Math 521 HW1

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1. If $X_n \to X$ a.s. then $\mathbb{P}(\{\omega \mid X_n(\omega) \to X(\omega)\}) = 1$. Now, since f is continuous, we know that if $X_n(\omega) \to X(\omega)$, then we must have $f(X_n(\omega)) \to f(X(\omega))$. This is because if we set $y_n = X_n(\omega)$ and $y = X(\omega)$, then $y_n \to y$, and by the definition of sequential continuity, we must have that $f(y_n) \to f(y)$. Thus, we know that

$$\{\omega \bigm| f(X_n(\omega)) \to f(X(\omega))\} \supseteq \{\omega \mid X_n(\omega) \to X(\omega)\}$$

Which shows that

$$1 \geq \mathbb{P}\Big(\{\omega \bigm| f(X_n(\omega)) \to f(X(\omega))\}\Big) \geq \mathbb{P}(\{\omega \mid X_n(\omega) \to X(\omega)\}) = 1$$

So indeed, we have that $f(X_n) \to f(X)$ a.s.

2. We shall show that if $\omega \in \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} E_n$, then $\liminf_{n\to\infty} \mathbb{1}_{E_n}(\omega) = 1$ and 0 otherwise. Let $\omega \in \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} E_n$. Then there is some N so that $\omega \in \bigcap_{n \geq N} E_n$. This means that $\mathbb{1}_{E_n}(\omega) = 1$ for all $n \geq N$, and hence $\inf_{n \geq m} \mathbb{1}_{E_n}(\omega) = 1$ for any $m \geq N$. This means that $\liminf_{n\to\infty} \mathbb{1}_{E_n}(\omega) = 1$. Conversely, if $\omega \notin \{E_n \text{ ev.}\}$, then for every $m \geq 1$ we can find some $n \geq m$ so that $\omega \notin E_n$. This means that $\mathbb{1}_{E_n}(\omega) = 0$. This also means that $\inf_{n \geq m} \mathbb{1}_{E_n}(\omega) = 0$ for all $m \geq 1$, and hence $\liminf_{n\to\infty} \mathbb{1}_{E_n}(\omega) = 0$. This completes the proof.

3. Fix $\varepsilon > 0$. For each ω with $X_n(\omega) \to X(\omega)$, there exists some ω_n so that if $N \geq \omega_n$ then $|X_N(\omega) - X(\omega)| < \varepsilon$. Let $A_n = \{\omega : X_n(\omega) \to X(\omega),\ \omega_n \leq n\}$. Immediately we have that $A_n \uparrow \{\omega : X_n(\omega) \to X(\omega)\}$. Also, if we let $B_n = \{\omega : \mid X_n(\omega) - X(\omega)| < \varepsilon\}$, we have that $A_n \subset B_n$. Thus, we have that:

$$\mathbb{P}(A_n) \leq \mathbb{P}(B_n)$$

And so,

$$1 = \mathbb{P}(\{\omega \mid X_n(\omega) \to X(\omega)\}) = \lim_{n\to\infty} \mathbb{P}(A_n) \leq \lim_{n\to\infty} \mathbb{P}(B_n) \leq 1$$

Thus $X_n \to X$ in probability too.

4. Suppose per the contrary that D was uncountable. I claim that $D \cap [a, a+1)$ is uncountable for some a. This is because if $D \cap [a, a+1)$ were always countable, then $D = \bigcup_{a\in\mathbb{Z}} D \cap [a, a+1)$ would be a countable union of countable sets and therefore itself countable, a contradiction. So assume WLOG that $D \cap [0, 1)$ were uncountable. We shall prove that $f(1)$ cannot possibly be finite. Define for each point of discontinuity a the following:

$$l_a = \inf_{x<a} [f(a) - f(x)] \quad \text{and} \quad r_a = \inf_{x>a} [f(x) - f(a)]$$

Both of these are finite being bounded below by 0. If $l_a = r_a = 0$, then f is continuous at a. Otherwise, we have $l_a > 0$ or $r_a > 0$. Define $j_a = \max(l_a, r_a)$ to be the size of the jump at a.

We shall show that there is an $\varepsilon > 0$ so that uncountably many of the j_a are greater than ε. Otherwise, define $A_n = \left\{a : j_a \geq \frac{1}{n}\right\}$. Then $D = \bigcup_{n=1}^{\infty} A_n$ is a countable union of countable sets, and therefore countable, a contradiction.

Suppose that $f(1) = M$, and assume WLOG that $f(0) = 0$. Choose N so large so that $N \cdot \varepsilon > M$. Choose N elements from $D \cap (0, 1)$ with $j_a > \varepsilon$ and put them in increasing order, $a_1, a_2, \ldots, a_N$. Now writing,

$$\begin{aligned} f(1) &= f(1) - f(a_N) + \sum_{i=2}^{N} \left[f(a_i) - f\left(\frac{a_i + a_{i-1}}{2}\right) + f\left(\frac{a_i + a_{i-1}}{2}\right) - f(a_{i-1}) \right] + f(a_1) - f(0) \\ &\geq r_{a_N} + \sum_{i=2}^{N} l_{a_i} + r_{a_{i-1}} + l_{a_1} = \sum_{i=1}^{N} r_{a_i} + l_{a_i} \geq \sum_{i=1}^{N} j_{a_i} > N \cdot \varepsilon > M \end{aligned}$$

A contradiction. Thus f must have only countably many points of discontinuity.

