Math 522 HW3

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Exercise 5.4.4.

Suppose h is harmonic on $\mathbb{R}^d$, i.e.,

$$h(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)dy$$

Let $\xi_1, \xi_2, \ldots$ be i.i.d. uniform on $B(0,1)$, and define $S_n = \xi_1 + \cdots + \xi_n$. Show that $X_n = f(x + S_n)$ is a martingale. (ii) Use this to conclude that in any dimension, bounded harmonic functions must be constant.

Answer. By the conditional probability and independence,

$$E(h(x + S_n + \xi_{n+1}) \mid \mathcal{F}_n) = \frac{1}{|B(x + S_n, 1)|} \int_{B(x+S_n,1)} h(y)dy = h(x + S_n)$$

So $h(x + S_n)$ is a martingale. Since h is bounded, the martingale convergence theorem implies that $h(x + S_n)$ converges a.s. The event $\lim_{n\to\infty} h(x + S_n) = \lambda$ is permutable, so by the hewitt-savage 0-1 law, $\lim_{n\to\infty} h(x + S_n) = \lambda$ a.s. Let η be independent of ξ. Clearly, $h(x + \eta + S_n)$ has the same distribution as $h(x + S_{n+1})$. By the independence of η, and the martingale property,

$$E(h(x + \eta + S_n) \mid \eta) = E(h(x + y + S_n))\Big|_{y=\eta} = h(x + \eta)$$

By the dominated convergence theorem, using that h is bounded,

$$h(x + \eta) = \lim_{n\to\infty} E(h(x + \eta + S_n) \mid \eta) = E(\lim_{n\to\infty} h(x + \eta + S_n) \mid \eta) = \lambda \text{ for a.s. } \eta$$

By the continuity of harmonic functions, we have that h is constant in the ball $B(x,1)$. By repeating the same argument for arbitrary r and sending $r \to \infty$, we have that h is constant everywhere.

5.6.5. Strong law for additive functionals.

Suppose p is irreducible and has stationary distribution π. Let f be a function with $\sum_y |f(y)|\pi(y) < \infty$. Let T_x^k be the time of the kth return to x. (i) Show that

$$V_k^f = f(X(T_x^k)) + \cdots + f(X(T_x^{k+1} - 1)), \quad k \geq 1 \text{ are i.i.d.}$$

with $E|V_k^f| < \infty$. (ii) Let $K_n = \inf\left\{k : T_x^k \geq n\right\}$ and show that

$$\frac{1}{n}\sum_{m=1}^{K_n} V_m^f \to \frac{EV_1^f}{E_x T_x} = \sum f(y)\pi(y) \quad P_\mu \text{ a.s.}$$

(iii) Show that $\max_{m \leq n} V_m^{|f|}/n \to 0$ and conclude

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum f(y)\pi(y) \quad P_\mu \text{ a.s.}$$

for any initial distribution μ.

Answer. We can write $V_k^f = V_1^f \circ \theta(T_x^k)$. In this setting, the first return to x will just be 0, since we start at T_x^k, so the formula for V_k^f above works out. We use a really nice trick to compute $E(V_k^f)$. Notice that $X(T_x^k), \ldots, X(T_x^{k+1} - 1)$ is a walk that starts from x and ends just before x. Then by i.i.d. we can write by the strong markov property, since we start at T_x so the initial distribution is just x:

$$E(V_k^f) = \sum_y f(y) E_x\left(\sum_{m=0}^{T_x - 1} 1_{X_m = y}\right)$$

since we just sum over the values we see, weighted by how many times we see them. Recall, $\mu(y) = E_x(\sum_{m=0}^{T_x-1} 1_{X_m=y})$ defines a stationary measure as long as x is recurrent, which follows since the markov chain is irreducible. Since a stationary distribution exists, as stated in the problem, we know that $\pi(y) = E_x(\sum_{m=0}^{T_x-1} 1_{X_m=y})/\pi(S)$. It is clear that $\pi(S) = E_x(\sum_{m=0}^{T_x-1} 1) = E_x(T_x)$. So we can conclude that:

$$E(V_k^f) = \sum_y f(y)\pi(y)E_x(T_x).$$

Applying this to $|f|$, we know that $E(|V_k^f|) \leq E(V_k^{|f|}) < \infty$. So V_k^f is integrable. Then by the strong law of large numbers, we know that:

$$\frac{1}{n}\sum_{m=1}^{n} V_m^f \to E(V_1^f) = \sum f(y)\pi(y)E_x(T_x) \quad P_\mu \text{ a.s.}$$

Now, we can describe K_n as the smallest number of visits that takes at least n time. By Theorem 5.6.1, we know that since p is irreducible, $N_n(x)/n \to 1/E_x(T_x)$ P_μ a.s. $N_n(x)$ is the number of visits by time n. Now, by our descriptions, $N_n(x) \le K_n \le N_n(x) + 1$. So $K_n/n \to 1/E_x(T_x)$ a.s. Since K_n increases to ∞,

$$\frac{1}{K_n}\sum_{m=1}^{K_n} V_m^f \to E(V_1^f)$$

So we have concluded that:

$$\frac{K_n}{n}\frac{1}{K_n}\sum_{m=1}^{K_n} V_m^f \to \frac{E(V_1^f)}{E_x(T_x)} = \sum f(y)\pi(y) \quad P_\mu \text{ a.s.}$$

For the last part, we use a really nice trick. With $S_n = V_1^{|f|} + \cdots + V_n^{|f|}$, we can write:

$$\frac{V_n^{|f|}}{n} = \frac{S_n}{n} - \frac{n-1}{n}\frac{S_{n-1}}{n-1}$$

sending $n \to \infty$ shows that $V_n^{|f|}/n \to 0$ a.s. Now, let a_n be a sequence of positive real numbers with $a_n/n \to 0$. For any $\varepsilon > 0$, there is n_0 so that if $n \ge n_0$, $|a_n/n| < \varepsilon$. Choosing $n_0 > 1/\varepsilon \max_{m \le n_0} a_m$, we have that $|\max_{m\le n} a_m/n| < \varepsilon$ for all $n \ge n_0$. So as long as $a_n/n \to 0$, $\max_{m\le n} a_m/n \to 0$. Applying this to $V_n^{|f|}$ shows that $\max_{m\le n} V_m^{|f|}/n \to 0$ a.s.

