Math 334: Problem Set 2

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1. Let $\mathbf{x} \neq 0 \neq \mathbf{y}$. If we have equality, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\|\|\mathbf{y}\| \iff \langle \mathbf{x}, \mathbf{y} \rangle^2 = \|\mathbf{x}\|^2\|\mathbf{y}\|^2$$

$$\iff 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^4}\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2$$

Now let $\lambda = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{y}\|^2$. By plugging λ in, we get

$$\|\mathbf{x}\|^2 - 2\lambda\langle \mathbf{x}, \mathbf{y} \rangle + \lambda^2\|\mathbf{y}\|^2 = 0 \iff \langle \mathbf{x} - \lambda\mathbf{y}, \mathbf{x} - \lambda\mathbf{y} \rangle = 0$$

$$\iff \|\mathbf{x} - \lambda\mathbf{y}\|^2 = 0 \iff \mathbf{x} - \lambda\mathbf{y} = 0 \implies \mathbf{x} = \lambda\mathbf{y} \quad \square$$

2. Note,

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle \\
&= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \\
&= 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)
\end{aligned}$$

$$\begin{aligned}
\frac{\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2}{4} &= \frac{1}{4} \cdot \left(\|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 - \left(\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2\right)\right) \\
&= \frac{1}{4} \cdot 4\langle \mathbf{x}, \mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{y} \rangle \quad \square
\end{aligned}$$

3. Let $\mathbf{x} \in \mathbb{R}^n$ with $n \geq 2$. If $\mathbf{x} = \mathbf{0}$, simply choose $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \neq \mathbf{0}$, and note that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. So now let $\mathbf{x} \neq \mathbf{0}$, say

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Then, because $\mathbf{x} \neq \mathbf{0}$, there are a_i and a_j not both 0. Then, choose

$$\mathbf{y} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -a_j \\ 0 \\ \vdots \\ 0 \\ a_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{where } -a_j \text{ is in the } i\text{-th row and } a_i \text{ is in the } j\text{-th row.}$$

and note that $\mathbf{y} \neq \mathbf{0}$. Also note that

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= 0a_1 + 0a_2 + \cdots + 0a_{i-1} + a_i \cdot (-a_j) + 0a_{i+1} + \cdots + 0a_{j-1} + a_j \cdot a_i + 0a_{j+1} + \cdots + 0a_n \\
&= -a_i a_j + a_i a_j \\
&= 0 \quad \square
\end{aligned}$$

Let $\mathbf{x_i}$ = the number of Chris Pratt movies in 2021-i, and $\mathbf{y_i}$ = the amount of wheat produced in China in 2021-i. If you take all the data and find the correlation, you get that the correlation is about 0.5273, which is much higher than I would've guessed.

The base case is $n = 2$, so let $\mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n$ with $\langle \mathbf{x_i}, \mathbf{x_j} \rangle = 0$ if $i \neq j$. Then

$$\begin{aligned}
\|\mathbf{x_1} + \mathbf{x_2}\|^2 &= \langle \mathbf{x_1} + \mathbf{x_2}, \mathbf{x_1} + \mathbf{x_2} \rangle \\
&= \|\mathbf{x_1}\|^2 + 2\langle \mathbf{x_1}, \mathbf{x_2} \rangle + \|\mathbf{x_2}\|^2 \\
&= \|\mathbf{x_1}\|^2 + \|\mathbf{x_2}\|^2 \text{ because of the property above.}
\end{aligned}$$

Let $x_1, x_2, \ldots x_m \in \mathbb{R}^n$ be such that if $i \neq j$, then $\langle \mathbf{x_i}, \mathbf{x_j} \rangle = 0$. Suppose now that there is some m so that

$$\|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_m}\|^2 = \|\mathbf{x_1}\|^2 + \|\mathbf{x_2}\|^2 + \cdots + \|\mathbf{x_m}\|^2$$

We see that if $x_{m+1} \in \mathbb{R}^n$ with the same property, then

$$\begin{aligned}
\|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_{m+1}}\|^2 &= \langle \mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_{m+1}}, \mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_{m+1}} \rangle \\
&= \|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_m}\|^2 + 2\langle \mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_m}, \mathbf{x_{m+1}} \rangle + \|\mathbf{x_{m+1}}\|^2 \\
&= \|\mathbf{x_1} + \mathbf{x_2} + \cdots + \mathbf{x_m}\|^2 + 2\langle \mathbf{x_1}, \mathbf{x_{m+1}} \rangle + \cdots + 2\langle \mathbf{x_m}, \mathbf{x_{m+1}} \rangle + \|\mathbf{x_m}\|^2
\end{aligned}$$

Then, by the induction hypothesis and because $\langle \mathbf{x_i}, \mathbf{x_j} \rangle = 0$ for all $i \neq j$, we see that the above equals

$$\|\mathbf{x_1}\|^2 + \|\mathbf{x_2}\|^2 + \cdots + \|\mathbf{x_{m+1}}\|^2 \quad \square$$

4. Let $\varepsilon > 0$.

Lemma 0.1. *If $v_1, v_2, \ldots v_n \in \mathbb{R}^m$ and $n > 1$, $\min\limits_{1 \leq i < j \leq n} \{\text{angle between } v_i \text{ and } v_j\} \leq 2\pi/n$.*

