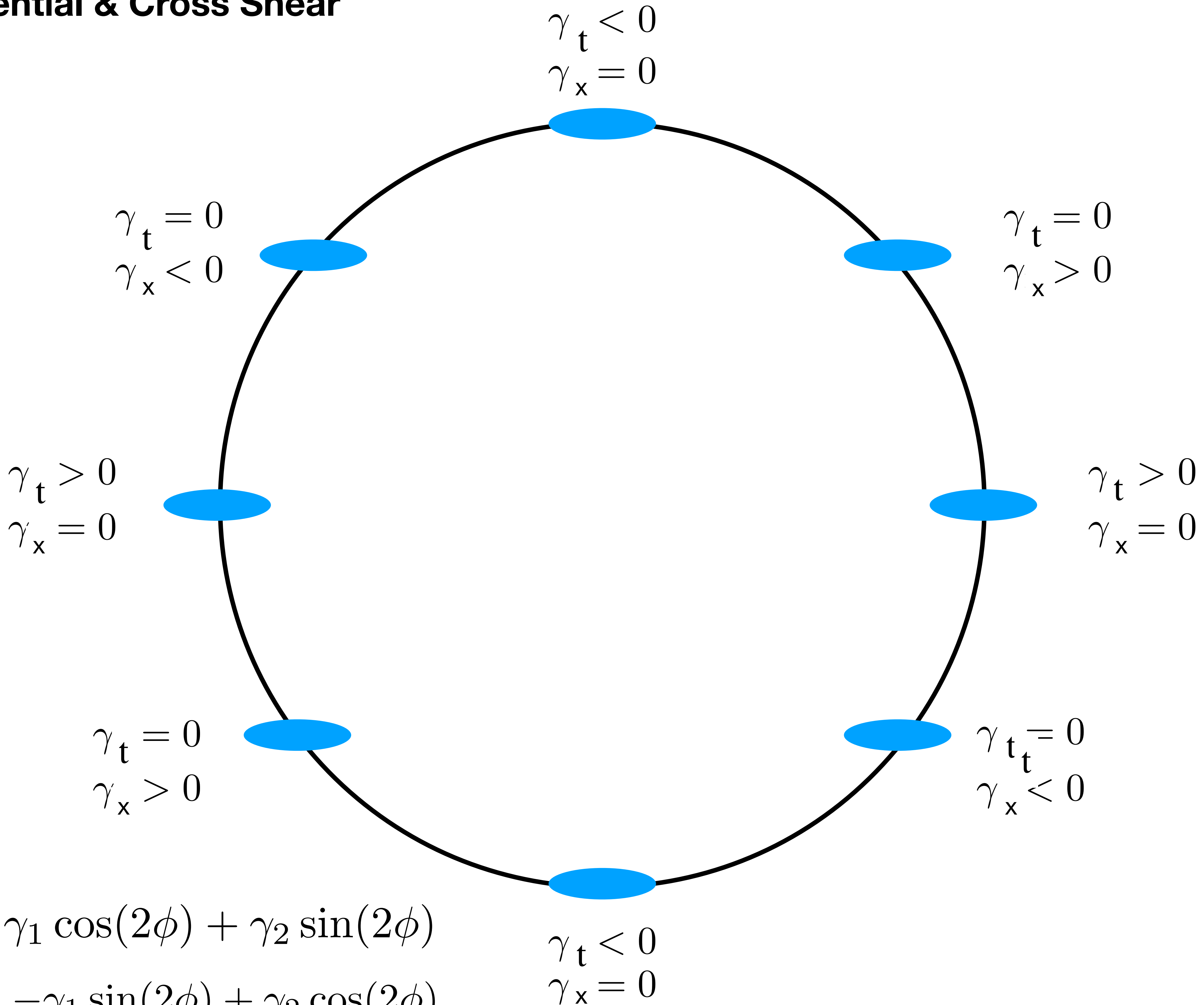
$$\begin{aligned}\hat{\boldsymbol{x}} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial x} &= \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}}\frac{\partial}{\partial r} - \frac{\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\phi}}}{r}\frac{\partial}{\partial \phi} \\ &= \cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{y}} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial y} &= \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{r}}\frac{\partial}{\partial r} - \frac{\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\phi}}}{r}\frac{\partial}{\partial \phi} \\ &= \sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\end{aligned}$$

POLAR COORDINATES

$$
\begin{aligned}
\nabla^2 &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} \\
&= \left(\cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\right)\left(\cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\right) \\
&+ \left(\sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\right)\left(\sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\right) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}
\end{aligned}
$$

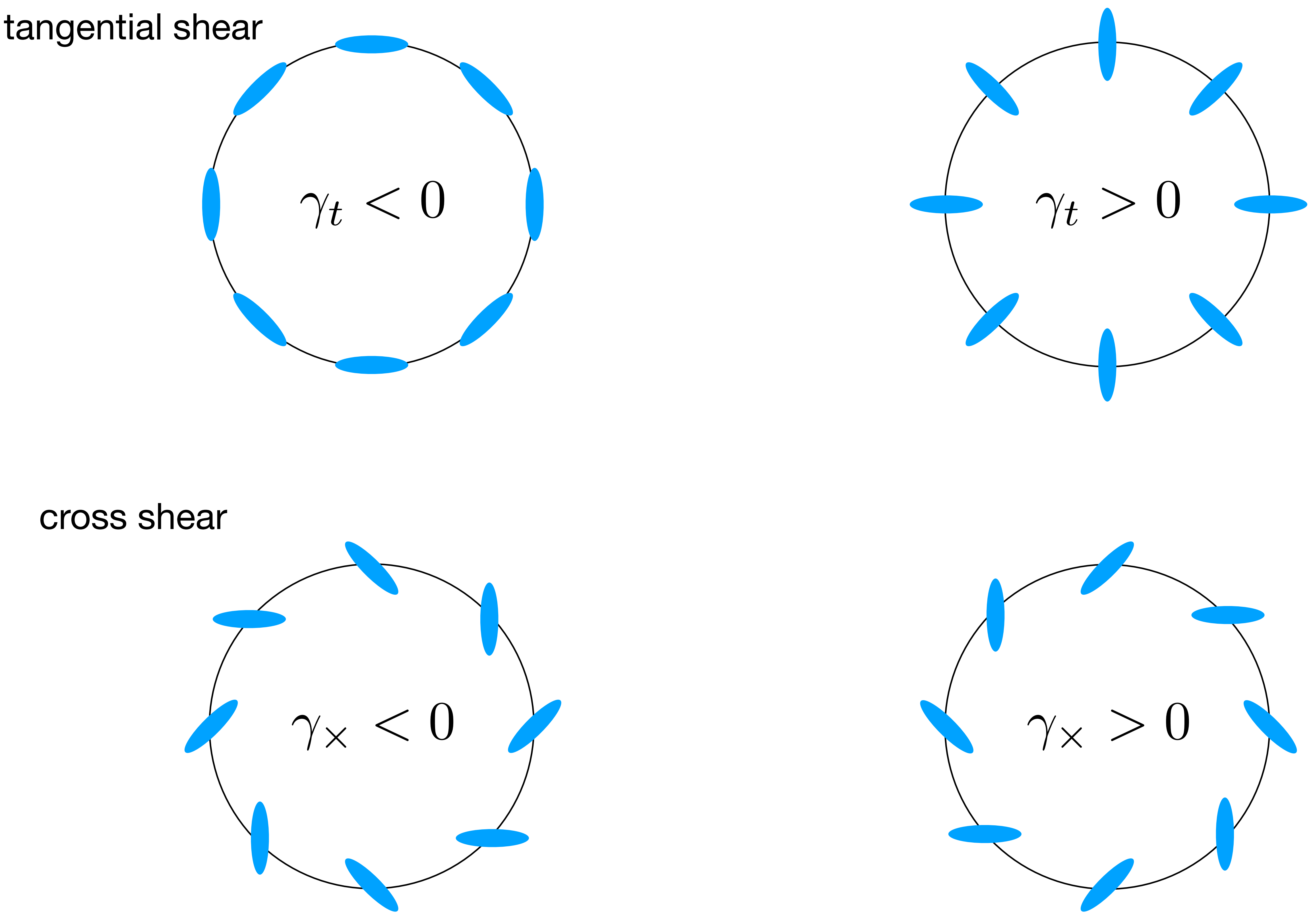
Tangential & Cross Shear



$$\gamma_t = \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi)$$

$$\gamma_\times = -\gamma_1 \sin(2\phi) + \gamma_2 \cos(2\phi)$$

SHEAR IN POLAR COORDINATES



SHEAR IN POLAR COORDINATES

$$\gamma_2 = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi = \left[\cos(\phi)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r}\frac{\partial}{\partial \phi}\right]\left[\sin(\phi)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r}\frac{\partial}{\partial \phi}\right]\psi$$

$$= \left[\cos(\phi)\sin(\phi)\frac{\partial^2}{\partial r^2} + \cos^2(\phi)\left(-\frac{1}{r^2}\frac{\partial}{\partial \phi} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \phi}\right) - \frac{\sin(\phi)}{r}\left(\cos(\phi)\frac{\partial}{\partial r} + \sin(\phi)\frac{\partial^2}{\partial r \partial \phi}\right) - \frac{\sin(\phi)}{r^2}\left(-\sin(\phi)\frac{\partial}{\partial \phi} + \cos(\phi)\frac{\partial^2}{\partial \phi^2}\right)\right]\psi$$

$$= \cos(\phi)\sin(\phi)\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi + \left(\cos^2(\phi) - \sin^2(\phi)\right)\left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} - \frac{1}{r^2}\frac{\partial}{\partial \phi}\right)\psi$$

$$= \frac{1}{2}\sin(2\phi)\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi + \cos(2\phi)\left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} - \frac{1}{r^2}\frac{\partial}{\partial \phi}\right)\psi$$

$$= \gamma_t \sin(2\phi) + \gamma_\times \cos(2\phi)$$

tangential shear operator

$$\Delta_t = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) = \frac{\partial^2}{\partial r^2} - \frac{1}{2}\nabla^2$$

cross shear operator

$$\Delta_\times = \left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} - \frac{1}{r^2}\frac{\partial}{\partial \phi}\right)$$

TANGENTIAL SHEAR THEOREM

$$\int_C ds\, \boldsymbol{\alpha} \cdot \boldsymbol{n} = 2 \int d^2x \;\kappa(\boldsymbol{x})$$

Gauss' Law

$$\int_0^{2\pi} d\phi \; r\, \alpha_r = 2 \int_0^{2\pi} d\phi \int_0^{r} dr' r' \kappa(r', \phi)$$

on a circle

$$\frac{\partial}{\partial r} \int_0^{2\pi} d\phi \; r\, \alpha_r = \frac{\partial}{\partial r} 2 \int_0^{2\pi} d\phi \int_0^{r} dr' \; r' \,\kappa(r', \phi)$$

taking the derivative of both sides

$$\int_0^{2\pi} d\phi \; \left(\alpha_r + r \frac{\partial \alpha_r}{\partial r} \right) = 2 \int_0^{2\pi} d\phi \; r\, \kappa(r, \phi)$$

$$\int_0^{2\pi} d\phi \; r \frac{\partial \alpha_r}{\partial r} = 2 \int_0^{2\pi} d\phi \; r\, \kappa(r, \phi) - \int_0^{2\pi} d\phi \; \alpha_r$$

rearrange it

$$\begin{aligned}
\int_C ds \; \frac{\partial \alpha_r}{\partial r} &= 4\pi r\, \overline{\kappa}(r) - \int_0^{2\pi} d\phi \; \alpha_r \\
&= 4\pi r\, \overline{\kappa} - \frac{1}{r} \int_C ds \; \alpha_r \\
&= 4\pi r\, \overline{\kappa} - \frac{2}{r} \int_V d^2x \;\kappa(\boldsymbol{x}) \\
&= 4\pi r\, \overline{\kappa} - 2\pi r \langle \kappa \rangle
\end{aligned}$$

Gauss' Law again

TANGENTIAL SHEAR THEOREM

$$
\begin{aligned}
\int_C ds\ \gamma_t = 2\pi r \overline{\gamma}_t &= \int_C ds\ \Delta_t \psi \\
&= \int_C ds\ \left[\frac{\partial^2}{\partial r^2} - \frac{1}{2}\nabla^2\right]\psi \\
&= \int_C ds\ \left[\frac{\partial \alpha_r}{\partial r} - \kappa\right] \\
&= 4\pi r\,\overline{\kappa} - 2\pi r \langle \kappa \rangle - \int_C ds\ \kappa \qquad \textit{using previous result} \\
&= 2\pi r\,\overline{\kappa} - 2\pi r \langle \kappa \rangle
\end{aligned}
$$

$$
\boxed{\overline{\gamma}_t = \overline{\kappa} - \langle \kappa \rangle}
$$

for any circle

SUMMARY OF AXIALLY SYMMETRIC LENSING

deflection angle : $$\alpha(r) = \frac{m(r)}{r}$$

$$m(r) = 2\int_0^r dr'\ r'\kappa(r') = \frac{M(r)}{\pi\Sigma_{\rm crit}}$$

convergence : $$\kappa(r) = \frac{m'(r)}{2r}$$

shear : $$\gamma(r) = \frac{m(r)}{r^2} - \frac{m'(r)}{2r}$$

$$\gamma_2 = \gamma(r)\sin(2\phi)$$

$$\gamma_1 = \gamma(r)\cos(2\phi)$$

inverse magnification : $$det\boldsymbol{A} = \frac{1}{\mu} = \left[1 - \frac{m(r)}{r^2}\right]\left[1 + \frac{m(r)}{r^2} - \frac{m'(r)}{r}\right]$$

tangential critical curve : $$\frac{m(r)}{r^2} = 1$$

radial critical curve : $$\frac{m'(r)}{r} - \frac{m(r)}{r^2} = 1$$

GRAVITATIONAL LENSING

13 - LENS MODELS: POWERLAW LENSES

R. Benton Metcalf

2022-2023

AXIALLY SYMMETRIC POWER-LAW LENSING

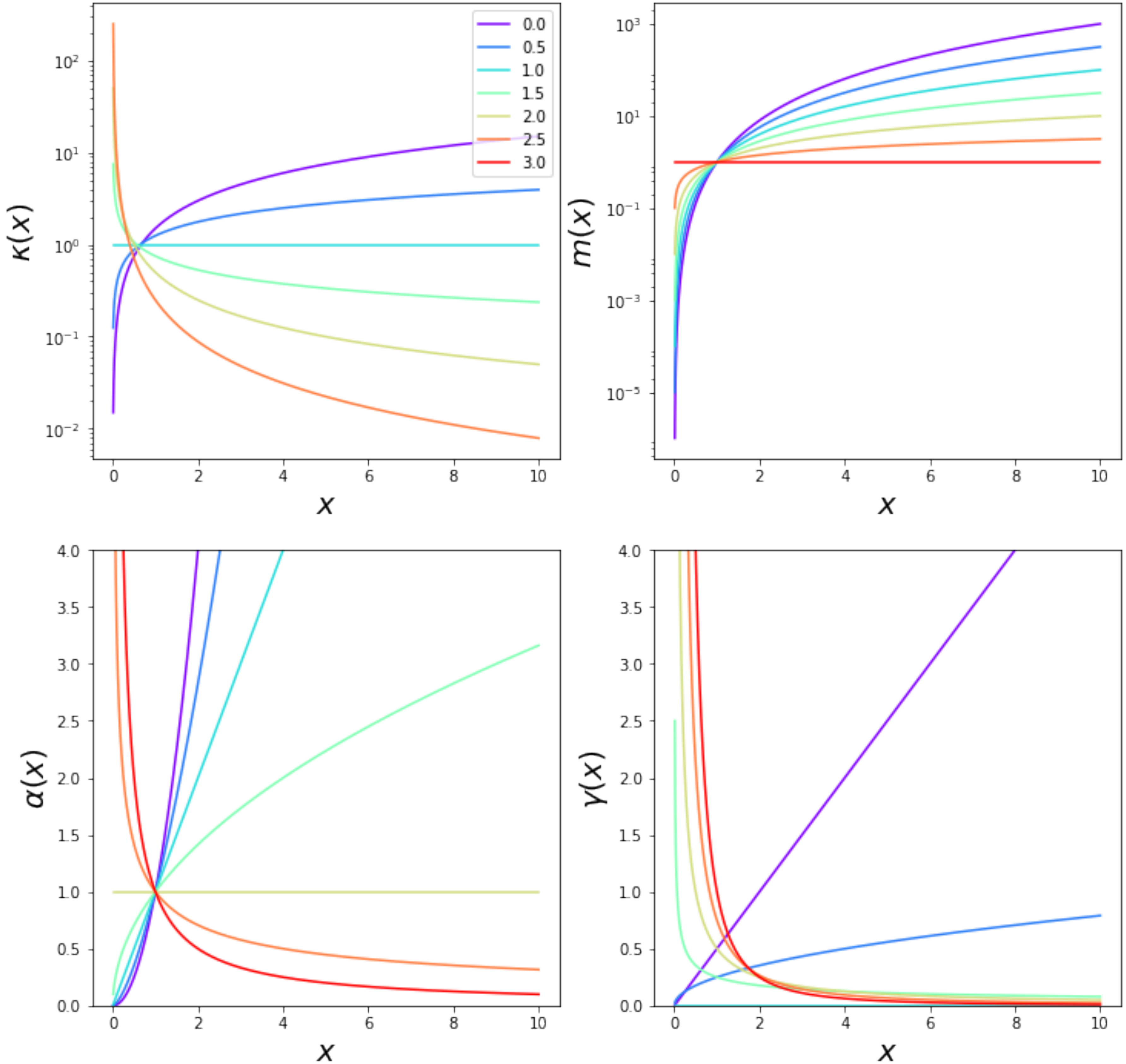
$$m(x) = x^{3-n}$$

$$\alpha(x) = \frac{m(x)}{x} = x^{2-n}$$

$$\kappa(x) = \frac{m'(x)}{2x} = \frac{3-n}{2}\, x^{1-n}$$

$$\gamma(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = x^{1-n} - \frac{3-n}{2}\, x^{1-n} = \frac{n-1}{2} x^{1-n}$$

POWER-LAW LENS



POWER-LAW LENS

- $n < 1$ convergence increases with x. This makes them not suitable for gravitationally bound objects, i.e. galaxies & clusters
- $n = 1$ convergence constant
- $1 < n < 2$ convergence decreasing function of x. The deflection is zero at the centre: $\alpha(x=0) = 0$
- $n = 2$ deflection is constant. $\alpha(x) = \mathrm{const.}$
- $2 < n < 3$ deflection diverges at origin. Lens potential diverges at origin.
- $n = 3$ corresponds to a point mass. $\alpha(x) = 1/x$
- $n > 3$ are unphysical because they produce 2D mass profiles, $M(r)$, that decrease with radius.

POWER-LAW LENS: CRITICAL LINES AND CAUSTICS

$$\lambda_t = 1 - \frac{m(x)}{x^2} = 0 = 1 - x^{1-n} \Rightarrow x_{crit,t} = 1$$

The tangential critical line has equation x=1 for any value of the slope parameter n.

This tells us that the reference angular scale used to define the dimensionless dimensions was the Einstein radius.

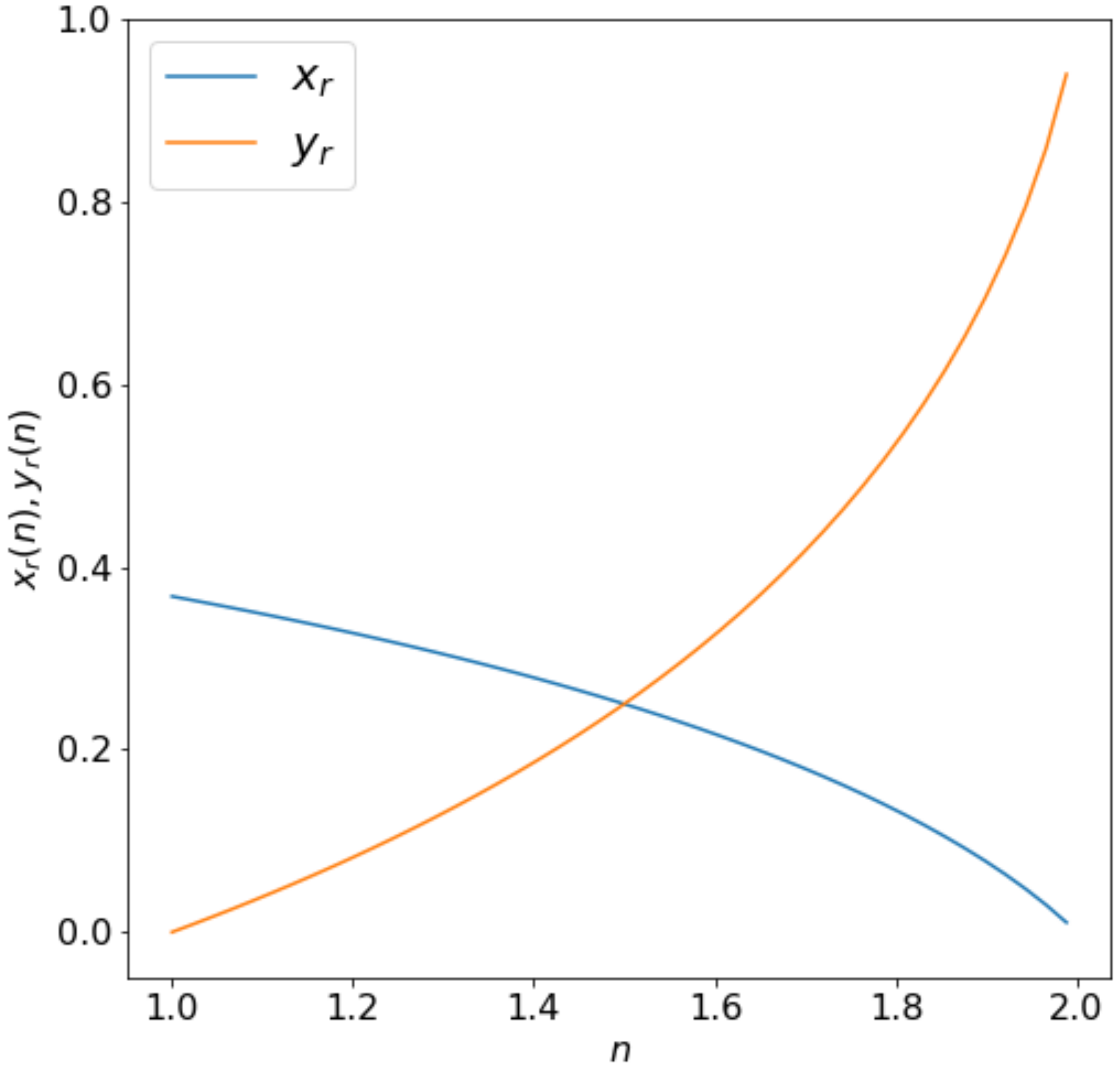
The caustic is the point y=0

$$y_{cau} = x_{crit} - \alpha(x_{crit}) \; ; \; \alpha(x) = x^{2-n} \;\; \Rightarrow y_{cau,t} = 0 \;\; \forall \, n$$

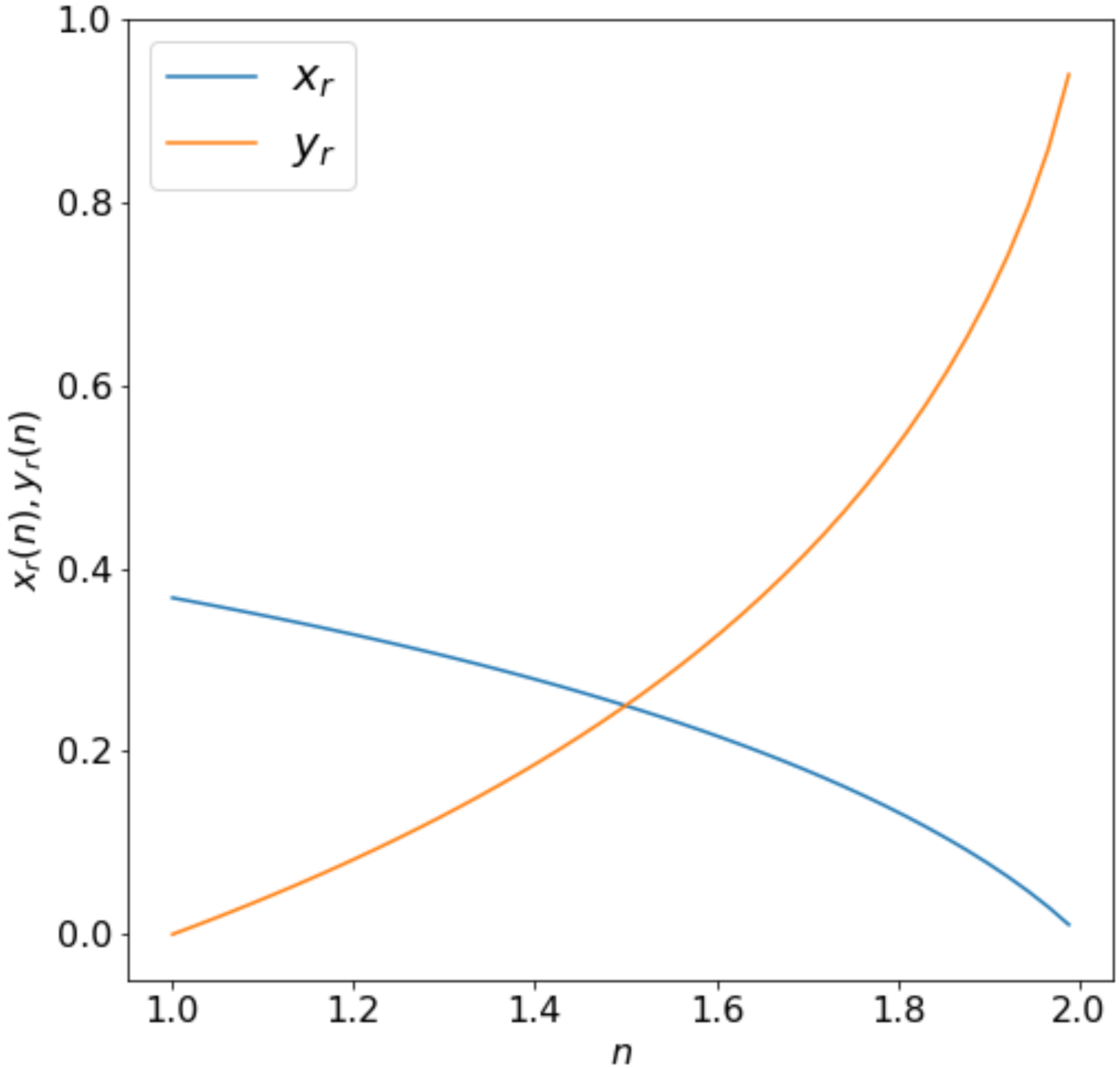
Actually, this is true for any axially symmetric lens!

$$\lambda_t = 1 - \frac{m(x)}{x^2} = 0 \;\Rightarrow y_{cau,t} = x_{crit,t} \left[1 - \frac{m(x_{crit,t})}{x^2_{crit,t}} \right] = 0$$

RADIAL CRITICAL LINE



RADIAL CRITICAL LINE



Large n, small radial critical line

AGAIN ON THE EXISTENCE OF THE RADIAL CRITICAL LINE

Another way to write the radial eigenvalue (see past lesson) is:

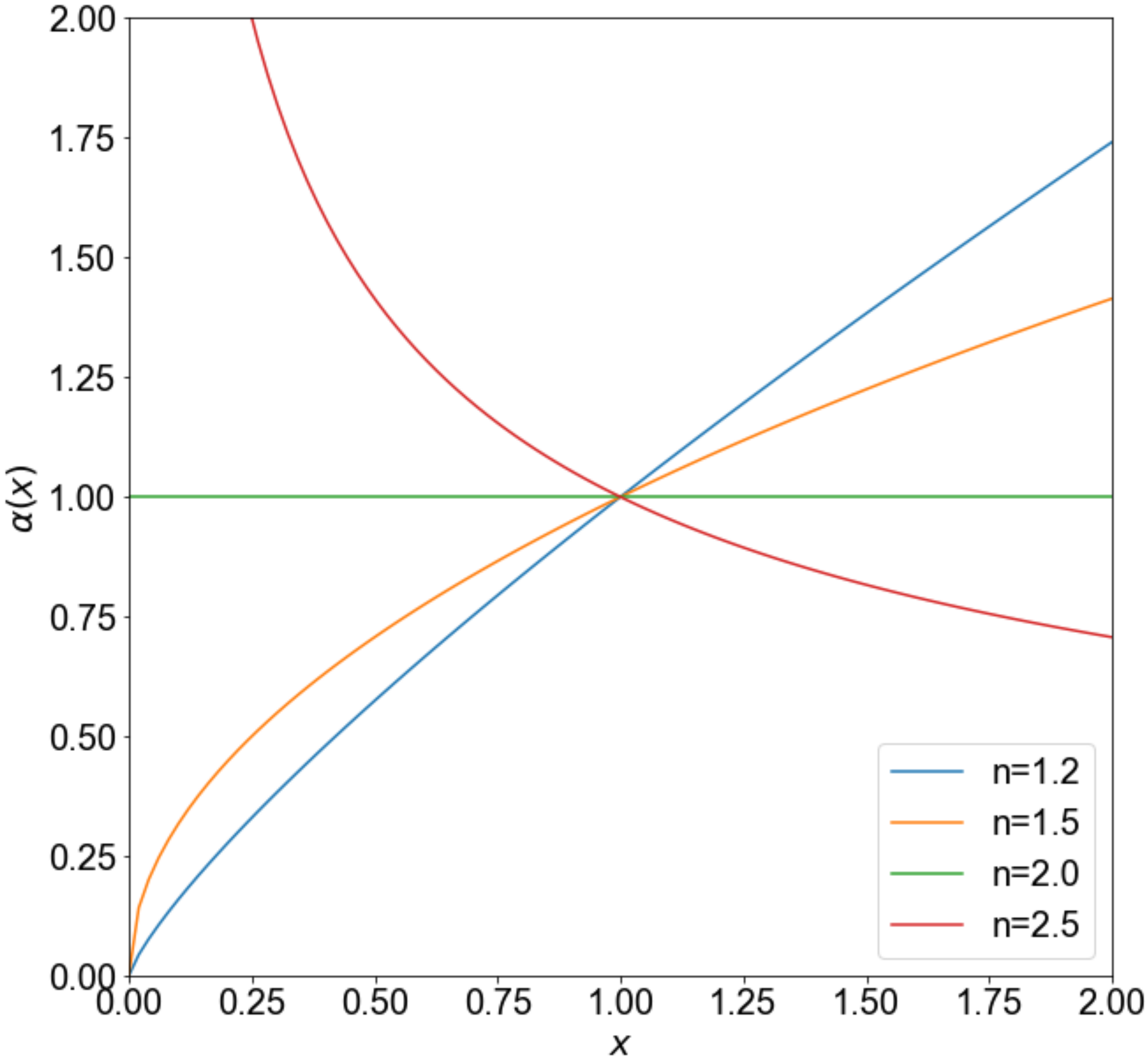
$$\lambda_r(x) = 1 - \alpha'(x)$$

This shows us that the radial critical line exists only if at some distance from the lens center

$\alpha'(x) = 1,$

i.e. the curve $\alpha(x)$ *is tangent to some line*

$f(x) = x + q$



AGAIN ON THE EXISTENCE OF THE RADIAL CRITICAL LINE

Another way to write the radial eigenvalue (see past lesson) is:

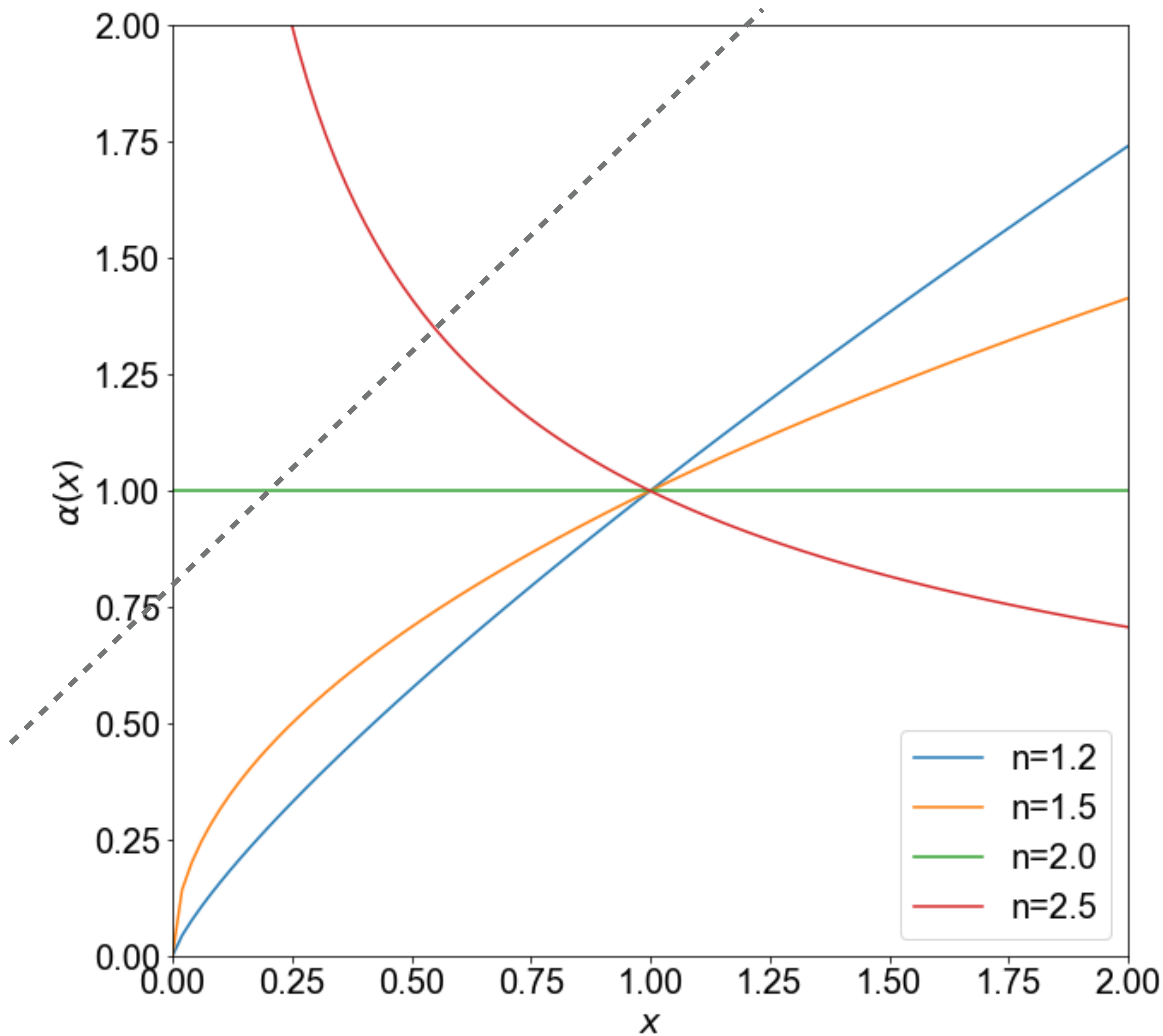
$$\lambda_r(x) = 1 - \alpha'(x)$$

This shows us that the radial critical line exists only if at some distance from the lens center

$\alpha'(x) = 1,$

i.e. the curve $\alpha(x)$ is tangent to some line

$f(x) = x + q$



AGAIN ON THE EXISTENCE OF THE RADIAL CRITICAL LINE

Another way to write the radial eigenvalue (see past lesson) is:

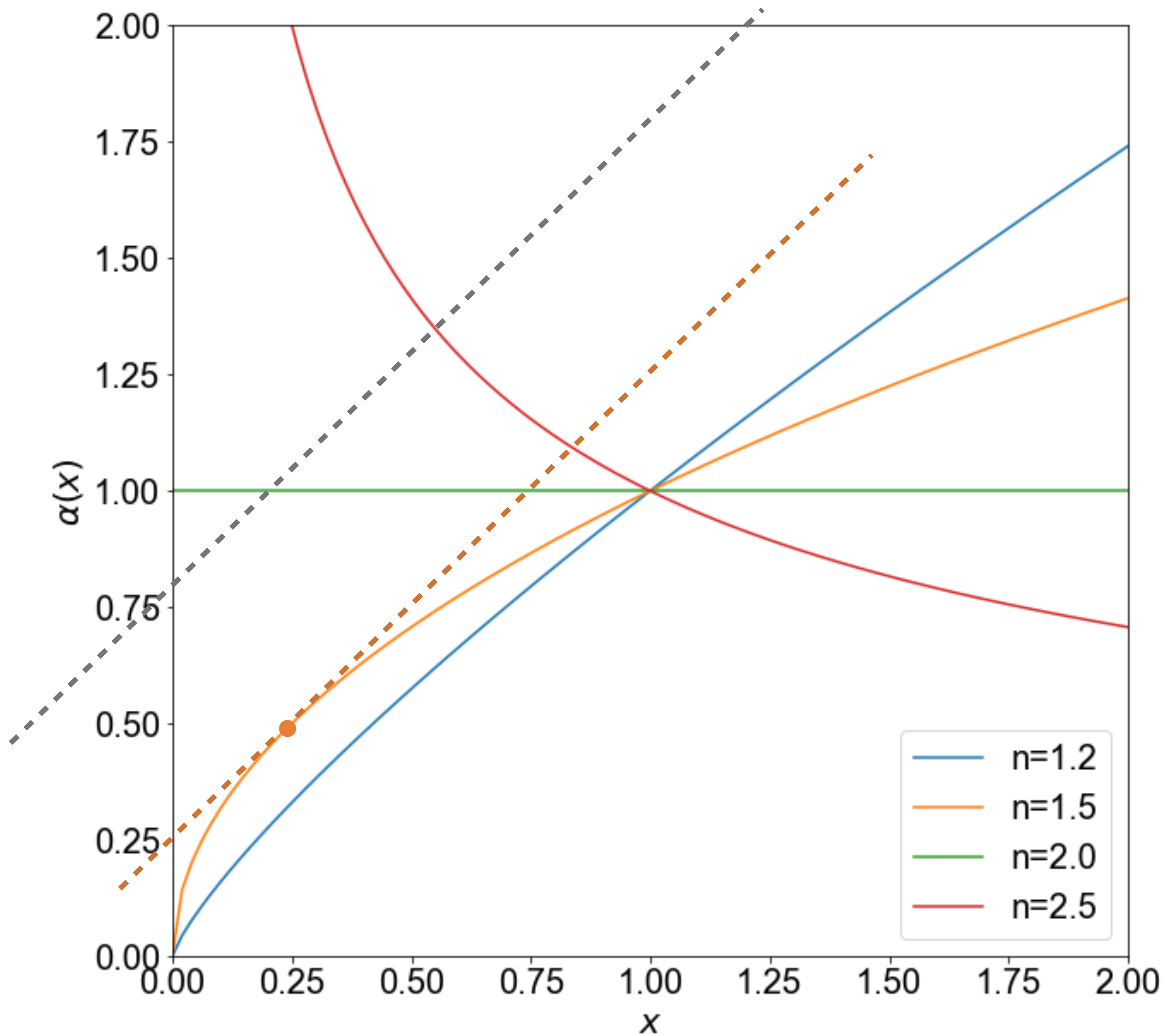
$$\lambda_r(x) = 1 - \alpha'(x)$$

This shows us that the radial critical line exists only if at some distance from the lens center

$\alpha'(x) = 1,$

i.e. the curve $\alpha(x)$ is tangent to some line

$f(x) = x + q$



AGAIN ON THE EXISTENCE OF THE RADIAL CRITICAL LINE

Another way to write the radial eigenvalue (see past lesson) is:

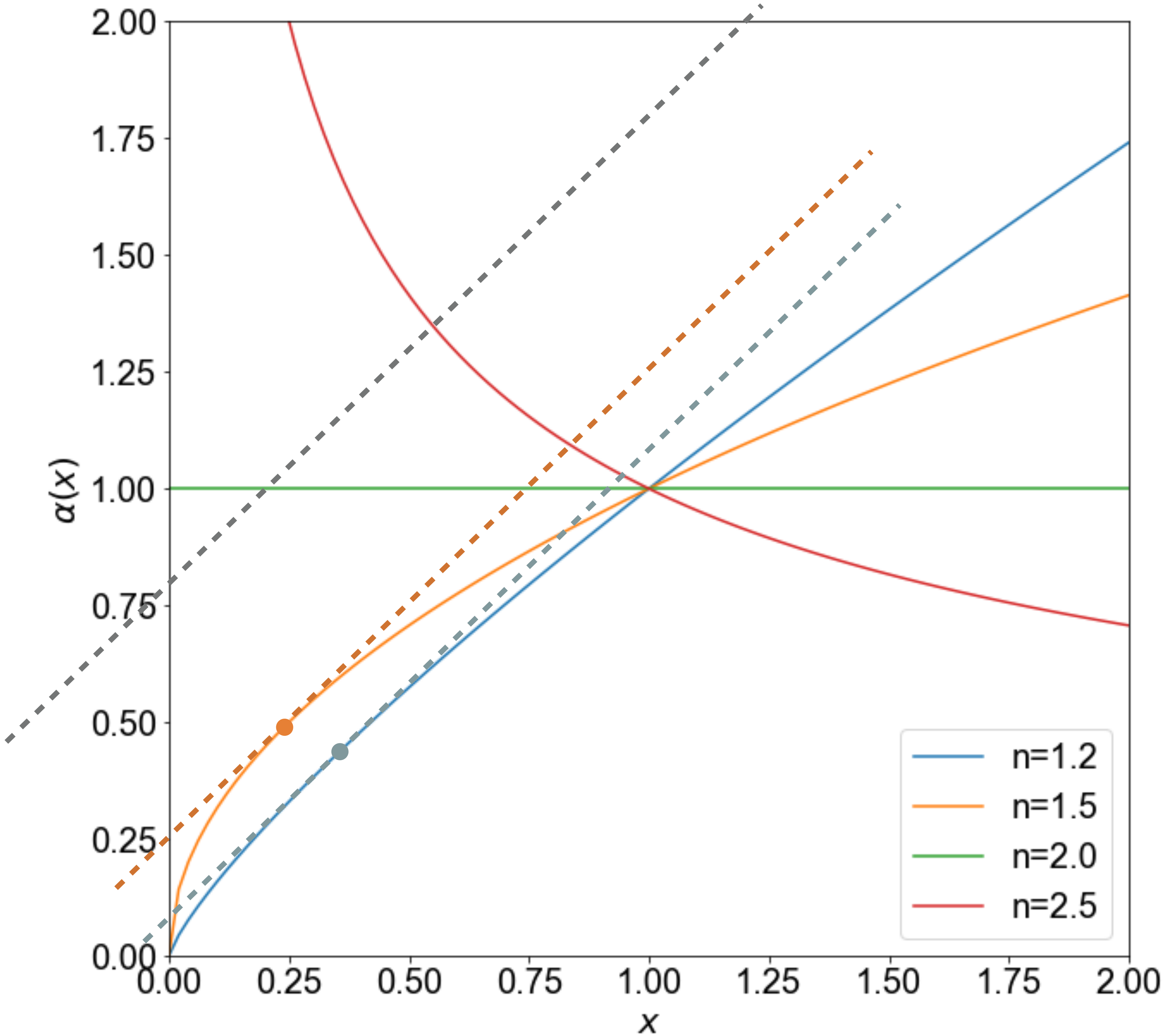
$$\lambda_r(x) = 1 - \alpha'(x)$$

This shows us that the radial critical line exists only if at some distance from the lens center

$\alpha'(x) = 1,$

i.e. the curve $\alpha(x)$ is tangent to some line

$f(x) = x + q$



HOW MANY IMAGES DOES A POWER-LAW LENS PRODUCE?

$$y = x - \alpha(x) \Rightarrow x^{2-n} - x - y = 0$$

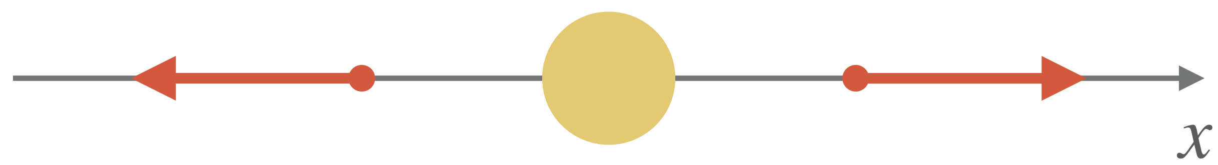
Apart from some special cases, solving analytically the lens equation is impossible.

However, we can determine graphically how many images a give source produce, by building the so called "image diagram"!

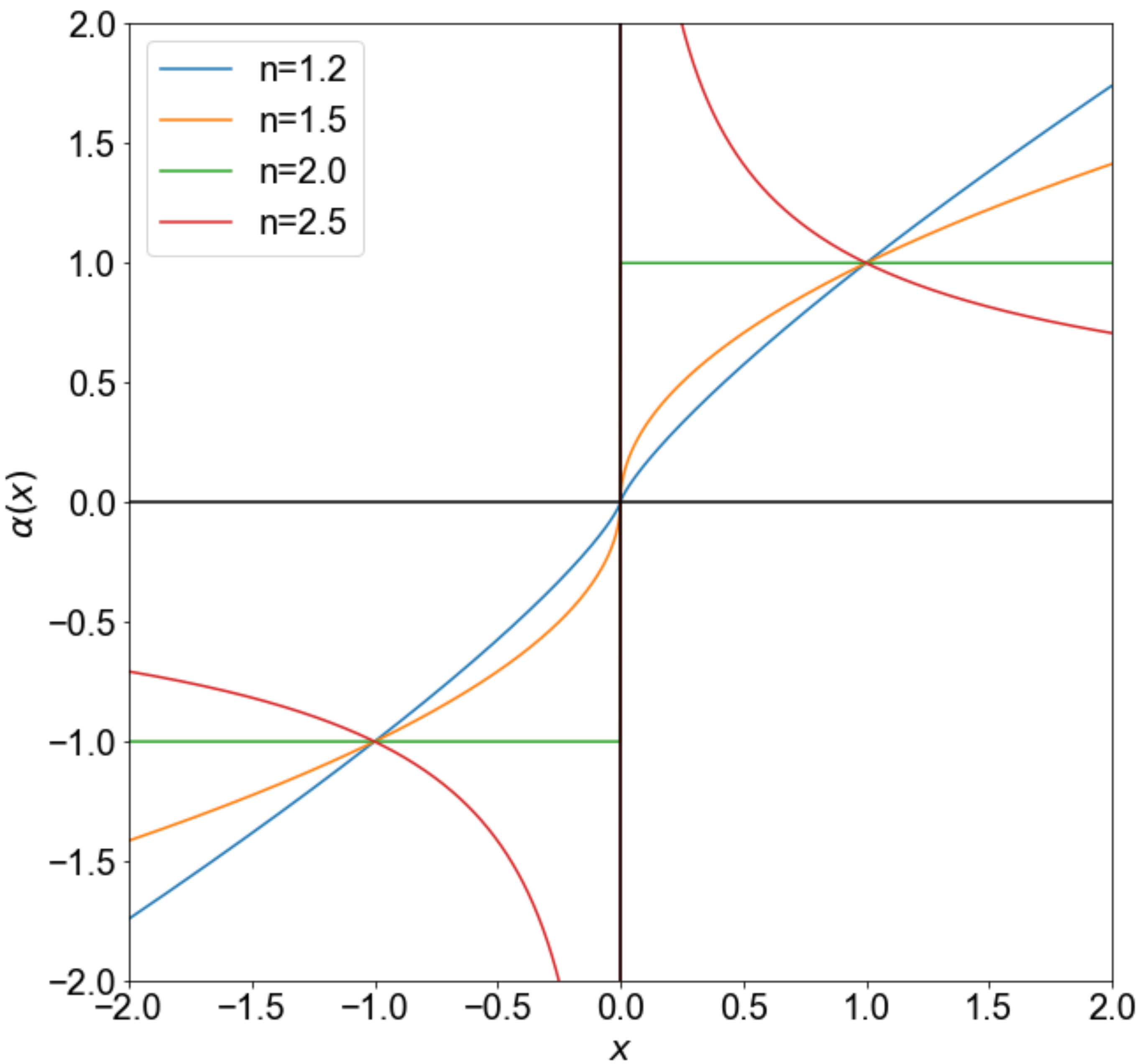
Method:

- Rewrite the lens equation as $\alpha(x) = x - y$
- It become obvious that, for a given source at y, the solutions of the lens equations are the points x where the line $f(x) = x - y$ intercepts the curve $\alpha(x)$

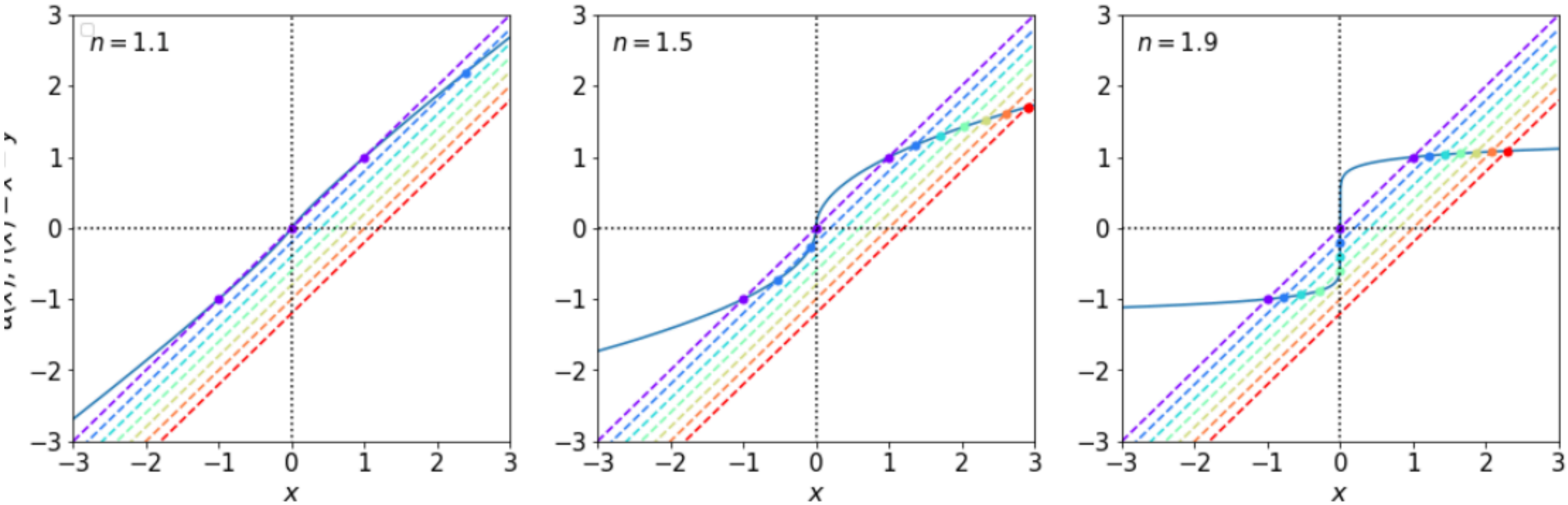
THE DOMAIN OVER WHICH $\alpha(x)$ IS DEFINED EXTENDS TO THE NEGATIVE x AXIS



$$\vec{\alpha}(x) = \alpha(x)\vec{e}_x$$



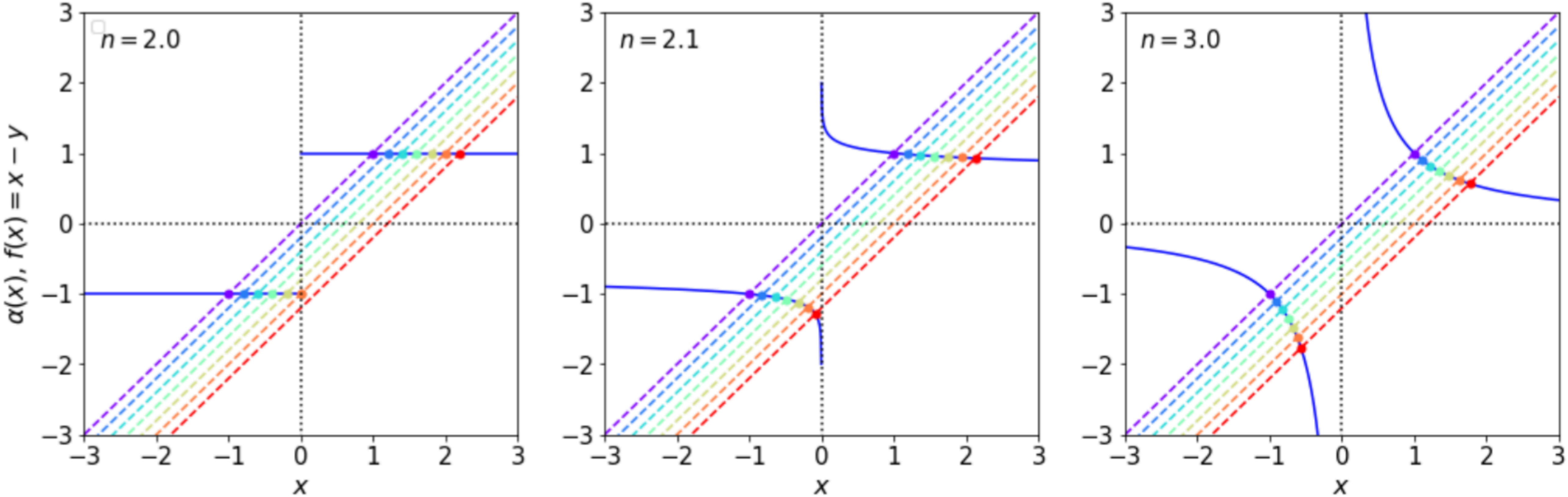
POWER-LAW LENSES : IMAGES



For $1 < n < 2$ there are 3 or 1 images.

3 images when the source is within the radial caustic.

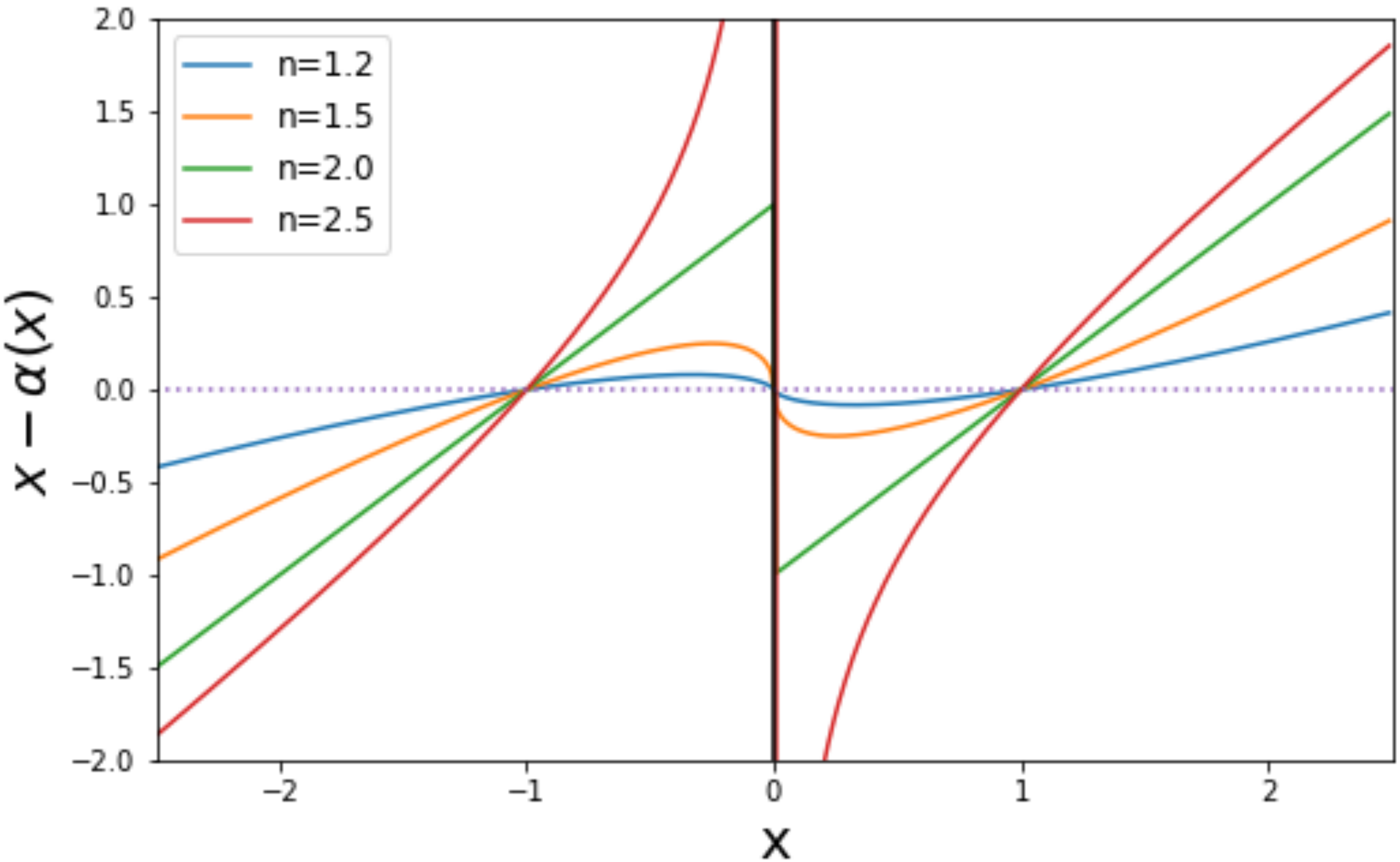
POWER-LAW LENSES : IMAGES



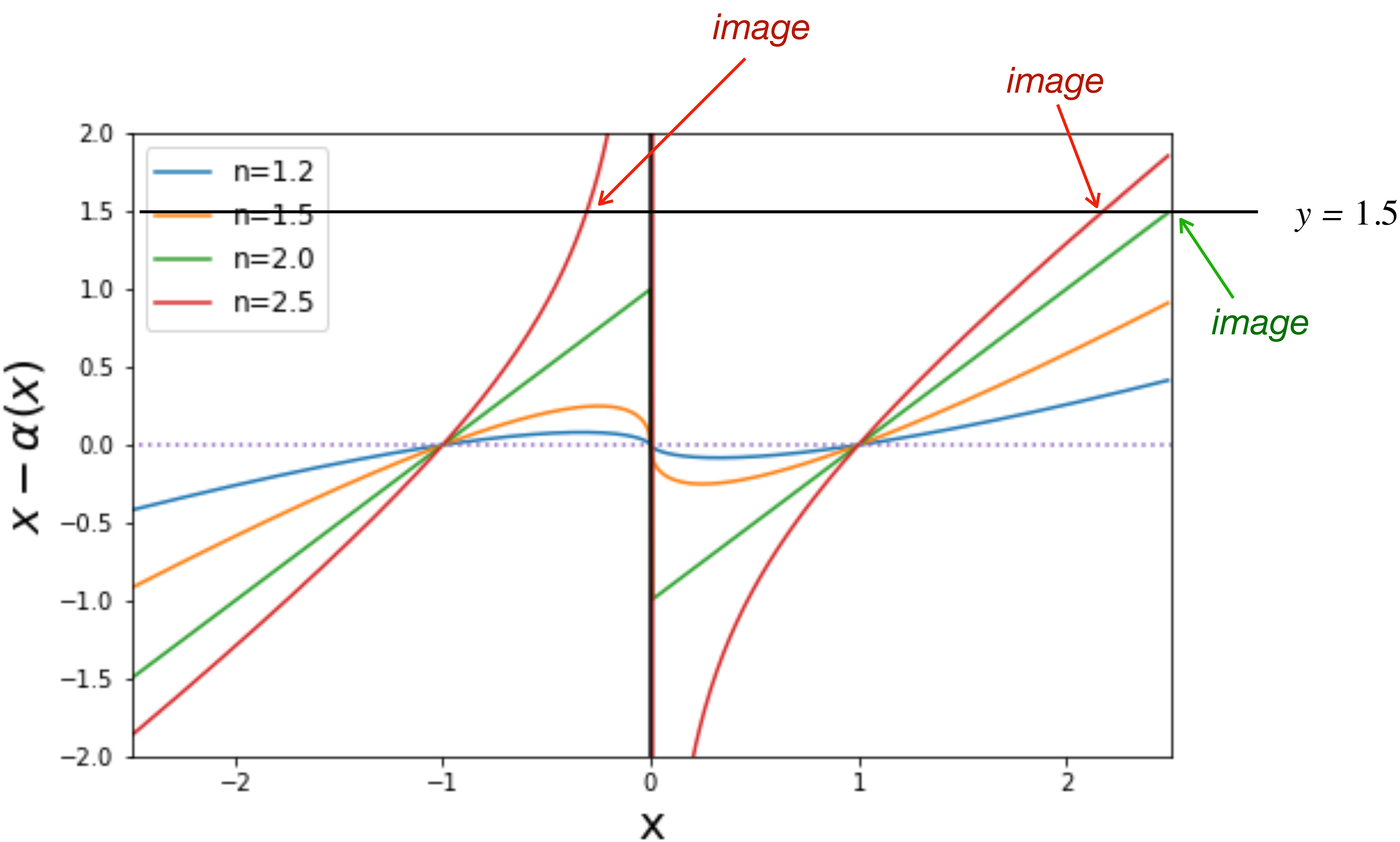
For $2 < n < 3$ there are always 2 images.

For $n = 3$ there are always 2 when $y < 1$ and 1 image when $y > 1$.

