Math HW5

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1. I claim that if B is an invertible matrix, then BA and A have the same kernel. This is because $BAx = 0 \iff Ax = 0$, the first direction following since B is invertible we must have $Ax = 0$, and the other direction is clear. Also, since the rank is the dimension of the image, equivalently the dimension of the rowspace or column space, we have $\text{rank}(A) = \text{rank}(A^T)$. So, $\text{rank}(AB) = \text{rank}(B^T A^T) = \text{rank}(A^T)\,\text{rank}(A)$ by above. So $\text{rank}(B^T AB) = \text{rank}(A)$ for invertible B. By the structure of a diagonal matrix, the rank of a diagonal matrix is just the number of zeros in the diagonal. So indeed, the theorem follows.

2. We prove the result by induction. The base case of $n = 2$ is as follows. Place the first vector v_1. The second vector has to have negative inner product with v_1, so in particular it has angle at least 90 degrees with v_1. Then place the third vector, which has angle at least 90 degrees with both v_1 and v_2. If somehow the fourth vector had angle greater than 90 degres, if we order the vectors counter-clockwise, we would get that a circle has angle greater than 360 degrees, which can't be.

   Now assume that we had $n+2$ vectors in $\mathbb{R}^n$ $\{v_1, \dots, v_{n+2}\}$ with $\|v_i\| = 1$ and $\langle v_i, v_j \rangle < 0$ for $i \neq j$. In particular, $\langle v_1, v_i \rangle < 0$ for $i > 1$. Project $v_2, \dots, v_{n+2}$ onto the $n-1$ dimensional subspace $\{\langle x, v_1 \rangle = 0\}$, i.e. take $v_i' = v_i - \langle v_i, v_1 \rangle v_1$. Consider:

$$\begin{aligned}
\langle v_i', v_j' \rangle &= \langle v_i - \langle v_i, v_1 \rangle v_1, v_j - \langle v_j, v_1 \rangle v_1 \rangle \\
&= \langle v_i, v_j \rangle - \langle v_i, v_1 \rangle \langle v_j, v_1 \rangle - \langle v_j, v_1 \rangle \langle v_i, v_1 \rangle + \langle v_i, v_1 \rangle \langle v_j, v_1 \rangle \\
&= \langle v_i, v_j \rangle - \langle v_i, v_1 \rangle \langle v_j, v_1 \rangle < 0
\end{aligned}$$

   Since, crucially, $\langle v_1, v_i \rangle$ and $\langle v_j, v_1 \rangle$ are both negative, their product is positive, which can only make $\langle v_i', v_j' \rangle$ smaller. By induction, this setup isn't possible, which completes the proof.

3. Since $(AB)_{ii} = \sum_{j=1}^{n} a_{ij}b_{ji}$ by expanding, we know that $\mathrm{Tr}(AB) = \sum_{i,j} a_{ij}b_{ji}$. Switching the order of summation and renaming the new i with j, we get that it also equals $\sum_{j,i} b_{ij}a_{ji} = \mathrm{Tr}(BA)$. Taking $B = A_2 \cdots A_k$ in the question proves the result.

4. Let $Ax = \lambda x$, and write $\lambda = a+bi$ and $x = y+zi$. Then $Ay+iAz = ay-bz+i(az+by)$. Matching real and imaginary parts, the kth row of this equation is just:

$$\sum_j A_{kj}y_j = ay_k - bz_k$$
$$\sum_j A_{kj}z_j = az_k + by_k$$

The only number we care about bounding is b, so we multiply the top equation by $-z_k$ and the bottom by y_k, add them together and get:

$$\sum_j A_{kj}(-y_jz_k + y_kz_j) = b(y_k^2 + z_k^2)$$

Summing over k yields:

$$\sum_{k,j} A_{kj}(-y_jz_k + y_kz_j) = b\sum_k (y_k^2 + z_k^2)$$

Sending $(k,j) \to (j,k)$, and adding the two equations up gives:

$$\sum_{k,j}(A_{kj} - A_{jk})(-y_jz_k + y_kz_j) = 2b\sum_k (y_k^2 + z_k^2)$$

Firstly,

$$\left(\sum_{k,j}(A_{kj} - A_{jk})(-y_jz_k + y_kz_j)\right)^2 = 4\left(\sum_{k<j}(A_{kj} - A_{jk})(-y_jz_k + y_kz_j)\right)^2$$

By Cauchy-Schwarz, we have that:

$$\left(\sum_{k<j}(A_{kj} - A_{jk})(-y_jz_k + y_kz_j)\right)^2 \leq \left(\sum_{k<j}(A_{kj} - A_{jk})^2\right)\left(\sum_{k<j}(-y_jz_k + y_kz_j)^2\right)$$

Each term in the first sum is bounded by $\max_{k,j} |A_{kj} - A_{jk}|$, and so:

$$b^2 \left(\sum_k (y_k^2 + z_k^2) \right)^2 \leq \binom{n}{2} \max_{k,j} |A_{kj} - A_{jk}|^2 \left(\sum_{k<j} (-y_j z_k + y_k z_j)^2 \right)$$

Adding back a factor of 2 (Recall that the diagonal terms are 0):

$$2b^2 \left(\sum_k (y_k^2 + z_k^2) \right)^2 \leq \binom{n}{2} \max_{k,j} |A_{kj} - A_{jk}|^2 \sum_{k,j} (-y_j z_k + y_k z_j)^2$$

Now we prove Langrange's identity. Notice that, for vectors $a, b \in \mathbb{R}^n$, we have that, by the circular property of the trace we proved before:

$$\begin{aligned}
\|ab^T - ba^T\|_F^2 &= \mathrm{Tr}\big((ab^T - ba^T)^T (ab^T - ba^T)\big) \\
&= \mathrm{Tr}\big(ba^T ab^T - ba^T ba^T - ab^T ab^T + ab^T ba^T\big) \\
&= \|a\|^2 \,\mathrm{Tr}\big(bb^T\big) - 2\langle a, b\rangle \langle b, a\rangle + \|b\|^2 \,\mathrm{Tr}\big(aa^T\big) \\
&= 2\|a\|^2 \|b\|^2 - 2\langle a, b\rangle^2
\end{aligned}$$

Writing this out gives:

$$\left(\sum_i a_i^2 \right) \left(\sum_i b_i^2 \right) - \left(\sum_i a_i b_i \right)^2 = \frac{1}{2} \sum_{i,j} (a_i b_j - a_j b_i)^2$$

In particular,

$$\sum_{k,j} (-y_j z_k + y_k z_j)^2 \leq 2 \left(\sum_k y_k^2 \right) \left(\sum_k z_k^2 \right) \leq \left(\sum_k (y_k^2 + z_k^2) \right)^2$$

Putting our findings together yields:

$$2b^2 \left(\sum_k (y_k^2 + z_k^2) \right)^2 \leq \binom{n}{2} \max_{k,j} |A_{kj} - A_{jk}|^2 \left(\sum_k (y_k^2 + z_k^2) \right)^2$$

Thus,

$$|b| \leq \sqrt{\frac{n(n-1)}{8}} \max_{k,j} |A_{kj} - A_{jk}|.$$