CSE 311 HW8

Rohan Mukherjee

March 8, 2023

1. (a) Let $a, b, c \in \mathbb{Z}^+$ be arbitrary positive integers. Suppose that $(a, b) \in R$ and $(b, c) \in R$. By the definition of being in R, we see that there exists integers $k, l \in \mathbb{Z}$ so that $\frac{a}{b} = 2k$, and $\frac{b}{c} = 2l$. We notice that, because $b \neq 0$,

$$\begin{aligned} \frac{a}{c} = \frac{\frac{a}{b}}{\frac{c}{b}} = \frac{2k}{\frac{1}{2l}} &= 4kl \\ &= 2(2kl) \end{aligned}$$

As $2kl \in \mathbb{Z}$, $\frac{a}{c}$ is an even integer. We see that by the definition of being in R, $(a, c) \in R$, so R is indeed a transitive relation. $\square$

(b) We notice that $(4, 2) \in R$, as $\frac{4}{2} = 2$ which is of course an even integer, but as the description of the problem states, $(2, 4) \notin R$.

2. (a) Let X, Y be arbitrary subsets of the naturals of size 2. Then $X = \{x_1, x_2\}$ for some $x_1 < x_2$ (it is not $\leqslant$ because then the set could potentially have only one element), and similarly $Y = \{y_1, y_2\}$ with $y_1 < y_2$. Suppose $X \preceq Y$ and $Y \preceq X$. Because $X \preceq Y$, we have that $x_1 \leqslant y_1$, and $x_2 \leqslant y_2$. Similarly, because $Y \preceq X$, we have that $y_1 \leqslant x_1$ and that $y_2 \leqslant x_2$. Because $\leqslant$ is antisymmetric, we see that both $x_1 = y_1$ and that $x_2 = y_2$. □

   (c) No. For example, the smaller element of X could be less than the smaller element of Y, while the larger element of X is greater than the larger element of Y. For example, take $X = \{1, 4\}$, and $Y = \{2, 3\}$. As the larger element of X is larger than the larger element of Y, $X \not\preceq Y$. Similarly, as the smaller element of Y is larger than the smaller element of X, we see that $Y \not\preceq X$, so we see that $\preceq$ is **not** a total order by counterexample (we just have to find one counterexample because we are disproving a $\forall$ statement).

   (d) Consider $X = \{1, 2\}$, $Y = \{-1, 3\}$, and $Z = \{0, 1\}$. Because the largest element of X is less than the largest element of Y, we have that $X \preceq Y$. Because the smaller element of Y is less than the smaller element of Z, we have that $Y \preceq Z$. But as the smaller element of X is strictly greater than the smaller element of Z, and the larger element of X is strictly greater than the larger element of Z, $X \not\preceq Z$. So we see our relation is not transitive by counterexample (once again, this is a $\forall$ statement, so we just have to find one counterexample).

5. (a) Let the domain of discourse be integers for the rest of this problem. We see that $\mathrm{TWIN-PRIME}(x) \coloneqq \mathrm{PRIME}(x) \wedge (\mathrm{PRIME}(x+2) \vee \mathrm{PRIME}(x-2))$.

   (b) $\forall N\ \exists x > N\ (\mathrm{TWIN-PRIME}(x))$.

   (c)

$$\begin{aligned}\neg(\forall N\ \exists x > N\ (\mathrm{TWIN-PRIME}(x))) &\equiv \exists N \neg(\exists x > N\ (\mathrm{TWIN-PRIME}(x)) \\ &\equiv \exists N\ \forall x > N\ (\neg \mathrm{TWIN-PRIME}(x))\end{aligned}$$

   (d) There is an integer so that every integer larger than it is not a twin prime.

6. (a) $\forall x \forall y([\text{Rational}(x) \wedge \neg\text{Rational}(y)] \rightarrow \neg\text{Rational}(x+y))$.

   (b) $\exists x \exists y([\text{Rational}(x) \wedge \neg\text{Rational}(y)] \wedge \text{Rational}(x+y))$.