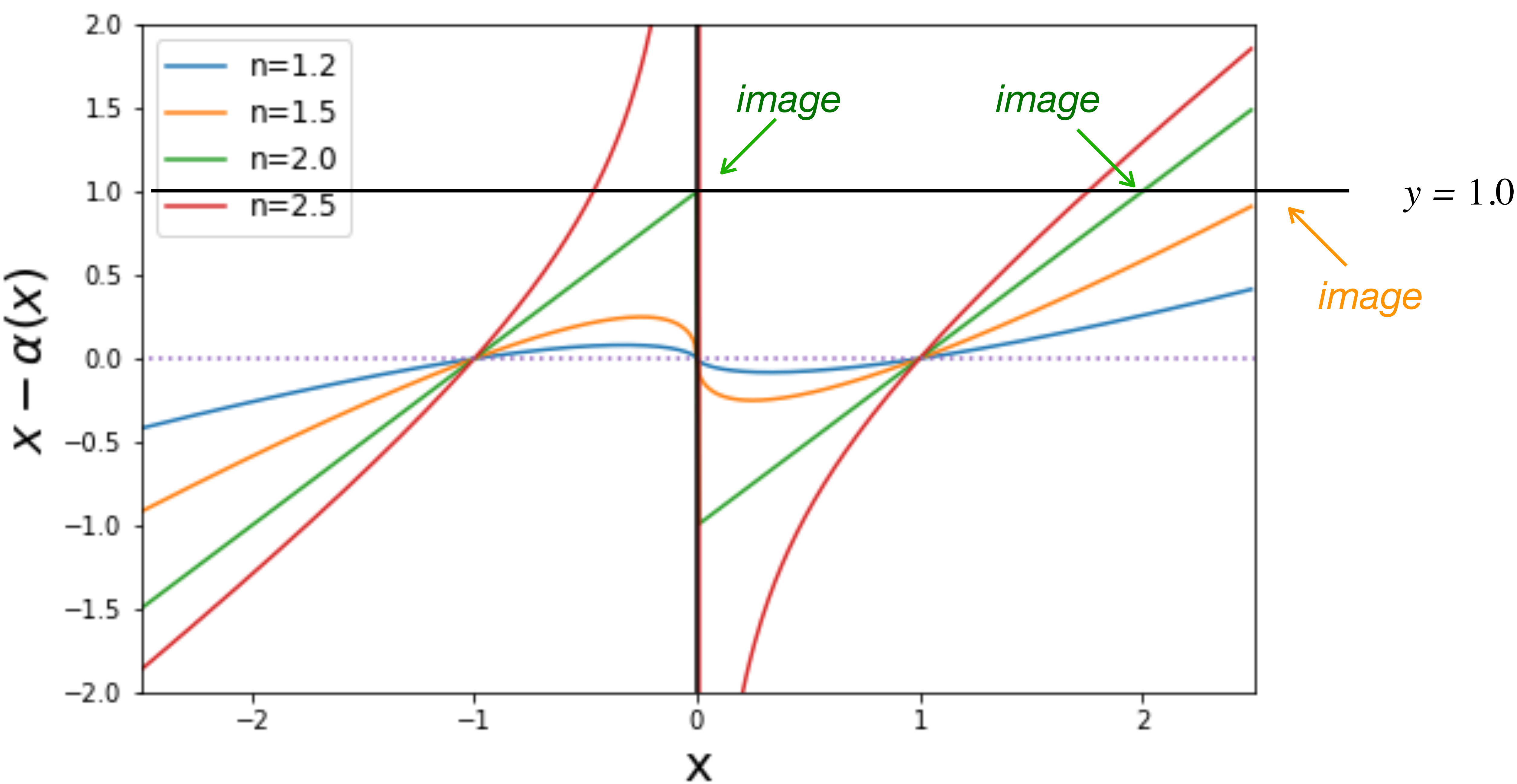
POWER-LAW LENSES : IMAGES



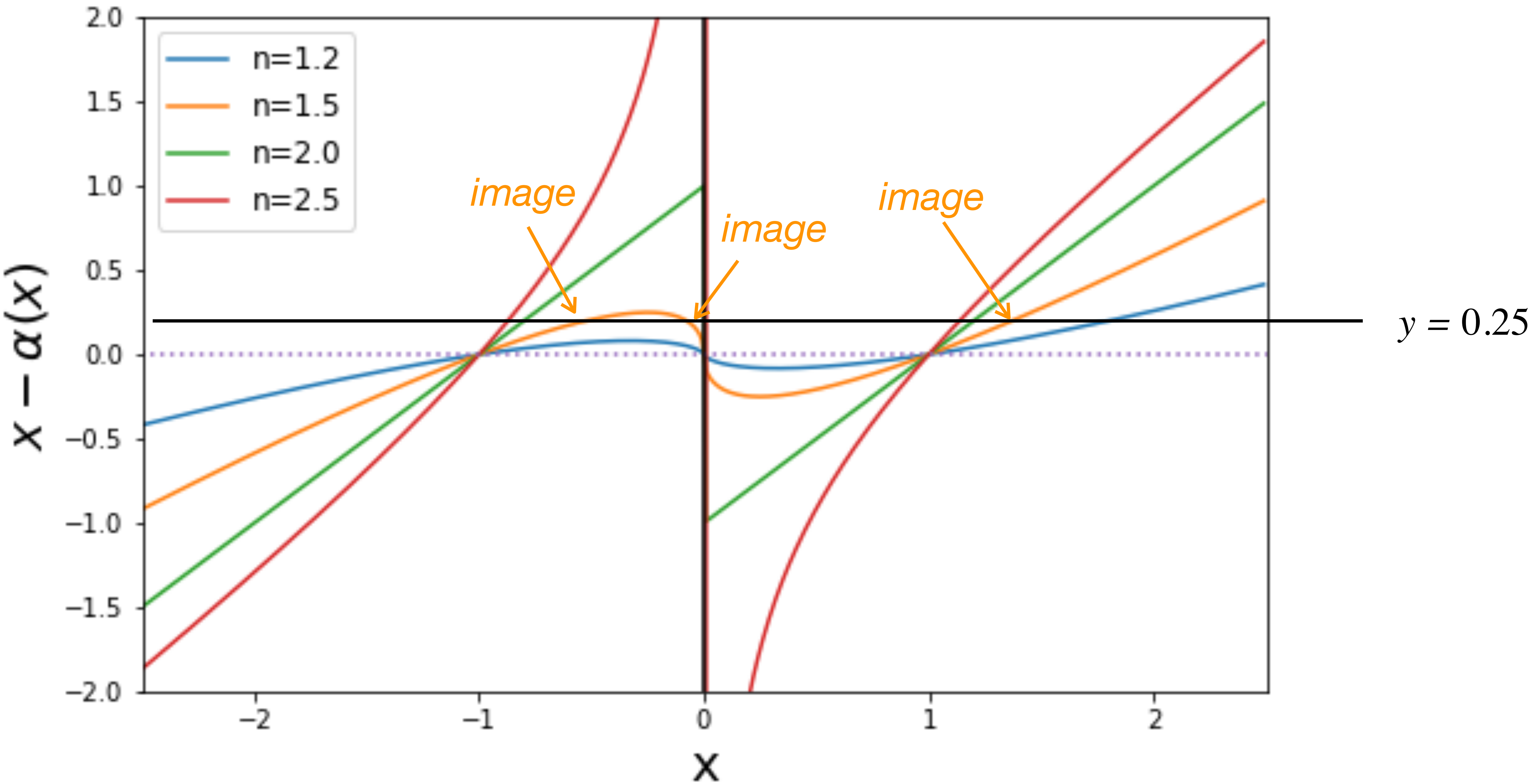
POWER-LAW LENSES : IMAGES



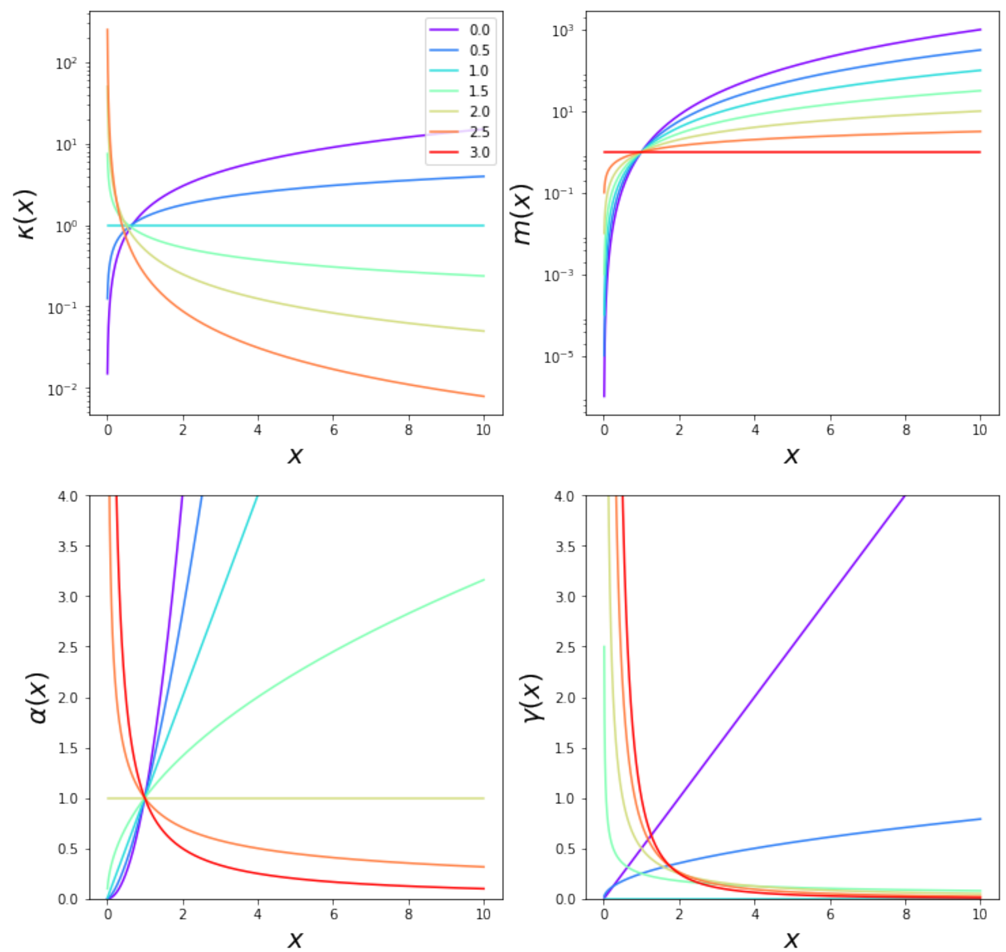
POWER-LAW LENSES : IMAGES

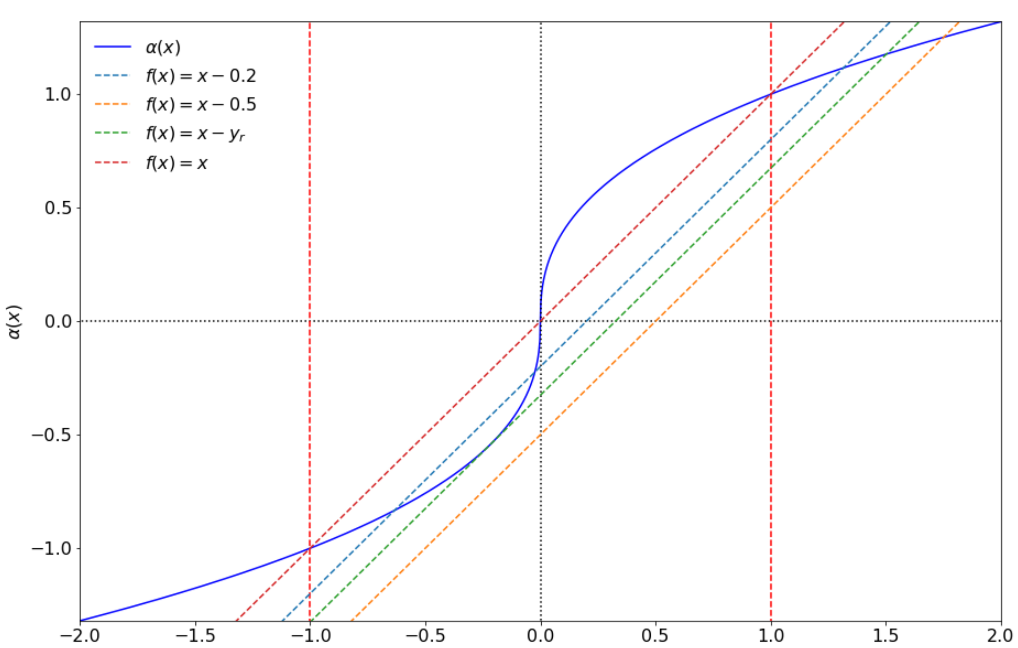


POWER-LAW LENSES : IMAGES



RECAP: POWER-LAW LENS (N<2)





$$m(x) = x^{3-n} \qquad \kappa(x) = \frac{m'(x)}{2x} = \frac{3-n}{2} x^{1-n}$$

$$\alpha(x) = \frac{m(x)}{x} = x^{2-n}$$

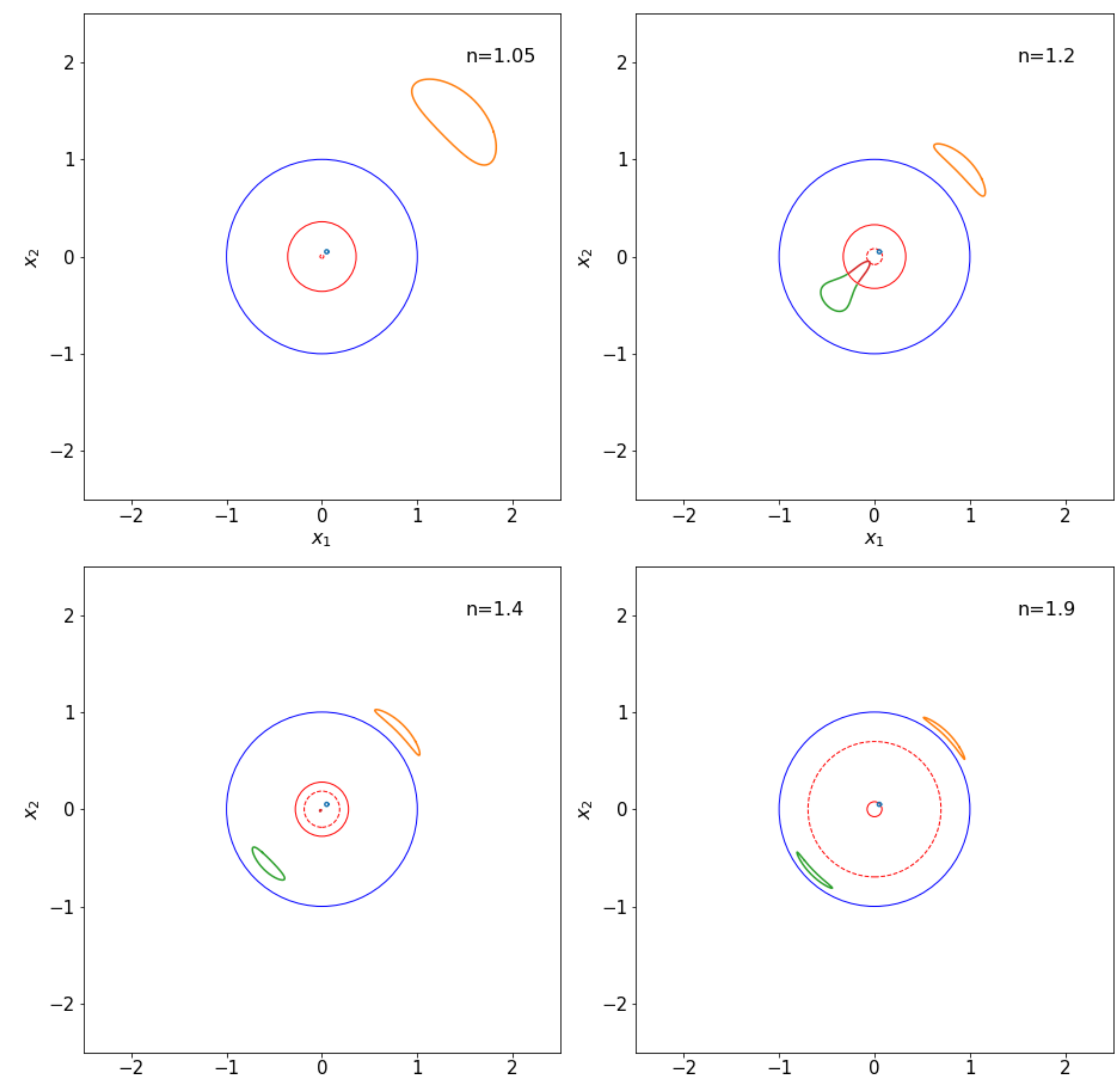
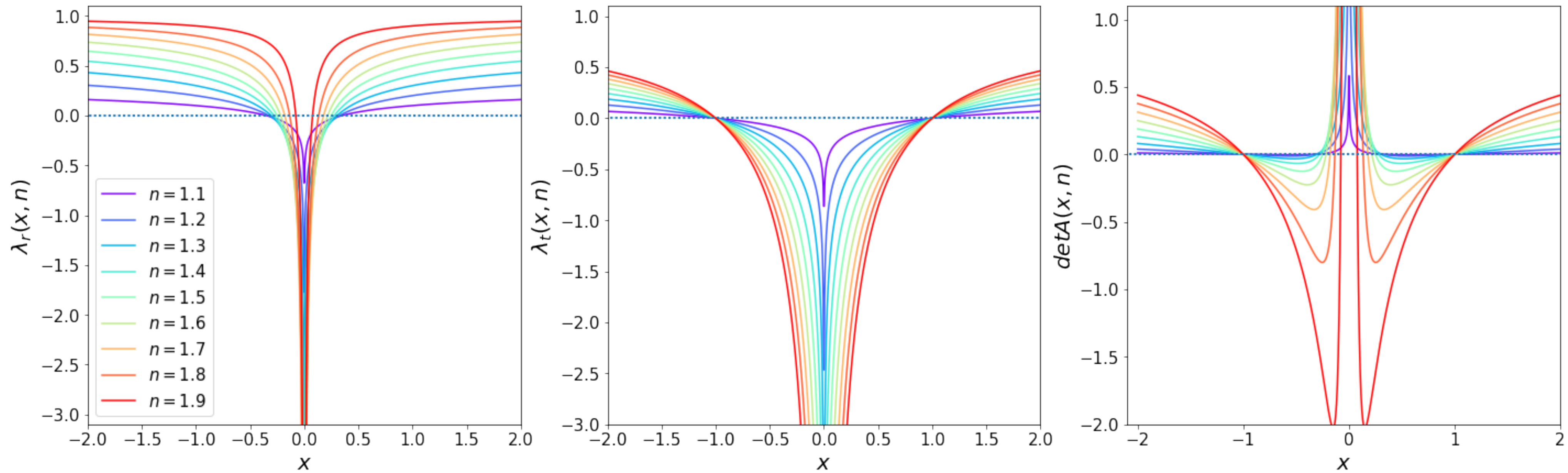


IMAGE PARITY



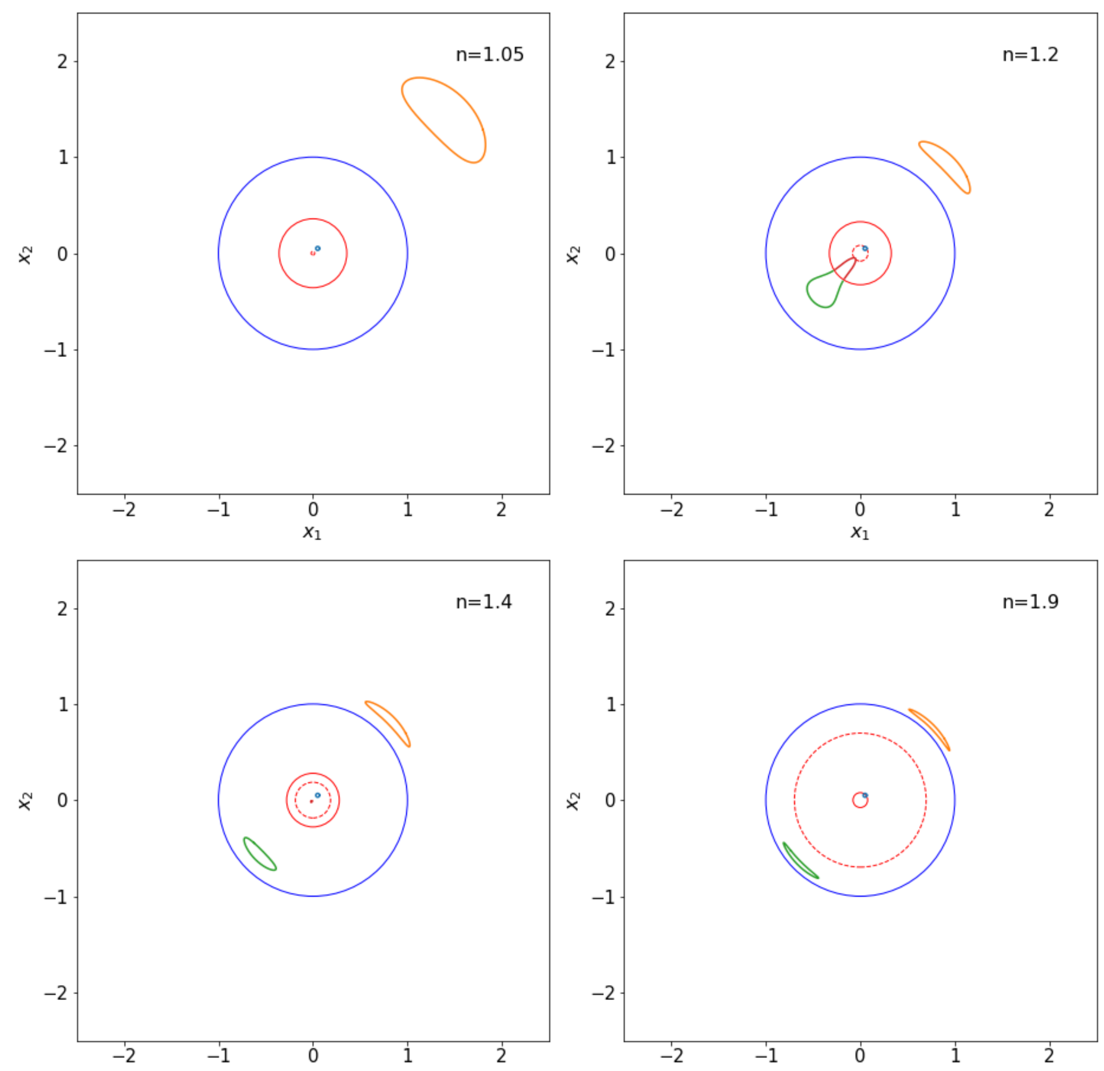
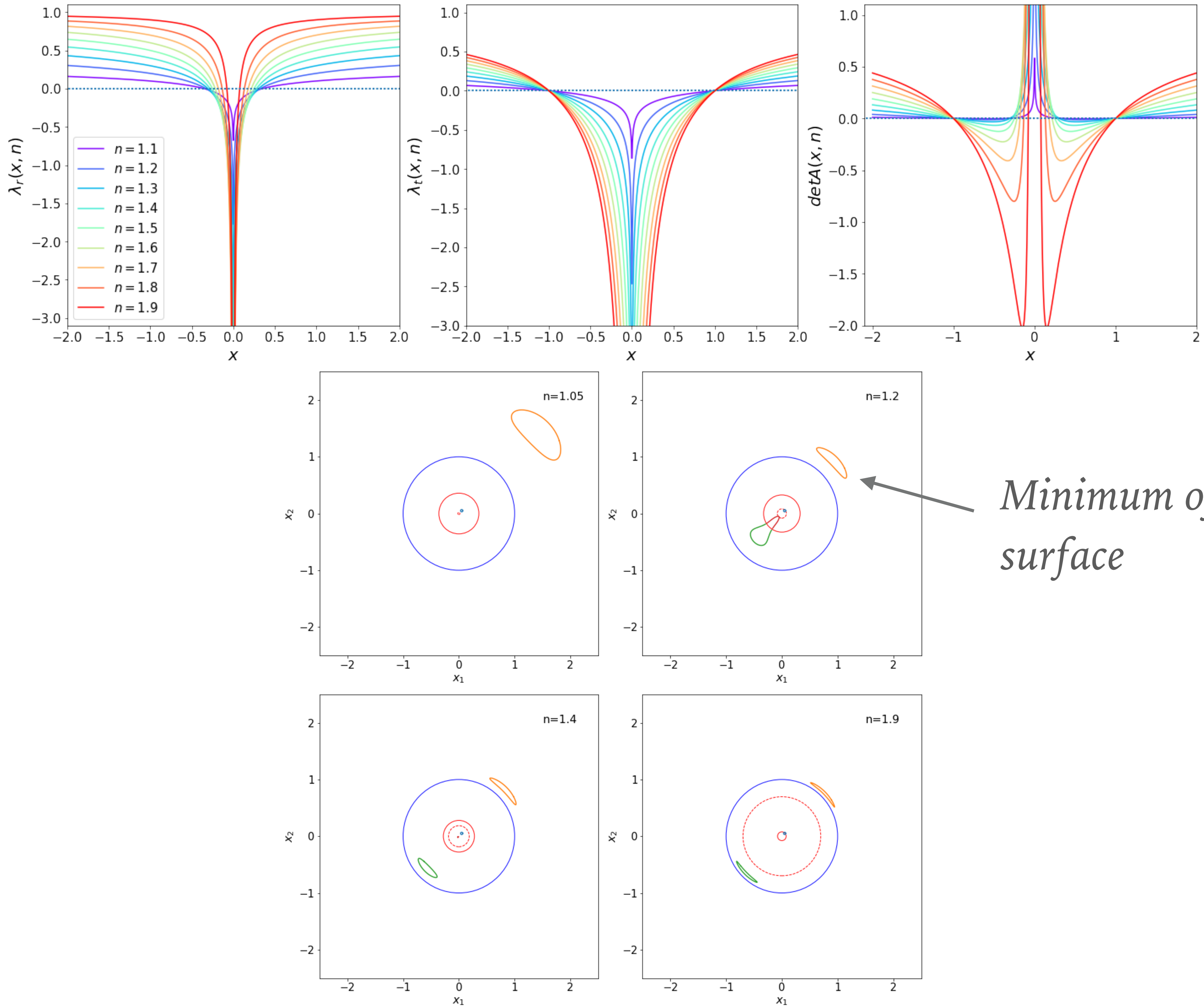


IMAGE PARITY



Minimum of TD surface

IMAGE PARITY

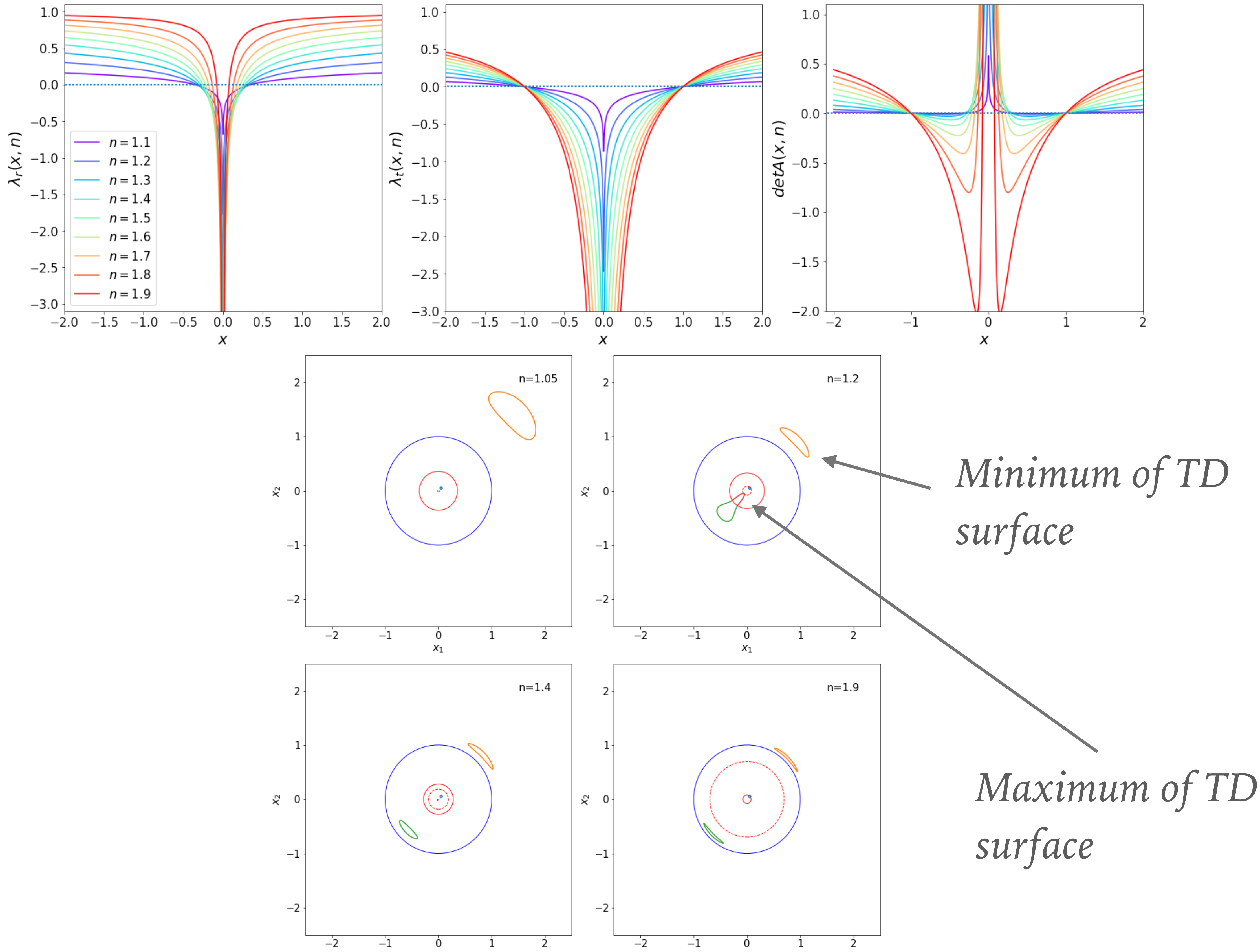
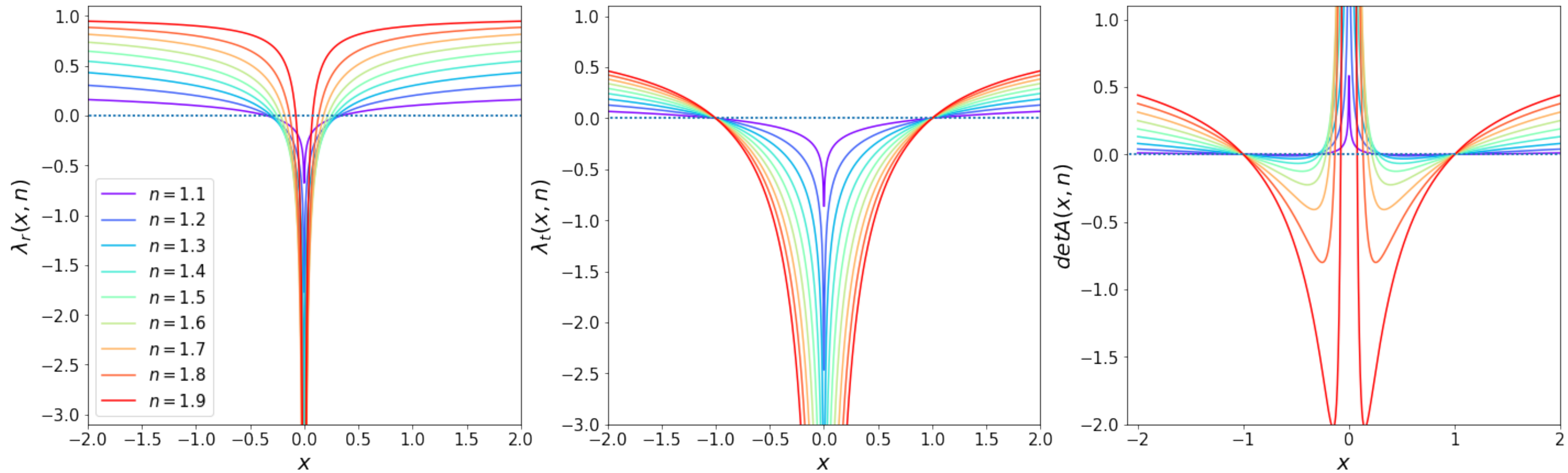
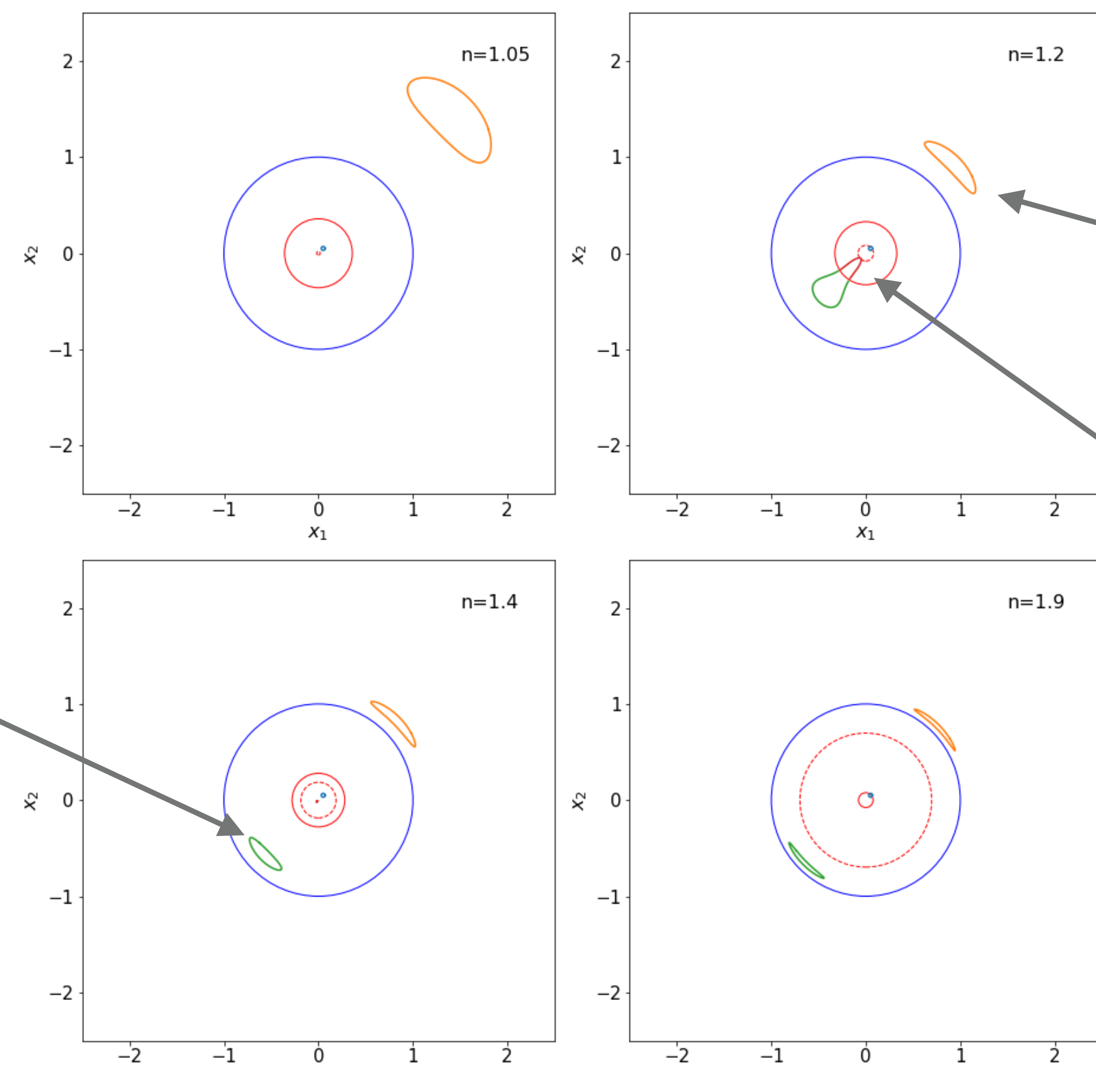


IMAGE PARITY





Minimum of TD surface

Saddle point of TD surface

Maximum of TD surface

POWER-LAW LENSES : IMAGE DISTORTION

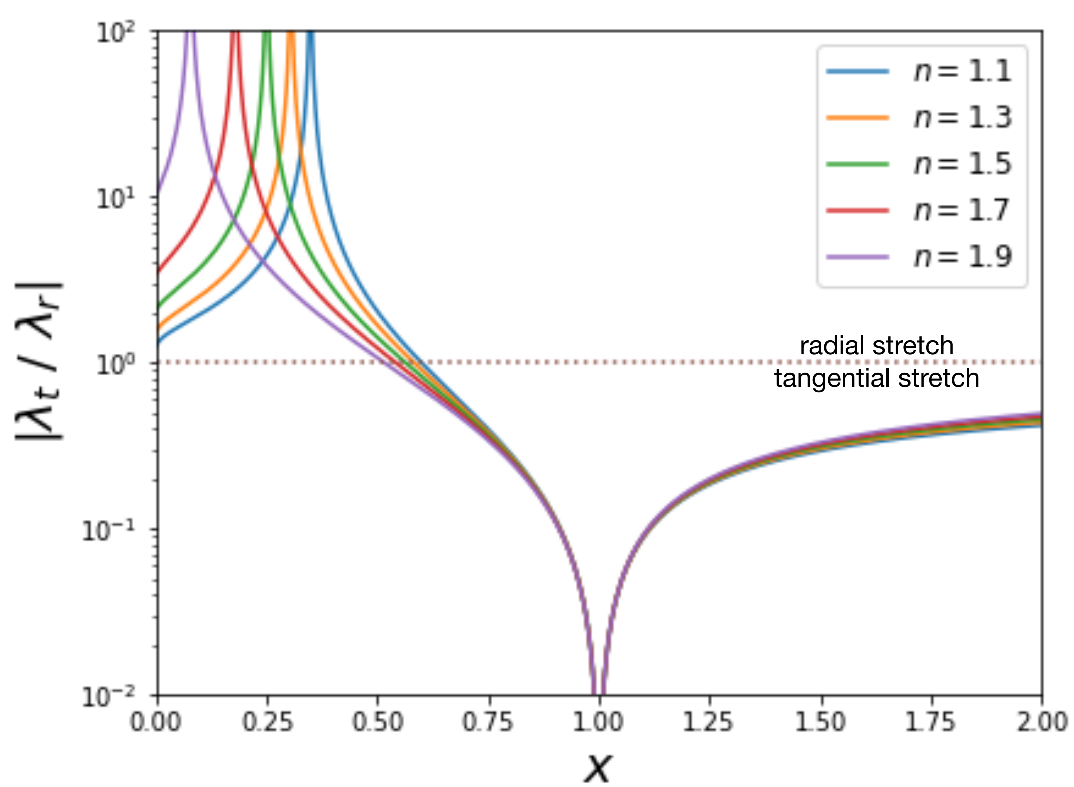
Distortion of infinitesimal images

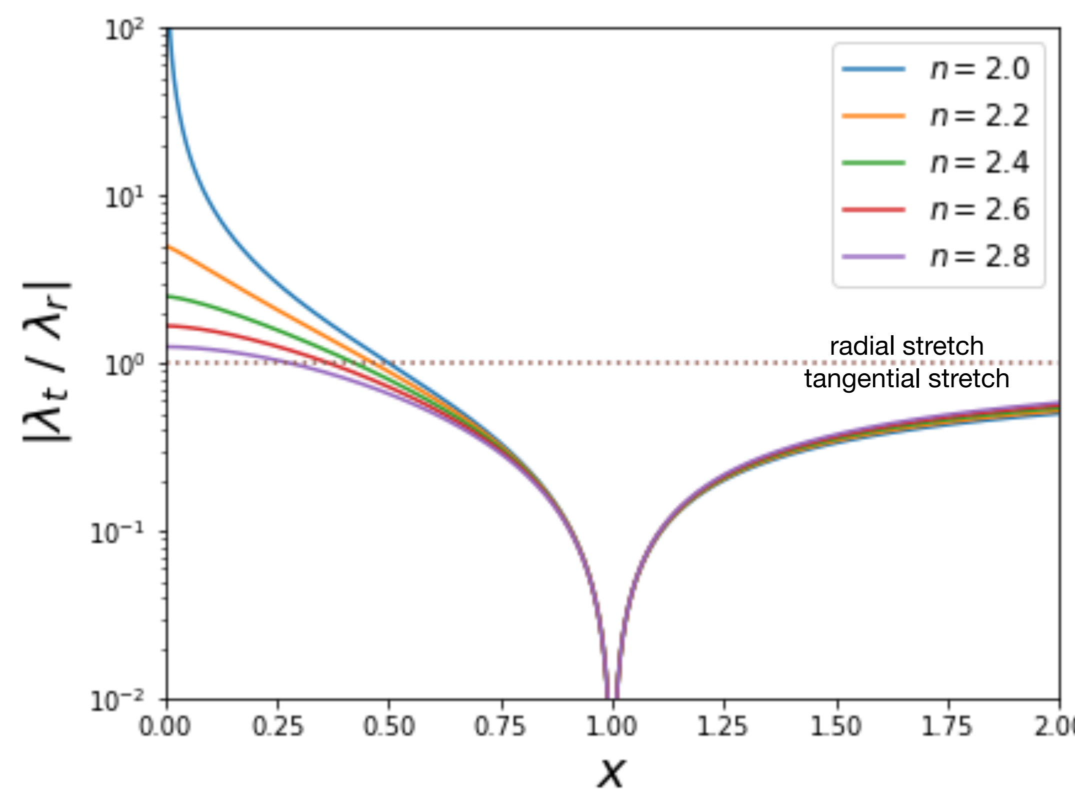
radial eigenvalue of A $\lambda_r = 1 - \alpha'(x)$

tangential eigenvalue of A $\lambda_t = 1 - \dfrac{\alpha(x)}{x}$

$\left|\dfrac{\lambda_t}{\lambda_r}\right| < 1$ *image is tangentially stretched*

$\left|\dfrac{\lambda_t}{\lambda_r}\right| > 1$ *image is radially stretched*





POWER-LAW LENSES : TIME-DELAY

$$t(\boldsymbol{x}) = \frac{(1+z_l)}{c}\frac{D_l D_s}{D_{ls}}\left(\frac{\xi_o}{D_l}\right)^2\left[\frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2 - \Psi(\boldsymbol{x})\right]$$

ξ_o *- units in which x,y are measured*

$$t(\boldsymbol{x}) = \frac{(1+z_l)}{c}\frac{D_l D_s}{D_{ls}}\left(\frac{\xi_o}{D_l}\right)^2\left[\frac{1}{2}|\boldsymbol{\alpha}(\boldsymbol{x})|^2 - \Psi(\boldsymbol{x})\right]$$

$$= \frac{(1+z_l)D_{\Delta t}}{c}\ \tau(\boldsymbol{x})$$

for a power-law lens $\Psi(\boldsymbol{x}) = \dfrac{x^{3-n}}{3-n}$ $\alpha(x) = x^{2-n}$

relative time-delay

$$\tau_{iA} = \tau_i - \tau_A = \left(\frac{\xi_o}{D_l}\right)^2\left[\frac{1}{2}\left(|x_i|^{2(2-n)} - |x_A|^{2(2-n)}\right) - \frac{1}{3-n}\left(|x_i|^{3-n} - |x_A|^{3-n}\right)\right]$$

Images that have the same separation will have a time delay that depends on the slope of the lens, n.

Generally, steeper mass profiles will have longer time-delays for the same image separation.

GRAVITATIONAL LENSING

14 - LENS MODELS: ISOTHERMAL SPHERE

R. Benton Metcalf

2022-2023

THE SINGULAR ISOTHERMAL SPHERE

The Singular Isothermal Sphere is a simple model to describe the distribution of matter in galaxies and clusters. It can be derived assuming that the matter content of the lens behaves like an ideal gas confined by a spherically symmetric gravitational potential. If the gas is isothermal and in hydrostatic equilibrium, its density profile is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

velocity dispersion of the gas particles

The profile has some good properties:

- *simple*
- *Reproduces the flat rotation curves of galaxies*

The profile is "unphysical"

- *singularity near the center*
- *mass is infinite*

THE SINGULAR ISOTHERMAL SPHERE

For lensing purposes, we are interested in the projection of this profile:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2} \qquad\qquad r^2 = \xi^2 + z^2$$

$$\Sigma(\xi) = 2\frac{\sigma_v^2}{2\pi G}\int_0^\infty \frac{dz}{\xi^2 + z^2} = \frac{\sigma_v^2}{\pi G \xi} \arctan \frac{z}{\xi}\Bigg|_0^\infty = \frac{\sigma_v^2}{2G\xi}$$

Using angular units: $\Sigma(\theta) = \dfrac{\sigma_v^2}{2GD_L\theta}$

Now we can compute the convergence:

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{cr}} \qquad\qquad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

$$\Rightarrow \kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{cr}} = \frac{\sigma_v^2}{2GD_L\theta}\frac{4\pi G}{c^2}\frac{D_L D_{LS}}{D_S} = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}\frac{1}{2\theta}$$

THE SINGULAR ISOTHERMAL SPHERE

$$\kappa(\theta) = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}\frac{1}{2\theta}$$

If we set $\theta_0 = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}$ *and* $x = \frac{\theta}{\theta_0}$, *then* $\kappa(x) = \frac{1}{2x}$

Remember that, for a power-law lens, $\kappa(x) = \frac{m'(x)}{2x} = \frac{3-n}{2}x^{1-n}$

Thus, the SIS lens is a power-law lens with n=2, and θ_0 *is the Einstein radius!*

$$\theta_{E,SIS} = \theta_0 = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}$$

Note that $\kappa(\theta_E) = 1/2$ *for a SIS lens.*

THE SINGULAR ISOTHERMAL SPHERE

We can use the usual formulas to derive all the relevant lens quantities. For example:

$$m(x) = x$$

We can compute the deflection angle as

$$\alpha(x) = \frac{m(x)}{x} = \frac{|x|}{x}$$

This has an easy form, so that we can solve the lens equation analytically. Indeed, the lens equation is: $y = x - \frac{|x|}{x}$ *,which can be split into*

$y = x - 1$ *for* $x>0$*, and* $y = x + 1$ *for* $x<0$

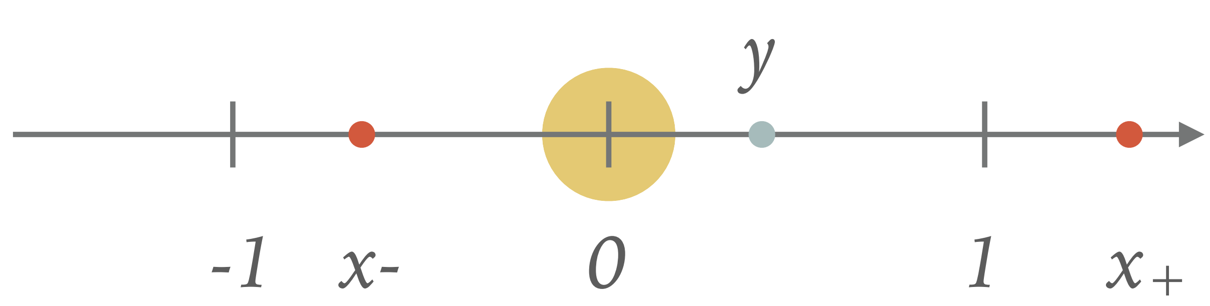
Two solutions exist only if $|y|<1$*!*

THE SINGULAR ISOTHERMAL SPHERE

$y = x - 1$ *for* $x>0$*, and* $y = x + 1$ *for* $x<0$

The solutions of the lens equation are:

$$x_+ = y + 1 \qquad\qquad x_- = y - 1$$

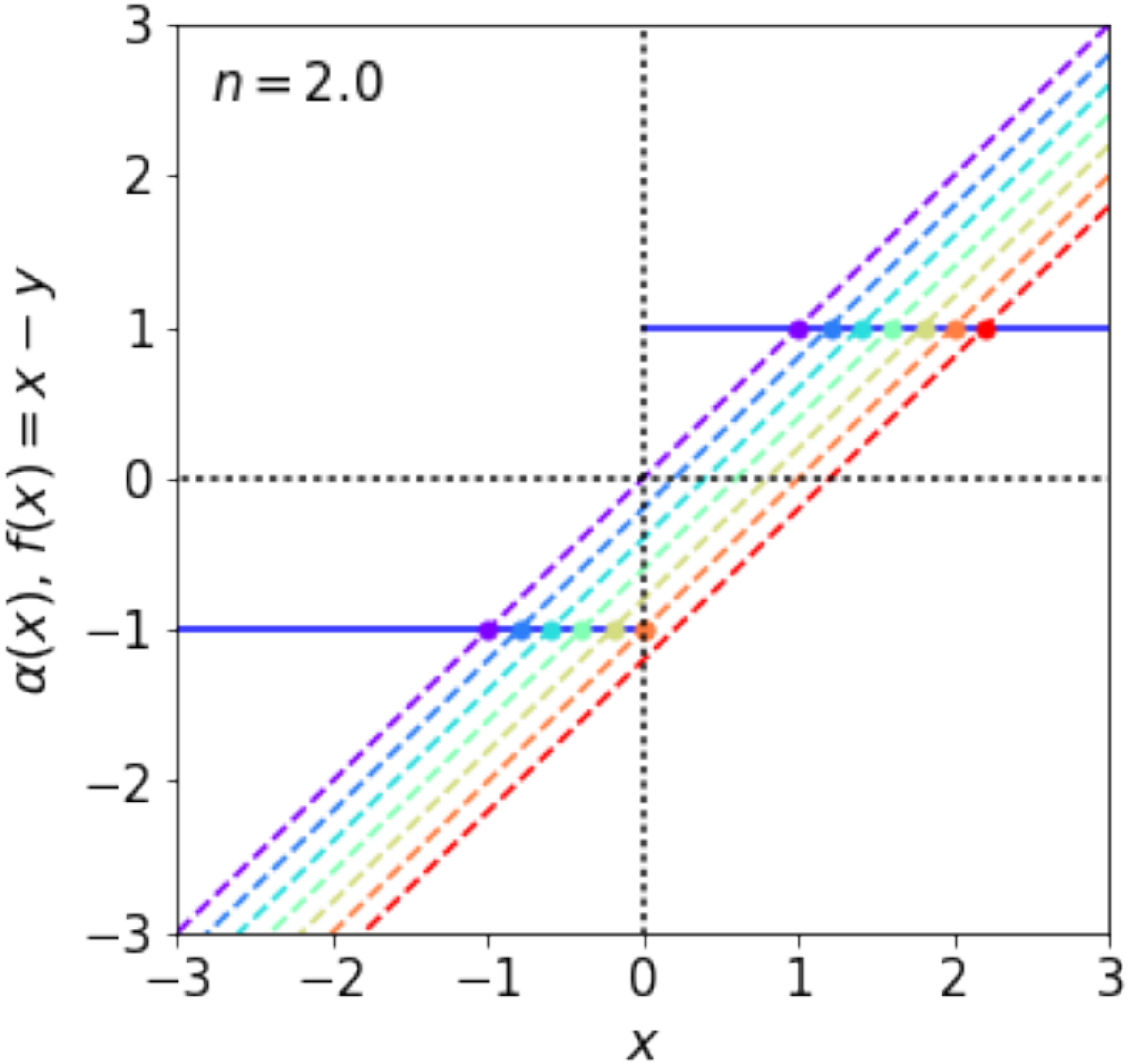


Going back to the angular units: $x = \frac{\theta}{\theta_E}$ $\quad y = \frac{\beta}{\theta_E}$

$$\theta_- = \beta - \theta_E$$

$$\theta_+ = \beta + \theta_E$$

IMAGE DIAGRAM (SIS)

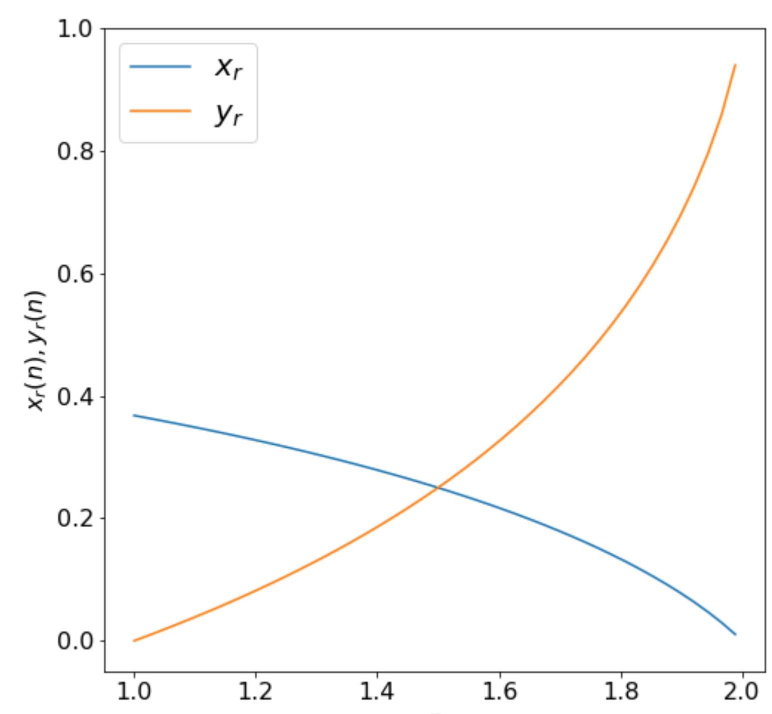


A limiting case of radial caustic for $n \to 2$

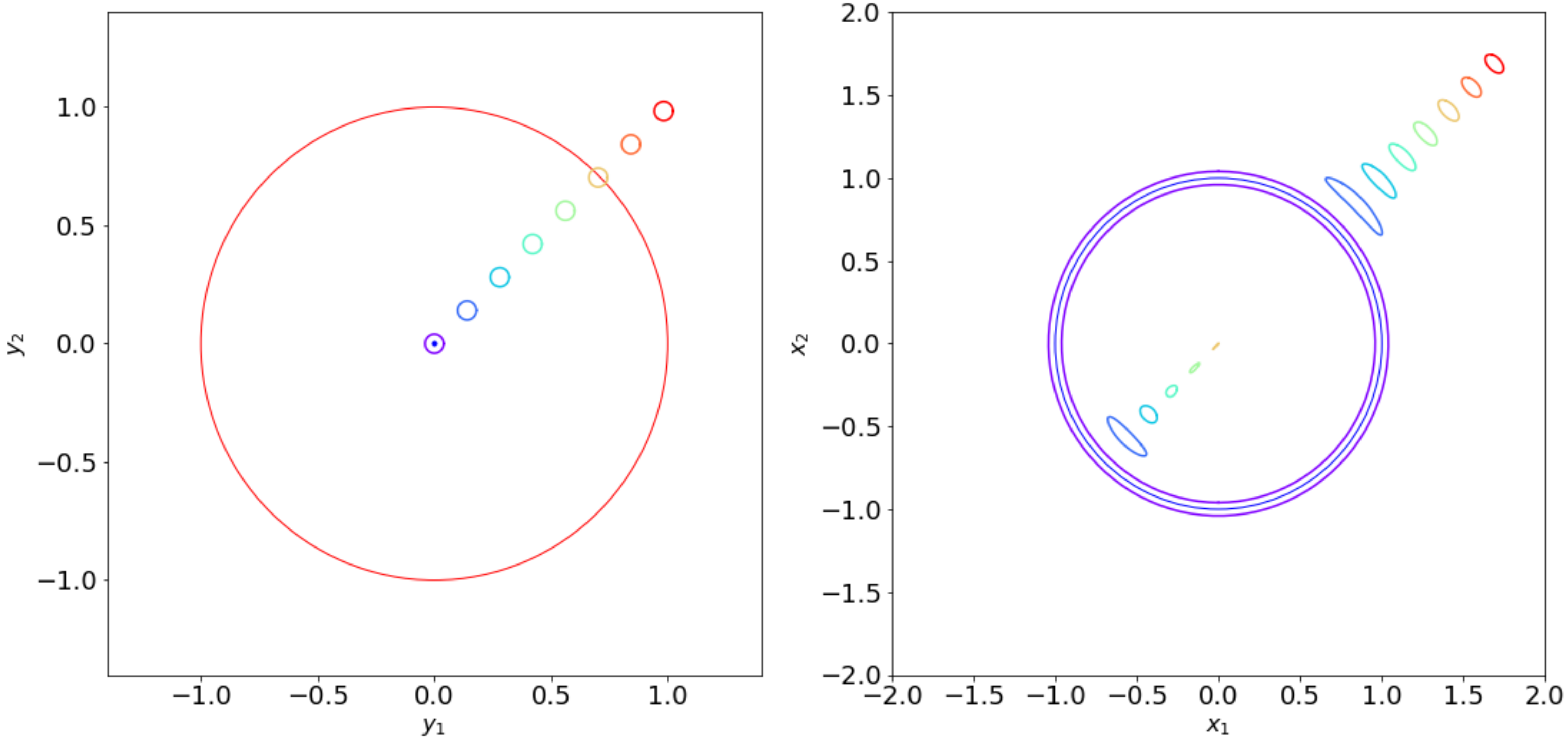
$$\frac{d\alpha}{dx} = 0 \ \ \forall\, x$$

There is no radial critical line!

However, there is a line that almost plays the role of the caustic (at least to determine the image multiplicity…)



THE SINGULAR ISOTHERMAL SPHERE



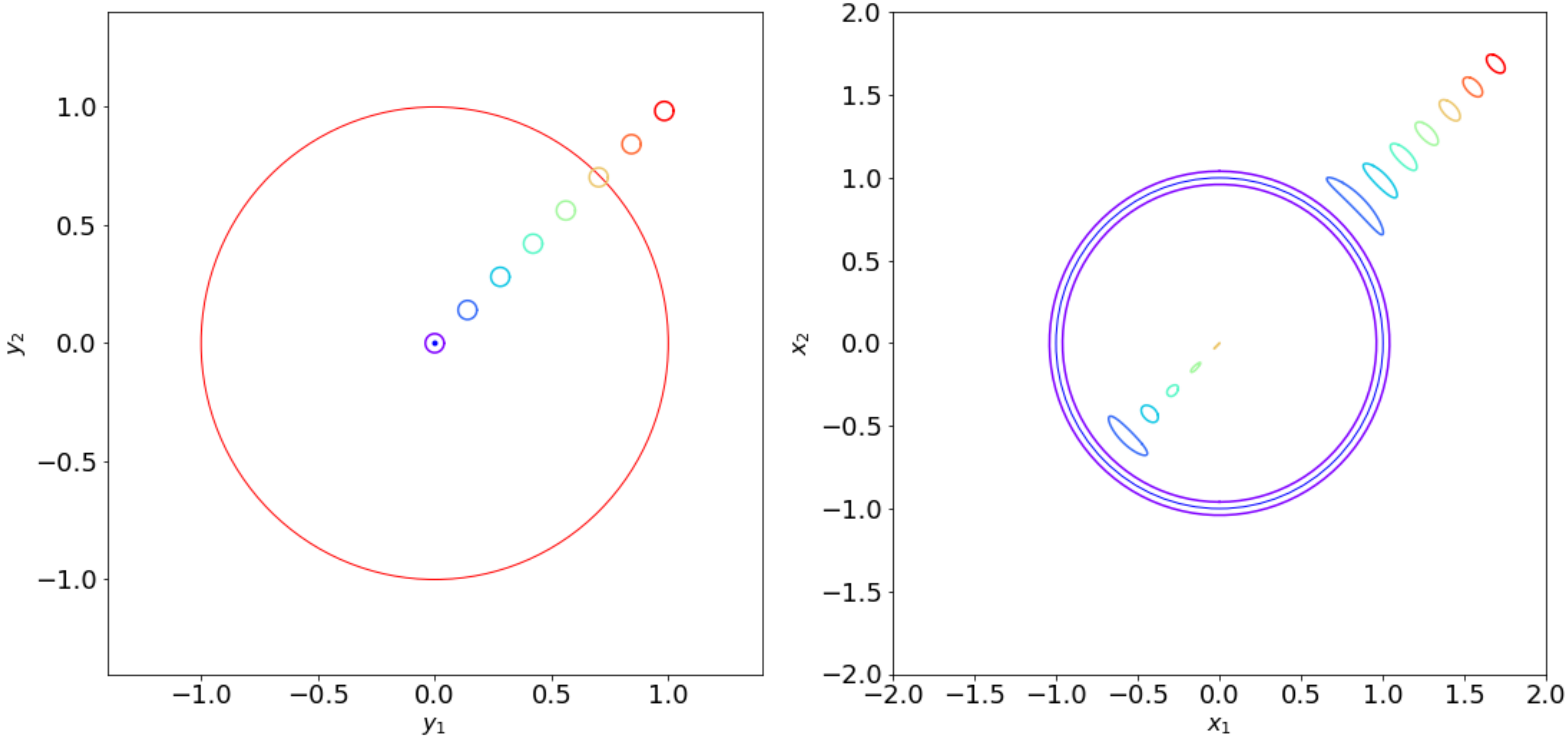
THE SINGULAR ISOTHERMAL SPHERE

$$\frac{d\alpha}{dx} = 0 \ \ \forall \ x$$

This implies that the radial eigenvalue of the Jacobian matrix is always $\lambda_r = 1$

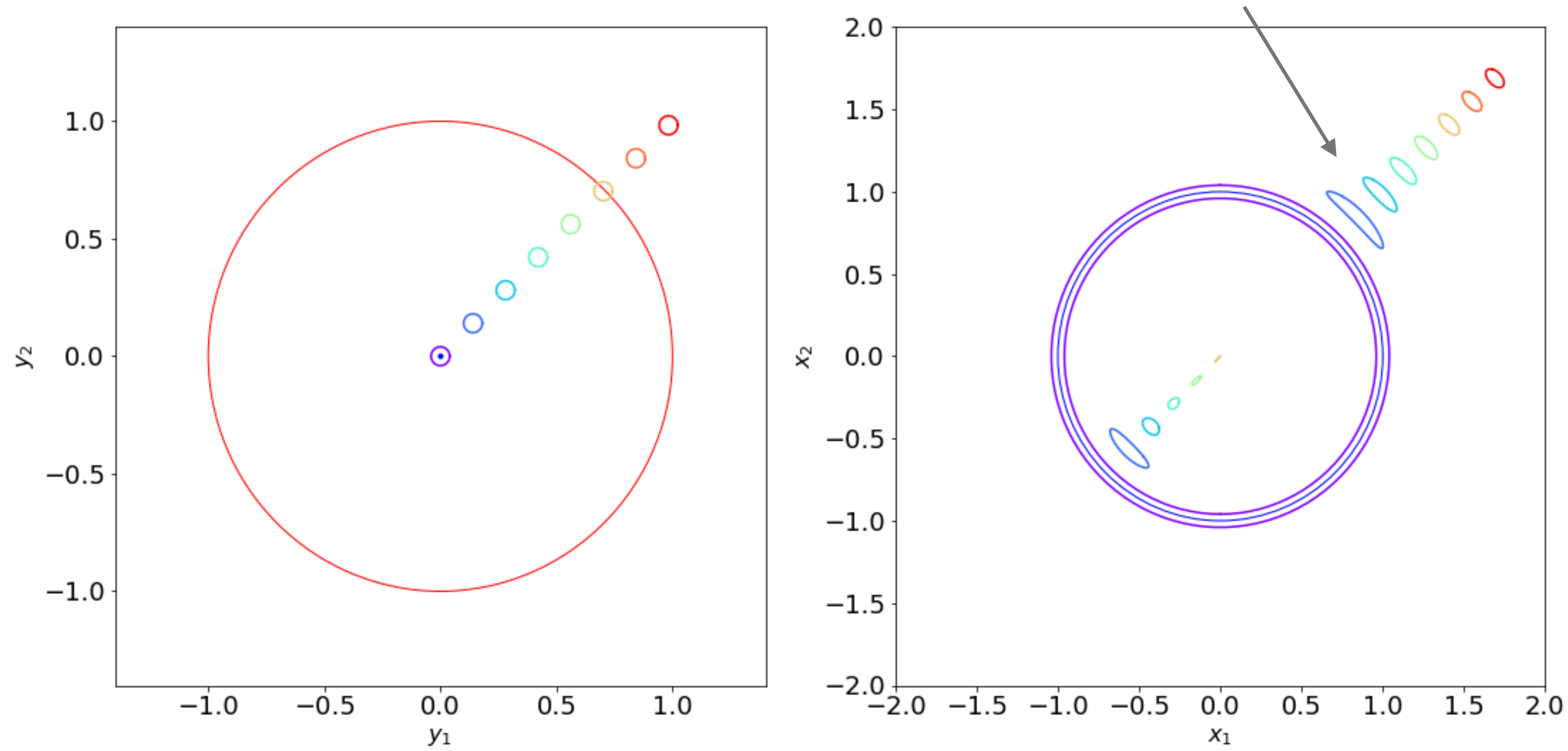
Thus, the SIS lens does not magnify, neither de-magnifies the images in the radial direction.

