Math Template

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1. By the X_i being independent from the $N(t)$ for all t, we get that:

$$\begin{aligned}
\mathbb{P}\left(\sum_{N(s)+1}^{N(t)} X_i \leq a, \sum_{1}^{N(s)} X_i \leq b\right) &= \sum_{m,n} \mathbb{P}\left(\sum_{N(s)+1}^{N(t)} X_i \leq a, \sum_{1}^{N(s)} X_i \leq b \mid N(s) = m, N(t) - N(s) = n\right) \cdot \\
&\quad \mathbb{P}(N(s) = m, N(t) - N(s) = n) \\
&= \sum_{m,n} \mathbb{P}\left(\sum_{m+1}^{m+n} X_i \leq a, \sum_{1}^{m} X_i \leq b\right) \mathbb{P}(N(s) = m, N(t) - N(s) = n)
\end{aligned}$$

By the independence of the X_i, we have that:

$$\mathbb{P}\left(\sum_{m+1}^{m+n} X_i \leq a, \sum_{1}^{m} X_i \leq b\right) = \mathbb{P}\left(\sum_{m+1}^{m+n} X_i \leq a\right) \mathbb{P}\left(\sum_{1}^{m} X_i \leq b\right)$$

By independence and the X_i being identically distributed, we have that:

$$\mathbb{P}\left(\sum_{m+1}^{m+n} X_i \leq a\right) = \mathbb{P}\left(\sum_{1}^{n} X_i \leq a \mid N(t) - N(s) = n\right)$$

and

$$\mathbb{P}\left(\sum_{1}^{m} X_i \leq b\right) = \mathbb{P}\left(\sum_{1}^{m} X_i \leq b \mid N(s) = m\right)$$

Putting these together and using that N, a poisson process, has independent increments, we get that:

$$\begin{aligned}
&\sum_{m,n}\mathbb{P}\left(\sum_1^n X_i \le a, N(t)-N(s)=n\right)\mathbb{P}\left(\sum_1^m X_i \le b, N(s)=m\right)\\
&\quad=\sum_n \mathbb{P}\left(\sum_1^{N(t)-N(s)} X_i \le a, N(t)-N(s)=n\right)\sum_m \mathbb{P}\left(\sum_1^m X_i \le b, N(s)=m\right)\\
&\quad=\mathbb{P}\left(\sum_1^{N(t)-N(s)} X_i \le a\right)\mathbb{P}\left(\sum_1^{N(s)} X_i \le b\right)
\end{aligned}$$

Finally, as $N(s)$ is independent of $N(t)-N(s)$, and all the X_i, we have that:

$$\mathbb{P}\left(\sum_1^{N(t)-N(s)} X_i \le a\right)=\sum_{m,n}\mathbb{P}\left(\sum_1^{N(t)-N(s)} X_i \le a \mid N(s)=n, N(t)-N(s)=m\right)\mathbb{P}(N(t)-N(s)=m, N(s)=n)$$

The middle probability is just (by independence and i.i.d. again, since these sums have the same number of terms and the m,n are fixed):

$$\begin{aligned}
\mathbb{P}\left(\sum_1^m X_i \le a\right)&=\mathbb{P}\left(\sum_{n+1}^{m+n} X_i\right)=\mathbb{P}\left(\sum_{n+1}^{m+n} \mid N(t)-N(s)=m, N(s)=n\right)\\
&=\mathbb{P}\left(\sum_{N(s)+1}^{N(t)} X_i \mid N(t)-N(s)=m, N(s)=n\right)
\end{aligned}$$

Plugging this back in to the first sum, we get that it equals:

$$\sum_{m,n}\mathbb{P}\left(\sum_{N(s)+1}^{N(t)} X_i \le a, N(t)-N(s)=m, N(s)=n\right)=\mathbb{P}\left(\sum_{N(s)+1}^{N(t)} X_i \le a\right)$$

This completes the proof.

2. If $A \in \mathcal{F}_L$ then $A \cap \{L \le n\} \in \mathcal{F}_n$ for all n. We take a look at the set $\{N \le n\}$. This is $(\{N \le n\} \cap A^c) \cup (\{N \le n\} \cap A)$. Using the definition of N, this equals $(\{M \le n\} \cap A^c) \cup (\{L \le n\} \cap A)$. The second set is in $\mathcal{F}_n$ by the assumption on A. For the first set,

$$\{M \le n\} \setminus A = \{M \le n\} \setminus (A \cap \{M \le n\}) = \{M \le n\} \setminus (A \cap \{L \le n\})$$

The above equality holds beacuse $L \le M$, so if $M \le n$ then $L \le n$ and so $\{M \le n\} \subset \{L \le n\}$. So by changing $A \cap \{M \le n\}$ to $A \cap \{L \le n\}$, we are making the set bigger, but still we can't remove things outside of $\{M \le n\}$, so the difference will be the same.

