Math 441 HW3

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1. Notice that $(A \times B)^c = \{(x, y) \in X \times Y \mid (x, y) \notin A \times B\} = \{(x, y) \in X \times Y \mid x \notin A \vee y \notin B\} = \{(x, y) \in X \times Y \mid x \in A^c\} \cup \{(x, y) \in X \times Y \mid y \in B^c\} = (A^c \times Y) \cup (X \times B^c)$. Since A is closed A^c is open, and since B is closed B^c is open. Thus the above is the union of two open sets (by definition of the product topology) and hence open, and hence $A \times B$ is closed.

2. (a) We recall that an arbitrary intersection of closed sets is closed. Since $A \subset B \subset \overline{B}$, $\overline{B}$ is a closed set containing A, and hence $\overline{A} \subset \overline{B}$ by definition of intersection.

   (b) Notice that $A \cup B \subset \overline{A} \cup \overline{B}$, thus $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$. Notice also that if C is closed, then C will be one of the sets in the intersection of all closed sets containing C, and thus $C = \overline{C}$, hence $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ as the finite union of closed sets is closed. For the reverse direction, notice that $A \subset A \cup B$, thus by part (a) $\overline{A} \subset \overline{A \cup B}$. Similarly, $\overline{B} \subset \overline{A \cup B}$, and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (c) By the above, since $A_\alpha \in \bigcup_{\alpha \in \Delta} A_\alpha$, we have that $\overline{A_\alpha} \subset \overline{\bigcup_{\alpha \in \Delta} A_\alpha}$, and since this holds for all $\alpha \in \Delta$, we have that $\bigcup_{\alpha \in \Delta} \overline{A_\alpha} \subset \overline{\bigcup_{\alpha \in \Delta} A_\alpha}$. A clean counterexample is as follows: $[1/n, 1]$ is closed in the standard topology of $\mathbb{R}$ for all $n \in \mathbb{N}$, thus $\overline{[1/n, 1]} = [1/n, 1]$ by the above, yet

   $$\bigcup_{n \in \mathbb{N}} [1/n, 1] = (0, 1]$$

   who's closure is $[0, 1]$, which is strictly larger than the LHS.

3. Given two distinct points (x_1, y_1) and (x_2, y_2) in $X \times Y$, since X is Hausdorff there exist an open $X_1 \subset X$ with $x_1 \in X_1$ and another open $X_2 \subset X$ with $x_2 \in X_2$ and $X_1 \cap X_2 = \emptyset$. Thus $(x_1, y_1) \in X_1 \times Y$ and also $(x_2, y_2) \in X_2 \times Y$. Finally, $(X_1 \times Y) \cap (X_2 \times Y) = (X_1 \cap X_2) \times Y = \emptyset \times Y = \emptyset$, and hence we are done, as both $X_1 \times Y$ and $X_2 \times Y$ are open in the product topology. We remark that we never used the condition that Y is Hausdorff.

4. Suppose that X is Hausdorff. Note that $\Delta^c = \{(x, y) \in X \times X \mid x \neq y\}$, and take any point $(x, y) \in \Delta^c$. By the Hausdorff condition there exists a $U_1 \subset X$ so that $x \in U_1$ and $U_2 \subset X$ so that $y \in U_2$ with $U_1 \cap U_2 = \emptyset$. I claim that $U_1 \times U_2 \subset \Delta^c$. Indeed, the condition $U_1 \cap U_2 = \emptyset$

shows that the x and y values of this product are never equal. By the union lemma Δ^c is open.

For the reverse direction, suppose that Δ^c is open, and let $x \neq y$ both in X. Since the product topology has basis

$$\mathcal{B} = \{U \times V \mid U \subset X \text{ open}, \ V \subset X \text{ open}\}$$

we may write $\Delta^c = \bigcup U_\alpha \times V_\alpha$. Since $(x, y) \in \Delta^c$, $(x, y) \in U \times V$ for some $U, \ V$ both open in X. Notice that $U \cap V = \emptyset$, since if there was some point $x \in U \cap V$, then $(x, x) \in U \times V$, which would say $(x, x) \in \Delta^c$, which can't be. Thus X is Hausdorff.

5. (a) Suppose there was an $a \in A^\circ \cap \partial A$. Since A° is open, we have found an open neighborhood of a fully contained in A, which contradicts the fact that a is a boundary point of A. Thus $A^\circ \cap \partial A = \emptyset$. Second, we recall that $\overline{A} = A \cup \partial A$. I claim that $A^\circ \cup \partial A = A \cup \partial A$. The forward direction is clear. Given any $a \in A \cup \partial A$, either $a \in A$, or $a \in \partial A$. The second case is trivial. Now, either there is an open neighborhood U of a so that $U \subset A$, or every open neighborhood U of a intersects A^c. In the first case, U is an open neighborhood contained in A, thus $U \subset A^\circ$ by definition. The second case states precisely that a is a boundary point of A (Note: any open neighborhood U of a intersects A since $a \in A$), thus $a \in \partial A$, and we are done.

   (b) I showed above $A = \overline{A}$ iff A is closed, and similarly, if A is open, then A is an open set contained in A, thus $A^\circ \supset A$. The reverse direction is by definition, hence $A = A^\circ$. Thus $A = A^\circ$ iff A is open (again, the interior is clearly open). If $\partial A = \emptyset$, then $\overline{A} = A^\circ$ by part (a). Since $A^\circ \subset A \subset \overline{A} = A^\circ$, we have $A = A^\circ = \overline{A}$, which shows that A is clopen. For the reverse direction, note that if A is clopen then $A = \overline{A} = A^\circ$, and hence $A^\circ = \overline{A} = A^\circ \cup \partial A$. Thus, $\emptyset = A^\circ \cap \partial A = (A^\circ \cup \partial A) \cap \partial A = (A^\circ \cap \partial A) \cup (\partial A \cap \partial A) = \emptyset \cup \partial A = \partial A$.