Proof. Suppose instead that $\min\limits_{1\leq i<j\leq n}\{\text{angle between } v_i \text{ and } v_j\} > 2\pi/n$. If we start at the origin, and move counterclockwise, the sum of the angles between two vectors that are next to each other will be equal to 2π. But this is a contradiction, because the sum of all these angles must be $> n \cdot 2\pi/n = 2\pi$. $\square$

Lemma 0.2. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, *we have that* $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos($*angle between* $\mathbf{x}$ *and* $\mathbf{y})$

Proof. If we treat $\mathbf{x}, \mathbf{y}$ as points in $\mathbb{R}^2$, we can say that the chord connecting them is equal to $\mathbf{y} - \mathbf{x}$. Now note that

$$\|\mathbf{x}\|^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}\rangle = \|\mathbf{x} - \mathbf{y}\|^2$$

Now if we treat all 3 vectors as line segments, you can use the law of cosines to get that

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\| \cos(\text{angle between } \mathbf{x} \text{ and } \mathbf{y})$$

Now if we match these equations and rearrange, we get that

$$\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\|\|\mathbf{y}\| \cos(\text{angle between } \mathbf{x} \text{ and } \mathbf{y})$$

The final thing to note is that this equation still works when these vectors don't make a triangle, i.e. when $\mathbf{y} = \mathbf{x}$ or $\mathbf{y} = -\mathbf{x}$. If $\mathbf{y} = \mathbf{x}$, then the angle between x, y is 0, so $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle = \|\mathbf{x}\|^2 \cdot 1 = \|\mathbf{x}\|^2 \cdot \cos(0)$. If $\mathbf{y} = -\mathbf{x}$, then the angle between $\mathbf{x}, \mathbf{y}$ is π, and $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, -\mathbf{x}\rangle = -\langle \mathbf{x}, \mathbf{x}\rangle = \|\mathbf{x}\|^2 \cdot (-1) = \|\mathbf{x}\|^2 \cdot \cos(\pi)$. $\square$

Because $\|\mathbf{v_i}\| = 1$ for all i, and $\langle \mathbf{v_i}, \mathbf{v_j}\rangle = \|\mathbf{v_i}\|\|\mathbf{v_j}\| \cos(\text{angle between } v_i \text{ and } v_j)$, we have that

$$\max_{1\leq i<j\leq n} \langle \mathbf{v_i}, \mathbf{v_j}\rangle = \cos(\min_{1\leq i<j\leq n}\{\text{angle between } v_i \text{ and } v_j\}) \geq \cos(2\pi/n)$$

This is because the inner product will be greatest when the two vectors are closest to each other. The inequality also gets flipped because $\cos(x)$ is decreasing.

First, if $\varepsilon \geq 1$, choose $n = 8$. We see that

$$\cos(2\pi/n) = \cos(\pi/4) = \sqrt{2}/2 > 0 \geq 1 - \varepsilon$$

Now, if $0 < \varepsilon < 1$, choose

$$n > \frac{2\pi}{\arccos(1-\varepsilon)}$$

We also see that

$$\begin{aligned} &\frac{2\pi}{n} < \arccos(1-\varepsilon) \\ \implies \cos&\left(\frac{2\pi}{n}\right) > \cos(\arccos(1-\varepsilon)) = 1 - \varepsilon \end{aligned}$$

because $\cos(x)$ is decreasing. Then by our discussion above, we have proven that this n works for every list $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. $\square$

5. If we let $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$, then we shall note that $\langle f, g\rangle = \langle g, f\rangle$, as multiplication is commutative, and that $\langle af(x) + bg(x), h(x)\rangle = a\langle f(x), h(x)\rangle + b\langle g(x), h(x)\rangle$ because the integral is known to be a linear operator. The final thing to note is that $\langle f, f\rangle \geq 0$, because the integral of a non-negative function is going to be non-negative. So now we can say the following. Let $f, g : [0, 1] \to \mathbb{R}$, $t \in \mathbb{R}$, and $\|f(x)\| \neq 0 \neq \|g(x)\|$. Then,

$$0 \leq \langle f(x) - tg(x), f(x) - tg(x)\rangle = \|f(x)\|^2 - 2t\langle f(x), g(x)\rangle + t^2\|g(x)\|^2$$

Now choose $t = \frac{\langle f(x), g(x)\rangle}{\|g(x)\|^2}$. Note that

$$\begin{aligned} 0 &\leq \|f\|^2 - 2\frac{\langle f(x), g(x)\rangle^2}{\|g(x)\|^2} + \frac{\langle f(x), g(x)\rangle^2}{\|g(x)\|^4}\|g(x)\|^2 \\ \implies &\langle f(x), g(x)\rangle^2 \leq \|f(x)\|^2\|g(x)\|^2 \end{aligned}$$

Now if we take the square root on both sides, and write out what all of these symbols mean, we get

$$\left|\int_0^1 f(x)g(x)dx\right| \leq \left(\int_0^1 f(x)^2dx\right)^{1/2} \left(\int_0^1 g(x)^2dx\right)^{1/2} \quad \square$$