Then finally, $\{M \leq n\} \in \mathcal{F}_n$, and $A \cap \{L \leq n\} \in \mathcal{F}_n$ by the hypothesis on A. So both of these two parts are in $\mathcal{F}_n$, so there union is too, which completes the proof.

3. I give three proofs for this first assertion. The first is by induction. The base case, that $\mathbb{P}(S_1 = X_1 \leq x) = x$ for $0 \leq x \leq 1$ is true by definition. Then suppose that for $0 \leq x \leq 1$, $\mathbb{P}(S_n \leq x) = \frac{x^n}{n!}$. Then for $0 \leq x \leq 1$, we have that:

$$\begin{aligned}
\mathbb{P}(S_{n+1} \leq x) &= \int_0^x \int_{-\infty}^{\infty} p_{X_{n+1}}(x-y) p_{S_n}(y) \mathrm{d}y \mathrm{d}x \\
&= \int_0^x \int_0^x \frac{s^{n-1}}{(n-1)!} \mathrm{d}x \mathrm{d}y \\
&= \int_0^x \frac{s^n}{n!} \mathrm{d}x \\
&= \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

In the middle we used that $p_{S_n}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(S_n \leq x) = \frac{x^{n-1}}{(n-1)!}$. This completes the induction.

Plugging in, we get $\mathbb{P}(S_n \leq n) = \frac{1}{n!}$.

The second proof is by taking cross-sections. First, we know that the area of an isocoles right triangle with side lengths x has area $\frac{x^2}{2}$. The horizontal cross sections of the tetrahedron defined by $x + y + z \leq 1$ are isocoles right triangles. By summing all of these, we get that the area of this tetrahedron is:

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \mathrm{d}z \mathrm{d}y \mathrm{d}x &= \int_0^1 \int_0^{1-x} (1-x-y) \mathrm{d}y \mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^2}{2} \mathrm{d}x \\
&= \frac{1}{6}
\end{aligned}$$

Let the volume of the n-simplex be V_n. Taking a cross-section of the $n+1$-simplex defined by $x_1 + \cdots + x_{n+1} \leq 1$ will yield an n-simplex. If one of the side lengths of this n-simplex is x, then by similarity with the bottom n-simplex who has side lengths of 1, the area of this n-simplex is $x^n V_n$. By summing all these up, we get that the volume of the $n+1$-simplex is:

$$\int_0^1 x^n V_n \mathrm{d}x = \frac{V_n}{n+1}$$

Induction yields that $V_n = \dfrac{1}{n!}$.

The third proof is a little bit of linear algebra, and my favorite one. Consider $S = \{0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$. Then for any permutation $\pi \in S_n$, the set $\pi(S) = \left\{0 \leq x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)} \leq 1\right\}$ has the same volume as S. Since the disjoint union of these sets is $[0,1]^n$, we get that the volume of S is just $\dfrac{1}{n!}$.

This relates to the original simplex in the following way. Let A be the linear map defined by $A(x_1, \ldots, x_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1})$. A simple exercise shows that $A(S) = \{x_1 + \cdots + x_n \leq 1\}$. Then by high-dimensional geometry, the volume of $A(S)$ is $\det(A) \cdot S$. The matrix of A is of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

This matrix is lower-triangular so has determinant 1. Thus the volume of $\{x_1 + \cdots + x_n \leq 1\}$ is $\dfrac{1}{n!}$.

Now knowing that $\mathbb{P}(T \geq n) = \dfrac{1}{n!}$, we can compute the expected value of T as:

$$\mathbb{E}[T] = \sum_{n=1}^{\infty} \mathbb{P}(T \geq n) = \sum_{n=1}^{\infty} \frac{1}{n!} = e$$

Notice that $S_T = \displaystyle\sum_{n=1}^{T} X_n = \sum_{n=1}^{\infty} X_n \mathbb{1}_{T \geq n}$. Then as $S_T \geq 0$ and Fubini-Tonelli theorem, we can switch the order of summation and integration to get:

$$\mathbb{E}\left[\sum_{n=1}^{\infty} X_n \mathbb{1}_{T\geq n}\right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n]\mathbb{P}(T \geq n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{n!} = \frac{e}{2}$$

4. Let S be the stopping time that is constantly equal to 1. Let $X_1 = 0$ equivalently, and $X_2, \ldots,$ be a sequence Radamacher random variables independent from X_1. Define $S_n = \inf\{n \geq 1 : S_n \geq 1\}$. Then clearly $S \geq 2$, so $T \leq S$. However, $\{S - T = 1\} = \{S = 2\} = \{X_1 + X_2 = 1\} = \{X_2 = 1\}$ which is certainly not in $\mathcal{F}_1$ (it is neither the empty set or the whole space). This is a counterexample.