5. By Holder's inequality, we know that:

$$\int \left| X_n \cdot \mathbb{1}_{|X_n| \geq k} \right| d\mathbb{P} \leq \left(\int |X|^p d\mathbb{P} \right)^{1/p} \left(\int \mathbb{1}^q_{|X| \geq k} d\mathbb{P} \right)^{1/q}$$

Where $1/p + 1/q = 1$. Notice that:

$$\int \mathbb{1}^q_{|X_n|\geq k} d\mathbb{P} = \mathbb{P}(|X_n| \geq k)$$

From here, by Markov's inequality,

$$\mathbb{P}(|X_n| \geq k) = \mathbb{P}(|X_n|^p \geq k^p) \leq \frac{\mathbb{E}[|X_n|^p]}{k^p}$$

Finally, using that $\sup_n \mathbb{E}[|X_n|^p] = C < \infty$, we have that:

$$\sup_n \mathbb{E}[|X_n \cdot \mathbb{1}_{|X_n|\geq k}|] \leq C^{1/p} \cdot \frac{C^{1/q}}{k^{p/q}} \overset{k\to\infty}{\longrightarrow} 0$$

which shows that X_n is uniformly integrable.

6. I claim that if $X : (\Omega_1, \mathcal{F}) \to (\Omega_2, \mathcal{G})$ is measurable, and $\mathcal{B} \subset \mathcal{F}$ is any σ-algebra, then

$$\left\{B : X^{-1}(B) \in \mathcal{B}\right\}$$

is too. This is because, given $A \in \left\{B : X^{-1}(B) \in \mathcal{B}\right\}$, we know that $X^{-1}(A^c) = X^{-1}(A)^c \in \mathcal{B}$, and if $\{A_i\}_{i=1}^{\infty} \subset \left\{B : X^{-1}(B) \in \mathcal{B}\right\}$, then $X^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} X^{-1}(A_i) \in \mathcal{B}$.

We now apply this result to $\mathcal{B} = \sigma(X^{-1}(\mathcal{A}))$. Clearly,

$$\mathcal{A} \subset \left\{B : X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))\right\}$$

Thus,

$$\mathcal{G} = \sigma(\mathcal{A}) \subset \left\{B : X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))\right\} \subset \mathcal{G}$$

So for any set $B \in \mathcal{G}$, we have that $X^{-1}(B) \in \sigma(X^{-1}(\mathcal{A}))$. This in turn shows that $X^{-1}(\mathcal{G}) = \sigma(X) \subset \sigma(X^{-1}(\mathcal{A}))$, which completes the proof.

7. a) If $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ then $(\bigcup_{n=1}^{\infty} A_n)^c \in A$, so let $A_0 = (\bigcup_{n=1}^{\infty} A_n)^c$. Then we have that

$$A_0 \cap A_n \subset A_0 \cap \bigcup_{n=1}^{\infty} A_n = \emptyset$$

so A_0 is disjoint from all the A_n as well. Thus we have found a sequence of disjoint sets $A_0, A_1, \ldots$ whose union is the whole space, $\{\pm 1\}^{\mathbb{N}}$. Since each element

$A \times \{\pm 1\}^{\mathbb{N}}$ for $A \subset \{\pm 1\}^n$ (for arbitrary n) can be writen as a disjoint union via:

$$\coprod_{a \in A} \{a\} \times \{\pm 1\}^{\mathbb{N}}$$

Thus we may assume by unrolling as above that each of the A_i are of the form $\{a_i\} \times \{\pm 1\}^{\mathbb{N}}$ for some finite string of ± 1s a_i.

Recall that $\{\pm 1\}$ with the discrete topology is compact. Tychnoff's theorem tells us that $\{\pm 1\}^{\mathbb{N}}$ is compact as well, endowed with the product topology. For a string $s = (s_1, \ldots, s_n)$, of ± 1s, we know that $\{\pm 1\}^{i-1} \times \{s_i\} \times \{\pm 1\}^{\mathbb{N}}$ is open by definition of the product topology (being the preimage of s_i under the projection onto the ith coordinate). In particular

$$\{s\} \times \{\pm 1\}^{\mathbb{N}} = \bigcap_{i=1}^{n} \{\pm 1\}^{i-1} \times \{s_i\} \times \{\pm 1\}^{\mathbb{N}}$$

is also open, being a finite intersection of open sets. Thus each of the A_i are open, or more importantly, each of the A_i^c are closed.

Recall a theorem saying that a topological space X is compact iff every family of closed sets having the finite intersection property has nonempty intersection. If we could find a finite subfamily of the A_i^c that have empty intersection, say $A_{n_1}^c, \ldots, A_{n_m}^c$, we already have (after taking complements) $\bigcup_{i=1}^m A_{n_i} = \{\pm 1\}^{\mathbb{N}}$, and we cannot add anymore disjoint sets, since every other set in $\mathcal{A}$ has to intersect with one of the A_{n_i}s. So for every $i \geq n_m$ we have $A_i = \emptyset$. In the other case, the family A_i has the finite intersection property and must not have empty intersection, a contradiction. This completes the proof.

b) If $\mathbb{P} : \mathcal{A} \to [0, 1]$ is a finitely additive probabilty on $\mathcal{A}$, let $A_1, A_2, \ldots$ be a sequence of disjoint sets in $\mathcal{A}$. Then by part (a) there is some $m \geq 1$ so that $A_i = \emptyset$ for all $i \geq m$. Thus we have that

$$\mathbb{P}\left(\coprod_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\coprod_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

by finite additivity and because $\mathbb{P}(A_i) = \mathbb{P}(\emptyset) = 0$ for all $i \geq m$. Thus $\mathbb{P}$ is countably additive.