Math 336 HW8

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1. If $f'(z) \neq 0$, then for $|w|$ sufficiently small,

$$f(z+w) = f(z) + f'(z)w + o(w)$$

We can find $|w|$ sufficiently small so that $|o(w)| \leq \frac{1}{2}|f'(z)||w|$ by definition. Then,

$$|f(z+w) - f(z)| \geq \frac{1}{2}|f'(z)||w| > 0$$

Hence, we can find a neighborhood of z so that f differs from it outside of z, hence f is injective. Now suppose that f is locally bijective in some neighborhood of z_0 and that $f'(z_0) = 0$. If f is constant then we are done, since it is obviously not injective. Else, expand f as a power series as:

$$f(z) = f(z_0) + \sum_{i=2}^{\infty} a_n(z-z_0)^n$$

Find the least such m so that $a_m \neq 0$ (it exists since f is not constant, and is ≥ 2 since it isn't 1 by hypothesis). We see then that

$$f(z) = f(z_0) + a_m(z-z_0)^m + \sum_{i=m+1}^{\infty} a_n(z-z_0)^n$$

If we look at the set of solutions to $f(z) = f(z_0) + \varepsilon$, we see that this says

$$a_m(z-z_0)^m = \varepsilon$$

Which has set of solutions $z = z_0 + \varepsilon a_m e^{\frac{2\pi i k}{m}}$ for $k \in \{0, \ldots, m-1\}$. One also notes that we may find a sufficiently small neighborhood of z so that $|\sum_{i=m+1}^{\infty} a_n(z-z_0)^n| \leq \frac{1}{2}|a_m(z-z_0)^m|$. Choosing ε so that $z_0 + \varepsilon a_m e^{\frac{2\pi i k}{m}}$ is in this neighrbood of z, it follows then that $a_m(z-z_0)^m = \varepsilon$ and $a_m(z-z_0)^m + \sum_{i=m+1}^{\infty} a_n(z-z_0)^n = \varepsilon$ have the same number of solutions in this neighborhood of z, which is $m \geq 2$. Since this setup works for all $\varepsilon > 0$, f cannot be injective in any neighborhood of z_0, a contradiction.

2. One notes that

$$F(G(z)) = \frac{i - i\frac{1-z}{1+z}}{i + i\frac{1-z}{1+z}} = \frac{i(1+z) - i(1-z)}{i(1+z) + i(1-z)} = \frac{2iz}{2i} = z$$

And similarly,

$$G(F(z)) = i\frac{1 - \frac{i-z}{i+z}}{1 + \frac{i-z}{i+z}} = i\frac{(i+z) - (i-z)}{(i+z) + (i-z)} = i\frac{-2iz}{2i} = z$$

Since F has a two sided inverse, it is bijective. F is the quotient of two holomorphic polynomials, who's denominator is never 0 on $\mathbb{H}$, so F is holomorphic on all of $\mathbb{H}$. F also maps into $\mathbb{D}$ since if we write $z = x + iy$ with $y > 0$, we see that

$$|F(z)| = \left|\frac{i - x - iy}{i + x + iy}\right| = \frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}$$

The hypothesis that $x^2 + (1-y)^2 < x^2 + (1+y)^2$ is equivalent to $y > 0$, so our claim holds. Since $|i| = 1$, G also maps into $\mathbb{D}$.

We notice that $G(0) = i$, and since $f(i) = 0$, we have that $f(G(0)) = 0$. Now, $f \circ G : \mathbb{D} \to \mathbb{C}$ with $|f(G(z))| \leq 1$ for all $z \in \mathbb{D}$, and hence we may apply Schwartz lemma to get

$$|f(G(z))| \leq |z|$$

Now given $w \in H$, taking $z = F(w)$ yields

$$|f(w)| = |f(G(F(w)))| \leq |F(w)| = \left|\frac{i-w}{i+w}\right|$$

I shall spare you the details, but the exact same proof as above would show that for any $\beta \in \mathbb{H}$, $F_\beta(z) = \frac{\beta - z}{\beta + z}$ and $G_\beta(w) = \beta\frac{1-w}{1+w}$ also have all the same properties as the above F, G (nice domains, holomorphic, bijective). Let f be a bijective holomorphic mapping from $\mathbb{H}$ to $\mathbb{D}$. Since f is onto, there is some $\beta \in \mathbb{H}$ so that $f(\beta) = 0$. We notice that $G_\beta(0) = \beta$, and hence $f(G_\beta(0)) = 0$. Also, $f(G_\beta(z)) : \mathbb{D} \to \mathbb{D}$ with $|f(G_\beta(z))| \leq 1$ (since it maps to the unit disk). Applying the Schwartz lemma gives us

$$|f(G_\beta(z))| \leq |z| \quad \text{for all } z \in \mathbb{D}$$

and that $|(f \circ G_\beta)'(0)| \leq 1$. Similarly, we may also apply the lemma to $G^{-1} \circ f^{-1}(z)$ to get

$$|(G^{-1} \circ f^{-1})'(z)| \leq 1$$

Since $(f \circ G)^{-1'}(0) = \frac{1}{(f \circ G)'(0)}$ (Note: $(f \circ G)^{-1}(0) = 0$), we may conclude that $|(f \circ G_\beta)'(0)| = 1$. Thus by the Schwartz lemma, $(f \circ G)(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$. Plugging in $z = F(w)$, we conclude that

$$f(w) = e^{i\theta}F(w) = e^{i\theta}\frac{\beta - w}{\beta + w}$$

Which completes the proof.