   (c) Suppose by way of contradiction that there were two real numbers x, y where x is rational, y is irrational, and $x+y$ is rational. Because x is rational, there are integers a, b where $b \neq 0$ so that $x = \frac{a}{b}$. Similarly, as $x+y$ is rational, there are integers c, d where $d \neq 0$ so that $x + y = \frac{c}{d}$. By plugging in these values, we notice that

$$\begin{aligned} \frac{a}{b} + y &= \frac{c}{d} \\ y &= \frac{c}{d} - \frac{a}{b} && \text{(Rearrange)} \\ y &= \frac{bc - da}{db} && \text{(Combine fractions)} \end{aligned}$$

   As $bc - da \in \mathbb{Z}$, $db \in \mathbb{Z}$, and as $d \neq 0$, and $b \neq 0$, $db \neq 0$, we see that by definition y is rational, which is a contradiction. $\square$

7. The author didn't assume the negation of the claim properly. He should've started with "Suppose both L_1 and L_2 are irregular and that $L_1 \cap L_2$ is regular." (The negation of $p \to q$ is $p \wedge \neg q$). Instead, assuming that the original statement can be written as $p \to q$, he assumed $p \to \neg q$, and showed that claim to be false. So, assuming his proof is correct, the negation of $p \to \neg q$ should be true, which the negation of this is going to be $p \wedge q$, which is definitely NOT equivalent to $p \to q$.

8. This assignment took me around 3.5 hours to complete. I don’t have any comments.

9. Assume by way of contradiction that $S = \{a^{311}b^{311}b^m c^m \mid m \geqslant 0\}$ was regular (rearrange the given set a little bit). Then the pumping lemma applies. So there exists a $p \in \mathbb{Z}$ so that every string w with length greater than p can be divided into three parts, $w = xyz$, with the length of xy not larger than p, the length of y not smaller than 1, and $\forall i \in \mathbb{N}\ xy^i z \in S$. y may be in one of four places:

   Case 1: y is in the $a^{311}b^{311}$ part of the string. Then y is of the form $a^k b^l$ for some $k, l \geqslant 0$ but not both zero. If $k \neq 0$, xy^2z has $311 + k$ a's, so therefore it can't be in S. If $k = 0$, then y is of the form b^l with $l \geqslant 1$. We notice that xy^2z is of the form $a^{311}b^{311+l}b^m c^m$, after rearrangement, which clearly isn't in S because the number of b's is $311 + m + l \neq 311 + m$ because $l \neq 0$.

   Case 2: y is in the b^{311+m} part. Then y is of the form b^l for some $l \neq 0$, and we see that the last part of the previous case applies, so this too is impossible.

   Case 3: y is in the c^m part. Then y is of the form c^l for some $l \neq 0$. Clearly then $xy^2z = a^{311}b^{311}b^m c^{m+l}$. We notice that $m + l + 311 \neq 311 + m$, as $l \neq 0$, so the number of b's is not 311 more than the number of c's, which is again a contradiction. Finally,

   Case 4: y is in the $b^m c^m$ part. Then y is of the form $b^k c^l$, where $k, l \neq 0$ (the above cases handled where they were already 0). We see that $xy^2z = a^{311}b^{311}b^{311-k}b^k c^l b^k c^l c^{311-l}$. This shows that there would be a b after a c, which is of course not in S. As these were the only four cases, we see that S cannot possibly be regular. $\square$

   I found that using the regular method with $S = \{a^{311}b^{311}b^m \mid m \geqslant 0\}$ would be a lot simpler of a proof.

10. We define function ends from the set of all regex to the set of all regex as follows.

 $\text{ends}(\varepsilon) = \varepsilon$, $\text{ends}(\emptyset) = \emptyset$, and $\text{ends}(a) = a \cup \varepsilon$. Then we define it structurally by, given regular expressions A, B, $\text{ends}(AB) = \text{ends}(A)B \cup \text{ends}(B)$, $\text{ends}(A \cup B) = \text{ends}(A) \cup \text{ends}(B)$, and finally $\text{ends}(A^*) = \text{ends}(A)A*$. Now, for a regular language L, we know there exists a regular expression R for it. Let $P(L) :=$ $\text{ends}(L)$ is recognized by $\text{ends}(R)$. We start by proving the base cases.

 The ends of the language that recognizes only the empty set is also empty, so $P(\emptyset)$ is true. The ends of the language that recognizes only ε is also just the language that include ε, because if it was something non-ε in Σ^*, then it would already not be recognized. And clearly if it is ε then it is recognized. Finally, ends of the language that recognizes a is just going to be a and ε, because we could add an a to ε. Suppose A, B are arbitrary languages and suppose $P(A), P(B)$. I tried finishing this, but I couldn't really figure out how to word this correctly. I do believe the bulk of the problem was finding the recursive definition of ends, so I am happy enough with this attempt.