GRAVITATIONAL LENSING

5- THE LENSING POTENTIAL & SHEAR

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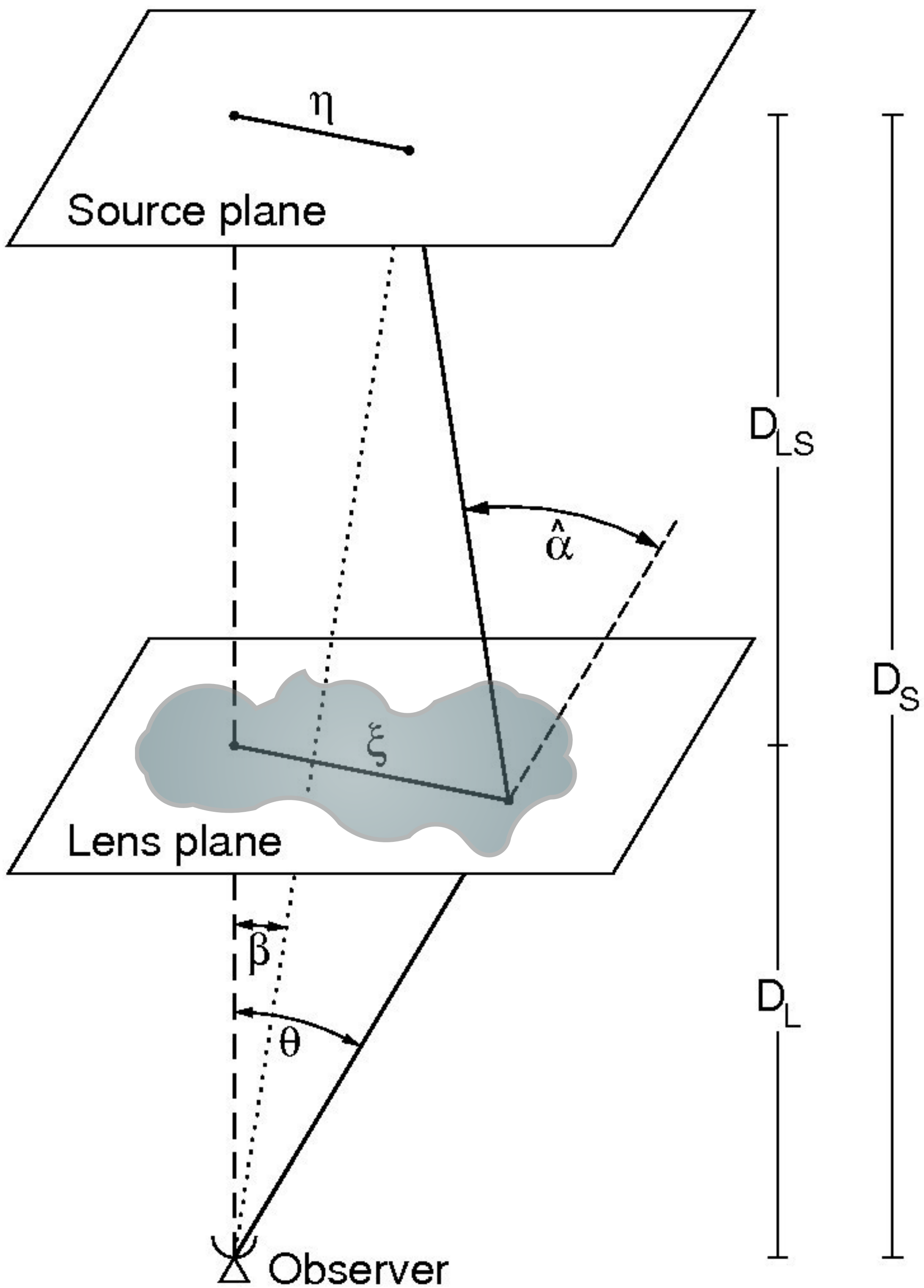
2022-2023

LENS EQUATION



Remember that:

1) *positions on the lens and source planes are defined by vectors*
2) *the deflection angle itself is a vector*

$$\vec{\theta} = \frac{\vec{\xi}}{D_L} \qquad \vec{\beta} = \frac{\vec{\eta}}{D_S}$$

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\vec{\hat{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

THE LENSING POTENTIAL

$$\psi(\theta) = \left(\frac{2}{c^2} \frac{D_{ls}}{D_s D_l} \right) \int_{-\infty}^{\infty} dz \ \phi_N(D_l\theta, z)$$

$$\boldsymbol{\nabla}_\theta \psi(\theta) = \boldsymbol{\alpha}(\theta)$$

$$\nabla^2_\theta \psi(\theta) = 2\kappa(\theta)$$

dimensionless surface density

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{\rm crit}}$$

critical surface density

$$\Sigma_{\rm crit} = \frac{c^2}{4\pi G} \frac{D_s}{D_{ls} D_l}$$

THE LENSING POTENTIAL

Green's functions relating the surface density to the potential and deflection.

$$\psi(\theta) = \frac{1}{\pi} \int d^2\theta' \ \kappa(\boldsymbol{\theta}') \ \ln\left(|\boldsymbol{\theta} - \boldsymbol{\theta}'|\right)$$

$$\kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_{\rm crit}}$$

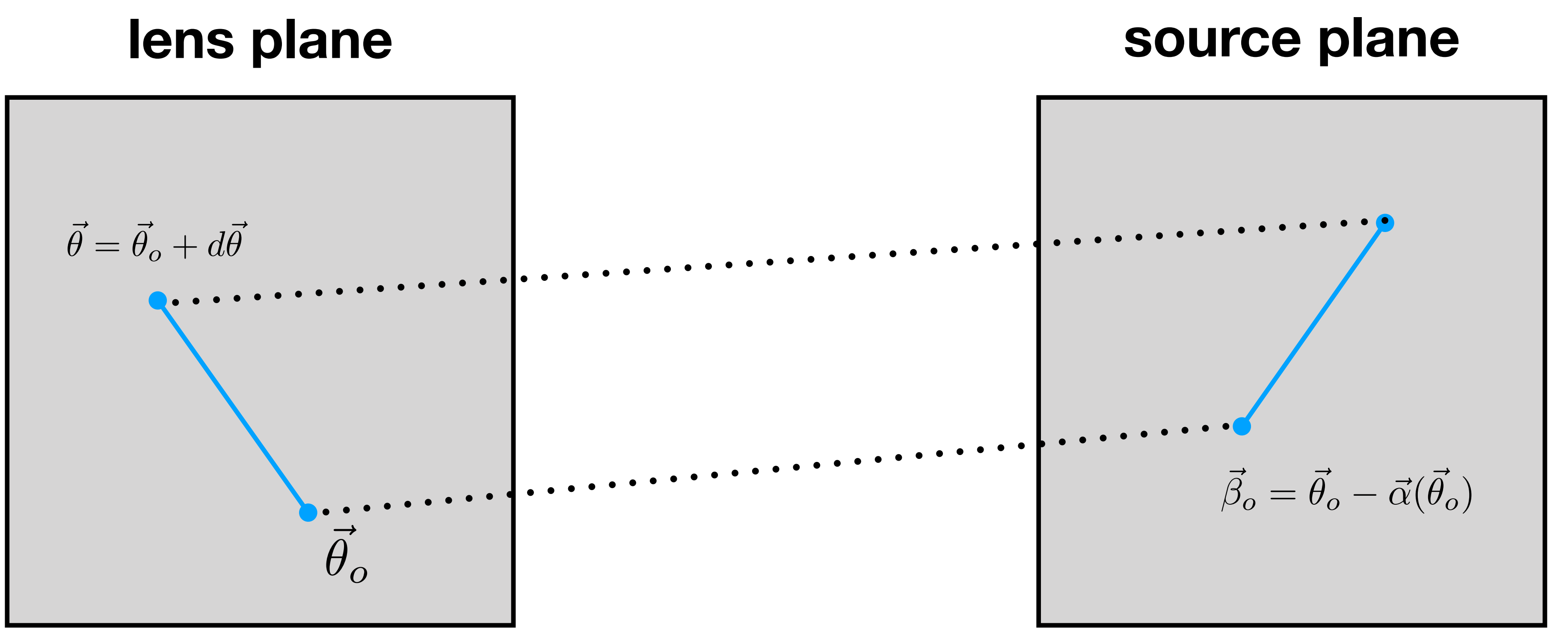
$$\boldsymbol{\alpha}(\theta) = \frac{1}{\pi} \int d^2\theta' \ \kappa(\boldsymbol{\theta}') \ \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}')}{|\boldsymbol{\theta} - \boldsymbol{\theta}'|^2}$$

$$\Sigma_{\rm crit} = \frac{c^2}{4\pi G} \frac{D_s}{D_{ls} D_l}$$

LENS MAPPING

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \int d^2\theta' \; \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \frac{\Sigma(\vec{\theta}')}{\Sigma_{crit}}$$



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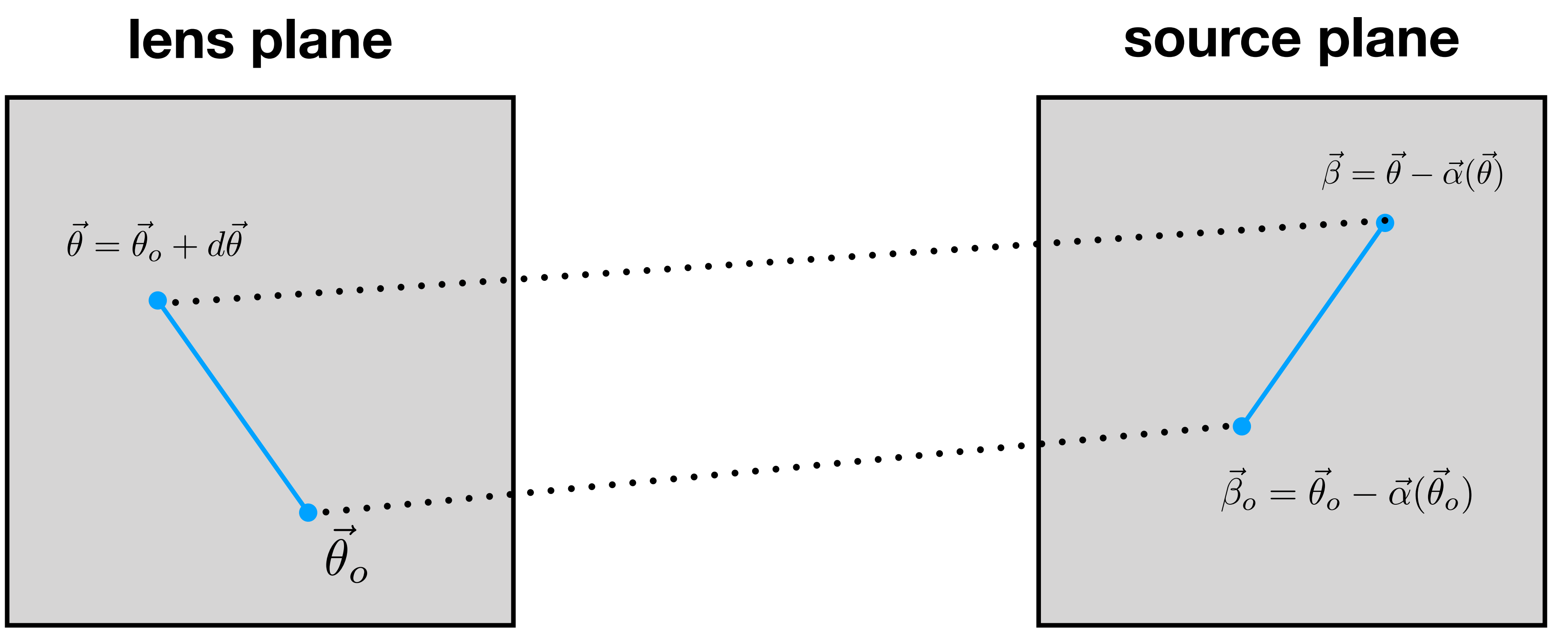

LENS MAPPING

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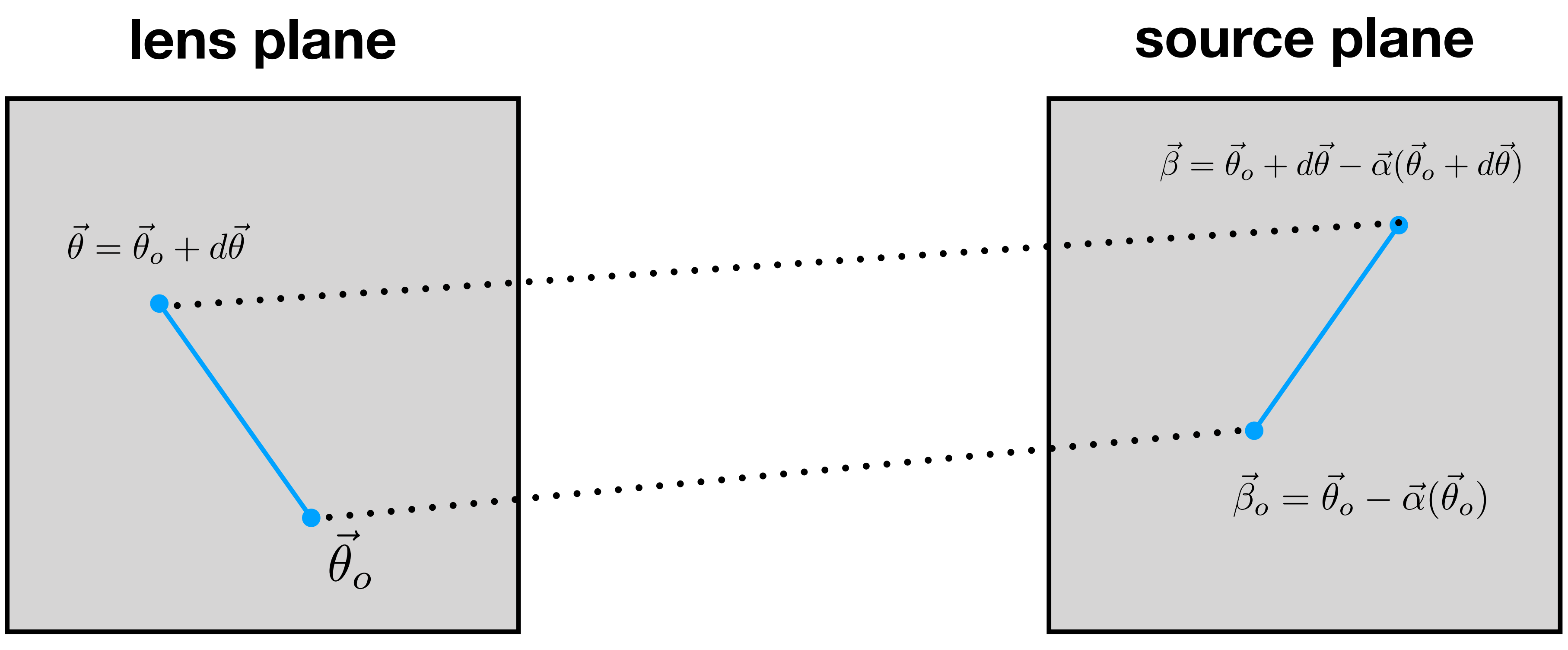

$$d\vec{\beta} = \vec{\beta} - \vec{\beta}_o = d\vec{\theta} - \left(\vec{\alpha}(\vec{\theta}_o + d\vec{\theta}) - \vec{\alpha}(\vec{\theta}_o)\right)$$

LENS MAPPING

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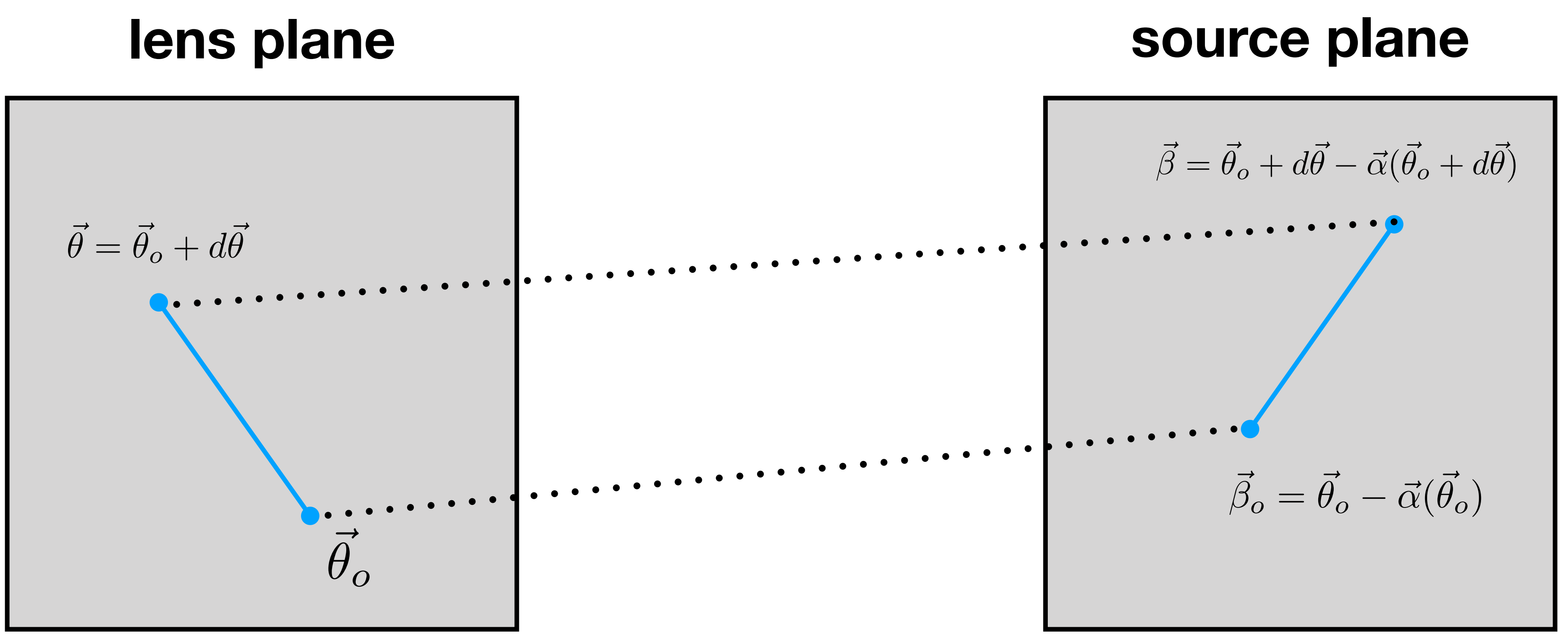

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$$= d\vec{\theta} - d\vec{\theta} \cdot \nabla_\theta \vec{\alpha}(\vec{\theta}_o)$$

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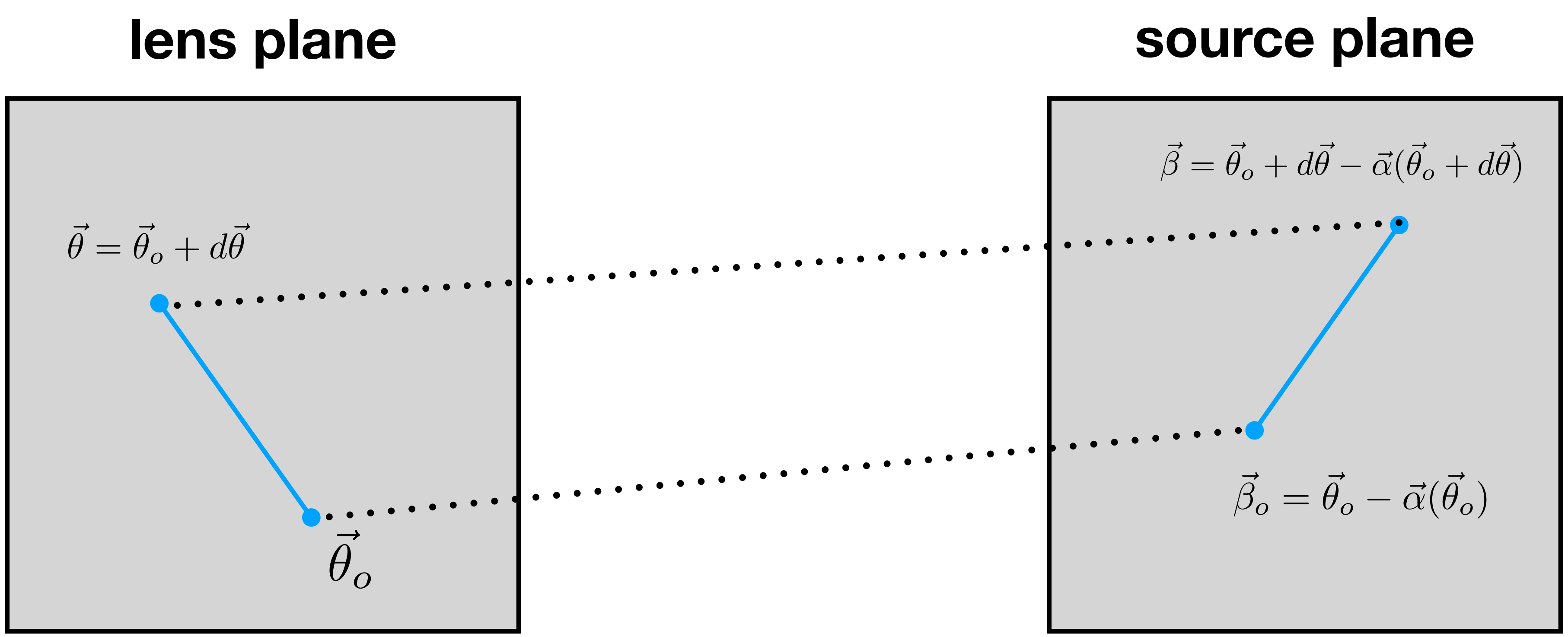

$$\begin{aligned} d\vec{\beta} = \vec{\beta} - \vec{\beta}_o &= d\vec{\theta} - \left(\vec{\alpha}(\vec{\theta}_o + d\vec{\theta}) - \vec{\alpha}(\vec{\theta}_o)\right) \\ &= d\vec{\theta} - d\vec{\theta} \cdot \nabla_\theta \vec{\alpha}(\vec{\theta}_o) \\ &= \boldsymbol{A}\, d\vec{\theta} \end{aligned}$$

LENS MAPPING : ROTATION OF SHEAR

$$R^T \Gamma R$$

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

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$$\begin{pmatrix} \gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2 \sin(\phi)\cos(\phi) & \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1 \sin(\phi)\cos(\phi) \\ \gamma_2(\cos(\phi)^2 - \sin(\phi)^2) - 2\gamma_1 \sin(\phi)\cos(\phi) & -\gamma_1(\cos(\phi)^2 - \sin(\phi)^2) + 2\gamma_2 \sin(\phi)\cos(\phi) \end{pmatrix}$$

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$$\begin{pmatrix} \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi) & \gamma_2 \cos(2\phi) - \gamma_1 \sin(2\phi) \\ \gamma_2 \cos(2\phi) - \gamma_1 \sin(2\phi) & -(\gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi)) \end{pmatrix}$$

LENS MAPPING : ROTATION OF SHEAR

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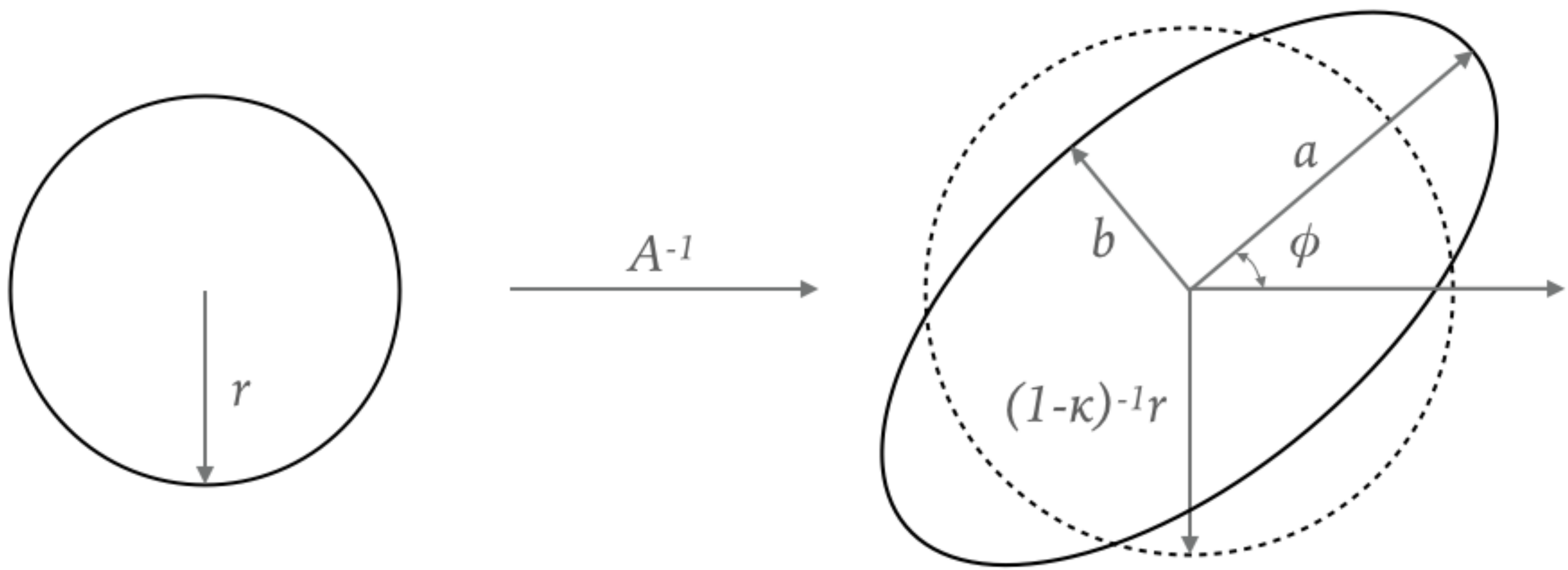
$$\boxed{\begin{aligned} \gamma_1' &= \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi) \\ \gamma_2' &= \gamma_2 \cos(2\phi) - \gamma_1 \sin(2\phi) \end{aligned}}$$

Not a vector!

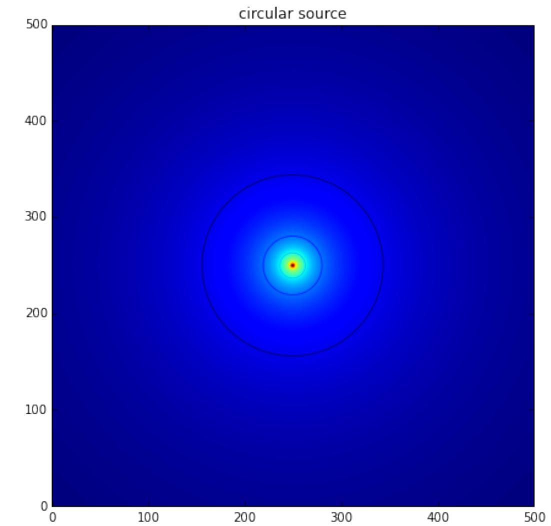
spin-2 object

LENS MAPPING



ON THE SPIN-2 NATURE OF SHEAR

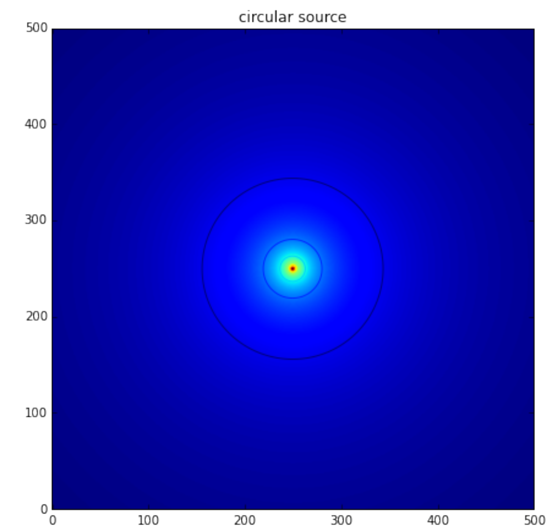
Consider a circular source



ON THE SPIN-2 NATURE OF SHEAR

Consider a circular source

How is it distorted if we apply a pure convergence transformation?



