Math 55a Midterm Exam

Sept. 28-Oct. 2, 2020 – due by end of day (US Eastern) on Friday October 2.

No collaboration allowed; no materials other than lecture notes, Artin, and Axler.

Directions

1. This take-home exam is **due by Friday October 2** (end of the day), online on Canvas. You are welcome to write your answers on paper and upload a scan or photos; LaTeX strictly optional. The exam is expected to take at most 2 hours for most of you; you are encouraged to take it in one sitting and submit as soon as done, but this is not mandatory.

2. In your solutions, you may refer to any of the results seen in class or in the relevant parts of the book. (You do not need to cite the exact source unless it's an obscure fact). Please show your reasoning (enough so that I can tell how you arrived at the answer). The material covered is what we've seen in class up to Lecture 10 (September 25) included; equivalently: Artin through section 4.4, Axler through chapter 5 (included).

3. **You may not use any external sources on this exam other than the course materials (class notes, Artin, and Axler).** You may not use any other textbooks or external websites. You may not discuss the contents of this midterm with anyone, including in office hours (exception: check with Prof. Auroux via e-mail if a clarification is necessary) until after the due date, *even if you have turned it in.*

4. **On your exam, please include the following statement, with your signature.** (If you are typing the exam, no need to print and sign: copy the statement and type your name.)

"I affirm my awareness of the standards of the Harvard College Honor Code. While completing this exam, I have not consulted any external sources other than class notes and the textbooks (Artin and Axler). I have not discussed the problems or solutions of this exam with anyone, and will not discuss them until after the due date."

Signed: ________________

There are **six problems**: three about groups and three about linear algebra. For multi-part problems, each question can be attempted independently (but you may need to use the results of the previous parts).

Problem 1. (4 points) Let G be a finite abelian group. Show that the subset of G consisting of those elements which have odd order is a subgroup.

Problem 2. (6 points) Give examples of the following subgroups in the symmetric group S_5, or explain why none exists.

(a) a cyclic subgroup of order 6;

(b) a non-cyclic subgroup of order 6;

(c) a subgroup of order 7;

(d) a subgroup of order 8.

(continued on next page)

Problem 3. (6 points) Let G be a group (not necessarily abelian), and assume that there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow G \xrightarrow{\varphi} \mathbb{Z}/k \longrightarrow 0,$$

i.e. there exists a surjective group homomorphism $\varphi : G \to \mathbb{Z}/k$ with $\operatorname{Ker}\varphi \simeq \mathbb{Z}/2$.

(a) Show that the nontrivial element of $\operatorname{Ker}\varphi$ is in the center $Z(G)$, i.e. commutes with every element of G.

(b) Show that, if $g \in G$ is mapped by φ to the generator $1 \in \mathbb{Z}/k$, then the order of g in G is either k or $2k$.

(c) Assume k is odd. Show that G is isomorphic to the cyclic group $\mathbb{Z}/2k$. (Hint: use (a) and (b)).

(d) Does the same conclusion ($G \simeq \mathbb{Z}/2k$) hold if k is even? (give a proof or a counterexample).

Problem 4. (4 points) Let $\phi : V \to W$ be a linear map of vector spaces over a field k. Show that if $\operatorname{Ker}(\phi)$ and $\operatorname{Im}(\phi)$ are both finite-dimensional then V is finite-dimensional.

Problem 5. (4 points) Let V be a finite-dimensional vector space over the field k, and $f, g : V \to k$ two linear maps. Show that if $\operatorname{Ker}(f) \subset \operatorname{Ker}(g)$ then there exists $\lambda \in k$ such that $g = \lambda\, f$.

Problem 6. (6 points) Let V be an n-dimensional vector space over a field k, and $f, g : V \to V$ two linear operators such that $f \circ g - g \circ f = g$.

(a) Show that g maps eigenvectors of f to eigenvectors of f (or to the zero vector).

(b) Show that, if $k = \mathbb{C}$, then g cannot be invertible.

(c) Give an example showing that, if k is a finite field, then g can be invertible.