3. Define $g(z) = \frac{1}{M} f(Rz)$. It is now clear that $g : \mathbb{D} \to \mathbb{D}$, since $|g| \leq \frac{1}{M}|f| \leq 1$. So, we can apply Schwartz-Pick to see that

$$M\left|\frac{f(Rz) - f(0)}{M^2 - \overline{f(Rz)} f(0)}\right| = \left|\frac{g(z) - \frac{1}{M} f(0)}{1 - \frac{1}{M}\overline{f(Rz)}\frac{1}{M} f(0)}\right| \leq \left|\frac{z - 0}{1}\right| = |z|$$

Letting $u = Rz$, we conclude that

$$\left|\frac{f(u) - f(0)}{M^2 - \overline{f(Rz)} f(0)}\right| \leq \frac{|u|}{RM}$$

Since every $w \in \mathbb{D}_R$ can be described by $R \cdot z$ with $z \in \mathbb{D}$, the inequality in the problem statement holds.

4. Fix $z \in \mathbb{D}$. By schwartz-pick, for every $w \in \mathbb{D} \setminus z$, we have that

$$\left|\frac{f(z) - f(w)}{1 - \overline{f(z)} f(w)}\right| \leq \left|\frac{z - w}{1 - \bar{z}w}\right|$$

Since $|z - w| > 0$, we can divide both sides by it to preserve equality and see that

$$\left|\frac{\frac{f(z)-f(w)}{z-w}}{1 - \overline{f(z)} f(w)}\right| \leq \left|\frac{1}{1 - \bar{z}w}\right|$$

Now, since this holds for every $w \neq z$, we can take a limit on both sides to get

$$\left|\lim_{w\to z} \frac{\frac{f(z)-f(w)}{z-w}}{1 - \overline{f(z)} f(w)}\right| = \lim_{w\to z}\left|\frac{\frac{f(z)-f(w)}{z-w}}{1 - \overline{f(z)} f(w)}\right| \leq \lim_{w\to z}\left|\frac{1}{1 - \bar{z}w}\right| = \left|\lim_{w\to z} \frac{1}{1 - \bar{z}w}\right|$$

One now notices that

$$\lim_{w\to z} \frac{f(z) - f(w)}{z - w} = f'(z)$$
$$\lim_{w\to z} f(w) = f(z)$$
$$\lim_{w\to z} w = z$$

And one recalls that $z\bar{z} = ||z||^2$. We conclude that

$$\left|\frac{f'(z)}{1 - |f(z)|^2}\right| \leq \left|\frac{1}{1 - |z|^2}\right|$$

Since $z \in \mathbb{D}$, and since $f(z) \in \mathbb{D}$, we have that $|z|^2 < 1$, and that $|f(z)|^2 < 1$. So, $|1 - |f(z)|^2| = 1 - |f(z)|^2$, and also that $|1 - |z|^2| = 1 - |z|^2$. We conclude that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

5. Consider $\theta = -\arg\left(\frac{1}{2\pi}\int_0^{2\pi}\gamma(e^{it})dt\right)$ where $\theta \in [0, 2\pi)$. We notice that $\arg(e^{i\theta}\frac{1}{2\pi}\int_0^{2\pi}\gamma(e^{it})dt) = \theta + \arg\left(\frac{1}{2\pi}\int_0^{2\pi}\gamma(e^{it})dt\right) = 0$, so $e^{i\theta}\frac{1}{2\pi}\int_0^{2\pi}\gamma(e^{it})dt = 1$ since it also has magnitude 1. Now, we also notice that, since we can match real parts,

$$1 = \Re\left(e^{i\theta}\frac{1}{2\pi}\int_0^{2\pi}\gamma(e^{it})dt\right) = \frac{1}{2\pi}\int_0^{2\pi}\Re\left(e^{i\theta}\gamma(e^{it})\right)dt$$

Also, one recalls that $\Re(z) \leq |z|$, so it follows that

$$\Re\left(e^{i\theta}\gamma(e^{it})\right) \leq |e^{i\theta}\gamma(e^{it})| = 1$$

Simultaneously,

$$\frac{1}{2\pi}\int_0^{2\pi}|\gamma(e^{it})|dt = 1$$

And hence

$$\frac{1}{2\pi}\int_0^{2\pi}\Re\left(e^{i\theta}\gamma(e^{it})\right) - 1dt = 0$$

The inside of this integral is nonnegative, and we showed long ago (334?) that if the inside took a negative value somewhere, the entire integral would be negative, which can't be. We conclude that

$$\Re\left(e^{i\theta}\gamma(e^{it})\right) - 1 = 0$$

and hence $\Re\left(e^{i\theta}\gamma(e^{it})\right) = 1$. If it were ever the case that $e^{i\theta}\gamma(e^{it})$ had nonzero imaginary part, $|e^{i\theta}\gamma(e^{it})| > 1$, which can't be. So $e^{i\theta}\gamma(e^{it}) = 1$, hence $\gamma(e^{it}) = e^{-i\theta}$, i.e. $\gamma(e^{it})$ is constant.