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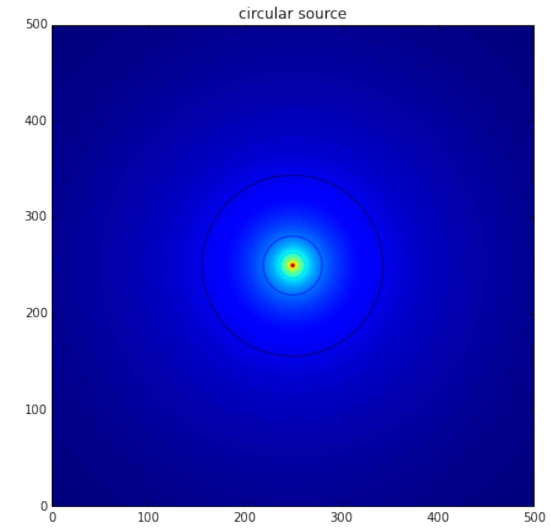

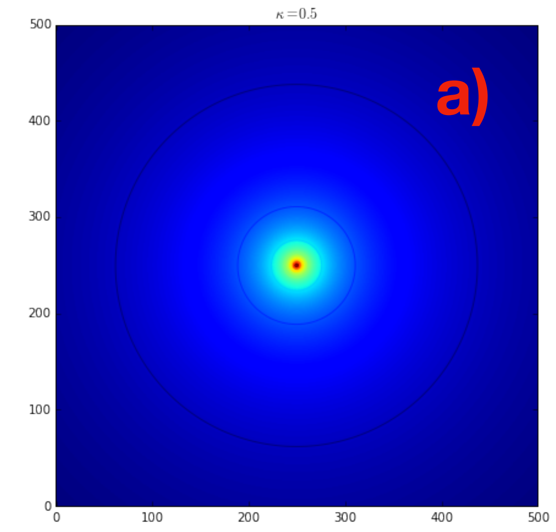

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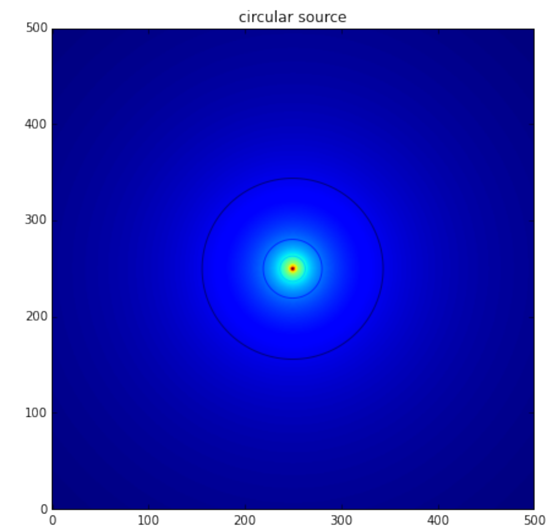

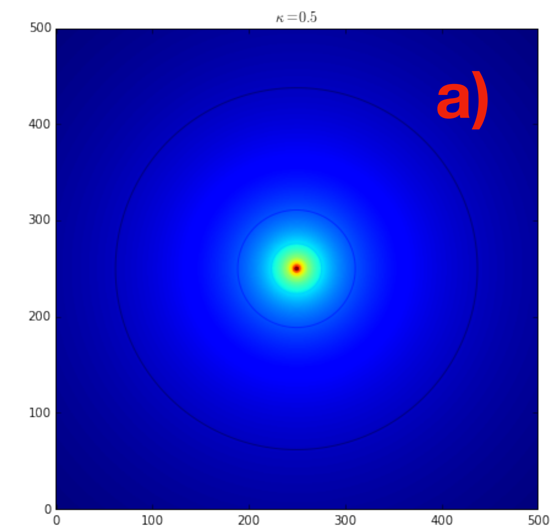

- $\gamma_2 = 0 \quad \gamma_1 > 0$

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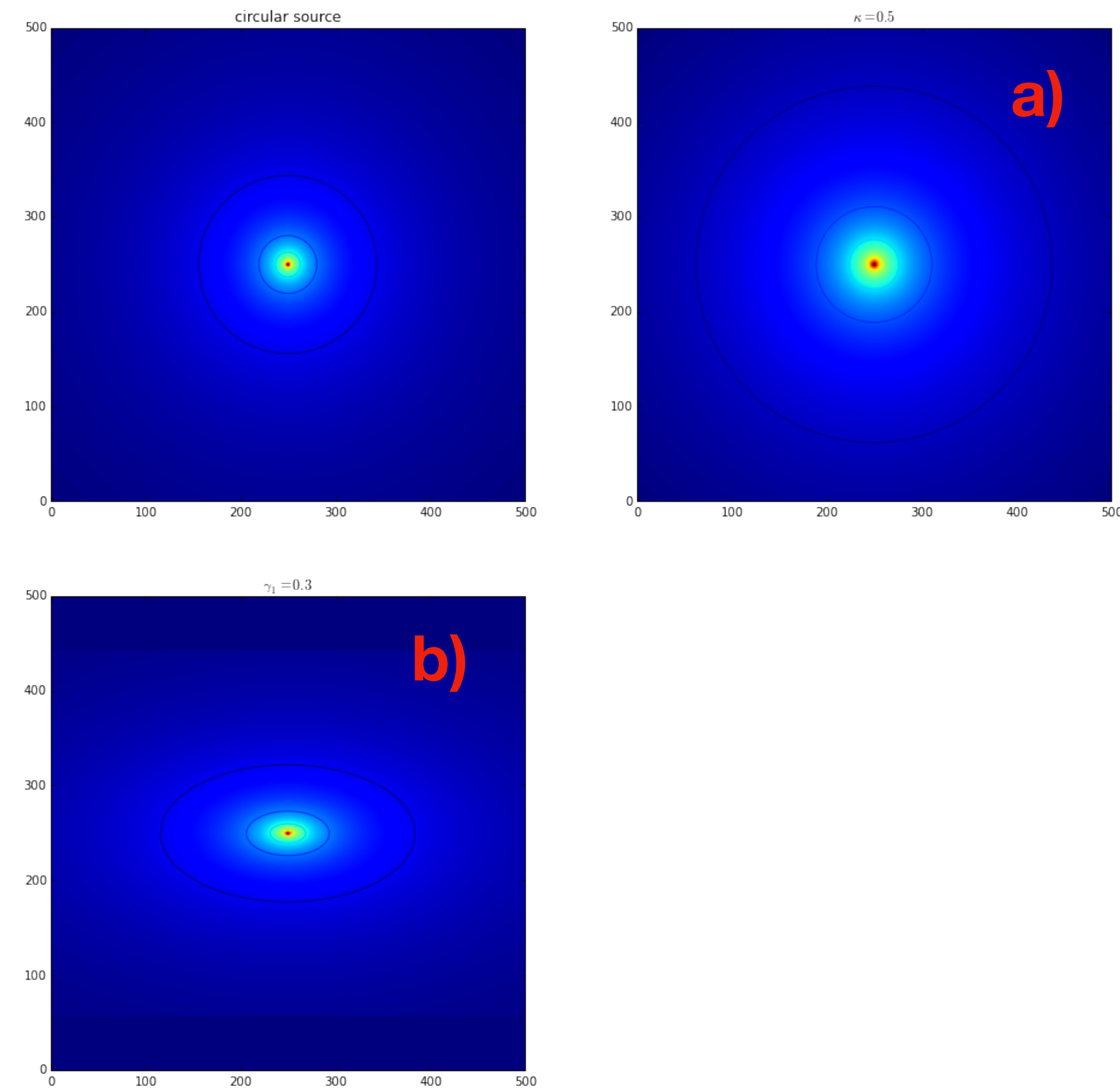

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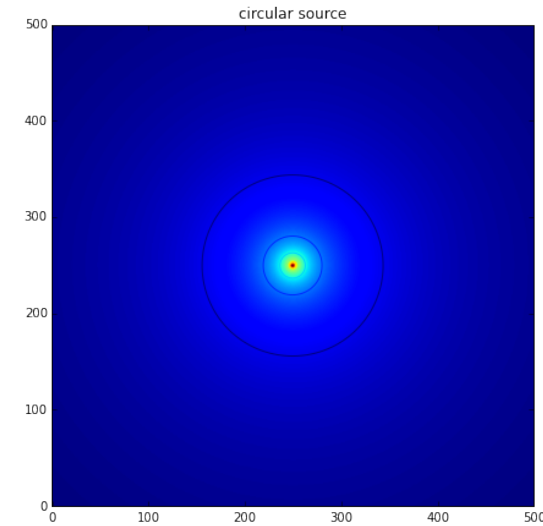

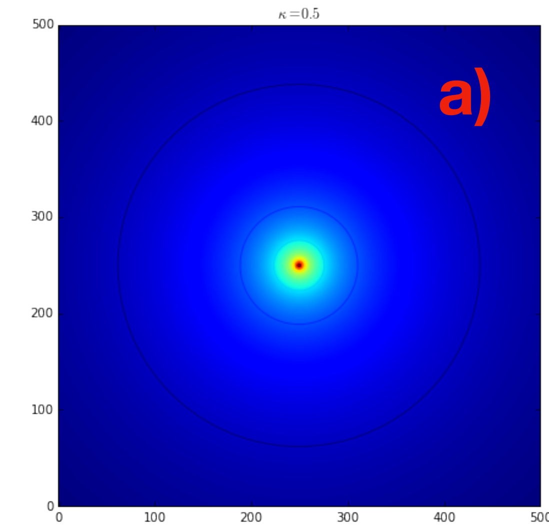

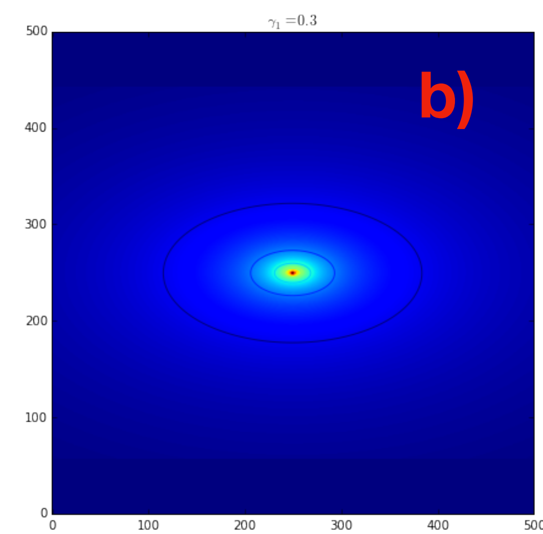

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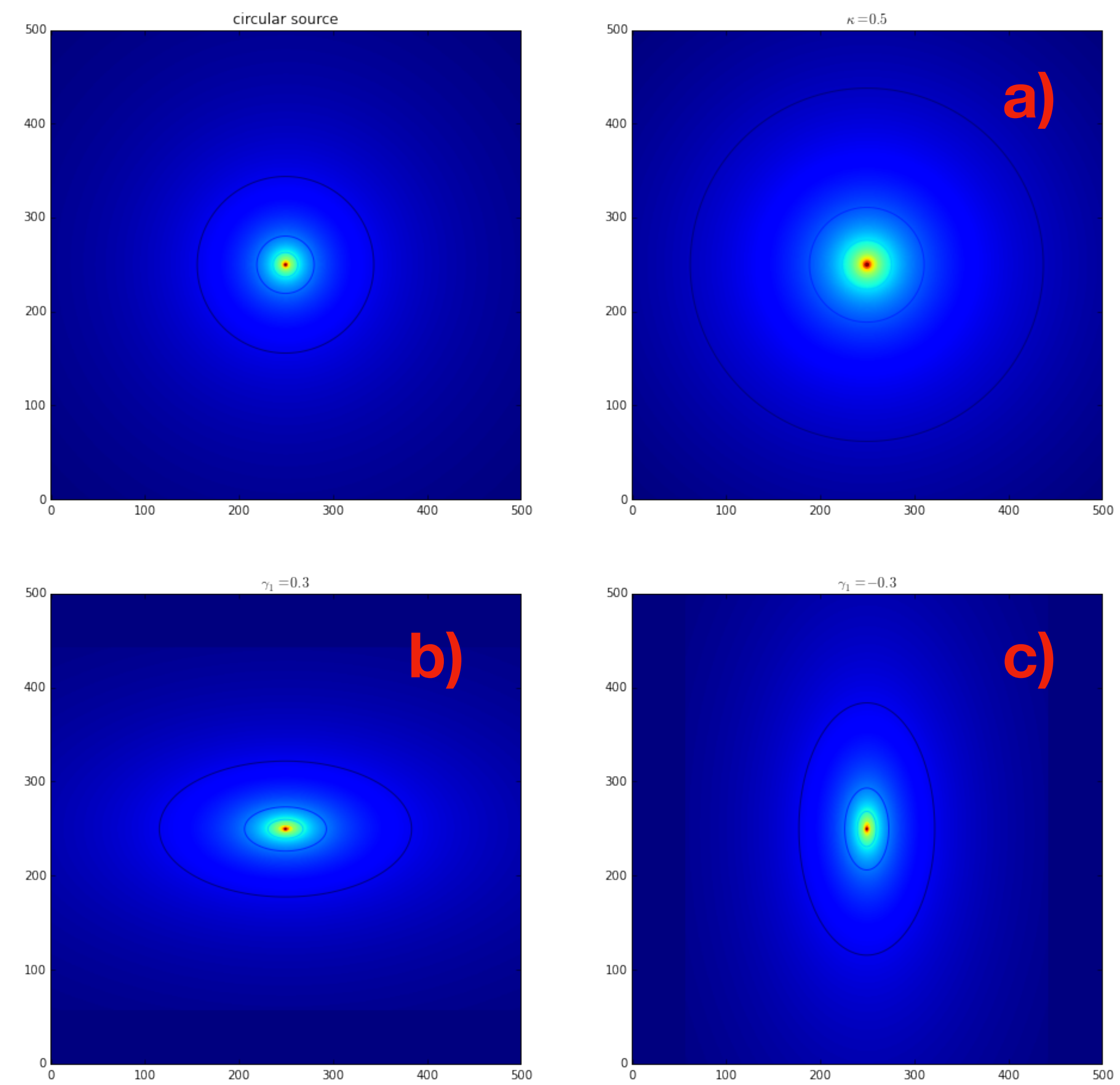

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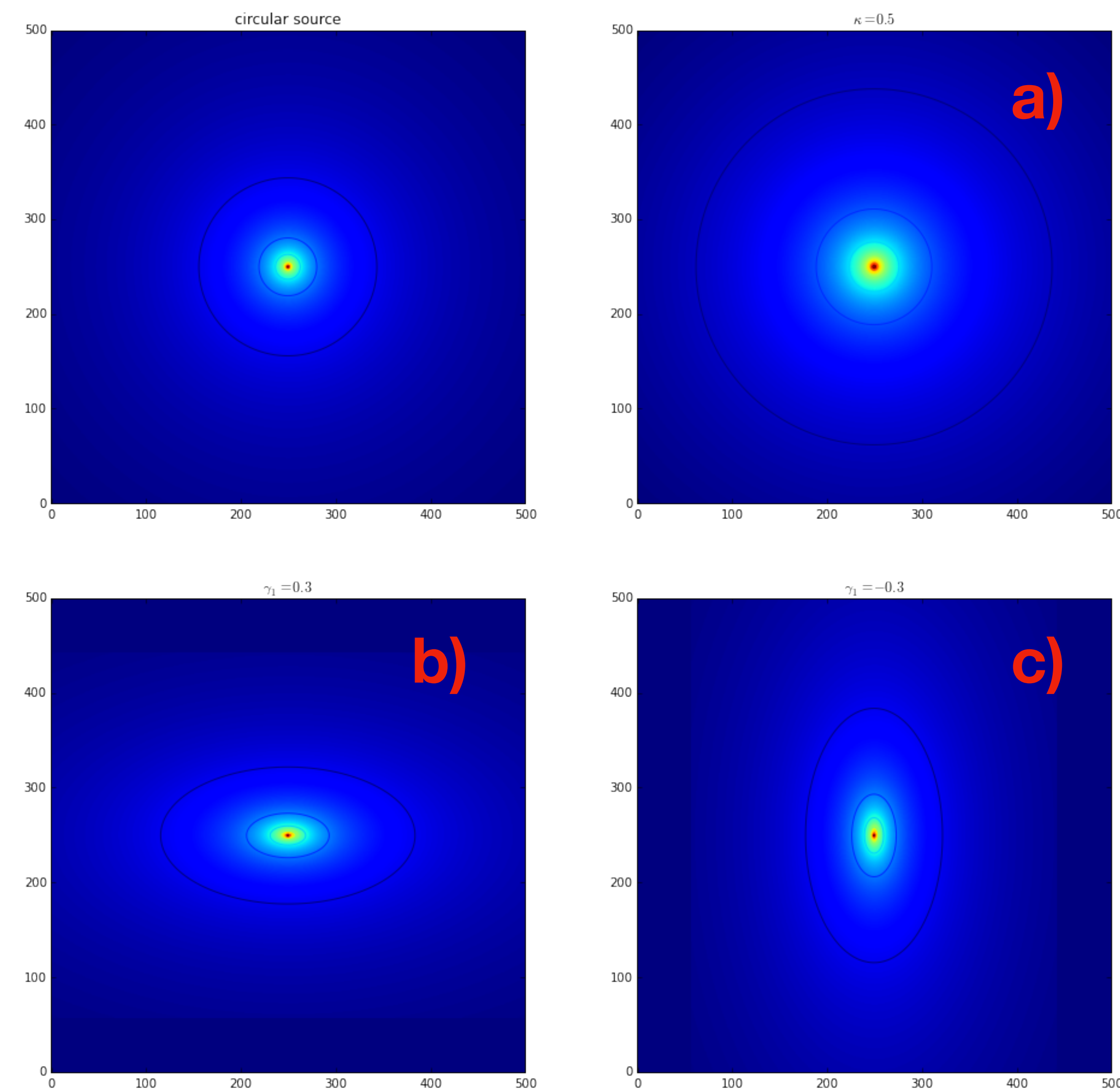

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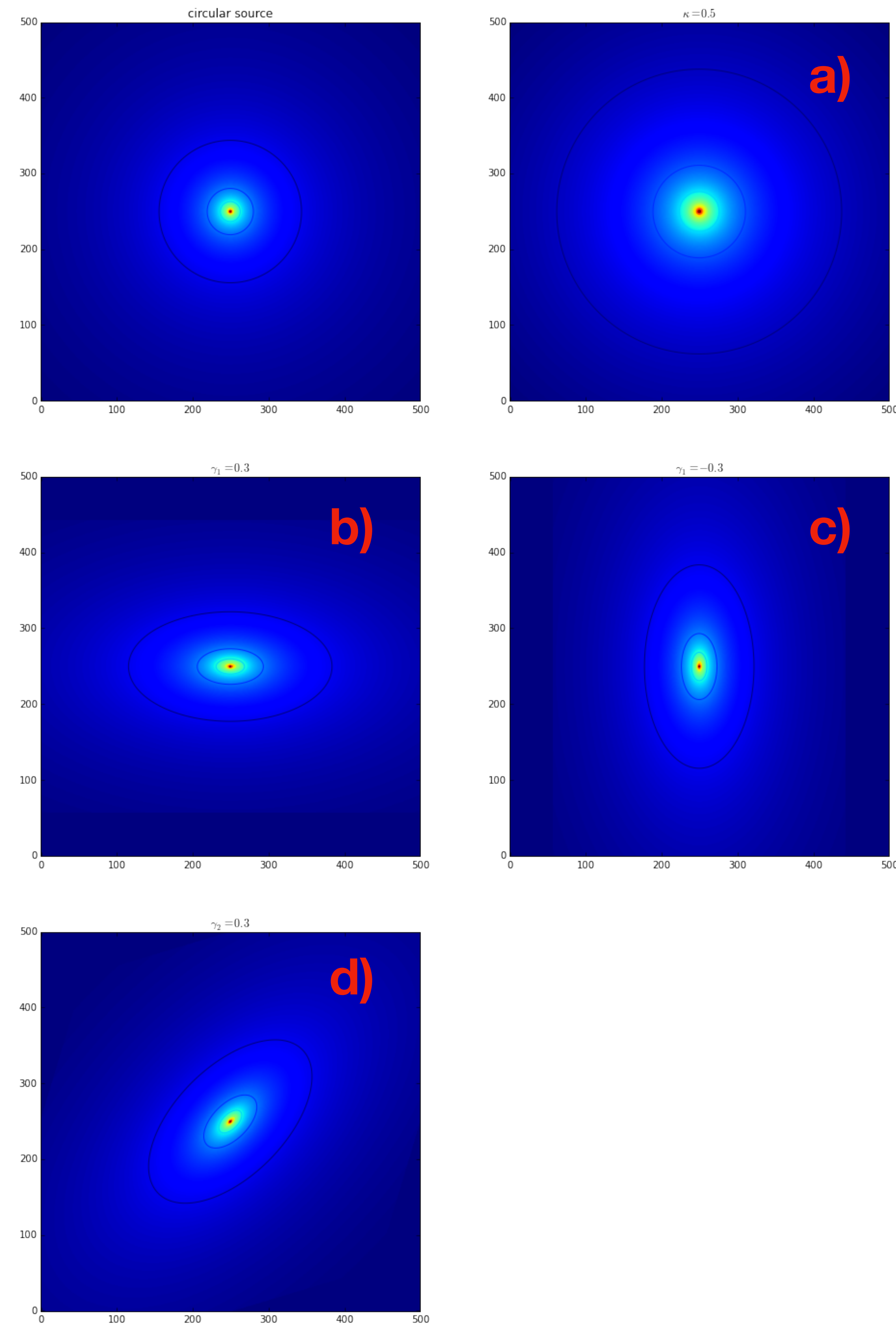

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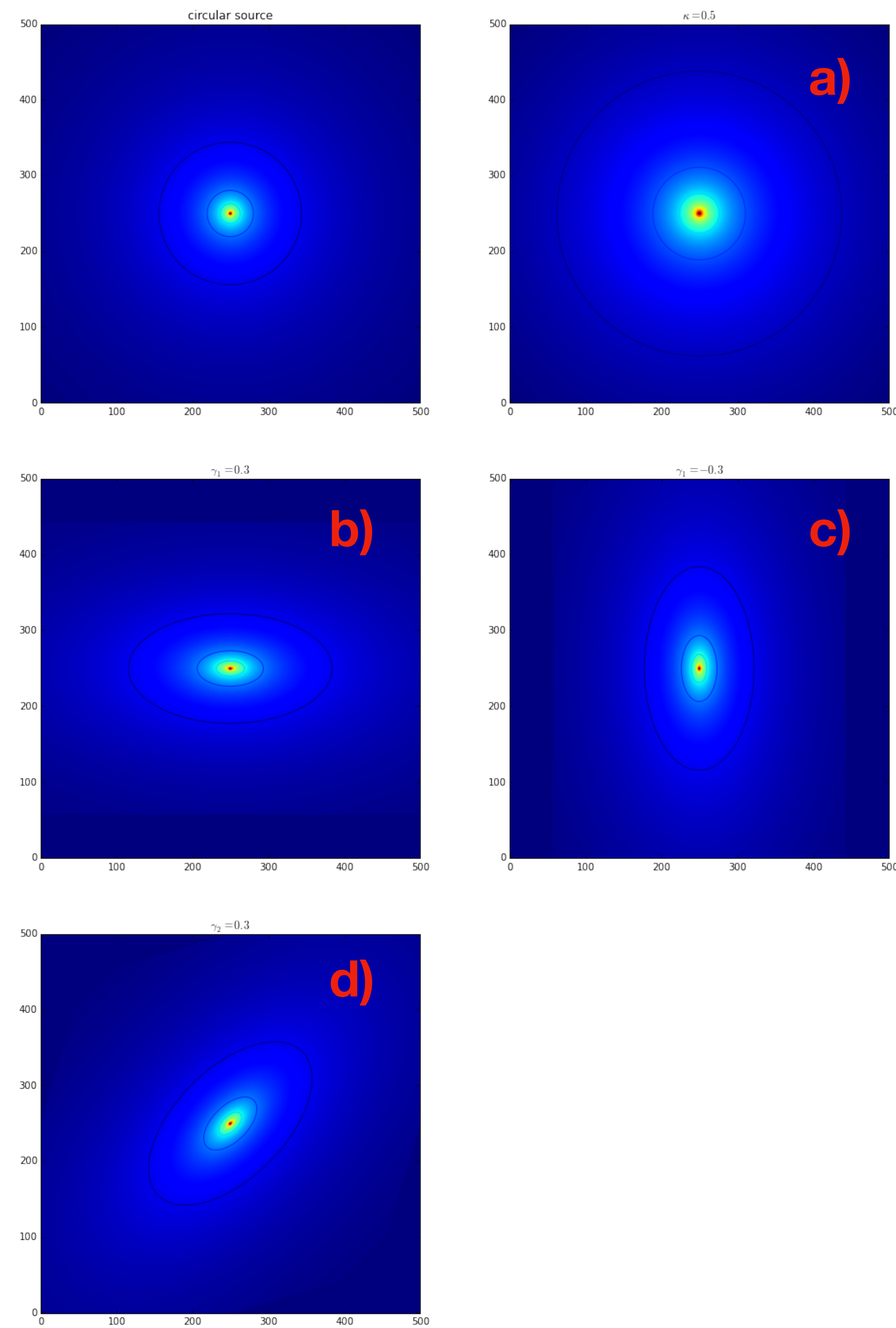

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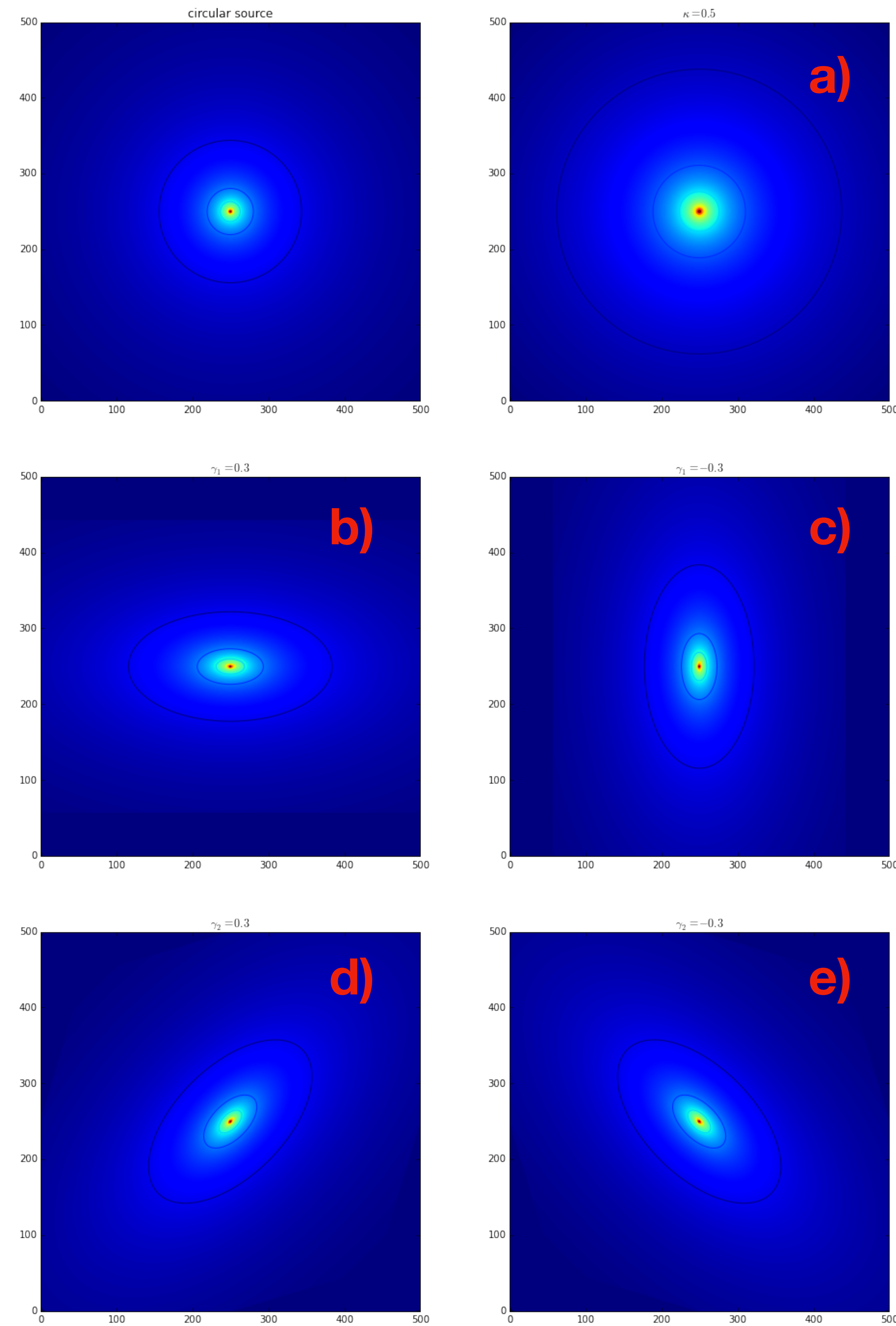
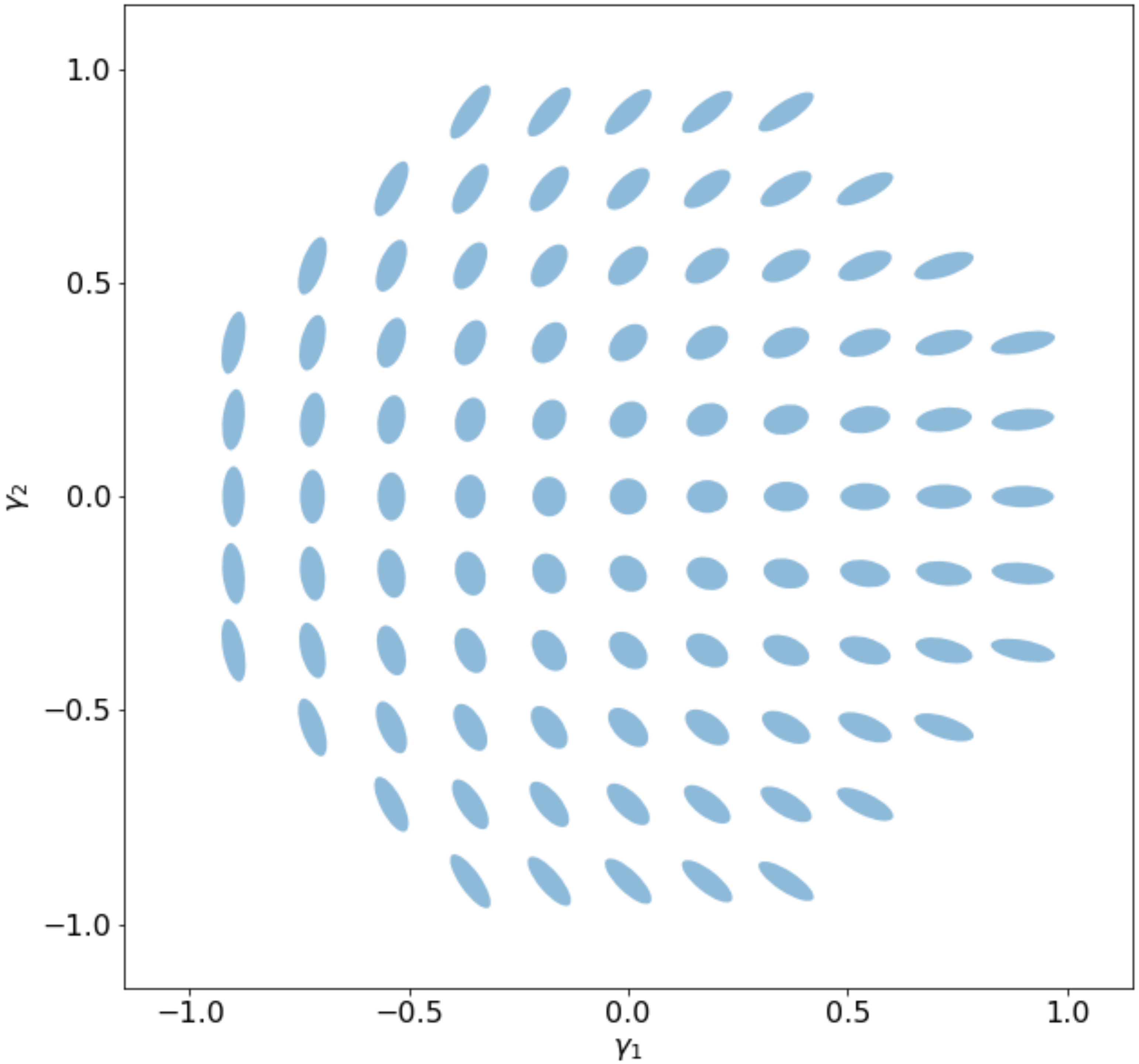
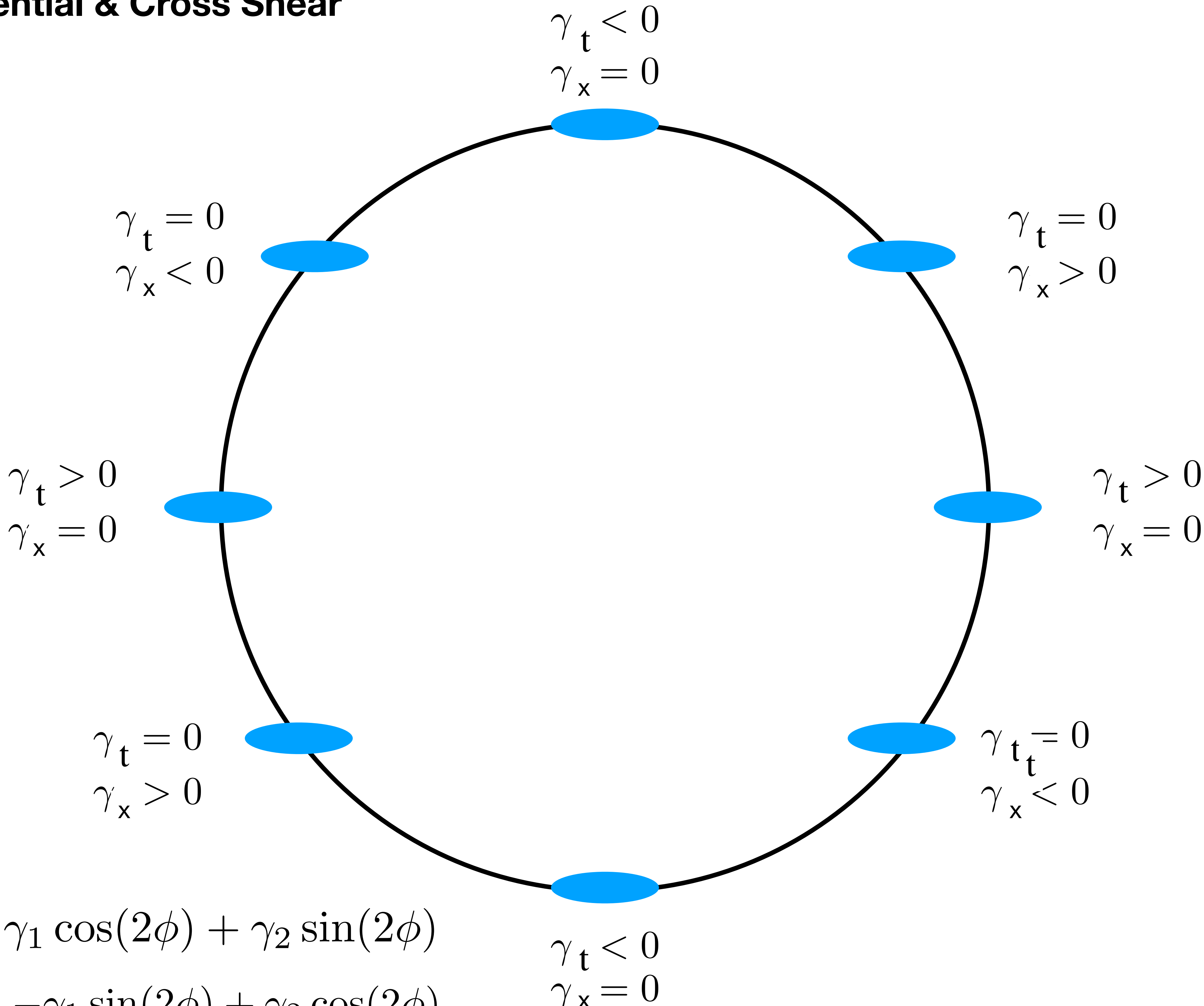


SHEAR DISTORTIONS



Tangential & Cross Shear



$$\gamma_t = \gamma_1 \cos(2\phi) + \gamma_2 \sin(2\phi)$$

$$\gamma_\times = -\gamma_1 \sin(2\phi) + \gamma_2 \cos(2\phi)$$

DEPENDENCE ON REDSHIFT

We have seen that the lensing potential, the deflection angle, the convergence, the shear… depend on a combination of distances.

