Math 441 HW1

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1. (a) This is false. $X - (A \cup B) = (X - A) \cap (X - B)$, not union. For example, take $X = \{1, 2\}$, $A = \{1\}$, and $B = \{2\}$. One sees that $X - (A \cup B) = \emptyset$, while $(X - A) \cup (X - B) = \{2\} \cup \{1\} = \{1, 2\} \neq \emptyset$.

   (b) This one is also not true. Consider $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Clearly $A \cup C = C \subseteq B \cup C = C$, while $A \not\subseteq B$.

2. Suppose that
$$x \in X \cap \bigcup_{\alpha \in A} Y_\alpha$$
By definition, $x \in X$ and $x \in \bigcup_{\alpha \in A} Y_\alpha$. The second statement says there exists $\beta \in A$ so that $x \in Y_\beta$. Thus,
$$x \in X \cap Y_\beta \subseteq \bigcup_{\alpha \in A} (X \cap Y_\alpha)$$
Which establishes the first direction. Next, if
$$x \in \bigcup_{\alpha \in A} (X \cap Y_\alpha)$$
then there exists $\beta \in A$ so that $x \in (X \cap Y_\beta)$. Thus $x \in X$, and $x \in Y_\beta \subseteq \bigcup_{\alpha \in A} Y_\alpha$. Hence,
$$x \in X \cap \bigcup_{\alpha \in A} Y_\alpha$$
which completes the proof.

3. Suppose that $B \subseteq A$ is infinite, and A is finite. If $B = A$, we are done, so suppose $B \subsetneq A$. Since A is finite, there is a bijective map $\varphi : A \to \{1, \ldots, n\}$. I claim that the restriction of φ to B is injective. Indeed, if this was not the case, then there would be some $x \neq y$ both in B with $\varphi(x) = \varphi(y)$. But since $B \subseteq A$, we would also have $x \neq y$ in A, and hence φ would not be an injection from A to $\{1, \ldots, n\}$, a contradiction. Letting $C = f(B)$, we see by definition that $\varphi|_B : B \to f(B)$ is bijective. Since $f(B) \subset \{1, \ldots, n\}$, we can write

$f(B) = \{k_1, \ldots, k_m\}$ with $m \leq n$. The map ψ that sends $k_i \to i$ is clearly bijective, and since the composition of bijective maps is bijective, we see that $\psi \circ \varphi|_B$ is a bijection from B to $\{1, \ldots, m\}$, a contradiction.

A second solution is as follows: since B is infinite, it is not empty. Since it is not equal to A, it is a proper subset of A. By theorem 6.2, there is a bijection from B to $\{1, \ldots, m\}$ for some $m < n$, a contradiction.

4. (a) Clearly $\emptyset$ and $\mathbb{R}$ are in $\mathcal{T}$. Let $U_1, U_2 \in \mathcal{T}$. If either are the empty set, their intersection is empty, and thus their intersection is in $\mathcal{T}$. If (WLOG) $U_1 = \mathbb{R}$, then $U_1 \cap U_2 = U_2 \in \mathcal{T}$, and we are done. So suppose that U_1 and U_2 are both not empty or all of $\mathbb{R}$. Then $U_1 = (-\infty, p)$ and $U_2 = (-\infty, q)$. Clearly $U_1 \cap U_2 = (-\infty, \min\{p, q\})$, and since $\min\{p, q\}$ is either p or q, this is in $\mathcal{T}$ by definition. Inductively continuing this process, we see that the intersection of any collection of finite sets from $\mathcal{T}$ lie in $\mathcal{T}$. Let $\{(-\infty, p)\}_{p \in I}$ denote any collection of open sets in $\mathcal{T}$ (WLOG, none of them are the empty set, since they contribute nothing to the union, and if all are the empty set then the empty set is clearly in $\mathcal{T}$, so you are already done, and if any are $\mathbb{R}$, then the union is all of $\mathbb{R}$, so you are also done). It is immediately clear that $\bigcup_{p \in I}(-\infty, p) \subset (-\infty, \sup_{p \in I} p)$, where the sup can potentially be $+\infty$ (In that case you would get all of $\mathbb{R}$), since if $a \in \bigcup_{p \in I}(-\infty, p)$, then $a < p$ for some p, and thus $a < \sup_{p \in I} p$ so $a \in (-\infty, \sup_{p \in I} p)$. Assuming that p is finite, given any $x \in (-\infty, \sup_{p \in I} p)$, $a < \sup_{p \in I} p$ and hence there is some $p \in I$ so that $a < p$ (taking $\varepsilon = (\sup_{p \in I} p - a)/2$). Thus $x \in (-\infty, a) \subseteq \bigcup_{p \in I}(-\infty, p)$, so set equality is attained. Since $\sup_{p \in I} p$ is either in $\mathbb{R}$ or is equal to $+\infty$, the proof is complete.

   (b) Let a_n denote the first n digits of the decimal expansion for $\sqrt{2}$ (e.g., $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, and so on). It is clear that $a_n \to \sqrt{2}$ in the limit, and also that $10^n \cdot a_n$ is an integer, by construction. Thus a_n is always rational as it may be expressed as the ratio of two integers, and hence $(-\infty, a_n) \in \mathcal{T}$ for every $n \in \mathbb{N}$. If $\mathcal{T}$ was a topology, $\bigcup_{n=1}^{\infty}(-\infty, a_n) = (-\infty, \sqrt{2}) \in \mathcal{T}$, but this is a contradiction since $\sqrt{2} \notin \mathbb{Q}$.