Math 334 HW 5

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1. **Part 2:** Suppose that $(x_n)_{n=1}^\infty \to x$ is a sequence of reals that converges to x. Then there is some $N > 0$ such that $\forall n > N$, $|x_n - x| < 1$. Clearly there are only finitely many elements in $\{x_n \mid 1 \leqslant n \leqslant N\}$, because N is a finite number. But every finite set is bounded, therefore there is some $R > 0$ such that $\{x_n \mid 1 \leqslant n \leqslant N\} \subset B(x, R)$. But clearly $\{x_n \mid n > N\} \subset B(x, 1)$, and so if we pick $R^* = \max\{1, R\}$, we have that $(x_n)_{n=1}^\infty \subset B(x, R^*)$, i.e. that $(x_n)_{n=1}^\infty$ is bounded. $\square$

   Part 3: Let $(x_{n_k})_{k=1}^\infty$ be a subsequence of $(x_n)_{n=1}^\infty \to x$, and let $\varepsilon > 0$. Then, because $(x_n)_{n=1}^\infty$ is convergent, we know that there is some $N > 0$ such that $|x_n - x| < \varepsilon$ for all $n > N$. Suppose on the contrary that $n_k < k$. It is clear that $n_1 \geqslant 1$, by the definition of a subsequence (the smallest value in $\mathbb{N}$ is certainly 1). Then we have that $1 \leqslant n_k < k$. If we choose $k = 1$, we see that $1 \leqslant n_1 < 1$, which is impossible. So $n_k \geqslant k$. Then for all $k > N$, we see that $n_k \geqslant k$ so $|x_{n_k} - x| < \varepsilon$, and so $(x_{n_k})_{k=1}^\infty \to x$. If all subsequences converge, then because $(x_n)_{n=1}^\infty$ is a subsequence of itself, it also converges. $\square$

   Part 1: We shall construct a subsequence of $(x_{2n})_{n=1}^\infty$, that is, $(x_{6n})_{n=1}^\infty$. Because this sequence is a subsequence of $(x_{2n})_{n=1}^\infty$, its limit must also go to α_1. But we notice that each $6n$ is divisible by 3, and so we see that $(x_{6n})_{n=1}^\infty$ is also a subsequence of $(x_{3n})_{n=1}^\infty$. As all subsequences must go to the same limit as the main sequence by part 2 (if the main sequence converges), we see that $(x_{6n})_{n=1}^\infty \to \alpha_2$. Because the limit is unique, we see that $\alpha_1 = \alpha_2$. Similarly, we define $(x_{3^n})_{n=1}^\infty$. Clearly this is a subsequence of $(x_{3n})_{n=1}^\infty$, as each index is certainly divisible by 3, and it is also a subsequence of $(x_{2n+1})_{n=1}^\infty$ because each 3^n is odd. By the same reasoning, we see that $\alpha_2 = \alpha_3$, so we conclude that $\alpha_1 = \alpha_2 = \alpha_3$. $\square$

2.

   Theorem 0.1. *If* $(a_n)_{n=1}^\infty \to a$, *then* $b_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^n a_k \to a$.

   Proof. Omitted (See Homework 4). $\square$

   Consider the sequence $(f(x_n))_{n=1}^\infty$. Because $f(x)$ is continuous, and because $(x_n)_{n=1}^\infty \to x$, we know that $f(x_n) \to f(x)$. Then, by Theorem 0.1, we see that $\dfrac{1}{n}\displaystyle\sum_{k=1}^n f(x_n) \to f(x)$. $\square$

3. One such function is
$$\frac{1}{\pi}\arctan(x)+\frac{1}{2}$$
This function is bijective because it has a (2-sided) inverse–namely $f^{-1}:(0,1)\to\mathbb{R}$ defined by $f^{-1}(x)=\tan\left(\pi\left(x-\frac{1}{2}\right)\right)$. Its continuity comes from $\arctan(x)$, $\frac{1}{\pi}$, and $\frac{1}{2}$ being continuous, and the sum/product/difference of continuous functions being continuous.

