Math 521 HW1

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1. Let (X, Y) be the random vector that takes on $(0,0)$, $(1,1)$, $(2,0)$ each with probability $1/3$. Then,

$$\begin{aligned}\mathbb{E}[XY] &= \frac{1}{3} \\ \mathbb{E}[X] &= 1 \\ \mathbb{E}[Y] &= \frac{1}{3}\end{aligned}$$

   Hence $\mathrm{Cov}(X, Y) = 0$. However, noticing that $Y \cdot \mathbb{1}_{|Y| \leq 1} = Y$,

$$\begin{aligned}\mathbb{E}\big[X \cdot \mathbb{1}_{|X|\leq 1} Y\big] &= 0 + \frac{1}{3} \cdot 1 \cdot 1 + 0 = \frac{1}{3} \\ \mathbb{E}\big[X \cdot \mathbb{1}_{|X|\leq 1}\big] &= \frac{1}{3} \\ \mathbb{E}[Y] &= \frac{1}{3}\end{aligned}$$

   thus, $\mathrm{Cov}(X \cdot \mathbb{1}_{|X|\leq 1}, Y) = \frac{1}{3} - \frac{1}{9}$.

2. We see that, by Fubini's theorem,

$$\begin{aligned}\mathbb{E}[f(X)] &= \int_{\mathbb{R}} f(x)\mu_X(dx) = \int_{\mathbb{R}} \int_0^x f'(y)dy\mu_X(dx) \\ &= \int_{\mathbb{R}} \int_y^\infty f'(y)\mu_X(dx)dy = \int_{\mathbb{R}} f'(y)\mathbb{P}(X \geq y)dy\end{aligned}$$

3. For $m \leq n$, we know that

$$\mathbb{E}\left[\frac{S_m}{S_n}\right] = \sum_{i=1}^m \mathbb{E}\left[\frac{X_i}{S_n}\right]$$

Now, notice that:

$$\mathbb{E}\left[\frac{X_1}{X_1+X_2}\right] = \int_{\mathbb{R}^2} \frac{x}{x+y} d(\mu_{X_1} \times \mu_{X_2}) = \int_{\mathbb{R}^2} \frac{x}{x+y} d(\mu_{X_2} \times \mu_{X_1}) = \mathbb{E}\left[\frac{X_2}{X_1+X_2}\right]$$

Since $\mu_{X_1} = \mu_{X_2}$. Extending this result to the case of n variables, we have, for $1 \le i \le n$,

$$\mathbb{E}\left[\frac{X_i}{S_n}\right] = \mathbb{E}\left[\frac{X_1}{S_n}\right]$$

Thus,

$$1 = \mathbb{E}\left[\frac{S_n}{S_n}\right] = \sum_{i=1}^{n} \mathbb{E}\left[\frac{X_i}{S_n}\right] = n\mathbb{E}\left[\frac{X_1}{S_n}\right]$$

And thus $\mathbb{E}\left[\frac{X_1}{S_n}\right] = \frac{1}{n}$. In conclusion we have that $\mathbb{E}[S_m/S_n] = m/n$.

On the other hand, if $m > n$, we have that:

$$\mathbb{E}\left[\frac{S_m}{S_n}\right] = \mathbb{E}\left[\frac{S_n}{S_n}\right] + \mathbb{E}\left[\frac{S_m - S_n}{S_n}\right] = 1 + \mathbb{E}\left[\frac{\sum_{i=n+1}^{m} X_i}{S_n}\right]$$

Now, each X_i for $i > n$ is independent of S_n, and thus we have that:

$$\mathbb{E}\left[\frac{\sum_{i=n+1}^{m} X_i}{S_n}\right] = \sum_{i=n+1}^{m} \mathbb{E}\left[\frac{X_i}{S_n}\right] = \sum_{i=n+1}^{m} \mathbb{E}[X_i]\mathbb{E}\left[\frac{1}{S_n}\right] = (m-n)\mathbb{E}[X_1]\mathbb{E}\left[\frac{1}{S_n}\right]$$

Which finally shows that:

$$\mathbb{E}[S_m/S_n] = 1 + (m-n)\mathbb{E}[X_1]\mathbb{E}[1/S_n]$$

4. Recall that $X_n \to X$ in probability iff given any subsequence X_{n_m} of X_n, there is a further subsequence $X_{n_{m_k}} \to X$ a.s. So, let $X_{n_m} + Y_{n_m}$ be a subsequence of $X_n + Y_n$. Find a subsequence of X_{n_m}, say $X_{n_{m_k}} \to X$ a.s.. Further, find a subsequence of $Y_{n_{m_k}}$ say $Y_{n_{m_{k_l}}} \to Y$ a.s.. Then, we have that $X_{n_{m_{k_l}}} + Y_{n_{m_{k_l}}} \to X + Y$ a.s., which shows that $X_n + Y_n \to X + Y$ in probability.

