# Math Template

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We pose the following question: Of all subsets $U \subset H = \{\pm 1\}^n$ with $|U| = 2^{n-1}$ containing no antipodal points, which maximizes

$$\left\| \sum_{x \in U} x \right\|?$$

We answer this question with the following theorem:

**Theorem 1.** *The best $U$ as above is just $U = \{x \in H : x_1 = 1\}$.*

The proof is as follows. Recall the structure theorem for the $m \times n$ matrices:

**Theorem 2** (Structure Theorem)**.** *Let $A$ be the matrix maximizing*

$$\beta(A) = \sum_{x \in D} \|Ax\|_\infty$$

*Define $W_i = \left\{x \in H \;\middle|\; \|Ax\|_\infty = |a_i^T x|\right\}$ to be the vertices row $a_i^T$ is useful to and $V_i = \left\{x \in W \;\middle|\; a_i^T x = |a_i^T x|\right\}$ as the positive half of $W_i$. Then,*

$$a_i = \sum_{x \in V_i} x$$

And our answer to the question for the $1 \times n$ case:

**Theorem 3.** *The optimal* $1 \times n$ *matrix is just* $(1, 0, \ldots, 0)$.

We now prove Theorem 1.

*Proof of Theorem 1.* I claim that finding the optimal $1 \times n$ matrix is equivalent to the above problem. With $u = \sum_{x \in U} x$, I claim that we can reduce our search space to only those $U$ such that $u^T x \geq 0$ for every $x \in U$. This follows because if there is an $x \in U$ with $u^T x < 0$, then $x$ is pointing away from the average direction of the rest of the points in $U$, so swapping $x$ with $-x$ will make the sum become larger (This needs more justification). Now,

$$\beta(u) = \sum_{x \in \{\pm 1\}^n} |u^T x| = \sum_{x \in U} u^T x - \sum_{x \in -U} u^T x = 2 \sum_{x \in U} u^T x = \sum_{x \in U} \sum_{y \in U} y^T x = \left\| \sum_{x \in U} x \right\|^2$$

By the structure theorem, finding the optimal matrix $A \in \mathbb{R}^{m \times n}$ is equivalent to finding first the optimal partition of $H$ into $m$ parts of even size, and then finding the optimal positive half for each partition. For the $1 \times n$ case, the optimal partition is just all of $H$, there is no choice. Then Theorem 3 tells us that the optimal positive half is just the positive half of $H$ associated to $(1, \ldots, 0)$, which is seen to be $\{x \in H : x_1 = 1\}$. This completes the proof. $\square$