Math 335 HW8

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November 14, 2023

1. We first note that

$$\int_{\partial S} -y dx = \int_{\partial S} \begin{pmatrix} -y \\ 0 \end{pmatrix} \cdot d\mathbf{x}$$

We can parameterize the boundary by 4 curves, but one must note that this parameterization traces clockwise. So we must multiply by a -1 in the integral, which actually ends up making things nicer.

$$\begin{aligned} \gamma_1(t) &= (a, t) & 0 \leq t \leq f(a) \\ \gamma_2(t) &= (t, f(t)) & a \leq t \leq b \\ \gamma_3(t) &= (b, f(b) - t) & 0 \leq t \leq f(b) \\ \gamma_4(t) &= (b - t, 0) & 0 \leq t \leq b - a \end{aligned}$$

We wish then to evalute the integral

$$\int_{-\partial S} \begin{pmatrix} y \\ 0 \end{pmatrix} \cdot d\mathbf{x}$$

where I put a $-$ in front of ∂S to signify that we are going clockwise (this is like swapping the bounds of the integral). By our parameterization above, we get that this integral equals

$$\int_0^{f(a)} \begin{pmatrix} t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \int_a^b \begin{pmatrix} f(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ f'(t) \end{pmatrix} dt + \int_0^{f(b)} \begin{pmatrix} f(b) - t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt \\ + \int_0^{b-a} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dt$$

A keen eye notices that the first, third, and fourth integral equal 0 (the dot product ends up being 0 in all of those cases). The resultant integral is precisely

$$\int_a^b f(t) dt$$

which is exactly what we wanted to show.

2. By noting that $\frac{\partial g}{\partial n} = \nabla g \cdot n$, where n is the outward normal vector to the surface, we see that

$$\int_{\partial S} f \frac{\partial g}{\partial n} ds = \int_{\partial S} f \cdot \nabla g \cdot n ds$$
$$= \int_{\partial S} \begin{pmatrix} f \cdot \frac{\partial g}{\partial x} \\ f \cdot \frac{\partial g}{\partial y} \end{pmatrix} \cdot n ds$$

By Corollary 5.17 in the book, we see that this integral equals

$$\int_S \frac{\partial}{\partial x} f \cdot \frac{\partial g}{\partial x} + \frac{\partial}{\partial y} f \cdot \frac{\partial g}{\partial y} dA$$
$$= \int_S \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial x^2} f + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial y^2} f dA$$
$$= \int_S f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) + \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{pmatrix} dA$$
$$= \int_S f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) + \nabla f \cdot \nabla g dA$$

which completes the proof.

3. We parameterize the ellipsoid by

$$G(\theta, \varphi) = (a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), b\cos(\phi))$$

This is just spherical coordinates but scaled up a little in two directions. Plugging this into the equation $x^2/a^2 + y^2/a^2 + z^2/b^2 = 1$ verifies that it works (it is also spherical coordinates). We then see clearly that $\theta \in [0, 2\pi]$, and that $\phi \in [0, \pi]$, by simply comparing this to spherical–we want to be integrating over the entire shape, but this time it is an ellipsoid. This does not change the angles. A very long calculation, which I have done but will not type up, shows that

$$\|\frac{\partial G}{\partial \theta} \times \frac{\partial G}{\partial \varphi}\| = a\sin(\varphi)\sqrt{a^2\cos^2(\phi) + b^2\sin^2(\phi)}$$

Our surface area is therefore

$$\int_0^{2\pi}\int_0^{\pi} a\sin(\varphi)\sqrt{a^2\cos^2(\phi) + b^2\sin^2(\phi)} d\phi = 2\pi a \int_0^{\pi} \sin(\varphi)\sqrt{a^2\cos^2(\phi) + b^2\sin^2(\phi)} d\phi$$

Letting $u = \cos(\varphi)$, and noting that this transformation is bijective on $\varphi \in [0, \pi]$, seeing that $\sin^2(\phi) = 1 - u^2$, and finally that $du = -\sin(\varphi)d\varphi$, we may conclude that

$$2\pi a \int_0^{\pi} \sin(\varphi)\sqrt{a^2\cos^2(\phi) + b^2\sin^2(\phi)} d\varphi = 2\pi \int_{-1}^{1} \sqrt{b^2 + (a^2 - b^2)u^2} du$$
$$= 2\pi ab \int_{-1}^{1} \sqrt{1 + \frac{a^2 - b^2}{b^2} u^2 du}$$

