

$$\boxed{b)} \quad N((a, u)(b, v)) = N(ab - u \cdot v, av + bu + u \times v)$$

$$= (ab - u \cdot v)^2 + (av + bu + u \times v) \cdot (av + bu + u \times v)$$

$$= a^2 b^2 + \underbrace{(u \cdot v)^2}_{-2ab(u \cdot v)} + a^2 |v|^2 + b^2 |u|^2 + \underbrace{(u \times v)^2}_{+2av \cdot (u \times v)} + 2bu \cdot (u \times v) + 2ab \cancel{u \cdot v}$$

(orthogonal)

$$\left(\begin{aligned} (u \cdot v)^2 &= |u|^2 |v|^2 \cos^2 \Theta \\ + |u \times v|^2 &= |u|^2 |v|^2 \sin^2 \Theta \end{aligned} \right)$$

$$= |u|^2 |v|^2$$

$$= a^2 b^2 + |u|^2 |v|^2 + a^2 |v|^2 + b^2 |u|^2$$

$$N((a, u)) \cdot N((b, v)) = (a^2 + |u|^2)(b^2 + |v|^2)$$

$$= a^2 b^2 + a^2 |v|^2 + b^2 |u|^2 + |u|^2 |v|^2 \quad \checkmark$$

$$\boxed{a)} ((a, u)(b, v))(c, w) = (ab - u \cdot v, av + bu + uxv)(c, w)$$

$$= (cab - cu \cdot v - w \cdot (av + bu + uxv), \\ (ab - u \cdot v)w + c(av + bu + uxv) \\ + (av + bu + uxv)xw)$$

$$= \left(\underline{cab} - \underline{cu \cdot v} - \underline{a \cdot \vec{v} \cdot \vec{w}} - \underline{b \cdot \vec{u} \cdot \vec{w}} - \underline{\vec{w} \cdot (\vec{u} \times \vec{v})}, \right. \\ \underline{ab \vec{w}} - \underline{(\vec{u} \cdot \vec{v}) \vec{w}} + \underline{c a \vec{v}} + \underline{c b \vec{u}} + \underline{c \vec{u} \times \vec{v}} \\ \left. + \underline{a \vec{v} \times \vec{w}} + \underline{b \vec{u} \times \vec{w}} + \underline{(\vec{u} \times \vec{v}) \times \vec{w}} \right)$$

vs. :

$$(a, u)(bc - v \cdot w, bw + cv + \vec{v} \times \vec{w}) \\ = (\underline{abc} - \underline{a \vec{v} \cdot \vec{w}} - \underline{\vec{u} \cdot (b \vec{w} + \vec{v} + \vec{v} \times \vec{w})}, \\ \underline{a b \vec{w}} + \underline{a c \vec{v}} + \underline{a \vec{v} \times \vec{w}} + \underline{b c \vec{u}} - \underline{(\vec{v} \cdot \vec{w}) \vec{u}} \\ + \underline{b \vec{u} \times \vec{w}} + \underline{c \vec{u} \times \vec{v}} + \underline{\vec{u} \times (\vec{v} \times \vec{w})})$$

Green: $\vec{u} \cdot (\vec{v} \times \vec{w}) \stackrel{?}{=} \vec{w} \cdot (\vec{u} \times \vec{v}) \checkmark$
(Standard identity)

Now: Does $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{v} \cdot \vec{w}) \vec{u}$
 $\stackrel{?}{=} (\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{v}) \vec{w}$

Recall: $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w}$

and: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

So: $(\vec{u} \times \vec{v}) \times \vec{w} = -\vec{w} \times (\vec{u} \times \vec{v})$

$= -((\vec{w} \cdot \vec{v}) \vec{u} - (\vec{w} \cdot \vec{u}) \vec{v})$

$= (\vec{w} \cdot \vec{u}) \vec{v} - (\vec{w} \cdot \vec{v}) \vec{u}$

So yes, associative.

□