CSE Template

Rohan Mukherjee

April 24, 2025

Problem 1.

Let $G = (A, B, E)$ be a bipartite graph with parts A and B. Since G is d-regular, $|A| \cdot d = |B| \cdot d$, both sides count the total number of edges leaving A and B. Thus $|A| = |B|$. For a subset S of A, we consider the quantity

$$d|S| = \sum_{a \in S} \deg a$$

Since G is d-regular, the degree of each vertex in B is d. Thus, in this sum, we count each neighbor of S at the very most d times. So, $d|S| \leq d|N(S)|$ meaning that $|S| \leq |N(S)|$ for each subset S of A. Hall's marriage theorem states that a perfect matching exists. Call this E_1, and remove all those edges. Repeat by induction to get d edge-disjoint subsets of E: $E_1, \ldots, E_d$ (which we color d different colors). Now suppose that every cycle in G has length at least L, to be chosen later. Independently choose a random edge X_i of each cycle C_i to be "corrupted" to the $(d + 1)$st color. Then define $A_{C_i \to e, C_j \to e'} = \left\{X_i = e, X_j = e'\right\}$ only for when $d(e, e') = 1$ and $e \in C_i, e' \in C_j$. Clearly, by our definition of the X_i,

$$P(A_{C_i \to e, C_j \to e'}) = P(X_i = e)P(X_j = e') = \frac{1}{|C_i||C_j|}$$

We must now look deeply into dependencies. This random event is fully measurable with respect to X_i and X_j. So it is not independent only on events that contain one of those variables.

We prove the following lemma: in a 2-regular graph, e is in ≤ 1 cycle. This is because a 2-regular graph is a union of disjoint cycles. Thus there are at most $\binom{d}{2}$ 2-colored cycles containing e, each cycle is found in a union of two of the E_i, which is a 2-regular graph. So, if a different event contains C_i, then we have $|C_i|$ choices for e, then at most $2d$ choices for e', and from our lemma above, there are at most $\binom{d}{2} \leq d^2$ 2-colored cycles containing e', yielding a total number of $2|C_i|d^3$.

The same is true for $|C_j|$. We now apply (Asymmetric) LLL. Take $x_{C_i \to e, C_j \to e'} = 2/|C_i||C_j|$. Then fixing A, we know that:

$$\sum_{A' \sim A} x_j = \sum_{A_{C_i \to f, C_k \to f'}} \frac{1}{|C_i||C_k|} + \sum_{A_{C_k \to f, C_j \to f'}} \frac{1}{|C_k||C_j|}$$

We have:

$$\sum_{A_{C_i \to f, C_k \to f'}} \frac{1}{|C_i||C_k|} \leq \sum_{A_{C_i \to f, C_k \to f'}} \frac{1}{|C_i|L} \leq \frac{2d^3}{L}$$

This is because there are around $|C_i|$ choices for f, and then $2d$ choices for f', with $\binom{d}{2} \leq d^2$ cycles containing f'. The same is true for $|C_j|$, so we get:

$$\sum_{A' \sim A} x_j \leq \frac{4d^3}{L}$$

To satisfy the conditions of LLL, we need:

$$\frac{2}{|C_i||C_j|} \prod_{A' \sim A} (1 - x_j) \geq \frac{1}{|C_i||C_j|}$$

Using $1 - x \approx e^{-x}$, we have:

$$\prod_{A' \sim A} (1 - x_j) = \exp\left(- \sum_{A' \sim A} x_j \right) \geq \exp\left(- \frac{4d^3}{L} \right)$$

It is this sufficient to take $\exp\left(-\frac{4d^3}{L}\right) \geq \frac{1}{2}$, or $L = O(d^3)$.