For example:

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_L D_S}\frac{2}{c^2}\int_0^{\infty} \Phi(D_L\vec{\theta})dz$$

$$\vec{\alpha}(\vec{\theta}) = \vec{\nabla}\hat{\Psi}(\vec{\theta}) = \frac{D_{LS}}{D_S}\hat{\alpha}(\vec{\theta})$$

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{cr}} = \frac{1}{2}\Delta_\theta\hat{\Psi}(\vec{\theta}) \qquad \gamma_1 = \frac{1}{2}(\hat{\Psi}_{11} - \hat{\Psi}_{22}) \qquad \gamma_2 = \hat{\Psi}_{12} = \hat{\Psi}_{21} \qquad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}}$$

The distance ratio $D_{LS}D_L/D_S$ is called "lensing distance".

Both the shear and the convergence, being second derivatives of the lensing potential, scale as the lensing distance

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o}\int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + (1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

critical density

$$\Omega_m = \frac{\rho}{\rho_{\rm crit}} \qquad \rho_{\rm crit} = \frac{3H_o^2}{8\pi G}$$

H_o Hubble parameter

comoving angular size distance

$$D_{CA}(z) = \begin{cases} R_{\rm curv}\sin\left(\frac{\chi}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda < 1 \\ \chi & \Omega_m + \Omega_\Lambda = 1 \\ R_{\rm curv}\sinh\left(\frac{\chi}{R_{\rm curv}}\right) & \Omega_m + \Omega_\Lambda > 1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o}\int_0^{z_s} \frac{dz}{\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda+(1-\Omega_m-\Omega_\Lambda)(1+z)^2}}$$

critical density

$$\Omega_m = \frac{\rho}{\rho_{\rm crit}} \qquad \rho_{\rm crit} = \frac{3H_o^2}{8\pi G} \qquad H_o$$

Hubble parameter

comoving angular size distance

$$D_{CA}(z_1,z_2) = \begin{cases} R_{\rm curv}\sin\left(\frac{\chi_1-\chi_2}{R_{\rm curv}}\right) & \Omega_m+\Omega_\Lambda<1 \\ \chi_1-\chi_2 & \Omega_m+\Omega_\Lambda=1 \\ R_{\rm curv}\sinh\left(\frac{\chi_1-\chi_2}{R_{\rm curv}}\right) & \Omega_m+\Omega_\Lambda>1 \end{cases}$$

curvature distance

$$R_{\rm curv} = \frac{c}{H_o\sqrt{1-\Omega_m-\Omega_\Lambda}}$$

COSMOLOGICAL DISTANCES

coordinate distance

$$\chi(z_s) = \frac{c}{H_o} \int_0^{z_s} \frac{dz}{\sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)(1+z)^2}}$$

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curvature distance

$$R_{\rm curv} = \frac{c}{H_o \sqrt{1 - \Omega_m - \Omega_\Lambda}}$$

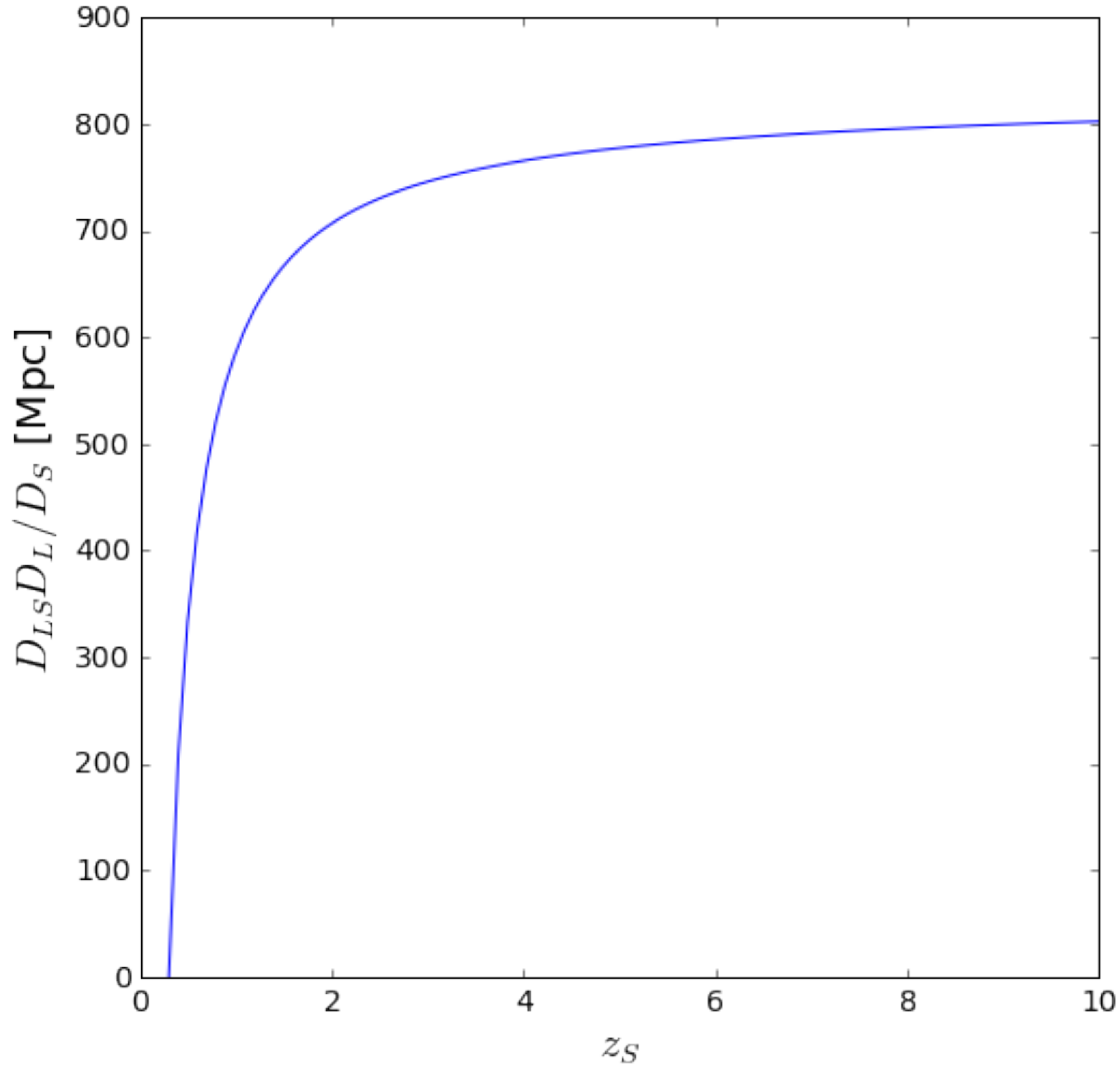
(proper) angular size distance

$$D_A(z, z_o) = \frac{D_{CA}(z, z_o)}{(1+z)}$$

luminosity distance

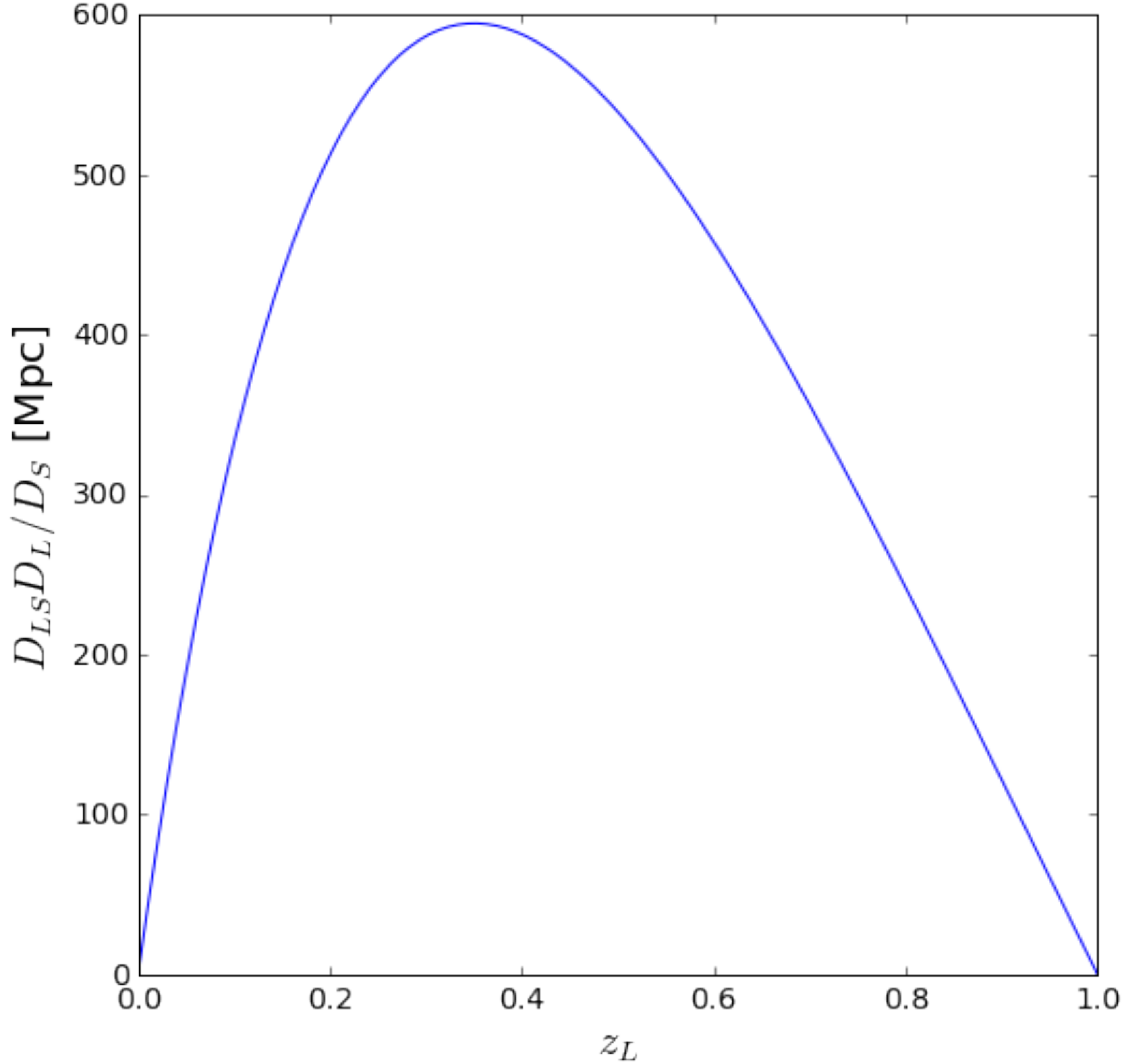
$$D_L(z) = (1+z) D_{CA}(z)$$

HOW DOES THE LENSING DISTANCE SCALE WITH SOURCE REDSHIFT?



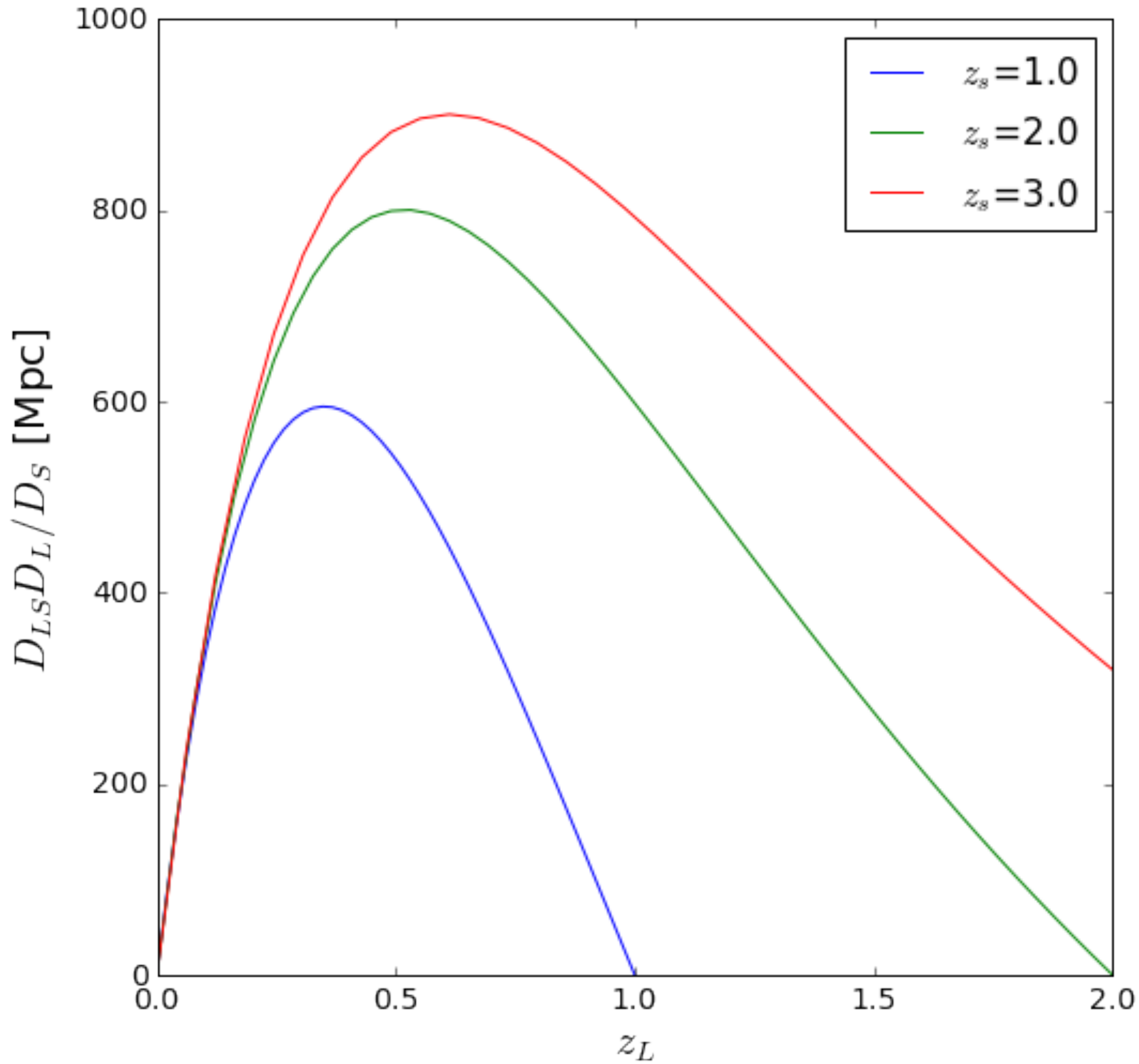
Note that if the lensing distance grows, the critical surface density decreases, the convergence and the shear grow!

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



The lensing distance peaks at ~half way between the source and the observer, meaning that there is an optimal distance where the lens produces its largest effects.

HOW DOES THE LENSING DISTANCE SCALE WITH LENS REDSHIFT?



Of course, the peak moves to larger distances as the distance to the source increases.

CONSERVATION OF SURFACE BRIGHTNESS

The source ***surface brightness*** *is*

$$I_\nu = \frac{dE}{dt dA d\Omega d\nu}$$

In phase space, the radiation emitted is characterized by the density

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p}$$

In absence of photon creations or absorptions, f is conserved (Liouville theorem)

$$dN = \frac{dE}{h\nu} = \frac{dE}{cp}$$

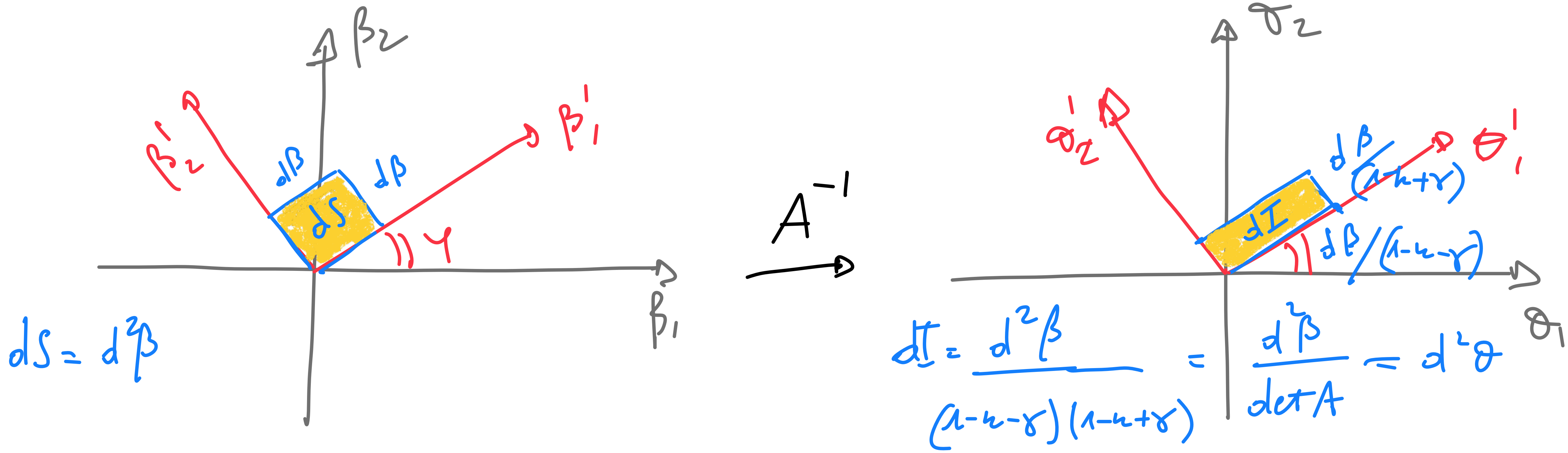
$$d^3x = c dt dA$$

$$d^3\vec{p} = p^2 dp d\Omega$$

$$f(\vec{x}, \vec{p}, t) = \frac{dN}{d^3x d^3p} = \frac{dE}{hcp^3 dA dt d\nu d\Omega} = \frac{I_\nu}{hcp^3}$$

Since GL does not involve creation or absorption of photons, neither it changes the photon momenta (achromatic!), surface brightness is conserved!

MAGNIFICATION

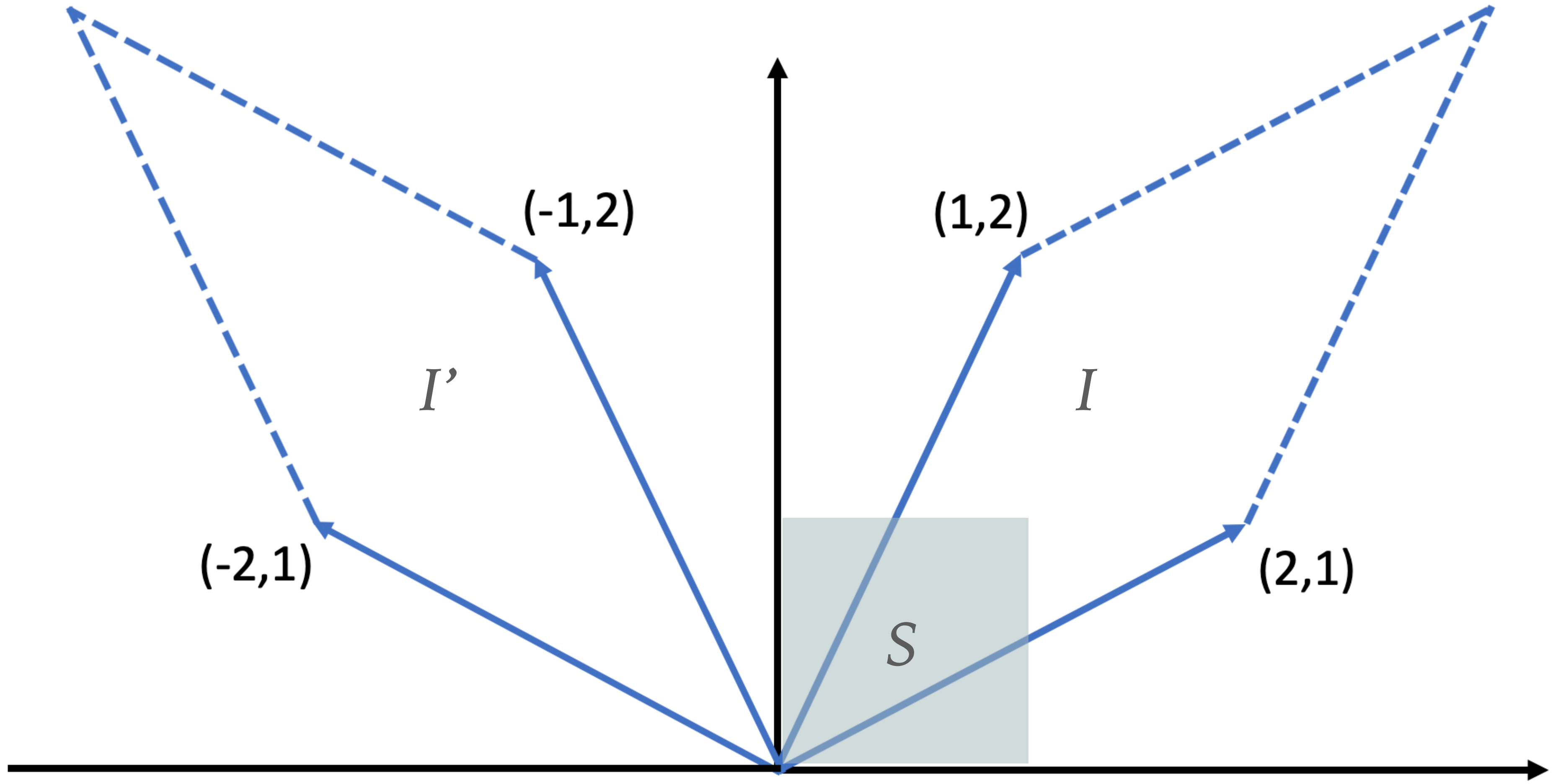


$$\mu(\vec{\theta}) = \frac{dI}{dS} = \frac{d^2\theta}{d^2\beta} = \det A^{-1}(\vec{\theta})$$

$$F_\nu = \int_I I_\nu(\vec{\theta})d^2\theta = \int_S I_\nu^S[\vec{\beta}(\vec{\theta})]\mu[\vec{\beta}(\vec{\theta})]d^2\beta$$

Lensing changes the amount of photons (flux) we receive from the source by changing the solid angle the source subtends

POSITIVE AND NEGATIVE MAGNIFICATION



$$I' = \det\begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix} = -3 \qquad I = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$$

GRAVITATIONAL LENSING

6 - THE LENS MAP, TIME-DELAYS

R. Benton Metcalf

2022-2023

CRITICAL LINES AND CAUSTICS

Both convergence and shear are functions of position on the lens plane:

$\kappa = \kappa(\vec{\theta})$

$\gamma = \gamma(\vec{\theta})$

The determinant of the lensing Jacobian is

$$\det A(\overrightarrow{\theta}) = [1 - \kappa(\overrightarrow{\theta}) - \gamma(\overrightarrow{\theta})][1 - \kappa(\overrightarrow{\theta}) + \gamma(\overrightarrow{\theta})]$$

The ***critical lines*** *are the lines made of the points where the eigenvalues of the Jacobian are zero:*

$$1 - \kappa(\overrightarrow{\theta}_t) - \gamma(\overrightarrow{\theta}_t) = 0 \qquad \textit{tangential critical line}$$
$$1 - \kappa(\overrightarrow{\theta}_r) + \gamma(\overrightarrow{\theta}_r) = 0 \qquad \textit{radial critical line}$$

Along these lines the magnification diverges!

Via the lens equations, they are mapped into the ***caustics****…* $\overrightarrow{\beta}_{t,r} = \overrightarrow{\theta}_{t,r} - \overrightarrow{\alpha}(\overrightarrow{\theta}_{t,r})$

TYPES OF IMAGES

There are three types of images:

eigenvalues

$$1 - \kappa - \gamma$$

$$1 - \kappa + \gamma$$

magnification

Both eigenvalues are positive.

$$|A| > 0 \quad \mu > 0$$

Normal image or
"minimum image"

One of the eigenvalues is negative.

$$|A| < 0 \quad \mu < 0$$

Inverted in one dimension.
"saddle point image"

Both of the eigenvalues is negative.

