Formula sheet - Probability Theory (FEB21005X/S)

Discrete Distributions

Binomial Distribution	Notation: $X \sim \mathrm{BIN}\,(n\,,p)$
$p_X(x;n,p) = \begin{cases} \binom{n}{x} p^x(1-p)^{n-x} & \text{for } x = 0,1,2,\dots \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}\,(X) = np$ $\mathrm{Var}\,(X) = np(1-p)$
$M_X(t) = (\,pe^t + (1-p)\,)^n$	$n \in \mathbb{N}, \quad 0 < p < 1$

Geometric Distribution	Notation: $X \sim \mathrm{GEO}\,(p)$
$p_X(x;p) = \begin{cases} p(1-p)^{x-1} & \text{for } x = 1,2,\dots \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}\,(X) = \frac{1}{p}$ $\mathrm{Var}\,(X) = \frac{(1-p)}{p^2}$
$M_X(t) = \dfrac{pe^t}{1-(1-p)e^t}$	$0 < p < 1$
Note: $\mathrm{GEO}\,(p) = \mathrm{NB}\,(r = 1\,, p)$	

Negative Binomial Distribution	Notation: $X \sim \mathrm{NB}\,(r\,,p)$
$p_X(x;r,p) = \begin{cases} \binom{x-1}{r-1} p^r(1-p)^{x-r} & \text{for } x = r, r+1,\dots \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}\,(X) = \frac{r}{p}$ $\mathrm{Var}\,(X) = \frac{r(1-p)}{p^2}$
$M_X(t) = \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r$	$r \in \mathbb{N}$ $0 < p < 1$
Note: $\mathrm{NB}\,(1\,,p) = \mathrm{GEO}\,(p)$	

Poisson Distribution	Notation: $X \sim \mathrm{POI}\,(\lambda)$
$p_X(x;\lambda) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^x}{x!} & \text{for } x = 0,1,2,\dots \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}\,(X) = \lambda$ $\mathrm{Var}\,(X) = \lambda$
$M_X(t) = \exp\{\lambda(e^t - 1)\}$	parameter: $\lambda > 0$

Multinomial distribution: Let $\mathbf{p} = (p_1, \ldots, p_k)$ be a k-dimensional vector, with $p_1, \ldots, p_k$ nonnegative numbers satisfying $p_1 + \ldots + p_k = 1$. A k-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_k)$ with a $\mathrm{MULT}_k\,(n\,,\mathbf{p})$ distribution has joint probability mass function

$$p_{\mathbf{X}}(x_1, \ldots, x_k; n, \mathbf{p}) = \begin{cases} \dfrac{n!}{x_1! \cdots x_p!}\, p_1^{x_1} \cdots p_k^{x_k} & \text{if } (x_1, \ldots, x_k) \in S_{k,n} \\ 0 & \text{otherwise} \end{cases}$$

with $S_{k,n} = \left\{ \mathbf{X} \in \mathbb{N}^k : x_1 + \ldots + x_k = n \right\}$
$\mathrm{E}\,(X_i) = np_i$, $\mathrm{Var}\,(X_i) = np_i(1 - p_i)$ and $\mathrm{Cov}\,(X_i\,,\,X_j) = -np_ip_j$ for $i \neq j$

Continuous Distributions

Uniform Distribution	Notation: $X \sim \mathrm{UNIF}\,(a\,,b)$
$f_X(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{otherwise} \end{cases}$ $F_X(x; a, b) = \begin{cases} 0 & \text{for } x \leq a \\ \dfrac{x-a}{b-a} & \text{for } a < x < b \\ 1 & \text{for } x \geq b \end{cases}$	$\mathrm{E}\,(X) = \frac{a+b}{2}$ $\mathrm{Var}\,(X) = \frac{(b-a)^2}{12}$
$M_X(t) = \dfrac{e^{bt} - e^{at}}{t(b-a)}$ which is not defined for $t = 0$	$a, b \in \mathbb{R}$ $a < b$
Note: $\mathrm{UNIF}\,(0\,,1) = \mathrm{BETA}\,(\alpha = 1\,,\beta = 1)$	