THE SINGULAR ISOTHERMAL SPHERE



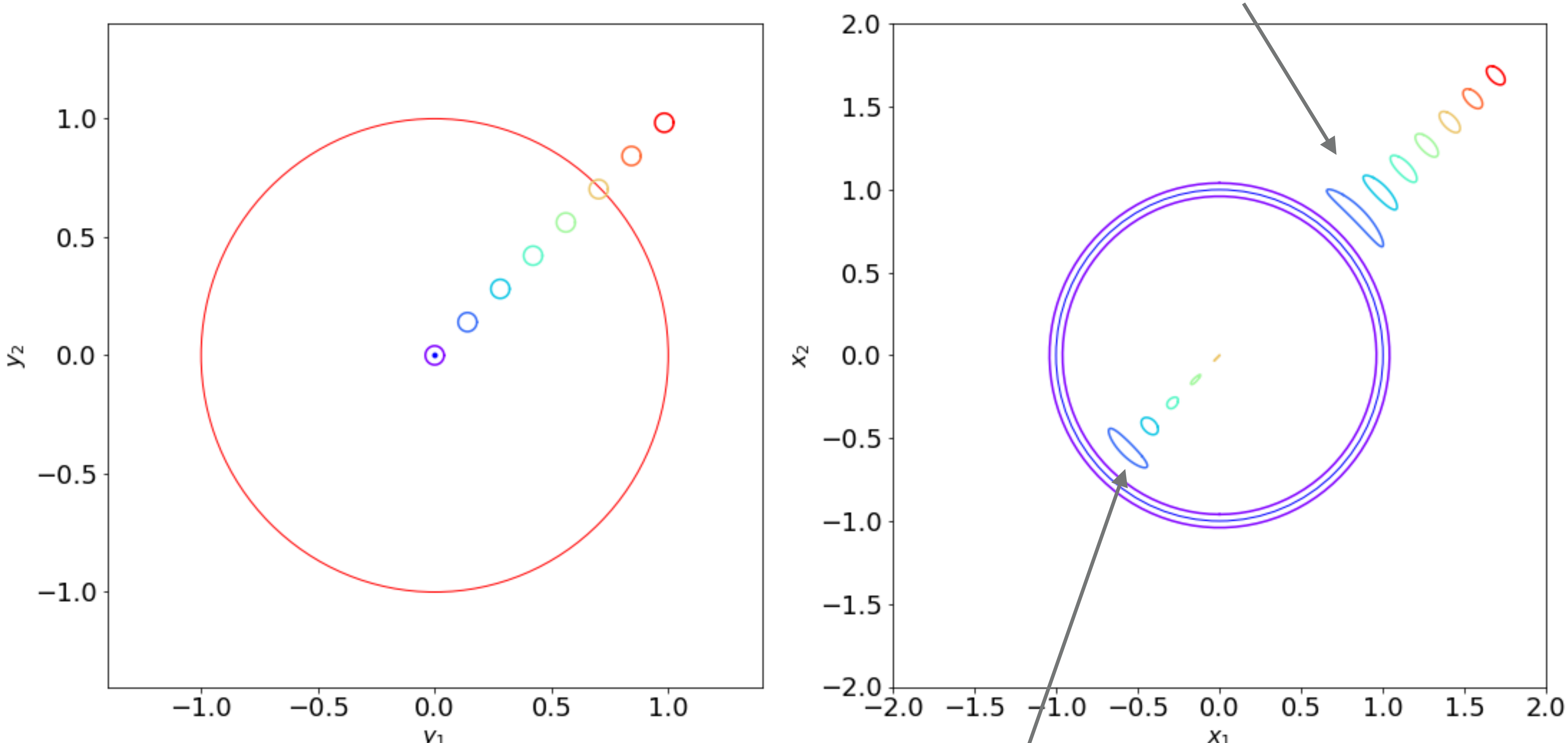
THE SINGULAR ISOTHERMAL SPHERE

Minimum of TD surface



THE SINGULAR ISOTHERMAL SPHERE

Minimum of TD surface



Saddle point of TD surface

THE SINGULAR ISOTHERMAL SPHERE

The shear and the convergence profiles are identical!

$$\gamma(x) = \bar{\kappa}(x) - \kappa(x) = \frac{m(x)}{x^2} - \frac{m'(x)}{2x} = \frac{1}{x} - \frac{1}{2x} = \frac{1}{2x} = \kappa(x)$$

Note that both $\kappa(x)$ and $\gamma(x)$ are defined to be positive! If x can assume negative values, we may write

$$\kappa(x) = \gamma(x) = \frac{1}{2|x|}$$

$$\gamma_1 = \frac{1}{2|x|}\cos 2\phi$$

$$\gamma_2 = \frac{1}{2|x|}\sin 2\phi$$

THE SINGULAR ISOTHERMAL SPHERE

We talked about the radial magnification already… what about the tangential one?

$$\mu(x) = [1 - \kappa(x) - \gamma(x)]^{-1} = \left[1 - \frac{1}{|x|}\right]^{-1} = \frac{|x|}{|x| - 1}$$

Which allows to calculate the magnification of the two images as

$$\mu_+ = \frac{y+1}{y} = 1 + \frac{1}{y} \textit{ and } \mu_- = \frac{|y-1|}{|y-1|-1} = \frac{-y+1}{-y} = 1 - \frac{1}{y}$$

Thus, the largest the projected distance of lens and source, the closest to unity is the magnification of the image x_+. On the contrary, the image at x_- has zero magnification when the source is located on the cut.

THE SINGULAR ISOTHERMAL SPHERE

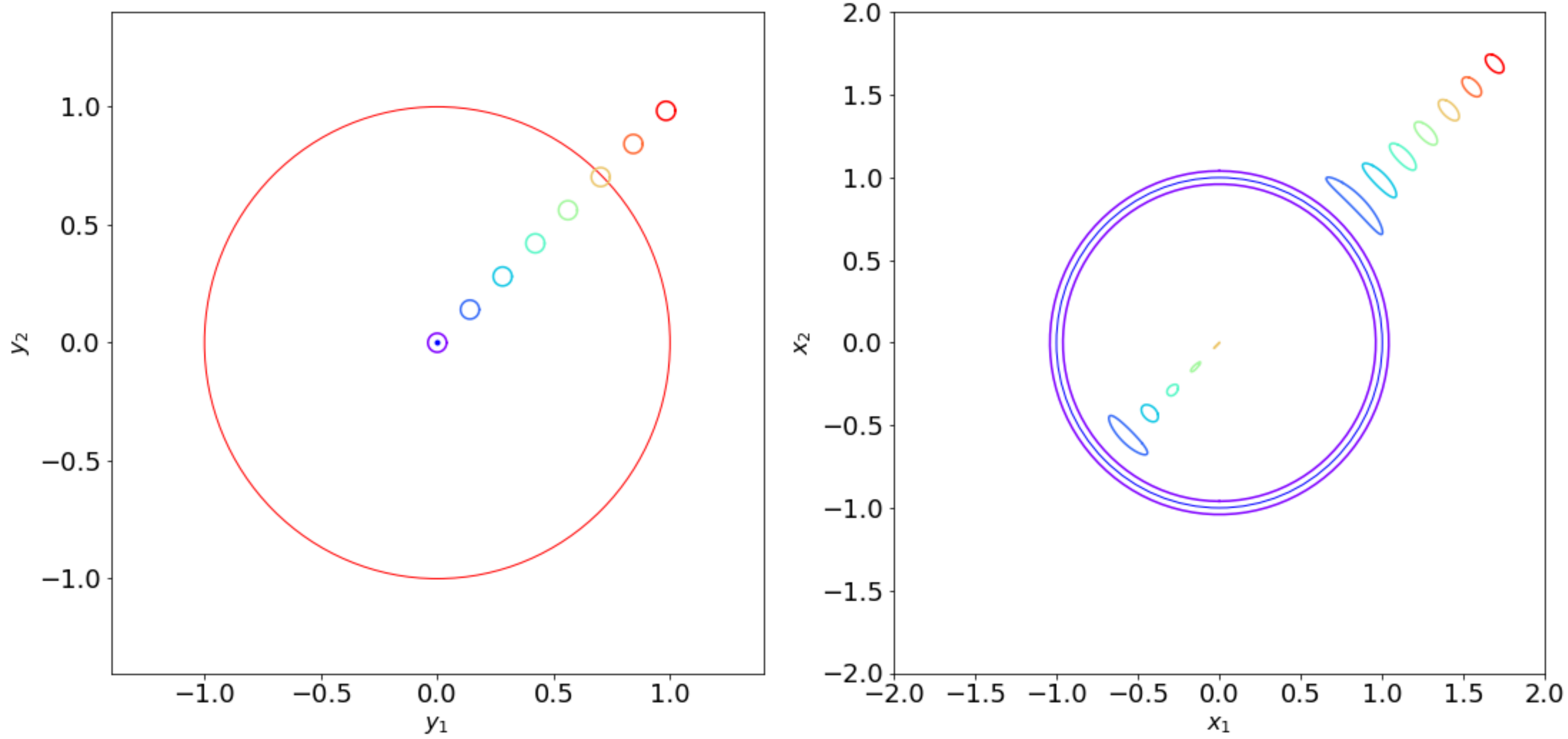
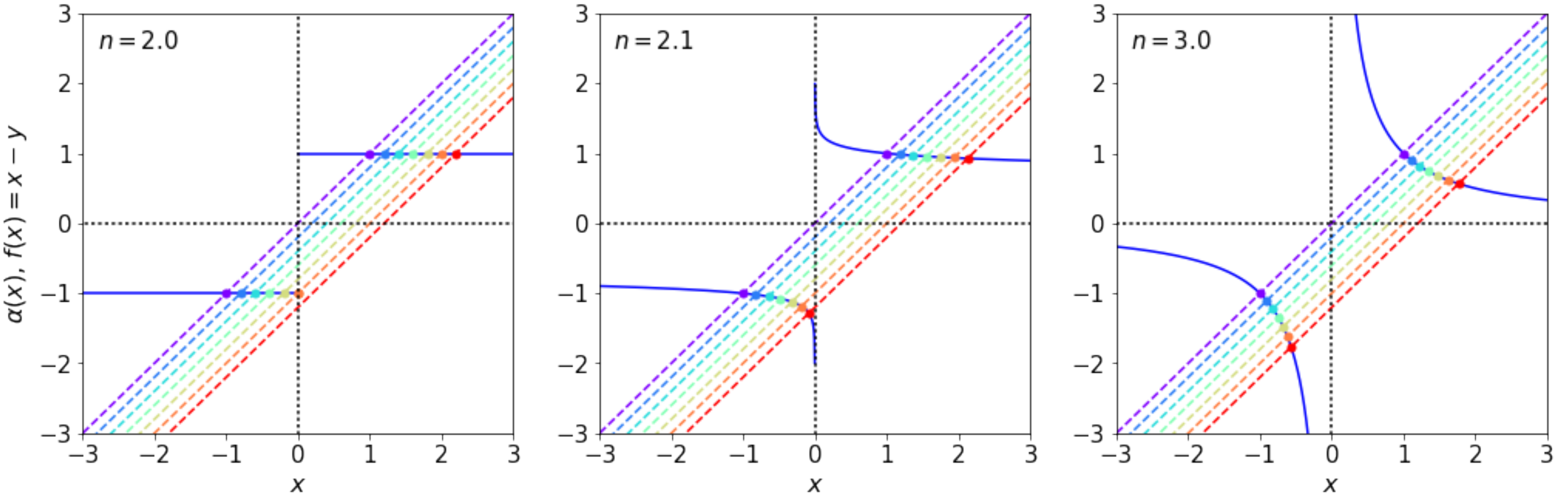


IMAGE DIAGRAM (N>=2)



PL lenses with n>2 always have 2 images, because the time delay surface is not continuously deformable.

In addition, the images are radially de-magnified!

SOFTENED PROFILES: THE NON-SINGULAR ISOTHERMAL SPHERE

The profiles considered so far have surface density profiles with a singularity at x=0. We consider another class of lenses which have a flat core.

Given the simplicity of the model, we investigate the effects of the core by modifying the SIS lens:

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\frac{1}{\sqrt{\xi^2+\xi_c^2}} = \frac{\Sigma_0}{\sqrt{1+\xi^2/\xi_c^2}}$$

$$\Sigma_0 = \frac{\sigma_v^2}{2G\xi_c}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

Choosing $\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \frac{1}{\sqrt{\xi^2 + \xi_c^2}} = \frac{\Sigma_0}{\sqrt{1 + \xi^2/\xi_c^2}}$$

$$\kappa(x) = \frac{1}{2\sqrt{x^2 + x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The mass profile is computed as follows

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2 + x_c^2} - x_c$$

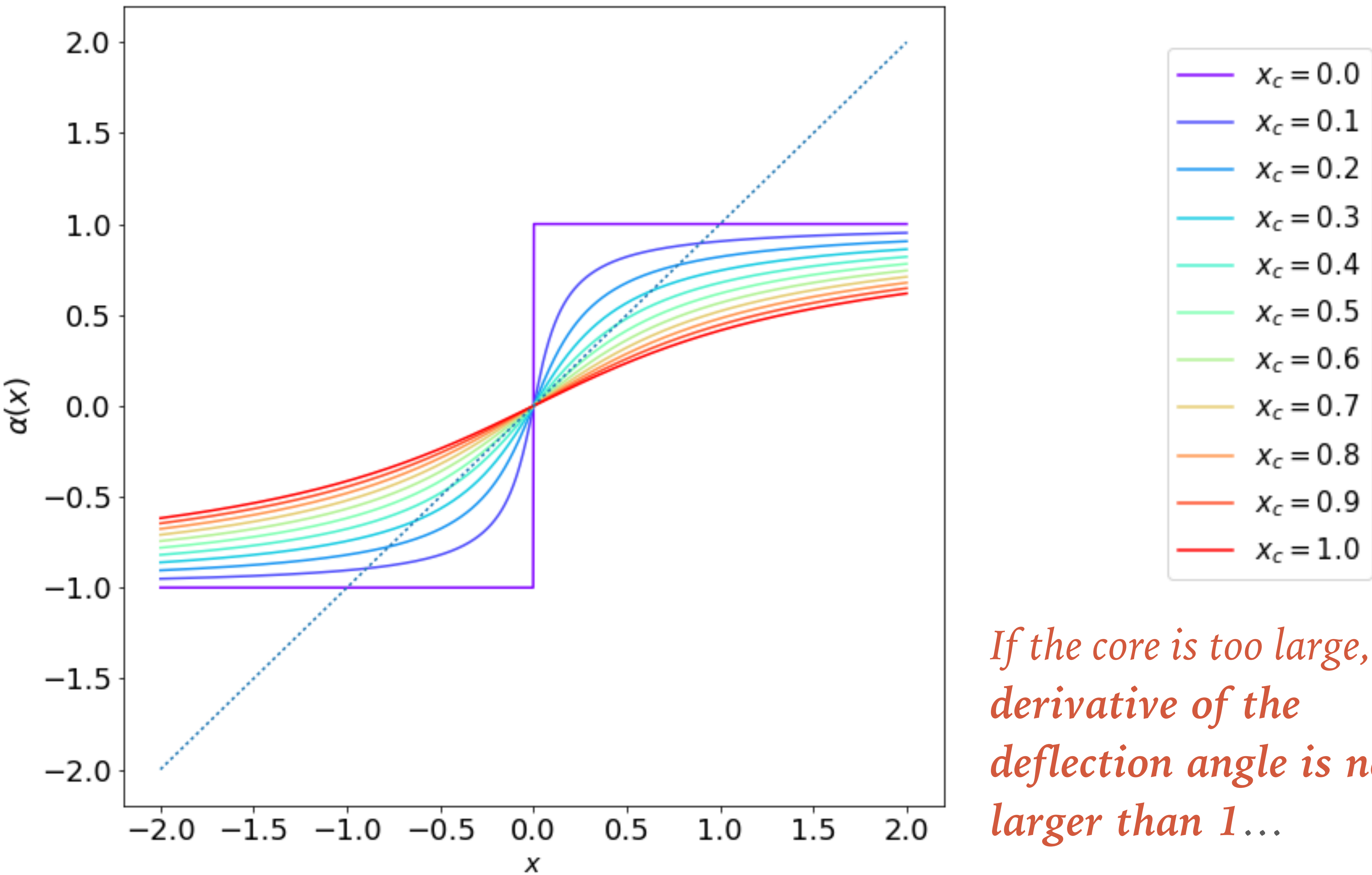
The deflection angle is

$$\alpha(x) = \frac{m(x)}{x} = \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

The shear is

$$\gamma(x) = \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2} - \frac{1}{2\sqrt{x^2 + x_c^2}}$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)



If the core is too large, ***the derivative of the deflection angle is never larger than 1…***

No radial critical line!

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The radius of the radial critical line can be found by solving the equation:

$$\left(1-\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right)=1+\frac{m(x)}{x^2}-2\kappa(x)=0$$

$$1+\frac{\sqrt{x^2+x_c^2}-x_c}{x^2}-\frac{1}{\sqrt{x^2+x_c^2}}=0$$

$$x_r^2=\frac{1}{2}\left(2x_c-x_c^2-x_c\sqrt{x_c^2+4x_c}\right)$$

$$x_r^2\geq 0 \text{ for } x_c\leq 1/2.$$

NON SINGULAR ISOTHERMAL SPHERE (NIS)

We can search for the tangential critical line:

$$m(x) = 2\int_0^x \kappa(x')x'dx' = \sqrt{x^2 + x_c^2} - x_c \qquad m(x)/x^2 = 1$$

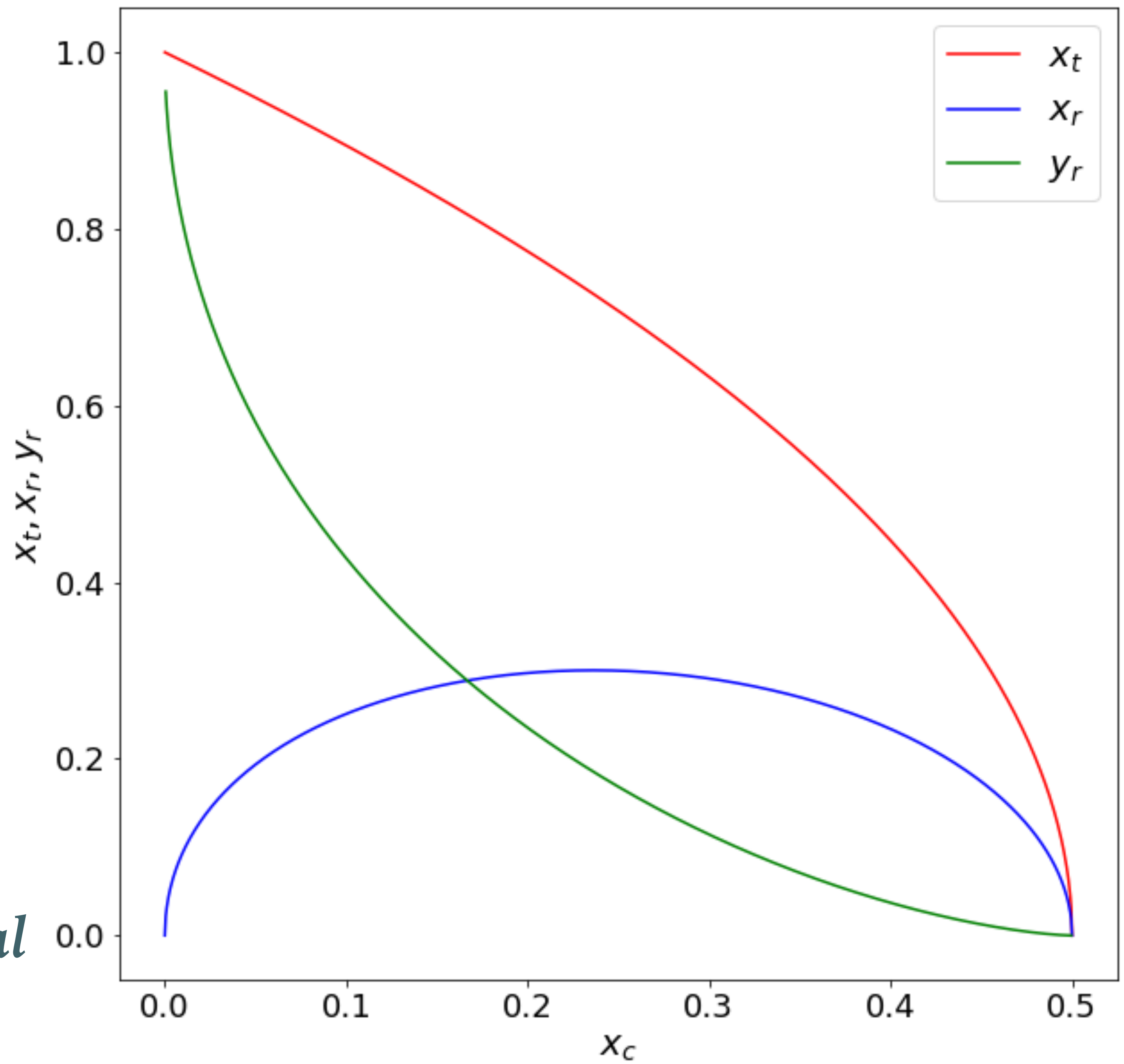
$$\sqrt{x^2 + x_c^2} - x_c = x^2$$

$$x^2(x^2 + 2x_c - 1) = 0$$

$$x_t = \sqrt{1 - 2x_c}$$

The tangential critical line also exists only if $x_c < 1/2$!

NON SINGULAR ISOTHERMAL SPHERE (NIS)



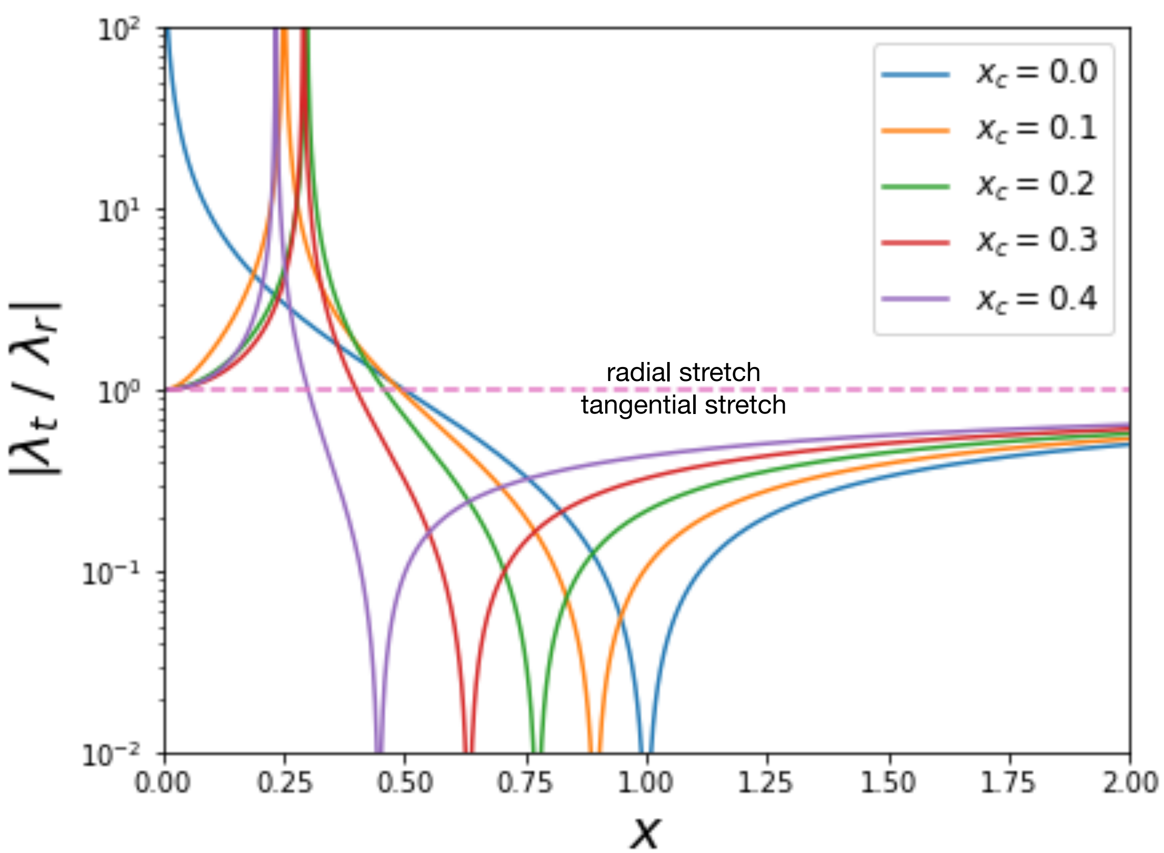
Small core, small radial critical line, large tangential critical line

NON SINGULAR ISOTHERMAL SPHERE (NIS)

Distortion of infinitesimal images

radial eigenvalue of A $\lambda_r = 1 - \alpha'(x) = 1 + \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2} - \frac{1}{\sqrt{x^2 + x_c^2}}$

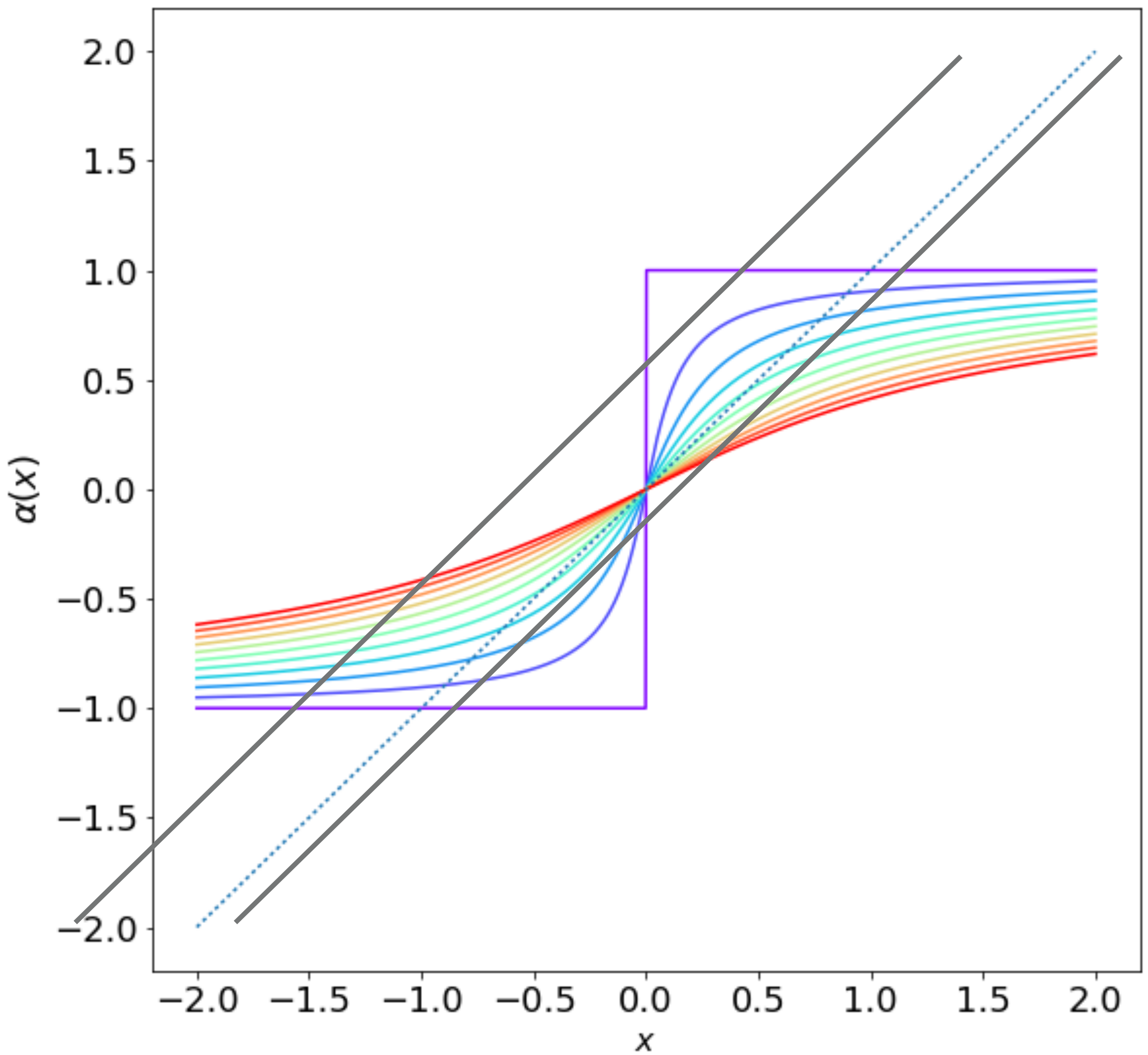
tangential eigenvalue of A $\lambda_t = 1 - \frac{\alpha(x)}{x} = 1 - \frac{\sqrt{x^2 + x_c^2} - x_c}{x^2}$



$\left|\frac{\lambda_t}{\lambda_r}\right| < 1$ *image is tangentially stretched*

$\left|\frac{\lambda_t}{\lambda_r}\right| > 1$ *image is radially stretched*

NON SINGULAR ISOTHERMAL SPHERE (NIS)



As you can see, this has implications also for the existence of multiple images…

NON SINGULAR ISOTHERMAL SPHERE (NIS)

The lens equation can be reduced to the form:

$$y = x - \frac{m(x)}{x} = x - \sqrt{1 + \frac{x_c^2}{x^2}} - \frac{x_c}{x}$$

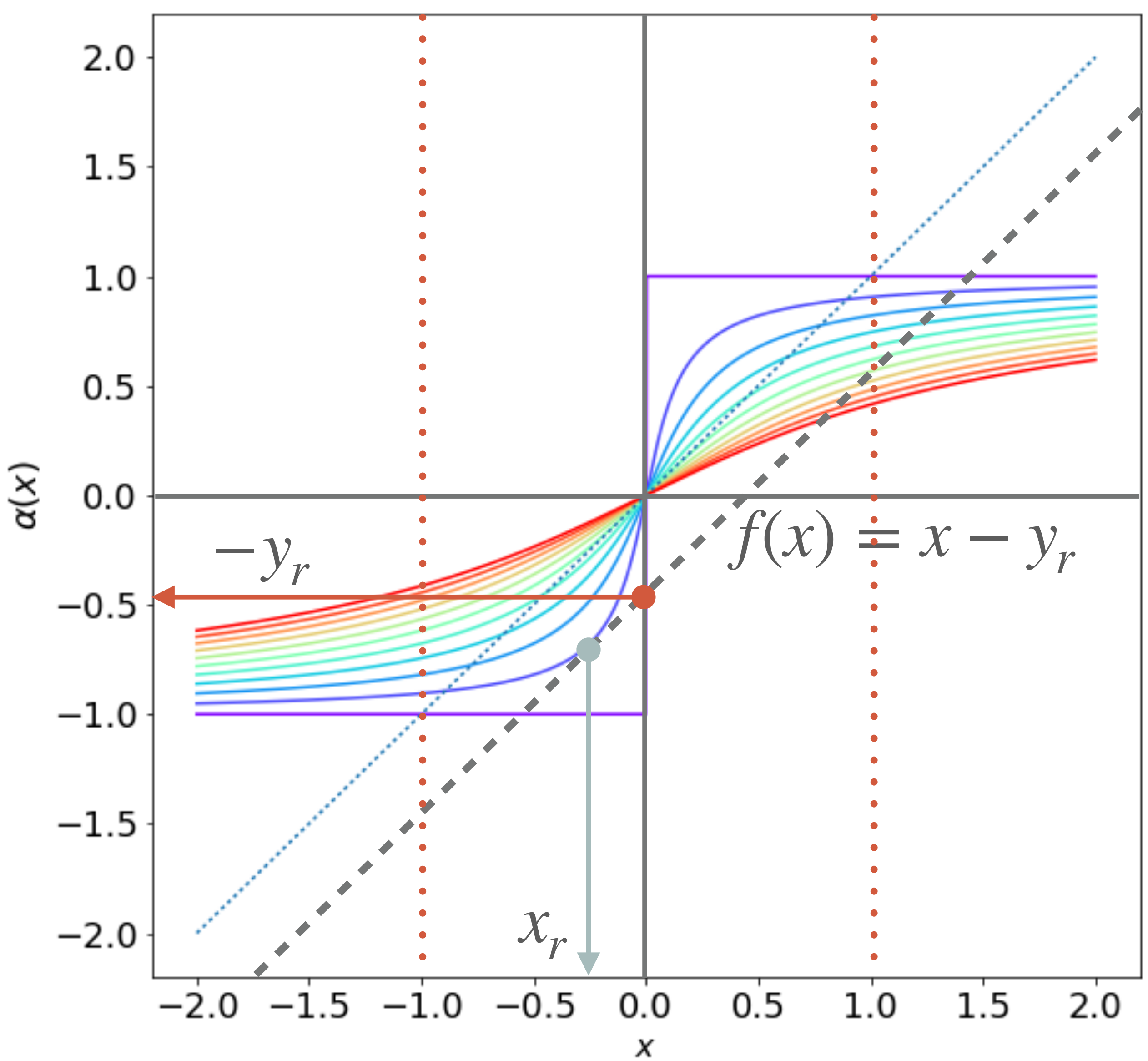
$$x^3 - 2yx^2 + (y^2 + 2x_c - 1)x - 2yx_c = 0\,.$$

There are up to three solutions, but, again the existence of multiple images depends on y and x_c*…*

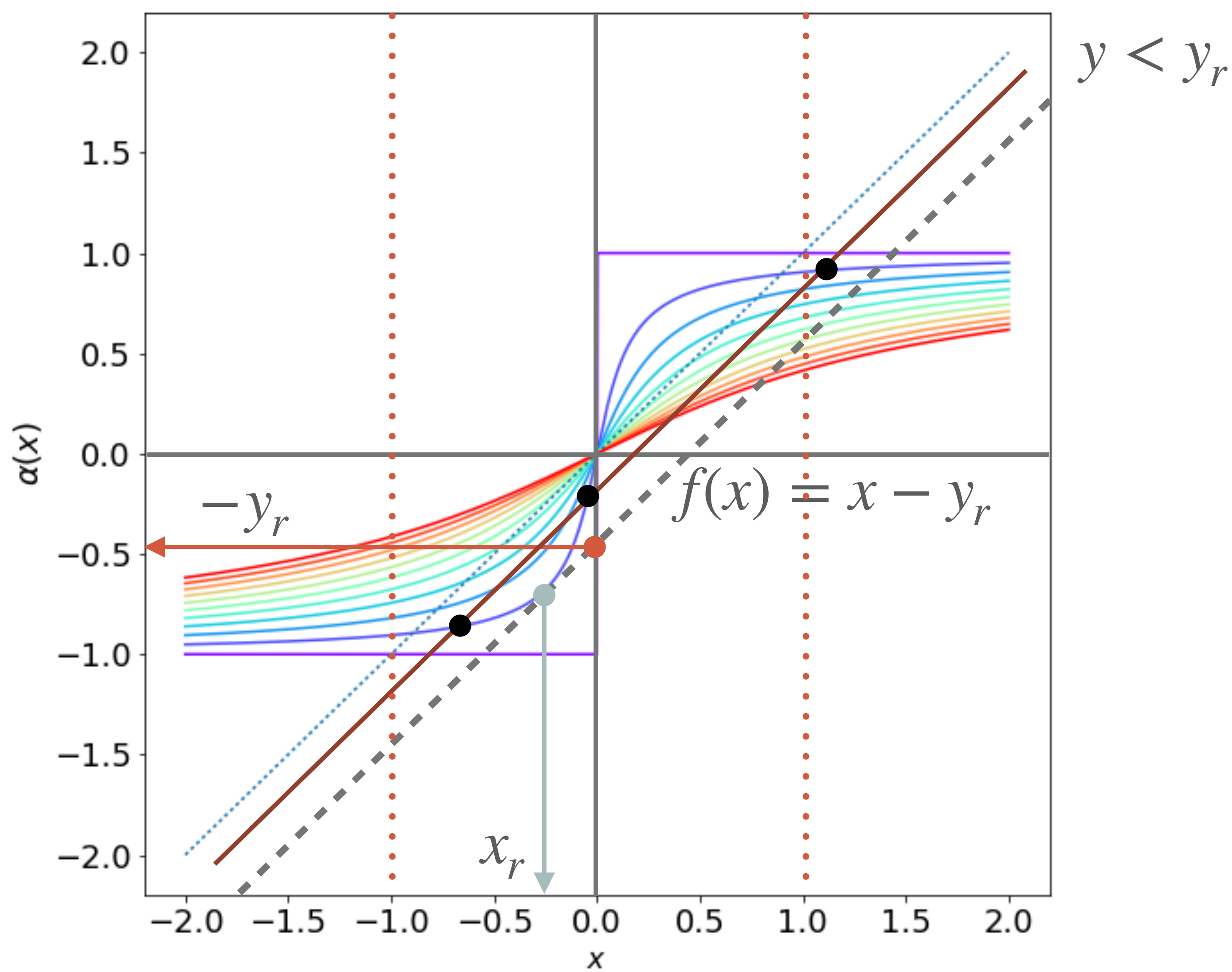
In particular on whether:

- *the radial caustic exists*
- *the source is inside or outside the radial caustic*

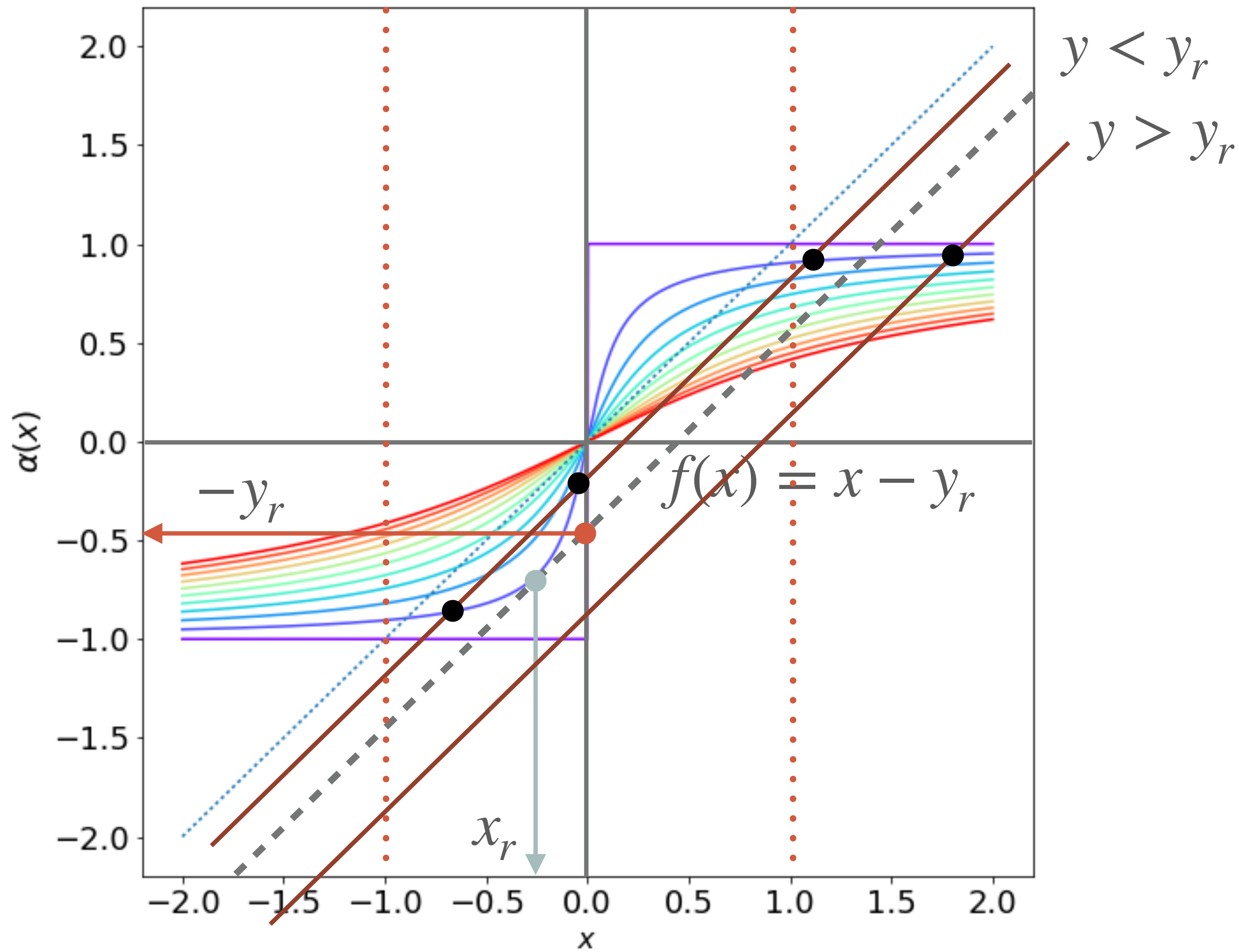
NON SINGULAR ISOTHERMAL SPHERE (NIS)



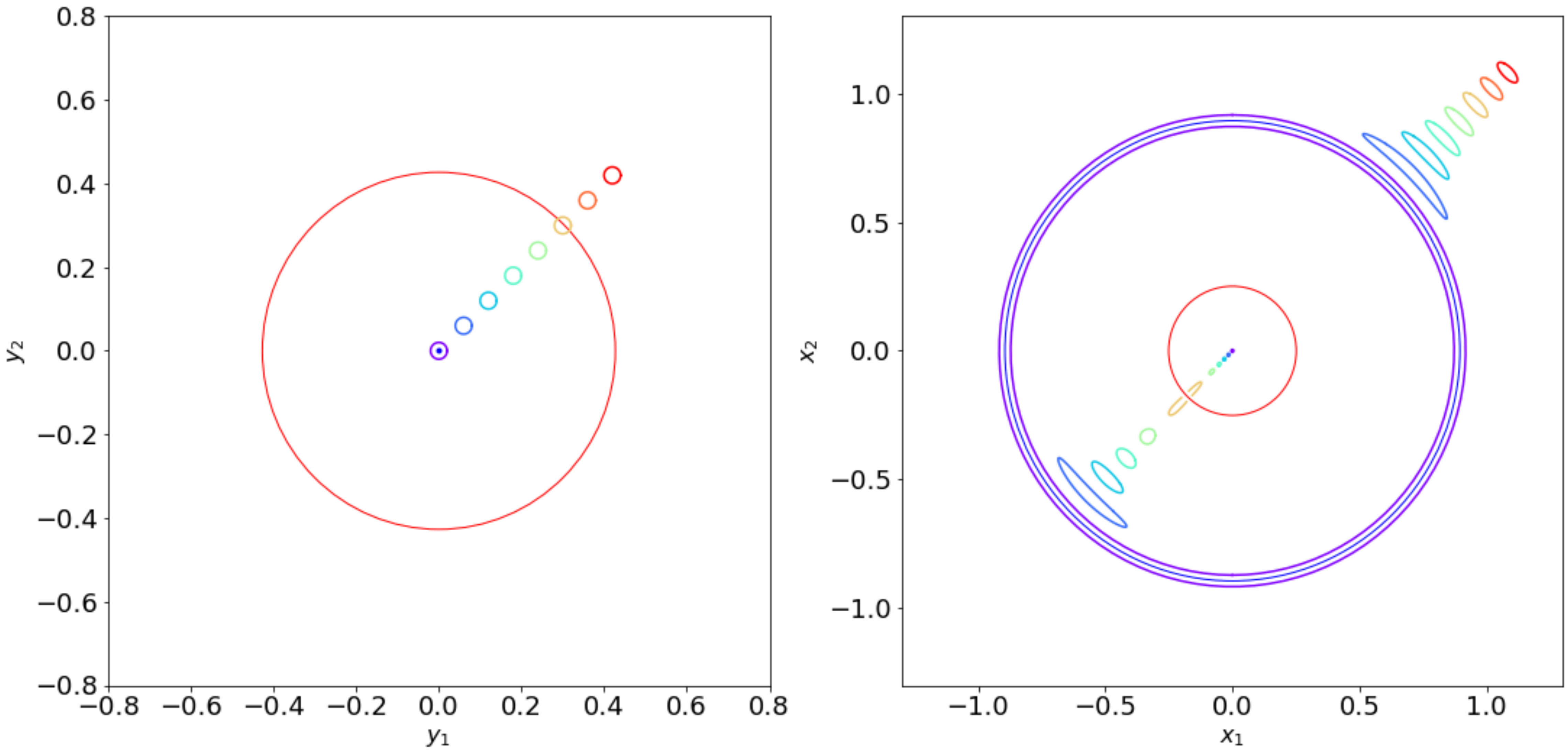
NON SINGULAR ISOTHERMAL SPHERE (NIS)



NON SINGULAR ISOTHERMAL SPHERE (NIS)



NON SINGULAR ISOTHERMAL SPHERE (NIS)



Three images if the source is inside the radial caustic; One image otherwise.

Parity: changes at each critical line (remember: maxima, minima, saddle points of TDS)

OTHER INTERESTING PROFILES: NFW DENSITY PROFILE

Numerical simulations in the framework of the Cold-Dark-Matter model show that halos develop a sort of "universal" density profile of the kind

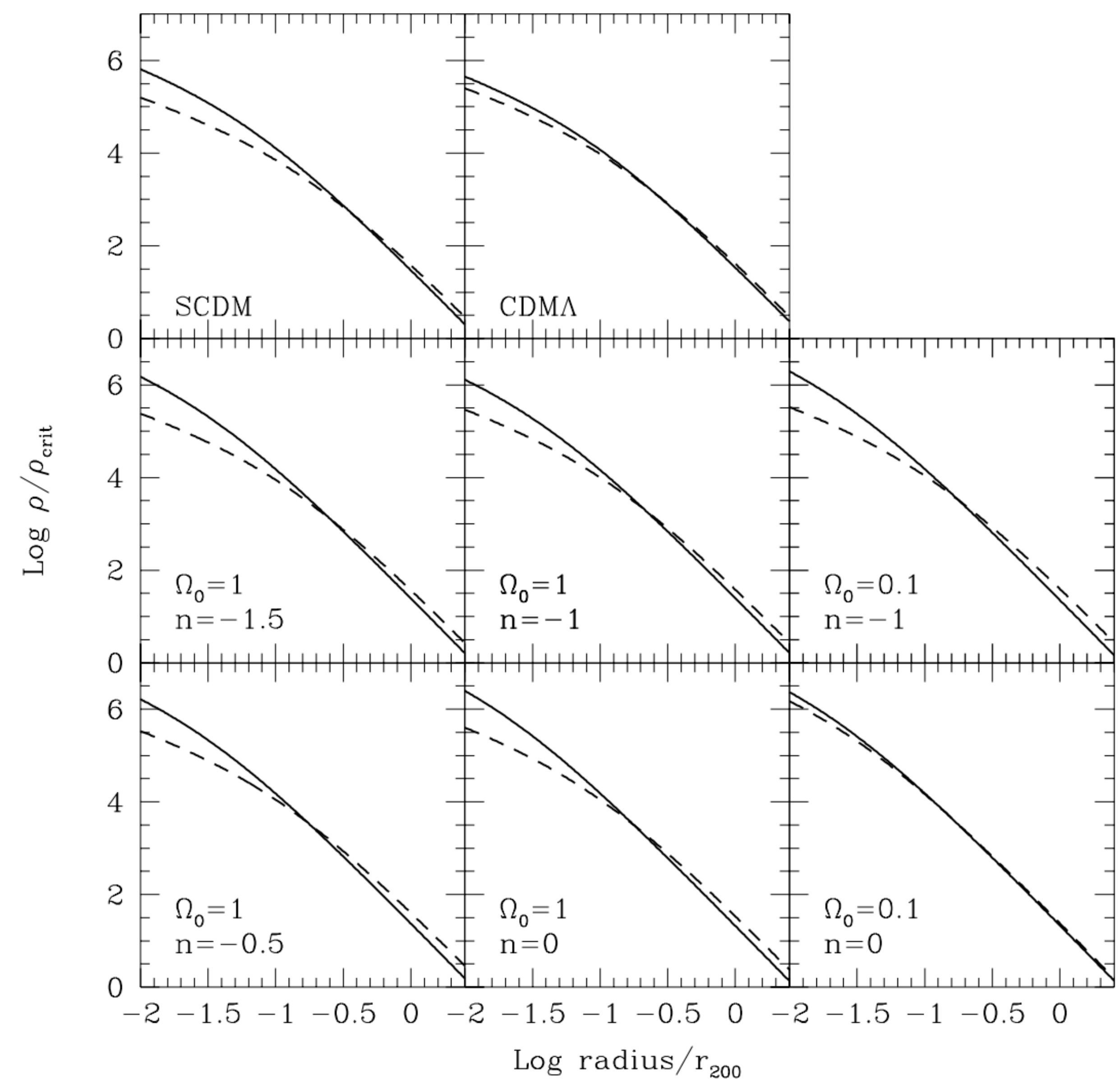
$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2}$$

The profile depends on two parameters (ρ_s, r_s). They can be expressed in terms of the mass and of the concentration of the halo:

$$r_{200} = 1.63 \times 10^{-2} \left(\frac{M_{200}}{h^{-1}M_\odot}\right)^{1/3} \left[\frac{\Omega_0}{\Omega(z)}\right]^{-1/3} (1+z)^{-1}h^{-1}$$

$$\rho_s = \frac{200}{3}\rho_{cr}\frac{c^3}{[\ln(1+c) - c/(1+c)]} \qquad c = \frac{r_{200}}{r_s}$$

Navarro, Frenk & White, 1997



LENSING PROPERTIES OF THE NFW PROFILE

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2-1} f(x) \,, \qquad x = \xi / r_s$$

with

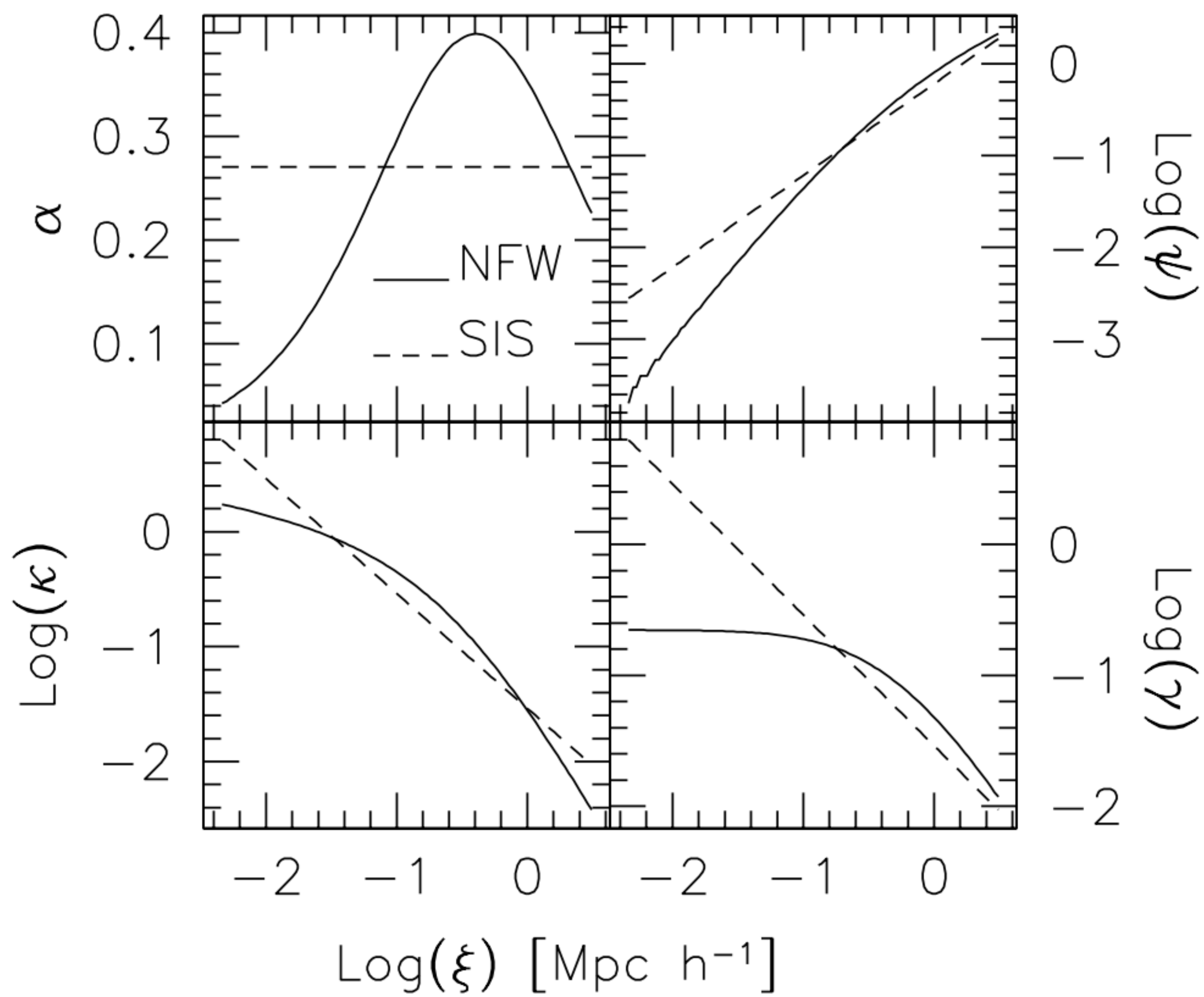
$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \arctan\sqrt{\frac{x-1}{x+1}} & (x>1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh}\sqrt{\frac{1-x}{1+x}} & (x<1) \\ 0 & (x=1) \end{cases}$$

$$\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2-1} \qquad \kappa_s \equiv \rho_s r_s \Sigma_{\rm cr}^{-1}$$

$$m(x) = 2\int_0^x \kappa(x') x' dx' = 4k_s h(x)$$

$$h(x) = \ln\frac{x}{2} + \begin{cases} \frac{2}{\sqrt{x^2-1}} \arctan\sqrt{\frac{x-1}{x+1}} & (x>1) \\ \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh}\sqrt{\frac{1-x}{1+x}} & (x<1) \\ 1 & (x=1) \end{cases}$$

$$\alpha(x) = \frac{4\kappa_s}{x} h(x)$$



LENSING PROPERTIES OF THE NFW PROFILE

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2 - 1} f(x) \ , \qquad x = \xi / r_s$$

with

$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \operatorname{arctan} \sqrt{\frac{x-1}{x+1}} & (x > 1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

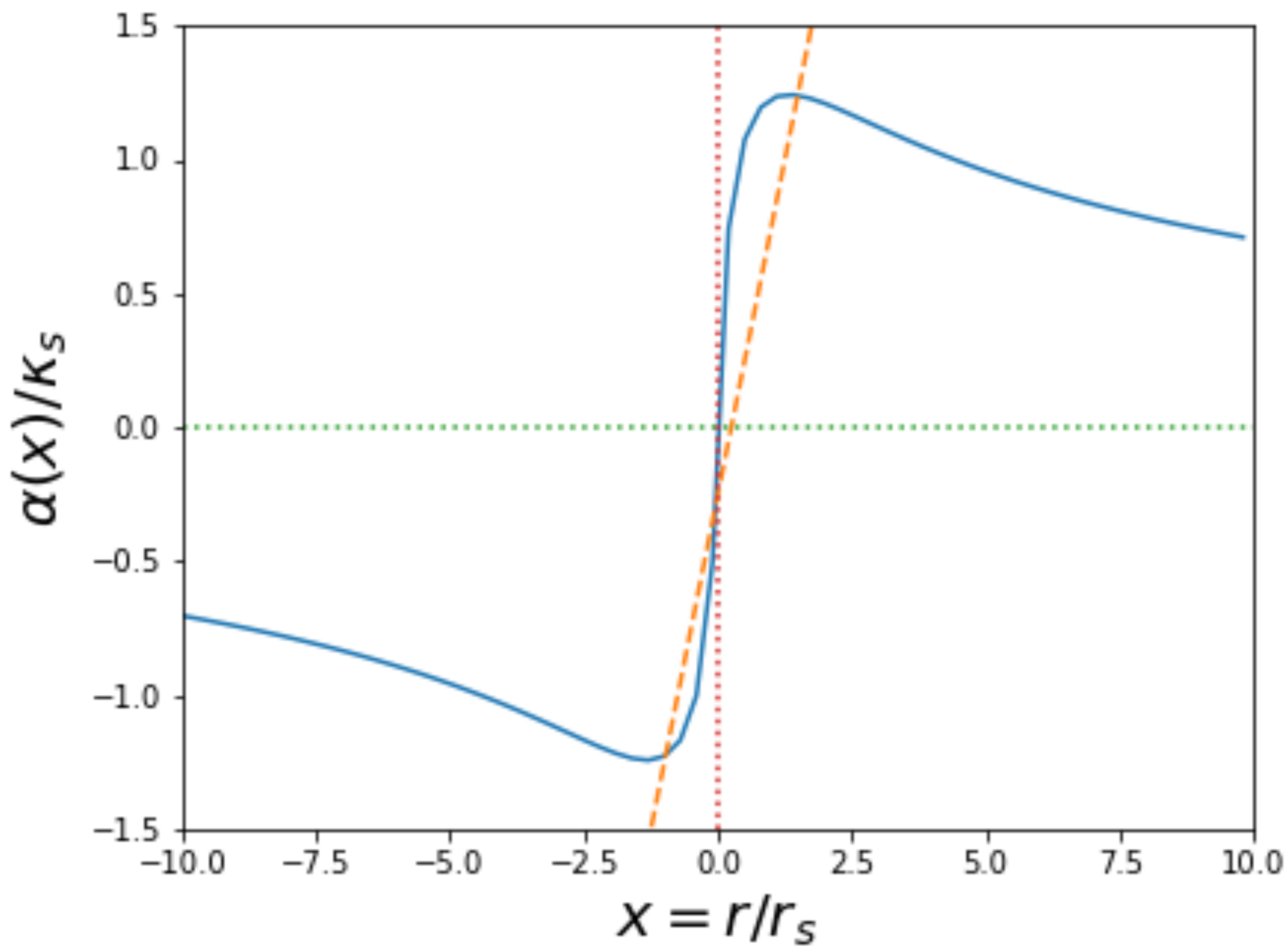
$$\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2 - 1} \qquad \kappa_s \equiv \rho_s r_s \Sigma_{\rm cr}^{-1}$$

$$m(x) = 2 \int_0^x \kappa(x') x' dx' = 4 k_s h(x)$$

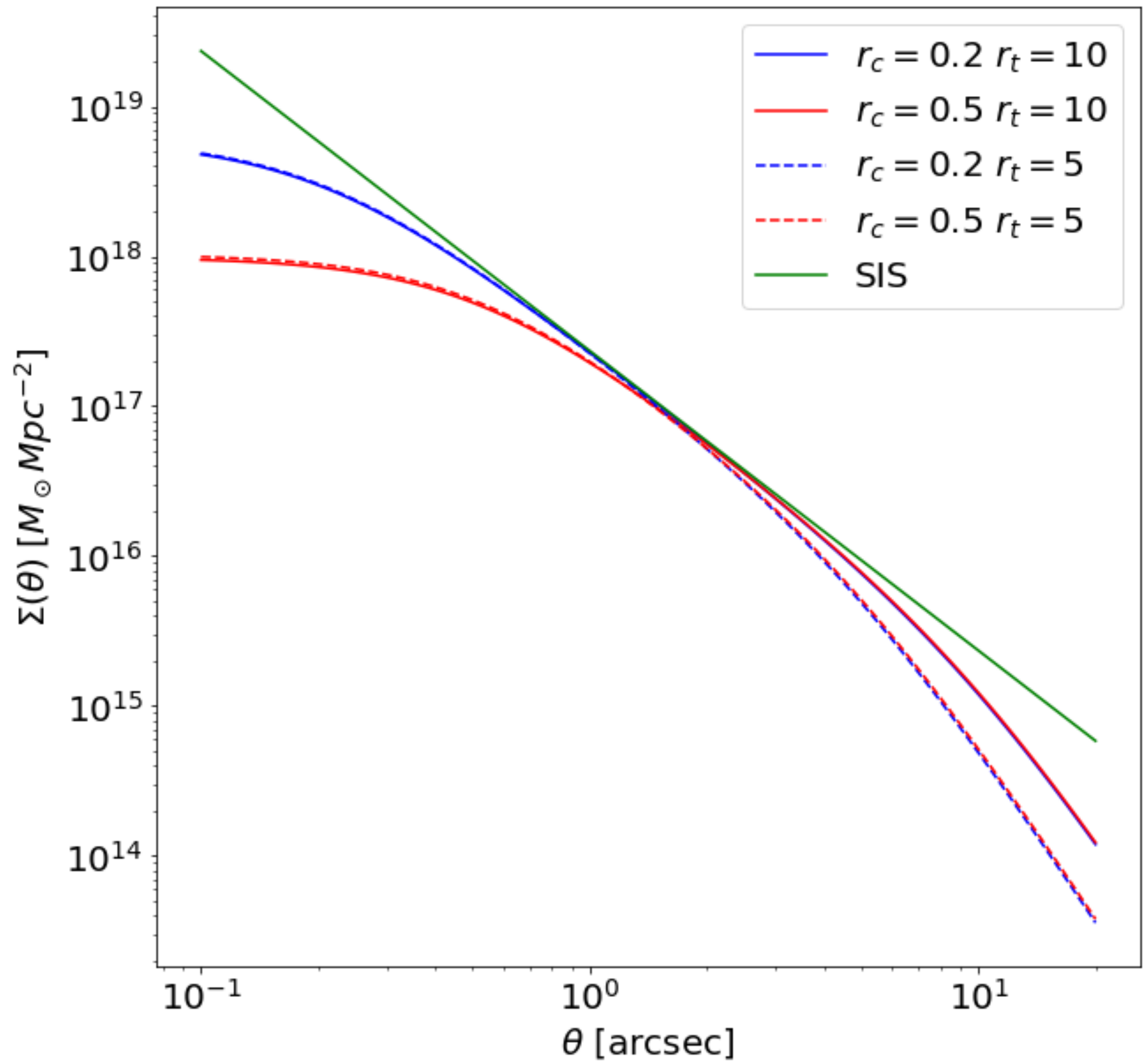
$$h(x) = \ln \frac{x}{2} + \begin{cases} \frac{2}{\sqrt{x^2-1}} \operatorname{arctan} \sqrt{\frac{x-1}{x+1}} & (x > 1) \\ \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 1 & (x = 1) \end{cases}$$

$$\alpha(x) = \frac{4\kappa_s}{x} h(x)$$

One or three images



PIEMD (OR DPIE) PROFILE



The PIEMD model is a particular mass profile which includes both a core and a truncation radius

$$\rho(r) = \frac{\rho_0}{\left(1 + \frac{r^2}{r_c^2}\right)\left(1 + \frac{r^2}{r_t^2}\right)}$$

$$\rho_0 = \frac{\sigma_0^2}{2\pi G}\frac{r_c + r_t}{r_c^2 r_t} \qquad r_t > r_c$$

The surface density is

$$\Sigma(\xi) = \frac{\sigma_0^2}{2G}\frac{r_t}{r_t - r_c}\left(\frac{1}{\sqrt{\xi^2 + r_c^2}} - \frac{1}{\sqrt{\xi^2 + r_t^2}}\right)$$

Thus, it is equivalent to the difference of two NIS models (see last lesson).