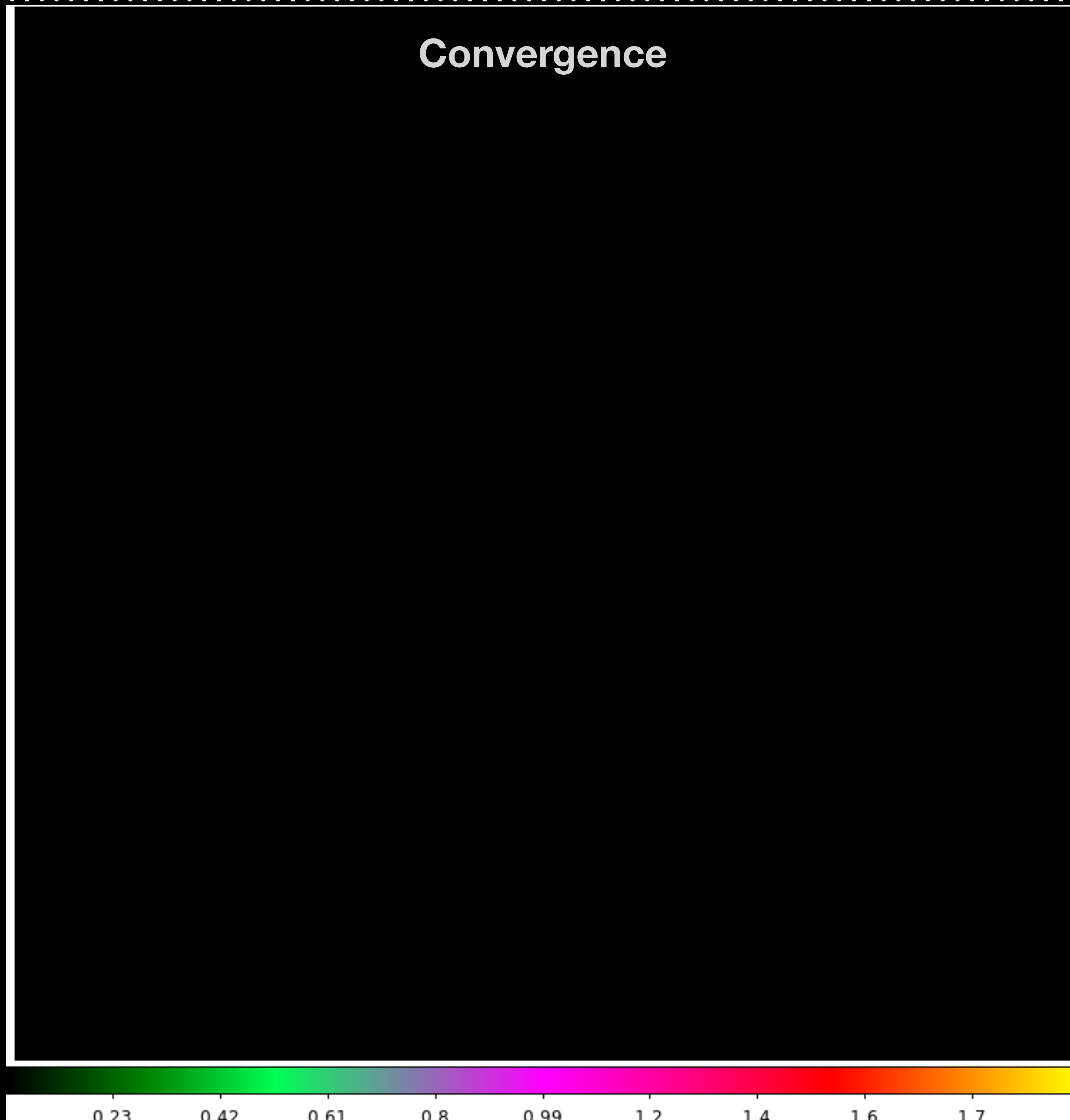
$$|A| > 0 \quad \mu > 0$$

Inverted in both dimension.
"maximum image"

LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

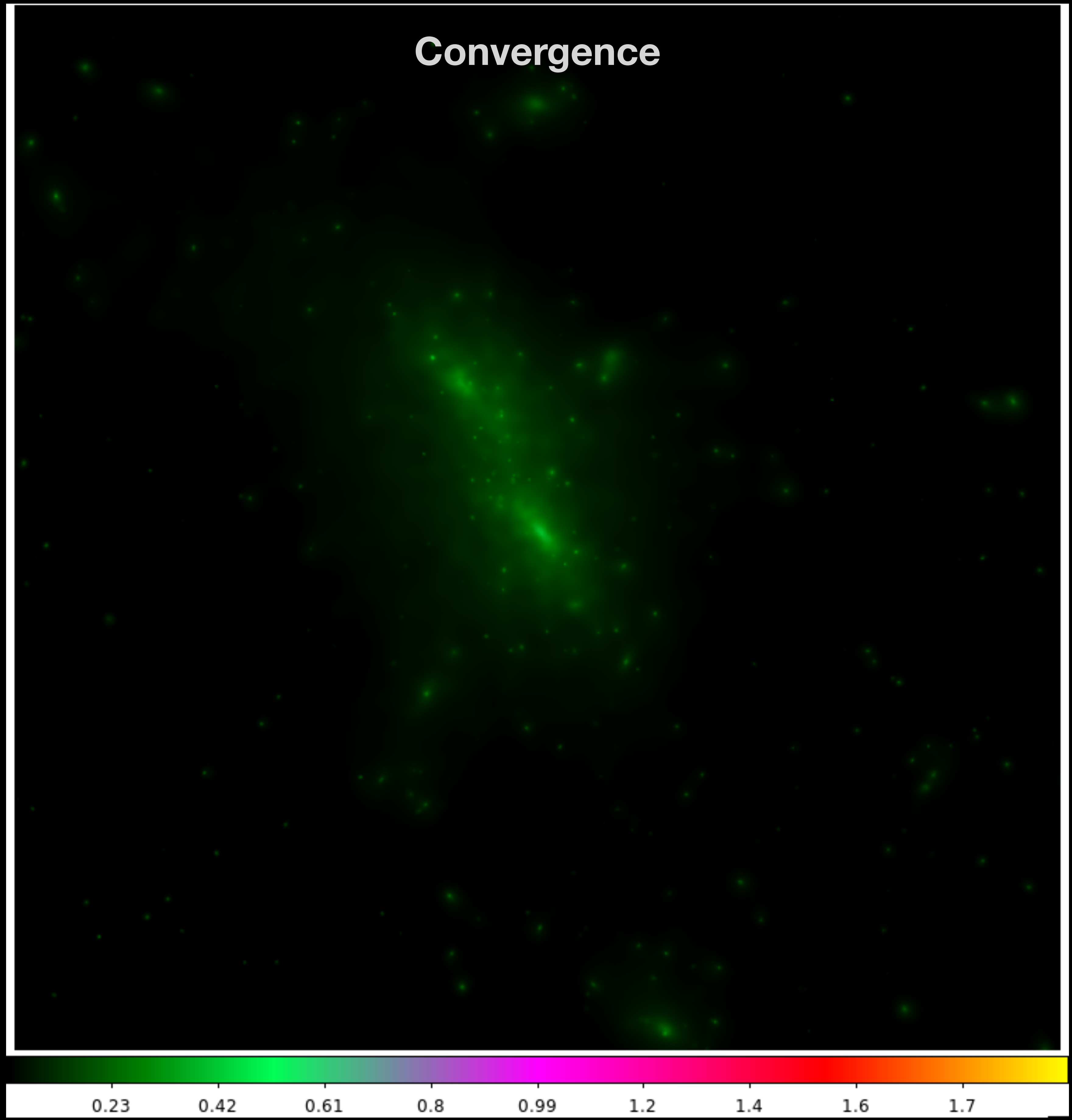
source z = 0.5



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

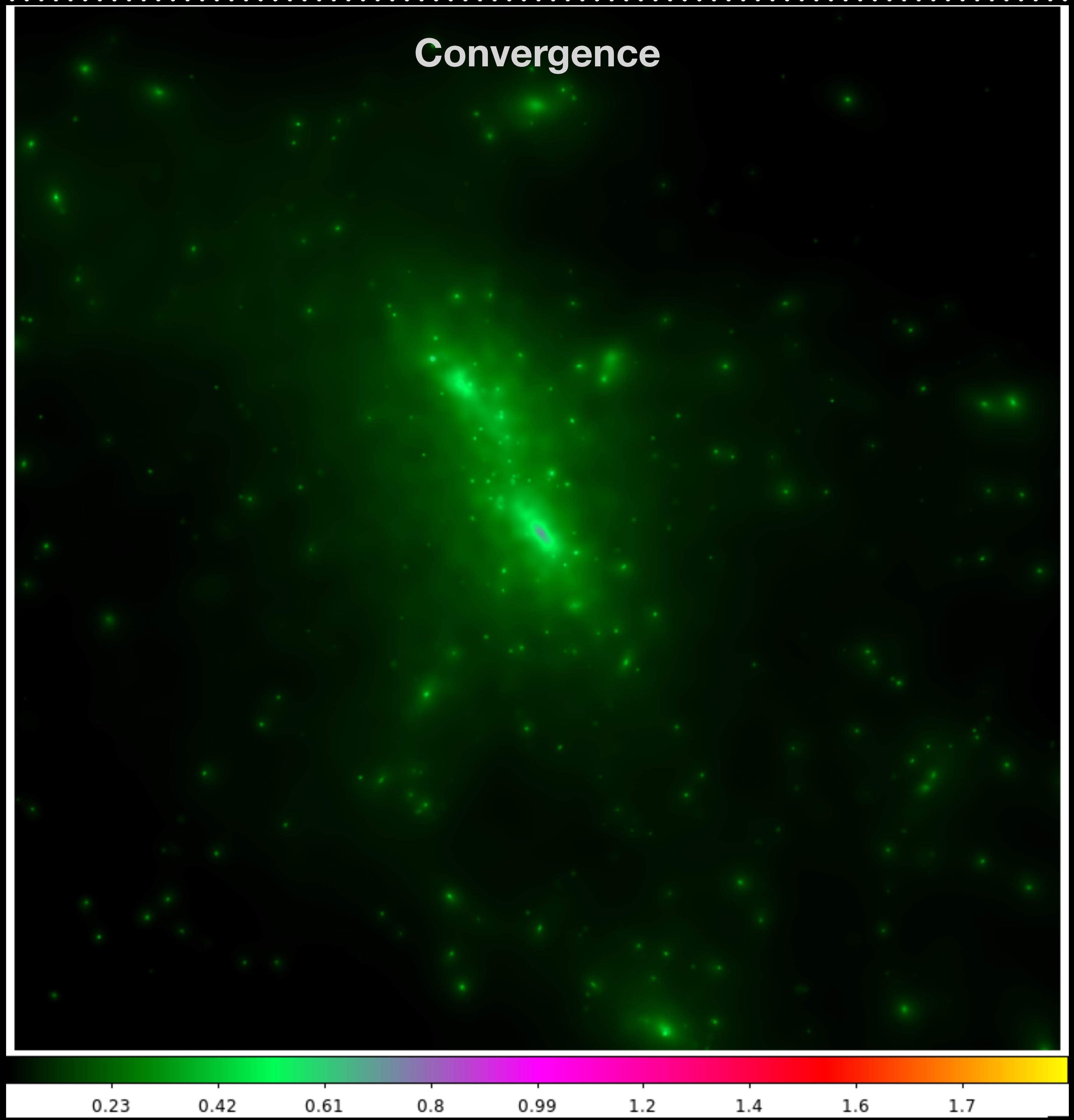
source z = 0.6



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

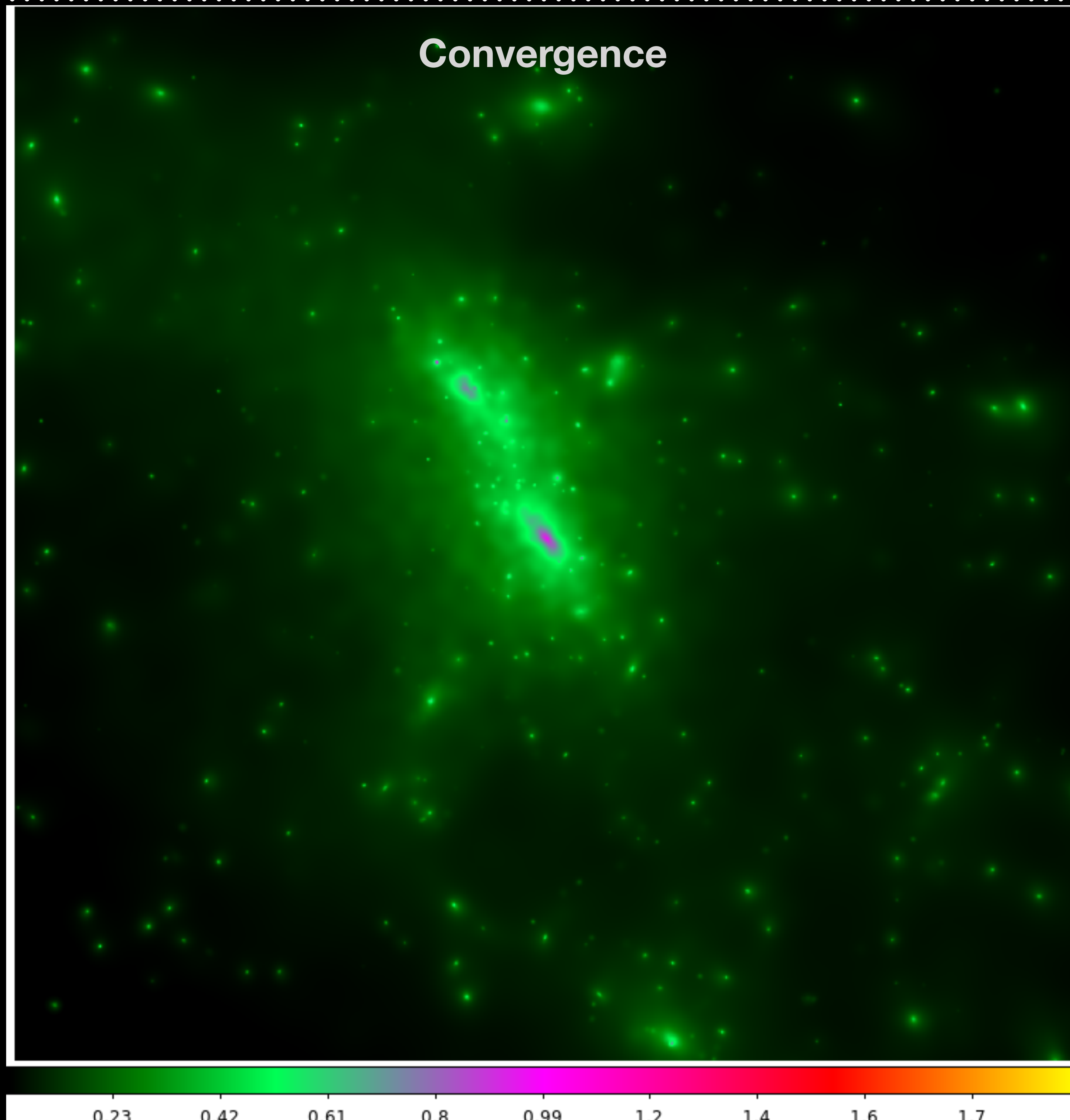
source z = 0.7



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

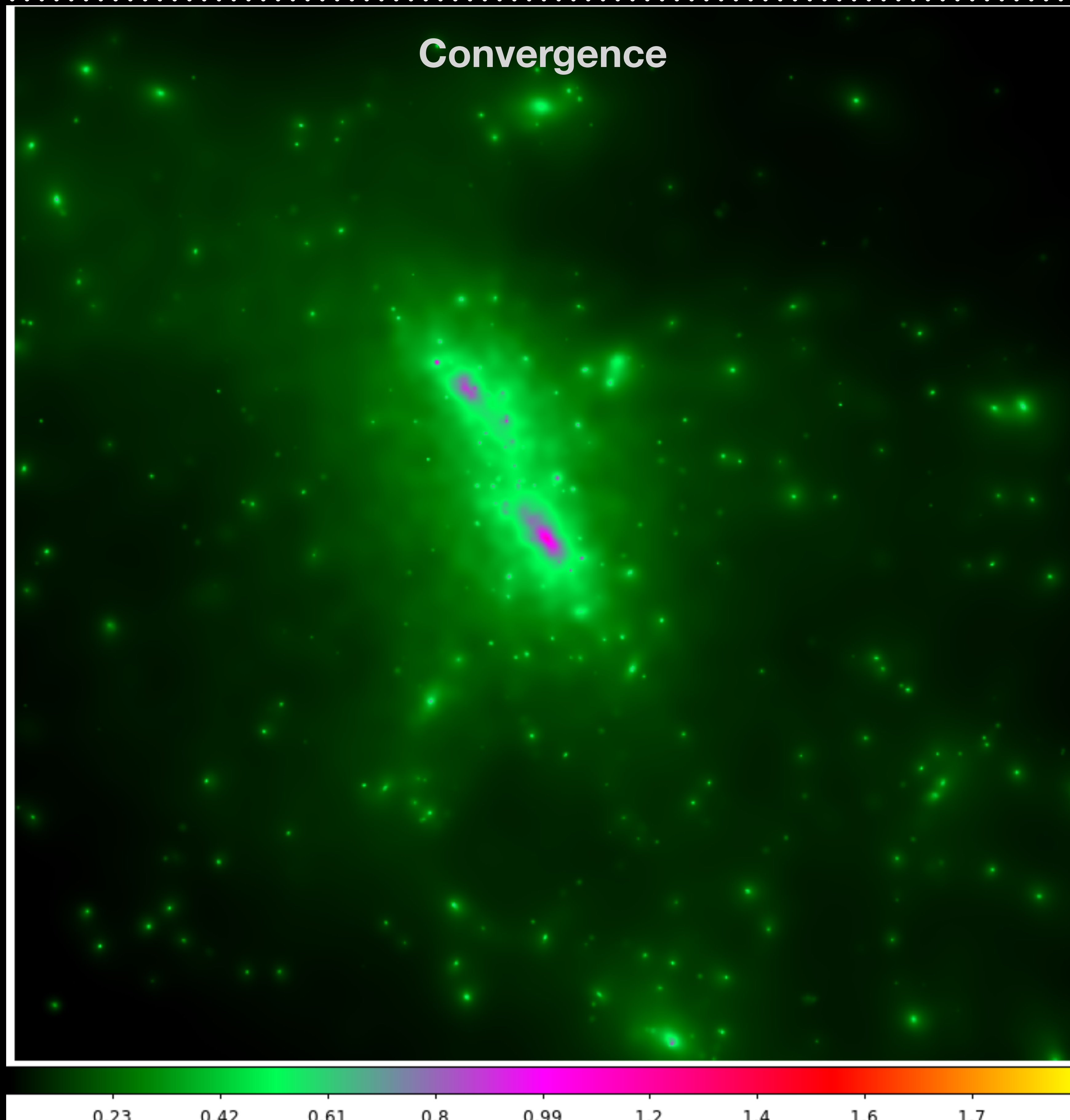
source z = 0.8



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

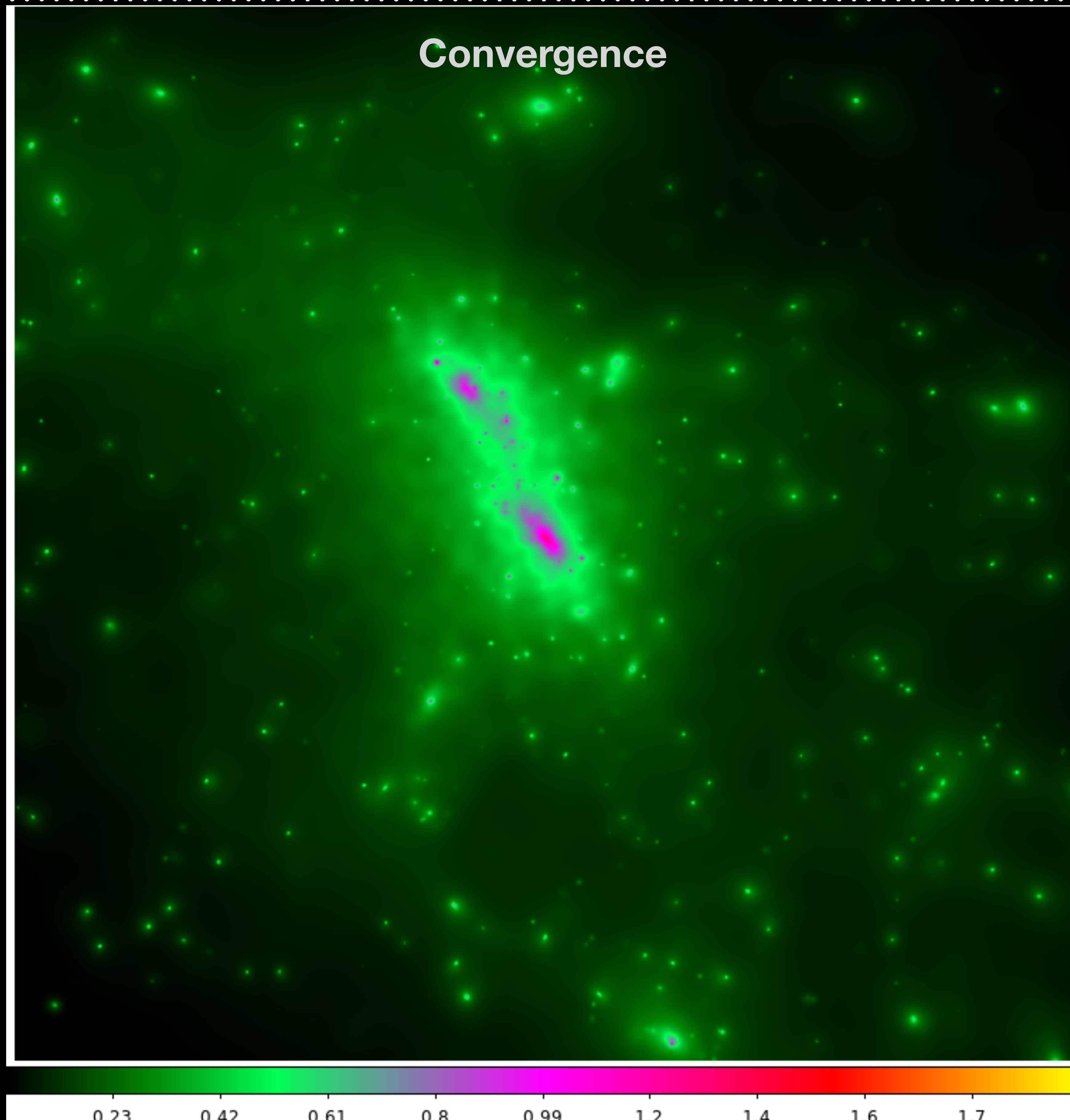
source z = 0.9



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

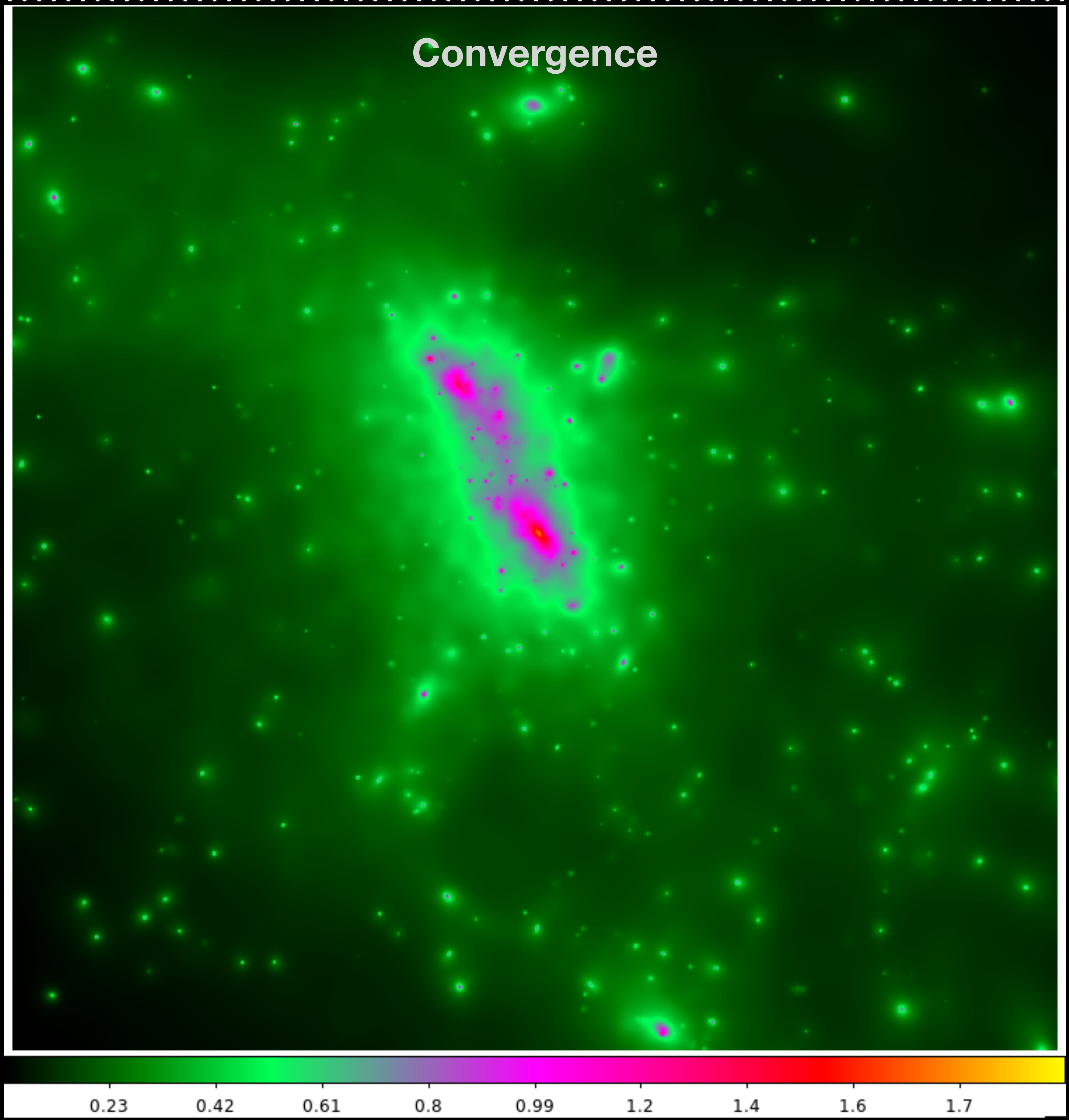
source z = 1.0



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

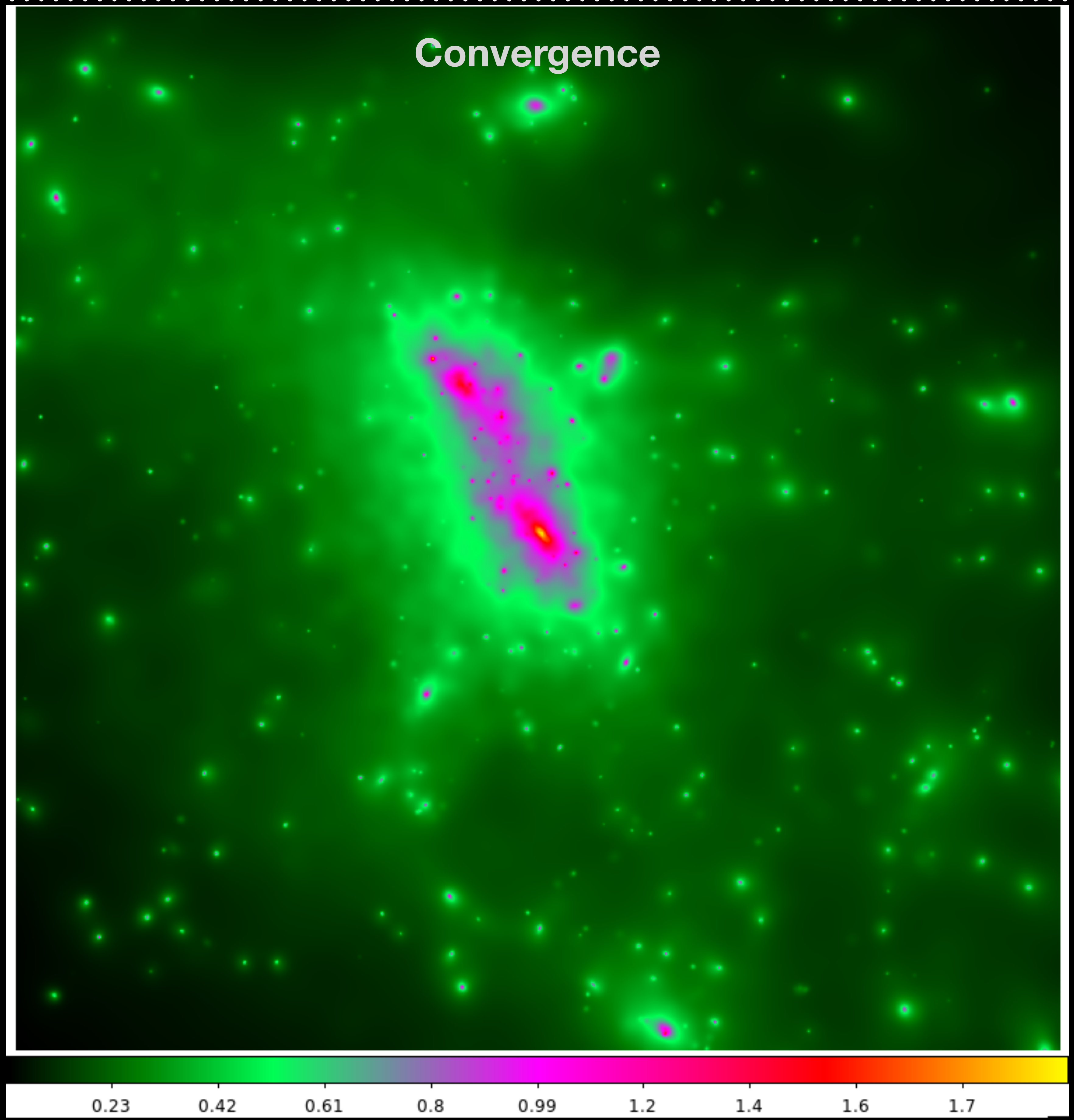
source z = 1.5



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

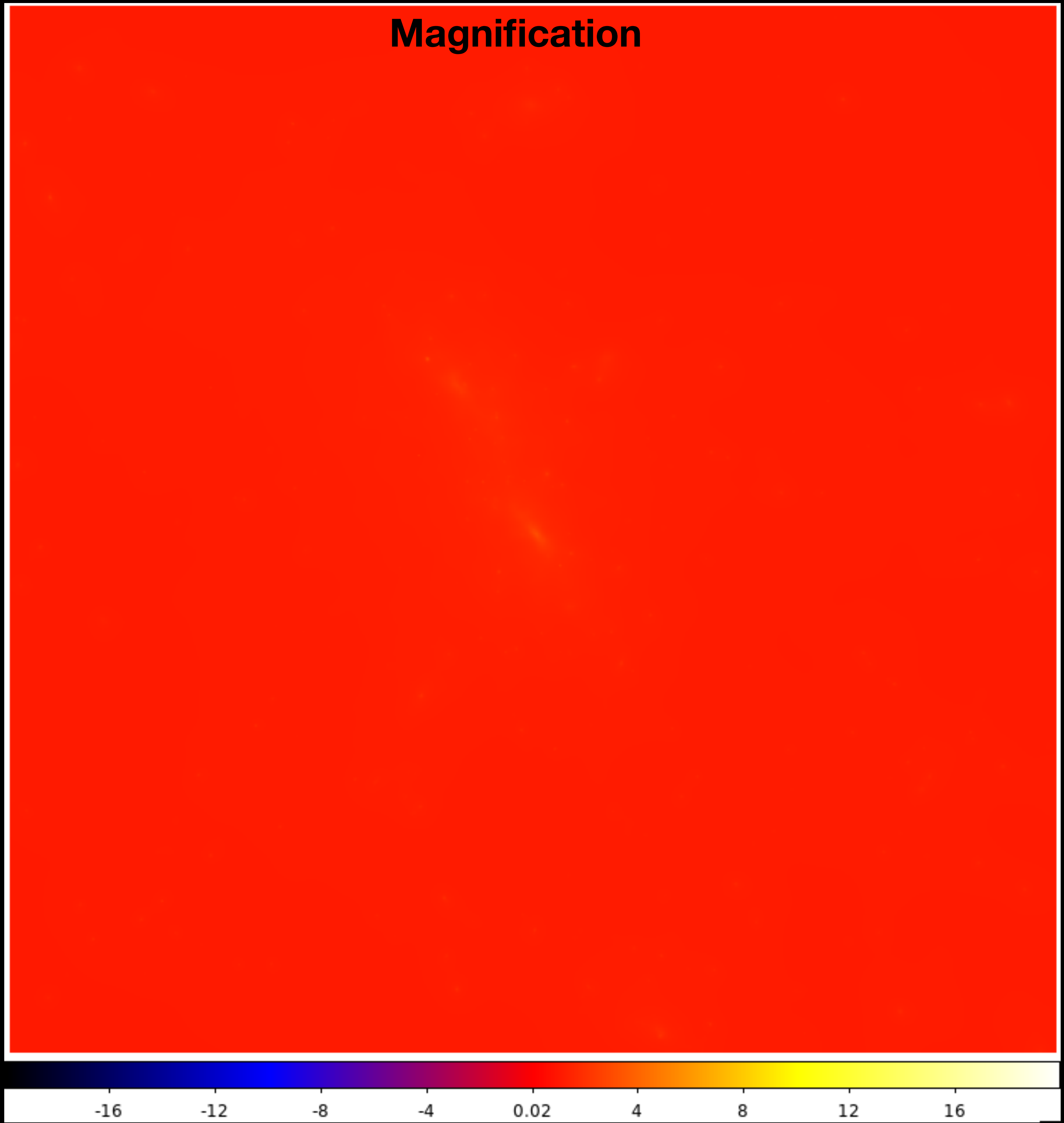
source z = 2.0



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

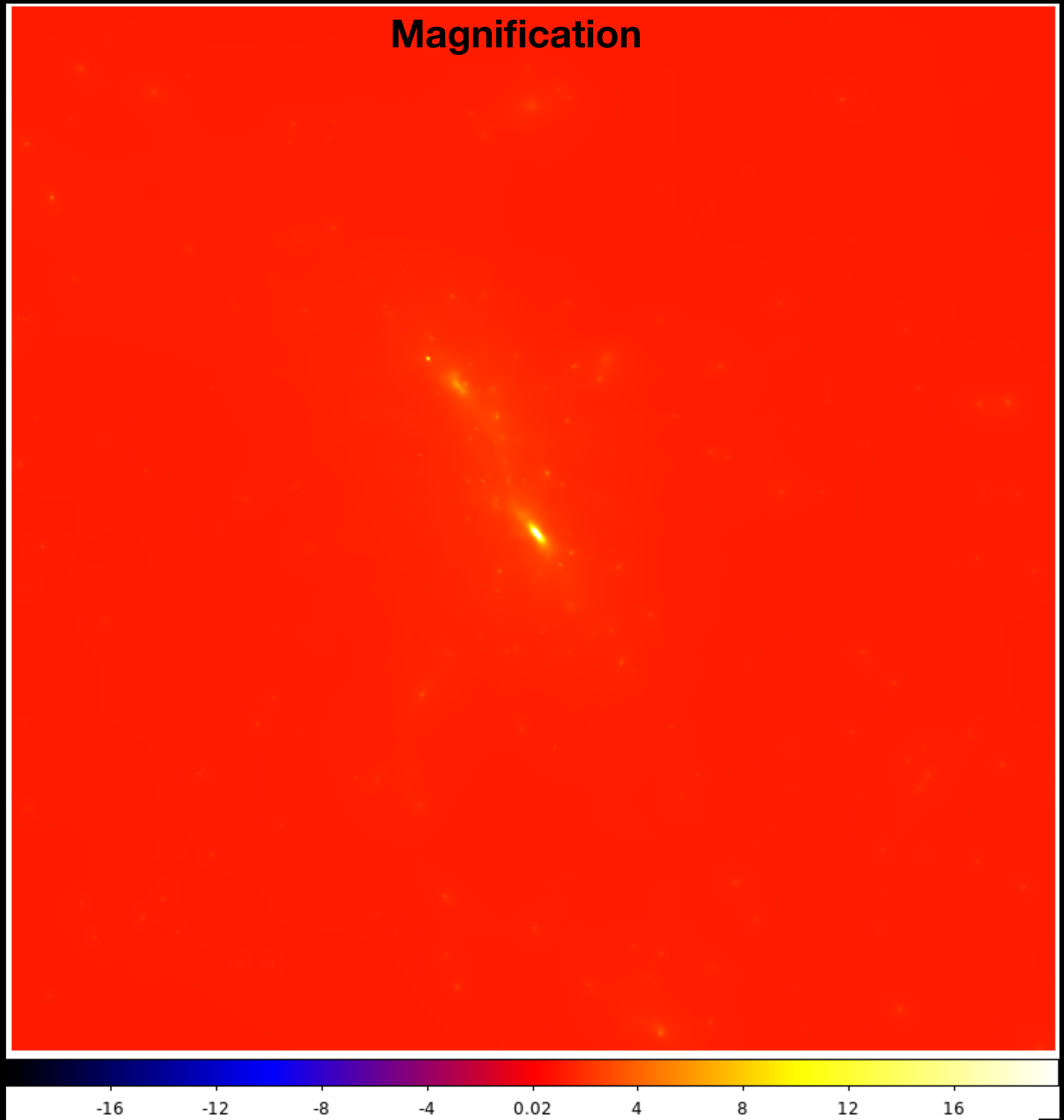
source z = 0.6



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

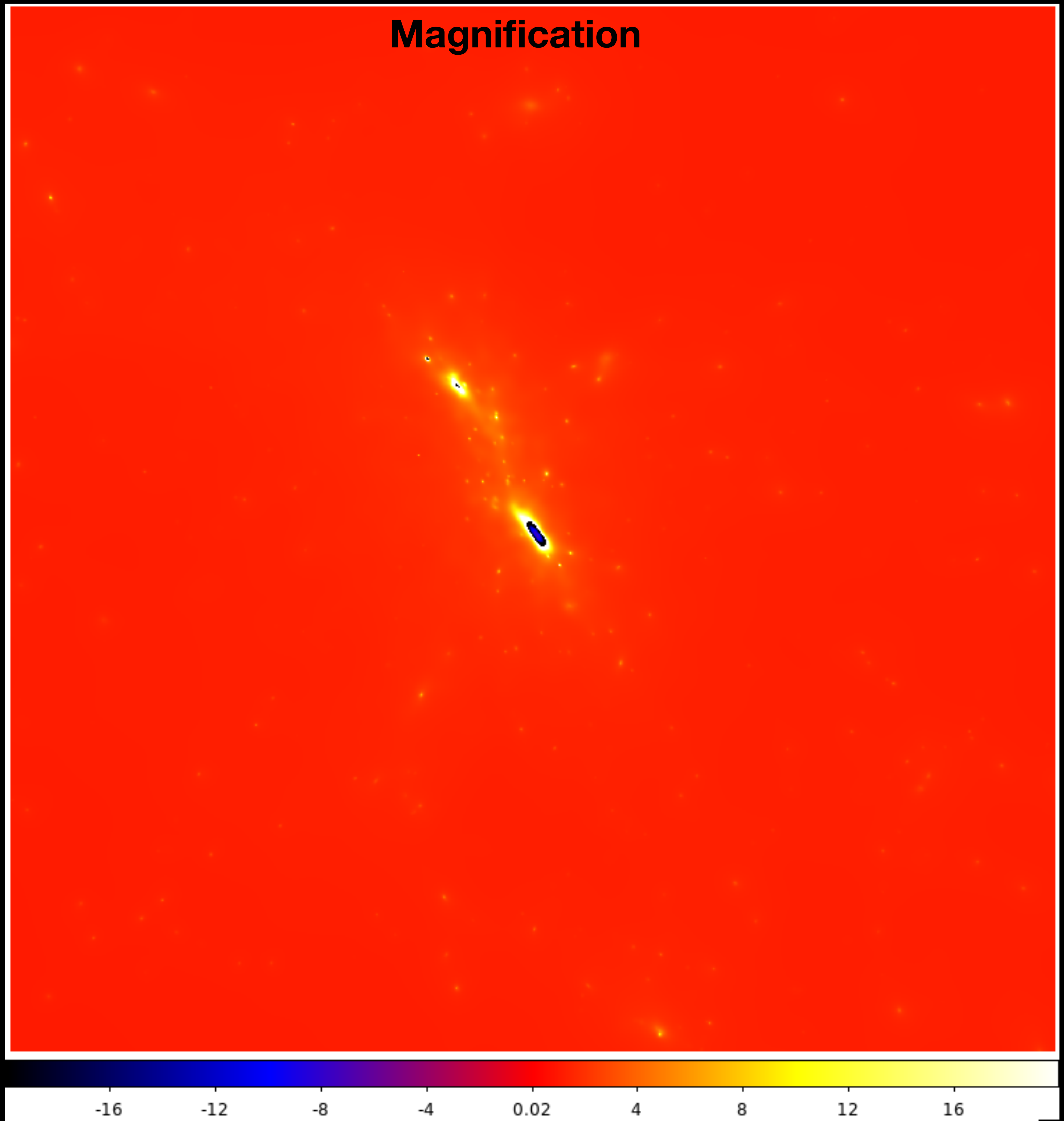
source z = 0.7



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

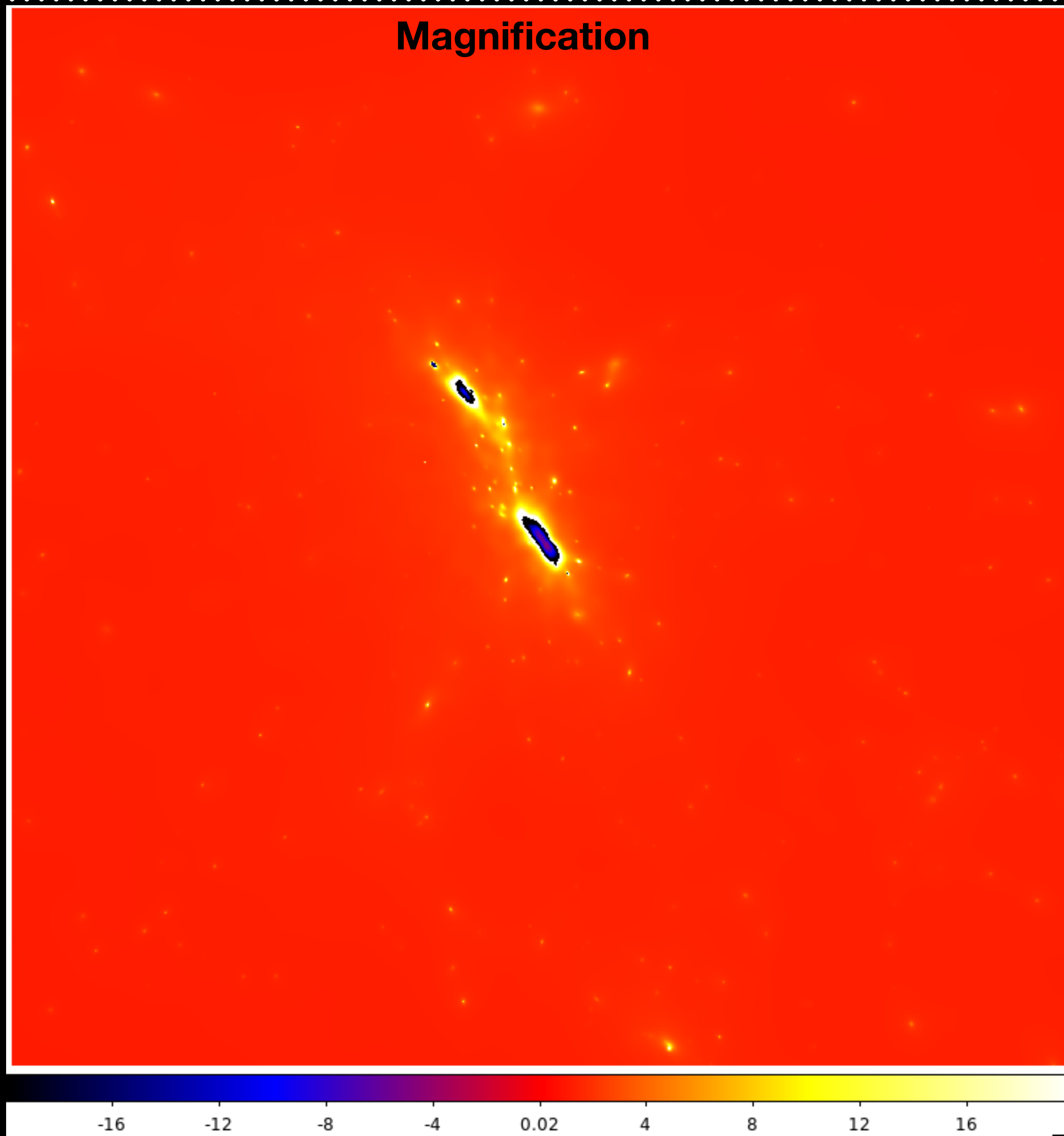
source z = 0.8



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

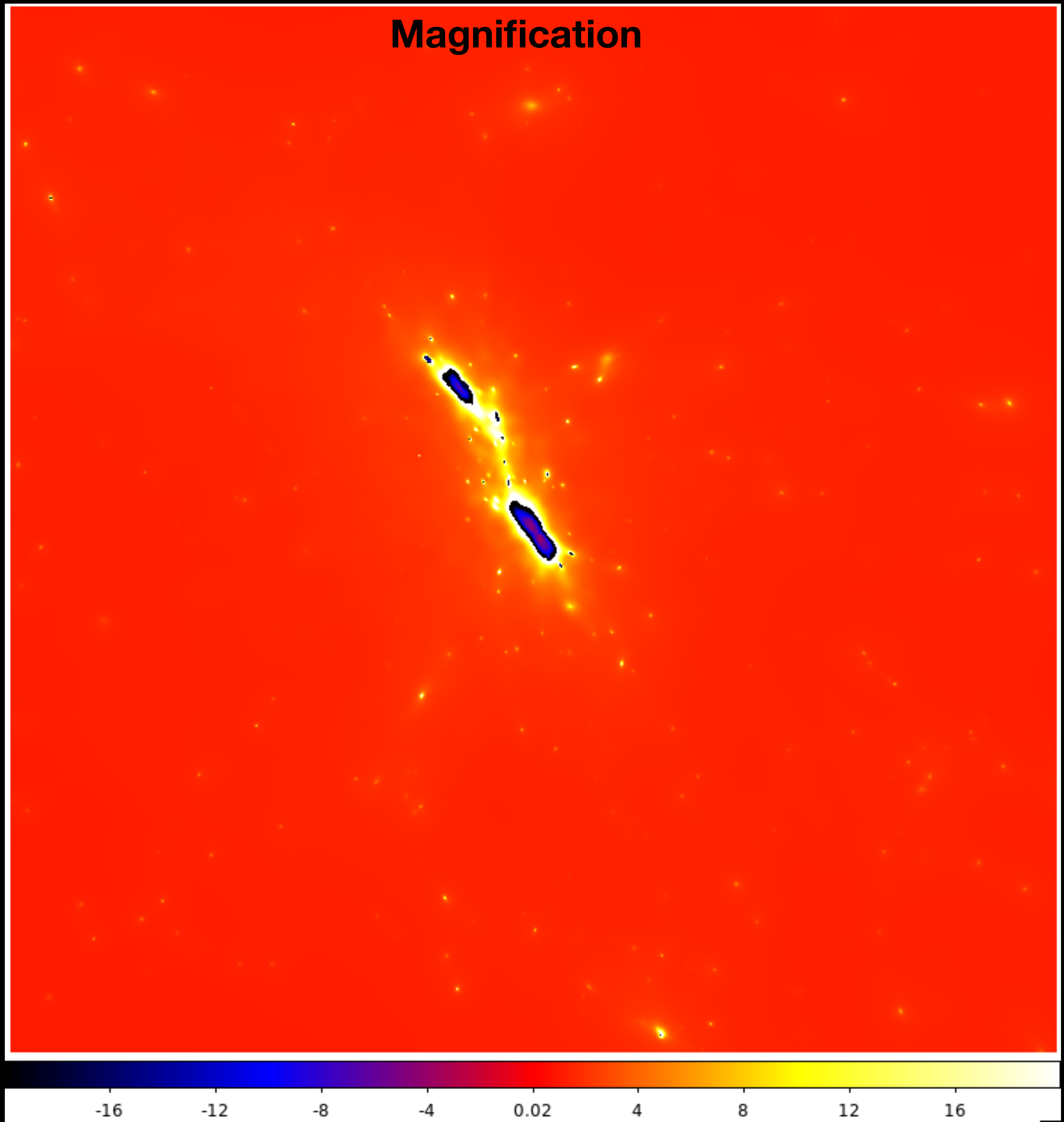
source z = 0.9



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

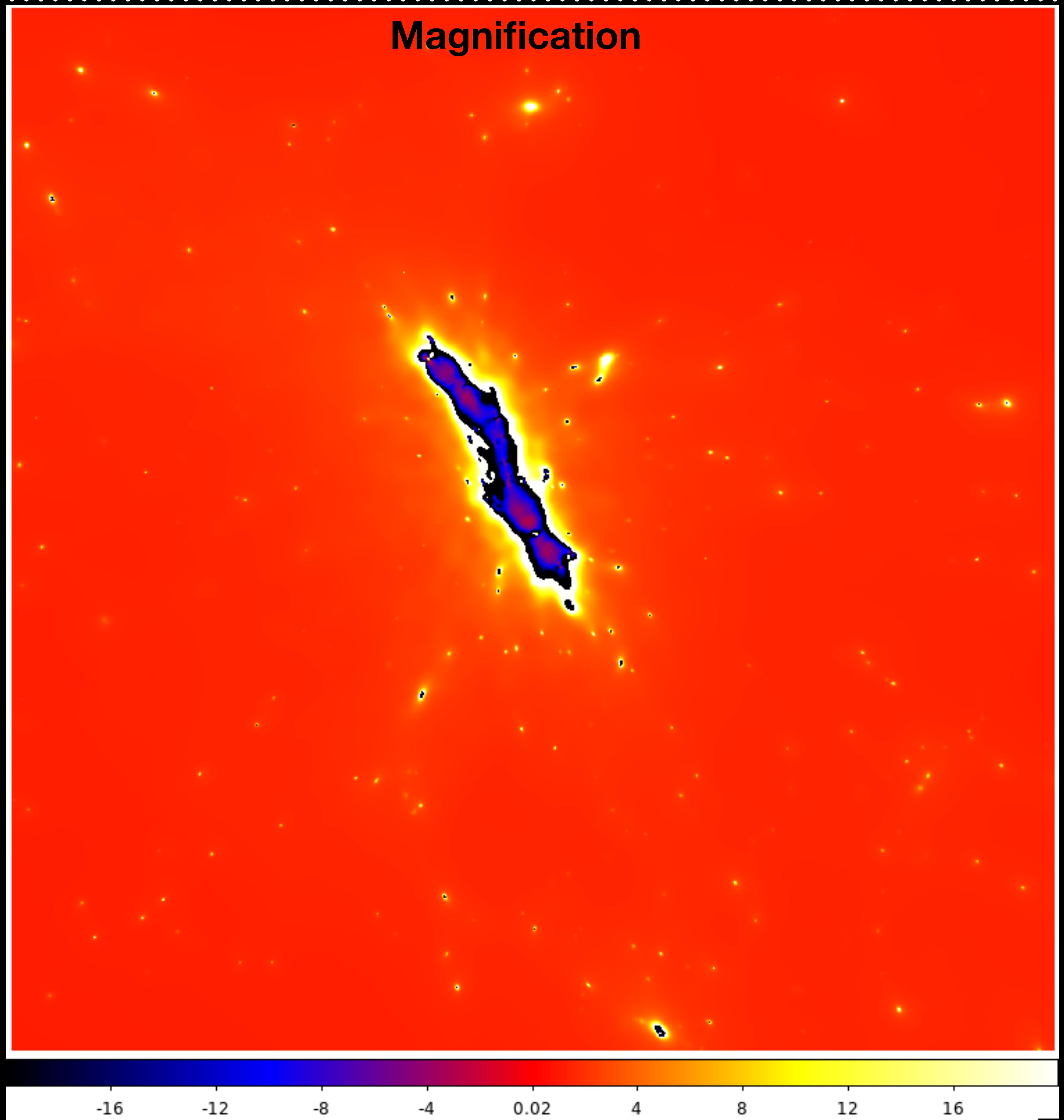
source z = 1.0



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

cluster z = 0.507

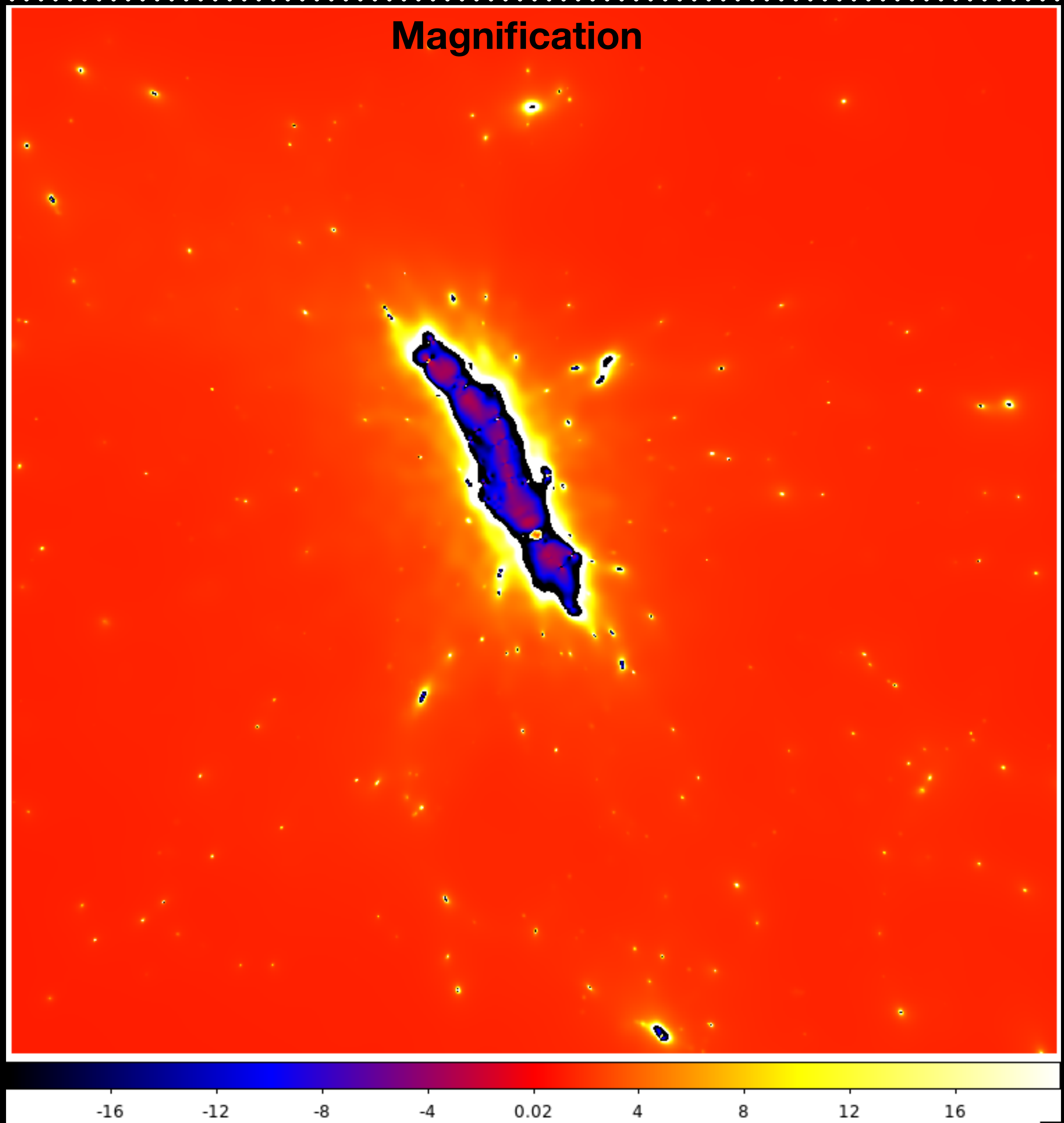
source z = 1.5



LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

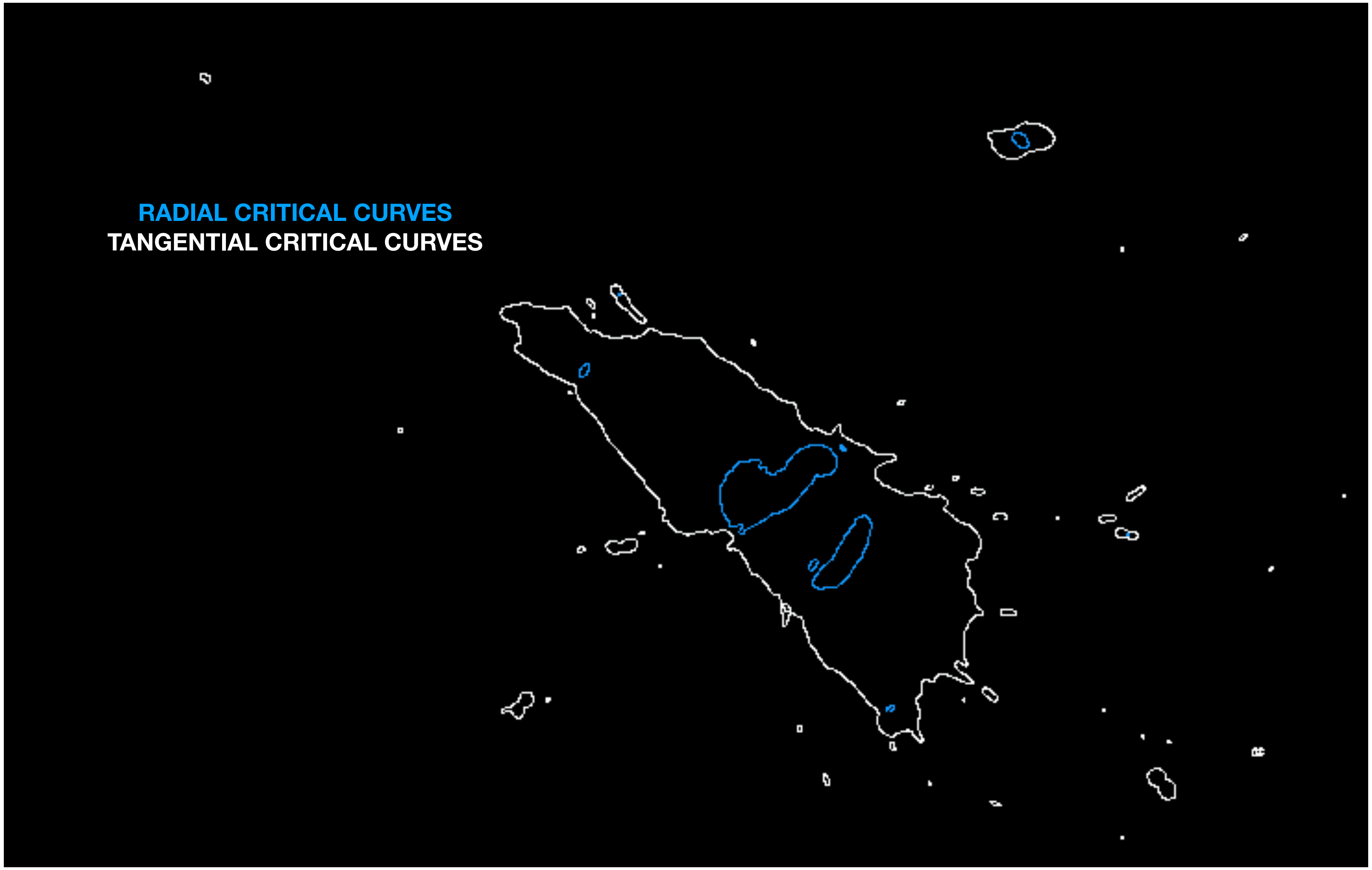
cluster z = 0.507

source z = 2.0



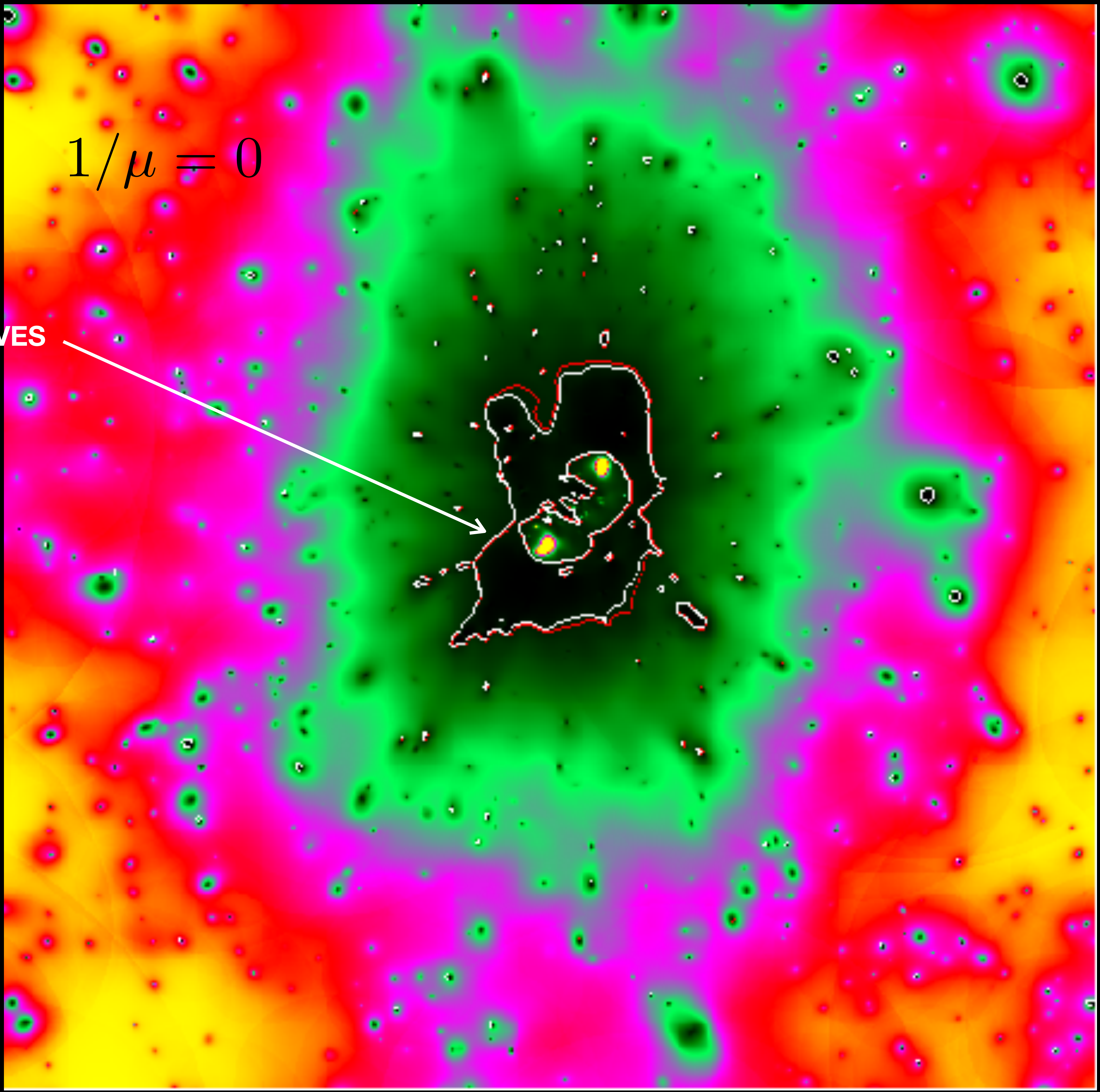
LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