5. (a) By Strong law of large numbers, we know that

$$\frac{S_n}{n} \to \mathbb{E}[X_1] = p - (1-p) = 2p - 1 > 0 \quad \text{a.s.}$$

Then for ω with probability 1, there exists n so that $S_n > 0$. Then for all those ω, $\inf\{m : S_m > 0\}$ is finite, which shows that $\mathbb{P}(\alpha < \infty) = 1$.

(b) First, notice that $\mathbb{P}(\inf S_n < -1) = \mathbb{P}(\beta < \infty)$, because these events are the same, because the infimum is < 0 if and only if there is an element that is < 0.

Define $\beta_k = \{\inf S_n \leq -k\}$. Then $\beta = \beta_1$, and β_k is a stopping time for every k. Suppose inductively that $\mathbb{P}(\beta_k < \infty) = \mathbb{P}(\beta < \infty)^k$. Then,

$$\mathbb{P}(\inf S_n \leq -(k+1)) = \sum_{l=1}^{\infty} \mathbb{P}(\inf S_n \leq -(k+1) \mid \beta_k = l)\mathbb{P}(\beta_k = l)$$

Conditioned on $S_l = -k$, we see that $\inf S_n \leq -(k+1)$ is equivalent to $\inf_{n\geq l} S_n - S_l \leq -1$ (we just need to go down 1 more step sometime afterwards). We make one more observation: $S_n - S_l = \sum_{l+1}^{n} X_i$, and $\{\beta_k = l\}$ is $\mathcal{F}_l$-measurable. Thus $S_n - S_l$, $\{\beta_k = l\}$ are independent. We conclude:

$$\begin{aligned}\mathbb{P}(\inf S_n \leq -(k+1)) &= \sum_{l=1}^{\infty} \mathbb{P}\left(\inf_{n\geq l} S_n - S_l \leq -1\right)\mathbb{P}(\beta_k = l) \\ &= \mathbb{P}(\beta < \infty)^k \cdot \mathbb{P}(\inf S_n \leq -1) \\ &= \mathbb{P}(\beta < \infty)^{k+1}\end{aligned}$$

The second equality coming from the inductive hypothesis and the assumption that the X_i are i.i.d.

Now, suppose that $\mathbb{P}(\beta < \infty) = 1$. Then $\mathbb{P}(\inf S_n \leq -k) = 1$ for every $k \geq 1$. Notice that:

$$\{\inf S_n \leq k\} \searrow \{\inf S_n = -\infty\}$$

And so, $\mathbb{P}(\inf S_n = -\infty) = 1$. Yet, recall that

$$\mathbb{P}(\liminf S_n \geq 0) \geq \mathbb{P}(\liminf S_n/n = 2p - 1) = 1.$$

Then for ω almost surely, $S_n \geq 0$ eventually, and so the infimum is finite. Then $\mathbb{P}(\inf S_n = -\infty) = 0$, a contradiction.

(c) We know that:

$$\begin{aligned}\mathbb{P}(\beta < \infty) &= \mathbb{P}(X_1 = -1) + \mathbb{P}\Big(X_1 = 1, \inf_{n\geq 2} S_n \leq -1\Big) \\ &= (1-p) + \mathbb{P}\Big(\inf_{n\geq 2} S_n - X_1 \leq -2 \mid X_1 = 1\Big)\mathbb{P}(X_1 = 1) \\ &= (1-p) + \mathbb{P}\Big(\inf_{n\geq 1} S_n \leq -2\Big)p\end{aligned}$$

The third equality follows since $S_n - X_1$ is independent of X_1 and since the X_i are identically distributed.

Therefore, letting $x = \mathbb{P}(\beta < \infty)$ and using the last part, we know that $x = (1-p) + xp^2$. Solving, we get $x = 1$ or $x = \dfrac{1-p}{p}$. Since $x < 1$, we have that $x = \dfrac{1-p}{p}$.