Finally, noting that (The proof of this theorem is left as a trivial exercise to the reader)

$$\int_{-1}^{1} \sqrt{1+cu^2}du = \sqrt{c+1} + \frac{\sinh^{-1}(\sqrt{c})}{\sqrt{c}}$$

We conclude that this integral equals

$$2\pi ab \cdot \left(\frac{a}{b} + \frac{b \cdot \sinh^{-1}(\sqrt{(a^2-b^2)/b^2})}{\sqrt{a^2-b^2}} \right) = 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2-b^2}} \sinh^{-1}\left(\frac{\sqrt{a^2-b^2}}{b} \right)$$

4. Because the upper half of the unit sphere has radial symmetry, it must necessarily have a center of mass on the z-axis (If it was elsewhere, this would mean its heavier i.e. more volume around a point that's not the z-axis, which doesn't make any sense). So what's left is to calculate the center of mass z-coordinate. We do this by parameterizing the unit sphere by

$$G(s,t) = (s, t, \sqrt{1-s^2-t^2})$$

The projection of the upper half of the unit sphere is going to be $B(0,1) \subset \mathbb{R}^2$, which is exactly what values (s,t) can take on, as the point (s,t) is the upper half of the unit sphere projected down. So $(s,t) \in B(0,1)$. It is now clear that

$$\frac{\partial G}{\partial s} = \left(1, 0, \frac{-s}{\sqrt{1-s^2-t^2}}\right)$$
$$\frac{\partial G}{\partial t} = \left(0, 1, \frac{-t}{\sqrt{1-s^2-t^2}}\right)$$

Taking the cross product of these two gives us

$$\frac{\partial G}{\partial s} \times \frac{\partial G}{\partial t} = \begin{pmatrix} \frac{s}{\sqrt{1-s^2-t^2}} \\ \frac{t}{\sqrt{1-s^2-t^2}} \\ 1 \end{pmatrix}$$

Who's magnitude is now

$$\sqrt{\frac{s^2+t^2+1-s^2-t^2}{1-s^2-t^2}} = \frac{1}{\sqrt{1-s^2-t^2}}$$

Where we did $1^2 = 1 = \frac{1-s^2-t^2}{1-s^2-t^2}$ and simplified. Our integral is now

$$\begin{aligned}
\left(\int_{\partial S} z dS\right) / \int_{\partial S} 1 dS &= \int_{B(0,1)} \frac{\sqrt{1-s^2-t^2}}{\sqrt{1-s^2-t^2}} dA / \int_{\partial S} 1 dS \\
&= \int_{B(0,1)} 1 dA / \int_{\partial S} 1 dS \\
&= \pi / \int_{\partial S} 1 dS \\
&= \pi / 2\pi \\
&= \frac{1}{2}
\end{aligned}$$

Where I converted to polar (the bounds are obvious are we are integrating over the unit ball in $\mathbb{R}^2$). Note that we also showed in class that the surface area of the upper hemisphere of the unit sphere is 2π. So the center of mass is going to be $(0, 0, 1/2)$, which is really nice, and even makes a lot of intuitive sense.

5. Writing $f = f(x, y, z)$, and $g = g(x, y, z)$, we see that $\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g = \begin{pmatrix} \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \end{pmatrix}$ just from the definition of the cross product. For a function $h(x, y, z)$, $\text{div}(h) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z}$. We see that the derivative of the first coordinate of $\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$ w.r.t. x would give us (by the product rule)

$$\frac{\partial f}{\partial xy}\frac{\partial g}{\partial z} + \frac{\partial g}{\partial xz}\frac{\partial f}{\partial y} - \frac{\partial g}{\partial yx}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial zx}\frac{\partial g}{\partial y}$$

Differentiating the second coordinate w.r.t. y, also through the product rule, gives us

$$\frac{\partial g}{\partial xy}\frac{\partial f}{\partial z} + \frac{\partial f}{\partial zy}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial xy}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial zy}\frac{\partial f}{\partial x}$$

Differentiating the third coordinate w.r.t. z gives us

$$\frac{\partial f}{\partial xz}\frac{\partial g}{\partial y} + \frac{\partial g}{\partial yz}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial xz}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial yz}\frac{\partial g}{\partial x}$$

Noting that the order of differentiation doesn't matter, if you look carefully, all terms cancel (after adding them) (for example, the 4th term in row 1 cancels with the 1st term in row 3), so $\text{div}(\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$.