CRITICAL CURVES OF A GALAXY CLUSTER



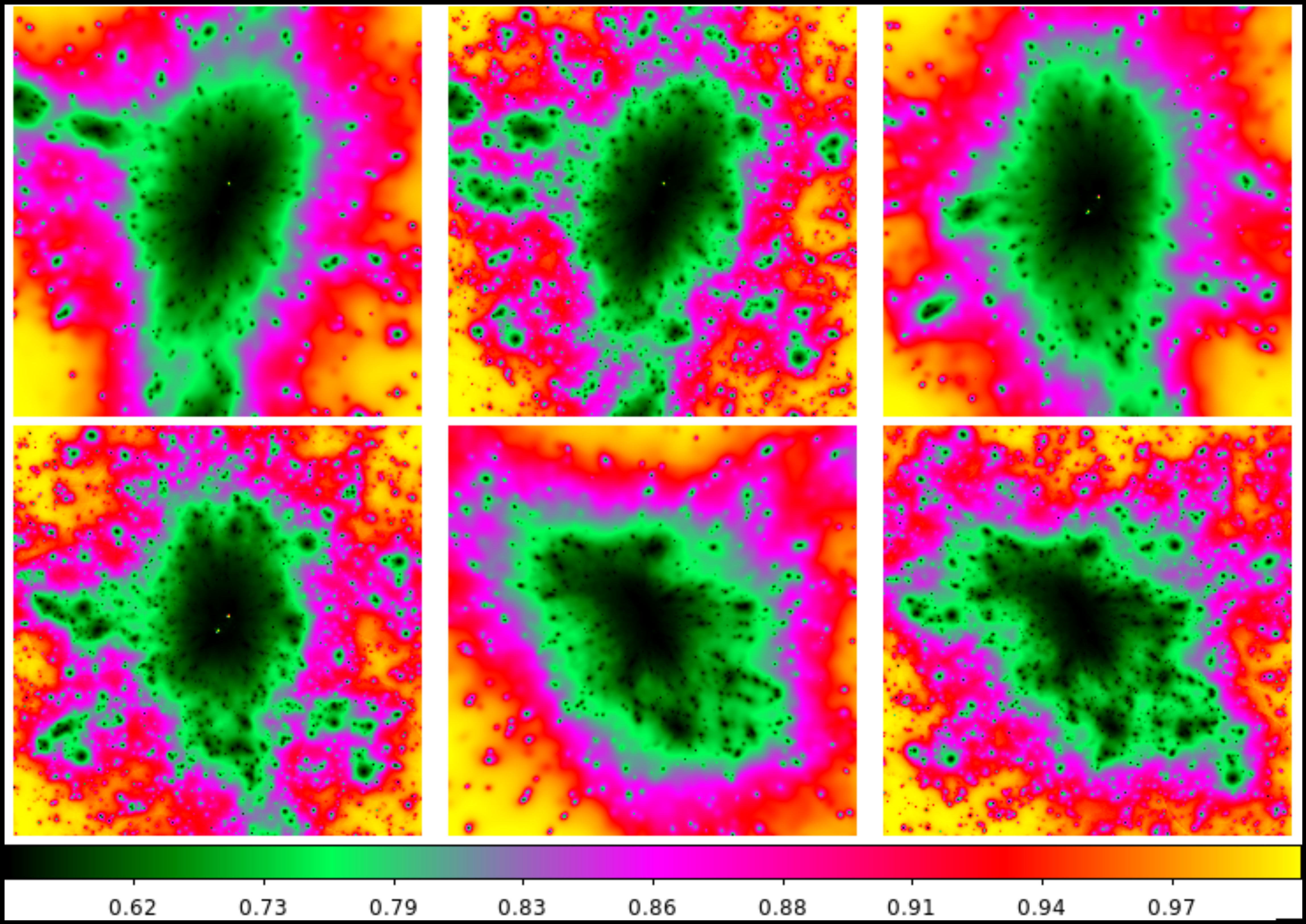
LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES

CRITICAL CURVES



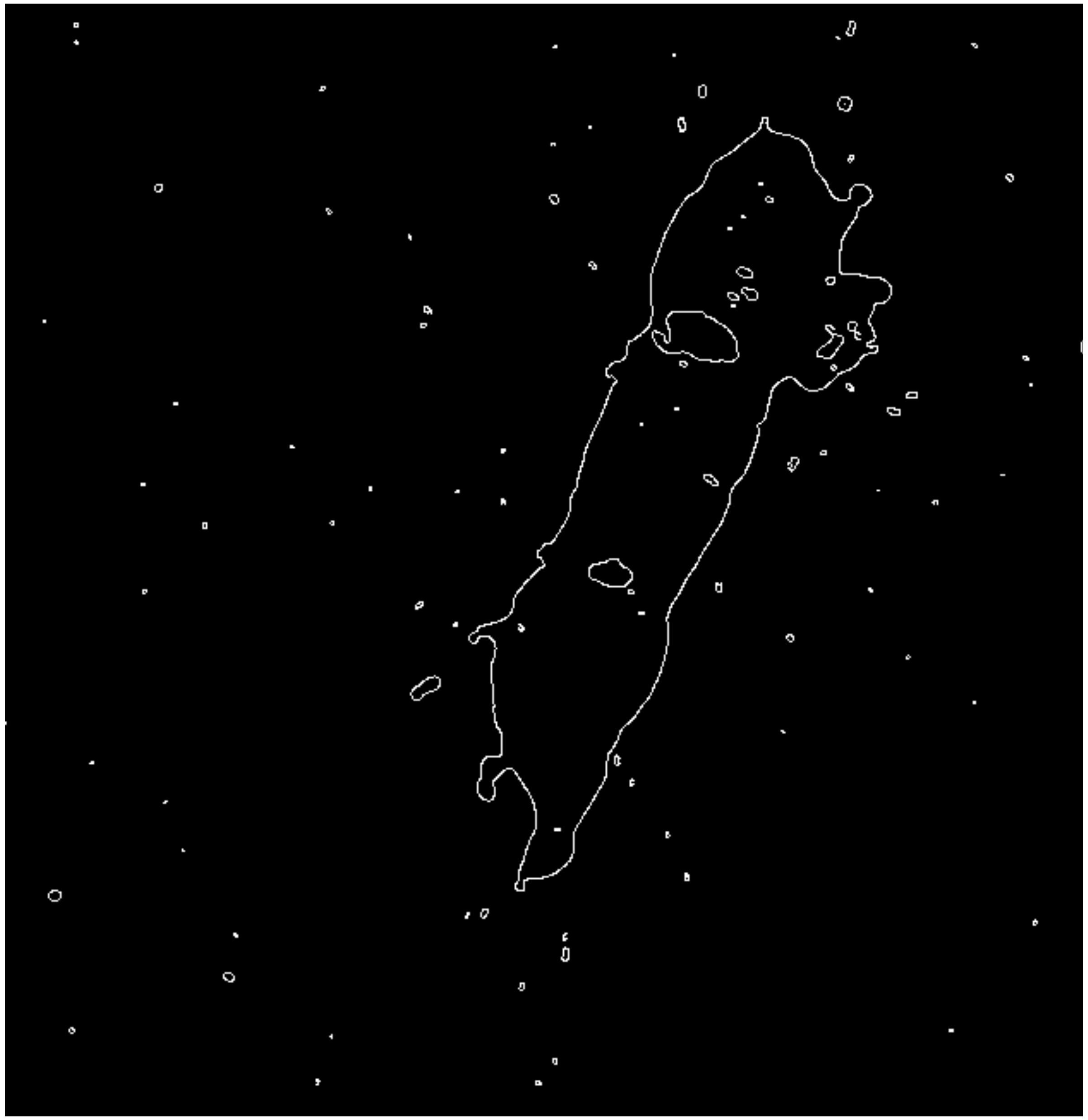
INVERSE MAGNIFICATION

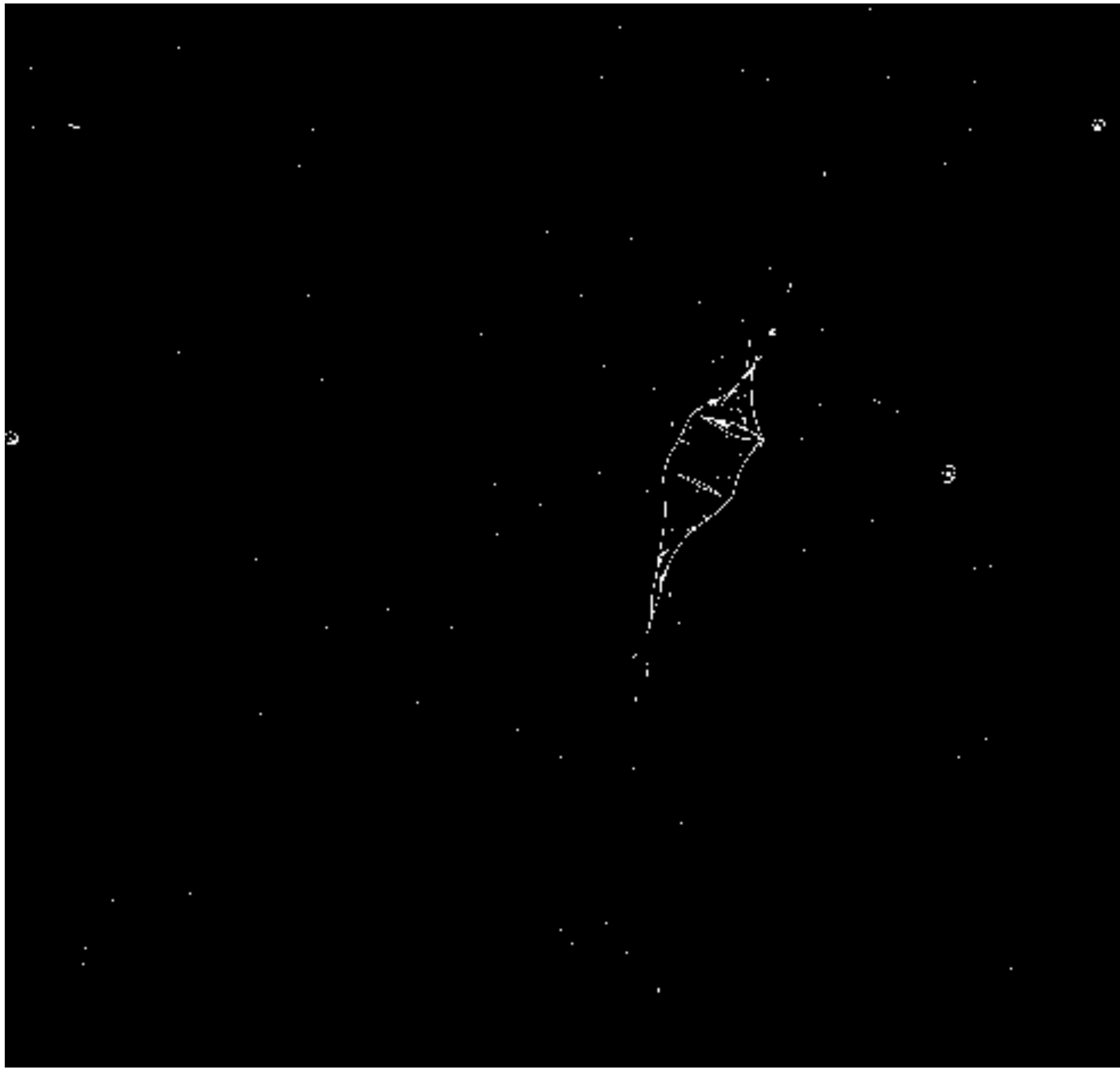
LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES



INVERSE MAGNIFICATION

LENS MAPPING : CRITICAL CURVES & CAUSTIC CURVES





POSITIVE AND NEGATIVE MAGNIFICATION

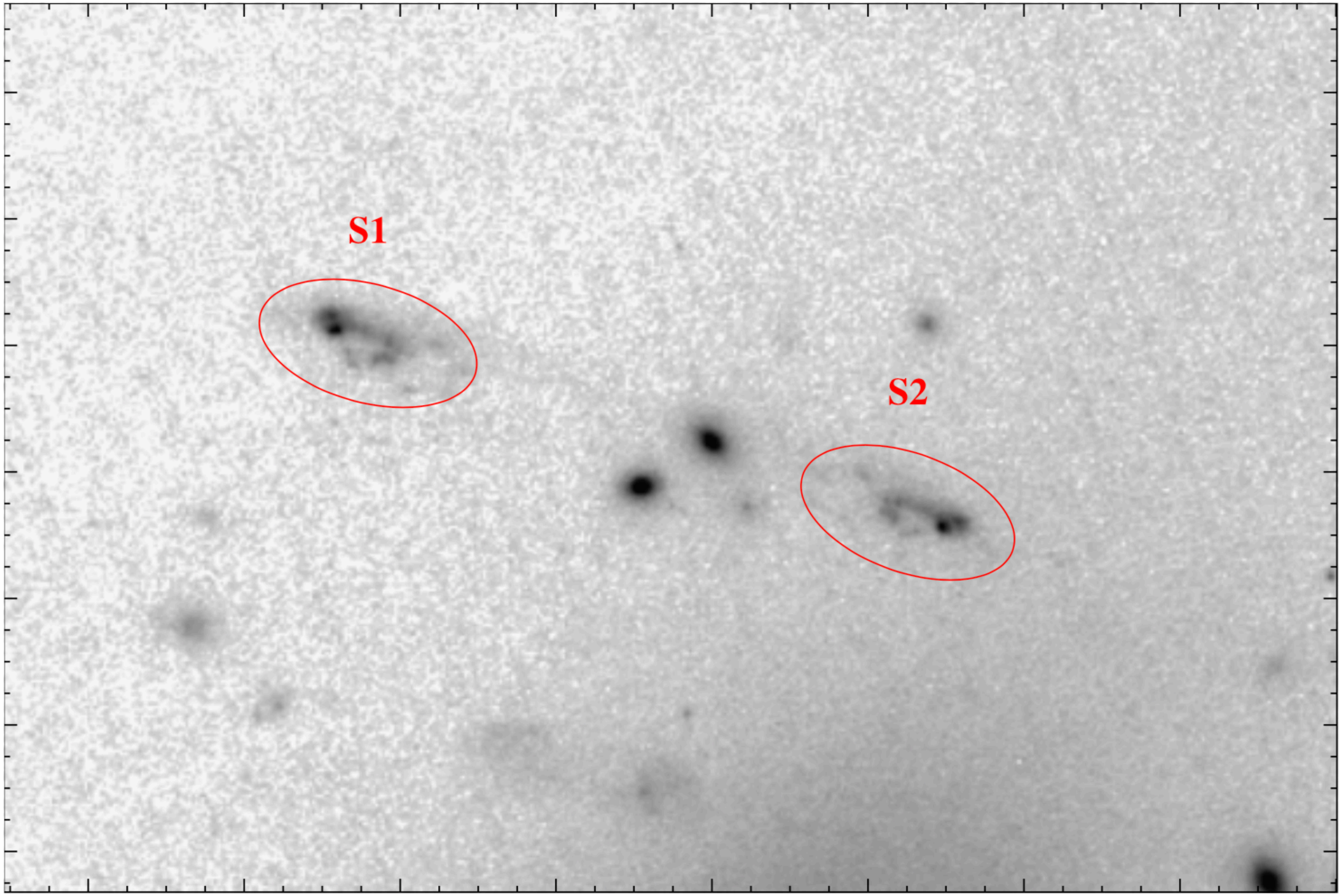
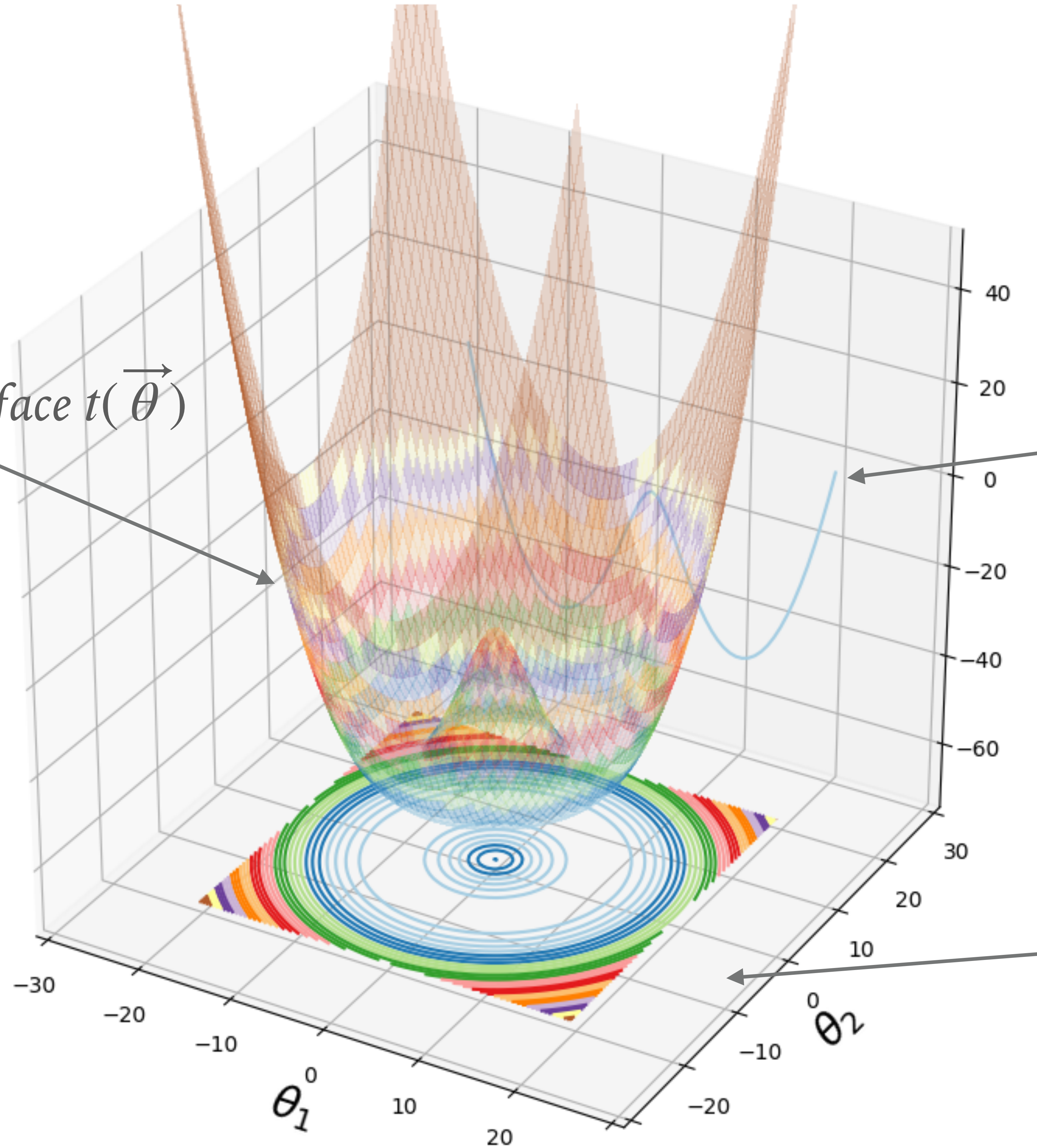


Fig. 10 The lensed pair S1–S2 in AC114. This galaxy at $z = 1.867$ displays the surprising morphology of a hook, with an obvious change in parity (Smail et al. 1995; Campusano et al. 2001)

Kneib & Natarajan 2012

EXAMPLE OF TIME DELAY SURFACE FOR A CIRCULAR LENS



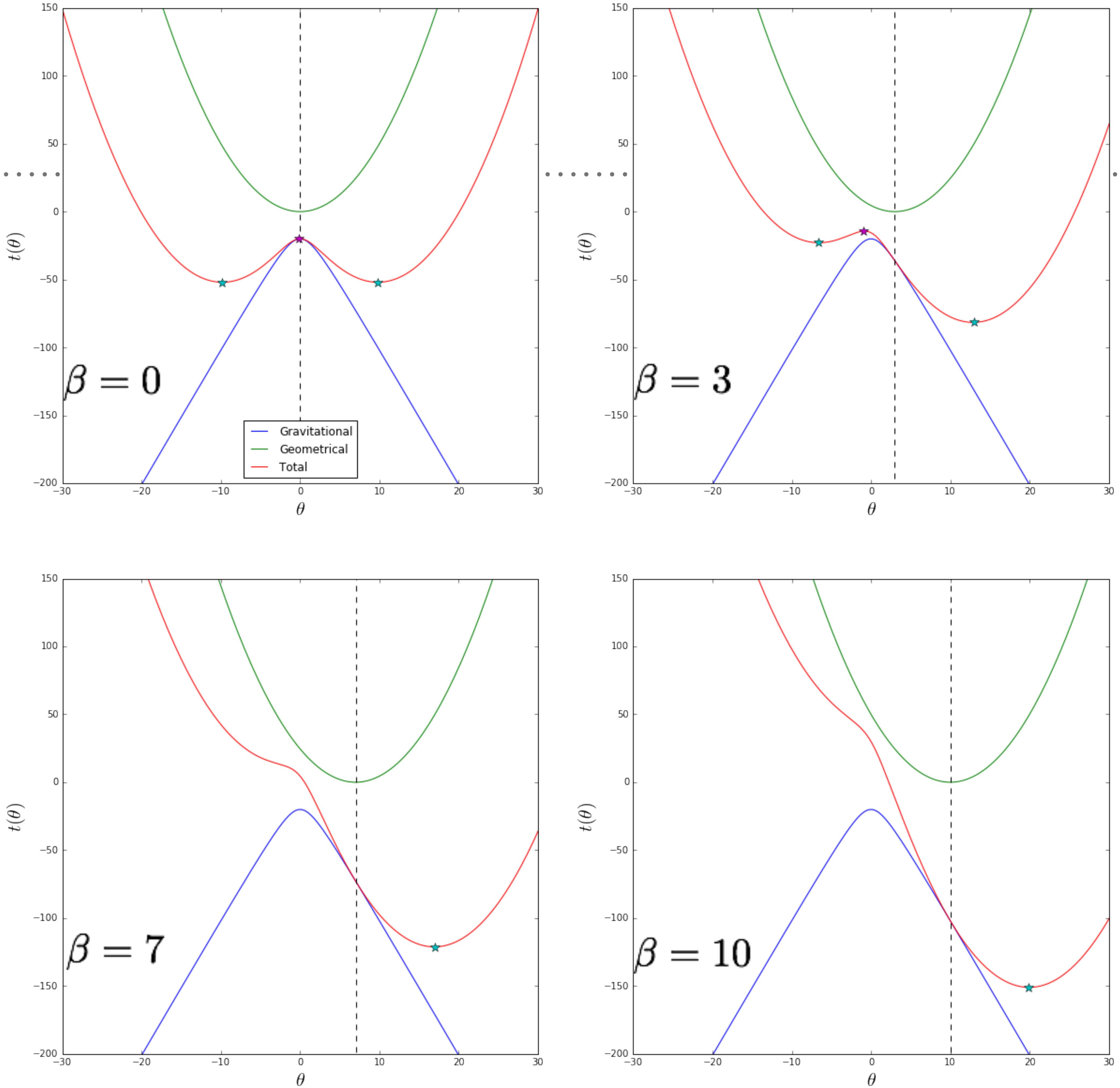
EXAMPLE OF TIME DELAY SURFACE

Toy potential:

$$\hat{\Psi}(\theta) = K\sqrt{\theta^2 + \theta_c^2}$$

Assuming axial-symmetry, we can discuss the time-delay function instead of the time delay surface.

$$t(\theta) \propto \frac{1}{2}(\theta - \beta)^2 - K\sqrt{\theta^2 + \theta_c^2}$$



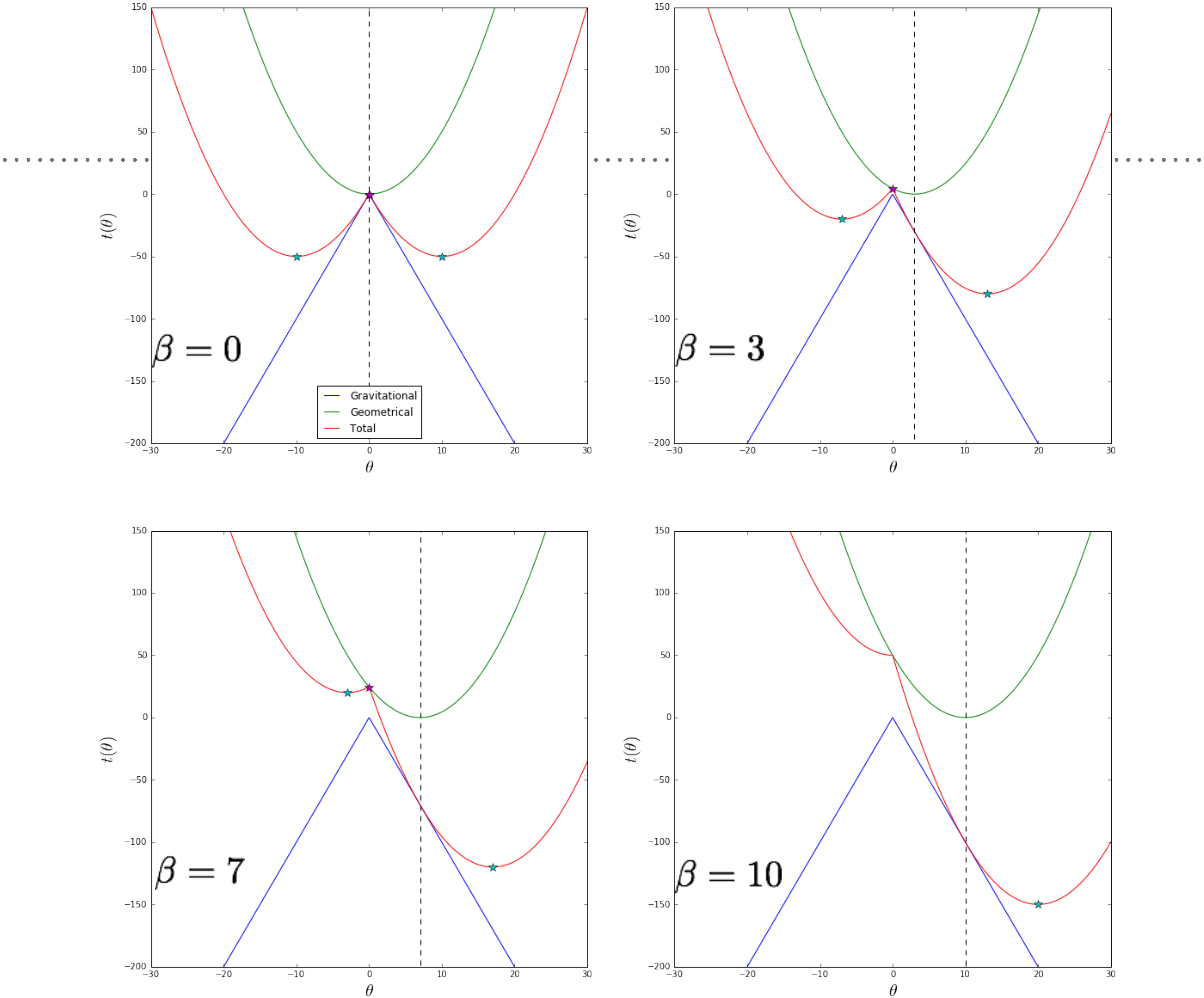
$t(\theta)$
θ
$\beta = 0$
$\beta = 3$
$\beta = 7$
$\beta = 10$
Gravitational
Geometrical
Total
150
100
50
0
−50
−100
−150
−200
−30
−20
−10
0
10
20
30