   Similarly, let $X_{n_m} Y_{n_m}$ be a subsequence of $X_n Y_n$. Find a subsequence of X_{n_m}, say $X_{n_{m_k}} \to X$ a.s.. Further, find a subsequence of $Y_{n_{m_k}}$ say $Y_{n_{m_{k_l}}} \to Y$ a.s.. Then, we have that $X_{n_{m_{k_l}}} Y_{n_{m_{k_l}}} \to XY$ a.s., which shows that $X_n Y_n \to XY$ in probability.

5. For a fixed k, we have that,

$$\mathbb{E}\left[(X_k - \overline{X})^2\right] = \mathbb{E}\left[X_k^2\right] - 2\mathbb{E}\left[X_k\overline{X}\right] + \mathbb{E}\left[\overline{X}^2\right]$$

First,

$$\mathbb{E}\left[X_k\overline{X}\right] = \frac{1}{n}\sum_i \mathbb{E}[X_k X_i] = \frac{1}{n}\mathbb{E}\left[X_k^2\right] + \frac{n-1}{n}\mathbb{E}[X_k]^2$$

Second,

$$\mathbb{E}\left[\overline{X}^2\right] = \frac{1}{n^2}\sum_{i,j}\mathbb{E}[X_i X_j] = \frac{1}{n^2}\left(n\mathbb{E}\left[X_k^2\right] + n(n-1)\mathbb{E}[X_k]^2\right) = \frac{1}{n}\mathbb{E}\left[X_k^2\right] + \frac{n-1}{n}\mathbb{E}[X_k]^2$$

Adding these together shows that,

$$\begin{aligned}\mathbb{E}\left[(X_k - \overline{X})^2\right] &= \mathbb{E}\left[X_k^2\right] - \frac{2}{n}\mathbb{E}\left[X_k^2\right] - \frac{2(n-1)}{n}\mathbb{E}[X_k]^2 + \frac{1}{n}\mathbb{E}\left[X_k^2\right] + \frac{n-1}{n}\mathbb{E}[X_k]^2 \\ &= \frac{n-1}{n}\mathbb{E}\left[X_k^2\right] - \frac{n-1}{n}\mathbb{E}[X_k]^2 = \frac{n-1}{n}\mathrm{Var}(X_1)\end{aligned}$$

Thus,

$$\mathbb{E}\left[\overline{V}_n\right] = \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}\left[(X_i - \overline{X})^2\right] = \sigma^2$$

6. We shall show that

$$(f(X) - f(Y))(g(X) - g(Y)) \geq 0$$

For each $\omega \in \Omega$, either $X(\omega) \leq Y(\omega)$ or vice versa. In the first case, both terms in the above product are ≤ 0 and the product is ≥ 0. In the second, both are ≥ 0, and so the product is ≥ 0. Thus,

$$\begin{aligned}\mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)g(Y)] - \mathbb{E}[f(Y)g(X)] + \mathbb{E}[f(Y)g(Y)]& \\ = \mathbb{E}[(f(X) - f(Y))(g(X) - g(Y))] &\geq 0\end{aligned}$$

Because X, Y are i.i.d., $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(X)]$, and $\mathbb{E}[f(Y)g(Y)] = \mathbb{E}[f(X)g(X)]$. Adding the two negative terms to both sides yields the desired inequality.

7. By Cauchy-Schwarz, we have that:

$$\mathbb{E}[X \cdot \mathbb{1}_{X>0}]^2 \leq \mathbb{E}\big[X^2\big]\mathbb{E}[1_{X>0}] = \mathbb{E}\big[X^2\big]\mathbb{P}(X > 0)$$

because $X \geq 0$, we know that $X \cdot \mathbb{1}_{X>0} = X$ (For if $X = 0$, then both sides are just 0). So we have that:

$$\frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \leq \mathbb{P}(X > 0)$$

8. Let ${}_mS_n = \sum_{k=m}^n \mathbb{1}_{A_k}$. Then we have that:

$$\mathbb{E}[{}_mS_n] = \sum_{k=m}^n \mathbb{P}(A_k) = \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{k=1}^{m-1} \mathbb{P}(A_k)$$

