Math Template

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1. If $f(x,y) = \langle x, y\rangle$, define $A_{i,j} = \langle e_i, e_j\rangle$. I then claim that $f(x,y) = x^T A y$. Writing $x = \sum x_i e_i$ and $y = \sum y_j e_j$, we know that $\langle x, y\rangle = \sum_{i,j} x_i y_j \langle e_i, e_j\rangle = \sum_{i,j} x_i y_j A_{i,j} = x^T A y$ by definition.

2. The data is as follows:

	Smartphones Sold	**EV Sales**	**Social Media Users**
Smartphones Sold	1.000000	-0.548806	-0.297813
EV Sales	-0.548806	1.000000	0.888376
Social Media Users	-0.297813	0.888376	1.000000

Table 1: Correlation Matrix for Smartphones Sold, EV Sales, and Social Media Users.

   Look at that. 0.88 correlation with social media useres with EV sales. That is super surprising.

3. If there were a magic inner product $\langle x, y\rangle$ so that $\langle x, x\rangle_M^{1/2} = ||x||_{\ell^3}$, then it would have to be linear, in particular. By some formula whose name I cannot recall:

$$4\langle x, y\rangle = ||x+y||_{\ell^3}^2 - ||x-y||_{\ell^3}^2.$$

   Then in particular, the following should hold:

$$\begin{aligned} 4\langle (1,1),(1,2)\rangle &= (2^3+3^3)^{2/3} - (1^3)^{2/3} \approx 9.6998 \\ &= 4\langle (1,1),(1,0)\rangle + 4\langle (1,1),(0,2)\rangle \\ &= (2^3+1^3)^{2/3} - (1^3)^{2/3} + (1^3+3^3)^{2/3} - (1^3+1^3)^{2/3} \approx 10.96 \end{aligned}$$

   This is a contradiction. For higher dimension values, we can just use these same vectors with padded 0s to get the same analysis.

4. For any configuration of 3 points on the sphere, draw the great circle through them (we may assume they are distinct). Since this is a great circle between 3 points on a sphere of radius 1, the great circle has a radius at most 1. However, if the optimal configuration of the points were to lie on a great circle of radius < 1, we could have that same configuration with the great circle on the x-axis, and this would scale the distances up by a constant factor, making the energy smaller.

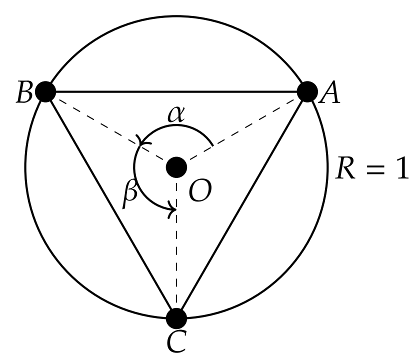
   Thus, the optimal point configuration lies on a circle of radius 1. Now there is no need for 3d, we may assume we have put 3 points on the unit circle in $\mathbb{R}^2$. Let the points be named A, B, C and a, b, c the corresponding sidelengths. Then by AM-GM,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3}{\sqrt[3]{abc}}$$

   By circumradius formula, $4R\text{Area}(ABC) = abc$ where R is the circumradius of the triangle formed by ABC. Thus, $abc = 4\text{Area}(ABC)$. This tells us that:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3}{\sqrt[3]{4\text{Area}(ABC)}}$$

   Now we answer the following: among all triangles with circumradius 1, which has the largest area? Draw this triangle on a circle, and label the center of the circle O. Connecting all three points to this middle point, we get three triangles.



   Call the angle between OA and OB α, and the angle between OB and OC β. By the area formula involving sines, $\text{Area}(ABC) = \frac{1}{2}(\sin(\alpha) + \sin(\beta) + \sin(2\pi - \alpha + \beta))$. Now we use Jensen's inequality. $f(x) = \sin(x)$ is concave on $[0, \pi]$, and therefore:

$$\frac{\sin(\alpha) + \sin(\beta) + \sin(2\pi - \alpha + \beta)}{3} \leq \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Thus, Area(ABC) $\leq \frac{3\sqrt{3}}{4}$, with equality being attained when $\alpha = \beta = \frac{2\pi}{3}$.
Plugging this back in shows that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3}{\sqrt[3]{4 \cdot (3\sqrt{3}/4)}} = \sqrt{3}.$$

with equality iff the triangle is equilateral, i.e. $a = b = c$. It is then easy to see that the points $(1, 0)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ sate this bound.