Math 441 HW2

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1. We recall that $\mathcal{B}$ is a basis of a topology $\mathcal{T}$ if every open $U \in \mathcal{T}$ can be written of the form $\bigcup_{\alpha \in I} B_\alpha$ with $B_\alpha \in \mathcal{B}$. Since $X \in \mathcal{T}$, write $X = \bigcup_{\alpha \in I} B_\alpha$. Now given any $x \in X$, we must have $x \in B_\alpha$ for some $\alpha \in I$, which completes (B1). One notices that B_1 and B_2 are unions of elements in $\mathcal{B}$ (namely, themselves), so they are both open. By the definition of a topology, $B_1 \cap B_2$ is open, and since $\mathcal{B}$ is a basis, we can write $B_1 \cap B_2 = \bigcup_{\alpha \in I} C_\alpha$ with $C_\alpha \in \mathcal{B}$. Now given any $x \in B_1 \cap B_2$, $x \in \bigcup_{\alpha \in I} C_\alpha$, which says that $x \in C_\alpha$ for some $\alpha \in I$. Thus $x \in C_\alpha \subset B_1 \cap B_2$, and we are done with this direction.

   For the other direction, consider $\mathcal{T}$, the set of all subsets of X of the form $\bigcup_{\alpha \in I} B_\alpha$ with $B_\alpha \in \mathcal{B}$ and I any indexing set along with the empty set. Given any $x \in X$ there exists some B_x so that $x \in B_x$. Since $B_x \subset X$, $X = \bigcup_{x \in X} B_x$, which shows that $X \in \mathcal{T}$. By construction of $\mathcal{T}$ the empty set is in $\mathcal{T}$. Given any $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$, WLOG we may assume that none of the U_α's are the empty set since they contribute nothing to the union. Write $U_\alpha = \bigcup B_\alpha$ with $B_\alpha \in \mathcal{B}$. One notices now that

$$\bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} \bigcup B_\alpha$$

   which is just an arbitrary union of things in $\mathcal{B}$ and hence in $\mathcal{T}$. Finally, let $U_1, U_2 \in \mathcal{T}$, and again write $U_1 = \bigcup_{\alpha \in \Delta_1} B_\alpha$ and $U_2 = \bigcup_{\beta \in \Delta_2} B_\beta$. From elementary set theory we see that

$$U_1 \cap U_2 = \bigcup_{\substack{\alpha \in \Delta_1 \\ \beta \in \Delta_2}} \left(B_\alpha \cap B_\beta\right)$$

   Now, for any $\alpha \in \Delta_1$, $\beta \in \Delta_2$, and any $x \in B_\alpha \cap B_\beta$, we can find a $C_x \in \mathcal{B}$ with $x \in B_x$ and $B_x \subset B_\alpha \cap B_\beta$. By the union lemma we can write $B_\alpha \cap B_\beta = \bigcup_{x \in B_\alpha \cap B_\beta} B_x$. Thus,

$$\bigcup_{\substack{\alpha \in \Delta_1 \\ \beta \in \Delta_2}} \left(B_\alpha \cap B_\beta\right) = \bigcup_{\substack{\alpha \in \Delta_1 \\ \beta \in \Delta_2}} \bigcup_{x \in B_\alpha \cap B_\beta} B_x$$

Which is just an arbitrary union of things in $\mathcal{B}$. By induction this holds for any finite number of sets, and thus $\mathcal{T}$ is a topology.

2. Given any $x \in \mathbb{R}$, $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 1)$, which establishes (B1). Given any $[n, n+1)$ and $[m, m+1)$ both in $\mathcal{B}$, either $n = m$ or $n \neq m$. In the first case the intersection is just $[n, n+1)$ which is in $\mathcal{B}$, so we are done. In the second case, WLOG $n < m$, and hence $[n, n+1) \cap [m, m+1) = \emptyset$, so we are also done. Thus $\mathcal{B}$ defines a topology on $\mathbb{R}$. This topology gives a hole on the real line next to every integer.

3. (a) Let U be an open subset of $\mathbb{R}$ (with the euclidean topology), and let $x \in U$. Since U is open, we can find an $\varepsilon > 0$ so that $(x - \varepsilon, x + \varepsilon) \subset U$. By truncating the decimal expansion of x, we can find an increasing sequence of rational numbers $a_n \to x$. Thus find $N \in \mathbb{N}$ so that $x - \varepsilon/2 < a_N \leq x$. By choosing $\varepsilon/3 < 1/K < \varepsilon/2$, we see that $(a_N - 1/K, a_N + 3/K)$ is a neighborhood of x with both endpoints in $\mathbb{Q}$. By the union lemma all open sets in $\mathbb{R}$ are just the union of intervals with rational endpoints. Since intervals with rational endpoints are open, we are done.