4. Note that $f'(x)=2-4x\geqslant 0$ on $[0,1/2]$, so $f(x)$ is increasing on $[0,1/2]$. Note that because $0<x_0<1$, we have that $2x>0$ and $1-x>0$, therefore $2x(1-x)>0$. The maximum of $2x(1-x)$ is $1/2$, which can be found by setting $f'(x)$ above to 0, using the second derivative test to see that $f(x)$ is concave down everywhere, and noting that the $f(x)$ goes to $-\infty$. We conclude that $0<x_1=f(x_0)\leqslant 1/2$. By the first line, we also see that for all $n\geqslant 1$, $x_n\leqslant f(x_n)=x_{n+1}$. Then by the monotone convergence theorem, $(x_n)_{n=1}^{\infty}\to x$ for some x. I claim that $(x_n)_{n=0}^{\infty}$ converges. Given any $\varepsilon>0$, there is some $N\geqslant 1$ such that for all $n>N$, $|x_n-x|<\varepsilon$ because $(x_n)_{n=1}^{\infty}$ converges. So $(x_n)_{n=1}^{\infty}$ converges as well. This argument is just saying that if we "throw out" the first value, the rest of the sequence behaves nicely. Because $(x_n)_{n=0}^{\infty}$ converges, all of its subsequences converge. $\square$

5. If $(x_n)_{n=1}^{\infty}$ does not converge to 0, then there is an ε^* such that for all $N\in\mathbb{N}$ there is some $n>N$ such that $|x_n|\geqslant\varepsilon^*$. First, note that $y_{n+1}=x_{n+1}^{12}+y_n\geqslant y_n$ because $x_{n+1}^{12}\geqslant 0$. Now suppose that $(y_n)_{n=1}^{\infty}$ is bounded. Then $y_n\to\sup_{n\in\mathbb{N}}\{y_n\}$ by the monotone convergence theorem. By the definition of the supremum, we see that there is some $l\in\mathbb{N}$ such that $\sup_{n\in\mathbb{N}}\{y_n\}-(\varepsilon^*)^{12}/2\leqslant y_l\leqslant\sup_{n\in\mathbb{N}}\{y_n\}$. Now because x_n does not converge to 0, we see that there is some $N>l$ such that $|x_N|\geqslant\varepsilon^*$. Then $y_N\geqslant(\varepsilon^*)^{12}+y_l\geqslant\sup_{n\in\mathbb{N}}\{y_n\}-(\varepsilon^*)^{12}/2+(\varepsilon^*)^{12}>\sup_{n\in\mathbb{N}}\{y_n\}$, a contradiction. Thus y_n is not bounded, and therefore doesn't converge. $\square$

6. We shall construct the maximal subset, and then claim that $N(\varepsilon)$ works on it, therefore it works on every subset. Note first that $0\leqslant p(x)\leqslant 4$ for all $x\in[0,1]$. The case where $a>0$ is very similar, as instead of saying "below the tangent line" you would say "above the tangent line" and get that (WLOG) $f(1)$ is too large, so the proof of that case has been omitted. So suppose $a\leqslant 0$ and that the maximum value of $p(x)$ on $[0,1/2]$ is $f(c)\geqslant 4$. Then the tangent line to $x=1/2$, which $p(x)$ is below, has slope less than (or equal to) $\frac{4-1}{c-1/2}$, as the maximum value $f(1/2)$ can be is 1. Then this secant line is going to be $y=\frac{3}{c-1/2}(x-1/2)+1$, and when you plug in $x=0$, you see that this quantity is <0, as the largest value of $c-1/2$ can be is $1/2$. Now we construct the maximal example. The quadratic closest to $y=0$ is $p(x)=0$ itself. The steepest quadratic has maximum value $\leqslant 4$, say d. I claim that the distance between any other polynomial and these two will be $\leqslant d/2$. Suppose that $q(x)\in A$ has distance $>d/2$ from both of these polynomials. We see that $q(max)\geqslant d/2$, but because the largest value of the polynomial has maximum value d, this maximum could at most be at the same x-coordinate, where the distance between them would be maximal, and less than $d/2$, a contradiction. We repeat this process and see

that if we have 2^n curves, the distance between them is $< 1/2^{n-4}$. So given any ε, choose $N(\varepsilon) = \lceil \log_2 1/\varepsilon + 4 \rceil$. We see that $\sup_{x\in[0,1]} |p(x) - q(x)| < \dfrac{1}{2^{N-4}} < \varepsilon$. $\square$