GRAVITATIONAL LENSING

15 - LENS MODELS: ELLIPTICAL LENSES

R. Benton Metcalf

2022-2023

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Elliptical contours with their major axis along the ξ_2 axis

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

Now we make the surface density contours of the SIS elliptical:

$$\xi \Rightarrow \sqrt{\xi_1^2 + f^2 \xi_2^2}$$

Ensures that area of ellipse is equal to the area of the circle of radius ξ

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi} \quad \Rightarrow \quad \Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Elliptical contours with their major axis along the ξ_2 axis

Surface density is constant on ellipses with minor axis ξ and major axis ξ/f

SINGULAR ISOTHERMAL ELLIPSOID

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Let's derive the convergence in dimensionless units:

SINGULAR ISOTHERMAL ELLIPSOID

Let's derive the convergence in dimensionless units:

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}} \frac{\xi_0}{\xi_0}$$

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\mathrm{L}} D_{\mathrm{LS}}}{D_{\mathrm{S}}}$$

$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2}}$$

SINGULAR ISOTHERMAL ELLIPSOID

Let's derive the convergence in dimensionless units:

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}} \frac{\xi_0}{\xi_0}$$

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c}\right)^2 \frac{D_{\rm L} D_{\rm LS}}{D_{\rm S}}$$

$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2}}$$

In polar coordinates:

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2}$$

$$\kappa(x, \varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

With the ansatz $\Psi(x,\varphi) := x\tilde{\Psi}(\varphi)$

$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Delta(\varphi) = \sqrt{\cos\varphi^2 + f^2 \sin\varphi^2} \qquad \kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)}$$

The lensing potential can be obtained by solving the Poisson equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{x}\frac{\partial\Psi}{\partial x} + \frac{1}{x^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 2\kappa = \frac{\sqrt{f}}{x\Delta(\varphi)}$$

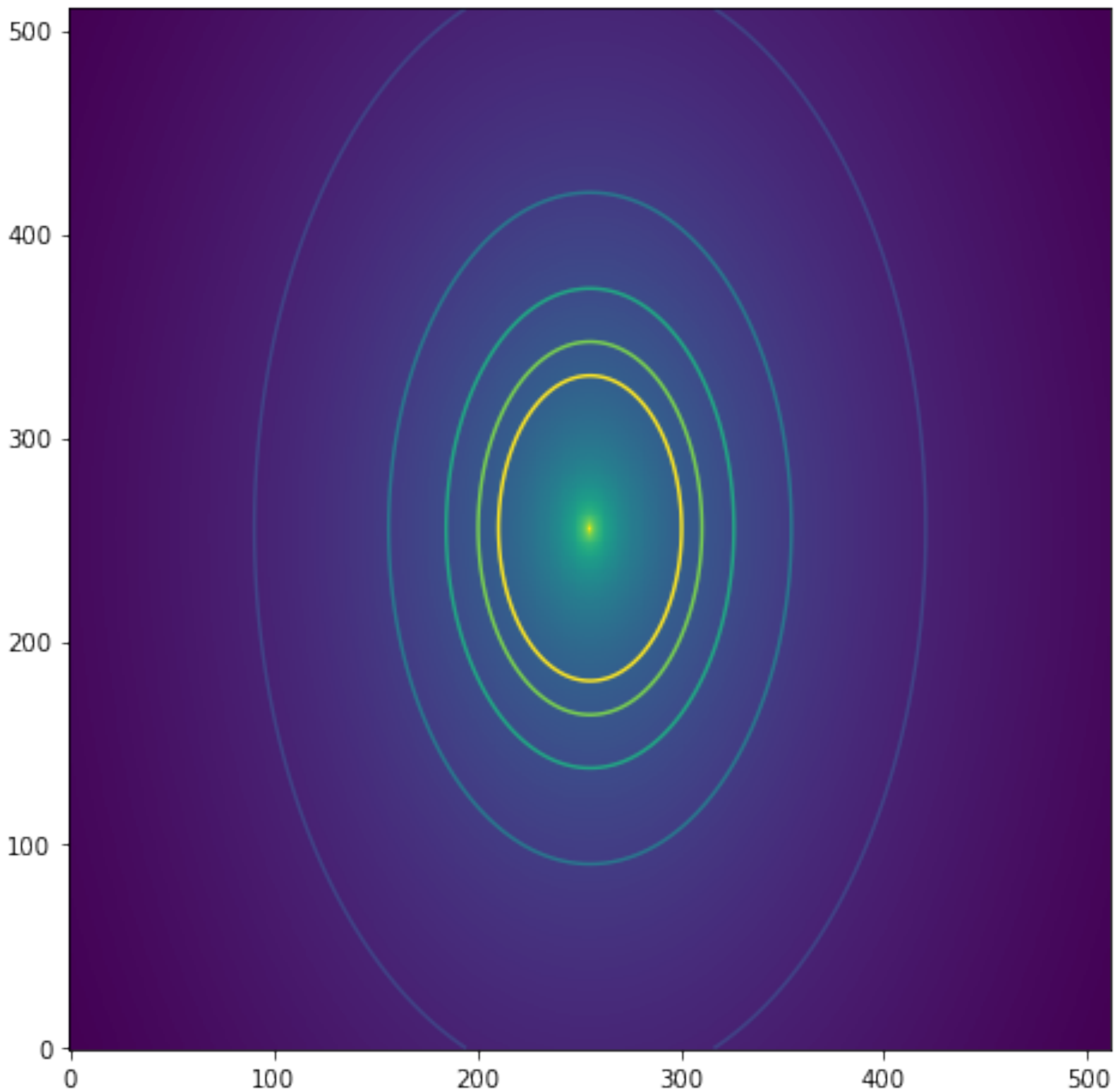
With the ansatz $\Psi(x,\varphi) := x\tilde{\Psi}(\varphi)$

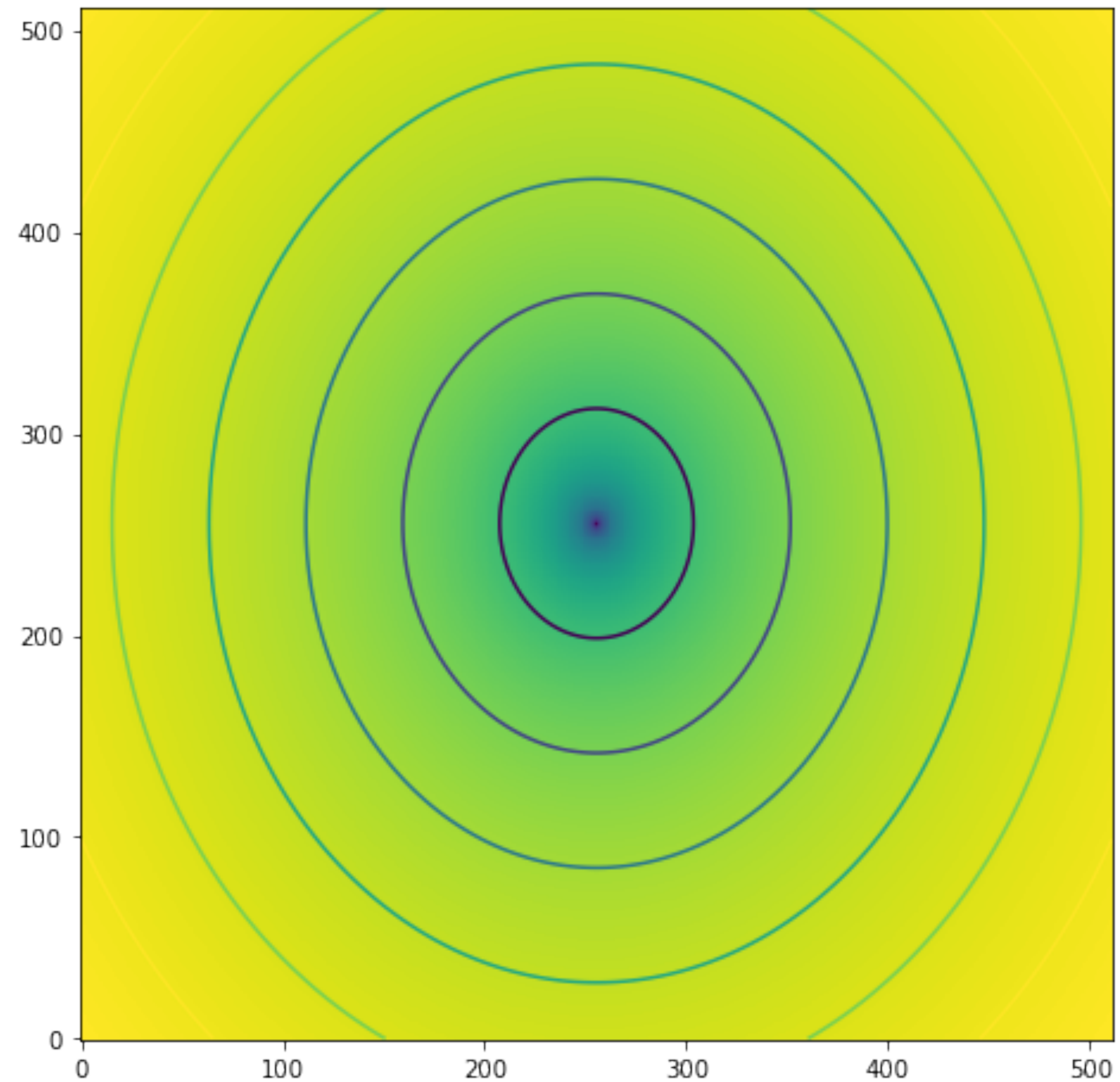
$$\tilde{\Psi}(\varphi) + \frac{d^2}{d\varphi^2}\tilde{\Psi}(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}$$

Solved with Green's method (Kormann et al. 1994):

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

CONVERGENCE AND POTENTIAL $f = 0.7$





SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}\left(f'/f\cos\varphi\right)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}(f'/f\cos\varphi)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}\left(f'/f\cos\varphi\right)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

$$\begin{aligned}
\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
\alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)
\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\Psi(x,\varphi) = x\frac{\sqrt{f}}{f'}\left[\sin\varphi \arcsin\left(f'\sin\varphi\right) + \cos\varphi \operatorname{arcsinh}\left(f'/f\cos\varphi\right)\right] \qquad f' = \sqrt{1-f^2}$$

Let's compute the deflection angle:

$$\frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial x} - \frac{\sin\varphi}{x}\frac{\partial}{\partial\varphi} \qquad \frac{\partial}{\partial x_2} = \sin\varphi\frac{\partial}{\partial x} + \frac{\cos\varphi}{x}\frac{\partial}{\partial\varphi}$$

$$\begin{aligned}
\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
\alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)
\end{aligned}$$

Analogy with the SIS: the deflection angle does not depend on x!

SINGULAR ISOTHERMAL ELLIPSOID

$$\alpha_1(\vec{x}) \quad = \quad \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

$$\alpha_2(\vec{x}) \quad = \quad \frac{\sqrt{f}}{f'} \arcsin(f'\sin\varphi)$$

The component of the shear:

SINGULAR ISOTHERMAL ELLIPSOID

$$\alpha_1(\vec{x}) = \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

$$\alpha_2(\vec{x}) = \frac{\sqrt{f}}{f'} \arcsin(f'\sin\varphi)$$

The component of the shear:

$$\gamma_1 = \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_2}{\partial x_2}\right)$$

$$\gamma_2 = \frac{\partial\alpha_1}{\partial x_2}$$

$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$

$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}
\alpha_1(\vec{x}) &= \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
\alpha_2(\vec{x}) &= \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)
\end{aligned}$$

The component of the shear:

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}\left(\frac{\partial\alpha_1}{\partial x_1} - \frac{\partial\alpha_2}{\partial x_2}\right) \\
\gamma_2 &= \frac{\partial\alpha_1}{\partial x_2}
\end{aligned}
\qquad
\begin{aligned}
\gamma_1 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi \\
\gamma_2 &= -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi
\end{aligned}$$

Similarly to the SIS: $\gamma = \kappa$

SINGULAR ISOTHERMAL ELLIPSOID

$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$

$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

We have now the ingredients to compute the lensing Jacobian matrix

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-2\kappa\sin^2\varphi & \kappa\sin 2\varphi \\ \kappa\sin 2\varphi & 1-2\kappa\cos^2\varphi \end{bmatrix}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\gamma_1 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\cos 2\varphi = -\kappa\cos 2\varphi$$
$$\gamma_2 = -\frac{\sqrt{f}}{2x\Delta(\varphi)}\sin 2\varphi = -\kappa\sin 2\varphi$$

We have now the ingredients to compute the lensing Jacobian matrix

$$A = \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ \gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-2\kappa\sin^2\varphi & \kappa\sin 2\varphi \\ \kappa\sin 2\varphi & 1-2\kappa\cos^2\varphi \end{bmatrix}$$

whose eigenvalues are:

$$\lambda_t = 1-\kappa-\gamma = 1-2\kappa$$
$$\lambda_r = 1-\kappa+\gamma = 1\,.$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,.\end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,.\end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\begin{aligned}\lambda_t &= 1-\kappa-\gamma = 1-2\kappa \\ \lambda_r &= 1-\kappa+\gamma = 1\,.\end{aligned}$$

As the SIS, the SIE does not have a radial critical line!

The tangential critical line is an ellipse, along which

$$\kappa = \frac{1}{2}$$

$$\kappa(x,\varphi) = \frac{\sqrt{f}}{2x\Delta(\varphi)} \quad \Rightarrow \quad \vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}\left[\cos\varphi, \sin\varphi\right]$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)}\,[\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}$$

SINGULAR ISOTHERMAL ELLIPSOID

$$\vec{x}_t(\varphi) = \frac{\sqrt{f}}{\Delta(\varphi)} [\cos\varphi, \sin\varphi]$$

The corresponding caustic can be found using the lens equation:

$$\begin{aligned} y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)} \cos\varphi - \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\ y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)} \sin\varphi - \frac{\sqrt{f}}{f'} \arcsin(f' \sin\varphi) \ . \end{aligned}$$

There is no radial caustic, but there is the cut, which can be computed as

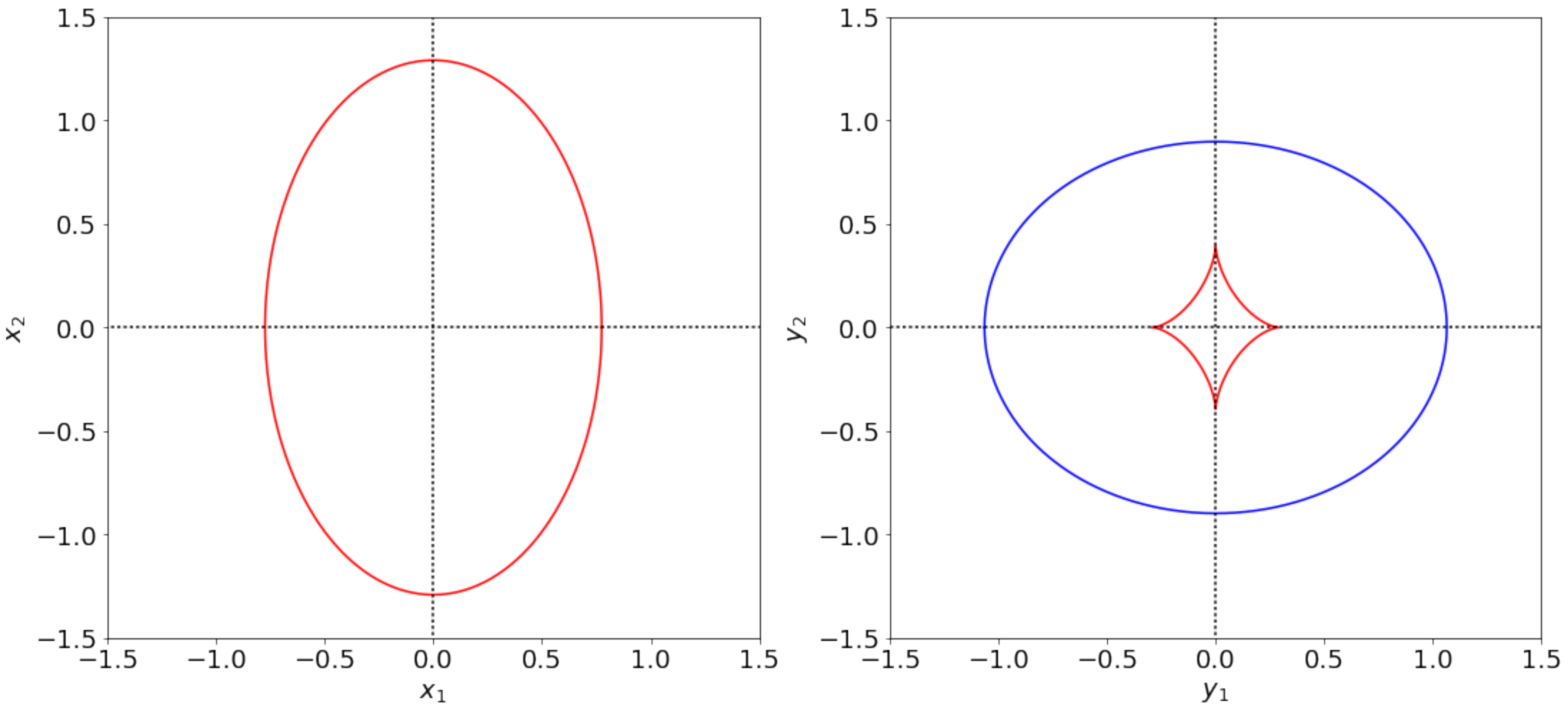
$$\vec{y}_c = \lim_{x \to 0} \vec{y}(x, \varphi) = -\vec{\alpha}(\varphi)$$

SINGULAR ISOTHERMAL ELLIPSOID

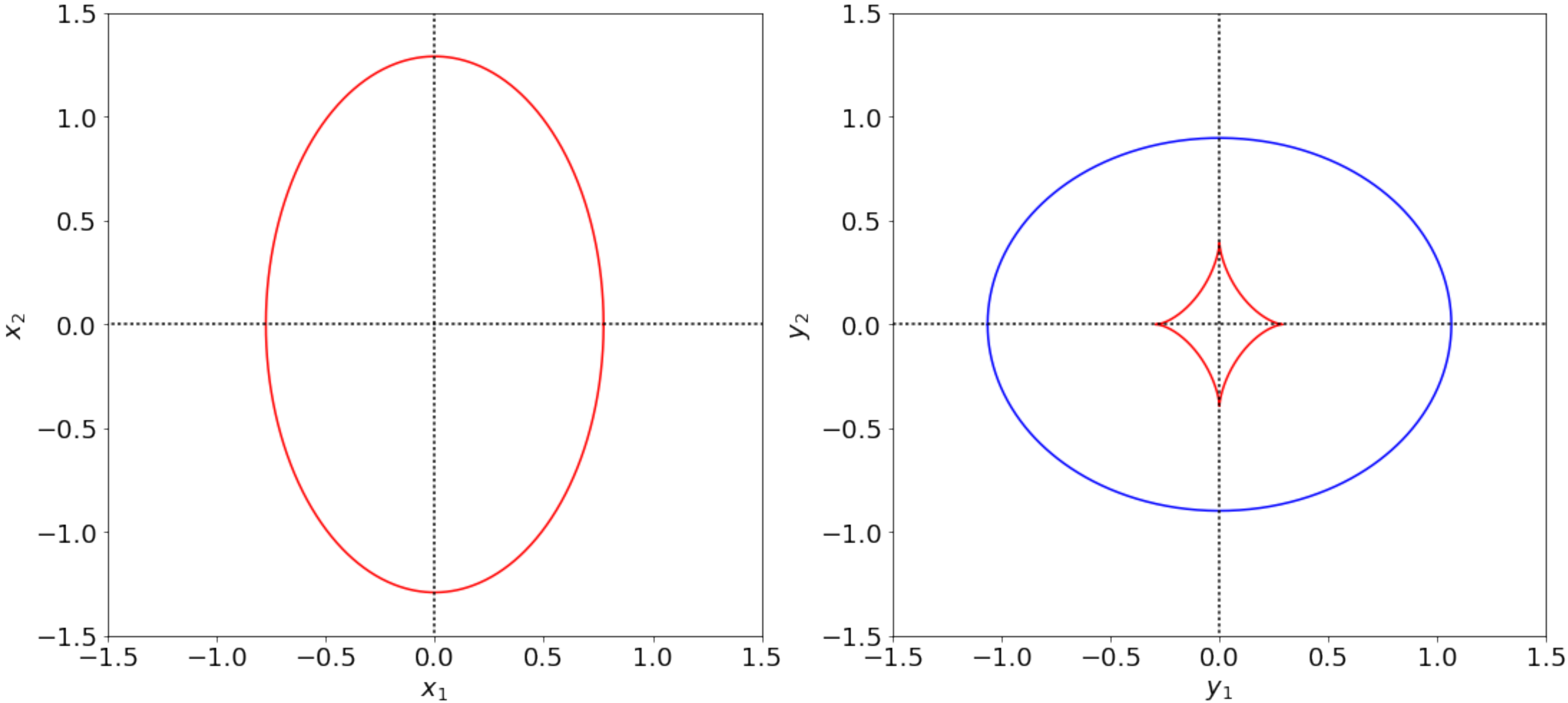
$$\vec{y}_c = \lim_{x\to 0} \vec{y}(x,\varphi) = -\vec{\alpha}(\varphi)$$

$$\begin{aligned}
y_{c,1} &= -\frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{c,2} &= -\frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}$$

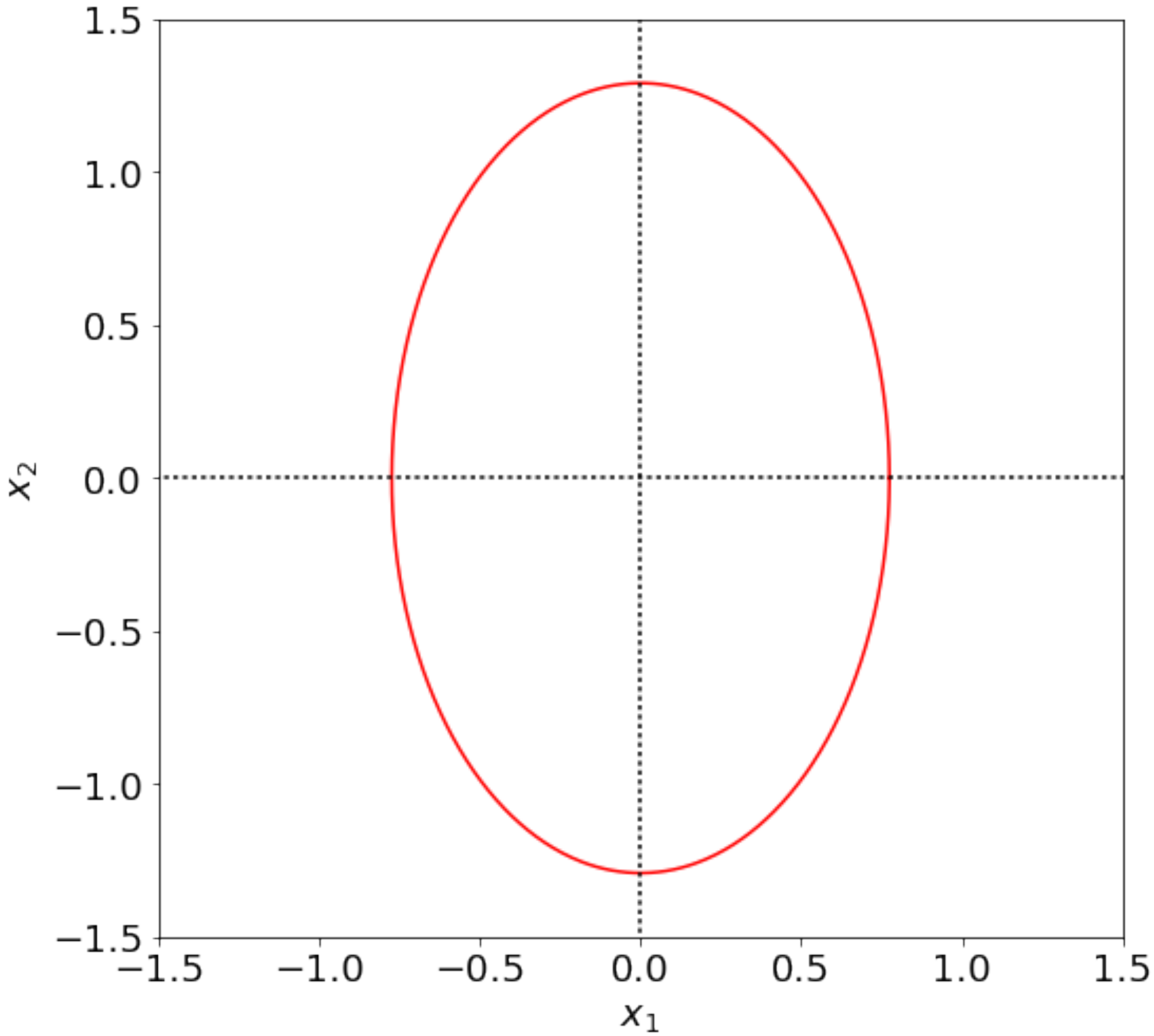
CRITICAL LINE, CUT, CAUSTIC ($f = 0.7$)

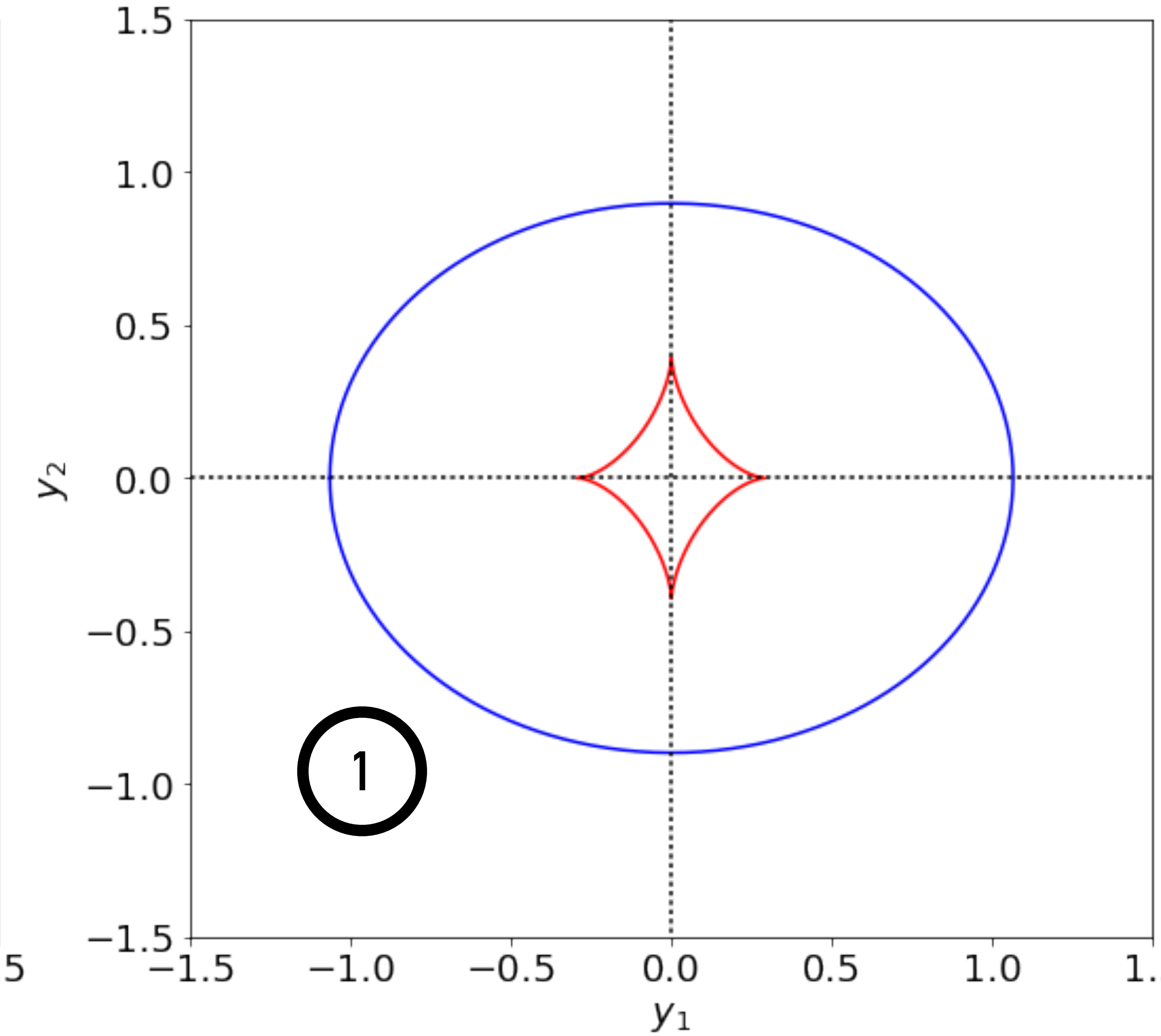


HOW MANY IMAGES?

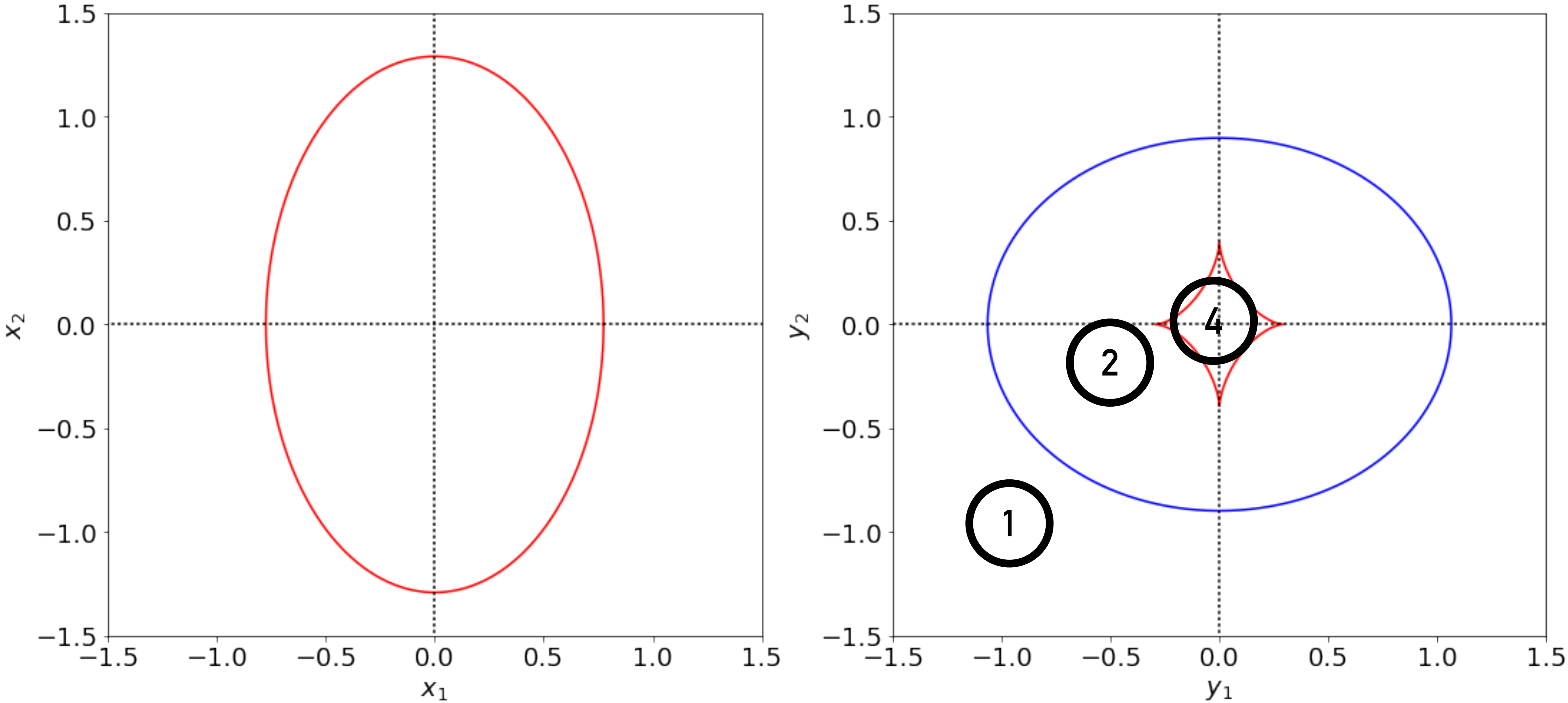


HOW MANY IMAGES?

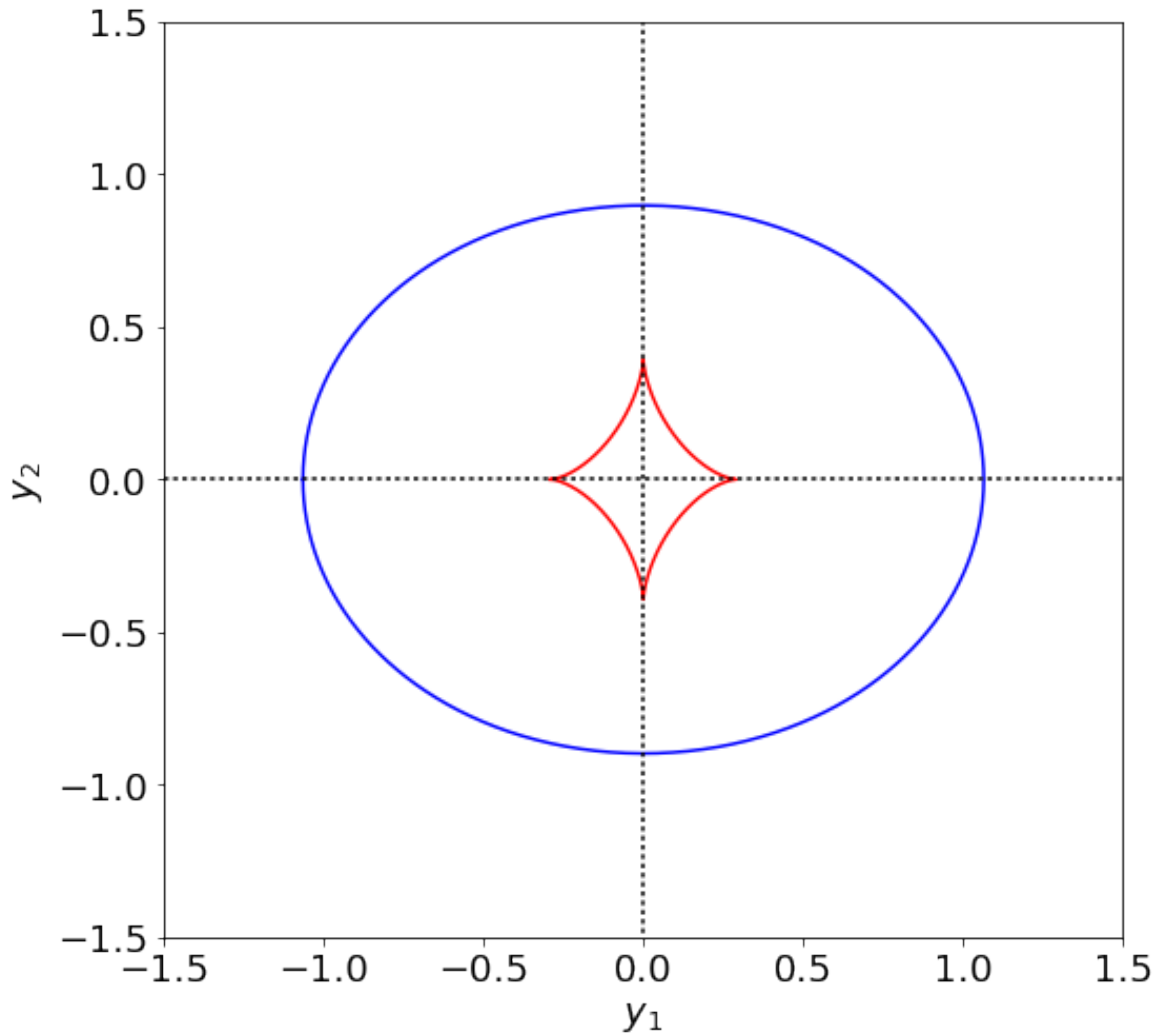




HOW MANY IMAGES?



CRITICAL LINE, CUT, CAUSTIC

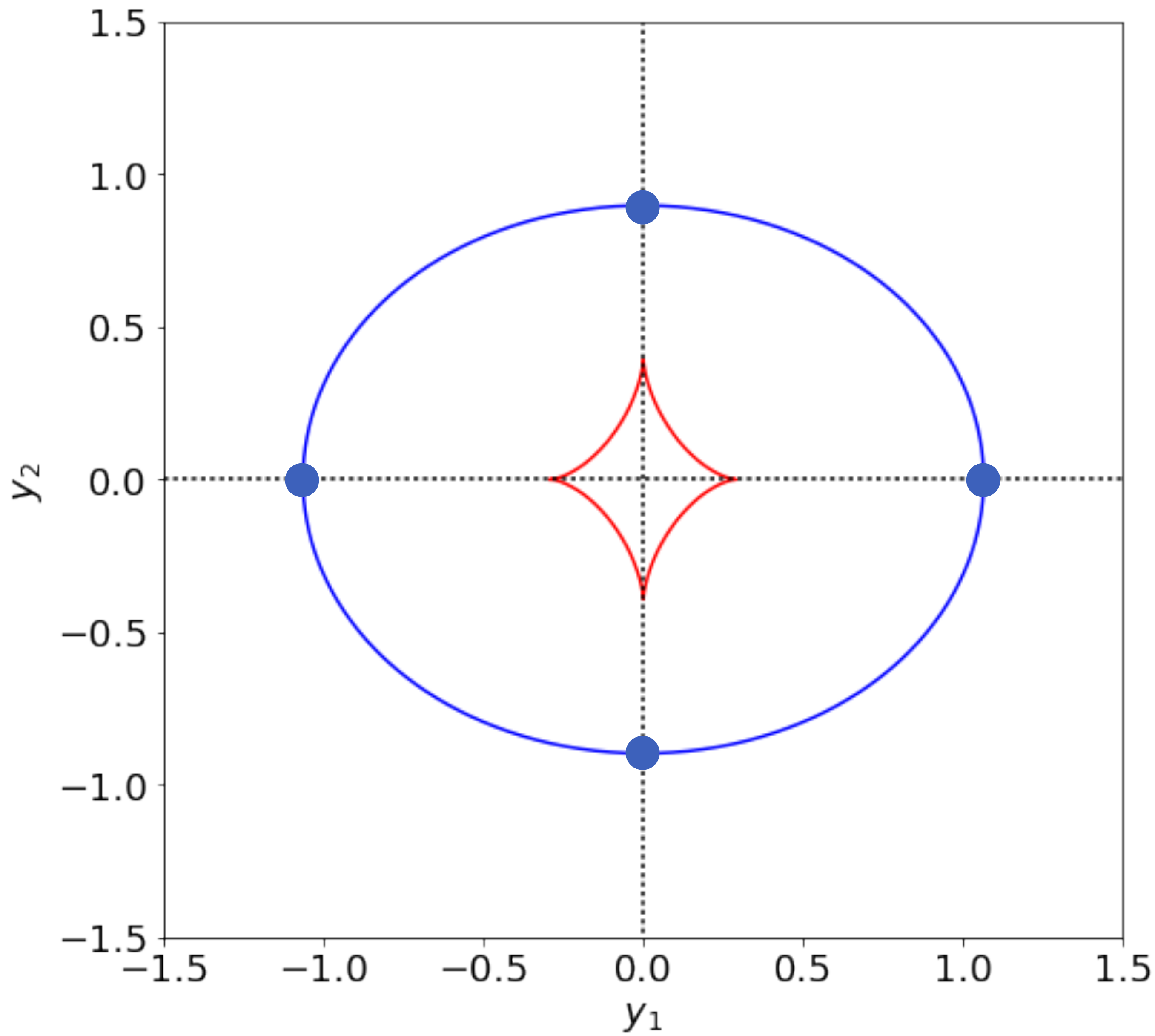


$$
\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\,.
\end{aligned}
$$

$$
\begin{aligned}
y_{c,1} &= -\frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{c,2} &= -\frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\,.
\end{aligned}
$$

$$f' = \sqrt{1-f^2}$$

CRITICAL LINE, CUT, CAUSTIC



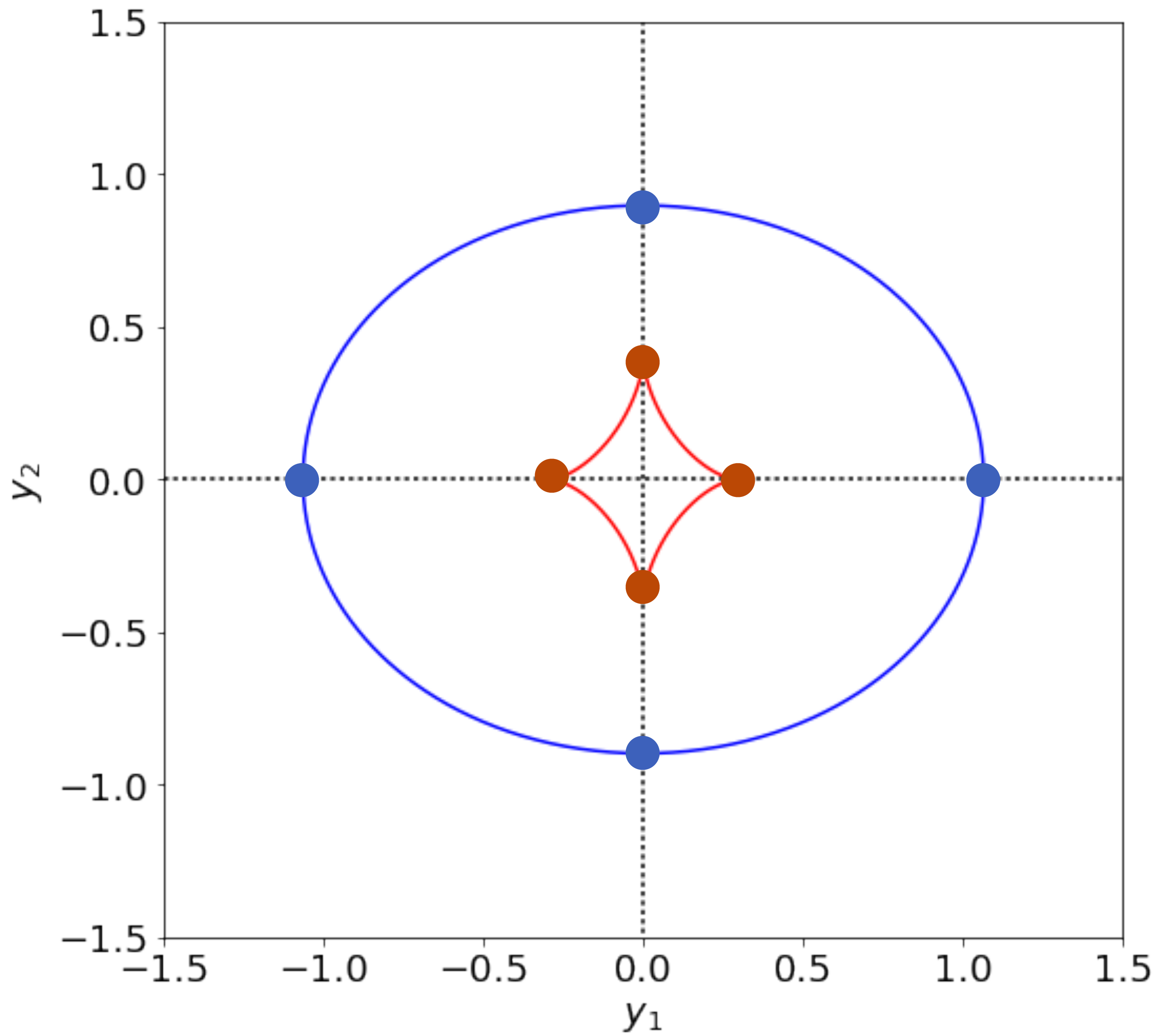
$$
\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}
$$

$$
\begin{aligned}
y_{c,1} &= -\frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{c,2} &= -\frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}
$$

$$f' = \sqrt{1-f^2}$$

$$
\begin{aligned}
s_{1,\pm,c} &= [y_{c,1}(\varphi = 0, \pi), 0]\ , \\
s_{2,\pm,c} &= [0, y_{c,2}(\varphi = \pi/2, -\pi/2)]
\end{aligned}
$$

CRITICAL LINE, CUT, CAUSTIC



$$
\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\,.
\end{aligned}
$$

$$
\begin{aligned}
y_{c,1} &= -\frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{c,2} &= -\frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\,.
\end{aligned}
$$

$$f' = \sqrt{1-f^2}$$

$$
\begin{aligned}
s_{1,\pm,c} &= [y_{c,1}(\varphi = 0,\pi), 0]\,, \\
s_{2,\pm,c} &= [0, y_{c,2}(\varphi = \pi/2, -\pi/2)]
\end{aligned}
$$

$$
\begin{aligned}
s_{1,\pm,t} &= [y_{t,1}(\varphi = 0,\pi), 0]\,, \\
s_{1,\pm,t} &= [0, y_{t,2}(\varphi = \pi/2, -\pi/2)]
\end{aligned}
$$

CRITICAL LINE, CUT, CAUSTIC

$$
\begin{aligned}
s_{1,\pm,c} &= [y_{c,1}(\varphi = 0, \pi), 0] \,, \\
s_{2,\pm,c} &= [0, y_{c,2}(\varphi = \pi/2, -\pi/2)]
\end{aligned}
$$

$$
\begin{aligned}
s_{1,\pm,t} &= [y_{t,1}(\varphi = 0, \pi), 0] \,, \\
s_{1,\pm,t} &= [0, y_{t,2}(\varphi = \pi/2, -\pi/2)]
\end{aligned}
$$

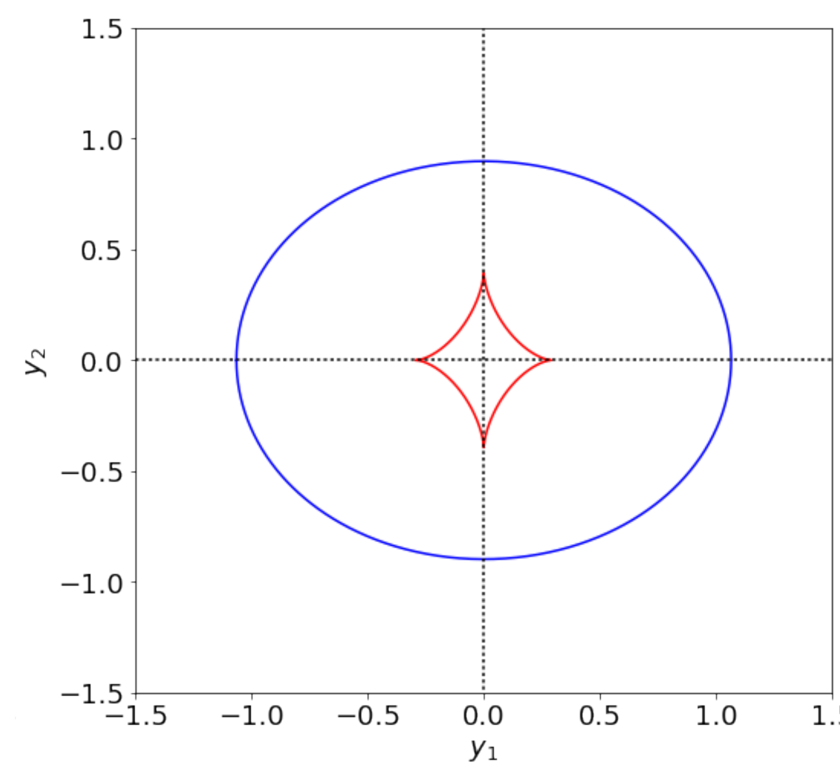
$$
\begin{aligned}
y_{c,1} &= -\frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{c,2} &= -\frac{\sqrt{f}}{f'} \arcsin(f'\sin\varphi) \,.
\end{aligned}
$$

$$
\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'} \arcsin(f'\sin\varphi) \,.
\end{aligned}
$$

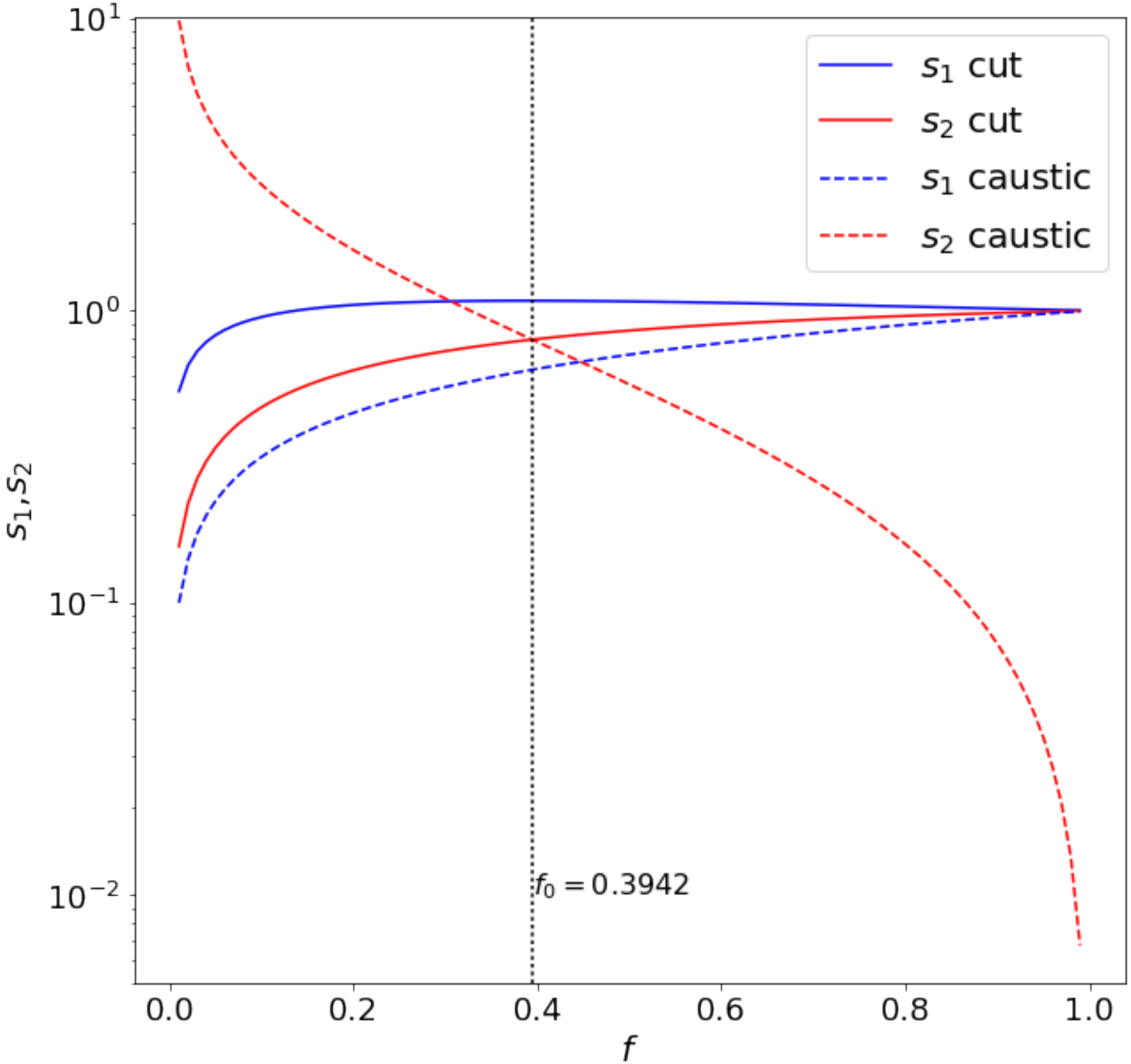
We can easily see that

$$s_{1,c} > s_{1,t}$$

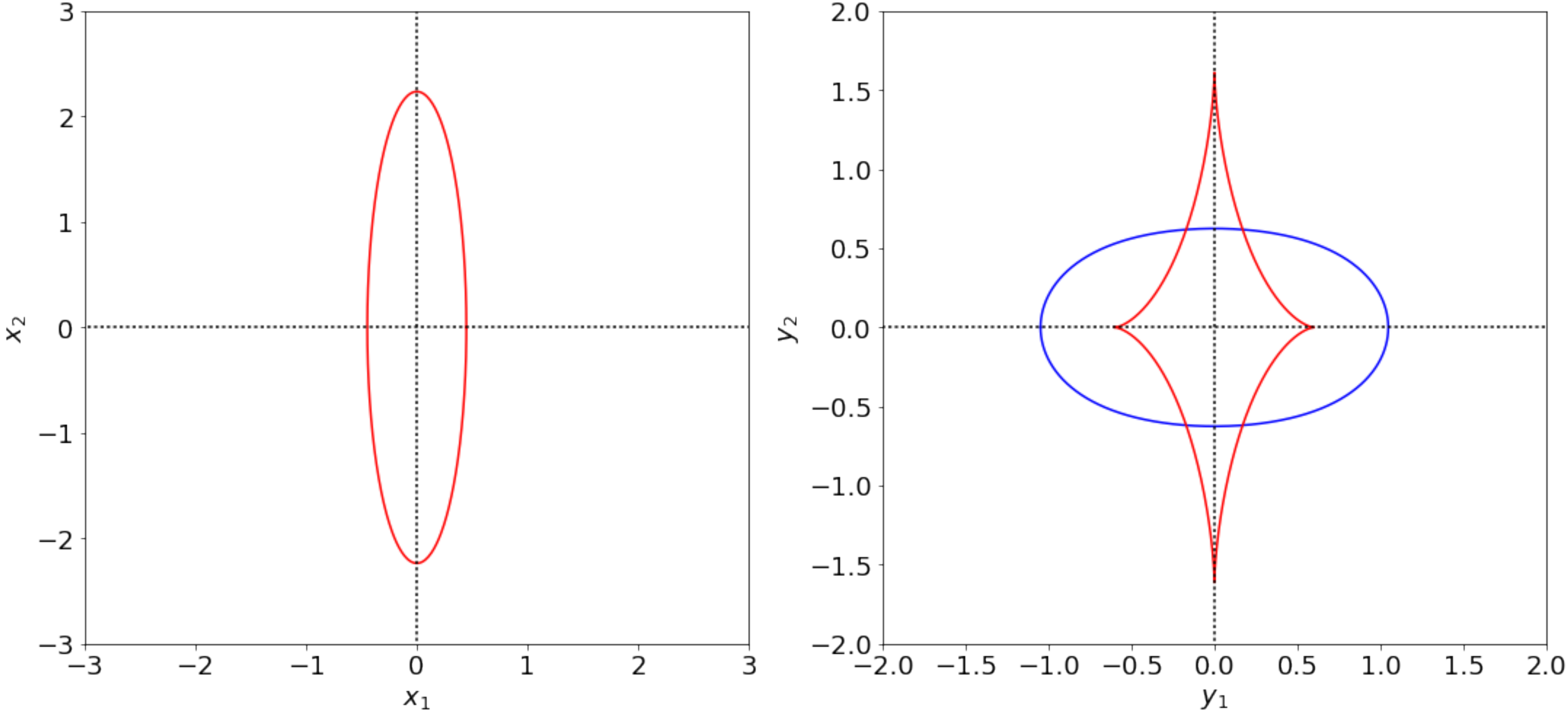
independent on f.



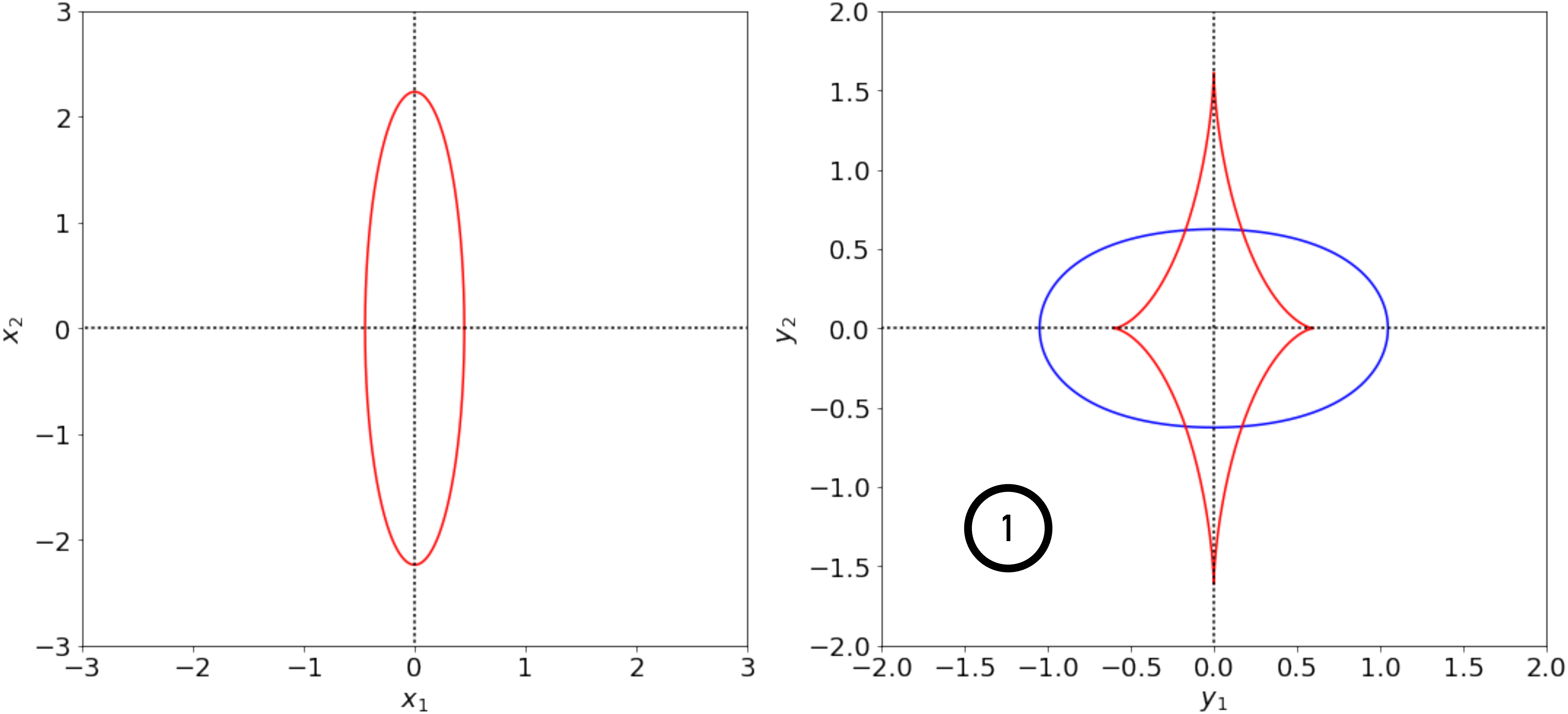
CRITICAL LINE, CUT, CAUSTIC



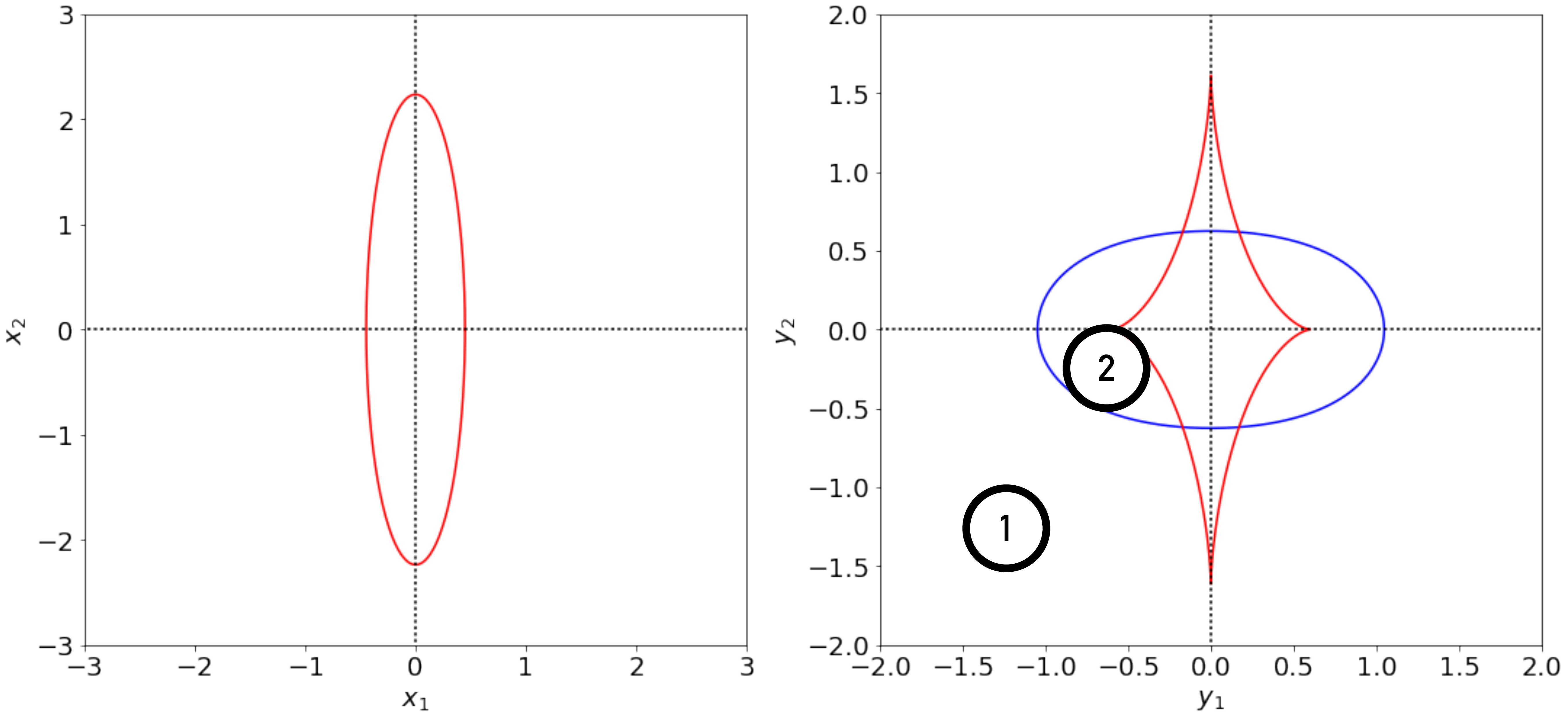
CRITICAL LINE, CUT, CAUSTIC



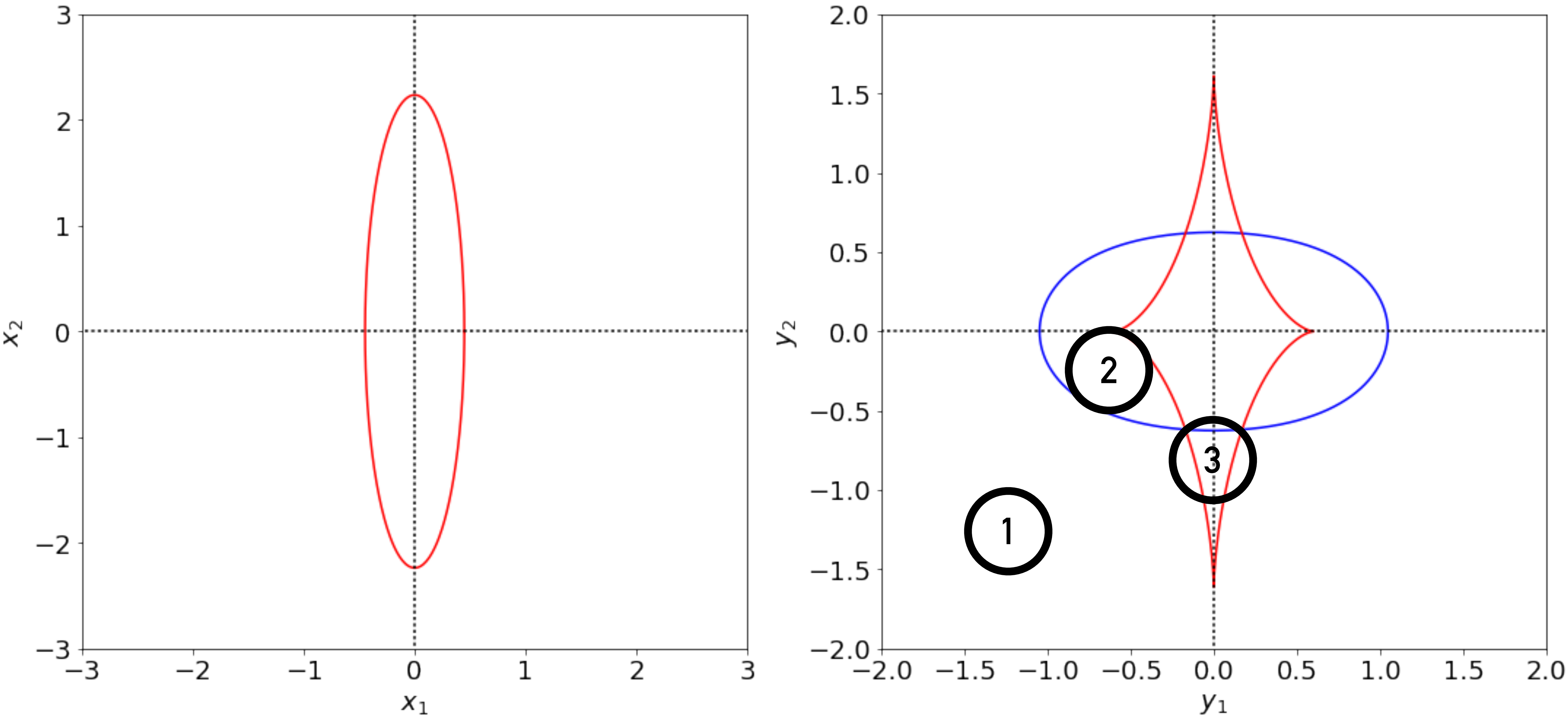
CRITICAL LINE, CUT, CAUSTIC



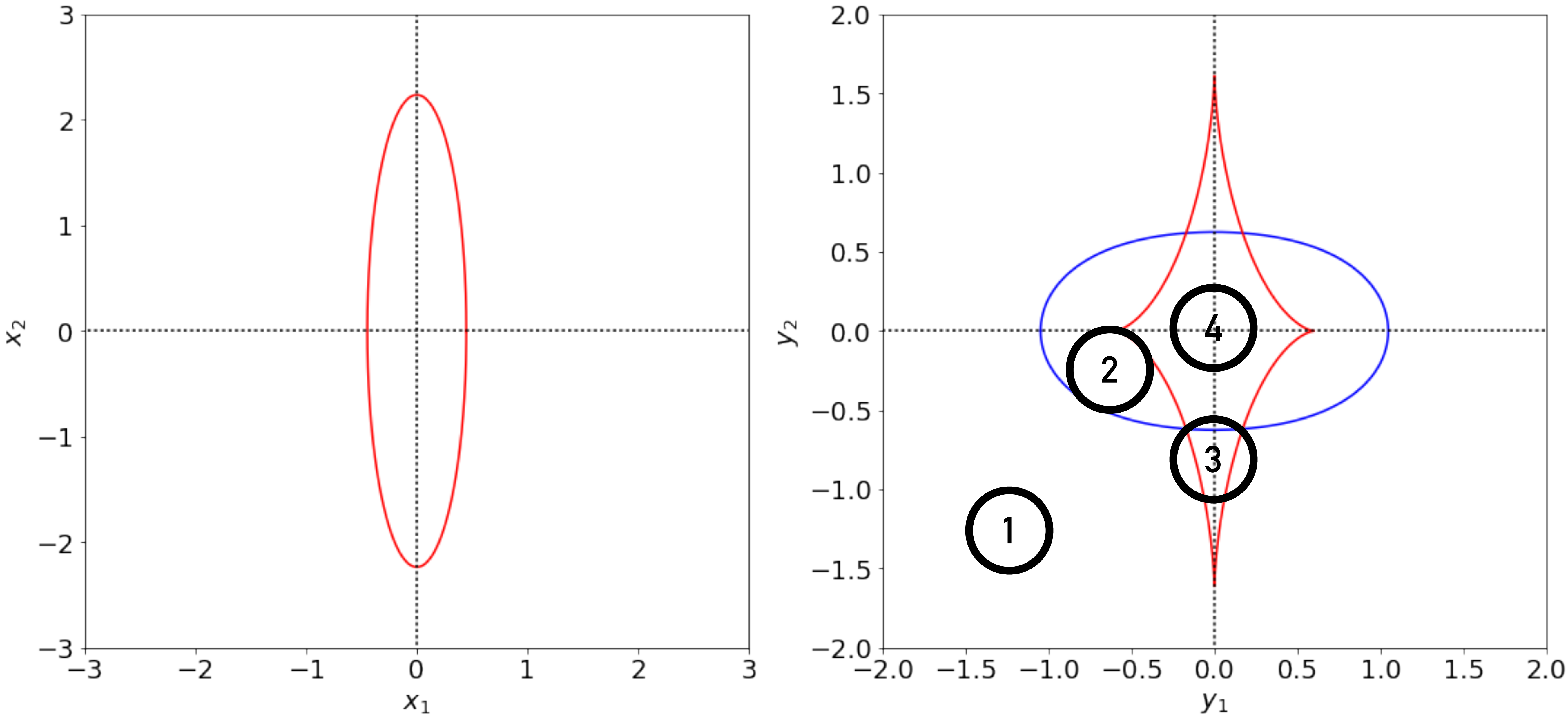
CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC



CRITICAL LINE, CUT, CAUSTIC



NON-SINGULAR-ISOTHERMAL-ELLIPSOID

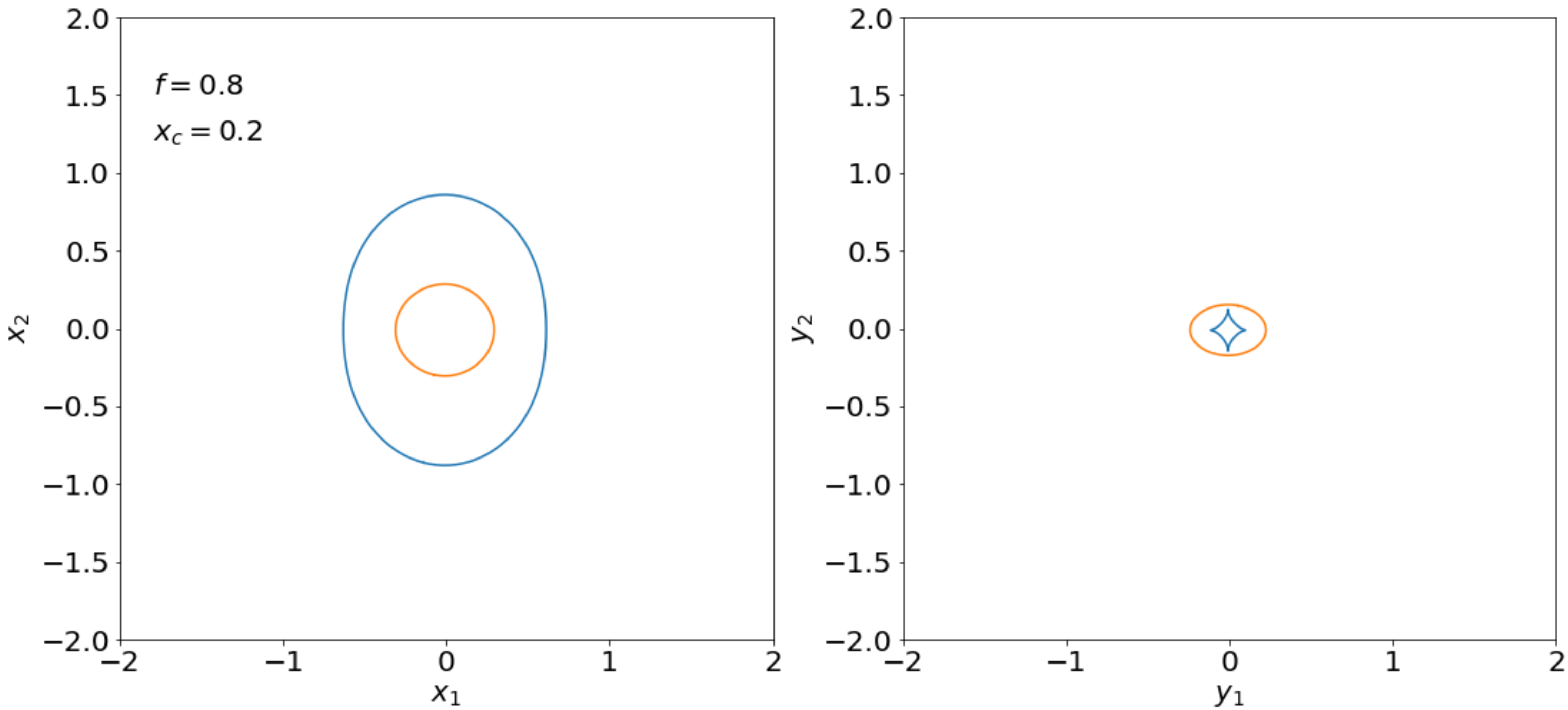
The SIE can be turned into a non-singular model by adding a core:

$$\Sigma(\vec{\xi}) = \frac{\sigma^2}{2G}\frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2\xi_2^2 + \xi_c^2}}$$

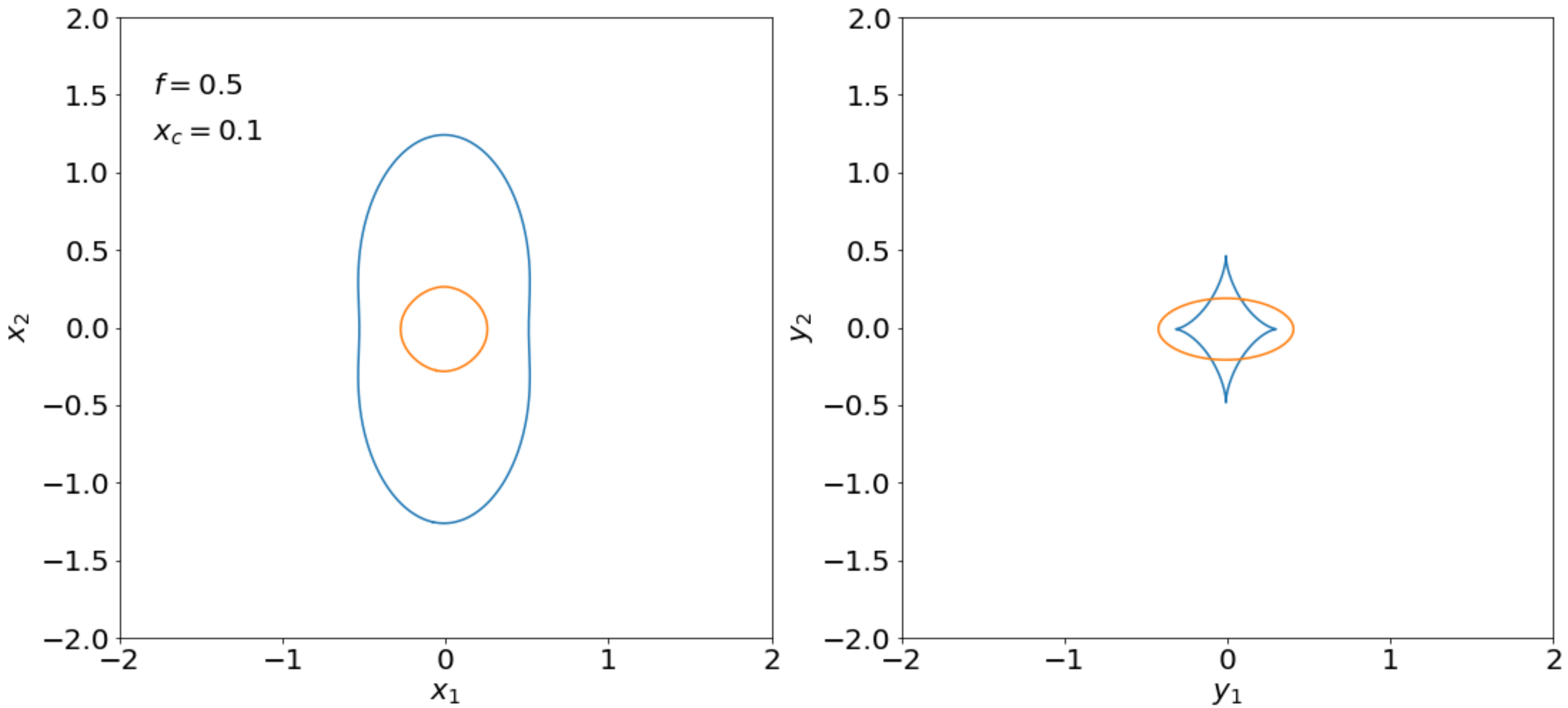
$$\kappa(\vec{x}) = \frac{\sqrt{f}}{2\sqrt{x_1^2 + f^2 x_2^2 + x_c^2}} .$$

In this case, the analytical treatment of the lens is much more complicated. We limit the discussion to the topology of the critical lines and caustics and infer information about the image multiplicities…

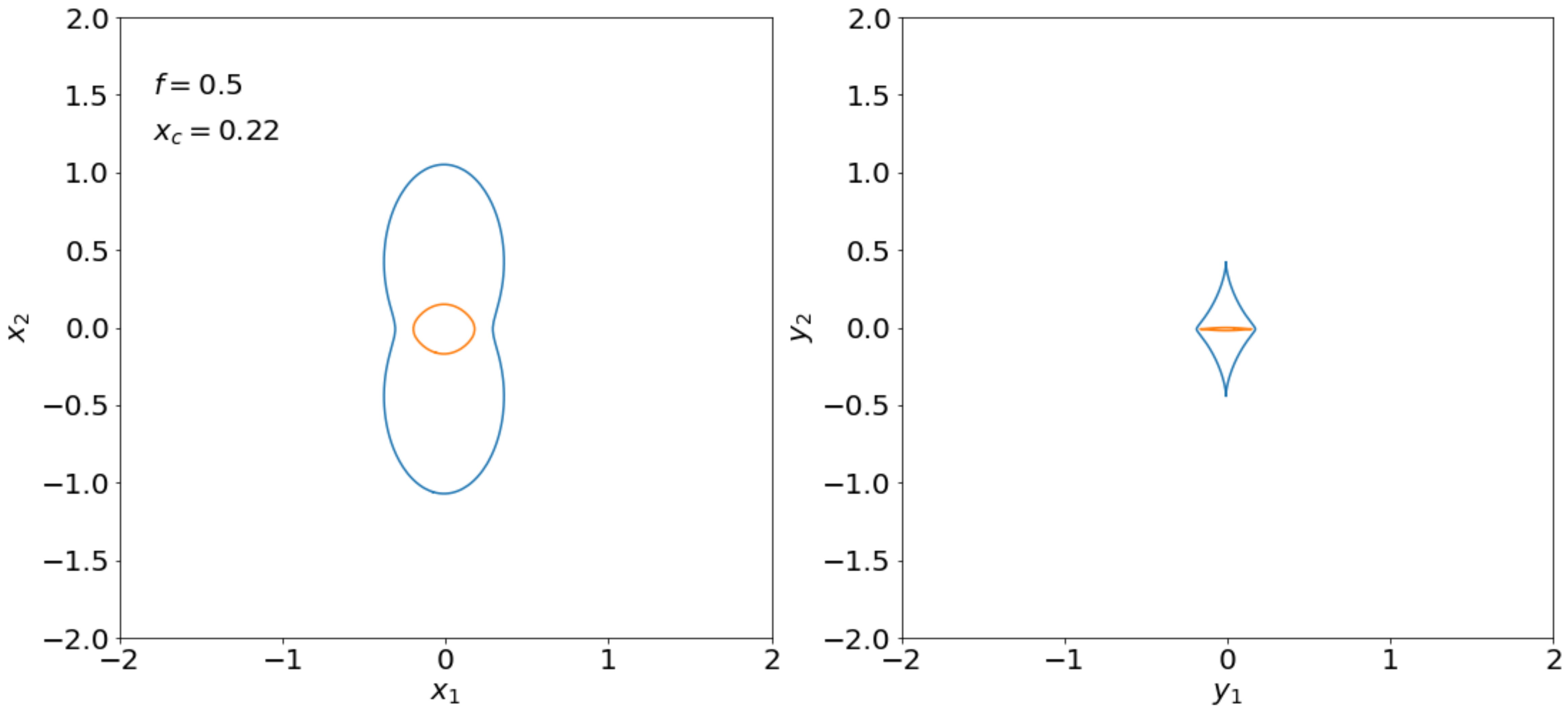
SMALL CORE RADIUS AND ELLIPTICITY



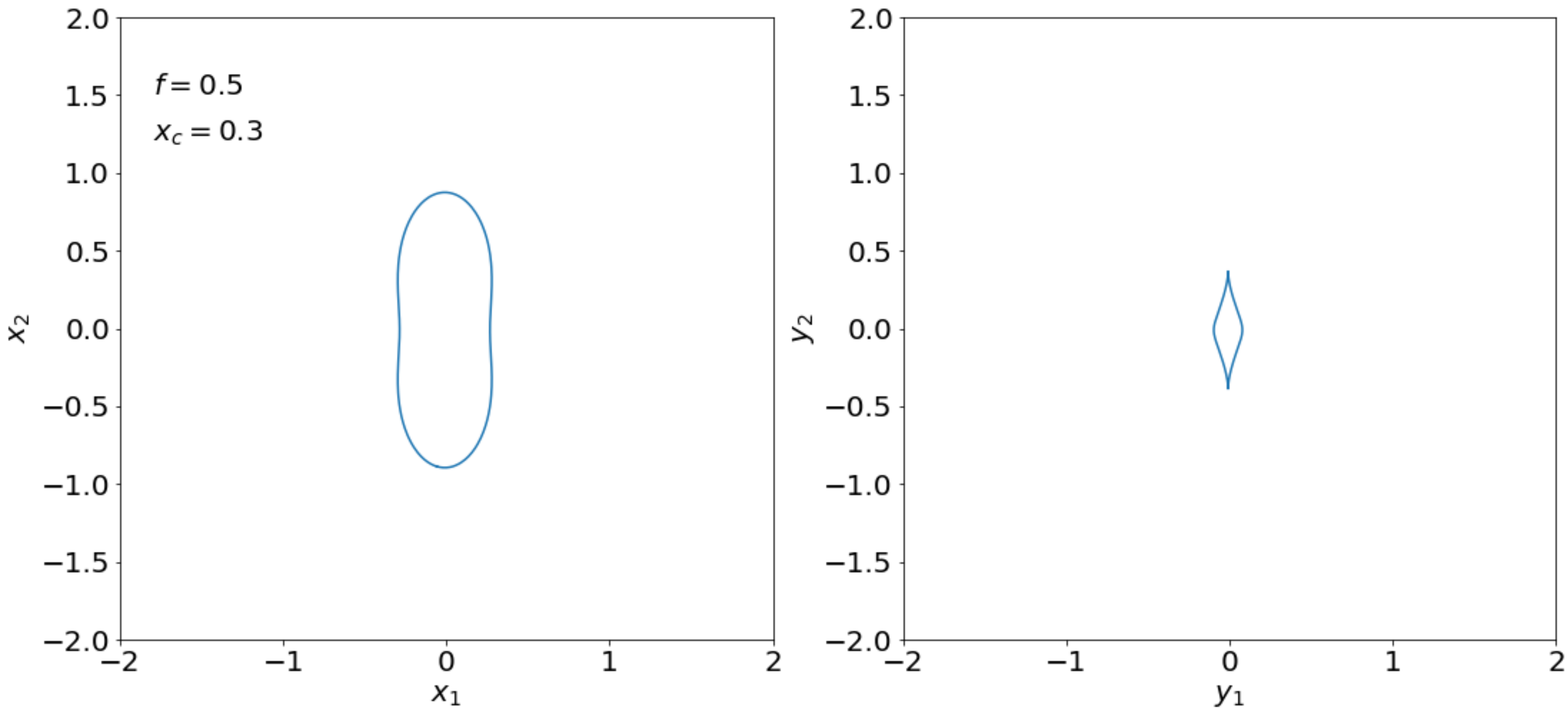
NAKED CUSP



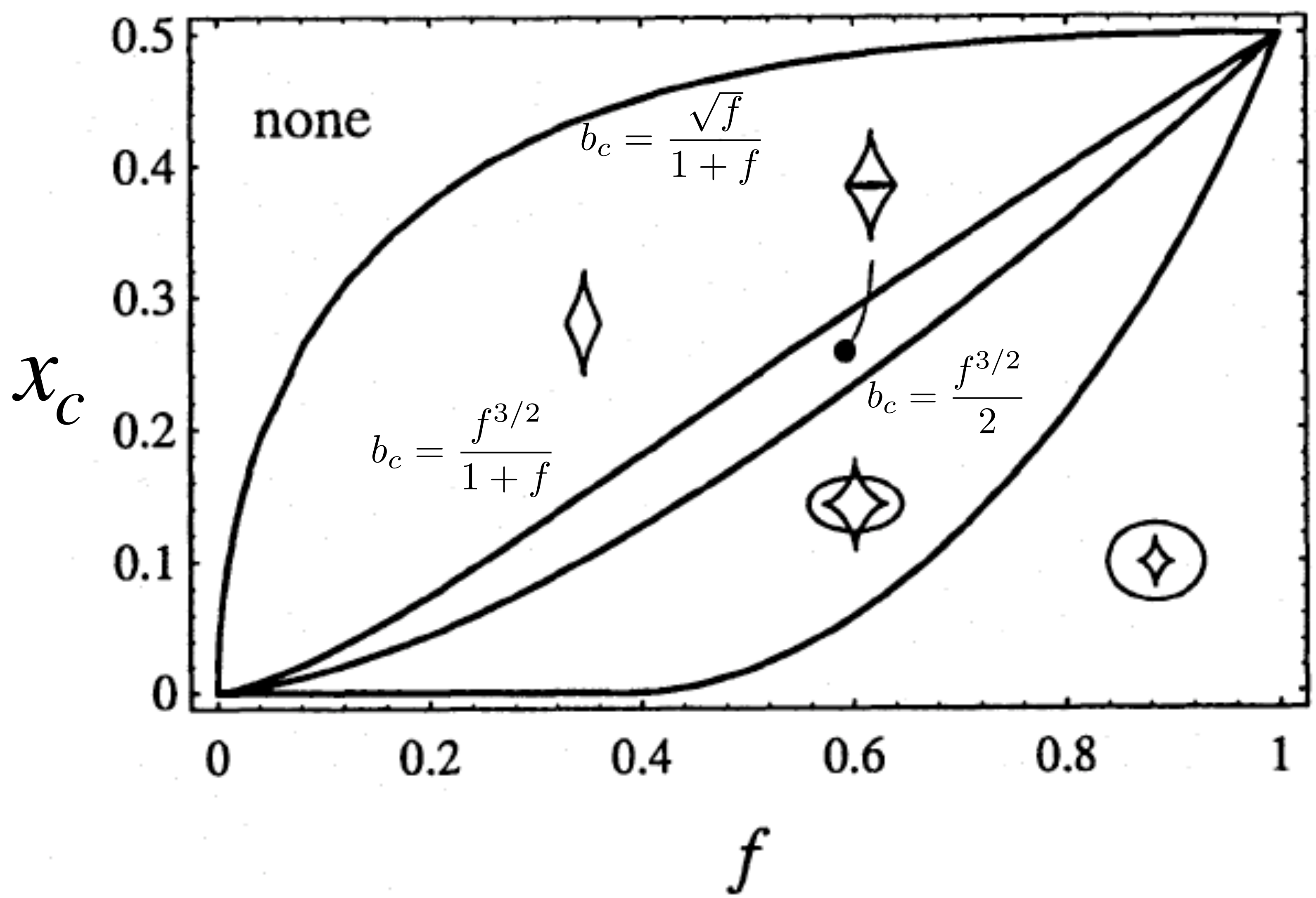
INCREASING THE CORE SIZE…



NO RADIAL CRITICAL LINE AND CAUSTIC



CAUSTIC TOPOLOGIES (SEE KORMANN, BARTELMANN & SCHNEIDER, 1994)



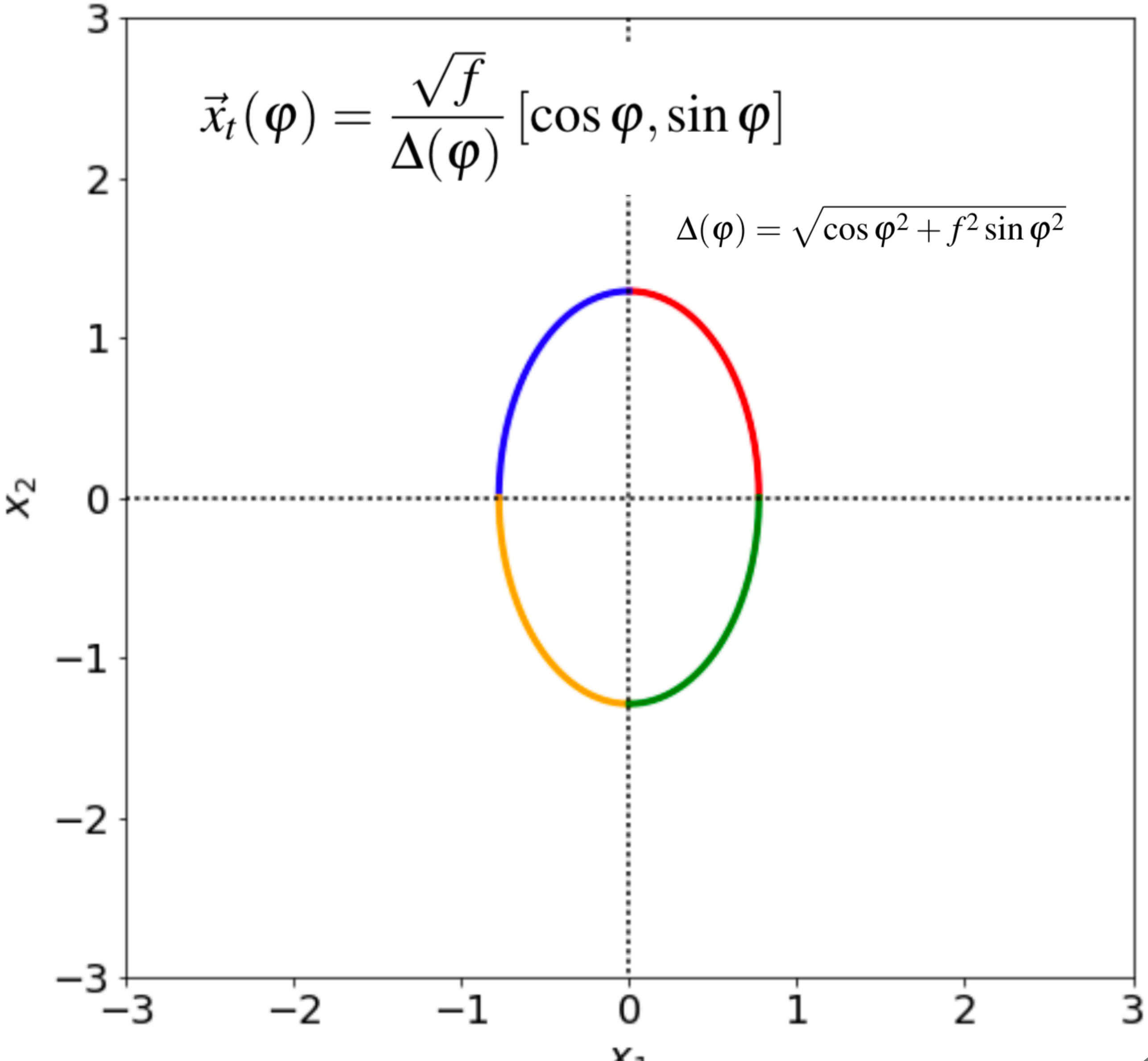
GRAVITATIONAL LENSING

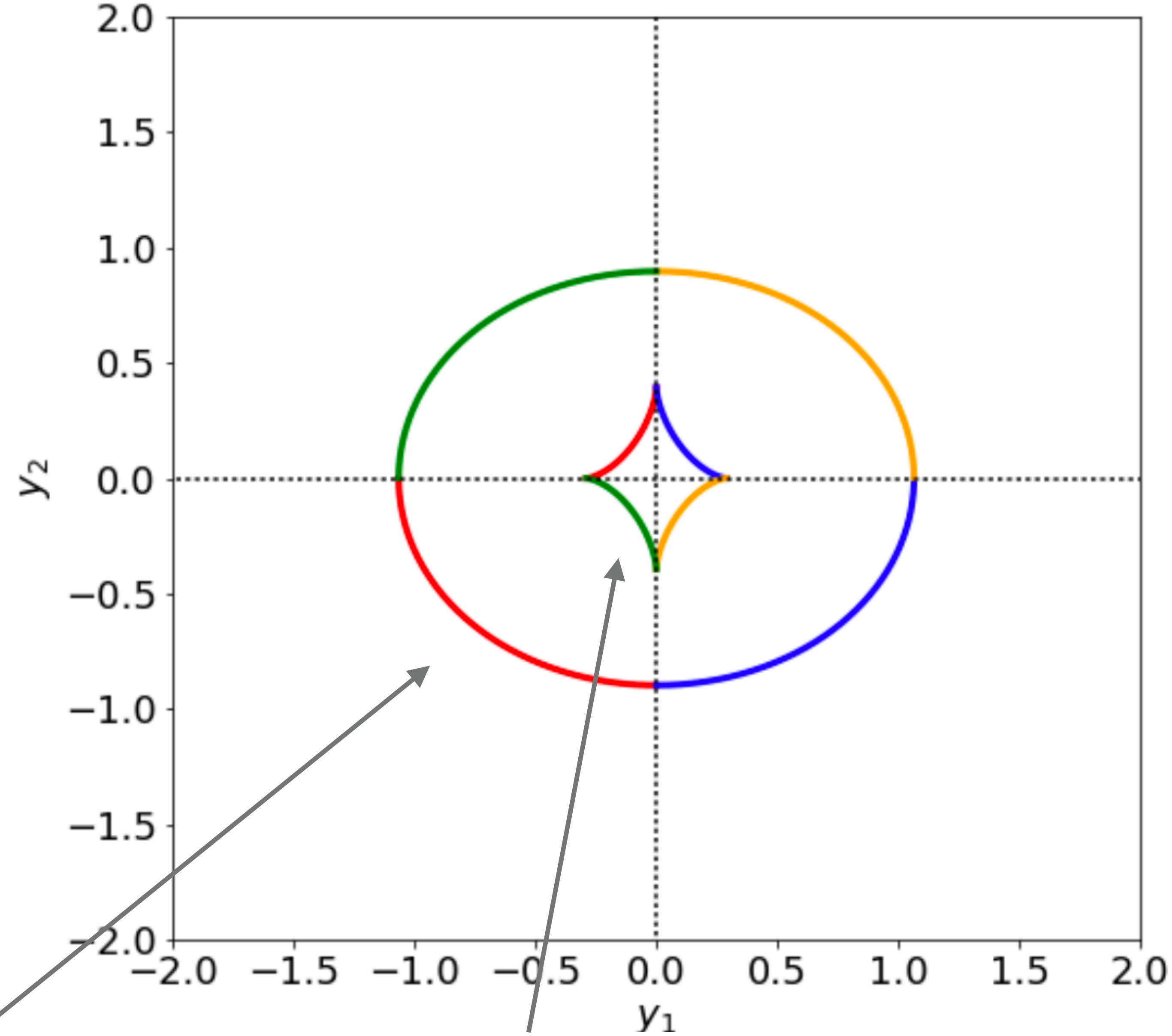
ELLIPTICAL MODELS II

R. Benton Metcalf

2022-2023

MAPPING RULES FOR CAUSTICS AND CUT

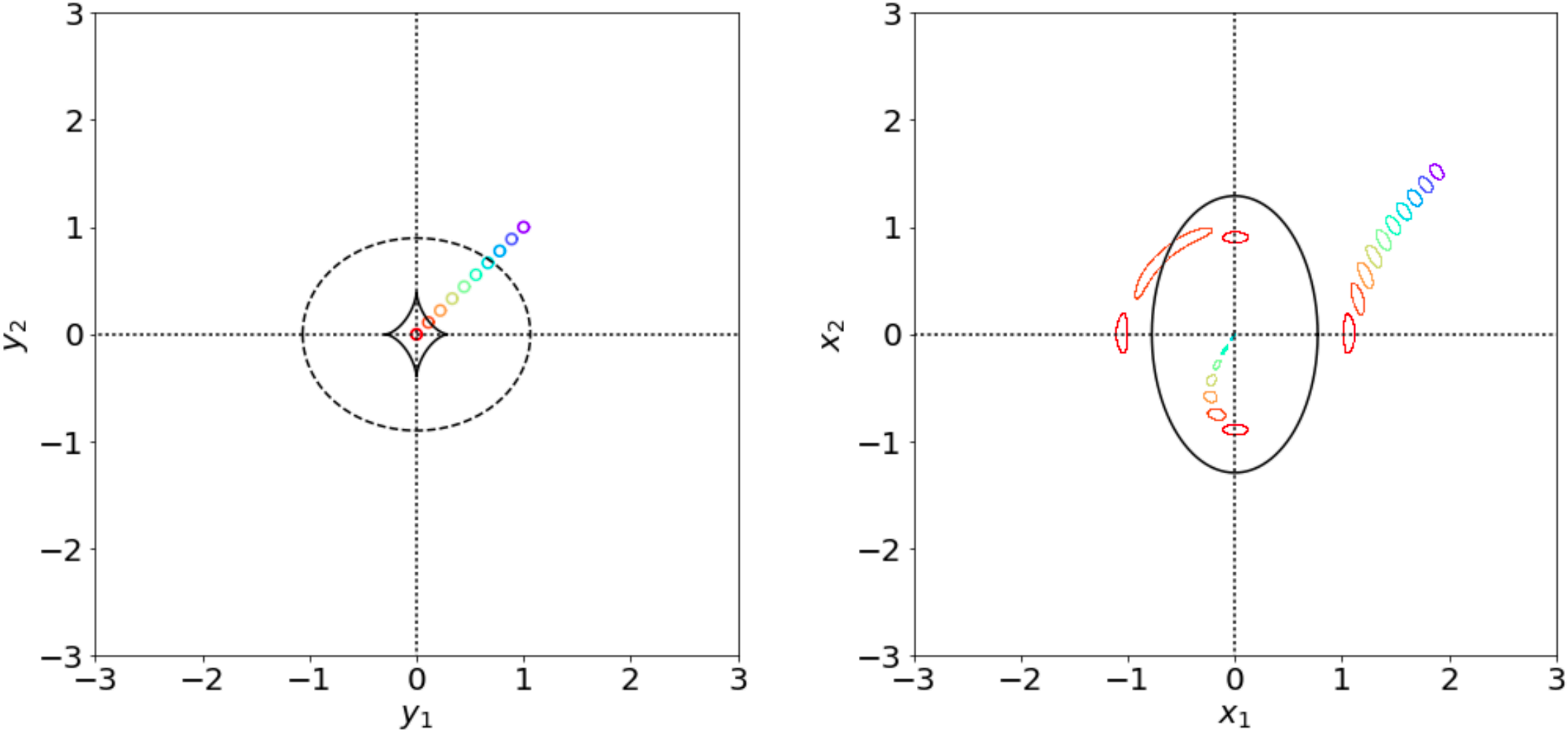




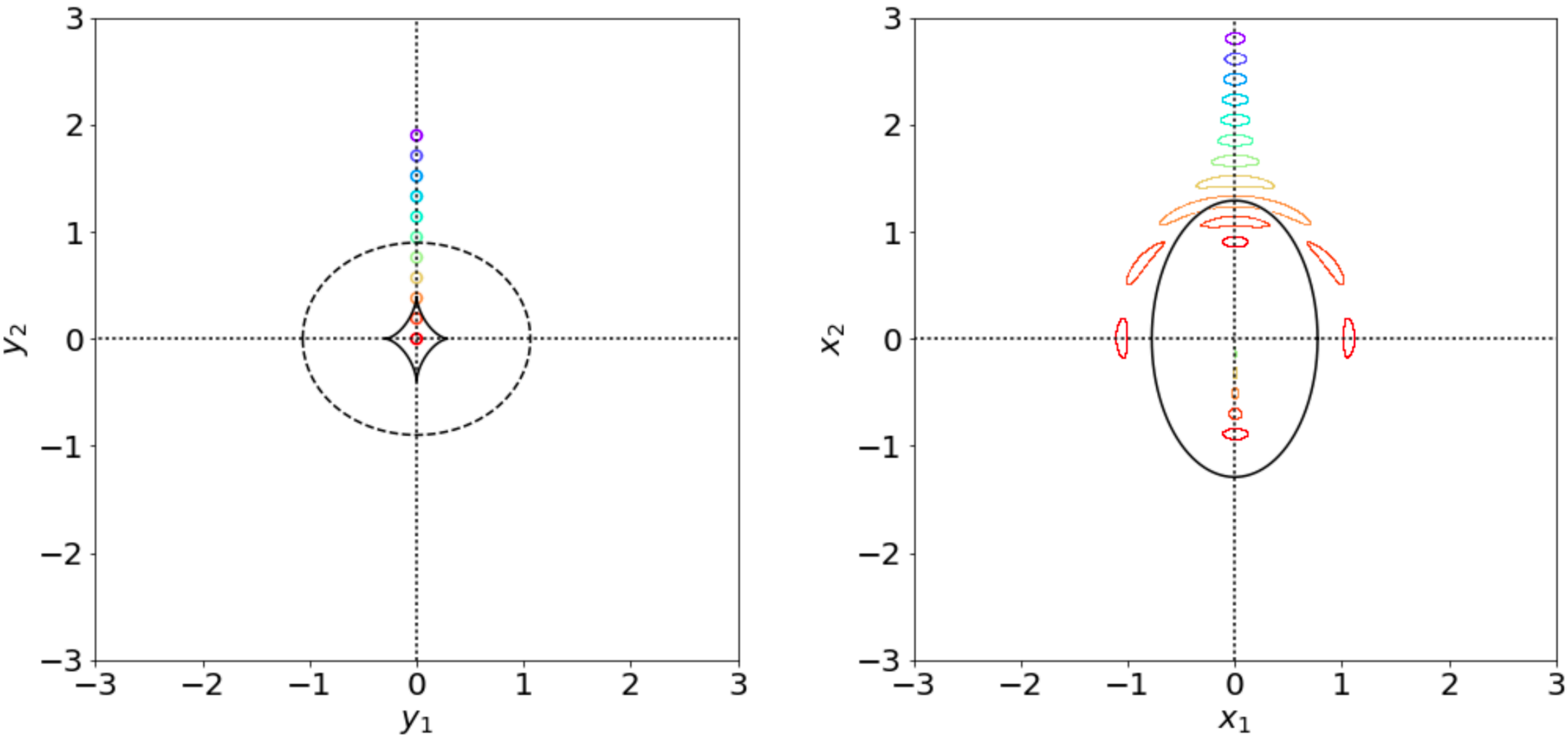
$$
\begin{aligned}
y_{c,1} &= -\frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{c,2} &= -\frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}
$$

$$
\begin{aligned}
y_{t,1} &= \frac{\sqrt{f}}{\Delta(\varphi)}\cos\varphi - \frac{\sqrt{f}}{f'}\operatorname{arcsinh}\left(\frac{f'}{f}\cos\varphi\right) \\
y_{t,2} &= \frac{\sqrt{f}}{\Delta(\varphi)}\sin\varphi - \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)\ .
\end{aligned}
$$

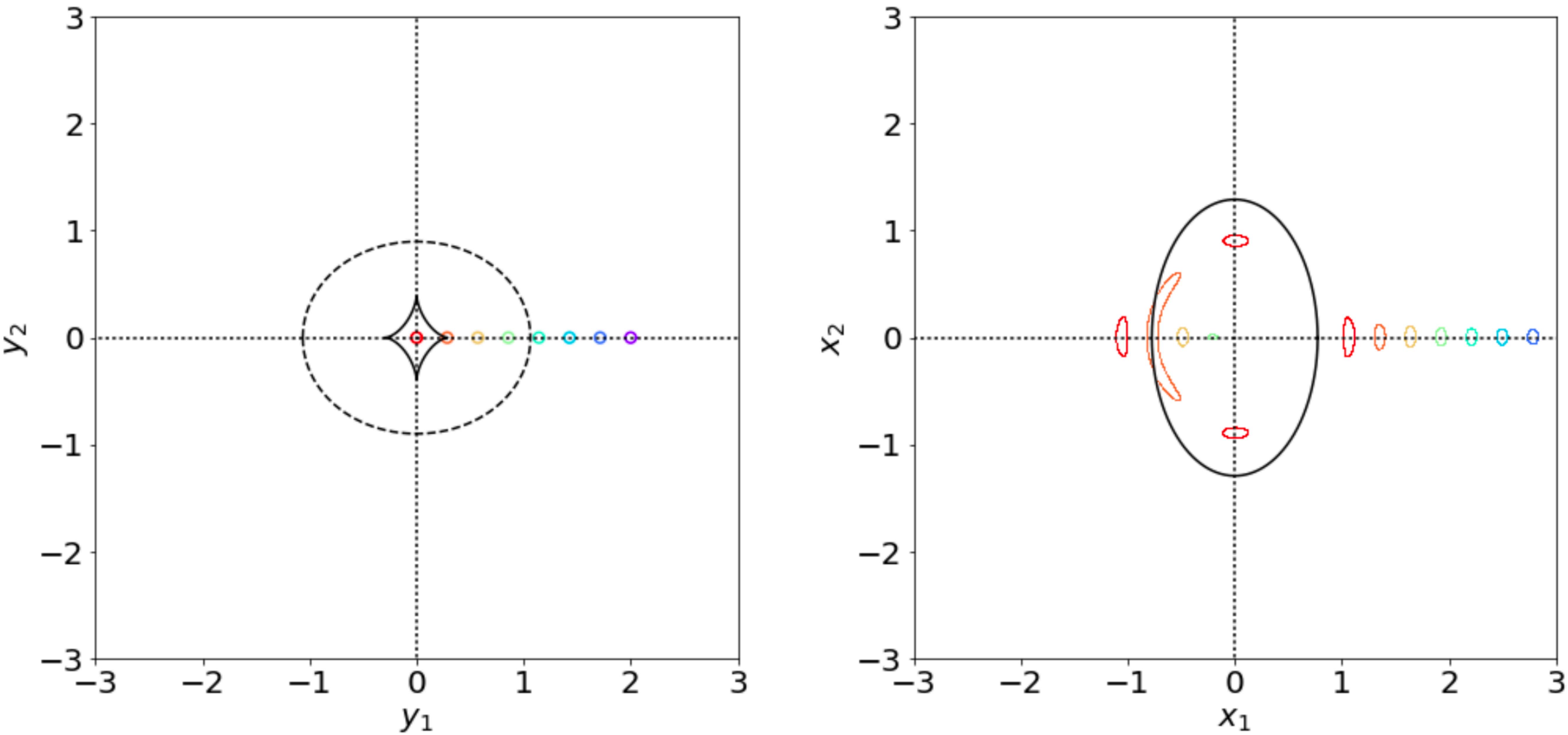
MULTIPLE IMAGES BY SINGULAR ISOTHERMAL ELLIPSOIDS



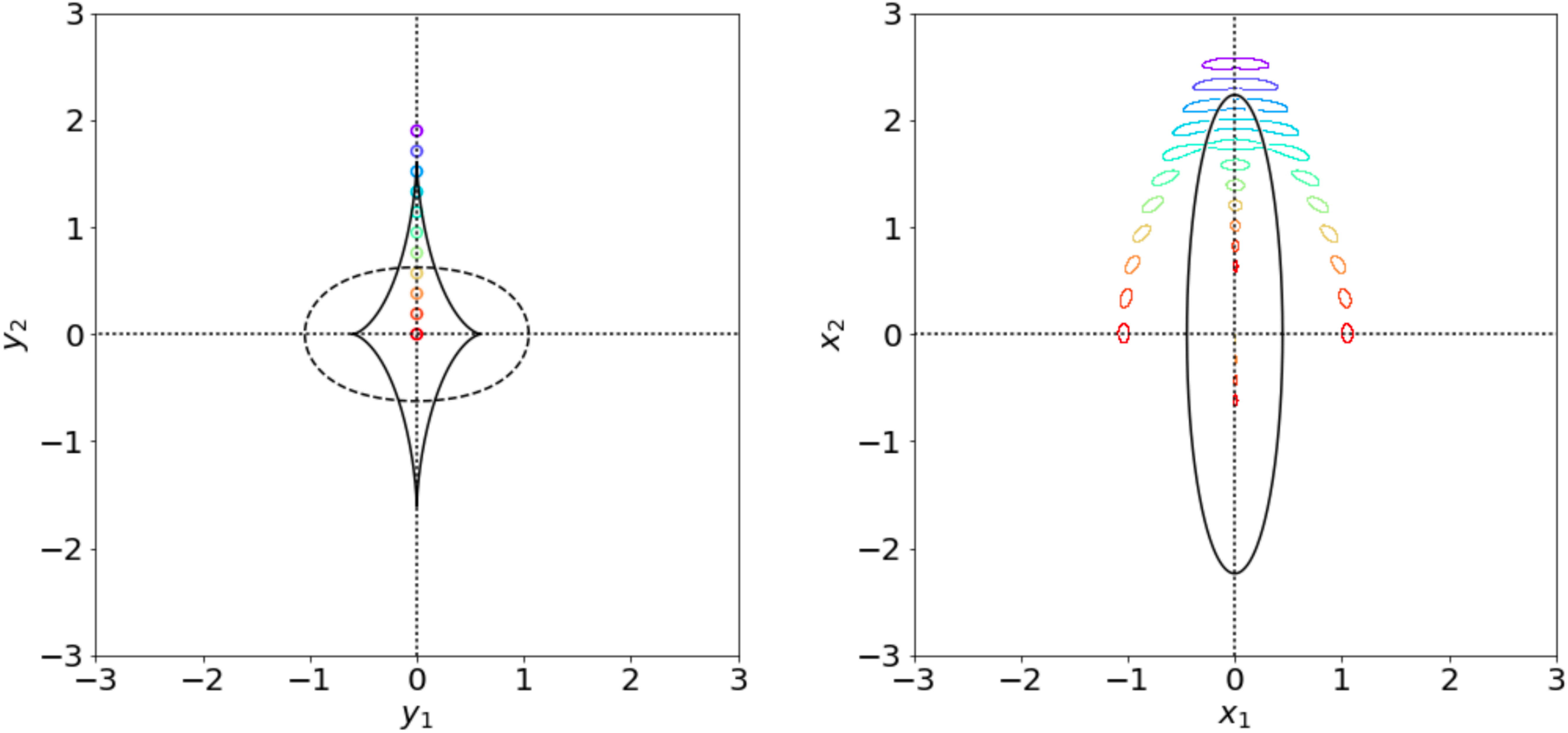
MULTIPLE IMAGES BY SINGULAR ISOTHERMAL ELLIPSOIDS



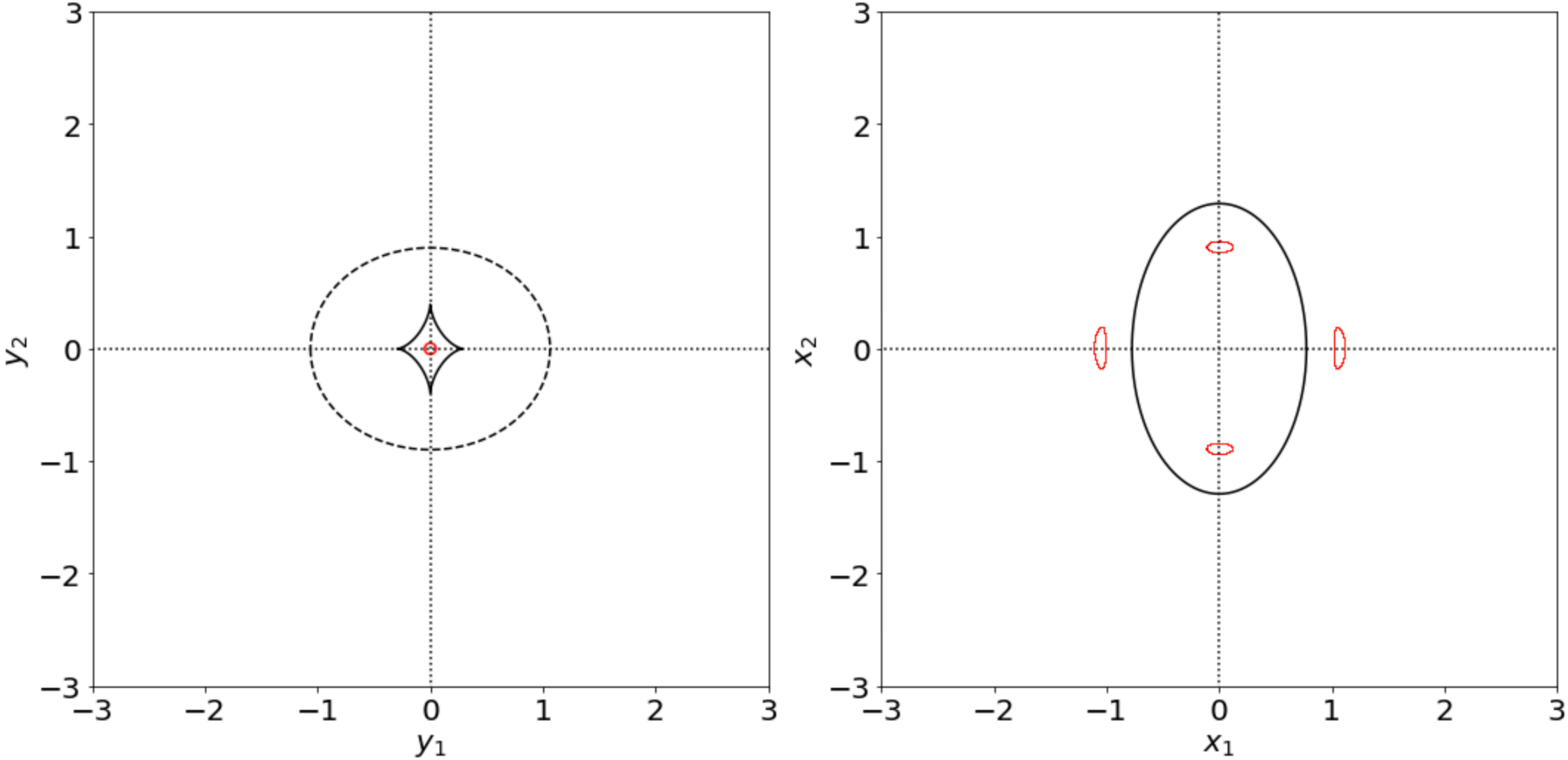
MULTIPLE IMAGES BY SINGULAR ISOTHERMAL ELLIPSOIDS



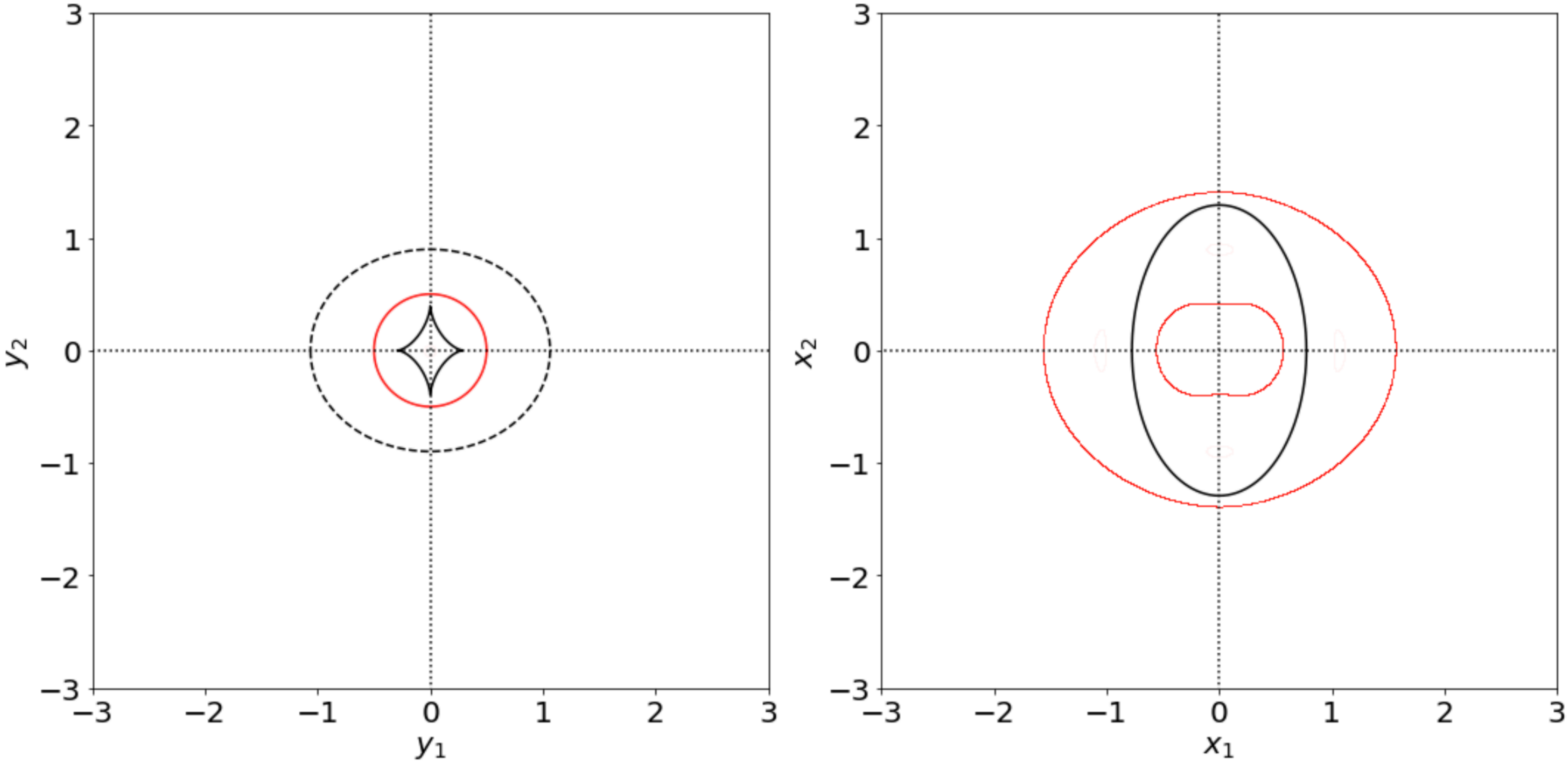
NAKED CUSPS



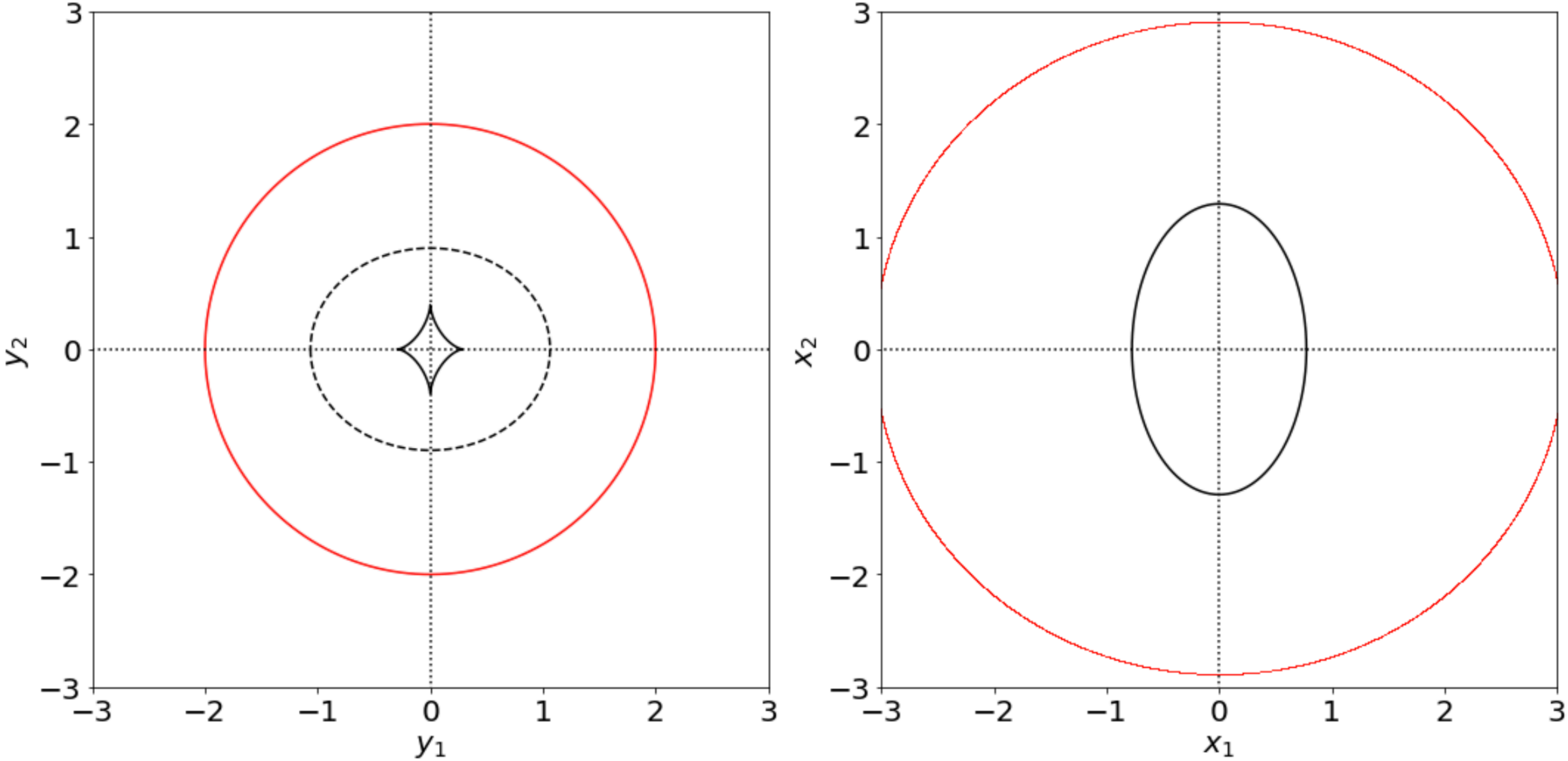
SOURCE SIZE EFFECTS



SOURCE SIZE EFFECTS



SOURCE SIZE EFFECTS



EXAMPLE 1

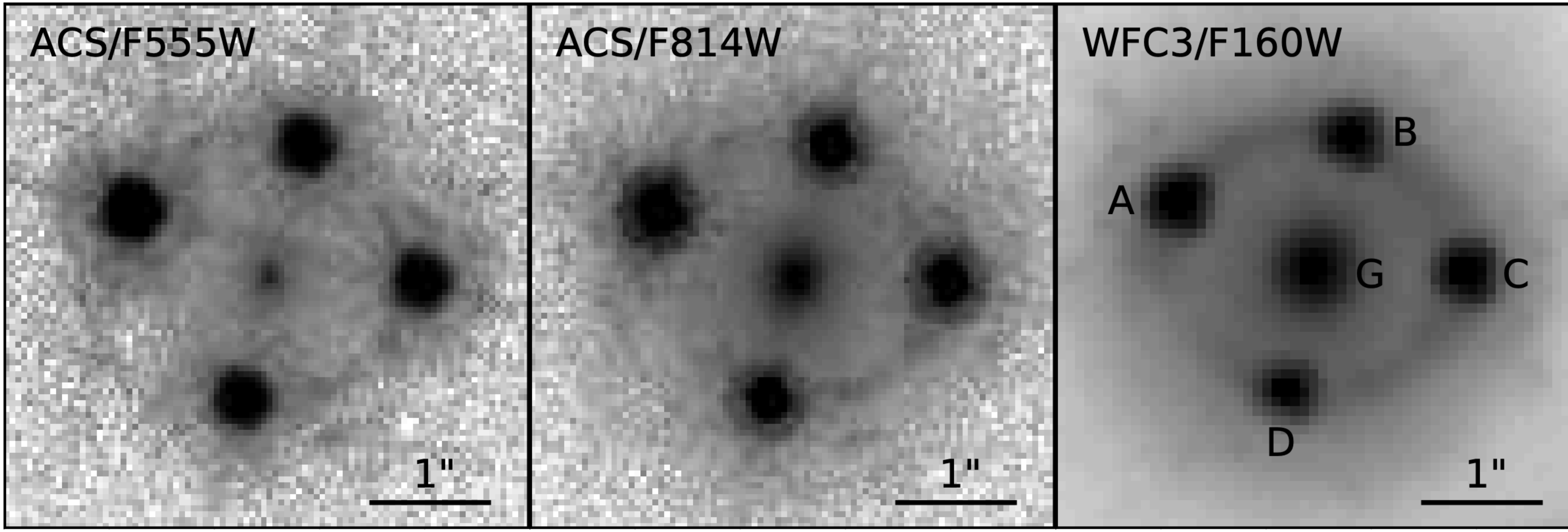
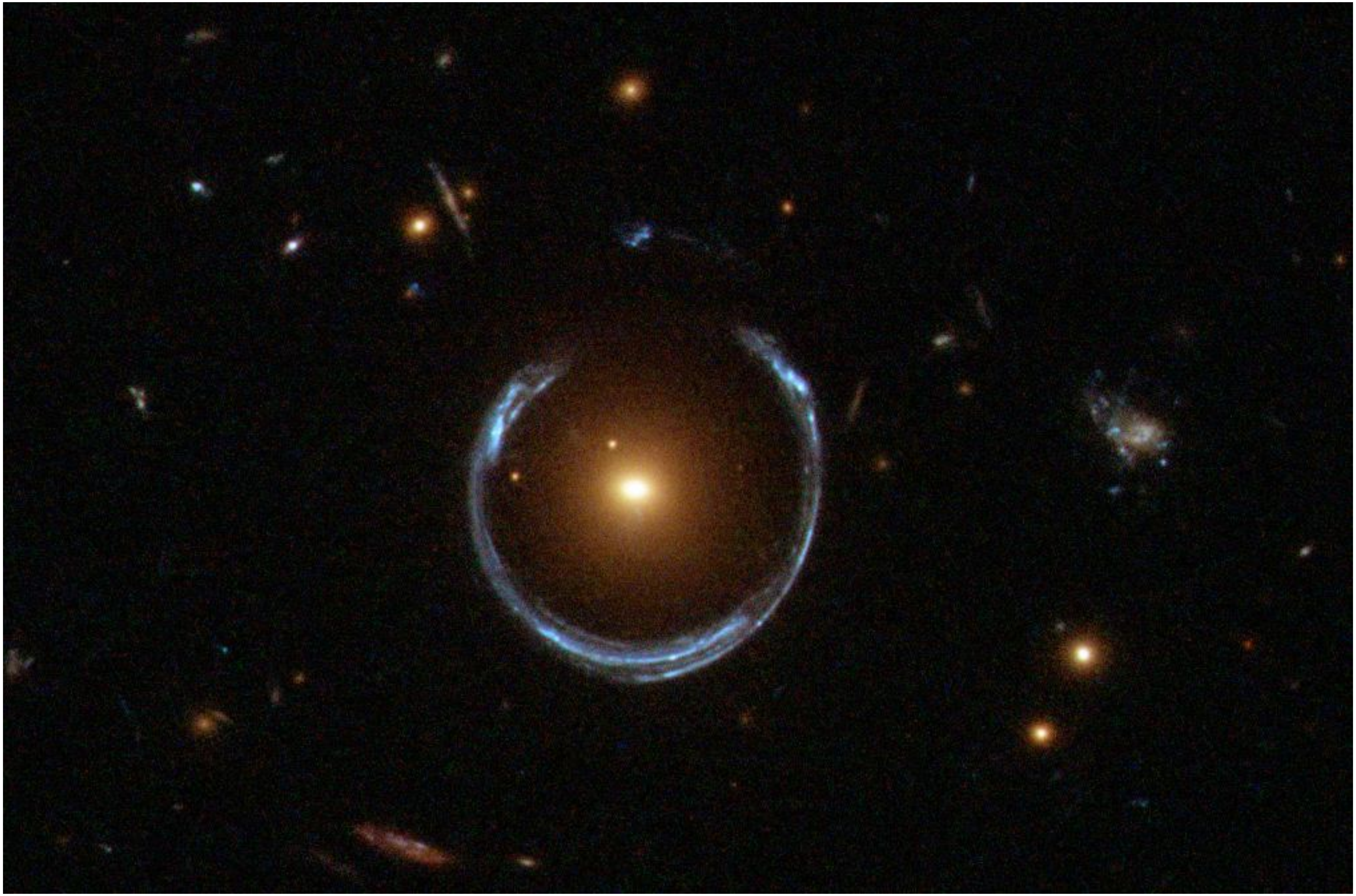
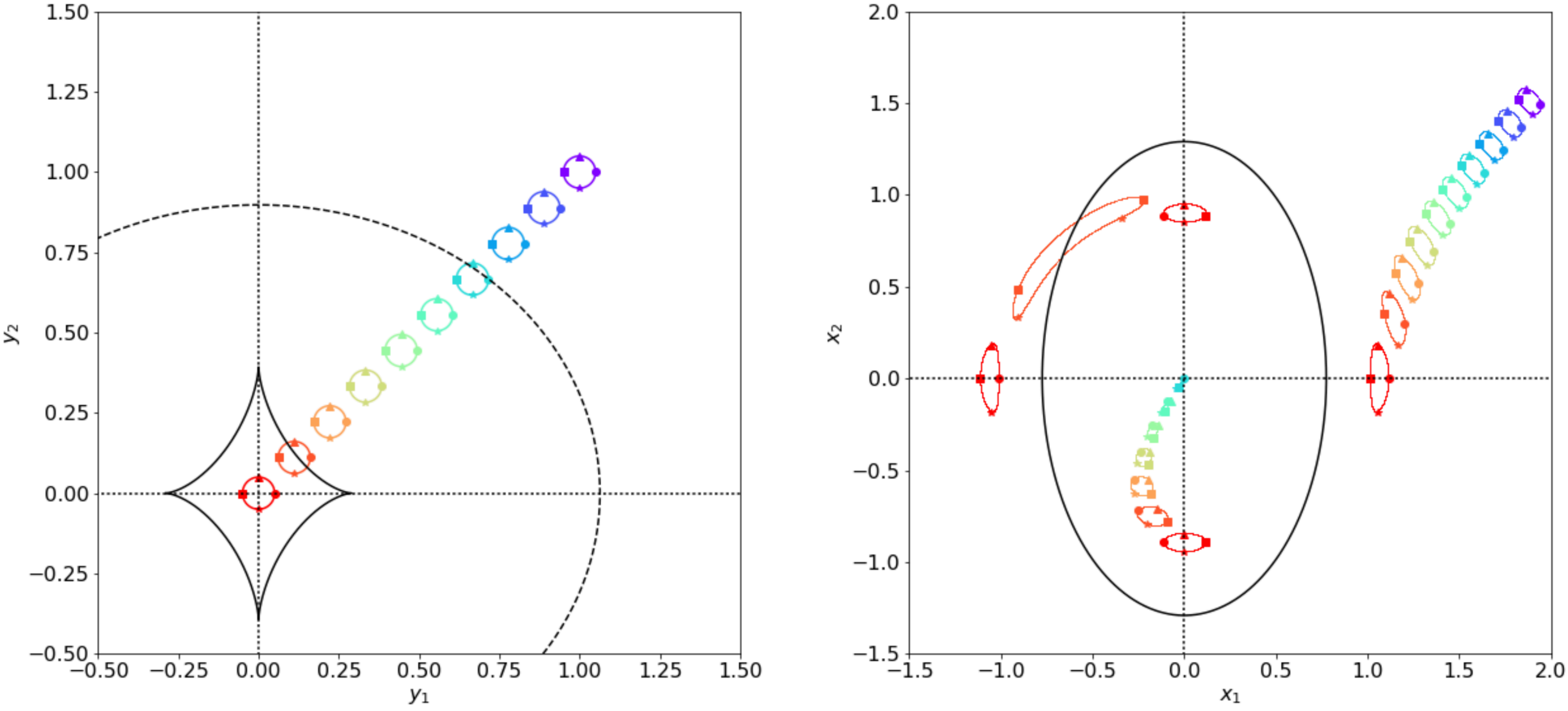


Figure 1. *HST* images of HE 0435−1223. Shown are cutouts of the lens system used for lens modeling in the ACS/F555W (left), ACS/F814W (middle), and WFC3/F160W (right) bands. The images are 4.″5 on a side. The scale is indicated in the bottom right of each panel. The main lens galaxy (G) and lensed quasar images (A, B, C, D) are marked.