EXAMPLE OF TIME DELAY SURFACE

Let's change potential:

$$\hat{\Psi}(\theta) = K|\theta|$$

The lens model is the same as before, but the core has been removed



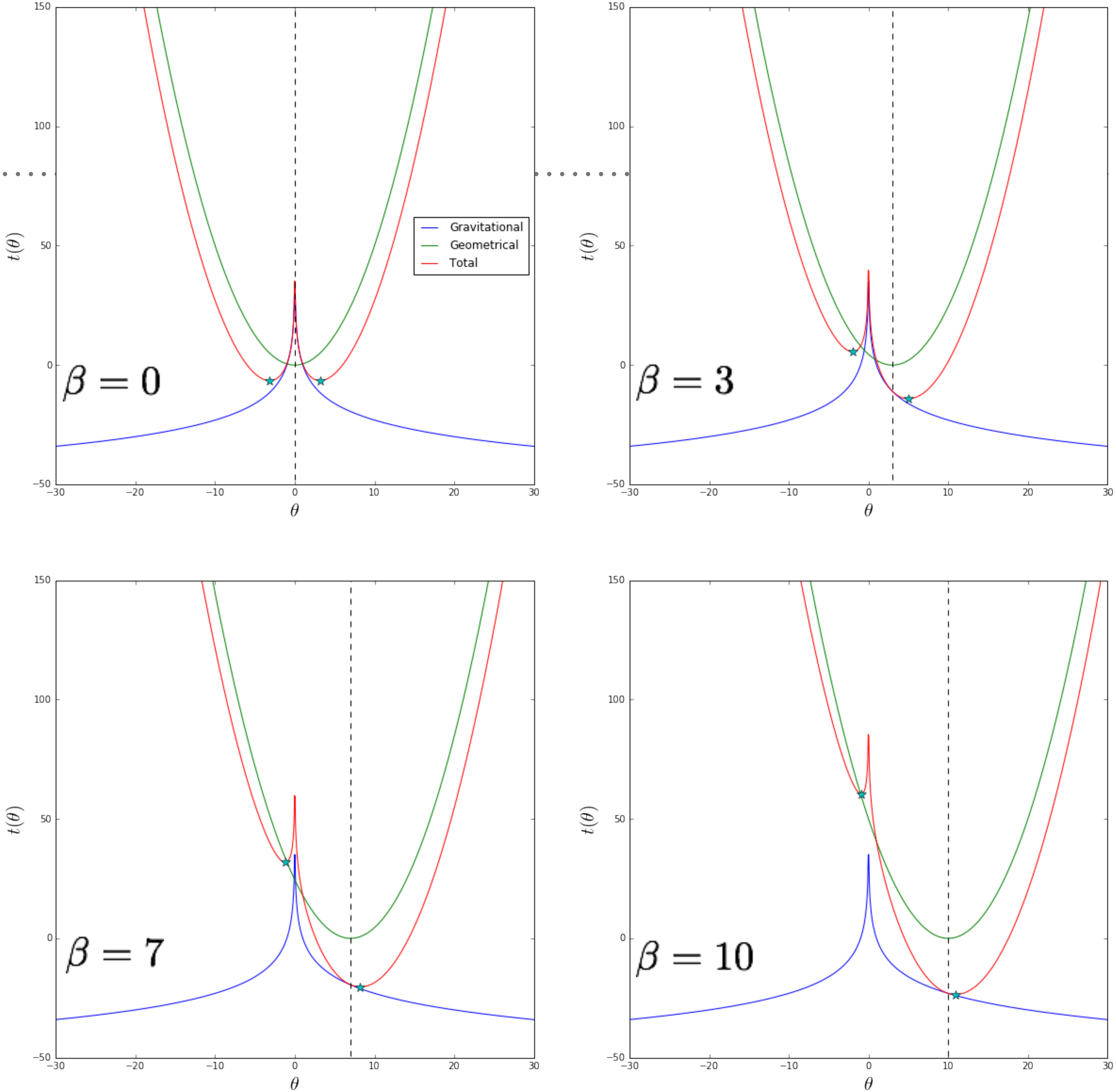
Gravitational
Geometrical
Total
$\beta = 0$
$\beta = 3$
$\beta = 7$
$\beta = 10$
$t(\theta)$
θ

EXAMPLE OF TIME DELAY SURFACE

Yet another potential

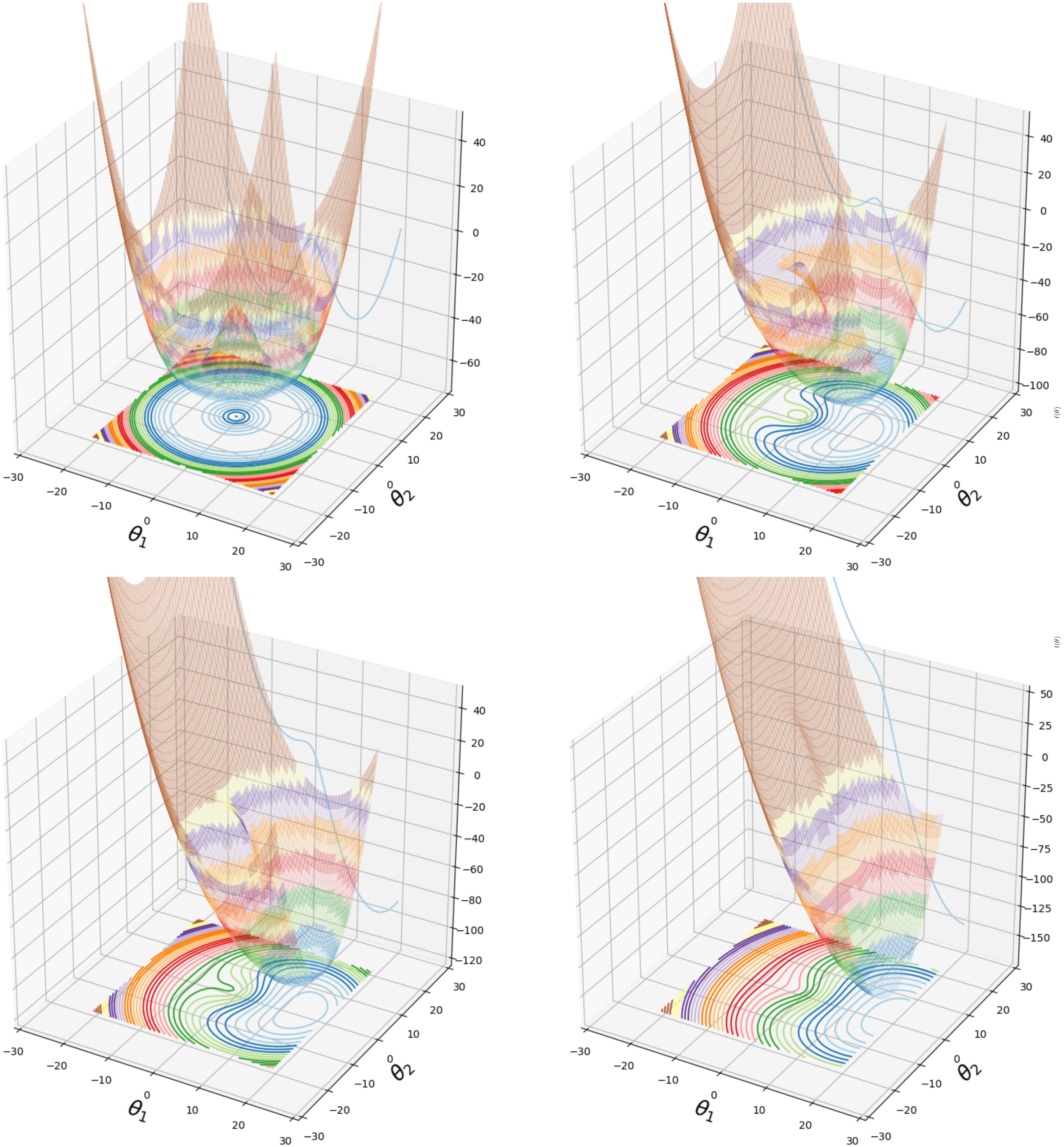
$$\hat{\Psi}(\theta) = K \ln|\theta|$$

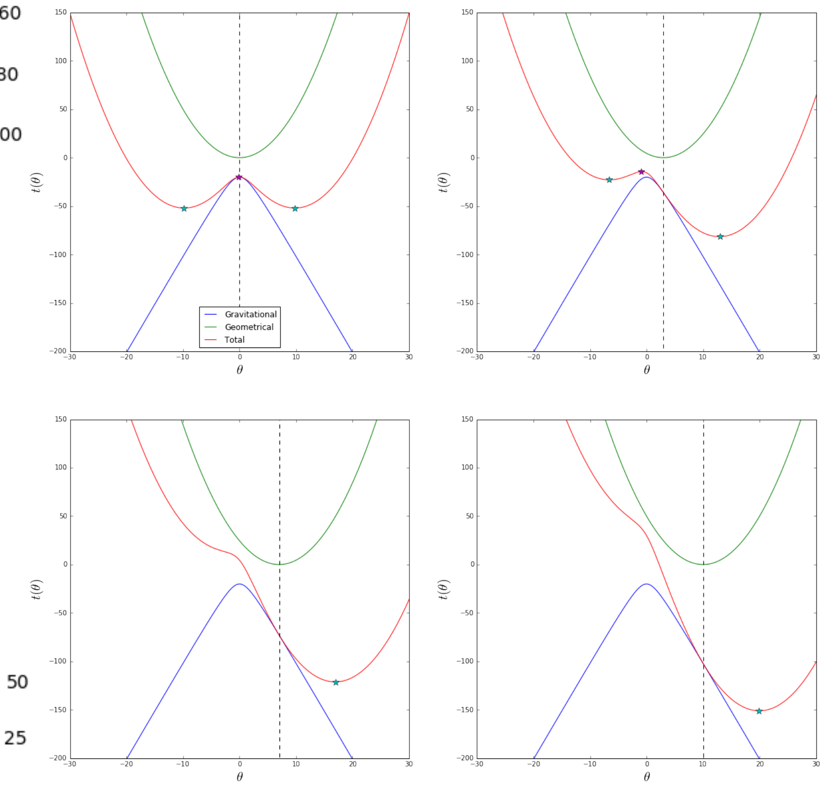
This is the lensing potential of the point mass…

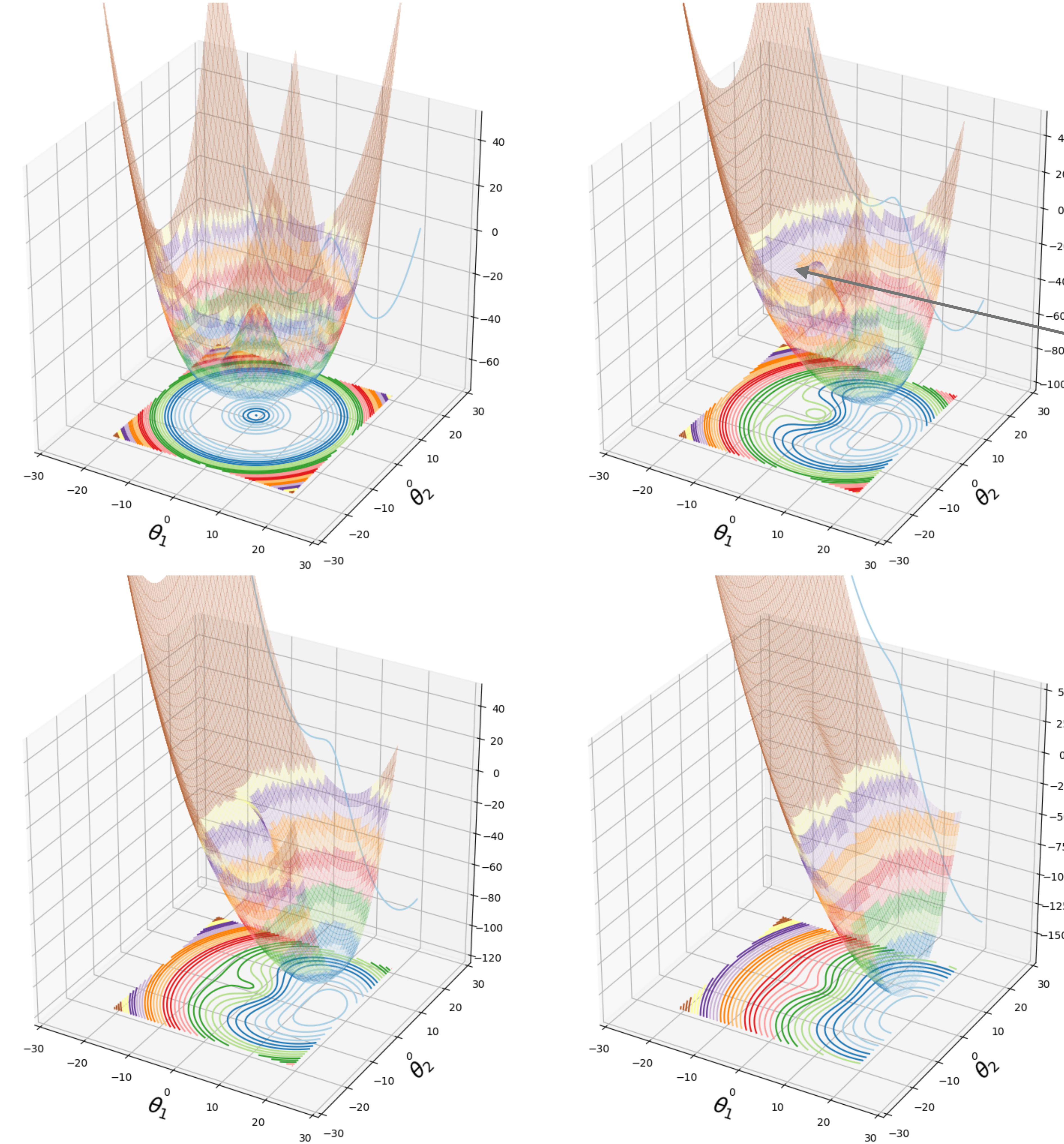


Gravitational
Geometrical
Total
$\beta = 0$
$\beta = 3$
$\beta = 7$
$\beta = 10$
$t(\theta)$
θ

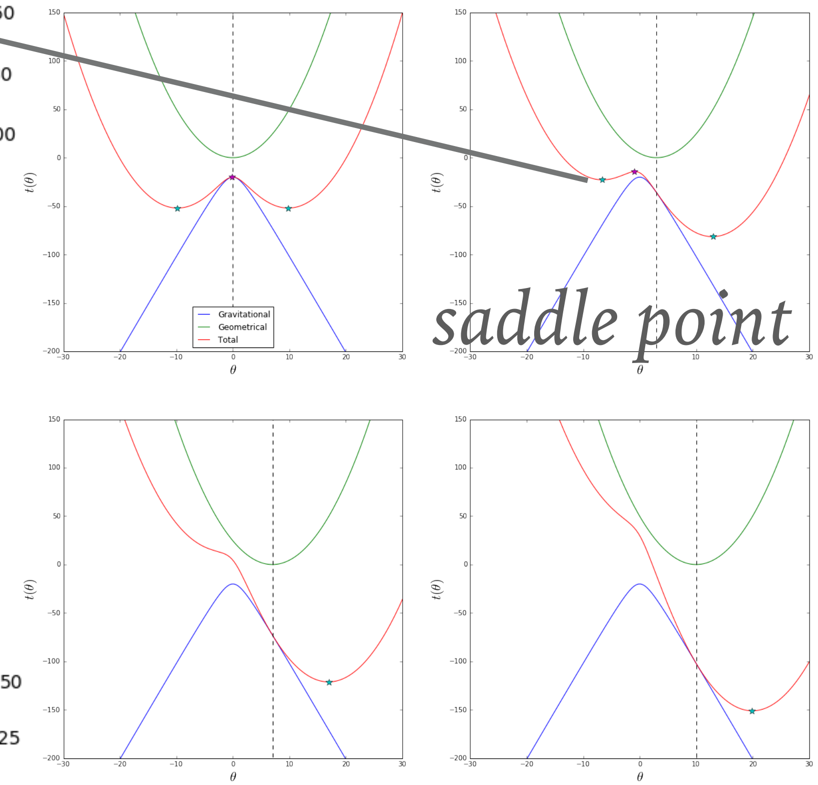
…but $t(\theta)$ is a surface!





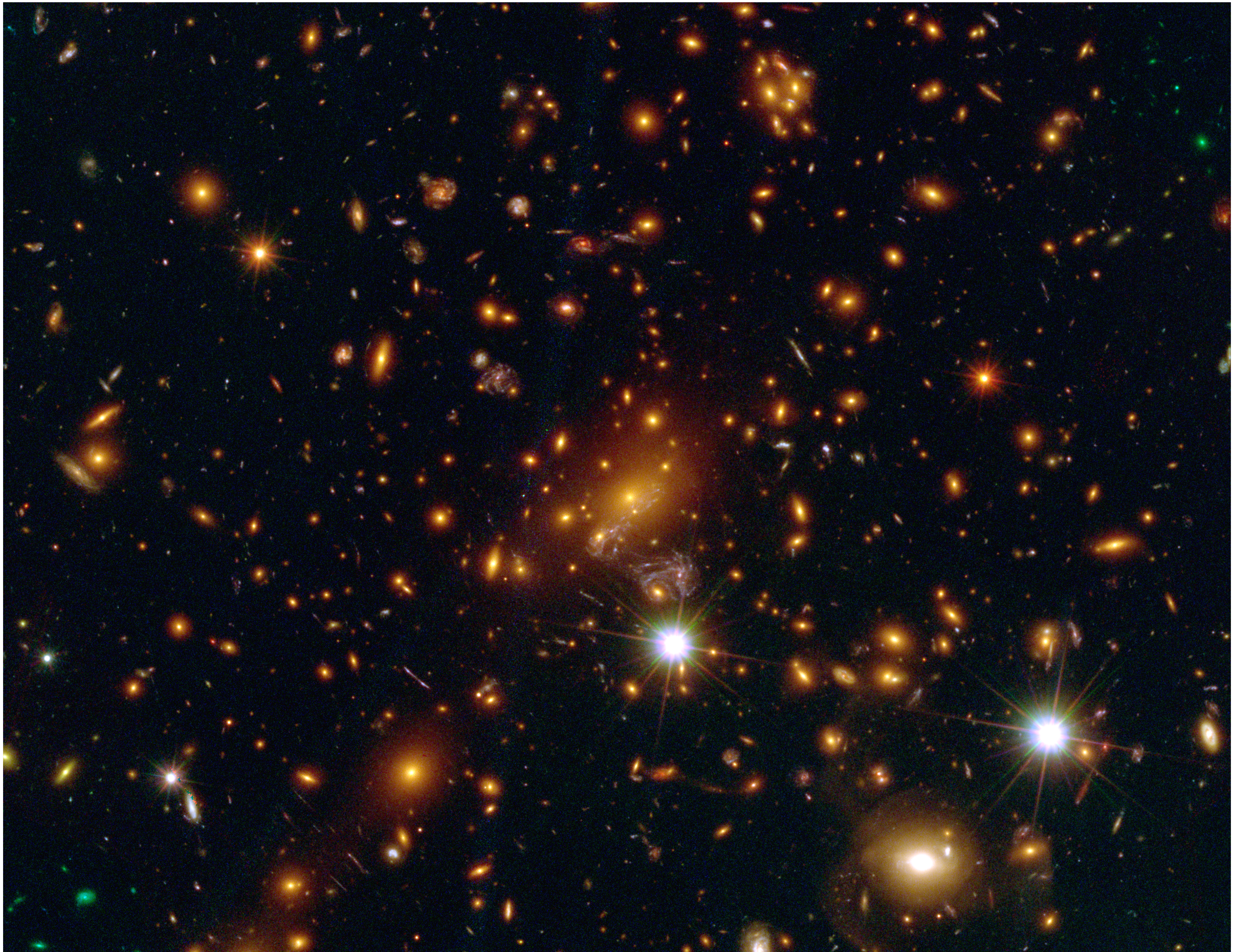


…but $t(\theta)$ is a surface!