And,

$$\begin{aligned}\mathbb{E}\big[{}_mS_n^2\big] &= \sum_{m\leq k,j\leq n} \mathbb{E}\big[\mathbb{1}_{A_k}\mathbb{1}_{A_j}\big] = \sum_{m\leq k,j\leq n} \mathbb{P}(A_k \cap A_j) \\ &= \sum_{1\leq k,j\leq n} \mathbb{P}(A_k \cap A_j) - 2\sum_{k=1}^{n}\sum_{j=1}^{m-1} \mathbb{P}(A_k \cap A_j) + \sum_{1\leq k,j\leq m-1} \mathbb{P}(A_k \cap A_j)\end{aligned}$$

We use the following lemma.

Lemma 1. *For* $a_n \uparrow \infty$ *and* $b_n \uparrow \infty$*, if* $\limsup_{n\to\infty} \frac{a_n^2}{b_n} = \alpha > 0$*, then* $\limsup_{n\to\infty} \frac{a_n}{b_n} = 0$ *(and hence* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$*).*

Proof. Suppose instead that $\limsup_{n\to\infty} \frac{a_n}{b_n} = \beta > 0$. Then, eventually $a_n/b_n \geq \beta/2$. Then eventually,

$$\frac{a_n^2}{b_n} \geq \frac{\beta}{2} a_n \uparrow \infty$$

a contradiction. $\square$

We use this to show that a number of terms are negligible in the limsup. First,

$$\limsup_{n\to\infty} \frac{\mathbb{E}[{}_1S_n]^2}{\mathbb{E}[{}_1S_n^2]} = \alpha > 0$$

So by the above lemma $\mathbb{E}[{}_1S_n]/\mathbb{E}[{}_1S_n^2] \to 0$. Also,

$$\sum_{k=1}^{n}\sum_{j=1}^{m-1}\mathbb{P}(A_k \cap A_j) \le \sum_{k=1}^{n}(m-1)\mathbb{P}(A_k) = (m-1)\mathbb{E}[{}_1S_n]$$

Combining the above, we have that:

$$\limsup_{n\to\infty}\frac{\mathbb{E}[{}_mS_n^2]}{\mathbb{E}[{}_1S_n^2]} = \limsup_{n\to\infty}\frac{\mathbb{E}[{}_1S_n^2] - 2\sum_{k=1}^{n}\sum_{j=1}^{m-1}\mathbb{P}(A_k \cap A_j) + \mathbb{E}\left[{}_1S_{m-1}^2\right]}{\mathbb{E}[{}_1S_n^2]} = 1$$

And also, because the bottom destroys all the lower order terms to the right of the first term on the top,

$$\begin{aligned}\limsup_{n\to\infty}\frac{\mathbb{E}[{}_mS_n]^2}{\mathbb{E}[{}_1S_n^2]} &= \limsup_{n\to\infty}\frac{\mathbb{E}[{}_1S_n]^2 - 2\mathbb{E}[{}_1S_n]\mathbb{E}[{}_1S_{m-1}] + \mathbb{E}[{}_1S_{m-1}]^2}{\mathbb{E}[{}_1S_n^2]} \\ &= \limsup_{n\to\infty}\frac{\left(\sum_{1\le k\le n}\mathbb{P}(A_k)\right)^2}{\sum_{1\le k,j\le n}\mathbb{P}(A_k\cap A_j)} = \alpha\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\frac{\mathbb{E}[{}_mS_n]^2}{\mathbb{E}[{}_mS_n^2]} = \alpha$$

after dividing top and bottom by $\mathbb{E}[{}_1S_n^2]$. By using the previous exercise, and noting that ${}_mS_n(\omega) > 0$ iff $\omega \in \bigcup_{k=m}^{n} A_k$, we have that:

$$\mathbb{P}\left(\bigcup_{k\ge m} A_k\right) = \mathbb{P}\left(\limsup_{n\to\infty}\bigcup_{k=m}^{n} A_k\right) \ge \limsup_{n\to\infty}\mathbb{P}\left(\bigcup_{k=m}^{n} A_k\right) \ge \limsup_{n\to\infty}\frac{\mathbb{E}[{}_mS_n]^2}{\mathbb{E}[{}_mS_n^2]} = \alpha$$

Since this holds for any m, and $\bigcup_{k\ge m} A_k \downarrow \{A_k \text{ i.o}\}$, we have that:

$$\mathbb{P}(A_k \text{ i.o.}) \ge \alpha.$$