EXAMPLE 2



PARITY INVERSION ACROSS THE CRITICAL LINES

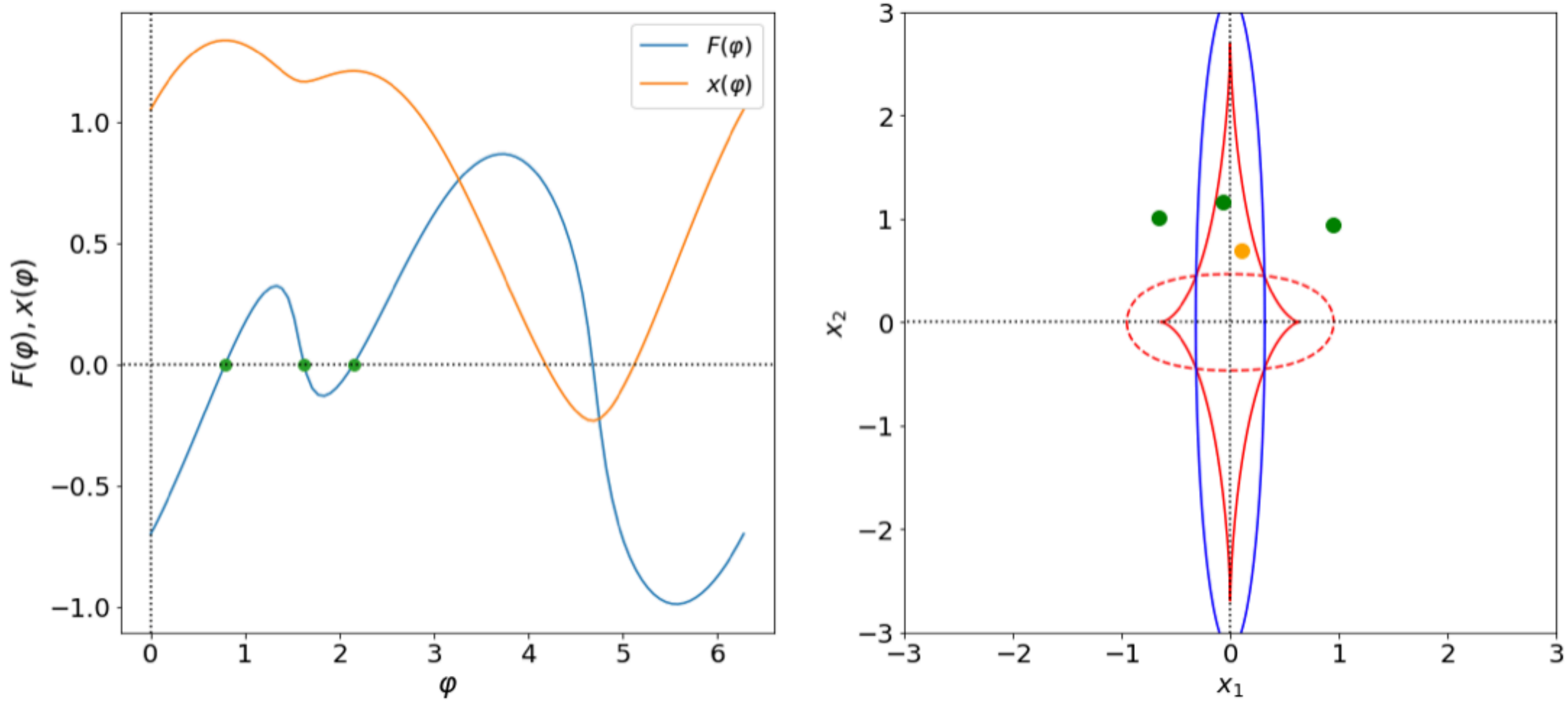


EXAMPLE 3

SIE LENSES : IMAGE POSITIONS

$$x = (y_1 + \alpha_1(\phi))\cos\phi + (y_2 + \alpha_2(\phi))\sin\phi$$

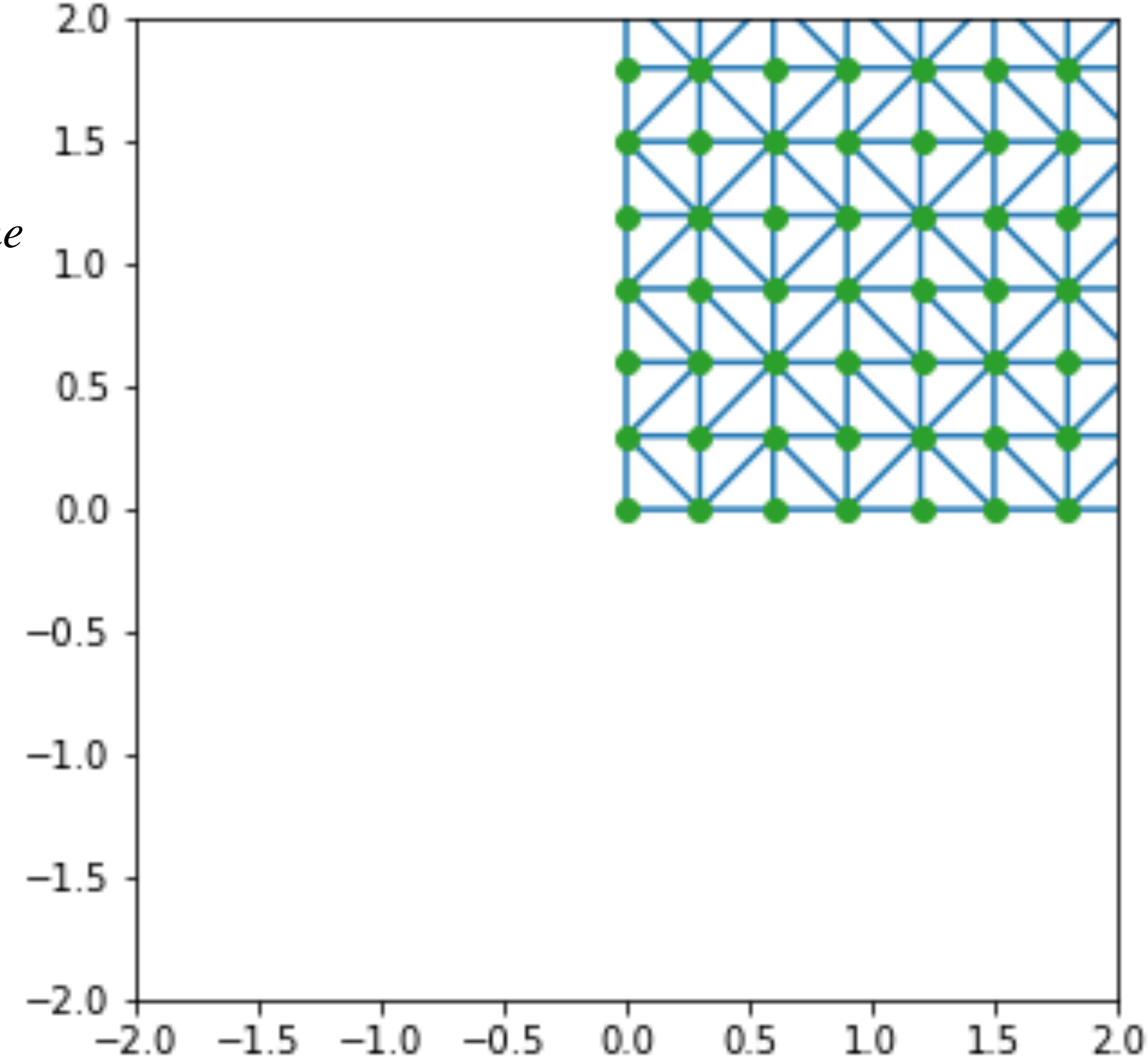
$$F(\phi) = (y_1 + \alpha_1(\phi))\sin\phi - (y_2 + \alpha_2(\phi))\cos\phi$$



NUMERICAL IMAGE POSITIONS

Tessellation of image plane.

Divide image plane into triangles.

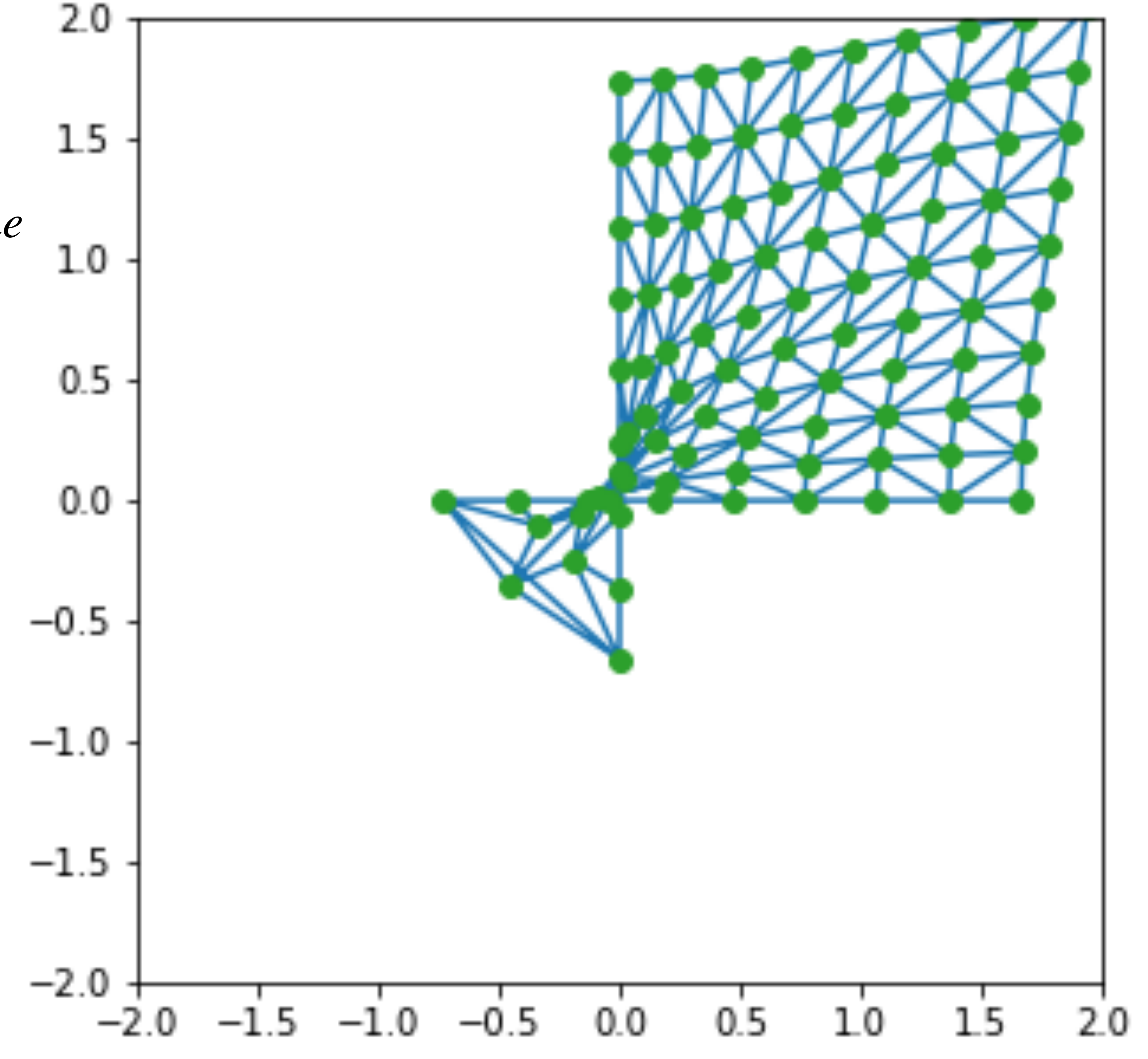


NUMERICAL IMAGE POSITIONS

Tessellation of image plane.

Divide image plane into triangles.

Find the source position of each vertex point by adding the deflection.



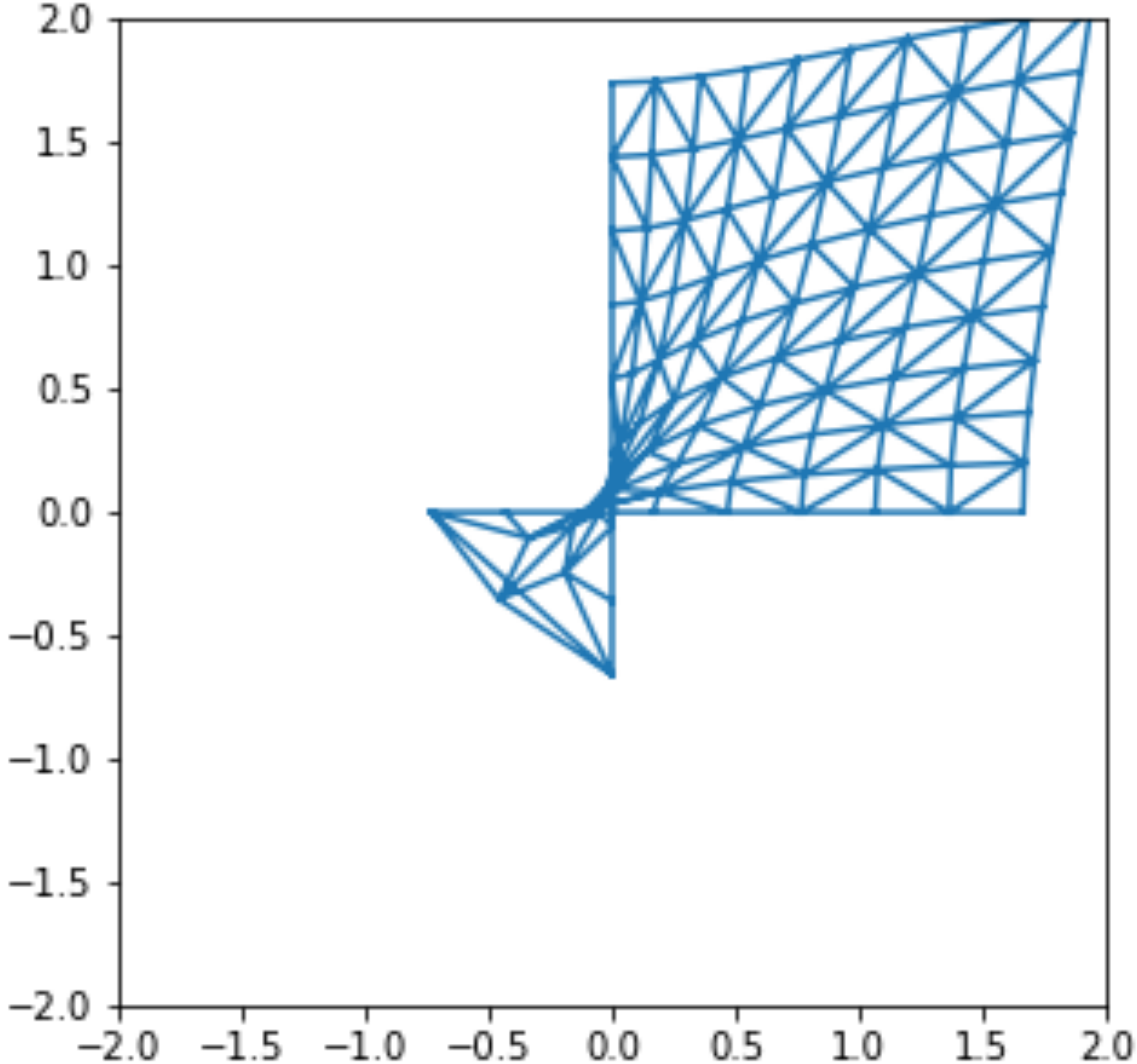
NUMERICAL IMAGE POSITIONS

Tessellation of image plane.

Divide image plane into triangles.

Find the source position of each vertex point by adding the deflection.

Find which triangle the source is in.



NUMERICAL IMAGE POSITIONS

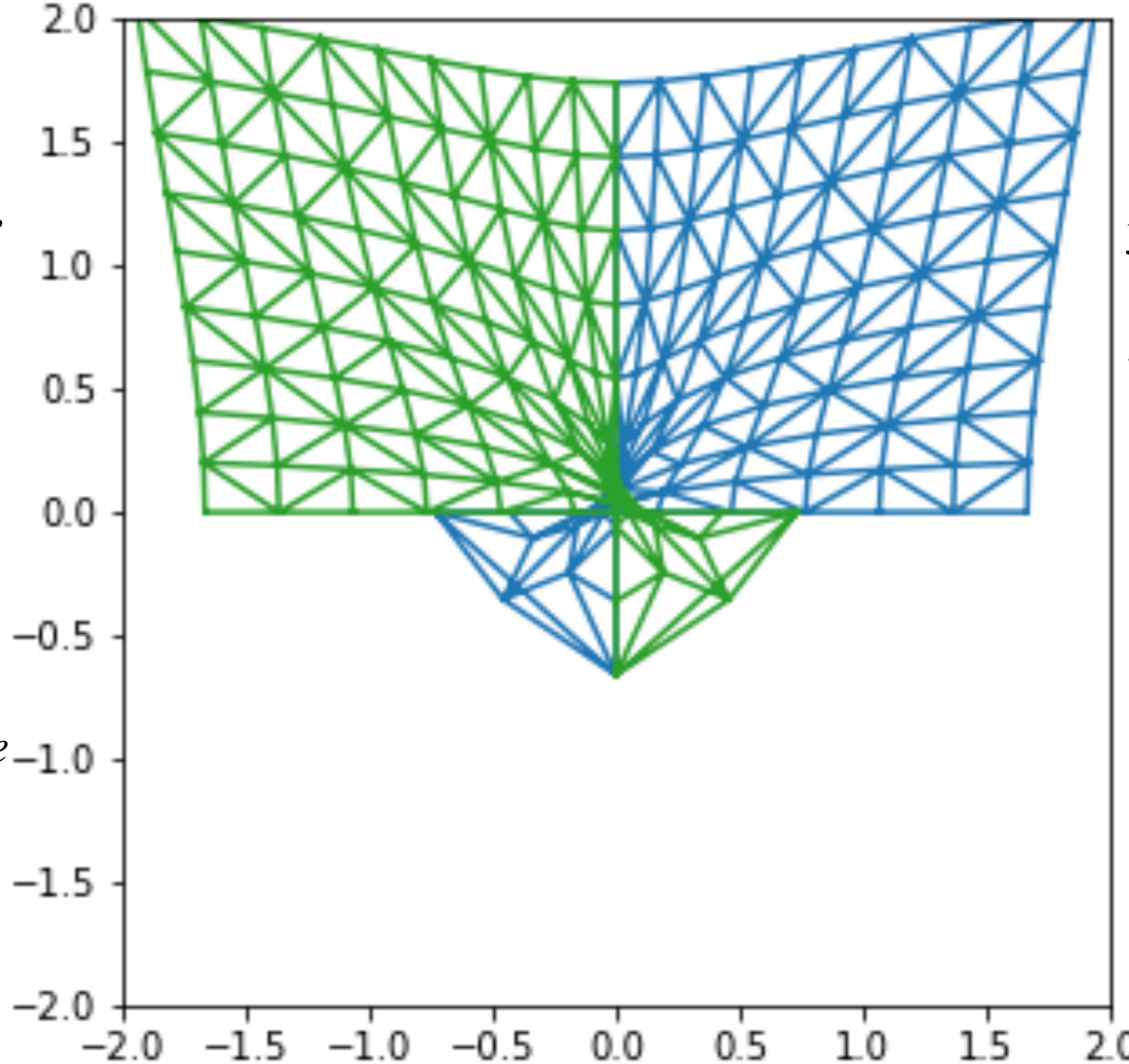
Tessellation of image plane.

Divide image plane into triangles.

Find the source position of each vertex point by adding the deflection.

Find which triangle the source is in.

Since the triangle will overlap, the source will be in more than one triangle.



Going back to the triangle on the image plane will tell you where the images are, to the resolution of the tessellation.

NUMERICAL IMAGE POSITIONS

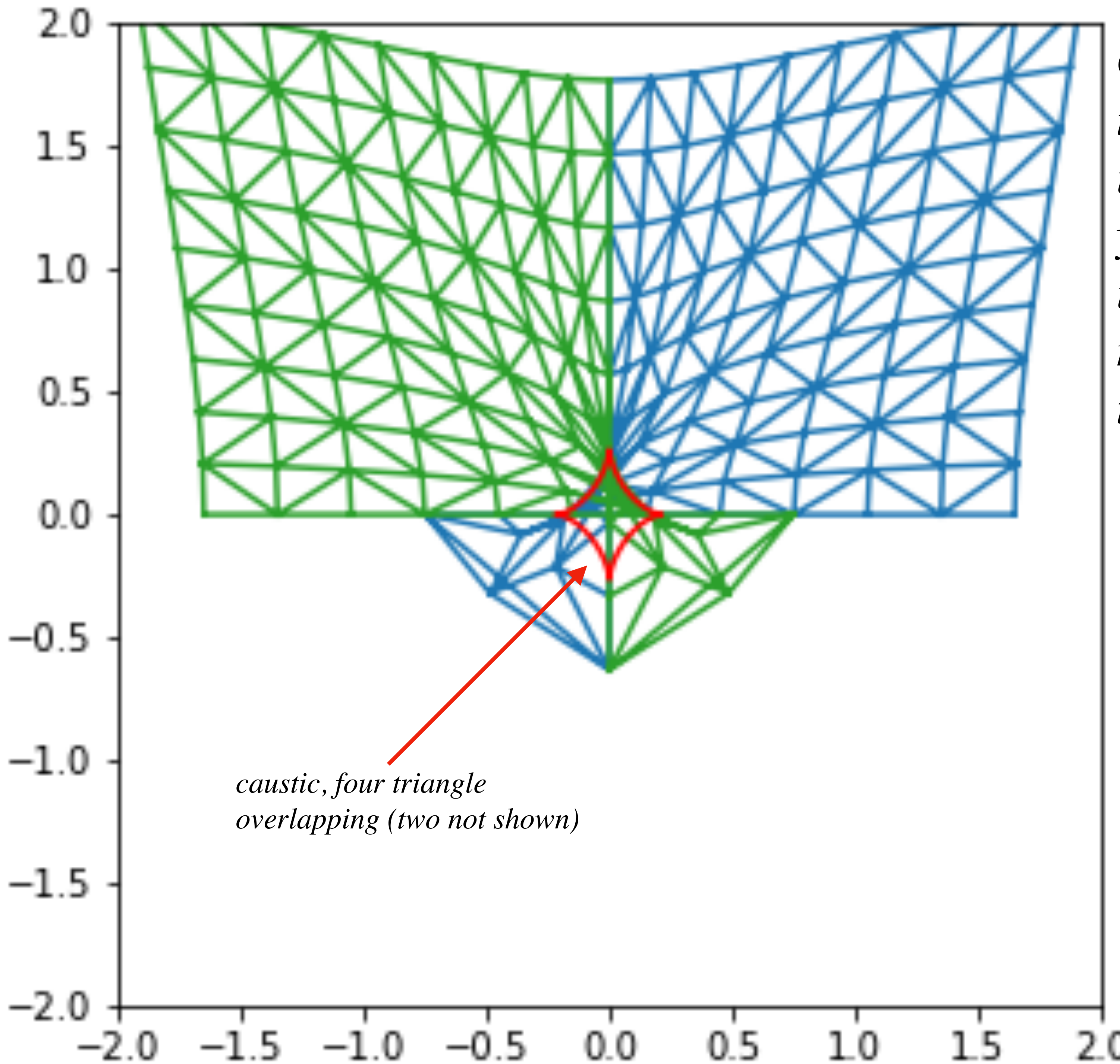
Tessellation of image plane.

Divide image plane into triangles.

Find the source position of each vertex point by adding the deflection.

Find which triangle the source is in.

Since the triangle will overlap, the source will be in more than one triangle.



Going back to the triangle on the image plane will tell you where the images are, to the resolution of the tessellation.

NUMERICAL IMAGE POSITIONS

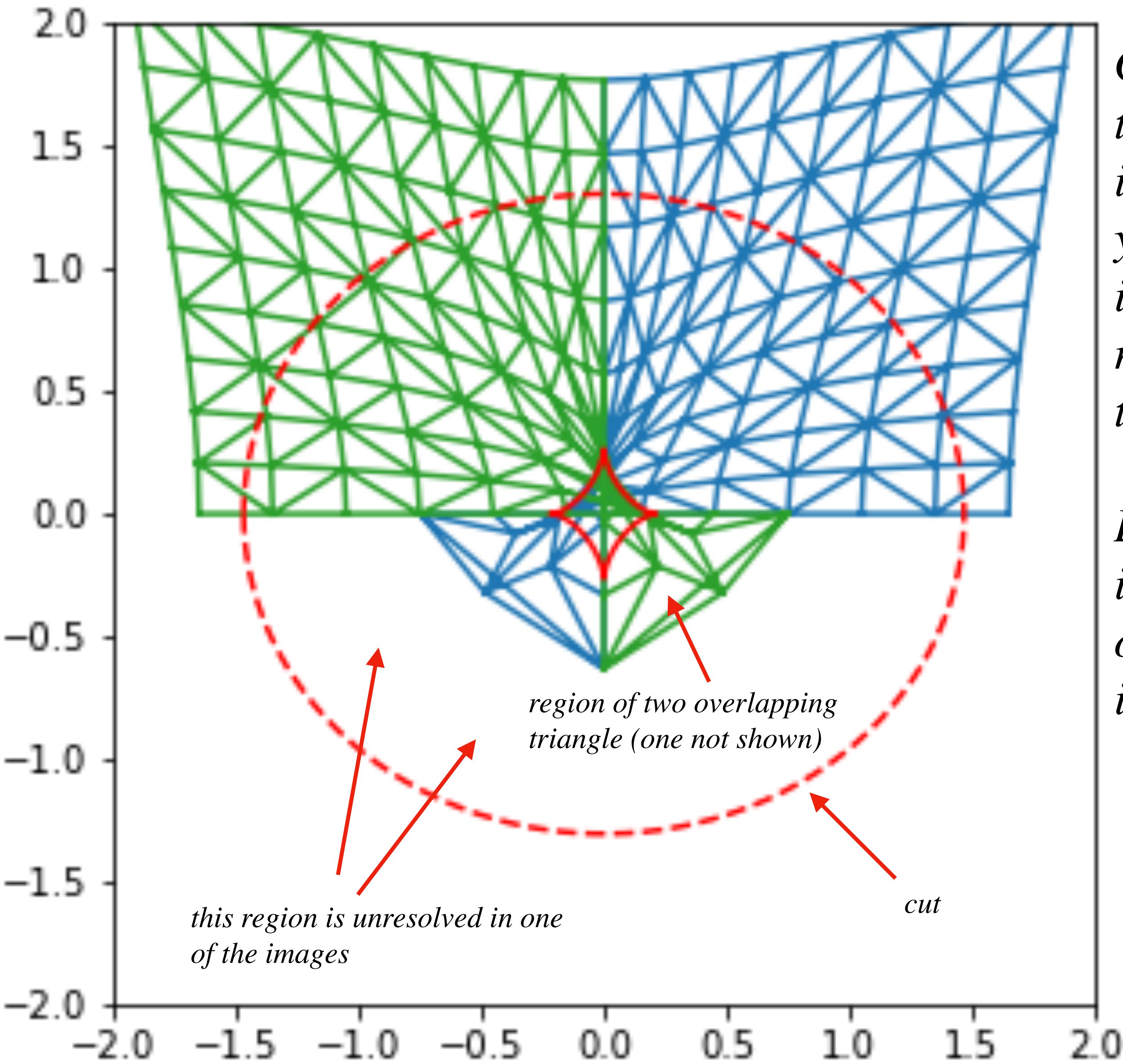
Tessellation of image plane.

Divide image plane into triangles.

Find the source position of each vertex point by adding the deflection.

Find which triangle the source is in.

Since the triangle will overlap, the source will be in more than one triangle.



Going back to the triangle on the image plane will tell you where the images are, to the resolution of the tessellation.

Low magnification images can be lost or spurious included.

ELLIPTICAL POWER-LAW LENS

In general, if $\kappa(x) = \kappa(R)$ $\quad R = \sqrt{q^2x^2 + y^2}$

$$z = x + iy$$

$$\kappa(z) = \frac{\partial \alpha *}{\partial z} \quad \Rightarrow \quad \alpha(z)^* = \frac{2}{qz}\int_0^{R(z)} dr \frac{r\ \kappa(r)}{\sqrt{1 - \frac{1-q^2}{q^2}\frac{r^2}{z^2}}}$$

For an elliptical power-law $\quad \kappa(x) = \frac{3-n}{2}\left(\frac{b}{R}\right)^{n-1}$

$$\alpha(z)^* = 2\frac{\partial \Psi}{\partial z} \quad \Rightarrow \quad \Psi(z) = \frac{1}{3-n}\frac{z\alpha^*(z) + z^*\alpha(z)}{2}$$

$$\Psi(x, y) = \frac{1}{3-n}(x\alpha_x + y\alpha_y)$$

ELLIPTICAL POWER-LAW LENS

In general, if $\kappa(x) = \kappa(R)$ $\qquad R = \sqrt{q^2x^2 + y^2}$

$$z = x + iy$$

$$\kappa(z) = \frac{\partial \alpha *}{\partial z} \quad \Longrightarrow \quad \alpha(z)^* = \frac{2}{qz}\int_0^{R(z)} dr \frac{r\ \kappa(r)}{\sqrt{1 - \frac{1-q^2}{q^2}\frac{r^2}{z^2}}}$$

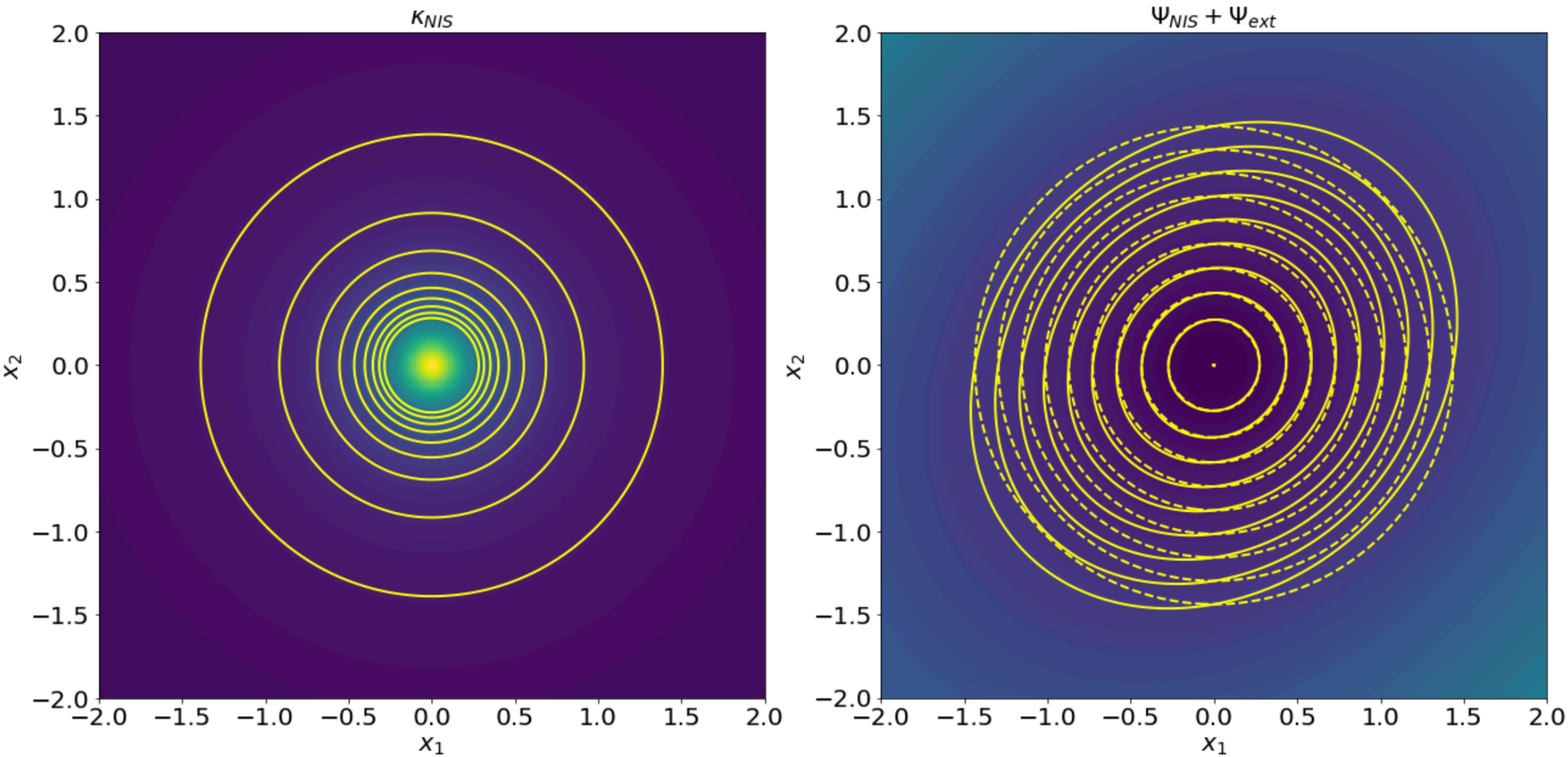
For an elliptical power-law $\quad \kappa(x) = \frac{3-n}{2}\left(\frac{b}{R}\right)^{n-1}$

$$\gamma^*(z) = \frac{\partial \alpha^*}{\partial z} = -\kappa(z)\frac{z^*}{z} + (2-n)\frac{\alpha^*}{z}$$


Q0957+561

EFFECTS OF EXTERNAL SHEAR

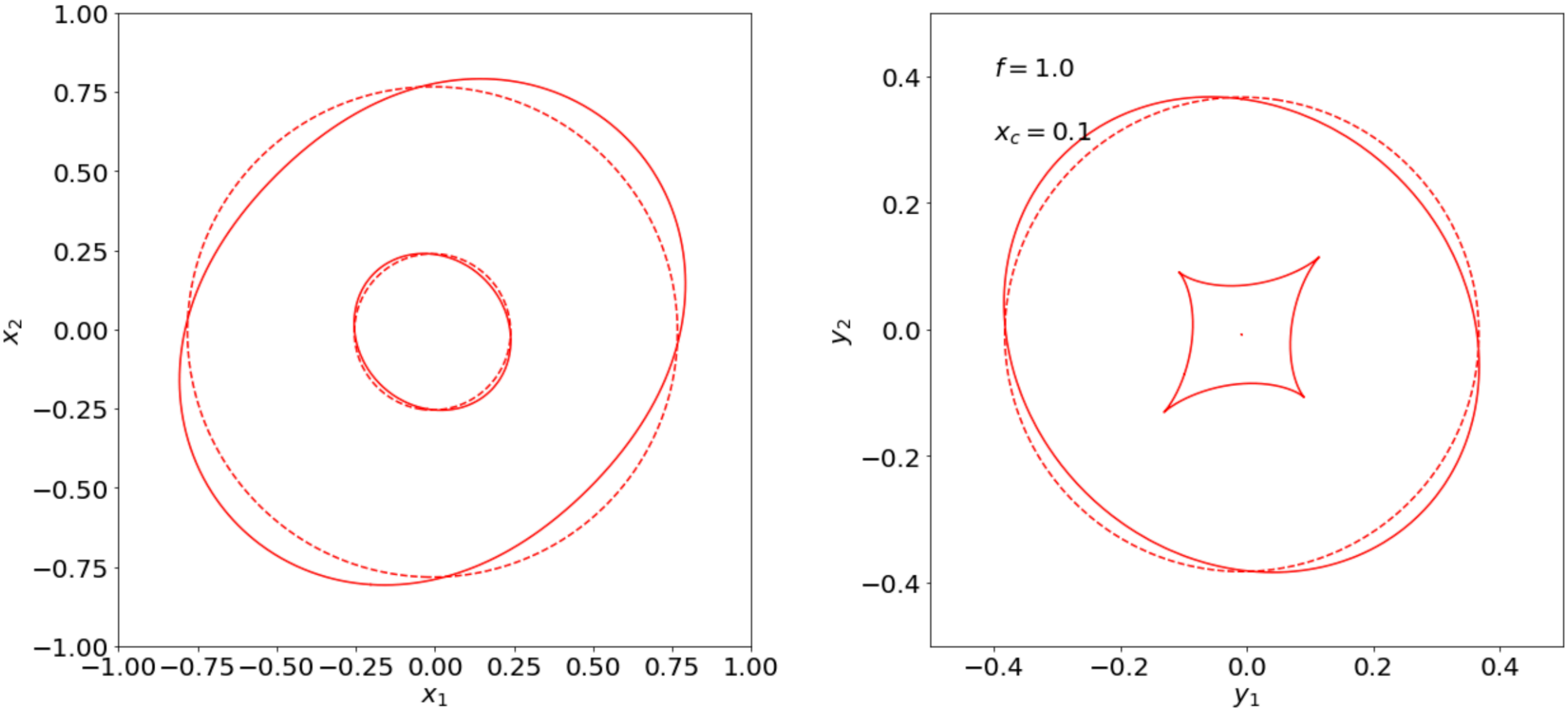
$$\Psi_\gamma(\overrightarrow{x}) = \Psi_{ext}(\overrightarrow{x}) = \frac{\gamma}{2}x^2 \cos 2(\phi - \phi_\gamma)$$



Introducing an external shear, the lensing potential becomes ~elliptical

EFFECTS OF EXTERNAL SHEAR

$$\Psi_\gamma(\overrightarrow{x}) = \Psi_{ext}(\overrightarrow{x}) = \frac{\gamma}{2}x^2\cos 2(\phi - \phi_\gamma)$$



We obtain a Pseudo-Elliptical lens!

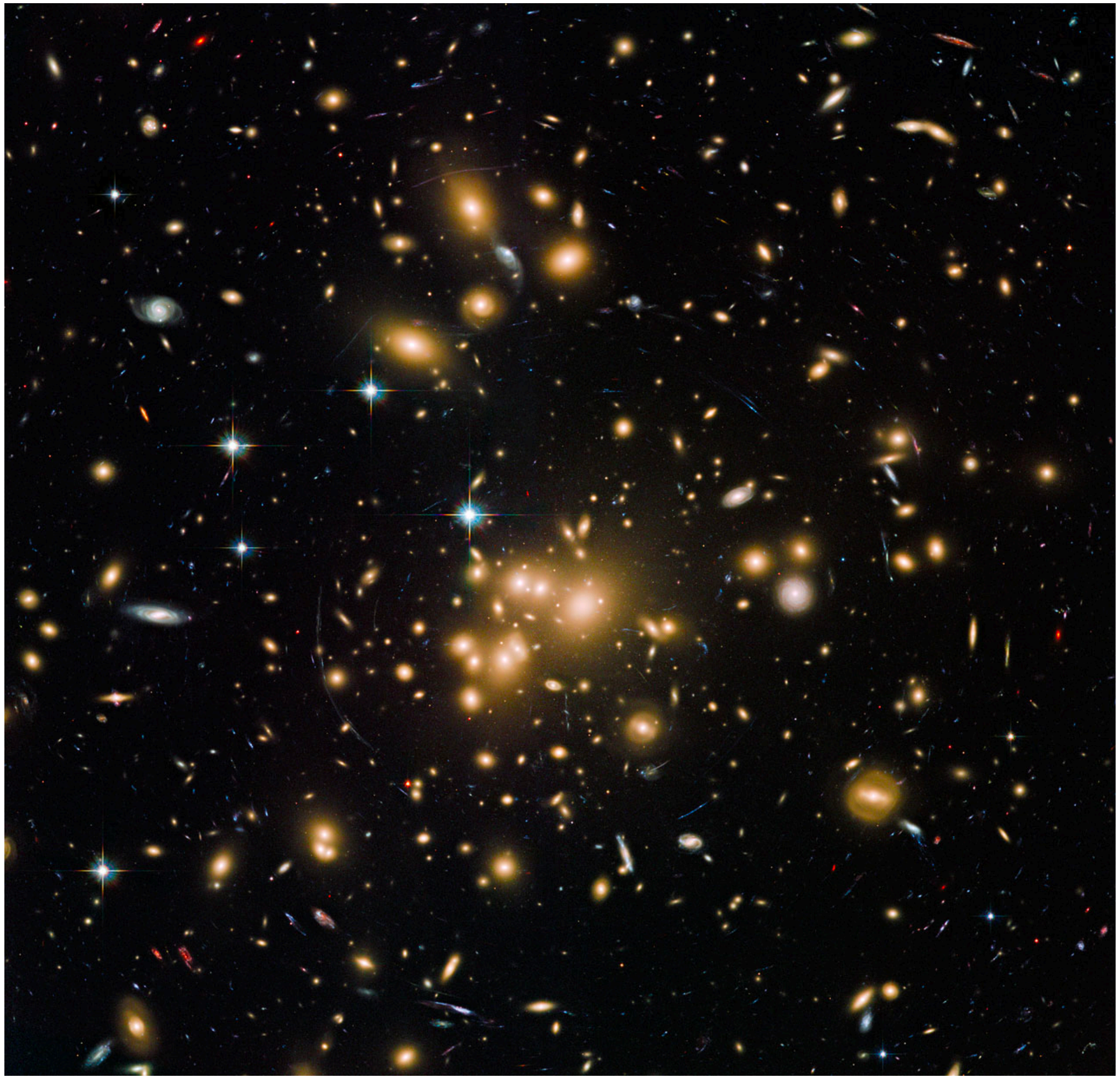
IN PRACTICE...

Realistic lens systems generally must include both intrinsic ellipticity and external shear.

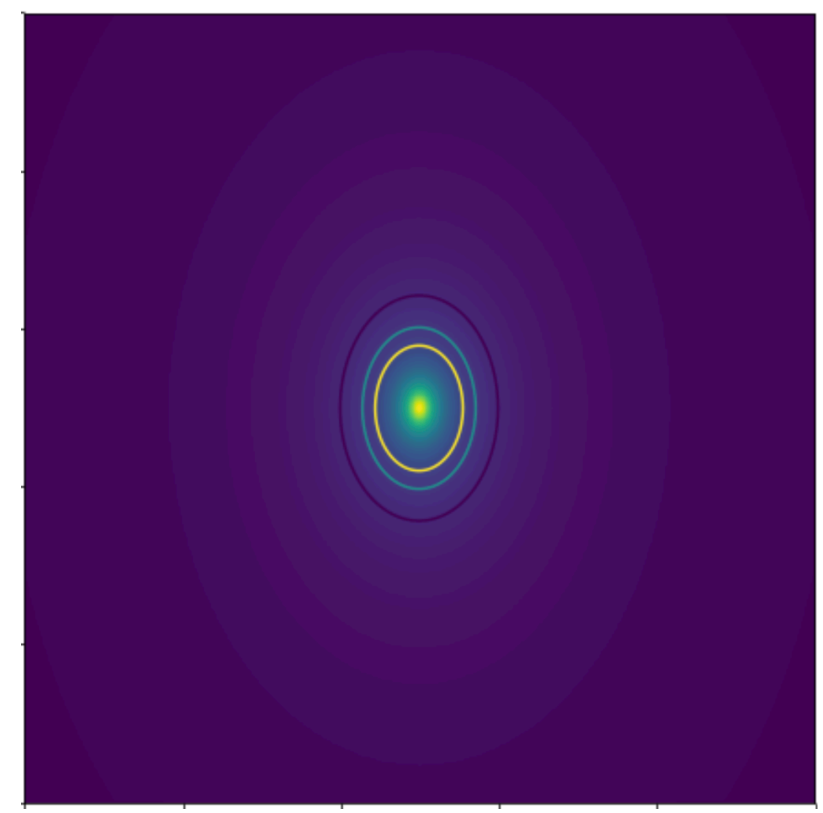
Uniform mass surface density from the lens' environment might exist, but is not detectable without further information. For example the velocity dispersion of the lens.

The direction of the external shear may not be related to the intrinsic ellipticity.

MULTIPLE MASS COMPONENTS



Often a simple lens model composed by a single, smooth mass distribution is not sufficient to describe realistic lenses

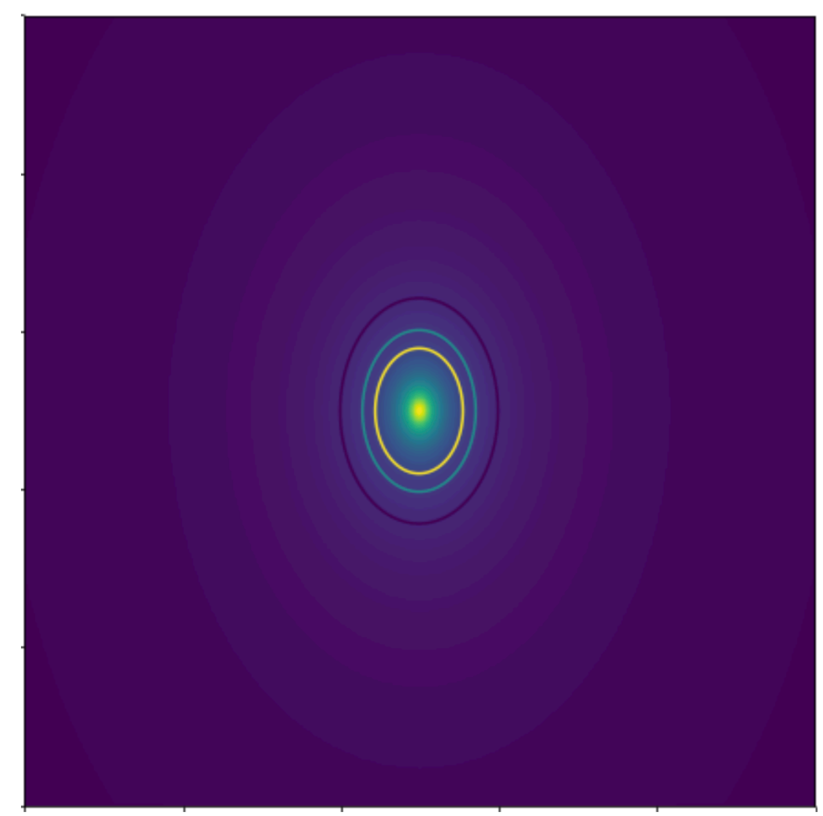


Hierarchy of mass components and combined potential:

$$\Psi(\vec{x}) = \sum_{i=1}^{n_{smooth}} \Psi_{smooth,i}(\vec{x} - \vec{x}_{smooth,i}) + \sum_{i=1}^{n_{sub}} \Psi_{sub,i}(\vec{x} - \vec{x}_{sub,i})$$

MULTIPLE MASS COMPONENTS

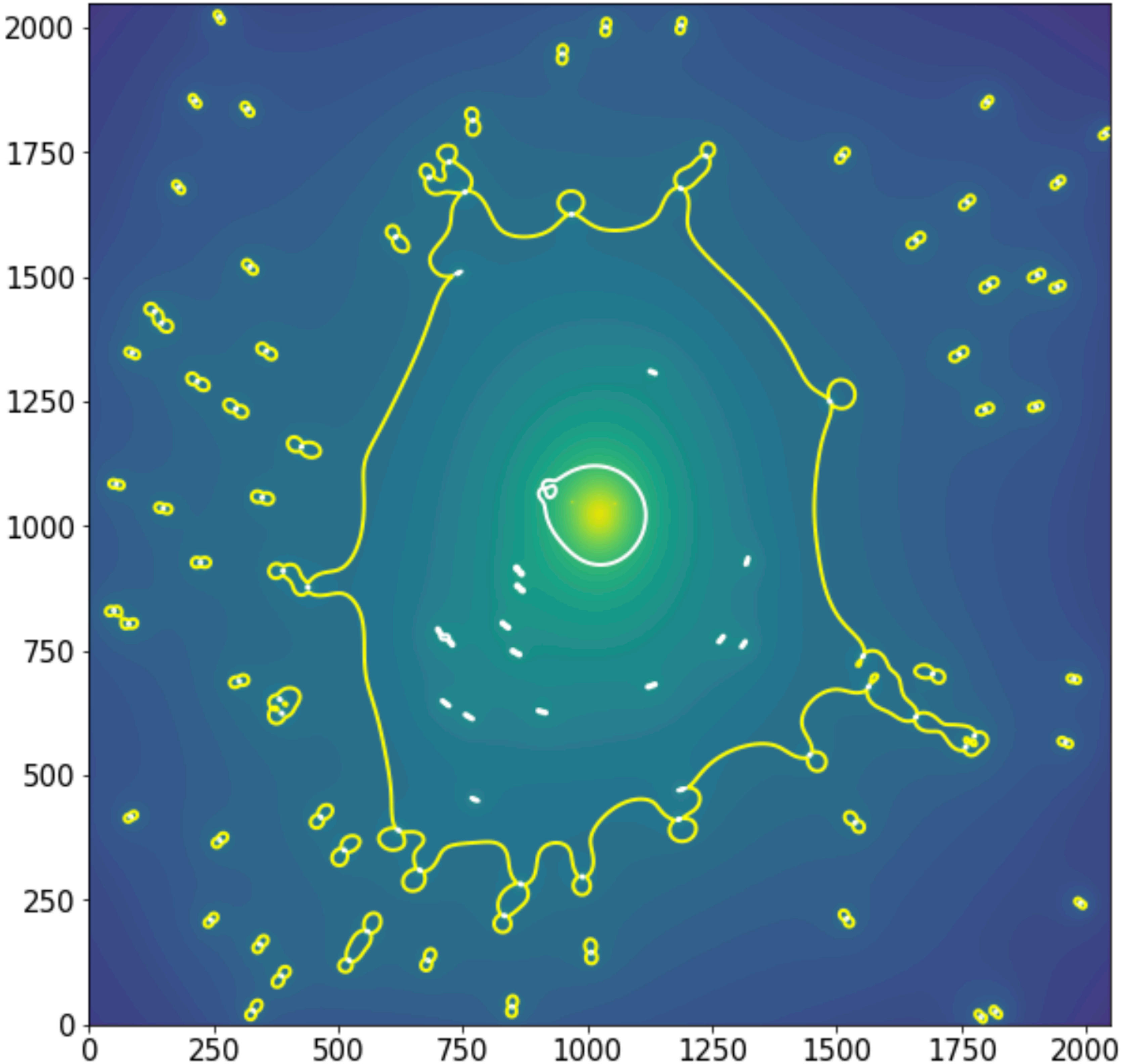
Often a simple lens model composed by a single, smooth mass distribution is not sufficient to describe realistic lenses



Hierarchy of mass components and combined potential:

$$\Psi(\vec{x}) = \sum_{i=1}^{n_{smooth}} \Psi_{smooth,i}(\vec{x} - \vec{x}_{smooth,i}) + \sum_{i=1}^{n_{sub}} \Psi_{sub,i}(\vec{x} - \vec{x}_{sub,i})$$

EXAMPLE



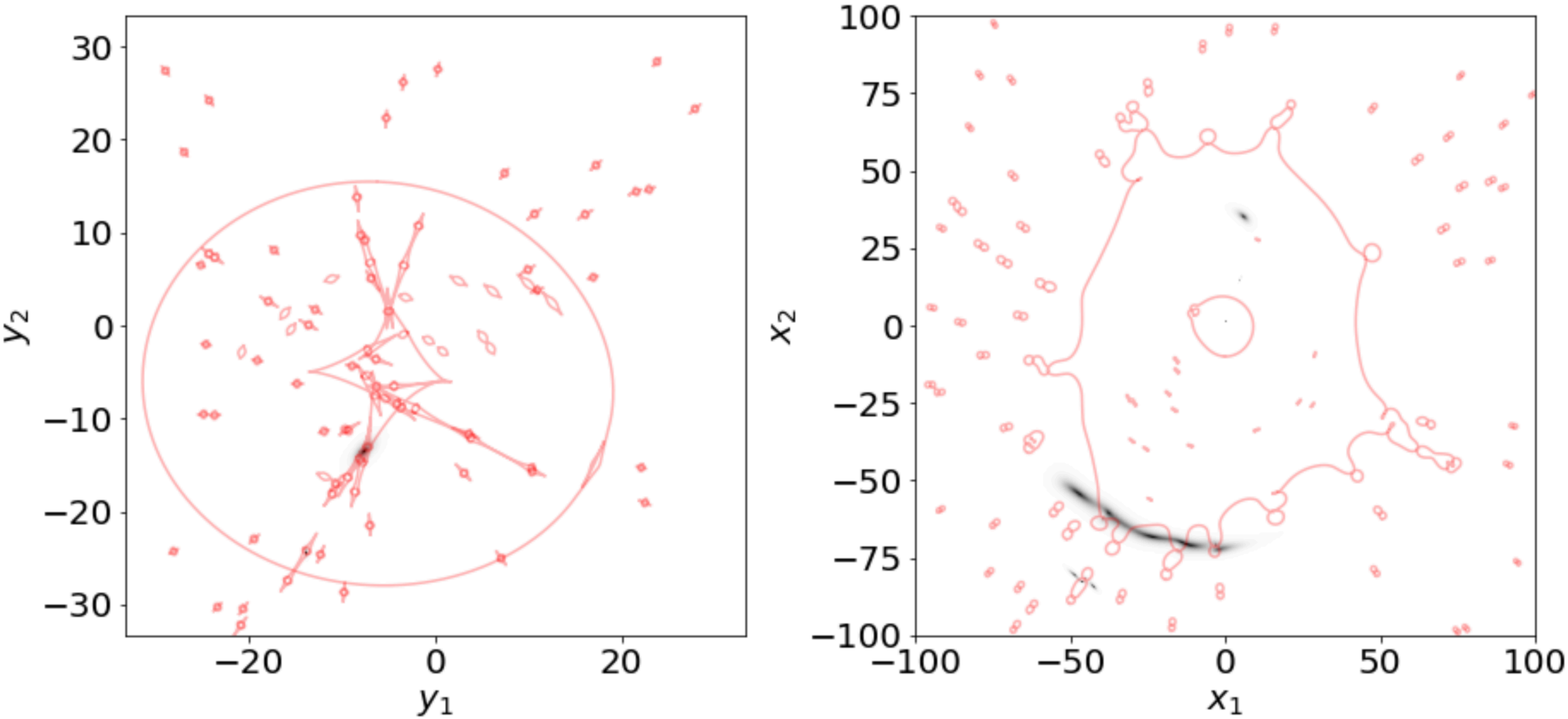
NIE lens + 100 substructures

Primary critical lines show a lot of wiggles

Many secondary critical lines

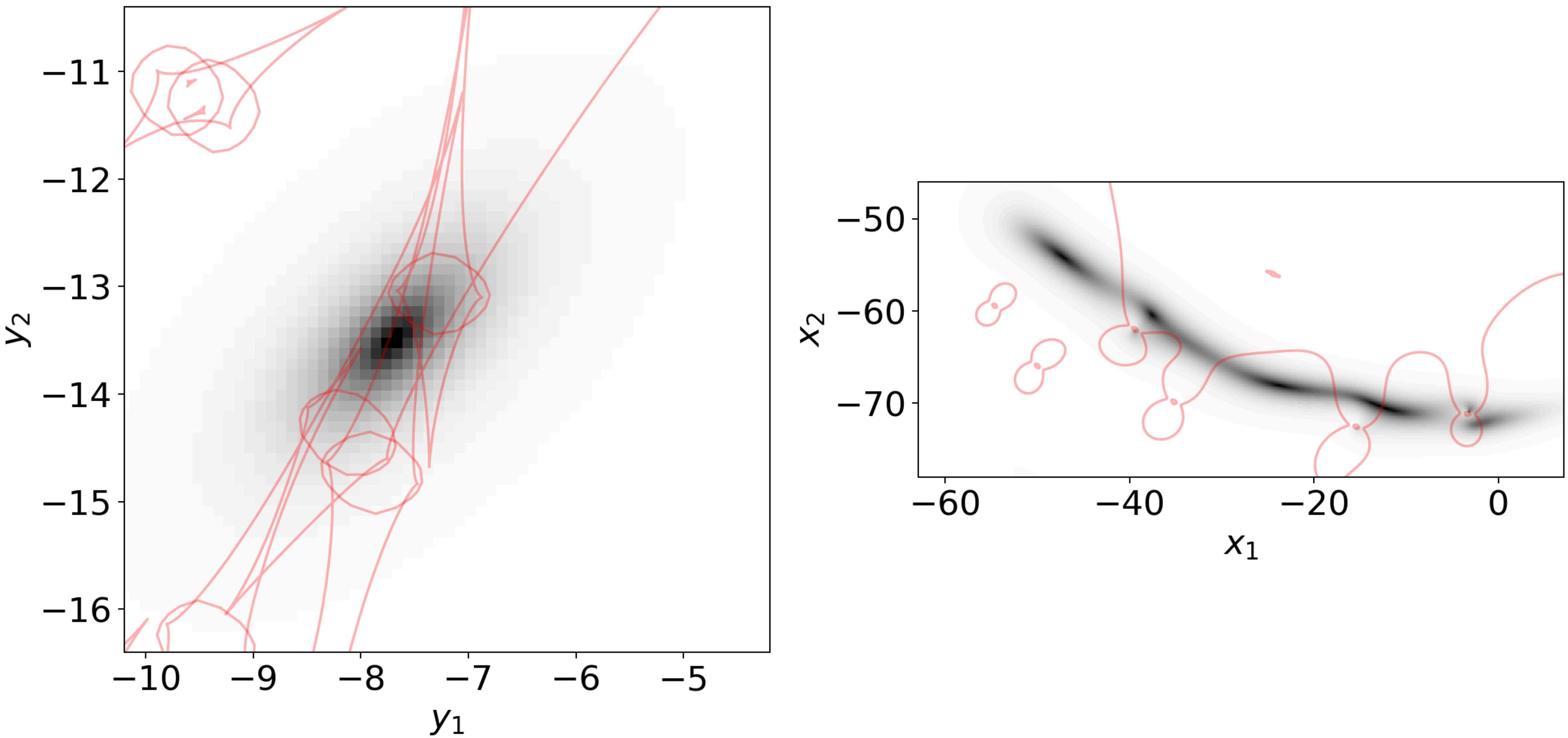
Effects of substructures can be of different kind…

EXAMPLE

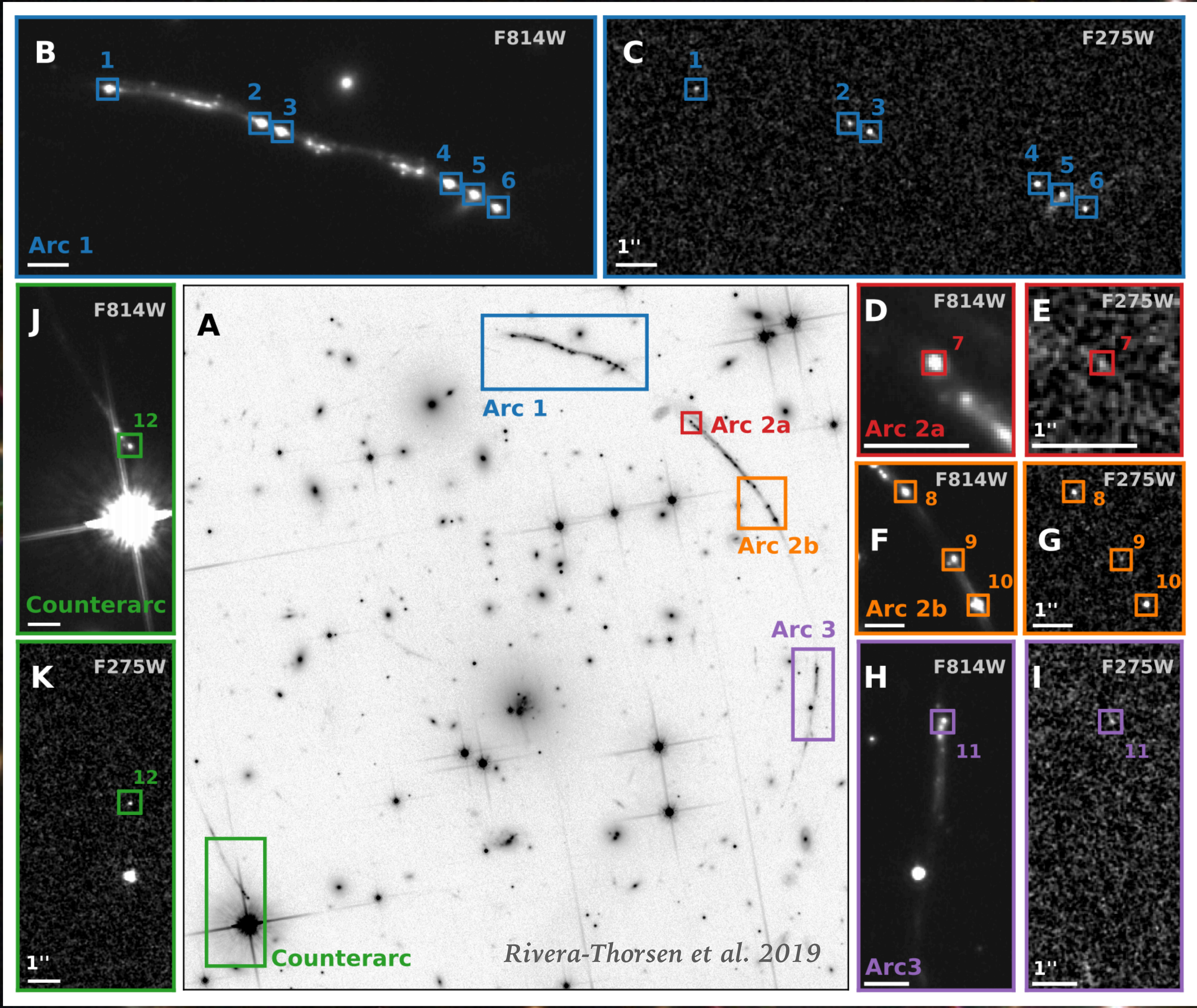


In this case, substructures cause a long cusp-arc (which would be made of 3 images) to split into several more multiple images

