Math 506 HW2

Rohan Mukherjee

April 9, 2024

1. We need only verify that $\mathrm{Hom}_R(M, M)$ has inverses. Let $T \neq 0 \in \mathrm{Hom}_R(M, M)$ be a linear map. Then $\ker T$ is a submodule of M. Since M is irreducible, and T is nonzero, we must have $\ker T = 0$, and similarly $\mathrm{Im}\, T = M$. Thus T has a two-sided inverse T^{-1} that is also a linear transformation, which shows that $\mathrm{Hom}_R(M, M)$ is a division ring.

2. We shall use the vertices $\{-1, 1\}^2$ of the square. Recall that $r \in D_8$ will result in a 90 degree rotation of the square, and s will result in a reflection about the x-axis. Thus, we calculate the image of r to be:

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

And similarly, flipping across the x-axis would be represnted by:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus we can define a (faithful) degree 2 representation of D_8 by sending $r \mapsto T$ and $s \mapsto S$.

3. Let $T = \pi(x^2)$. Since x has order 4 in Q_8, and π is an embedding, we have that $T^2 - I = 0$, meaning that the minimal polynomial of T divides $x^2 - 1$. It cannot be $x - 1$ for obivous reasons, and we wish to show that it is $x + 1$, so suppose by contradiction that it were $x^2 - 1$. Then T is similar to the following matrix:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Recall that $Z(Q_8) = \langle x^2 \rangle$. Let $T = PDP^{-1}$. Then $\varphi = P\pi P^{-1}$ is another embedding of Q_8 into $\mathrm{GL}_2(\mathbb{R})$, and now we have $\varphi(x^2) = D$. Let $\varphi(y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have:

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \varphi(x^2)\varphi(y)\varphi(x^2) = \varphi(y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This in particular shows that $b = c = 0$, hence $\varphi(y)$ is diagonal. However, the condition that $\varphi(y) \neq I$ and $\varphi(y^4) = I$ would show that $a \neq 1$ is a fourth root of 1, which is a contradiction. Thus the minimal polynomial of T is $x + 1$, and $\varphi(x) = -I$. In particular, $\pi(x) = P^{-1}(-I)P = -I$ as well. Thus the minimal polynomial of $\pi(x)$ is $x^2 + 1 = 0$, and it's rational canonical form is:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Replacing π with a conjugate shows that $\pi(x)$ is the above matrix. A quick calculation shows that if $T^2 = -I$, then T is of the form:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

So let $\pi(y)$ equal the above matrix for a suitable choice of a, b, c, d. Rewriting the last relation for the quarternion group, we get $y = x^{-1}yx^{-1}$. Applying π from above gives us the following equation:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}$$

This of course tells us that $b = c$. Thus $\pi(y^2)$ is just

$$(a^2 + b^2)I$$

We cannot possibly have $a^2 + b^2 = -1$ when $a, b \in \mathbb{R}$, which completes the proof. I note that the above contradictions could easily be used to find a degree 2 representation $\pi : Q_8 \to \mathrm{GL}_2(\mathbb{C})$.

4. Assume by contradiction that V was decomposable, i.e. $V = V_1 \oplus V_2$ for some nonzero

submodules V_1, V_2. Let $S = \operatorname{span}\{\, v_i \mid i \in \mathbb{Z} \,\}$. Define the following projection:

$$\pi : V \to S \quad \text{by} \quad v_i \mapsto v_i,\ w_i \mapsto 0$$

And let $\sum b_i v_i \in \pi(V_1) \cap \pi(V_2)$. Finding $\sum a_i v_i + \sum c_i w_i \in V_1$ and $\sum a_i' v_i' + \sum c_i' w_i' \in V_2$ with the above image, by applying our projection we see that

$$\sum b_i v_i = \sum a_i v_i = \sum a_i' v_i'$$

Which shows that terms of each sum is equal by linear independence of the v_i. Since $V_1 + V_2 = V$, it must be that $\sum b_i v_i \in V_1$ or V_2, so suppose without loss of generality it is the first case. Then we have the updated set of equations:

$$\sum b_i v_i = \pi\Big(\sum b_i v_i\Big) = \pi\Big(\sum b_i v_i + \sum c_i' w_i'\Big)$$

Where $\sum b_i v_i \in V_1$ and $\sum b_i v_i + \sum c_i' w_i' \in V_2$. Once again, if $\sum c_i' w_i' \in V_1$, we get a contradiction since then $\sum b_i v_i + \sum c_i' w_i' \in V_1 \cap V_2$, where the sum would be nonzero since $\sum b_i v_i \neq 0$ and the v_i, w_i are linearly independent. Similarly if $\sum c_i' w_i' \in V_2$, we get a contradiction. Thus we have verified that $S = P_1 \oplus P_2$.

Now note that $x - 1, y - 1 \neq 0$, so it acts as an invertible linear transformation. In particular, $(x-1)P_1 \cap (x-1)P_2 = (x-1)(P_1 \cap P_2) = (y-1)P_1 \cap (y-1)P_2 = 0$ by injectivity. Also note that for each $w_i \in T$, w_i is the image of $(x-1)v_{i-1}$ and $(y-1)v_i$ per a simple calculation. These facts together show that $T = (x-1)P_1 \oplus (x-1)P_2 = (y-1)P_1 \oplus (y-1)P_2$.

Now take $\sum a_i w_i \in (x-1)P_1 \cap (y-1)P_2$, and suppose that (WLOG) $\sum a_i w_i \in V_1$. Then $\sum a_i w_i = (x-1)\sum a_i v_{i-1}$ where $\sum a_i v_{i-1} \in P_1$, and $\sum a_i w_i = (y-1)\sum a_i v_i$ where $\sum a_i v_i \in P_2$. Since the projections trivially intersect, it must be that $\sum a_i v_{i-1} \in V_1$ and similarly $\sum a_i v_i \in V_2$. Since V_1, V_2 are K-stable, $(y-1)\sum a_i v_i \in V_2$ and not in V_1, showing that $\sum a_i w_i = 0$. Then $(x-1)P_1 = T \cap (x-1)P_1 = (y-1)P_1 \cap (x-1)P_1$, showing that $(x-1)P_1 \subset (y-1)P_1$. Applying this logic in reverse shows that $(x-1)P_1 = (y-1)P_1$ and that $(x-1)P_2 = (y-1)P_2$.

Since $P_1 \oplus P_2 = S$, suppose WLOG that $v_0 \in P_1$. By the above equality, we have that $(x-1)v_0 = w_1 \in (y-1)P_1$. Thus $(y-1)^{-1}w_1 = v_1 \in P_1$. Similarly, $(y-1)v_0 = w_0 \in (x-1)P_1$, showing that $(x-1)^{-1}w_0 = v_{-1} \in P_1$. By induction $v_i \in P_1$ for all $i \in \mathbb{Z}$, which shows that $P_1 = S$. At last, if any of the $v_i \in V_2$, then $v_i \in P_1 \cap P_2$ a contradiction, so V contains all the v_i. Since V is K-stable, $(y-1)v_i = w_i \in V_1$ for all i, which shows that $V_1 = V$, a contradiction. Thus V is indecomposable.