Exponential Distribution	Notation: $X \sim \mathrm{EXP}\,(\theta)$
$f_X(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$ $F_X(x; \theta) = \begin{cases} 1 - e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}\,(X) = \theta$ $\mathrm{Var}\,(X) = \theta^2$
$M_X(t) = (1 - t\theta)^{-1}$ for $t < \frac{1}{\theta}$	scale parameter: $\theta > 0$
Note: $\mathrm{EXP}\,(\theta) = \mathrm{GAM}\,(\theta\,,\kappa = 1)$	

Weibull Distribution	Notation: $X \sim \mathrm{WEI}(\theta\,,\beta)$
$f_X(x;\theta,\beta) = \begin{cases} \dfrac{\beta}{\theta^\beta}\, x^{\beta-1} e^{-(x/\theta)^\beta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}(X) = \theta\,\Gamma(1+\frac{1}{\beta})$ $\mathrm{Var}(X) = \theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right]$
$F_X(x;\theta,\beta) = \begin{cases} 1 - e^{-(x/\theta)^\beta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$	shape parameter: $\beta > 0$ scale parameter: $\theta > 0$
Note: $\mathrm{WEI}(\theta\,,\beta = 1) = \mathrm{EXP}(\theta)$	

Gamma function: for every $\kappa > 0$ the Gamma function is defined by

$$\Gamma(\kappa) = \int_0^\infty x^{\kappa-1} e^{-x}\, dx$$

Properties:

- $\Gamma(1) = 1$
- if $\kappa > 1$ then $\Gamma(\kappa) = (\kappa - 1)\cdot\Gamma(\kappa - 1)$
- if $\kappa \in \mathbb{N} \setminus \{0\}$ then $\Gamma(\kappa) = (\kappa - 1)!$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Gamma Distribution	Notation: $X \sim \mathrm{GAM}(\theta\,,\kappa)$
$f_X(x;\theta,\kappa) = \begin{cases} \dfrac{1}{\theta^\kappa \cdot \Gamma(\kappa)}\, x^{\kappa-1} e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}(X) = \kappa\theta$ $\mathrm{Var}(X) = \kappa\theta^2$
$M_X(t) = (1 - t\theta)^{-\kappa}$ for $t < \frac{1}{\theta}$	shape parameter: $\kappa > 0$ scale parameter: $\theta > 0$
Note: $\mathrm{GAM}(\theta = 2\,,\kappa) = \chi^2(2\kappa)$ and $\mathrm{GAM}(\theta\,,\kappa = 1) = \mathrm{EXP}(\theta)$	

Chi-Square Distribution	Notation: $X \sim \chi^2(\nu)$
$f_X(x;\nu) = \begin{cases} \dfrac{1}{2^{\nu/2} \cdot \Gamma(\nu/2)}\, x^{\nu/2-1} e^{-x/2} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}(X) = \nu$ $\mathrm{Var}(X) = 2\nu$
$M_X(t) = (1 - 2t)^{-\nu/2}$ for $t < \frac{1}{2}$	degrees of freedom: $\nu > 0$
Note: $\chi^2(\nu) = \mathrm{GAM}(\theta = 2\,,\kappa = \nu/2)$	

Normal Distribution	Notation: $X \sim \mathrm{N}(\mu, \sigma^2)$
$f_X(x;\mu,\sigma) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right\}$	$\mathrm{E}(X) = \mu$ $\mathrm{Var}(X) = \sigma^2$
$M_X(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}$	$\sigma^2 > 0$

Beta Distribution	Notation: $X \sim \mathrm{BETA}(\alpha, \beta)$
$f_X(x;\alpha,\beta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{E}(X) = \dfrac{\alpha}{\alpha+\beta}$ $\mathrm{Var}(X) = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
	shape parameter: $\alpha > 0$ shape parameter: $\beta > 0$
Note: $\mathrm{BETA}(\alpha = 1, \beta = 1) = \mathrm{UNIF}(0, 1)$	

Student t Distribution	Notation: $X \sim \mathrm{t}(\nu)$
$f_X(x;\nu) = \dfrac{1}{\sqrt{\nu\pi}} \cdot \dfrac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)} \left(1 + \dfrac{x^2}{\nu}\right)^{-(\nu+1)/2}$	$\mathrm{E}(X) = 0$ (if $\nu > 1$) $\mathrm{Var}(X) = \dfrac{\nu}{\nu-2}$ (if $\nu > 2$)
$M_X(t)$ does not exist	degrees of freedom: ν