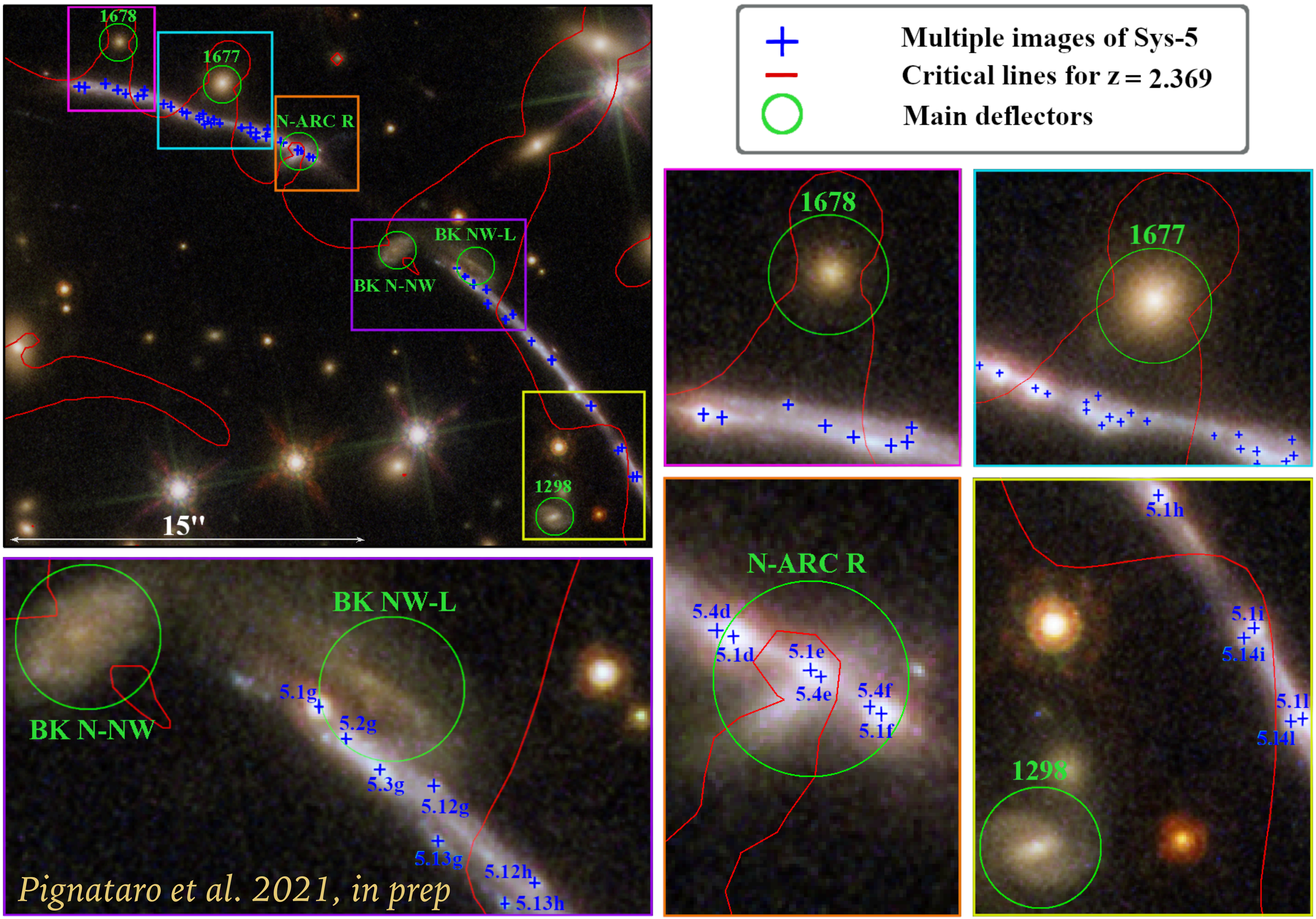
EXAMPLE



In this case, substructures cause a long cusp-arc (which would be made of 3 images) to split into several more multiple images



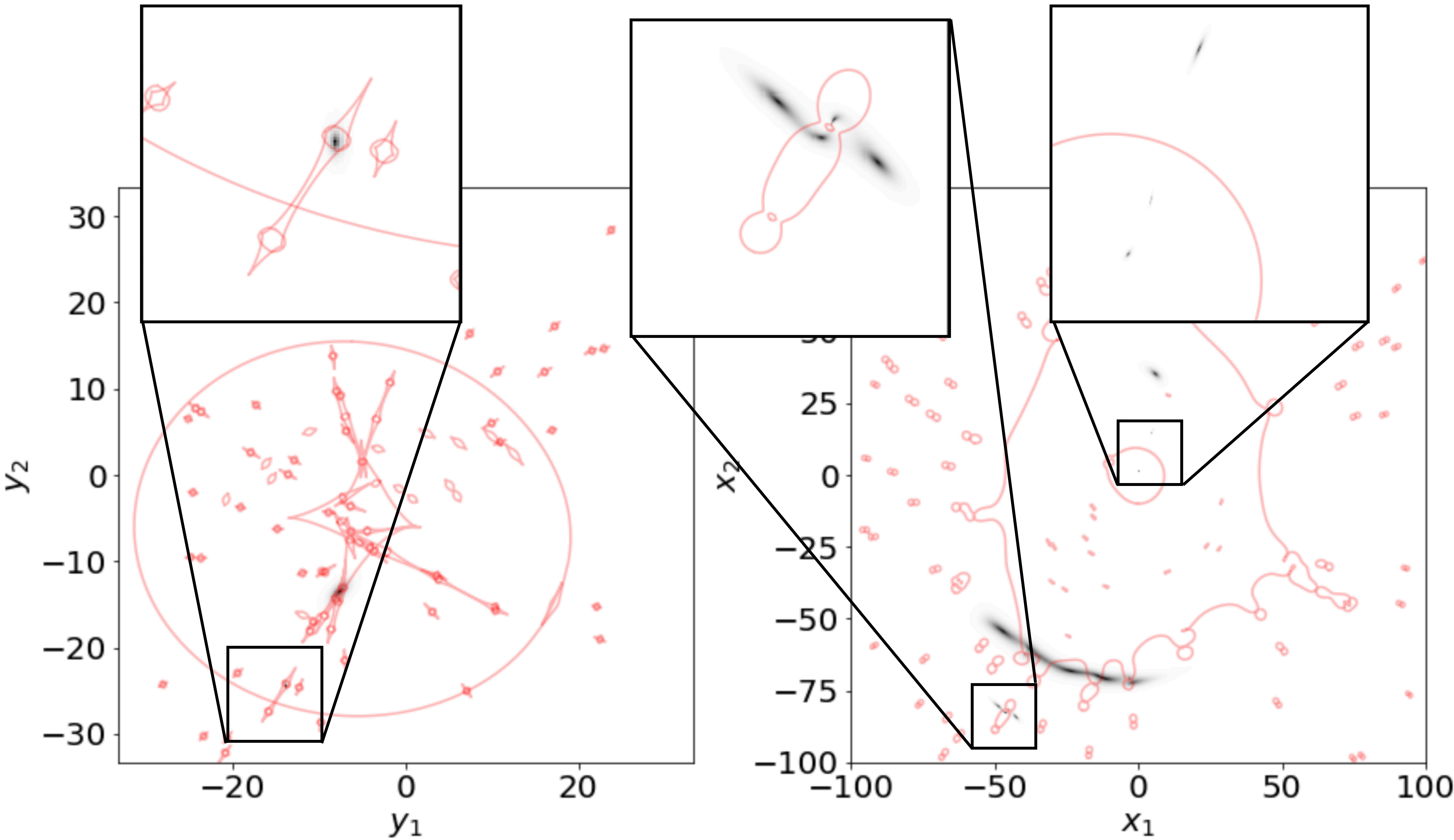
B
F814W
1
2
3
4
5
6
Arc 1
C
F275W
1''
J
F814W
12
Counterarc
K
F275W
A
Arc 1
Arc 2a
Arc 2b
Arc 3
Counterarc
Rivera-Thorsen et al. 2019
D
F814W
7
Arc 2a
E
F275W
F
8
9
10
Arc 2b
G
H
11
Arc3
I

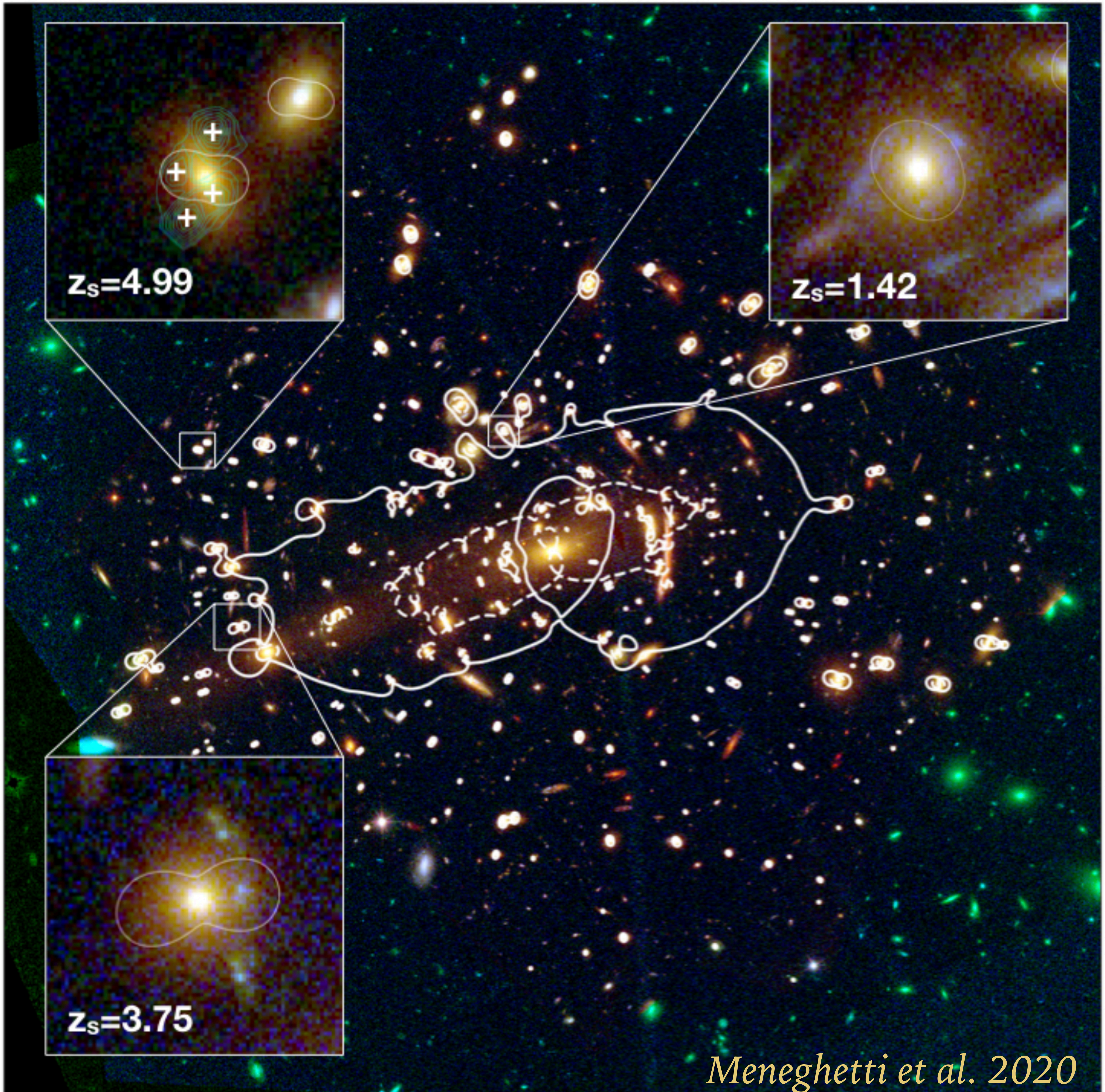


1678
1677
N-ARC R
BK NW-L
BK N-NW
1298
15"
Multiple images of Sys-5
Critical lines for z = 2.369
Main deflectors
5.4d
5.1d
5.1e
5.4e
5.4f
5.1f
5.1h
5.1i
5.14i
5.1l
5.14l
5.1g
5.2g
5.3g
5.12g
5.13g
5.12h
5.13h
Pignataro et al. 2021, in prep

EXAMPLE

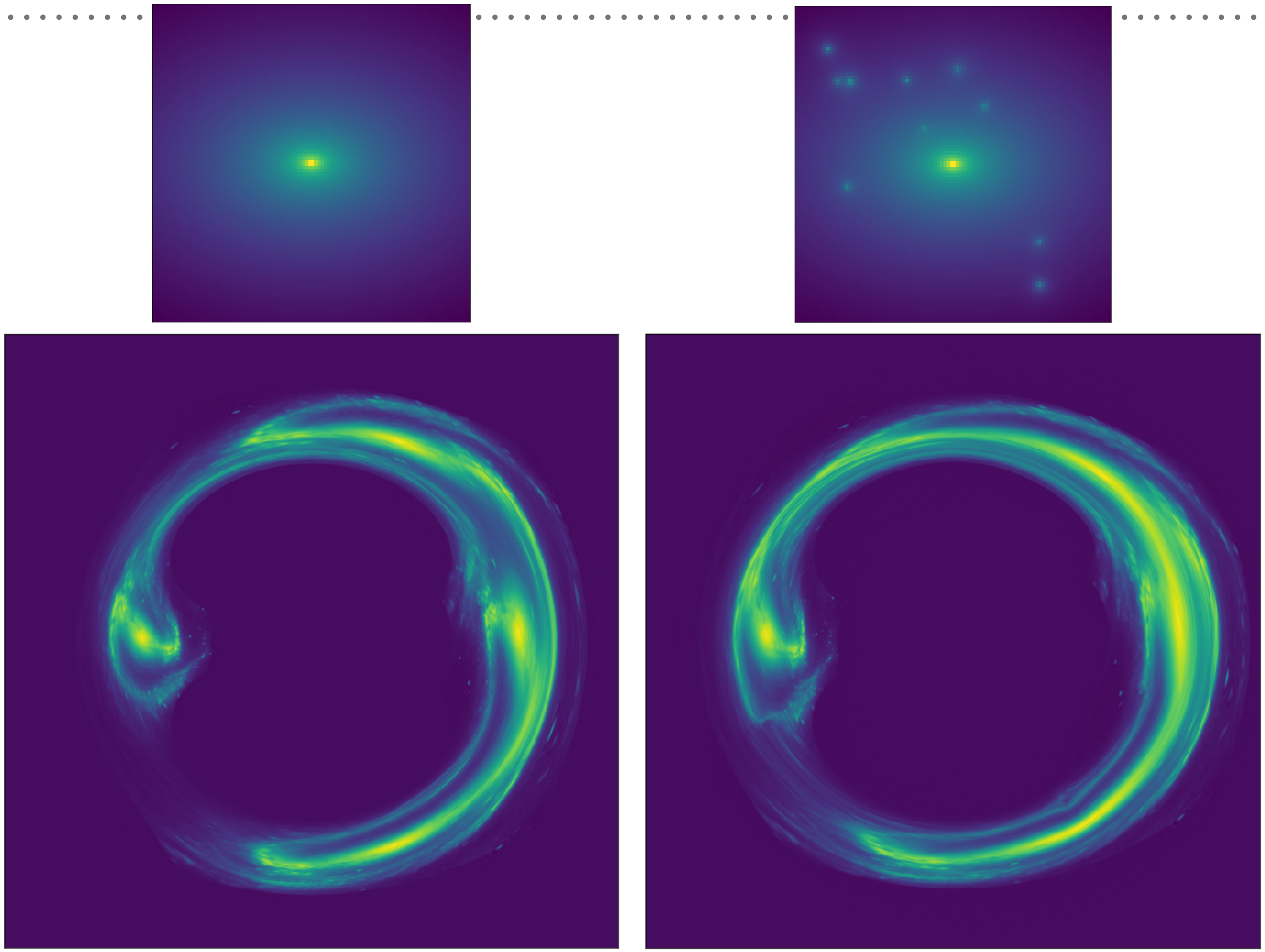
Substructures can individually act as strong lenses





Meneghetti et al. 2020

EXAMPLE



If they are much smaller than the size of the source the effects can be shifts or variation of surface brightness.

GRAVITATIONAL LENSING

LENS MODELLING I

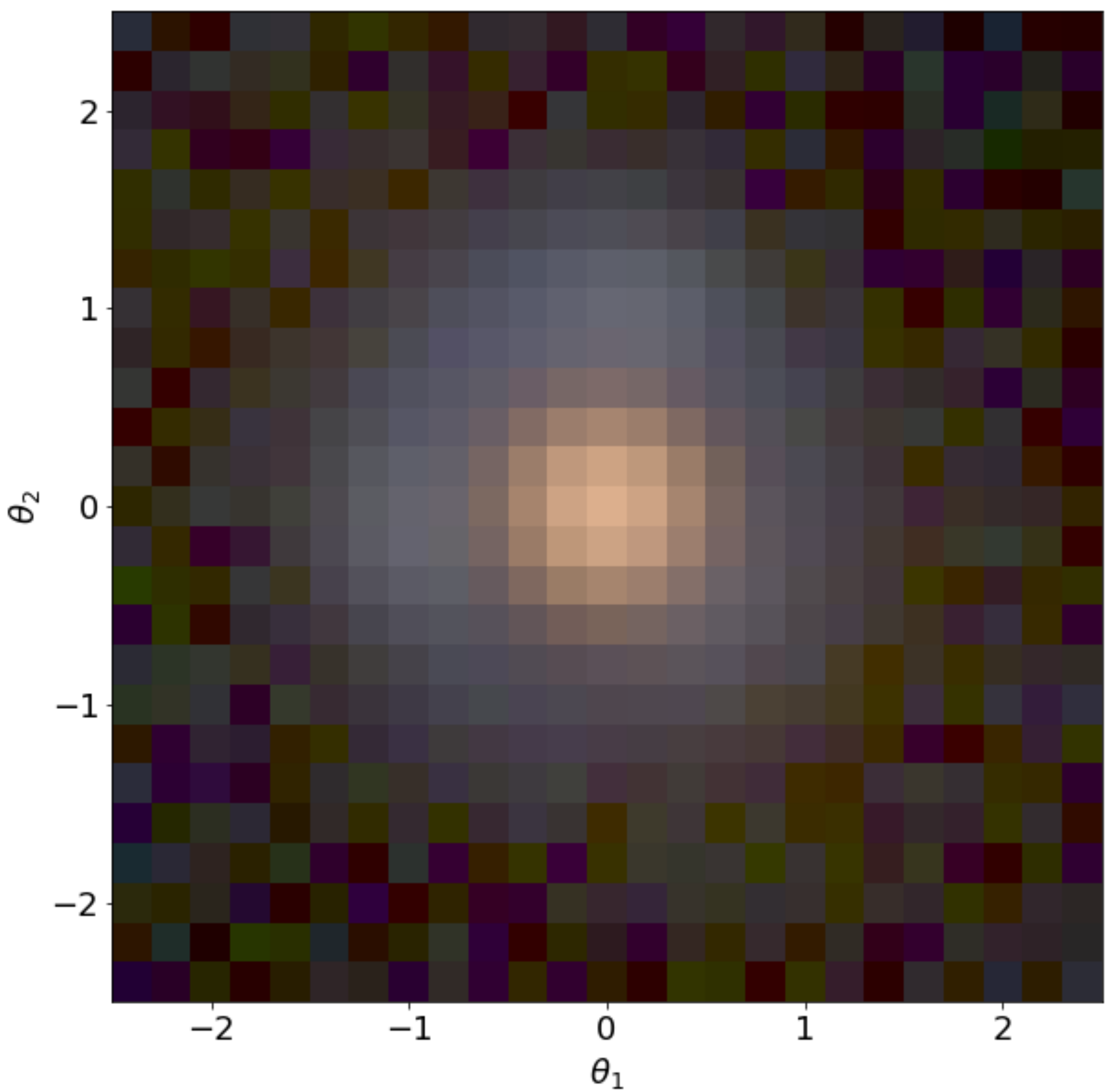
R. Benton Metcalf

2022-2023

MODELING A GALAXY USING STRONG LENSING

- Once a lens has been found, we may want to use the strong lensing observables to investigate what is its content
- Possible observables are:
  - Image positions
  - Image fluxes/distortions
  - Image time-delays
- Note that these observables provide different constraints on the potential and on its derivatives
- Strategies: **parametric** or free-form

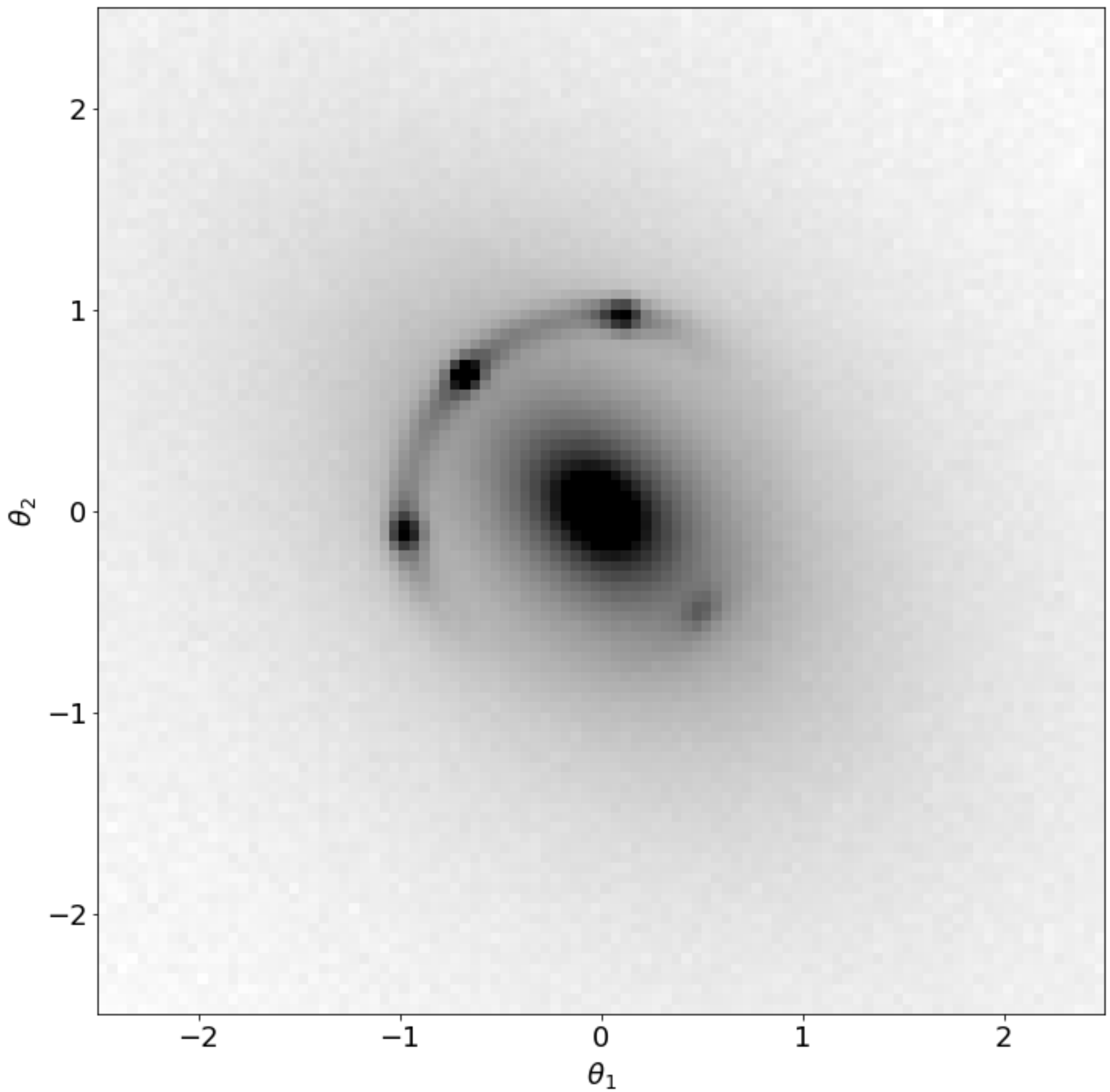
PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



Example: suppose we have discovered a lens candidate like the one in the Figure on the left.

We perform HST follow-up observations and we confirm the lensing nature of the object

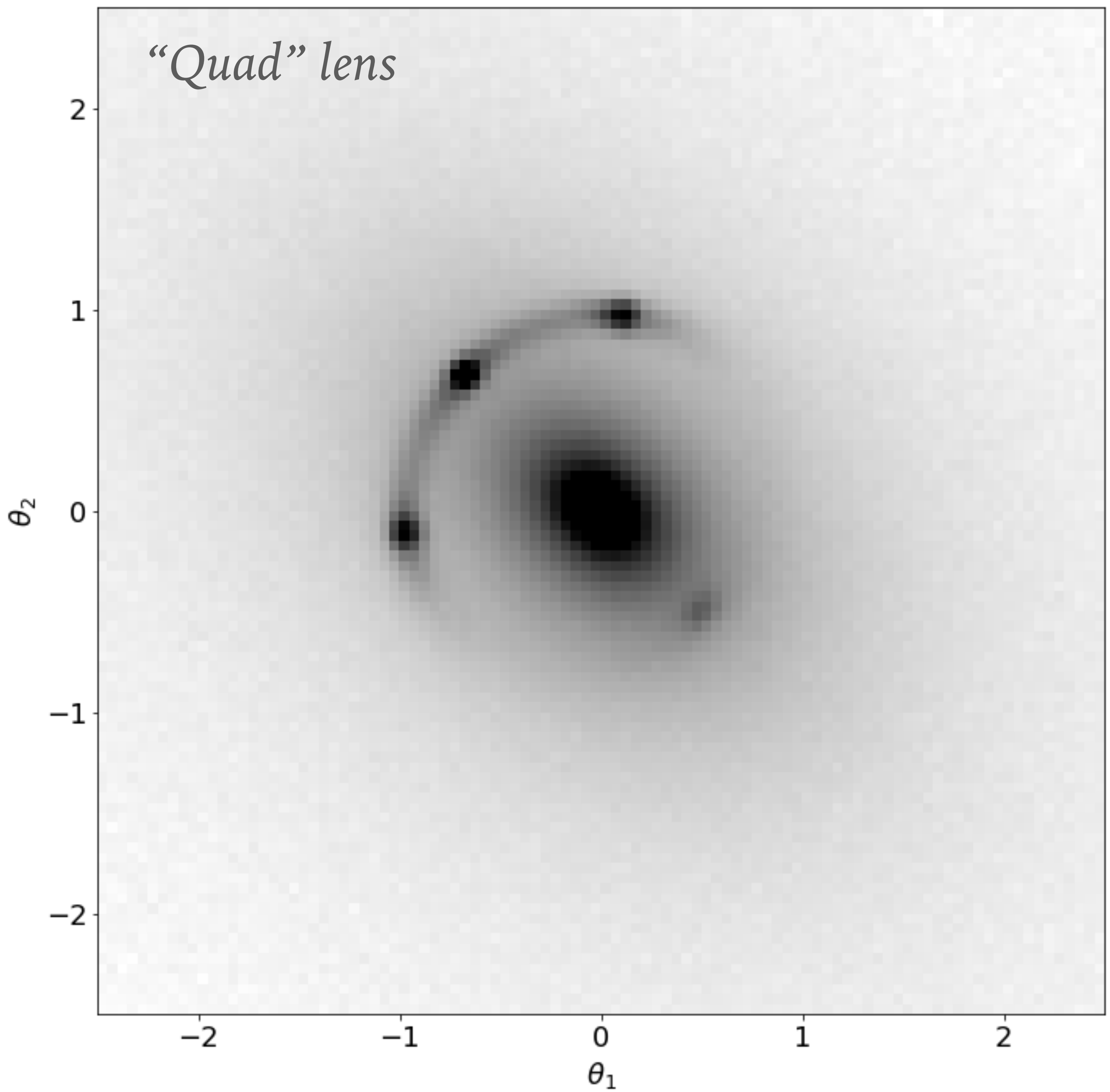
PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



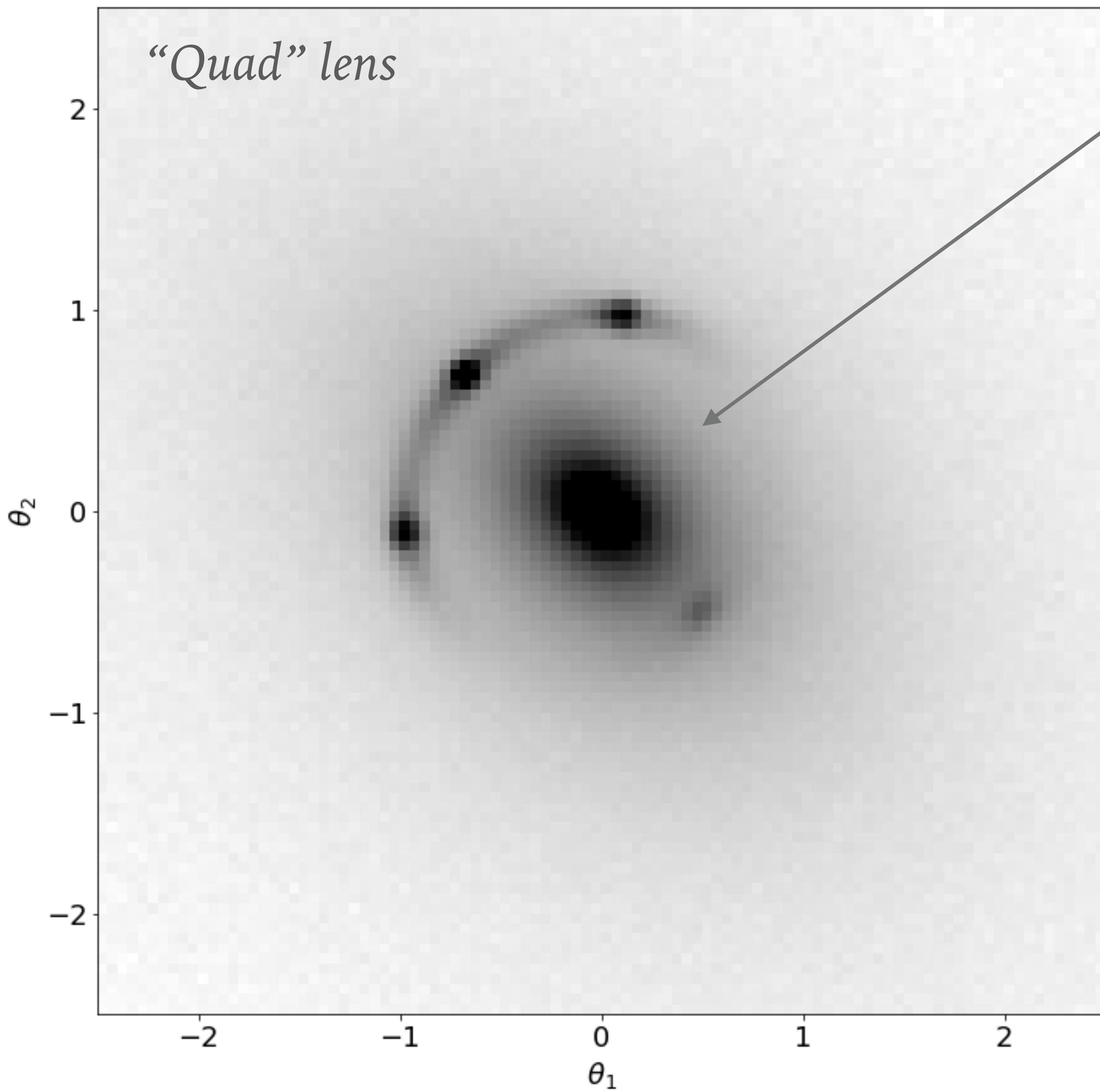
Example: suppose we have discovered a lens candidate like the one in the Figure on the left.

We perform HST follow-up observations and we confirm the lensing nature of the object

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY

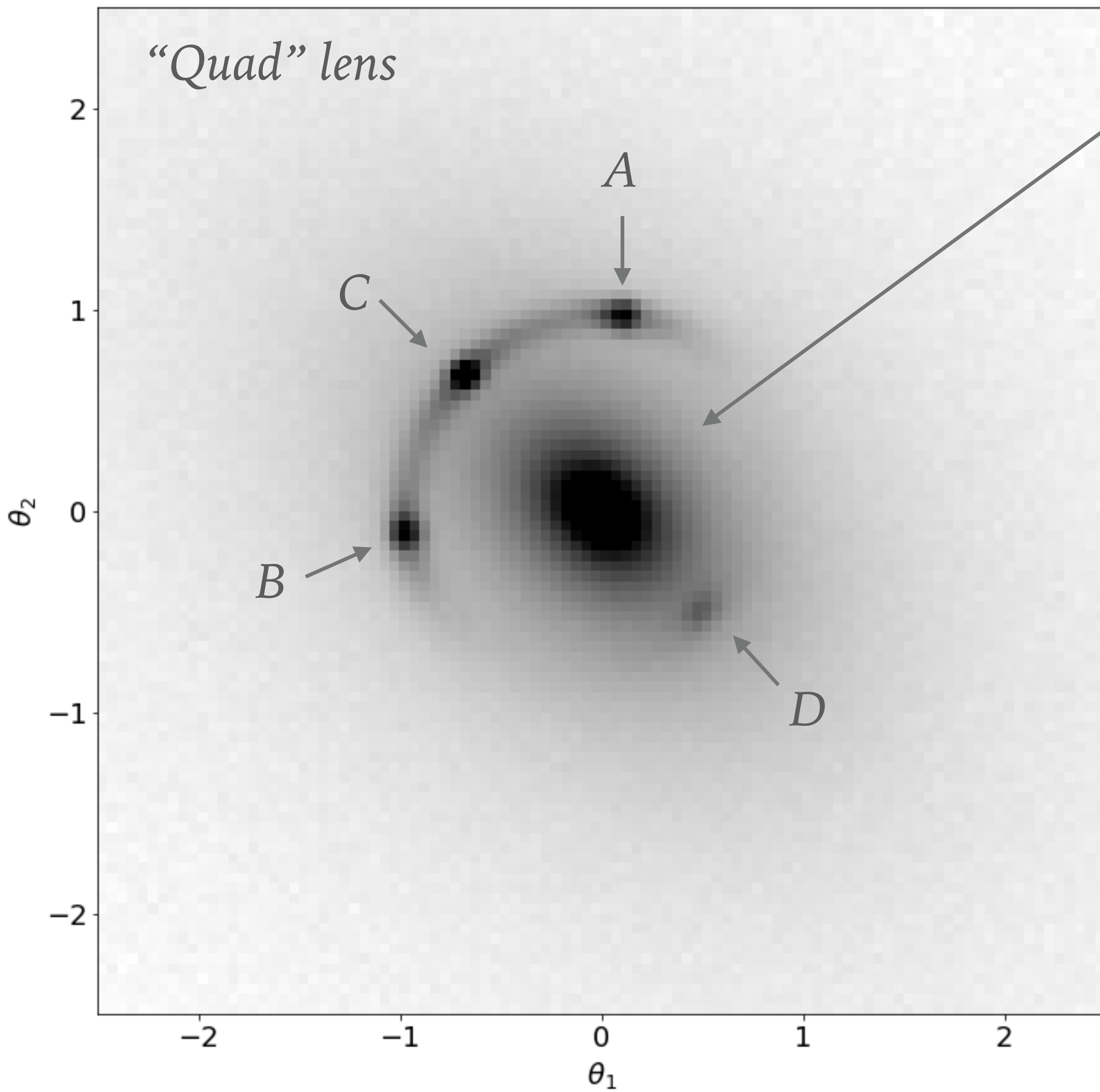


PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



The lens is an early type galaxy

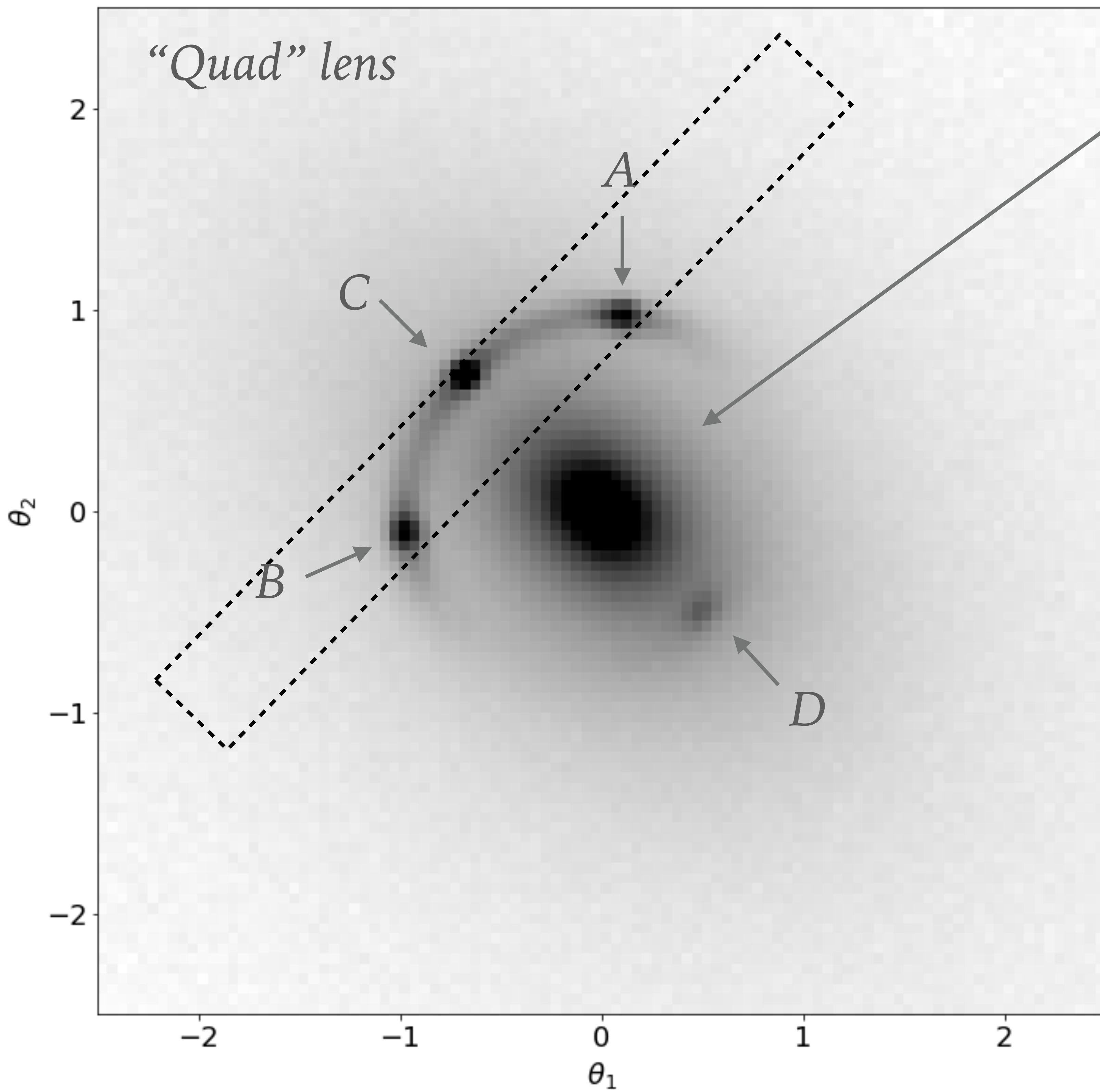
PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



The lens is an early type galaxy

We see four images of a QSO and a lensed galaxy host

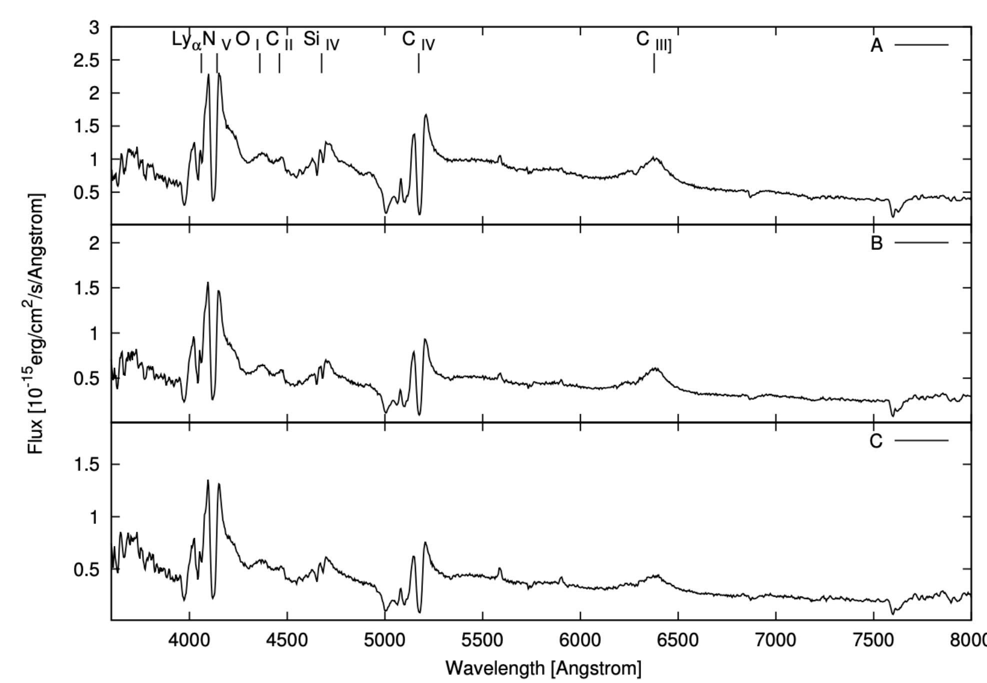
PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



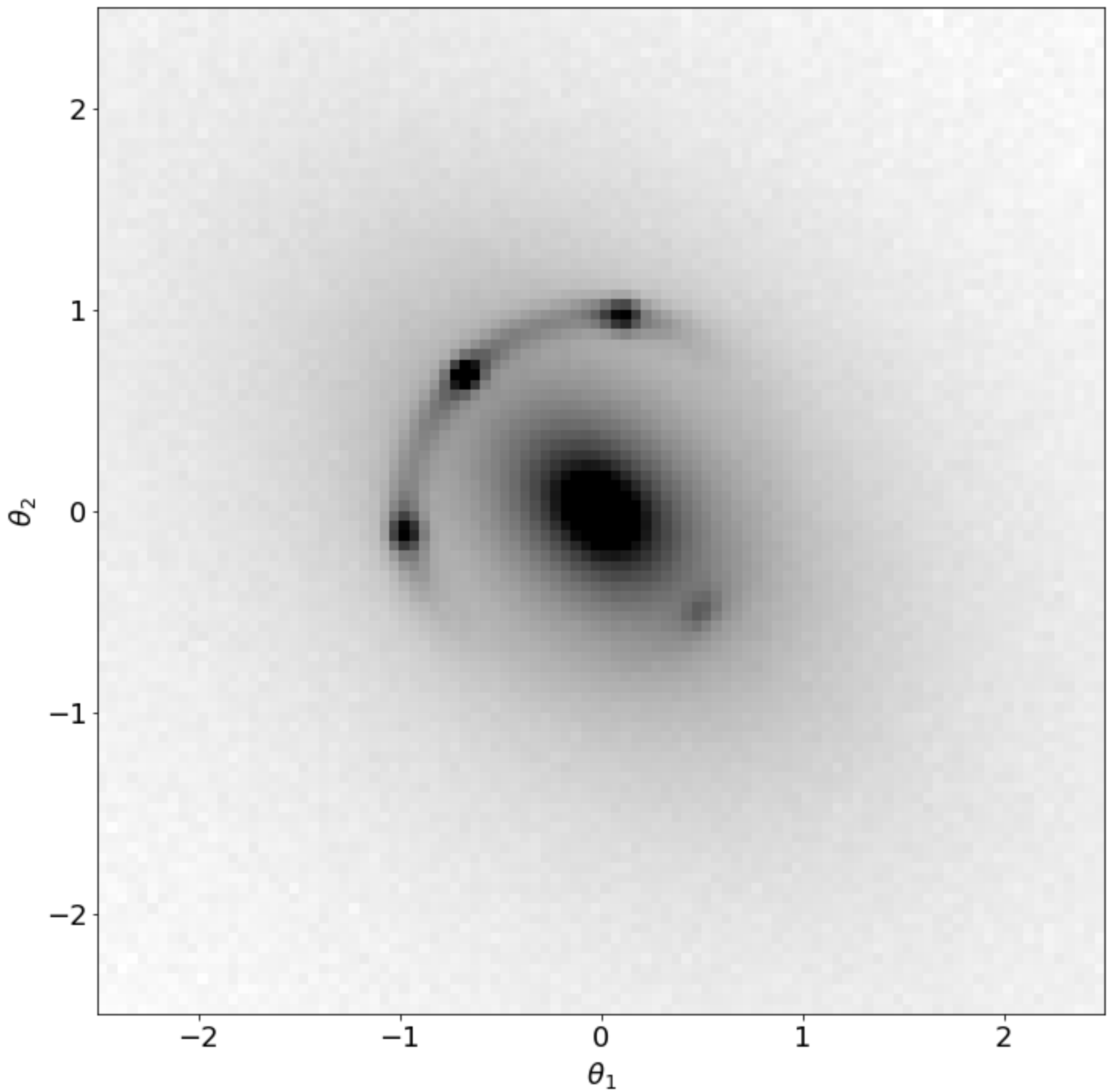
The lens is an early type galaxy

We see four images of a QSO and a lensed galaxy host

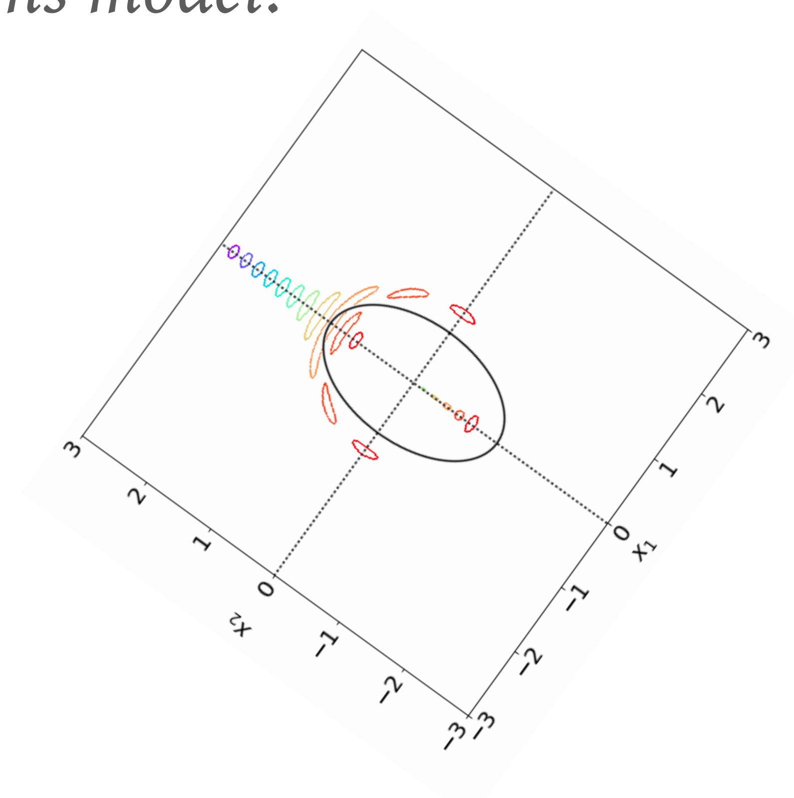
Spectroscopic confirmation and redshifts: $z_L = 0.3$, $z_S = 2$



PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY

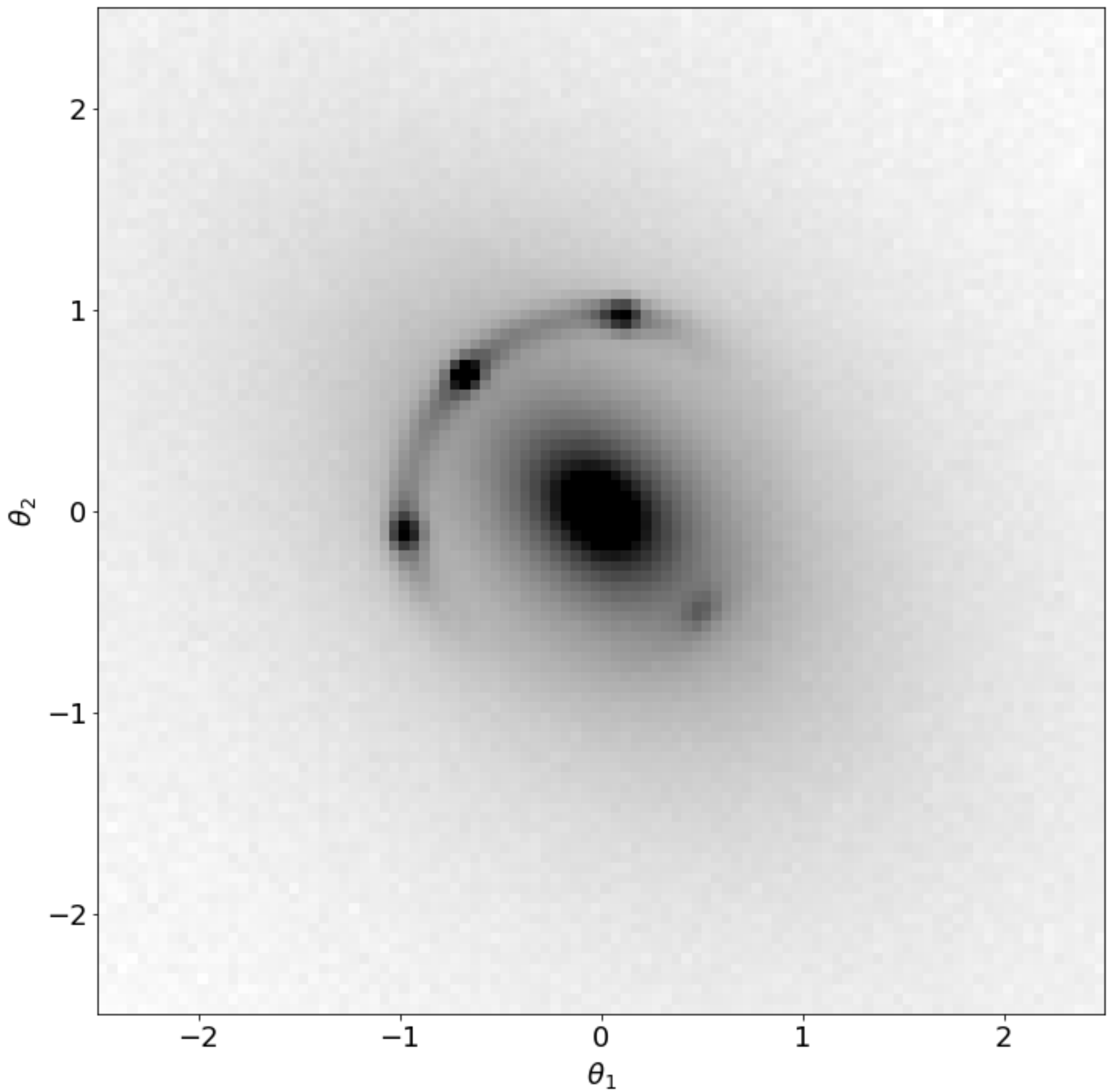


Looking at the lens, we realise that the image configuration is very similar to what we could reproduce with an elliptical lens model.



We decide to model the lens with a SIE

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY

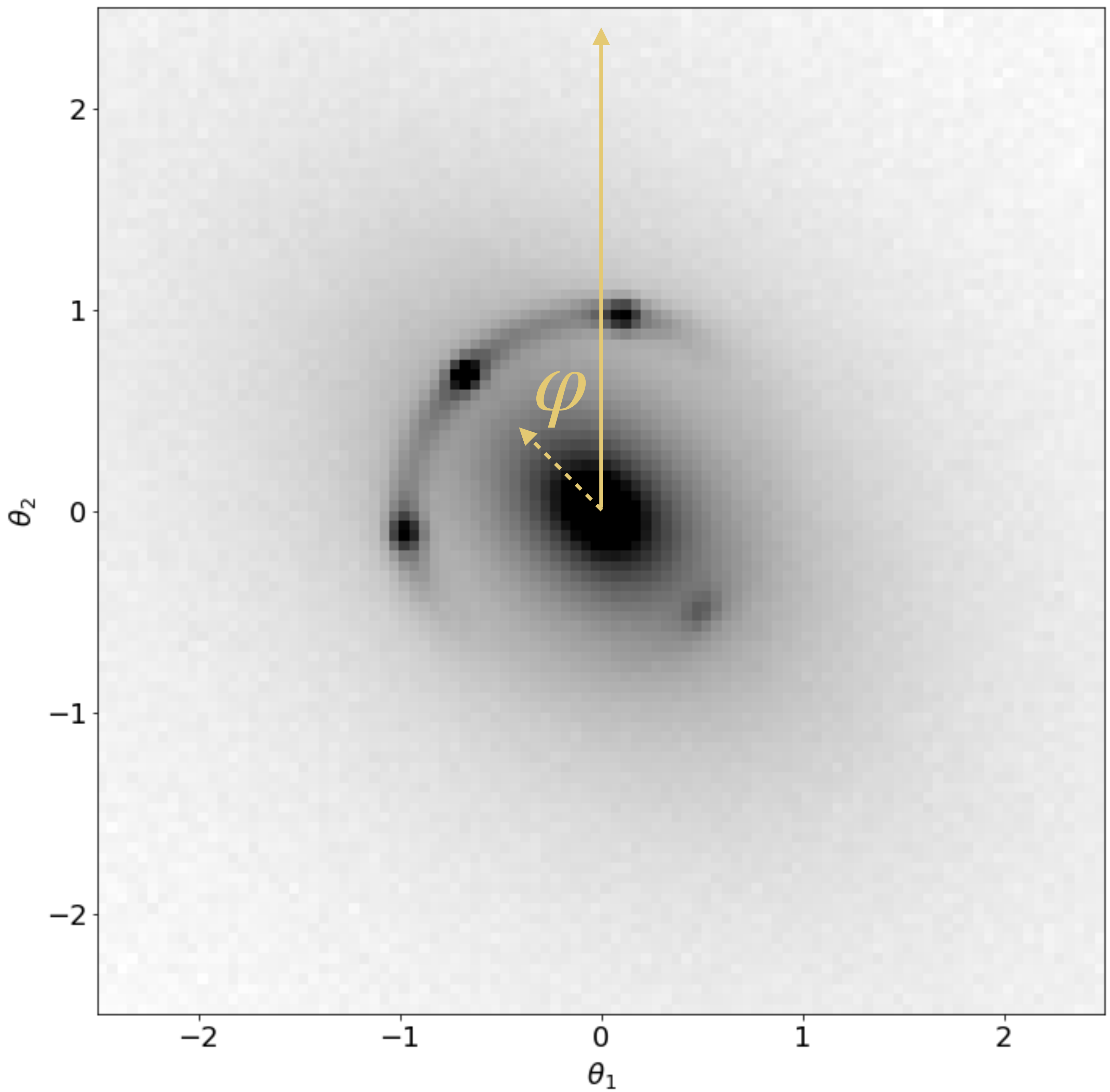


We decide to model the lens with a SIE

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G}\frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2\xi_2^2}}$$

Two model parameters: σ_v, f + orientation φ

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY

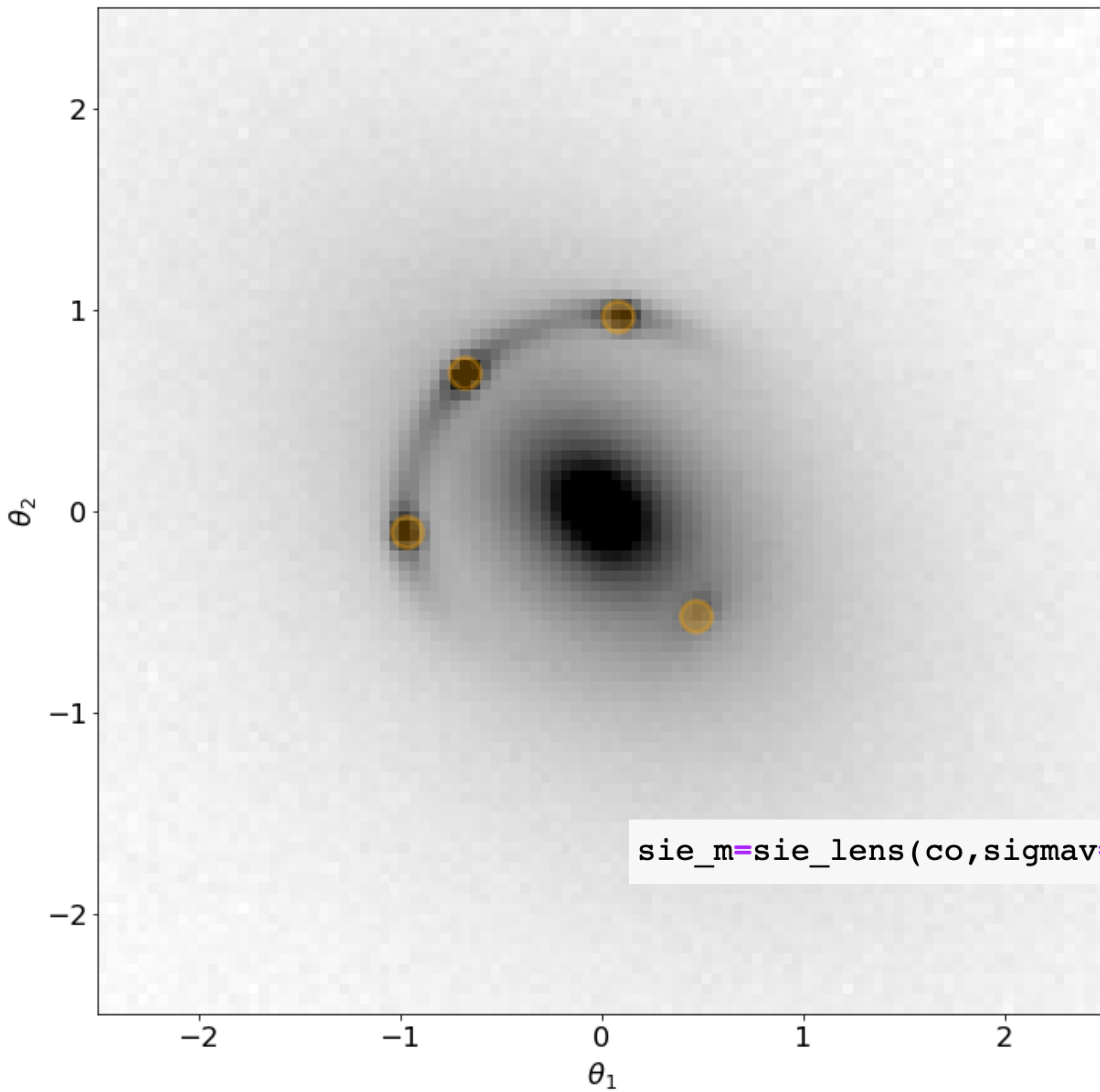


We decide to model the lens with a SIE

$$\Sigma(\vec{\xi}) = \frac{\sigma_v^2}{2G} \frac{\sqrt{f}}{\sqrt{\xi_1^2 + f^2 \xi_2^2}}$$

Two model parameters: σ_v, f + *orientation* φ

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



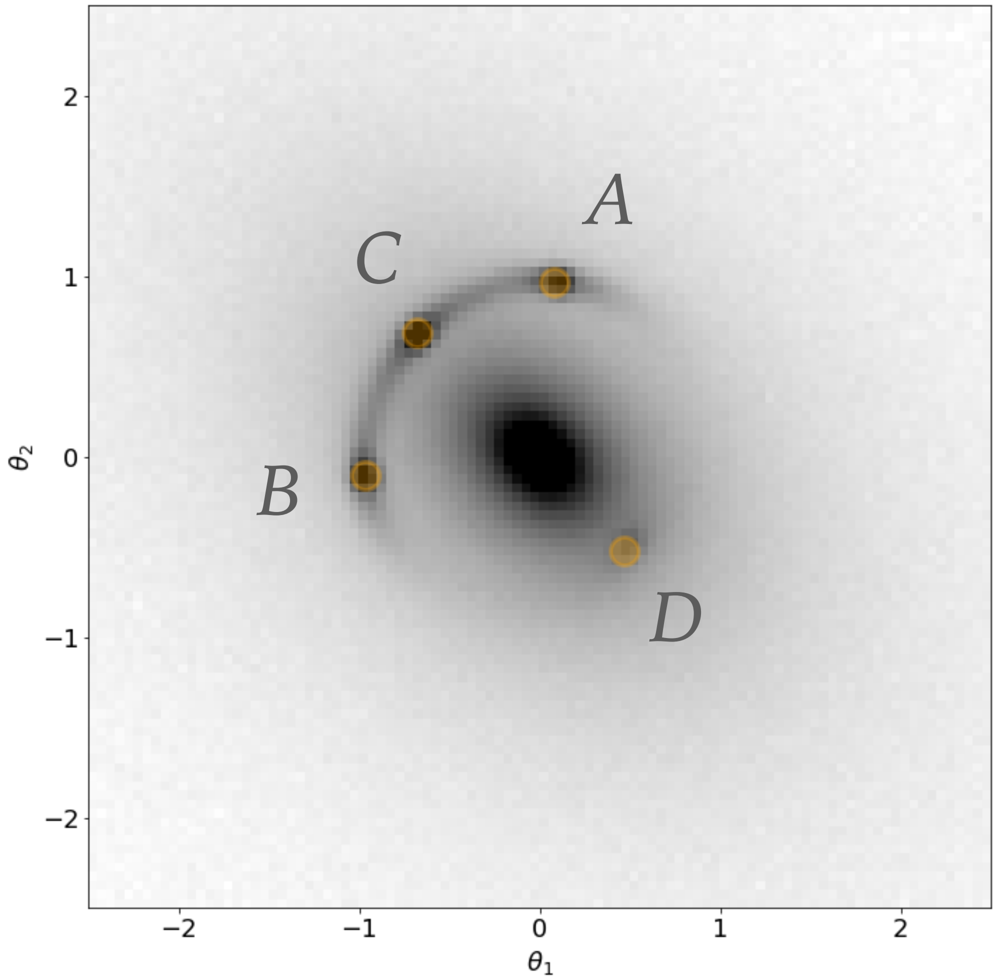
We start by using the image positions:

```
[ 0.07890278  0.96488912]
[-0.67984394  0.68741001]
[-0.96560759 -0.10242888]
[ 0.46841478 -0.51791398]
```

And we make a guess about the lens parameters:

```
sie_m=sie_lens(co,sigmav=180.0,zl=0.3,zs=2.0,f=0.3,pa=40.0)
```

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



Use the model to de-lens the images:

```
sie_m=sie_lens(co,sigmav=180.0,zl=0.3,zs=2.0,f=0.3,pa=40.0)
```