Now,

$$\frac{1}{n}\sum_{m=1}^{K_n} V_m^f = \frac{1}{n}\sum_{m=1}^{n} f(X_m) + \frac{1}{n}\Big(f(X_{n+1}) + \cdots + f(X(T_x^{K_n+1} - 1))\Big)$$

Since $T_x^{K_n-1} < n$ by definition, that last quantity can only contain terms from $V_{K_n-1}^f$ and $V_{K_n}^f$. By triangle inequality it is bounded by $2\max_{m\le n} V_m^{|f|}$. By what we just proved, this shows the error term goes to 0 as $n \to \infty$, and we are done.

5.6.6. Central limit theorem for additive functionals.

Suppose in addition to the conditions in the Exercise 5.6.5 that $\sum f(y)\pi(y) = 0$, and $E_x(V_k^{|f|})^2 < \infty$.
(i) Use the random index central limit theorem to conclude that for any initial distribution μ

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{K_n} V_m^f \to c\chi \text{ under } P_\mu$$

(ii) Show that $\max_{m \le n} V_m^{|f|}/\sqrt{n} \to 0$ in probability and conclude

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{n} f(X_m) \to c\chi \text{ under } P_\mu$$

Answer. First we prove the random index central limit theorem. Let $X_1, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$, and let $S_n = X_1 + \cdots + X_n$. Let N_n be a sequence of nonnegative integer-valued random variables and a_n a sequence of integers with $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. We shall show that $Y_n = S_{N_n}/\sigma\sqrt{a_n} \to \chi$. By the central limit theorem, $Z_n = S_{a_n}/\sigma\sqrt{a_n} \Rightarrow \chi$. For fixed $\varepsilon > 0$, define the event $A_n = \{(1-\varepsilon)a_n \le N_n \le (1+\varepsilon)a_n\}$. For N_n in this range, notice that:

$$|S_{N_n} - S_{a_n}| \le |S_{N_n} - S_{(1-\varepsilon)a_n}| + |S_{(1-\varepsilon)a_n} - S_{a_n}|$$

Now, for $N_n \in [(1-\varepsilon)a_n, (1+\varepsilon)a_n]$,

$$|S_{N_n} - S_{(1-\varepsilon)a_n}| \le \max_{1 \le k \le 2\varepsilon a_n} |S_{(1-\varepsilon)a_n + k} - S_{(1-\varepsilon)a_n}| := M_n$$

So, $|S_{N_n} - S_{a_n}| \le 2M_n$. Kolmogorov's maximal inequality implies that:

$$P\left(\max_{1 \le k \le 2\varepsilon a_n} |S_{(1-\varepsilon)a_n + k} - S_{(1-\varepsilon)a_n}| \ge x\sigma\sqrt{a_n}\right) \le \frac{\mathrm{Var}\big(S_{(1+\varepsilon)a_n} - S_{(1-\varepsilon)a_n}\big)}{x^2\sigma^2 a_n}$$

Now, $\mathrm{Var}\big(S_{(1+\varepsilon)a_n} - S_{(1-\varepsilon)a_n}\big) = 2\varepsilon a_n \sigma^2$. So,

$$P(M_n \ge x\sigma\sqrt{a_n}) \le 2x^{-2}\varepsilon$$

Thus,

$$P(|S_{N_n} - S_{a_n}| \ge x\sigma\sqrt{a_n}) \le P(A_n^c) + P(2M_n \ge x\sigma\sqrt{a_n}) \le P(A_n^c) + 8x^{-2}\varepsilon$$

So,

$$\limsup_{n\to\infty} P(|S_{N_n} - S_{a_n}| \geq x\sigma\sqrt{a_n}) \leq 8x^{-2}\varepsilon$$

As this holds for every $\varepsilon > 0$, we can conclude that $(S_{N_n} - S_{a_n})/\sigma\sqrt{a_n} \to 0$ in probability. By the coming together lemma, we conclude that $S_{N_n}/\sigma\sqrt{a_n} \Rightarrow \chi$.

Using this lemma with $N_n = K_n$ and $a_n = n/E_x(T_x)$, we conclude that, for some constant c,

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{K_n} V_m^f \Rightarrow c\chi$$

I claim that $nP(V_n^{|f|} \geq \varepsilon\sqrt{n}) \to 0$. This is because:

$$nP(V_n^{|f|} \geq \varepsilon\sqrt{n}) = \frac{1}{\varepsilon^2} E\left(\varepsilon^2 n 1_{(V_n^{|f|})^2 \geq \varepsilon^2 n}\right)$$

This last quantity converges pointwise to 0 and is dominated by $(V_n^{|f|})^2$. So by the dominated convergence theorem it converges to 0. Now,

$$P(\max_{m\leq n} V_m^{|f|}/\sqrt{n} \geq \varepsilon) \leq nP(V_n^{|f|} \geq \varepsilon\sqrt{n}) \to 0$$

So $\max_{m\leq n} V_m^{|f|}/\sqrt{n}$ converges to 0 in probability. By the same argument as before, and the coming together lemma again, we conclude that:

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{n} f(X_m) \Rightarrow c\chi.$$