   (c) Since A is open, by the above $\overline{A} = A^\circ \cup \partial A = A \cup \partial A$. Since $A^\circ \cap \partial A = A \cap \partial A = \emptyset$, we have that $\partial A \subset A^c$, thus $(A \cup \partial A) \setminus A = (A \cup \partial A) \cap A^c = (A \cap A^c) \cup (\partial A \cap A^c) = \partial A \cap A^c = \partial A$, thus $\overline{A} \setminus A = \partial A$. For the reverse direction, if $\partial A = \overline{A} \setminus A$, let $x \in A$. By construction $x \notin \partial A$. This says that there is a neighborhood U of x so that $V \cap A = \emptyset$ or $U \cap A^c = \emptyset$. The first condition is obviously false, since $x \in A$, thus $U \cap A^c = \emptyset$, or $U \subset A$. By the union lemma A is open, and we are done.

   (d) This is not true. Consider $X = \{1, 2\}$ with the topology $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}\}$. It suffices to prove finite intersection on two sets. So let $U, V \in \mathcal{T}$. If either is empty, their intersection is empty. If neither are empty, they are either both $\{1\}$, both $\{1, 2\}$, or one is $\{1\}$ and the other is $\{1, 2\}$. In the first and last case the intersection is $\{1\}$, and in the middle case the intersection is $\{1, 2\}$. For arbitrary union, either one of the sets is $\{1, 2\}$, or all are not $\{1, 2\}$. In the first case the union is $\{1, 2\}$, in the second case, if all sets are the empty set, the union is the empty set. Else, at least one is $\{1\}$, thus the union is $\{1\}$, since it can't be any bigger by the last sentence. Now, the only closed sets are $\{\emptyset, \{2\}, \{1, 2\}\}$. The only one of those containing $\{1\}$ is $\{1, 2\}$, thus $\overline{\{1\}} = \{1, 2\}$. Since $\{1, 2\}$ is open, $\{1, 2\}^\circ = \{1, 2\}$. This is strictly bigger than $\{1\}$, so we have disproven the claim.

6. The constant function 1 is never 0, thus $V(1) = \emptyset$. The constant function 0 is always 0, so $V(0) = \mathbb{R}^n$. If $x \in V(f) \cap V(g)$, then $f(x) = 0$ and $f(g) = 0$. Thus $x \in V(f, g)$. If $x \in V(f, g)$, then $f(x) = 0$ and $g(x) = 0$, thus $x \in V(f)$ and $x \in V(g)$, hence $x \in V(f) \cap V(g)$. Finally, if $x \in V(f) \cup V(g)$, then $f(x) = 0$ or $g(x) = 0$. WLOG the first case is true, thus $(f \cdot g)(x) = f(x)g(x) = 0 \cdot g(x) = 0$. If $(f \cdot g)(x) = 0$, then since $\mathbb{R}$ is an integral domain $f(x) = 0$ or $g(x) = 0$. Thus $x \in V(f) \cup V(g)$.

7. One recalls that nonconstant polynomials of finite degree in one variable have only finitely many roots. Thus, for any polynomial $f \in \mathbb{R}[x]$, we have three cases: $V(f) = \mathbb{R}$, $V(f) = \emptyset$, or $V(f)$ is a finite set. I claim that $V(x^2 + y^2 - 1)$, the circle, is not closed in $\mathbb{R} \times \mathbb{R}$. First we shall show that $V(x^2 + y^2 - 1) \not\subset V(f) \times V(g)$, unless $V(f) = V(g) = \mathbb{R}$. WLOG $V(f) \neq \mathbb{R}$. If $V(f) = \emptyset$, we are done, since the product of the empty set with anything is empty. Else, $V(f)$ is finite. The circle has uncountably many points at different x-values, thus will most certainly have a point with x value not in $V(f)$. Now, suppose that $V(x^2 + y^2 - 1) = \bigcap_{f_\alpha, g_\alpha \in \Delta} V(f_\alpha) \times V(g_\alpha)$. If all $V(f_\alpha) \times V(g_\alpha) = \mathbb{R}^2$, we certainly don't have equality, so there exists one $V(f) \times V(g) \neq \mathbb{R}^2$. This would say that $V(x^2 + y^2 - 1) \subset V(f) \times V(g)$, with not both $\mathbb{R}$, which we proved above impossible. In the general case where we have $V(T) \times V(G)$, we could indeed run the same argument and say there must be one that is not all of $\mathbb{R}^2$, which then $V(x^2 + y^2 - 1) \subset V(T) \times V(G) \subset V(f) \times V(g)$ for some $f \in T$ and $g \in G$ not both equivalently 0 but we proved that false. Notice that $V(T) \cup V(G) = \bigcap_{f \in T} V(f) \cup \bigcap_{g \in G} V(g) = \bigcap_{f \in T, g \in G} V(f) \cup V(g) = \bigcap_{f \in T, g \in G} V(f) \cup V(g) = \bigcap_{f \in T, g \in G} V(fg)$, and if we let $U = \{fg \mid f \in T, g \in G\}$, we would have this union equal to $V(fg)$. Thus finite union may be expressed as another $V(T)$. So if we had $V(x^2 + y^2 - 1) =$ finite union, we would have it equal to $V(T)$, which we showed above impossible (from the arbitrary intersection).