   (b) Suppose that $[\sqrt{2}, 3)$ could be written as the union of things in the given basis. Thus $[\sqrt{2}, 3) = \bigcup [a, b)$. We see that $a \geq \sqrt{2}$ for every a, since otherwise the union would contain too many elements. Thus $a > \sqrt{2}$, since $\sqrt{2}$ isn't rational. We also see that $\sqrt{2} \in [a, b)$ for some a, b rational. But this can't be, since $a > \sqrt{2}$, a contradiction. Question 2 shows that this basis satisfies (B1). Given two sets in this collection $[a, b)$ and $[c, d)$, either these two intervals are disjoint or they are not. If they are not disjoint, $[a, b) \cap [c, d) = [\max\{a, c\}, \min\{b, d\})$, which is in the collection, so we are done.

4. Given any $(x, y) \in \mathbb{R}^2$, $(x, y) \in x \times (y - 1, y + 1)$, which shows (B1). Given any two $\{a\} \times (b, c)$ and $\{d\} \times (e, f)$, either $a = d$ or not. If not, then the sets are disjoint. If they equal, their intersection is $\{a\} \times (\max\{b, e\}, \min\{c, f\})$, which is in the vertical interval topology, hence this defines a basis for a topology. This is like being attached to a line. Everything is super close to each other on the x-axis, and and looks like a normal real line on the y-axis.

5. Notice that given any $x \in \mathbb{Z}$, $\{x\} = \mathbb{Z} \cap (x - 1/2, x + 1/2)$, hence all singletons are open. Since any subset of $\mathbb{Z}$ is just the union of singletons, all subsets of $\mathbb{Z}$ are open. Thus the subspace topology on $\mathbb{Z}$ is the discrete topology.

6. In the first case, the open sets in Z are of the form $U \cap Z$ where $U \in \mathcal{T}$. In the second case, open sets in Z are of the form $V \cap Z$ where V is of the form $Y \cap U$ where U is in $\mathcal{T}$. Thus the open sets are of the form $U \cap Y \cap Z = U \cap Z$. So the open sets are exactly of the same form, and we are done. (Equivalently, in the first case you are also equal to $U \cap Z \cap Y$, and in the second case you are equal to $U \cap Z$, as stated.)

7. The subspace topology is the collection of sets of the form $Y \cap U$, where $U \in \mathcal{T}_X$. Since the only things in $\mathcal{T}_X$ are $\emptyset$, X, the only things in $\mathcal{T}_Y$ are $Y \cap X = Y$, and $Y \cap \emptyset = \emptyset$, thus the subspace topology on Y is the indiscrete topology. One notices that the subspace topology on $\{1\} \subset \mathbb{R}$ ($\mathbb{R}$ with the euclidean topology) is the indiscrete topology (since the only subsets of $\{1\}$ are $\emptyset$, and $\{1\}$), but that the euclidean topology is of course not indiscrete.

8. Notice that if $\{U_\alpha\}_{\alpha\in\Delta} \subset \mathcal{T}$,

$$\left(\bigcup_{\alpha\in\Delta} U_\alpha\right)^C = \bigcap_{\alpha\in\Delta} U_\alpha^c$$

Since U_α^C is bounded, we can find an $M > 0$ so that $\|x\| \leq M$ for every $x \in U_\alpha^C$. By inclusion this holds for everything in the above intersection, so arbitrary intersections are in $\mathcal{T}$. Similarly, if $U_1, U_2 \in \mathcal{T}$, find M_1, M_2 so that if $x \in U_1^c$, $y \in U_2^c$, then $\|x\| \leq M_1^c$, and $\|y\| \leq U_2^c$. Thus $(U_1 \cap U_2)^c = U_1^c \cup U_2^c$ is bounded by $\max\{M_1, M_2\}$, and by induction this holds for any finite number of sets. Finally, $\emptyset \in \mathcal{T}$ by definition, and $\mathbb{R}^2 \setminus \mathbb{R}^2 = \emptyset$ is bounded vacuously, so it is also in our collection. Thus we have a topology. Since smaller open neighborhoods are those that have a huge hole in the middle, I'm going to say that it would be in the shape of an ice cream cone with a filled inside.