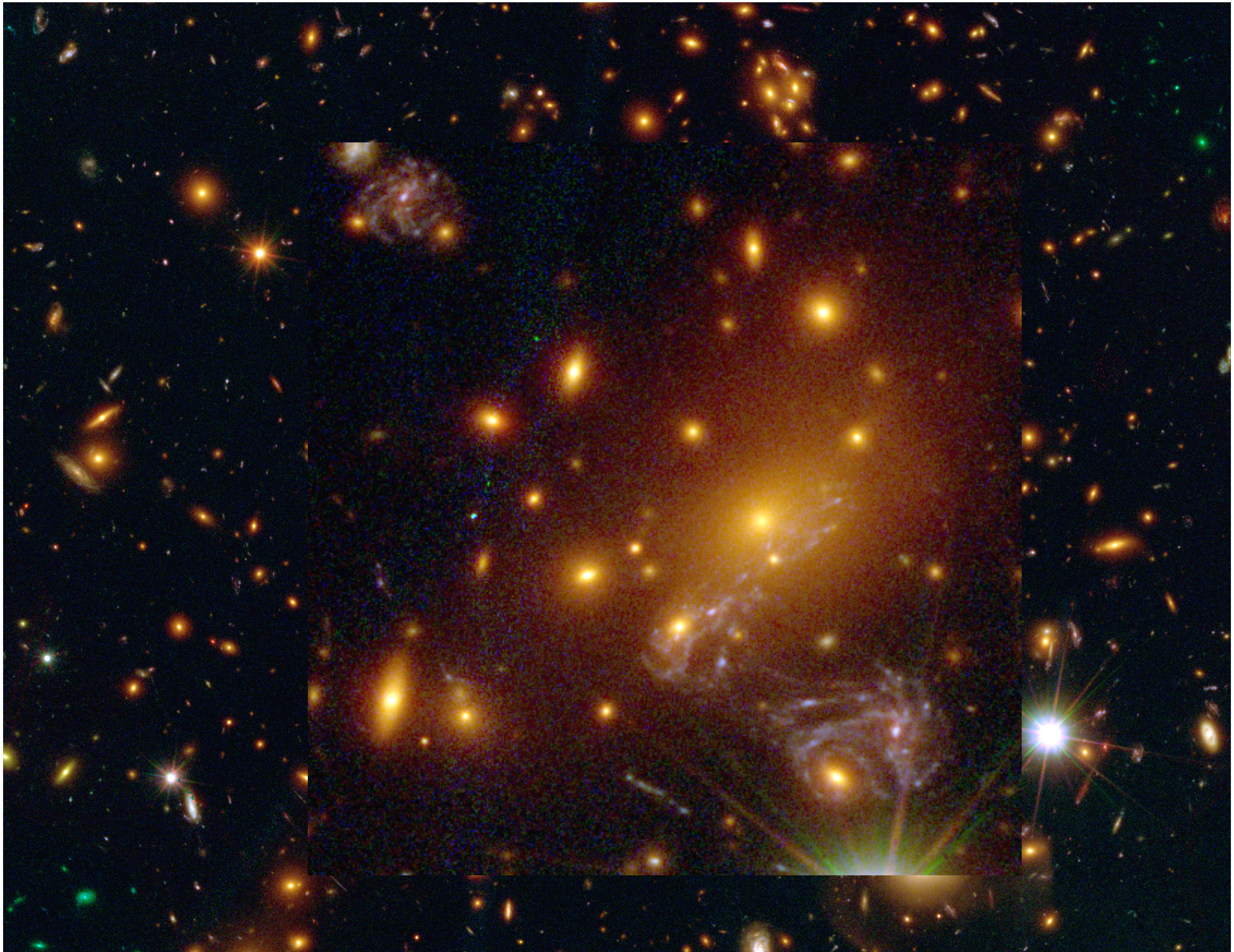


saddle point

SN REFSDAL IN MACS 1149

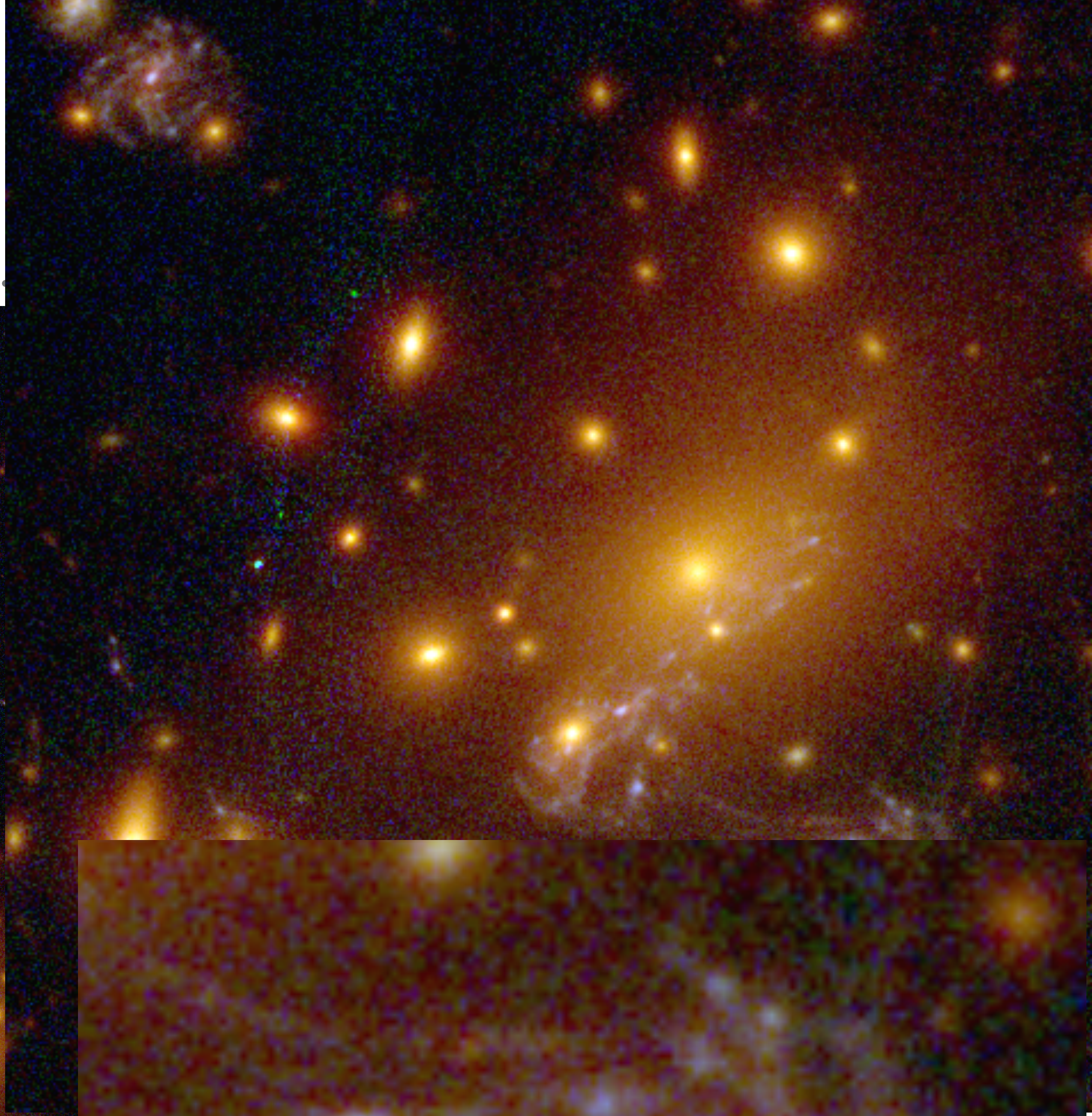


SN REFSDAL IN MACS 1149

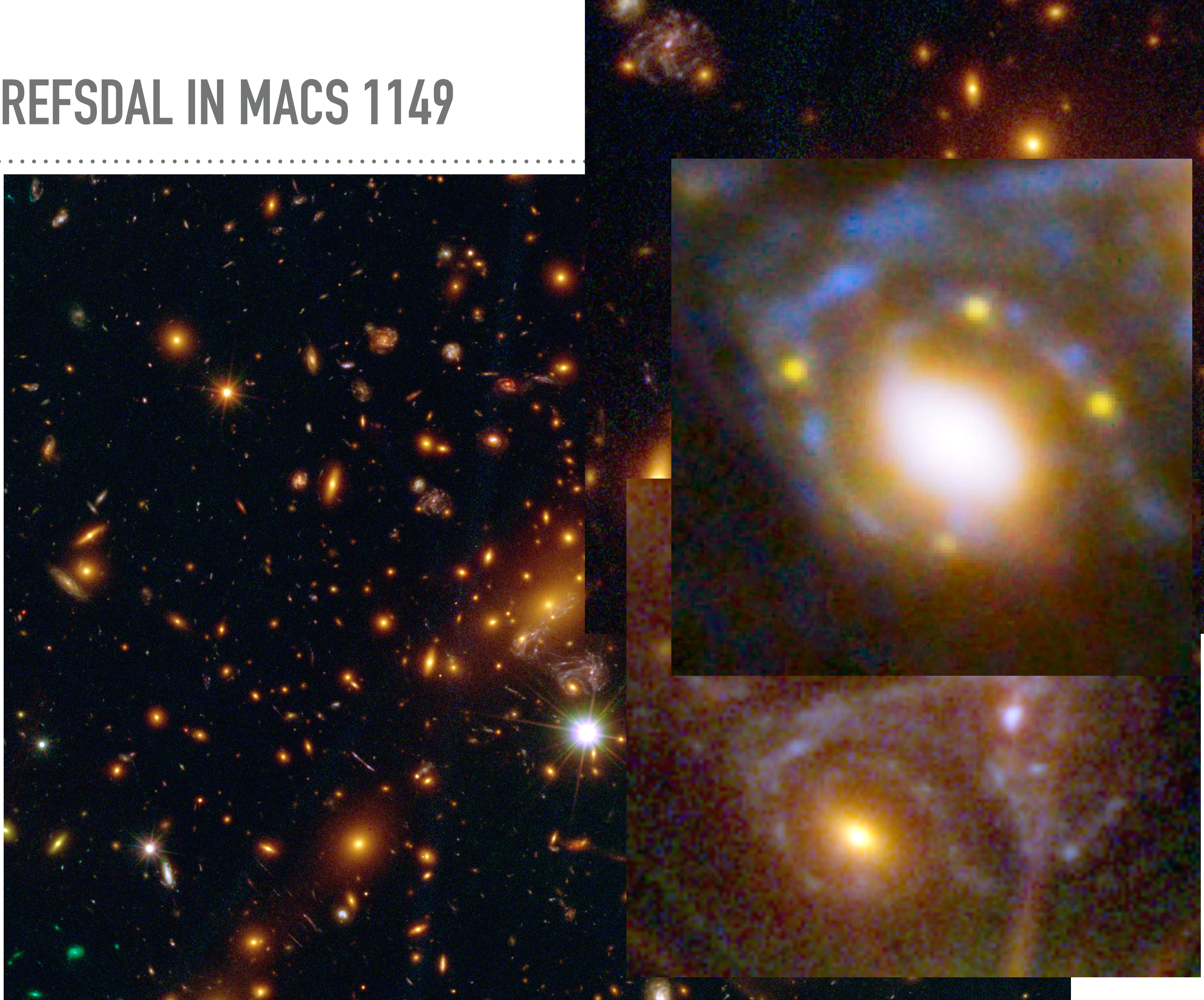


SN REFSDAL IN MACS 1149

SN REFSDAL IN MACS 1149

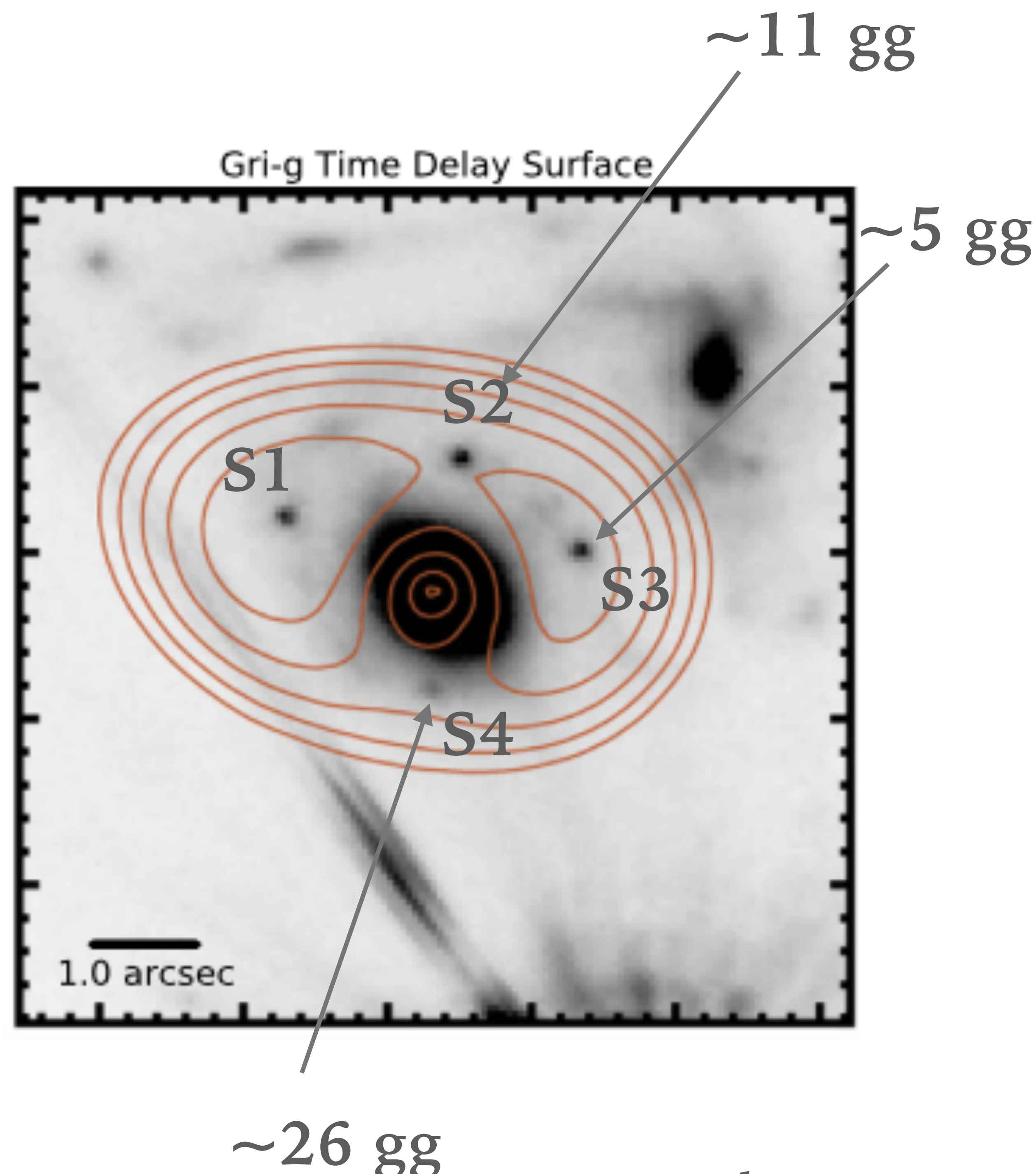


SN REFSDAL IN MACS 1149



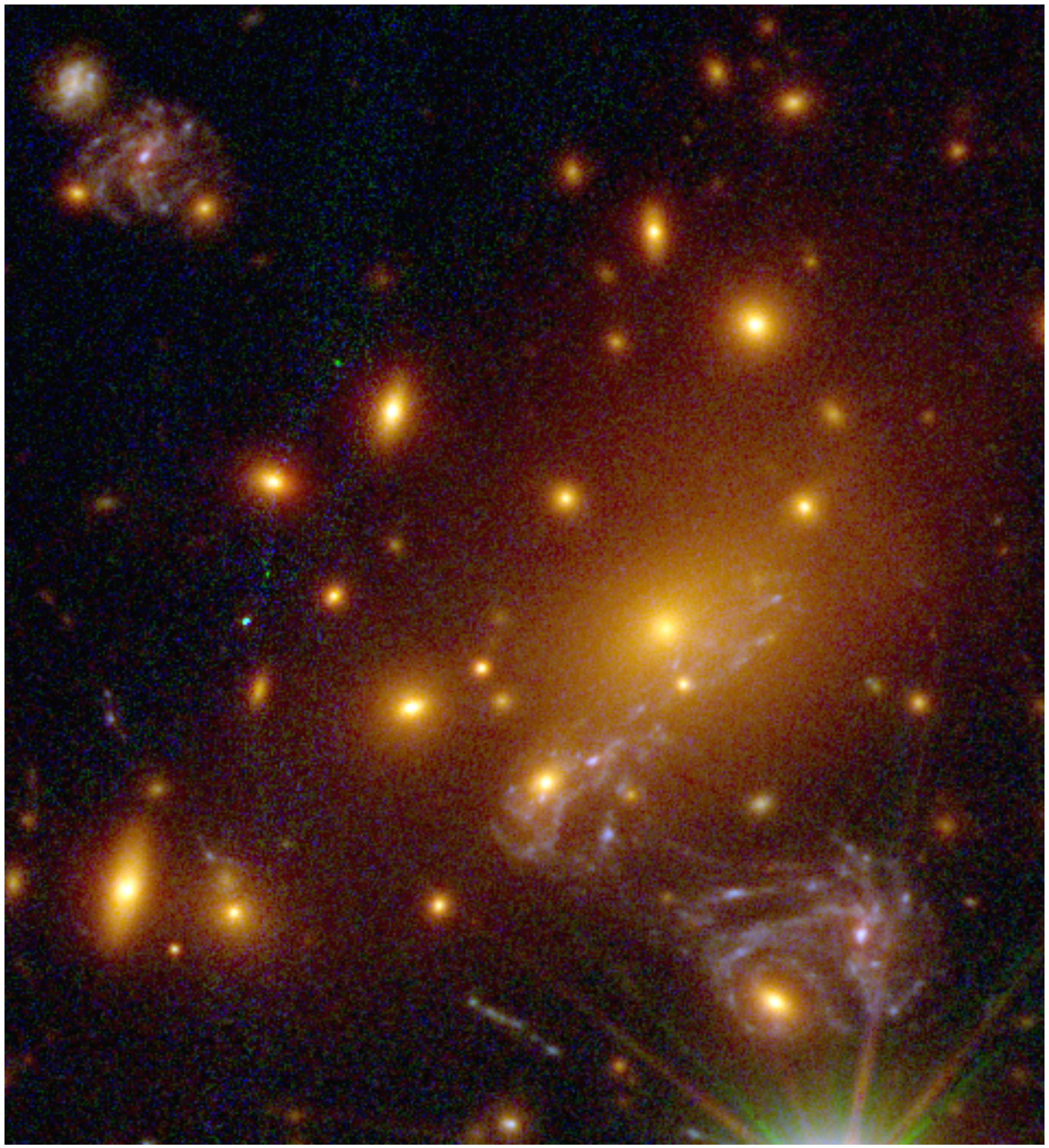
SN REFSDAL IN MACS 1149

Nov. 2014 (Kelly et al.)

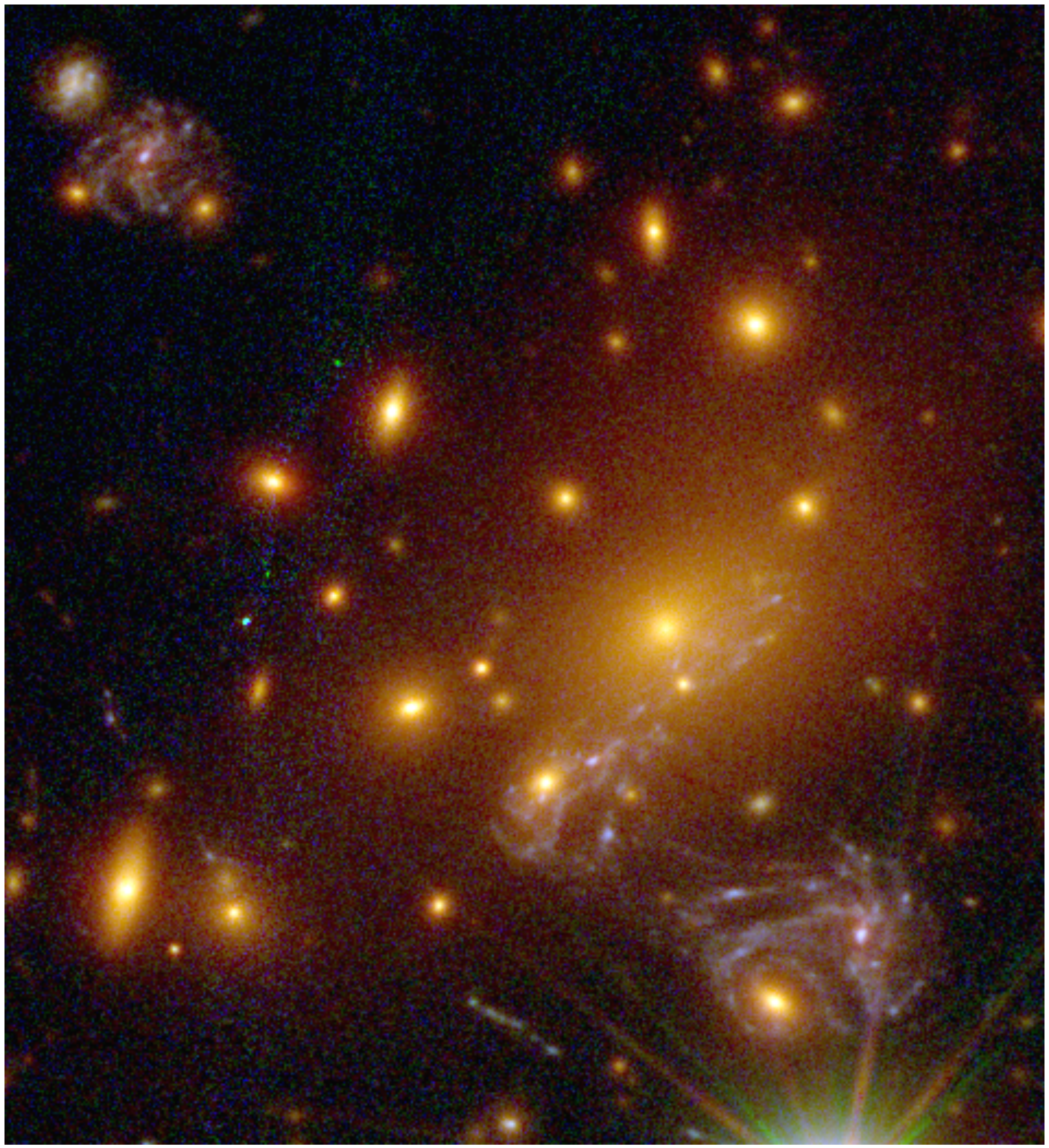


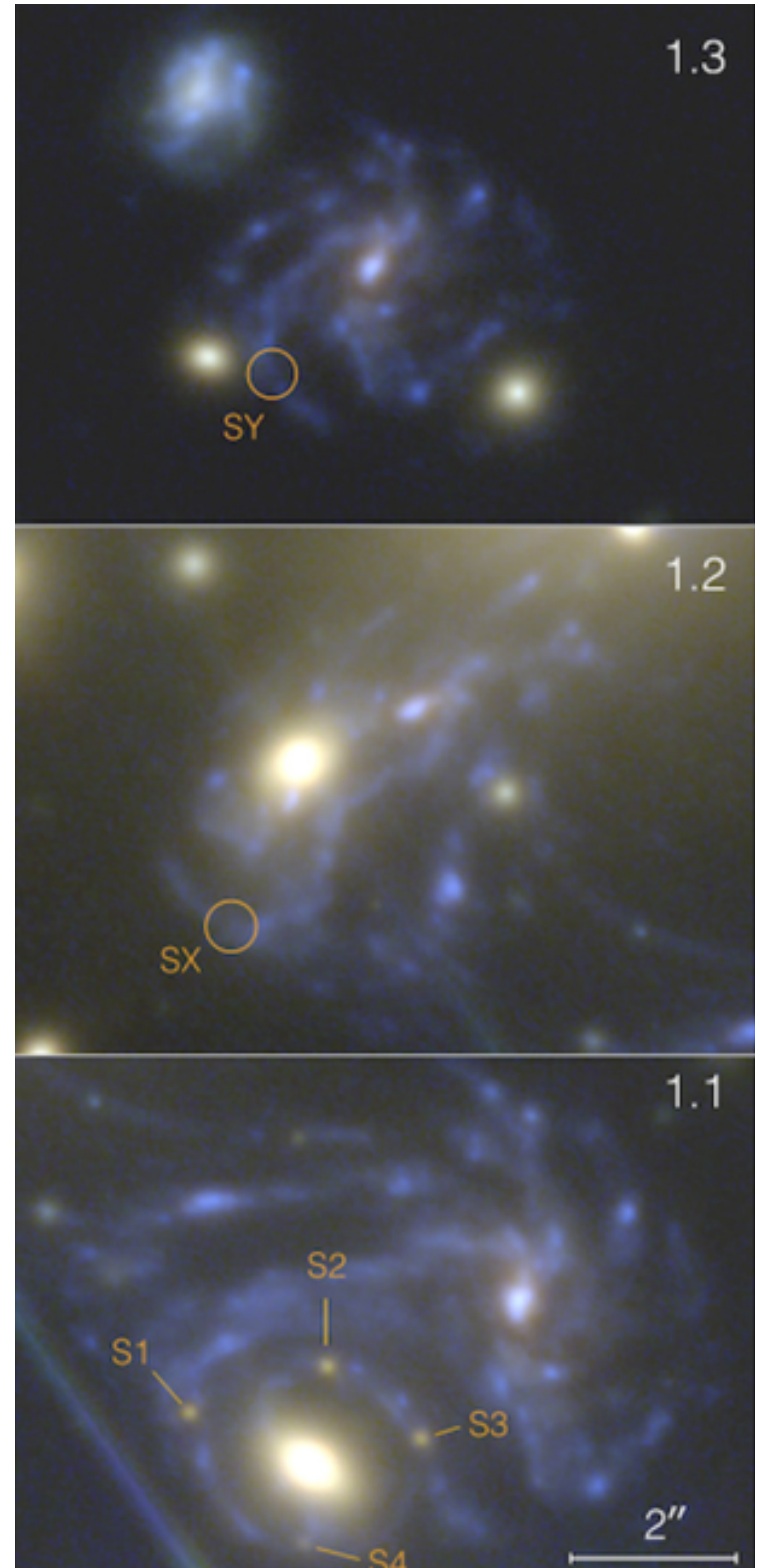
Treu et al. 2016

SN REFSDAL IN MACS 1149



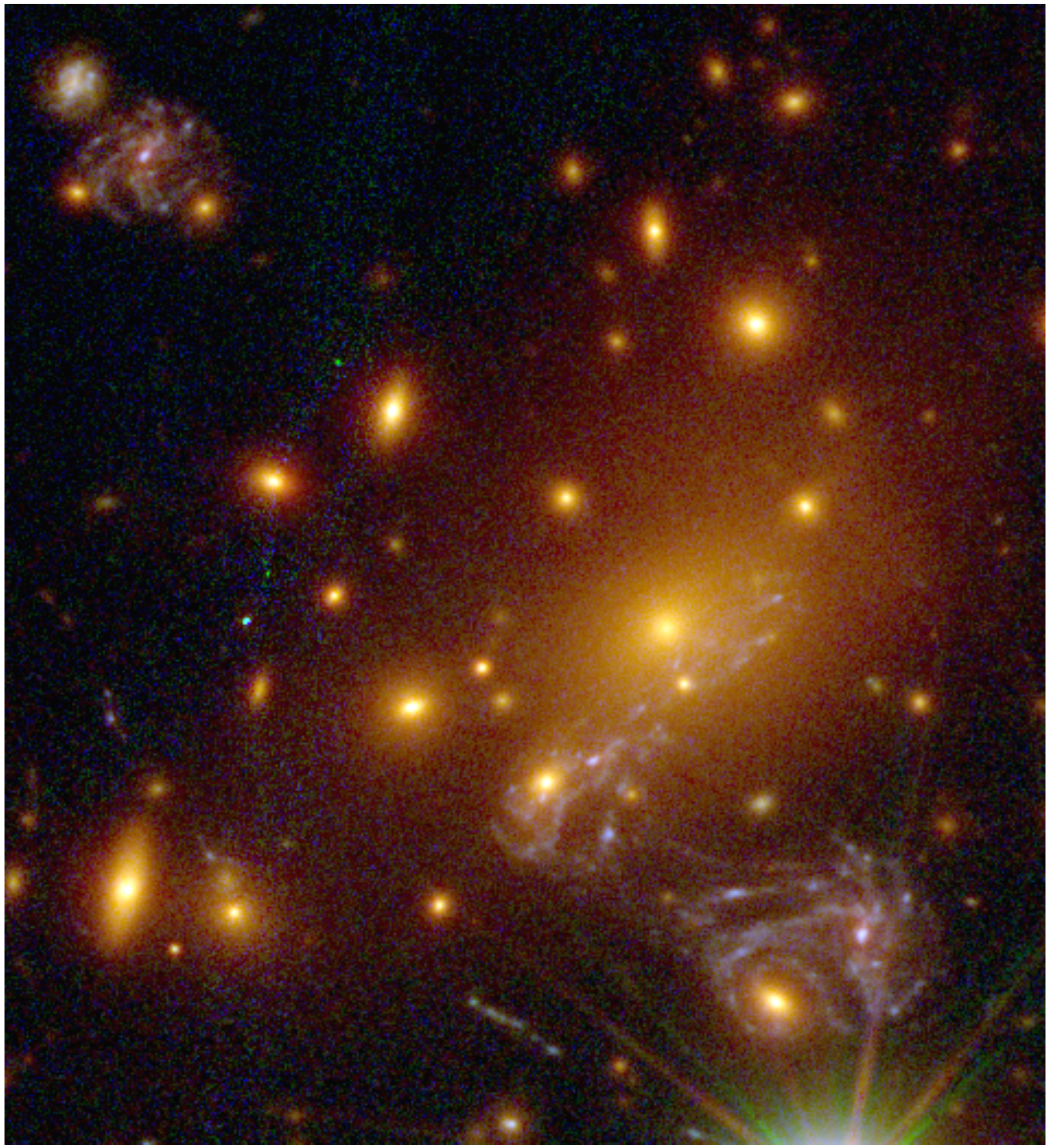
SN REFSDAL IN MACS 1149

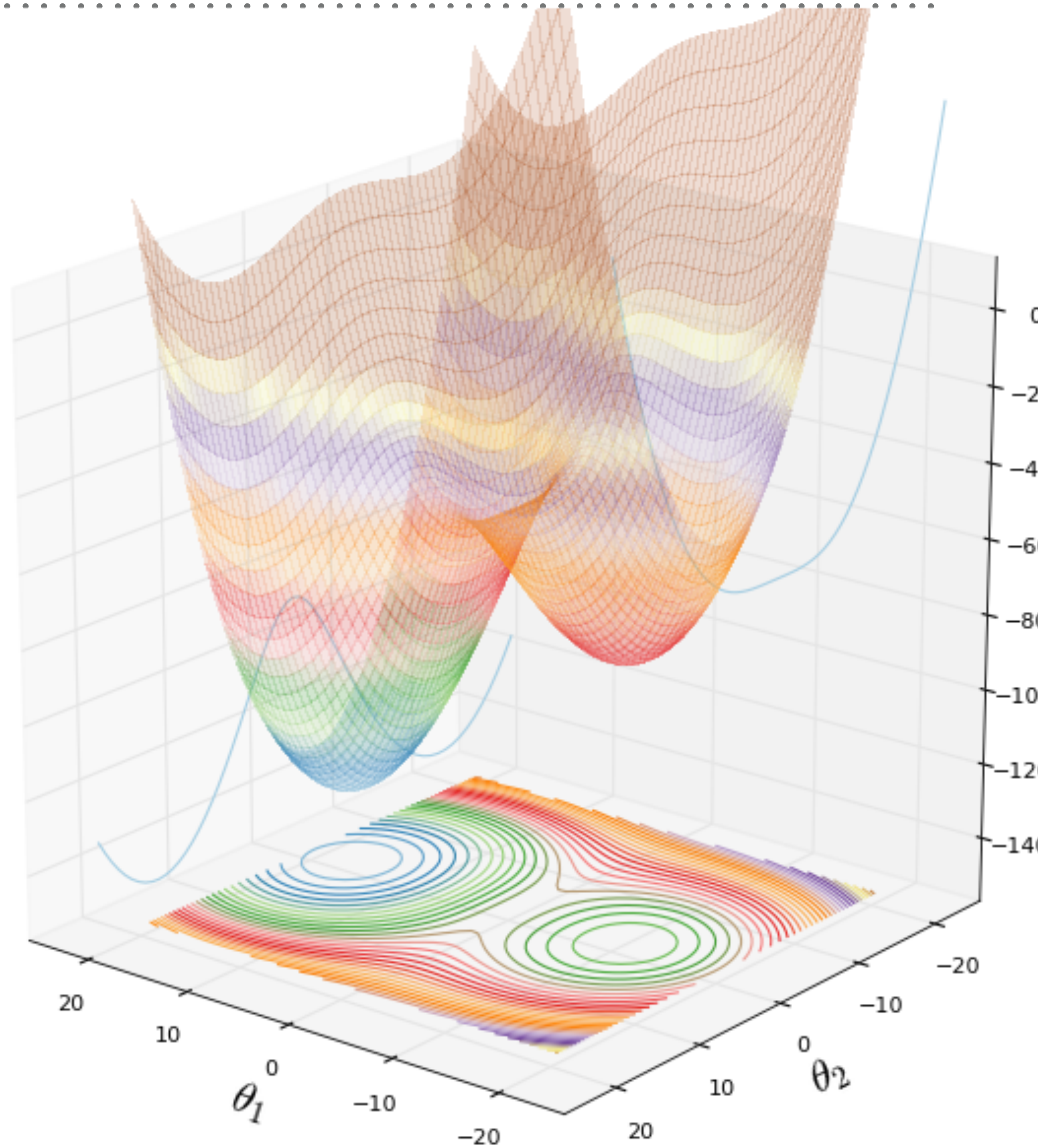




?

SN REFSDAL IN MACS 1149





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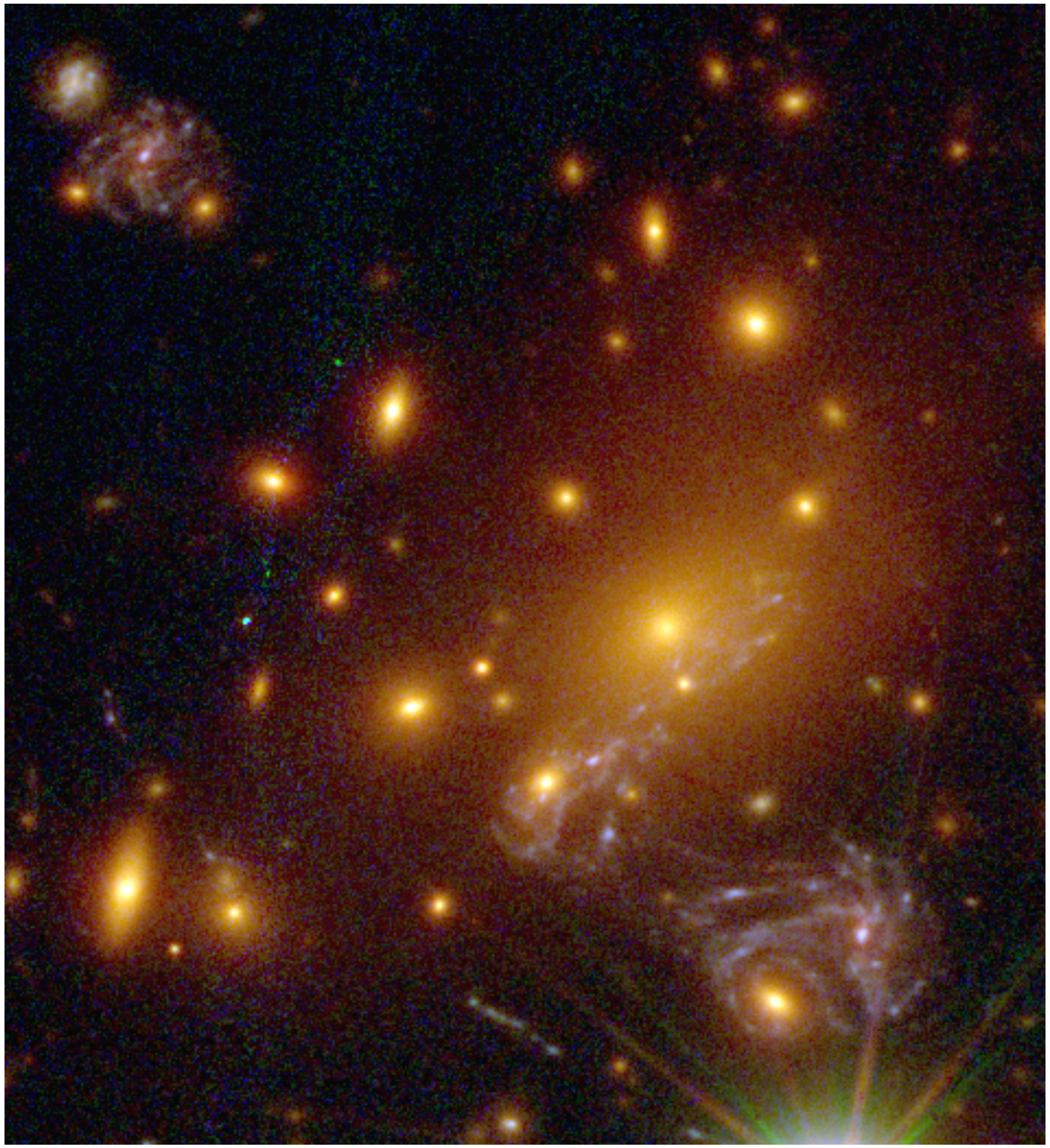

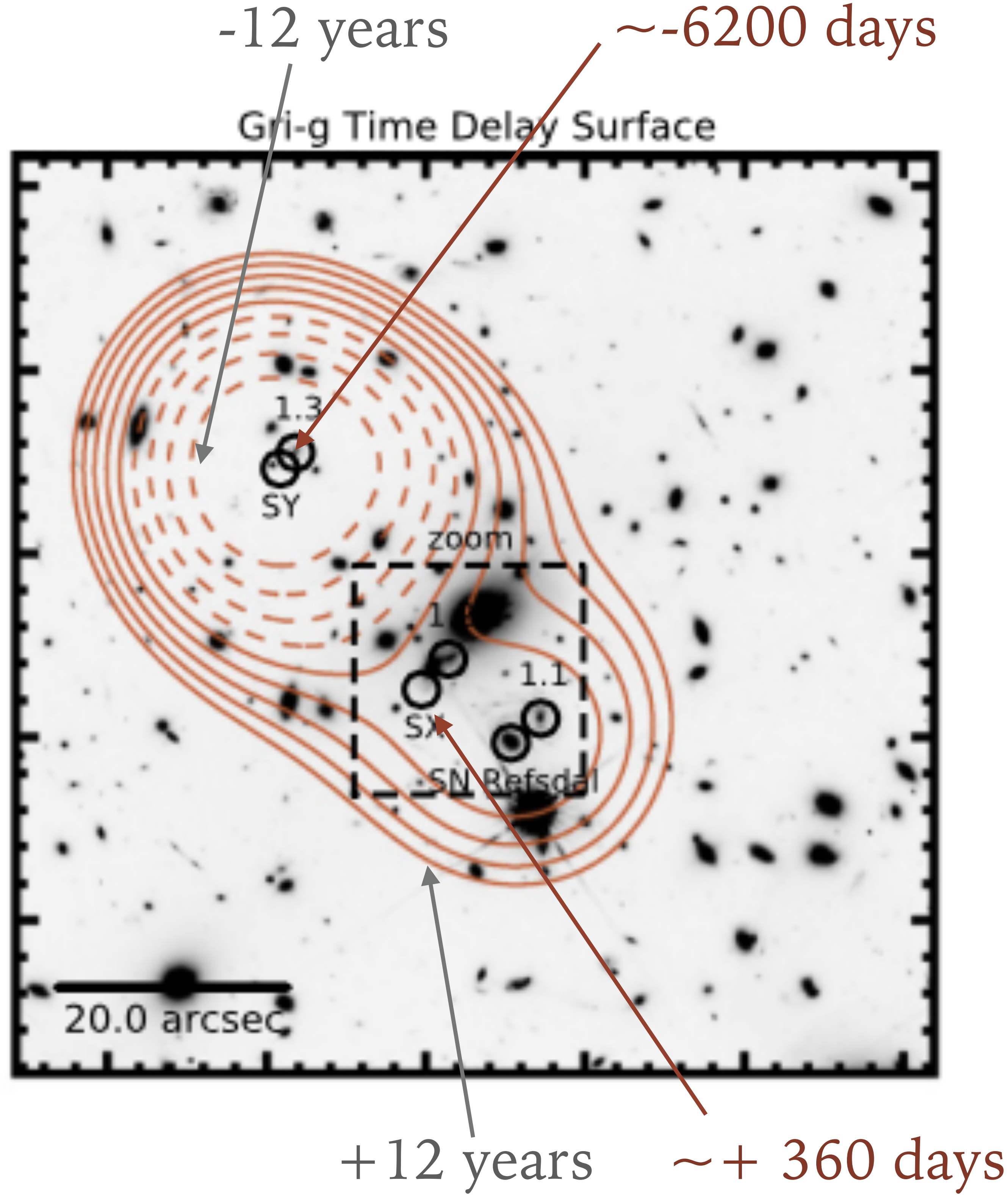

SN REFSDAL IN MACS 1149

16/12/2016…

Time delay

(SX- S1)
345+-10gg