(d) Notice that:

$$\begin{aligned}\mathbb{E}[S_{\alpha\wedge n}] &= \mathbb{E}[S_{\alpha\wedge n}; \alpha \leq n] + \mathbb{E}[S_\alpha; \alpha > n, \inf S_m \leq -k] + \mathbb{E}[S_n; \alpha > n, \inf S_m > -k] \\ &\geq \mathbb{E}[S_{\alpha\wedge n}; \alpha \leq n] - k\mathbb{P}(\alpha > n) + \mathbb{E}[S_n; \alpha > n, \inf S_m \leq -k]\end{aligned}$$

The idea is that we want the third term to be bounded as $n \to \infty$. Since $\mathbb{P}(\inf S_n \leq -k) = x^k$, where $x = \mathbb{P}(\beta < \infty)$, we know that $\mathbb{P}(\inf S_n = -k) = x^k - x^{k+1}$. If $\alpha > n$, and $\inf S_m = -\ell$, then $|S_1|, \ldots, |S_{\alpha\wedge n}| \in [0, \ell]$. By Fubini-Tonelli, we get that:

$$\begin{aligned}\mathbb{E}[|S_{\alpha\wedge n}|;\alpha>n,\inf S_m\le -k] &= \sum_{\ell=k}^{\infty}\mathbb{E}[|S_{\alpha\wedge n}|;\alpha>n,\inf S_m=-\ell]\\ &\le \sum_{\ell=k}^{\infty}\ell\mathbb{P}(\inf S_m=-\ell)\\ &= \sum_{\ell=k}^{\infty}\ell(x^{\ell}-x^{\ell+1}) = (1-x)\sum_{\ell=k}^{\infty}\ell x^{\ell}\end{aligned}$$

By noticing that $\mathbb{E}[S_{\alpha\wedge n};\alpha\le n]=\mathbb{E}[1;\alpha\le n]=\mathbb{P}(\alpha\le n)\to 1$, and since $k\mathbb{P}(\alpha>n)\to 0$ as $n\to\infty$ as α is finite almost surely, sending $n\to\infty$ shows that:

$$\liminf_{n\to\infty}\mathbb{E}[S_{\alpha\wedge n}]\ge 1-(1-x)\sum_{\ell=k}^{\infty}\ell x^{\ell}$$

Sending $k\to\infty$ and using that $\sum_{\ell=k}^{\infty}\ell x^{\ell}$ is the tail of a convergent power series shows that $\liminf_{n\to\infty}\mathbb{E}[S_{\alpha\wedge n}]\ge 1$.

The upper bound is much easier. By definition of α, $S_1,\ldots,S_{\alpha\wedge n-1}\le 0$. Thus $S_{\alpha\wedge n}\le 1$ since $X_i\le 1$. Thus, $\limsup_{n\to\infty}\mathbb{E}[S_{\alpha\wedge n}]\le 1$. Finally, by Monotone convergence theorem, we know that $\alpha\wedge n\nearrow\alpha$ as $n\to\infty$ and hence $\mathbb{E}[\alpha\wedge n]\to\mathbb{E}[\alpha]$. Hence we conclude that $\mathbb{E}[S_\alpha]=1$. By Wald's equation, $\mathbb{E}[X_1]\mathbb{E}[\alpha]=\mathbb{E}[S_\alpha]$. Thus $\mathbb{E}[\alpha]=\dfrac{1}{2p-1}$.

The question is as follows: Let $X_1,\ldots,X_n$ be i.i.d. with $\mathbb{E}[X_i]=\mu$ and $\mathrm{Var}(X_i)=\sigma^2$. Let $\Theta=\frac{1}{n}\sum_{i=1}^{n}X_i$. Then $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\Theta)^2\right]=\left(1-\frac{1}{n}\right)\sigma^2$.

Assume WLOG that X_i have mean 0 (notice that subtracting μ from each of the random variables does not change our desired expectation). Then:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{n}\sum_i (X_i - \Theta)^2\right] &= \frac{1}{n}\sum_i \mathbb{E}\left[(X_i - \Theta)^2\right] \\
&= \frac{1}{n}\sum_i \left(\mathbb{E}\left[X_i^2\right] - 2\mathbb{E}[X_i\Theta] + \mathbb{E}\left[\Theta^2\right]\right) \\
&= \mathbb{E}[X_1]^2 - 2\mathbb{E}[X_1\Theta] + \mathbb{E}\left[\Theta^2\right]
\end{aligned}$$

Now, by identically distributed,

$$\mathbb{E}\left[\Theta^2\right] = \mathbb{E}\left[\frac{1}{n}\sum_i X_i\Theta\right] = \frac{1}{n}\sum_i \mathbb{E}[X_i\Theta] = \mathbb{E}[X_1\Theta]$$

Lastly, by mean 0,

$$\mathbb{E}[X_1\Theta] = \frac{1}{n}\sum_j \mathbb{E}\left[X_1X_j\right] = \frac{1}{n}\sigma^2$$

And so our answer is $\left(1 - \frac{1}{n}\right)\sigma^2$.