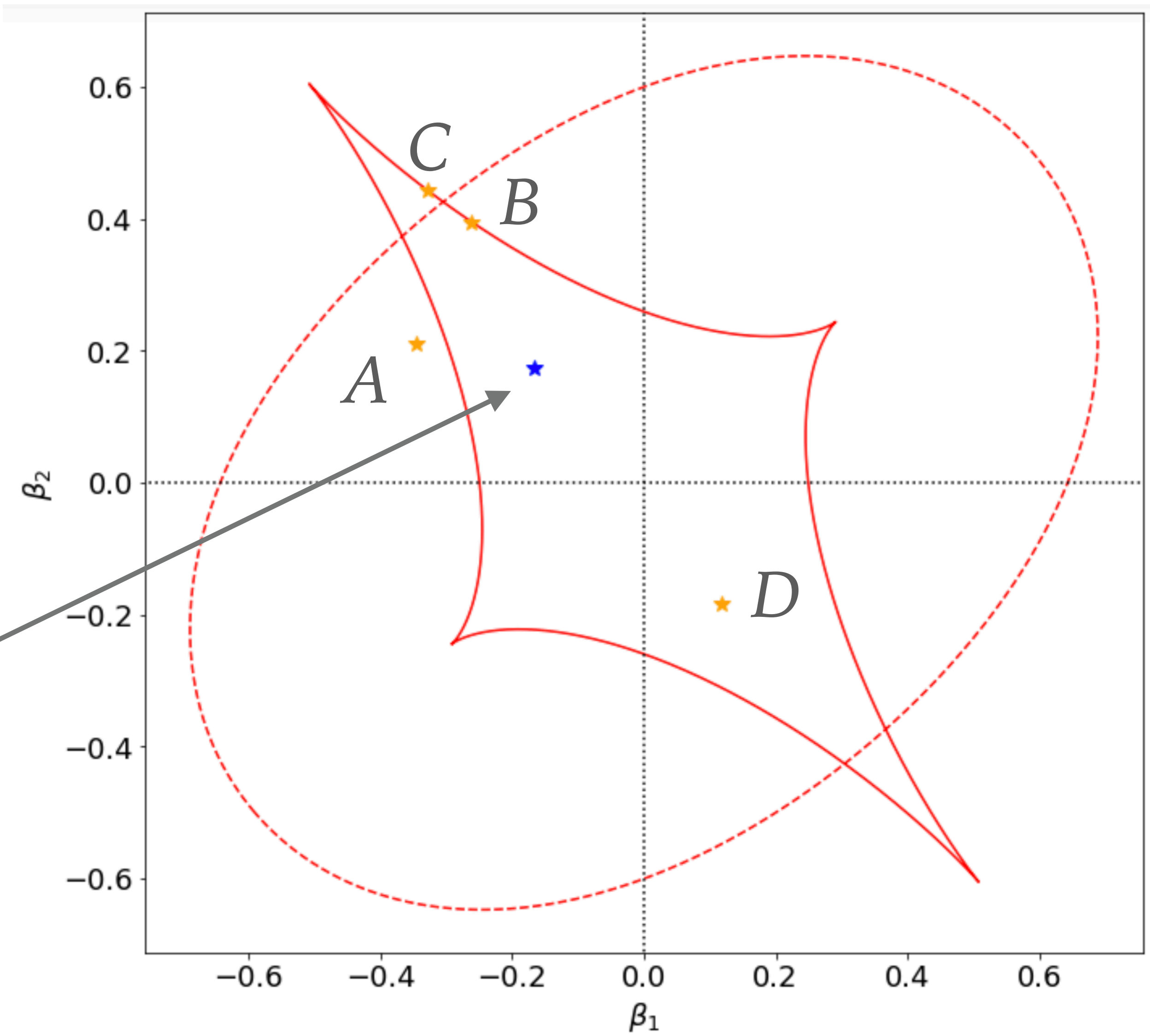
$$\alpha_1(\vec{x}) = \frac{\sqrt{f}}{f'}\mathrm{arcsinh}\left(\frac{f'}{f}\cos\varphi\right)$$

$$\alpha_2(\vec{x}) = \frac{\sqrt{f}}{f'}\arcsin(f'\sin\varphi)$$

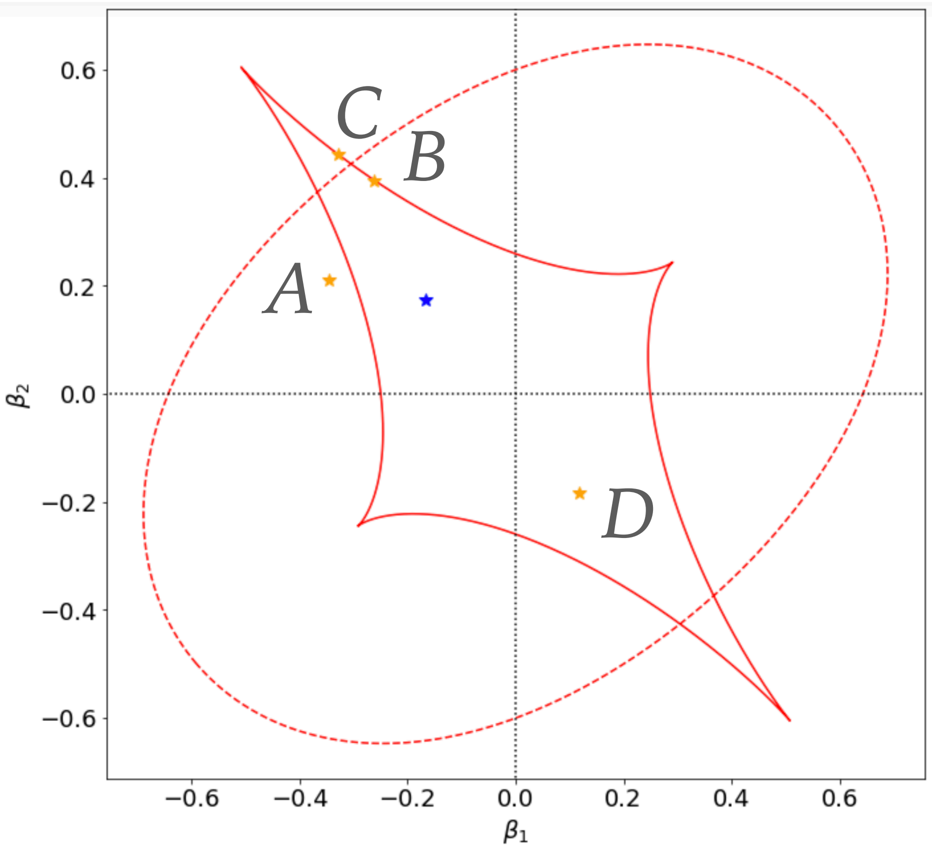
$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x})$$

Images end up in different "source" positions…

Let's take the average source position

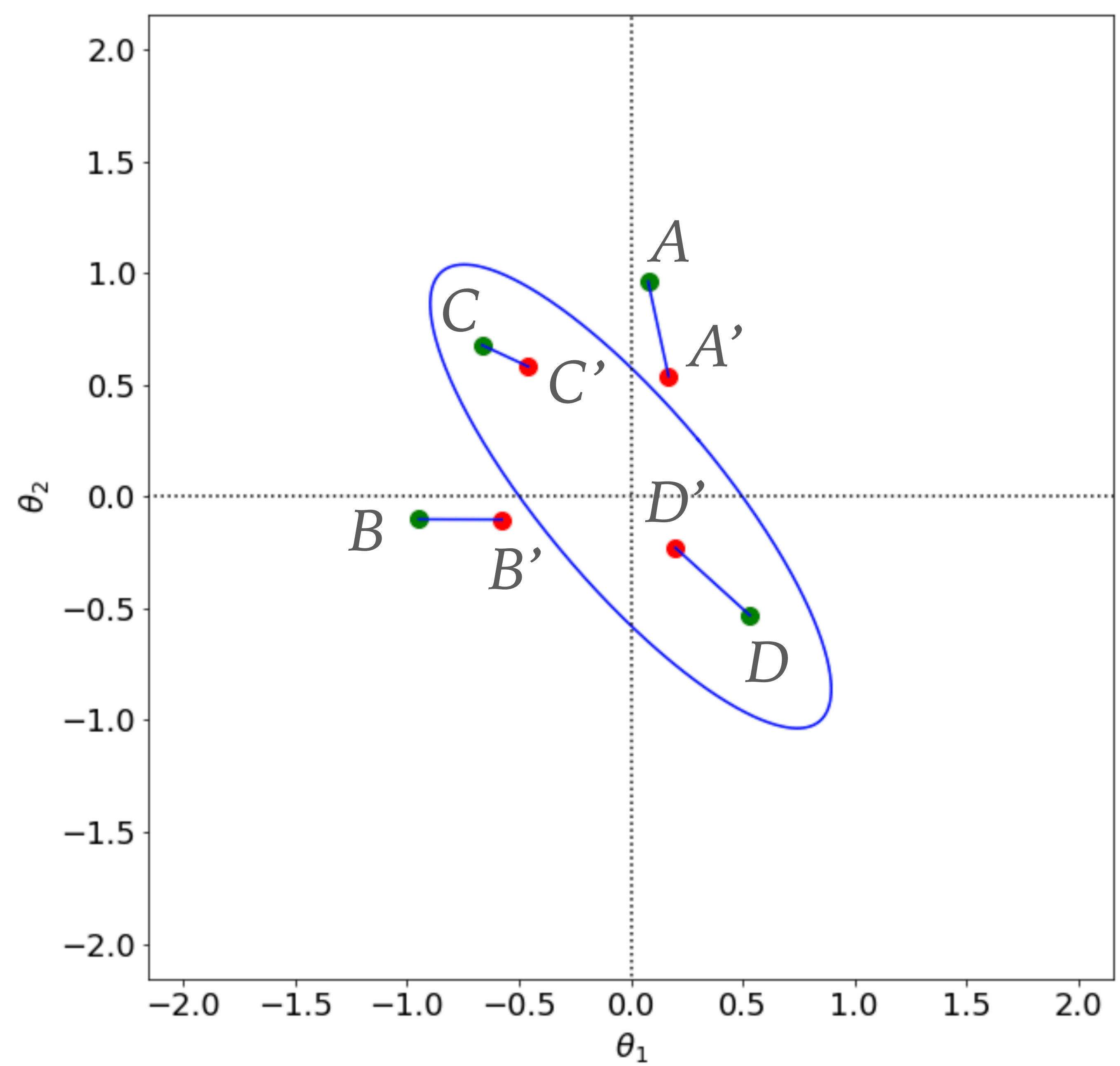


PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



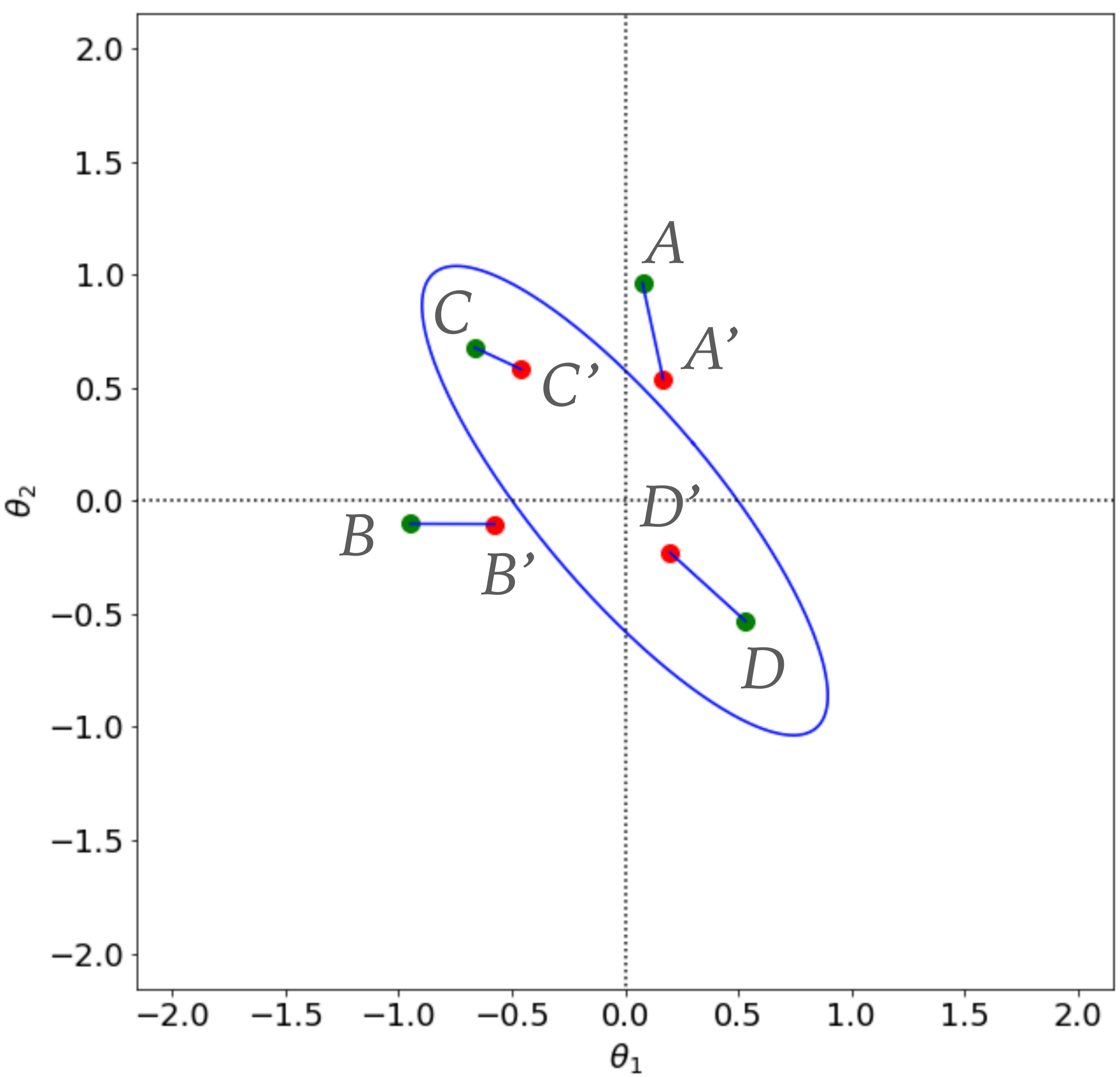
What kind of images would the average source produce?

Let's solve the lens equation… (see notebook 19)



Predicted images (red) are different from the observed images (green)!

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



Likelihood of the model:

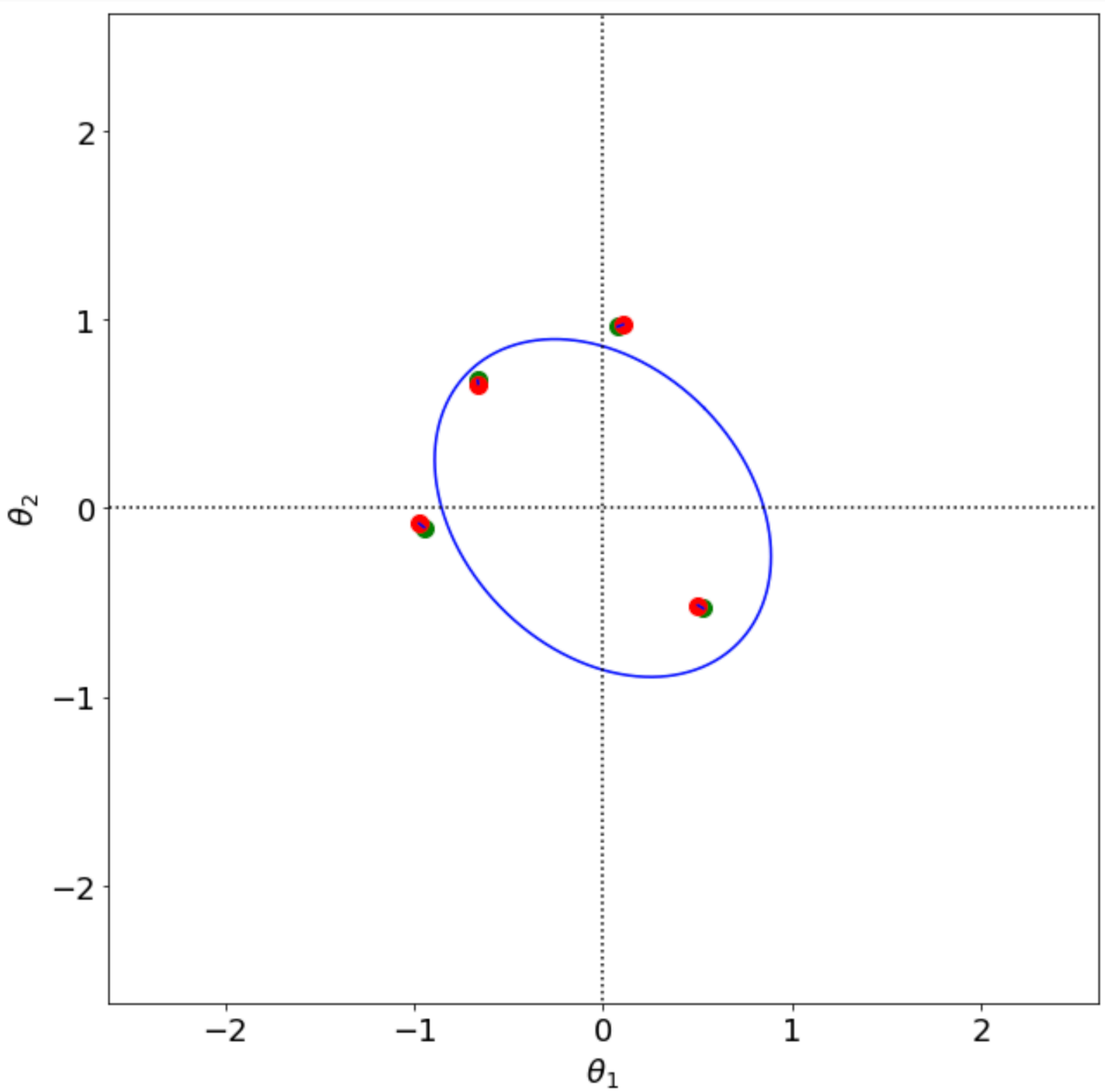
$$\Pr(\mathrm{D}\,|\,\overrightarrow{\mathrm{p}}) = \frac{1}{\Pi_{j=1}^{n}\sigma_j\sqrt{2\pi}}\exp -\frac{\chi^2}{2}$$

$$\chi^2 = \sum_{j=1}^{n}\frac{[\overrightarrow{\theta}_j - \overrightarrow{\theta}'_j(\overrightarrow{p})]^2}{\sigma_j^2}$$

$$\overrightarrow{p} = \{\sigma_v, f, \varphi\}$$

We need to find the best combination of parameters $\overrightarrow{p}$ such to minimise the lengths of the blue sticks, or, in other words, the χ^2. This also maximises the likelihood of the model!

PARAMETRIC RECONSTRUCTION OF A STRONG LENSING GALAXY



Likelihood of the model:

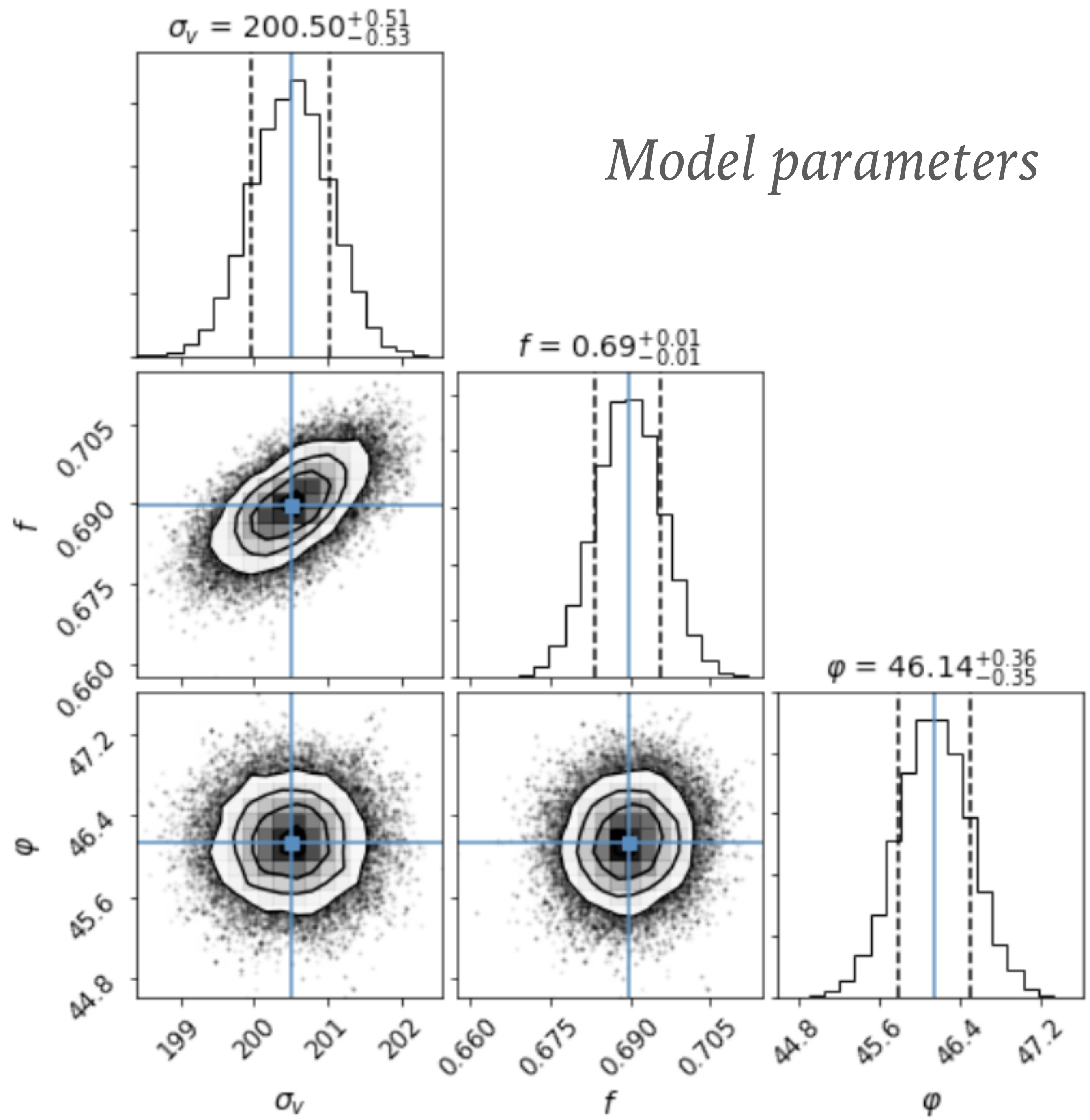
$$\Pr(D \mid \overrightarrow{p}) = \frac{1}{\Pi_{j=1}^{n}\sigma_j\sqrt{2\pi}} \exp - \frac{\chi^2}{2}$$

$$\chi^2 = \sum_{j=1}^{n} \frac{[\overrightarrow{\theta}_j - \overrightarrow{\theta}'_j(\overrightarrow{p})]^2}{\sigma_j^2}$$

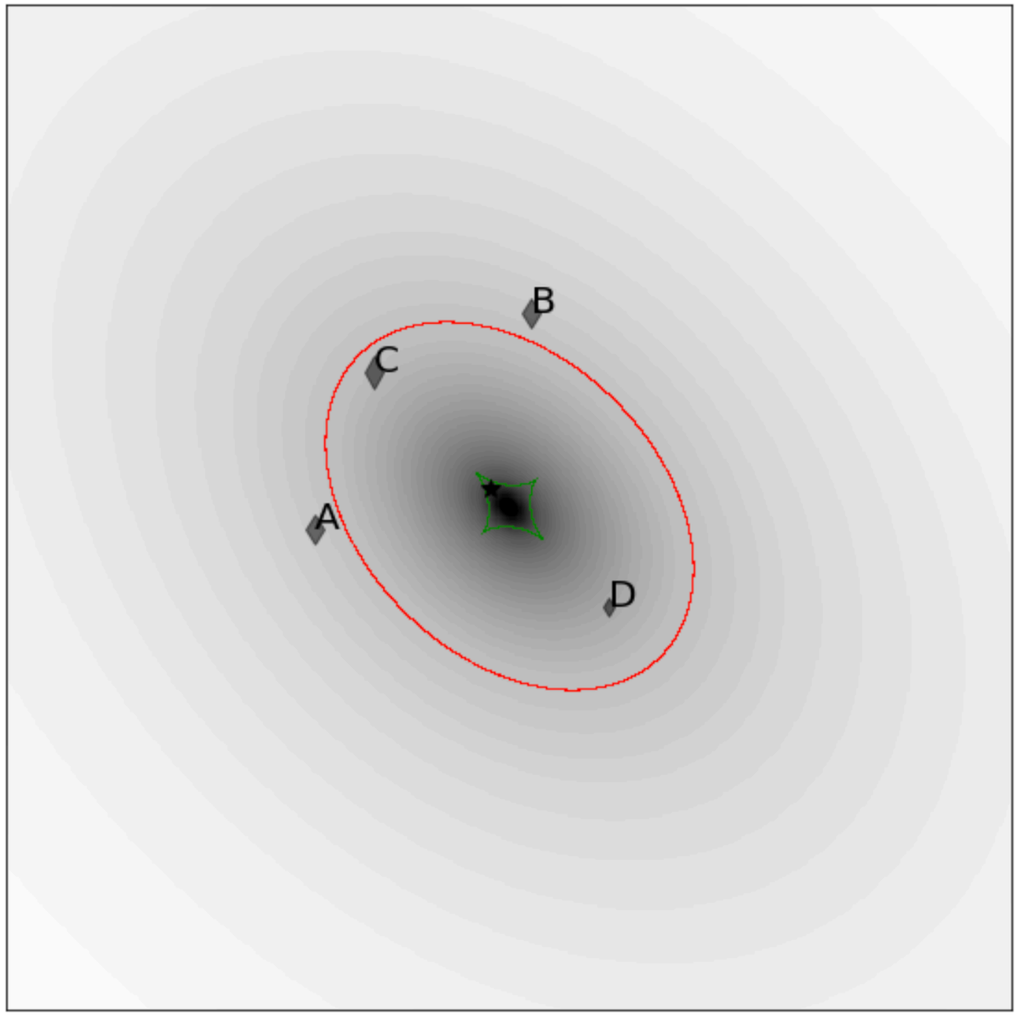
$$\overrightarrow{p} = \{\sigma_v, f, \varphi\}$$

We need to find the best combination of parameters $\overrightarrow{p}$ such to minimise the lengths of the blue sticks, or, in other words, the χ^2. This also maximises the likelihood of the model!

FIT RESULTS



Model parameters



Mass map, mass profile, etc.

LENS PLANE OPTIMIZATION

1. Using the observed images , thing their source positions.

$$\boldsymbol{y}_i = \boldsymbol{x}_i - \alpha(x_i; \boldsymbol{p})$$

2. Using the average source position find the image positions.

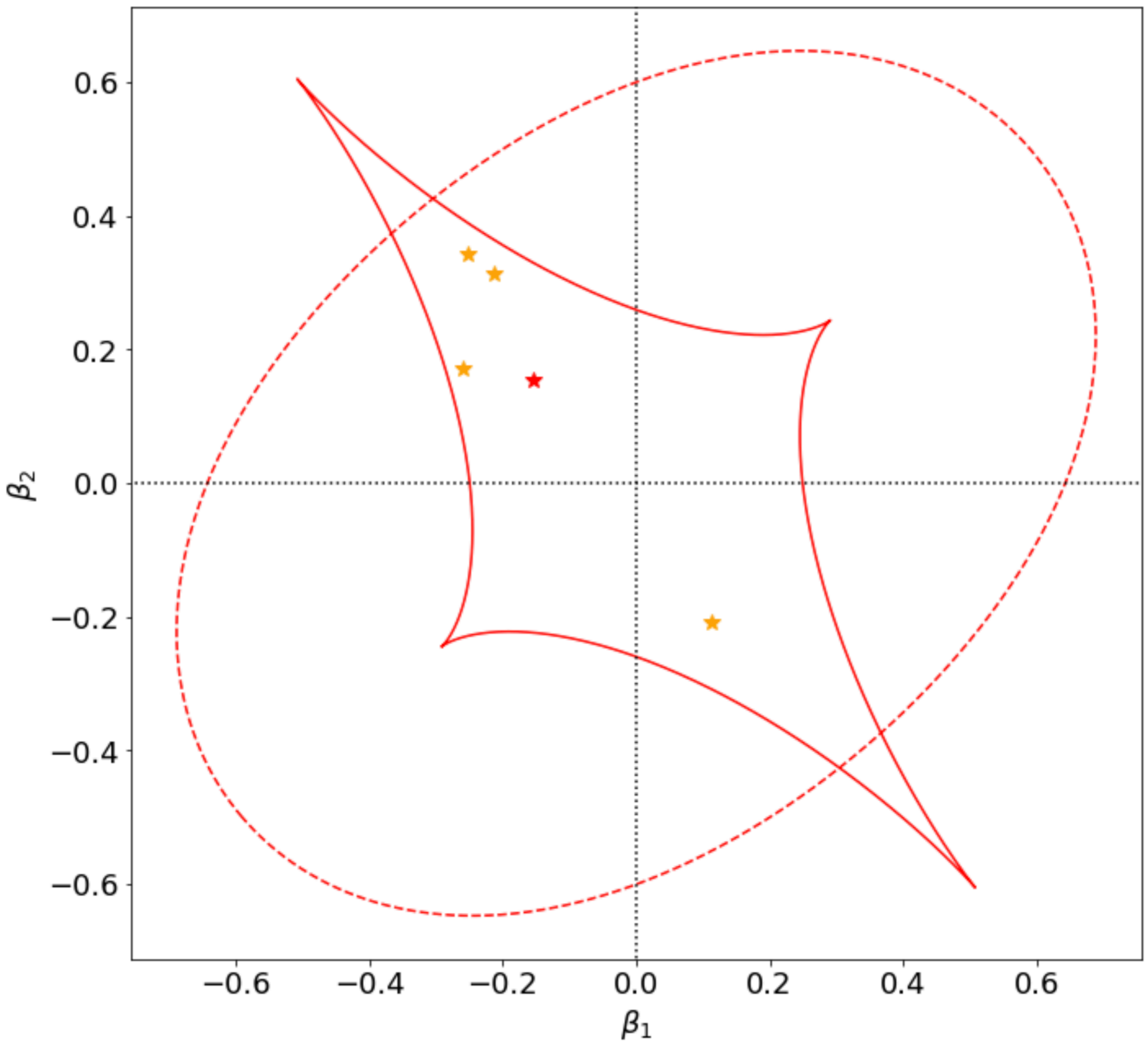
$$\overline{\boldsymbol{y}} = \frac{1}{n}\sum_i^n \boldsymbol{y}_i \qquad\qquad \overline{\boldsymbol{y}} \to \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\}$$

3. Calculate $\chi^2(\boldsymbol{p})$ *for the images.*

$$\chi^2(\boldsymbol{p}) = \sum_i^n \frac{|\boldsymbol{x}_i^{ob} - \boldsymbol{x}_i(\overline{\boldsymbol{y}}, \boldsymbol{p})|^2}{\sigma_i^2}$$

4. numerically minimize $\chi^2(\boldsymbol{p})$ *with respect to the lens parameters*

ALTERNATIVE STRATEGY: OPTIMISATION IN THE SOURCE PLANE

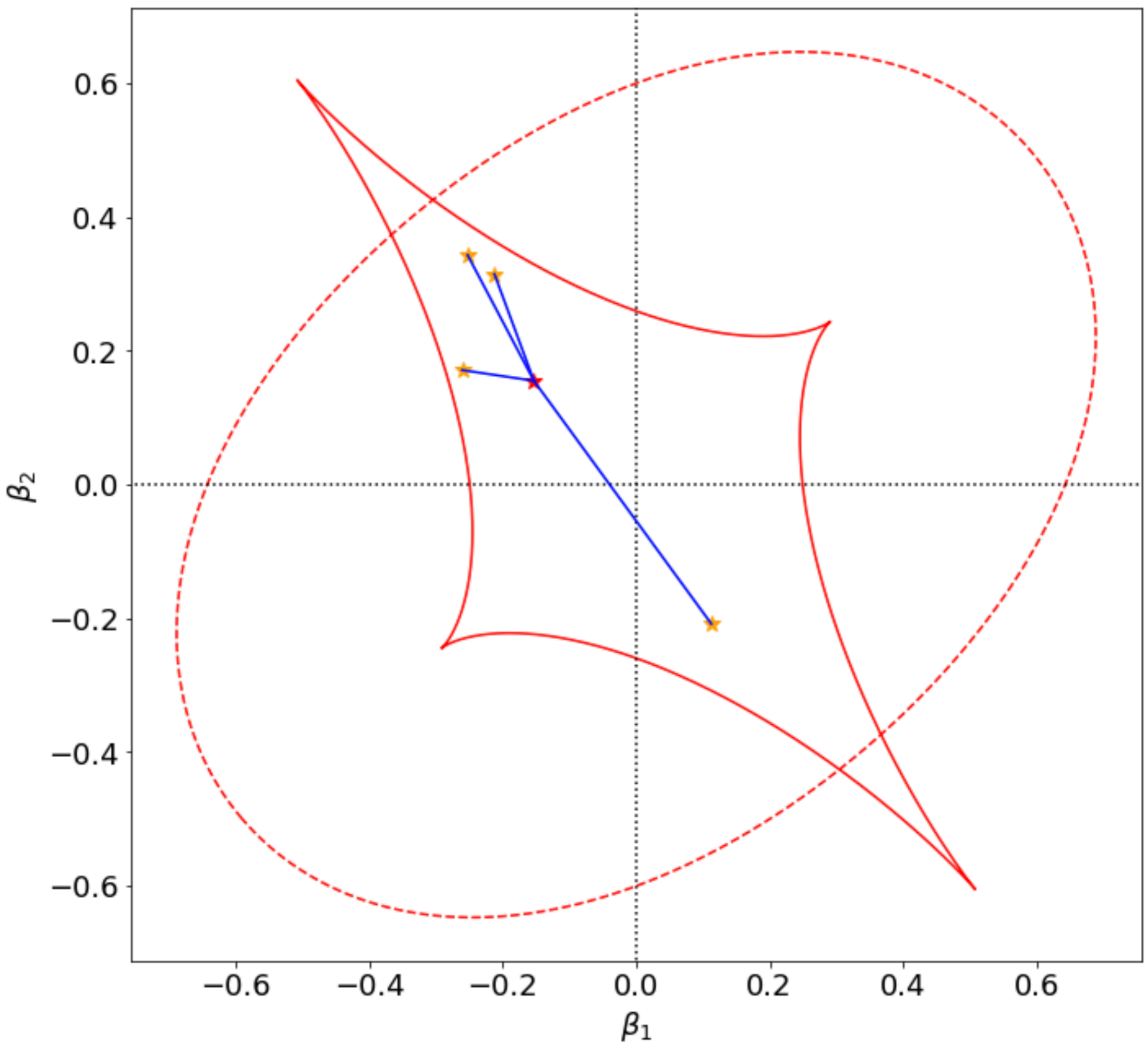


Instead of minimising the distance between observed and predicted image position, one could try to find the parameters that bring all the images onto the same source position.

$$\chi^2(\boldsymbol{p}) = \sum_i^n \frac{\mu_i^2 |\boldsymbol{y}_i - \overline{\boldsymbol{y}}|^2}{\sigma_i^2}$$

- ***No need to solve the lens equation: much faster!***
- ***But prone to introduce biases: higher ellipticity, shallower slope***

ALTERNATIVE STRATEGY: OPTIMISATION IN THE SOURCE PLANE



Instead of minimising the distance between observed and predicted image position, one could try to find the parameters that bring all the images onto the same source position.

$$\chi^2(\boldsymbol{p}) = \sum_i^n \frac{\mu_i^2 |\boldsymbol{y}_i - \overline{\boldsymbol{y}}|^2}{\sigma_i^2}$$

- ***No need to solve the lens equation: much faster!***
- ***But prone to introduce biases: higher ellipticity, shallower slope***

SOURCE PLANE OPTIMIZATION

1. *Using the observed images, thing their source positions.*

$$\boldsymbol{y}_i = \boldsymbol{x}_i - \alpha(x_i; \boldsymbol{p})$$

2. *Calculate a* $\chi^2(\boldsymbol{p})$ *that forces the source positions to be the same*

$$\overline{\boldsymbol{y}} = \frac{1}{n}\sum_i^n \boldsymbol{y}_i \qquad \chi^2(\boldsymbol{p}) = \sum_i^n \frac{\mu_i^2 |\boldsymbol{y}_i - \overline{\boldsymbol{y}}|^2}{\sigma_i^2}$$

3. *numerically minimize* $\chi^2(\boldsymbol{p})$ *with respect to the lens parameters*

SOURCE PLANE OPTIMIZATION

Advantage:

No need to solve the lens equation! faster

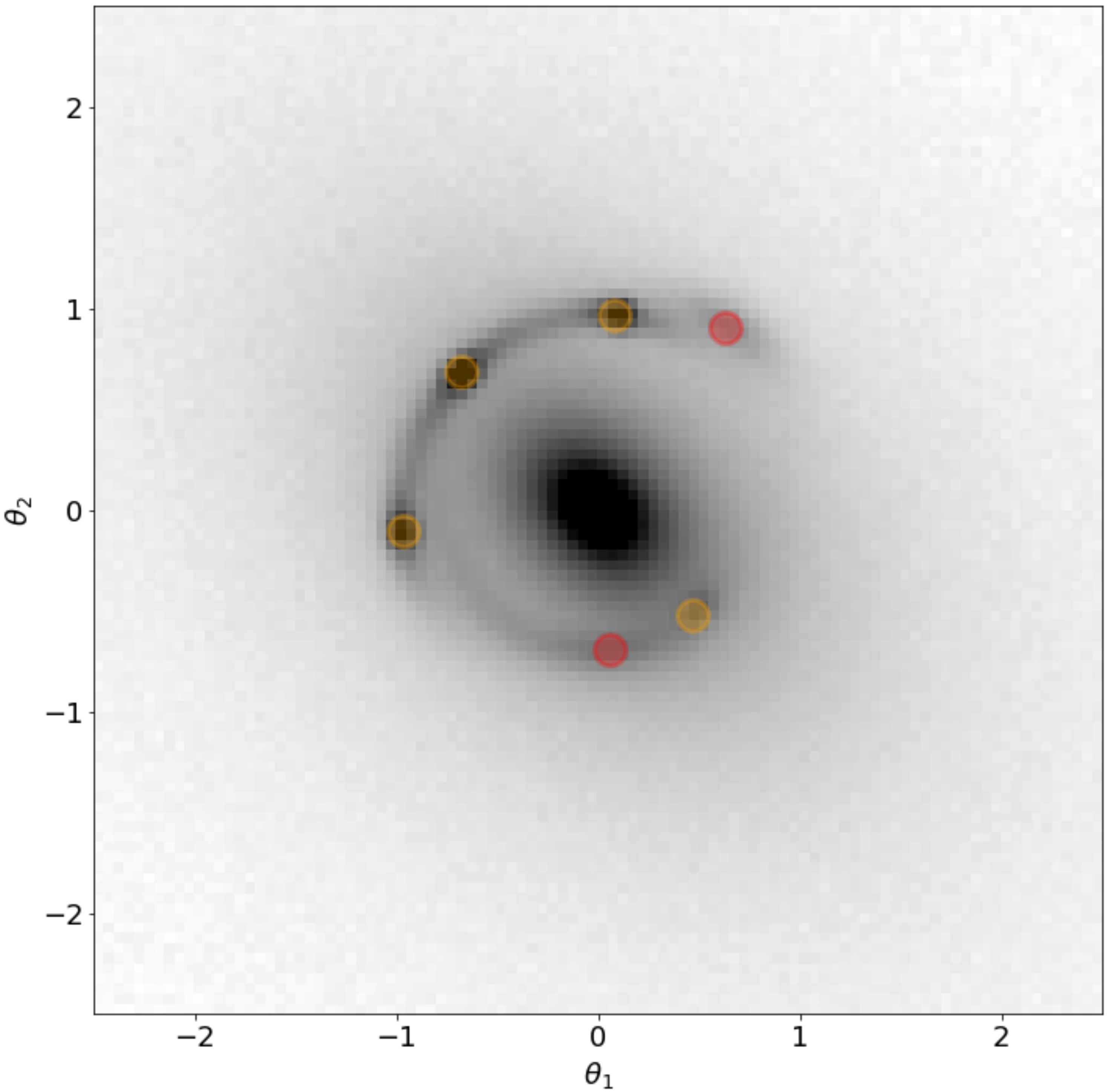
Disadvantages:

Prone to introduce biases.

Chi-squared is not well motivated from a statistical point of view.

Can find a model that has images that are not observed.

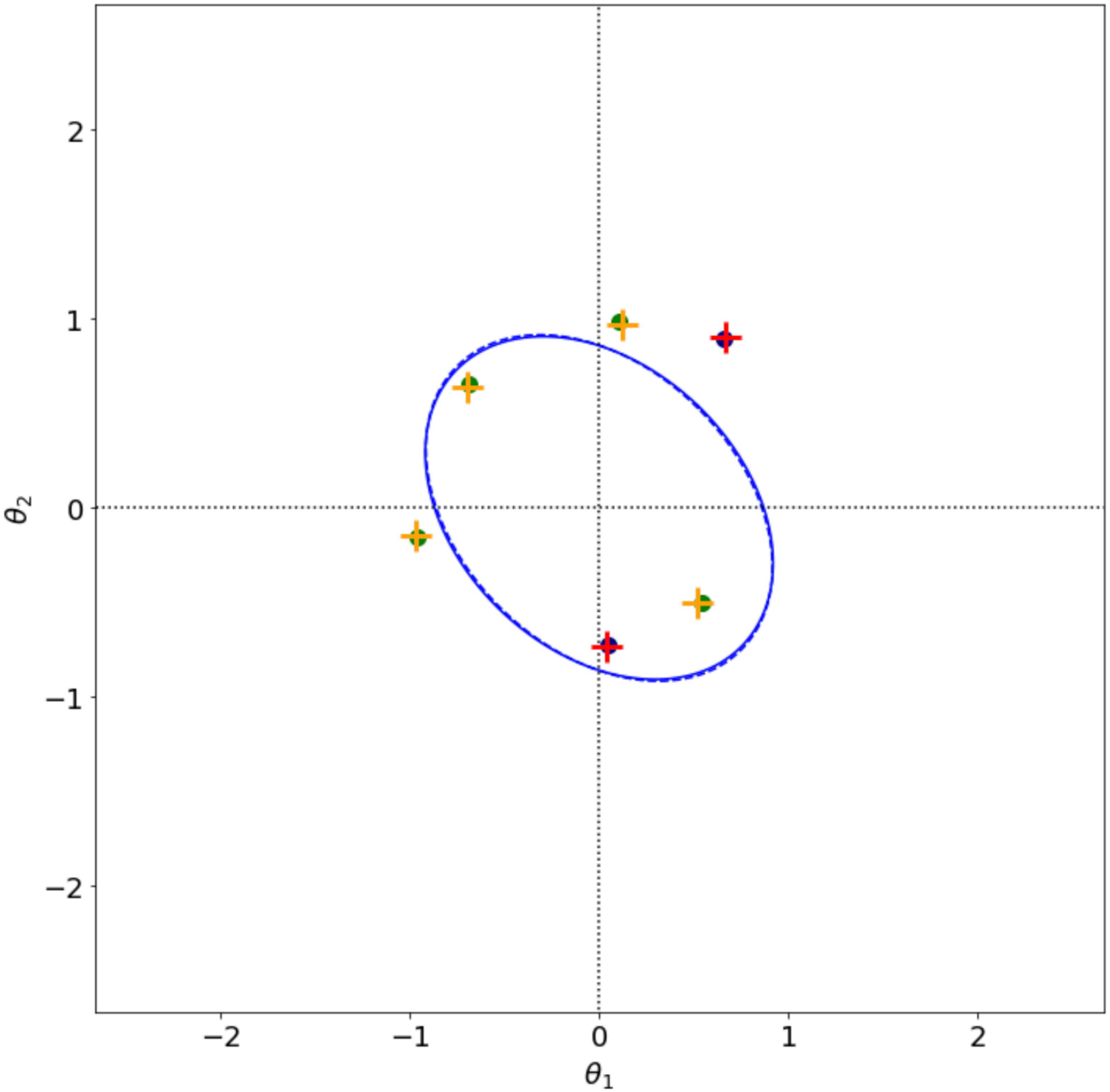
ADDING OTHER CONSTRAINTS: MULTIPLE SOURCES



*If there are more families of multiple images (**compound lenses**), they can be fitted jointly:*

$$\chi^2_{tot} = \sum_{i=1}^{N_{fam}} \sum_{j=1}^{n} \frac{[\overrightarrow{\theta}_{ij} - \overrightarrow{\theta}'_{ij}(\overrightarrow{p})]^2}{\sigma^2_{ij}}$$

ADDING OTHER CONSTRAINTS: MULTIPLE SOURCES



*If there are more families of multiple images (**compound lenses**), they can be fitted jointly:*

$$\chi^2_{tot} = \sum_{i=1}^{N_{fam}} \sum_{j=1}^{n} \frac{[\overrightarrow{\theta}_{ij} - \overrightarrow{\theta}'_{ij}(\overrightarrow{p})]^2}{\sigma^2_{ij}}$$

ADDING OTHER CONSTRAINTS: FLUX RATIOS AND/OR TIME DELAYS

To add other constraints into the model, we just have to rewrite the likelihood function:

$$\chi^2_{tot} = \chi^2_{pos} + \chi^2_{flux} + \chi^2_{\Delta t}$$

Where the χ^2 terms are:

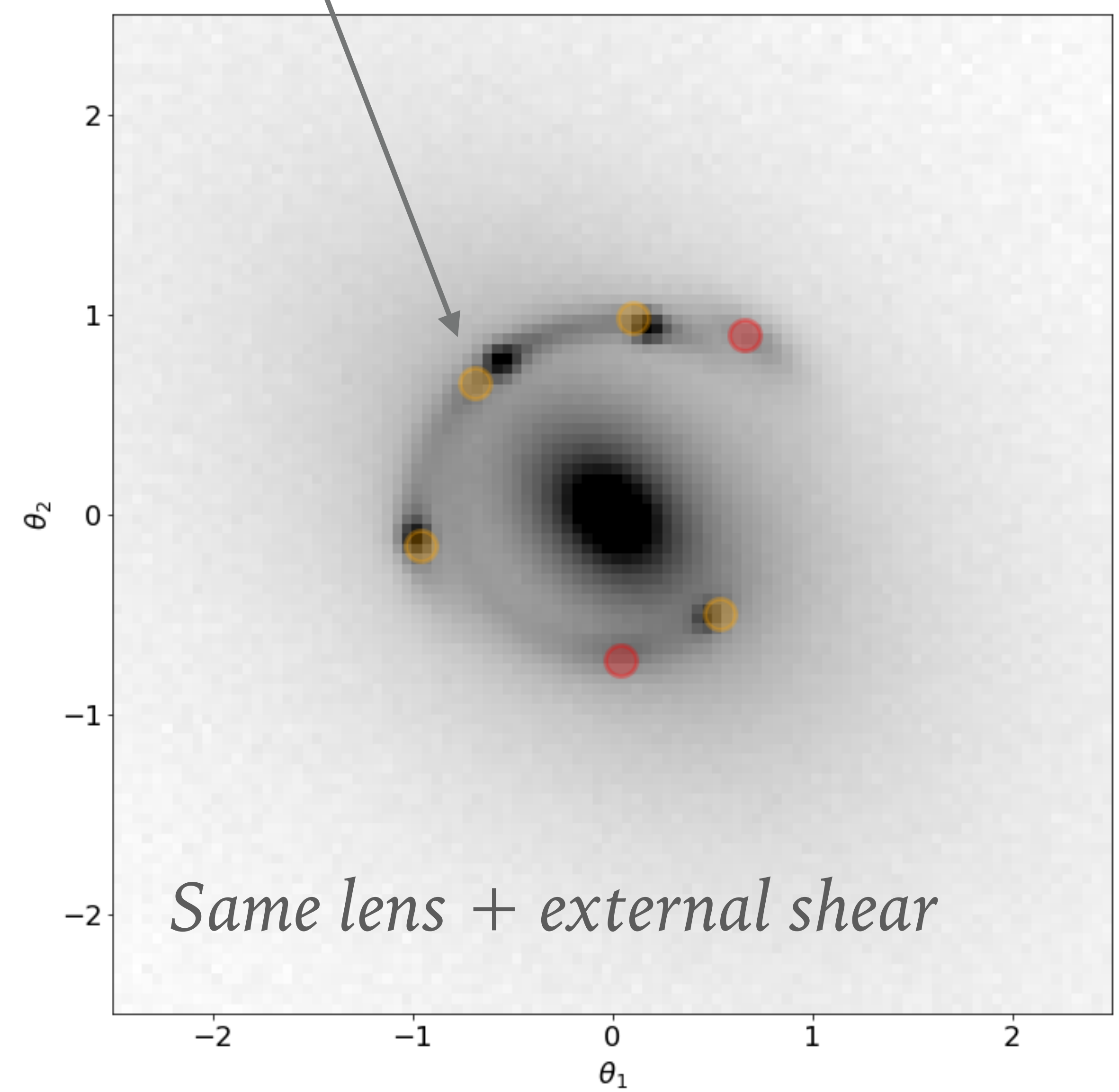
$$\chi^2_{flux} = \sum_{j=1}^{n} \frac{[F_j - F'_j(\overrightarrow{p})]^2}{\sigma^2_{F,j}} \qquad \textit{Flux-ratios}$$

$$\chi^2_{\Delta t} = \sum_{j=1}^{n} \frac{[\Delta t_j - \Delta t'_j(\overrightarrow{p})]^2}{\sigma^2_{\Delta t,j}} \qquad \textit{Time delays}$$

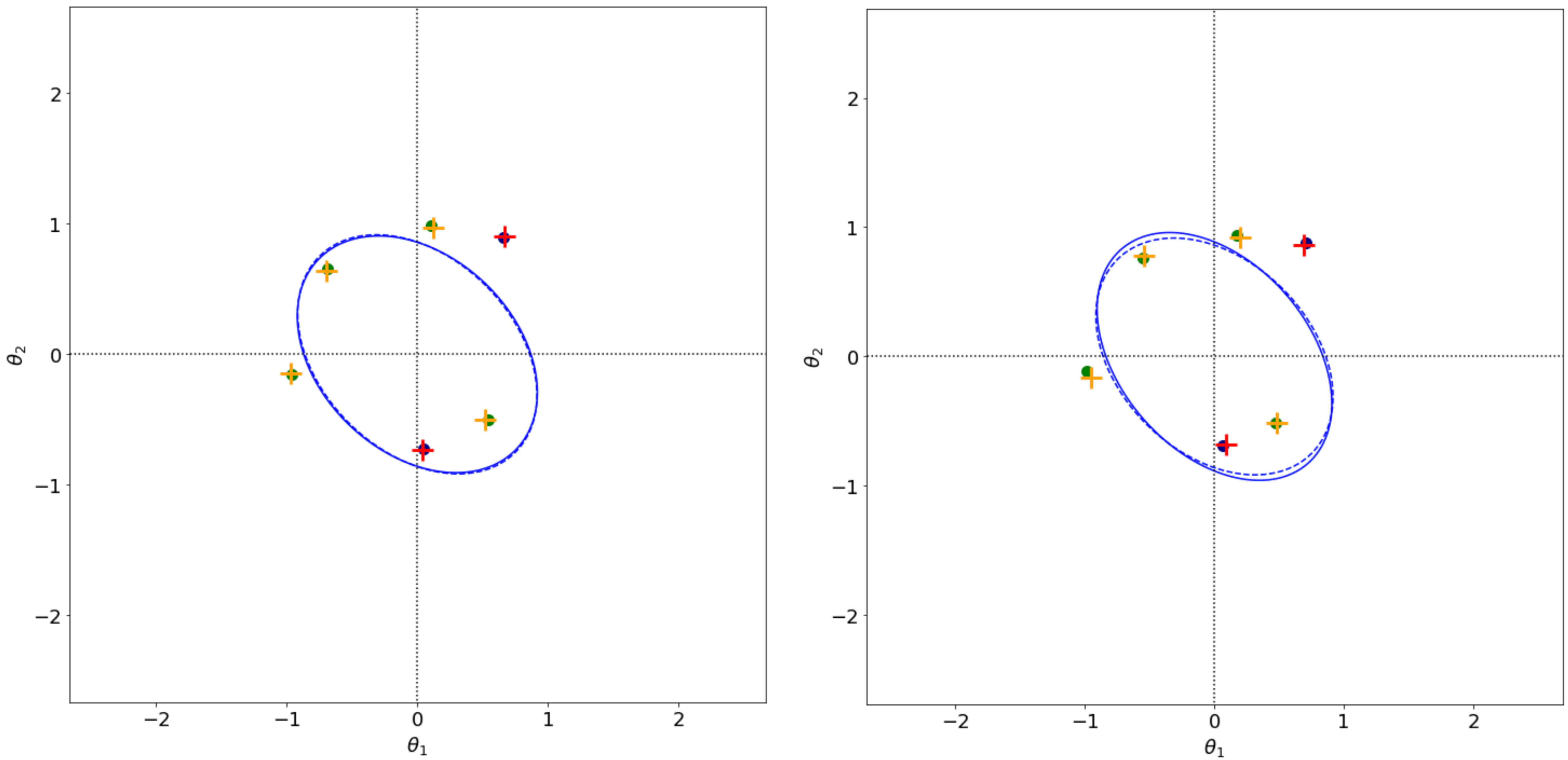
ADDITIONAL REMARKS

- The choice of the lens model is arbitrary: we chose to use a SIE because of the simplicity of the lens and of the number of multiple images (no 5th image!)
- However, additional mass components may be necessary! Example: external shear
- Or other profiles might work better! Example: power-law, softened profiles
- Caveat: the complexity of the model is limited by the number of constraints available

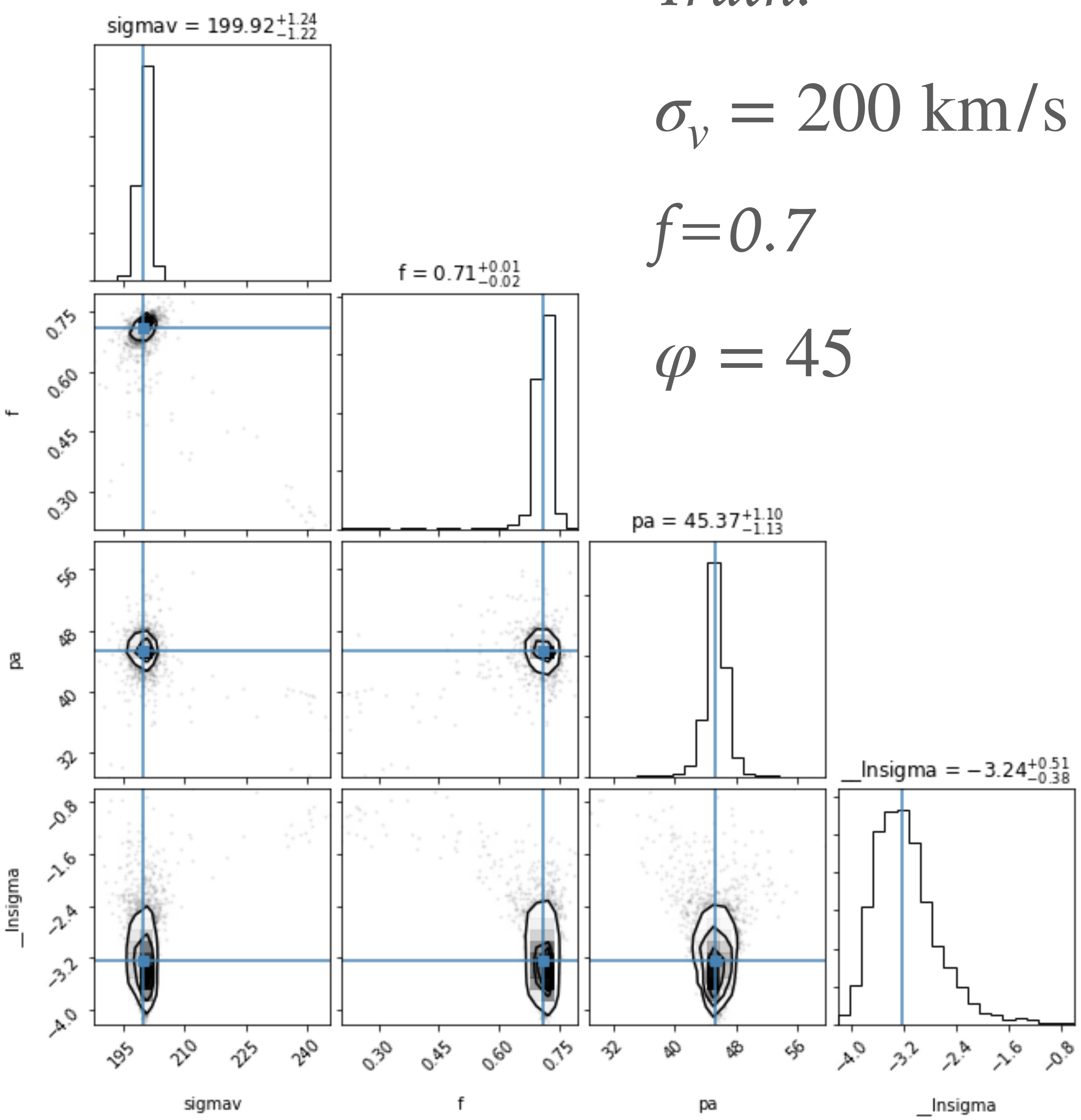
Note the shifts in the image positions!



EXAMPLE: NEGLECTING INGREDIENTS



EXAMPLE: NEGLECTING INGREDIENTS

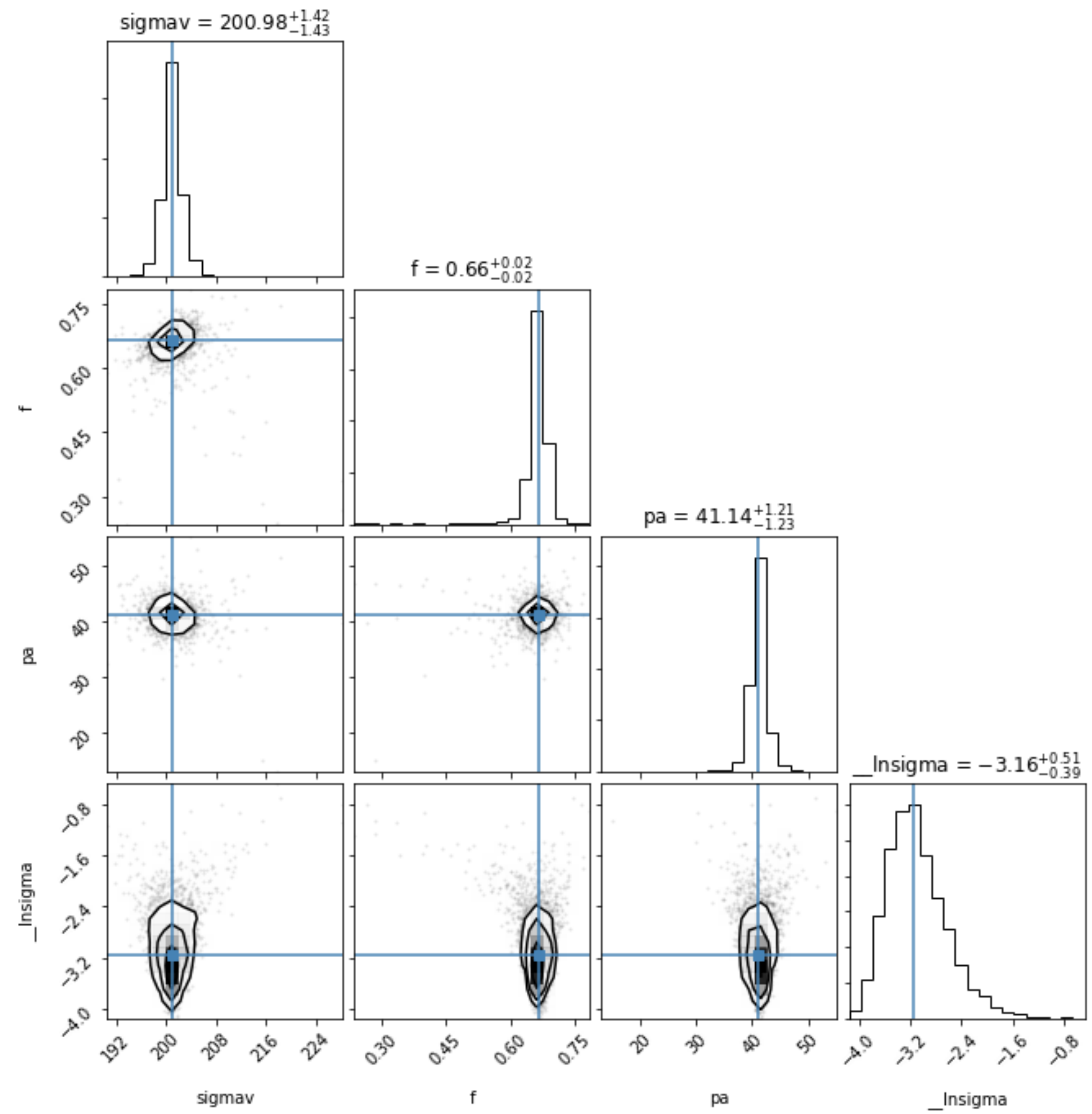


Truth:

$\sigma_v = 200$ km/s

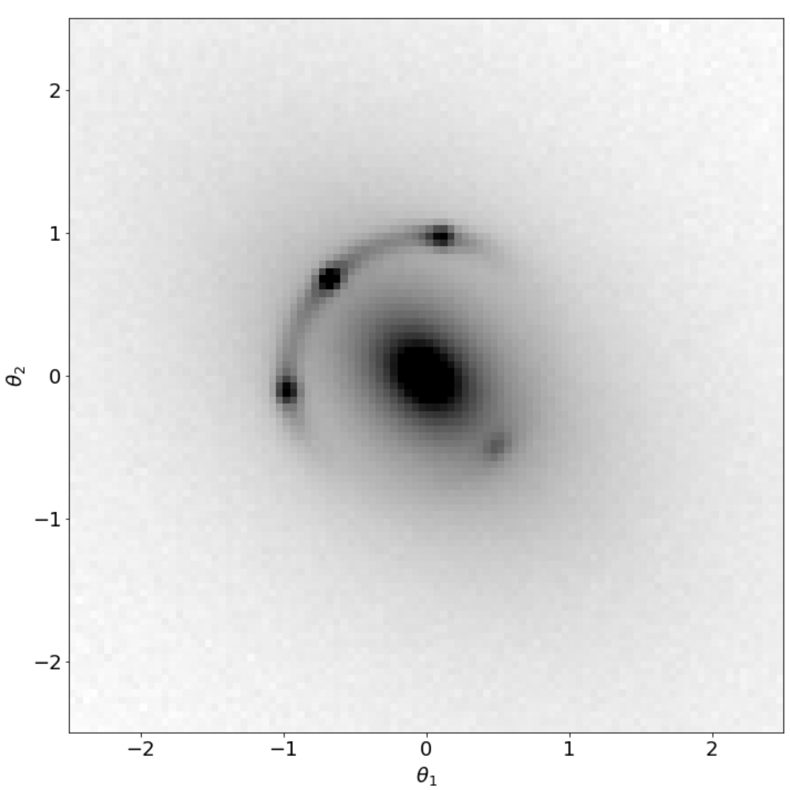
f=0.7

$\varphi = 45$



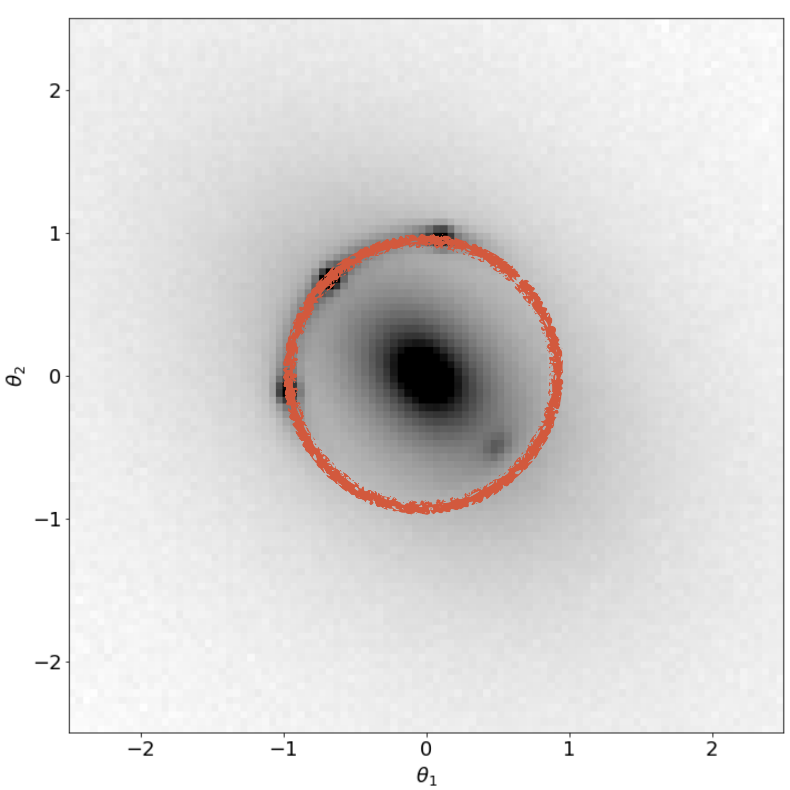
VERY IMPORTANT

- Image positions, magnifications, time delays: all depend on lens and source distances
- Without knowing the distances, we cannot measure physical quantities!
- To measure the distances, we must measure redshifts and assume a cosmological model



VERY IMPORTANT

- Image positions, magnifications, time delays: all depend on lens and source distances
- Without knowing the distances, we cannot measure physical quantities!
- To measure the distances, we must measure redshifts and assume a cosmological model



$$\textit{Einstein radius: } \theta_{E,SIS} = \frac{4\pi\sigma_v^2}{c^2}\frac{D_{LS}}{D_S}$$

$$\textit{convergence: } \bar{\kappa}(<\theta_E) = \frac{\bar{\Sigma}(\theta)}{\Sigma_{cr}} = 1$$

$$\textit{mass: } M(<\theta_E) = \pi\theta_E^2 D_L^2 \Sigma_{cr} = \pi\theta_E^2 D_L^2 \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$