Math 33x Final Homework

Rohan Mukherjee

November 14, 2023

1. Plugging $z = \frac{1}{2}$ into this formula gives us

$$\frac{1}{\pi} = \frac{1}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)$$

Which says (Note: $\prod \frac{1}{a_n} = \frac{1}{\prod a_n}$),

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{1}{1-\frac{1}{4n^2}} = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty}\frac{2n \cdot 2n}{(2n+1)(2n-1)}$$

We wish to find N so that

$$1 \leq \prod_{k=N}^{\infty}\left(1 + \frac{1}{4k^2-1}\right) \leq 1 + \varepsilon$$

Taking logs on both sides gives

$$0 \leq \sum_{k=N}^{\infty}\log\left(1 + \frac{1}{4k^2-1}\right) \leq \log(1+\varepsilon)$$

Now, $\log(1+\varepsilon) = \varepsilon + O(\varepsilon^2)$, and also, we are going to forget about the 4, since all that will do is add a constant. So, this is equivalent to asking when,

$$0 \leq \sum_{k=N}^{\infty}\log\left(1 + \frac{1}{k^2}\right) \leq \varepsilon$$

Which is equivalent to asking when

$$0 \leq \sum_{k=N+1000}^{\infty}\log\left(1 + \frac{1}{k^2}\right) \leq \varepsilon$$

This sum is bounded above by (Note: we used IBP),

$$\int_N^\infty \log\left(1+\frac{1}{x^2}\right)dx = x\log\left(1+\frac{1}{x^2}\right)\Big|_N^\infty + \int_N^\infty \frac{2x}{x^3+x}dx \leq x\log\left(1+\frac{1}{x^2}\right)\Big|_N^\infty + \int_N^\infty \frac{2}{x^2}dx$$

One notes that

$$\lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-2}{x^3+x}}{-\frac{1}{x^2}} = \frac{2x^2}{x^3+x} \to 0$$

And also that

$$\int_N^\infty \frac{2}{x^2}dx = \frac{2}{N}$$

And so the furthest RHS equals

$$N\log\left(1+\frac{1}{N^2}\right)+\frac{2}{N}$$

Taylor expanding the log gives

$$\log\left(1+\frac{1}{N^2}\right) = \frac{1}{N^2} + O\left(\frac{1}{N^4}\right)$$

And so,

$$N\log\left(1+\frac{1}{N^2}\right)+\frac{2}{N} = N\left(\frac{1}{N^2}+O\left(\frac{1}{N^4}\right)\right)+\frac{2}{N} = \frac{3}{N}+O\left(\frac{1}{N^3}\right)$$

$O\left(\frac{1}{N^3}\right) \ll \frac{1}{N}$, so it suffices to take $N \approx \frac{1}{4}\varepsilon^{-1}$.

2. Take $a_1 = -1$ and $a_n = \frac{1}{n^2}$ for $n \in \mathbb{N}_{>1}$. Clearly

$$\prod_{n=1}^{\infty}(1+a_n) = 0$$

While the sum obviously converges to $\frac{\pi^2}{6} - 2$. For the second part, take $a_n = (-1)^n$. It is clear that

$$\prod_{n=1}^{\infty}(1+a_n) = 0$$

While the sum diverges.

3. After playing around with the first couple of factors, I came up with the correct conjecture that

$$\prod_{k=0}^{n}\left(1+z^{2^k}\right) = \sum_{k=0}^{2^{n+1}-1} z^k$$

Proof. The base case is clear. Suppose it is true for $n-1$. Then

$$\left(1+z^{2^n}\right)\cdot\sum_{k=0}^{2^n-1} z^k = \sum_{k=0}^{2^n-1} z^k + \sum_{k=0}^{2^n-1} z^{k+2^n} = \sum_{k=0}^{2^n-1} z^k + \sum_{k=2^n}^{2^n+2^n-1} z^k = \sum_{k=0}^{2^{n+1}-1} z^k$$

□

Letting $n\to\infty$ shows that

$$\prod_{k=0}^{\infty}\left(1+z^{2^k}\right) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

4. First, notice that for any $0<r<1$, $\overline{\mathbb{D}_r}\subset\mathbb{D}$. If f has a zero in 0, we may find an $m>0$ so that f/z^m is holomorphic everywhere and doesn't vanish in 0 by the power series expansion, and also that f is not equivalently 0. In the sum, this shall translate to adding m 1s, which obviously does not affect convergence. Thus, we shall show that if g satisfies the hypothesis, the theorem works on it, and that shall prove the general case from what we talked about above. So suppose g is holomorphic in $\mathbb{C}$ and extends continuously to the unit disc, is bounded, and has zeros $(z_n)_{n=1}^{\infty}$ inside the unit disc none of which are 0. By Jensen's formula, for any $0<r<1$, g only has finitely many roots in $\overline{\mathbb{D}_r}$ (choose r so that no roots are on the boundary), and hence

$$\log|g(0)| = \sum_{i=1}^{N_r}\log\left(\frac{|z_i|}{r}\right) + \int_0^{2\pi}\left|\log\left(re^{i\theta}\right)\right|d\theta$$

Rearranging gives

$$\log|g(0)| - \int_0^{2\pi}\left|\log\left(re^{i\theta}\right)\right|d\theta = \sum_{i=1}^{N_r}\log\left(\frac{|z_i|}{r}\right)$$

Taking a limit as $r\to 1$ on both sides, noting that the integral will exist since f is continuously defined on the boundary of the unit circle, shows that

$$\log|g(0)| - \int_0^{2\pi}\left|\log\left(e^{i\theta}\right)\right|d\theta = \sum_{i=1}^{\infty}\log(|z_i|)$$

The LHS is just a number, and hence

$$\sum_{i=1}^{\infty}\log(|z_i|)$$

converges. We showed long ago that if the sum converges, then it's terms go to 0. Using this gives that

$$\log(|z_i|)\to 0$$

And hence,

$$|z_i| \to 1$$

from taking the exponential on both sides, noting that e^x is continuous everywhere. Now, for the really good idea. One notices that

$$\lim_{x\to 1} \frac{-\log(x)}{1-x} = \lim_{x\to 1} \frac{-\frac{1}{x}}{-1} = 1$$

Now, since $|z_i| < 1$ always, it follows immediately that $\log|z_i| < 0$. That is, $-\log|z_i| > 0$. Thus,

$$\sum_{i=1}^{\infty} -\log|z_i|$$

is a positive termed series. Also, $1 - |z_i| > 0$ as well, and hence

$$\sum_{i=1}^{\infty} 1 - |z_i|$$

is another positive termed series. Now, defined a new function

$$f(x) = \begin{cases} \frac{-\log(x)}{1-x}, \ x > 0, \ x \neq 1 \\ 1, \ x = 1 \end{cases}$$

The limit calculation we did above shows this function is continuous on $\mathbb{R}_{>0}$. Consider the limit

$$\lim_{i\to\infty} \frac{-\log(|z_i|)}{1-|z_i|} = \lim_{i\to\infty} f(|z_i|) = f(\lim_{i\to\infty}|z_i|) = f(1) = 1$$

We may now apply the limit comparison test to conclude that

$$\sum_{i=1}^{\infty} 1 - |z_i|$$

also converges. **Q.E.D.**

5. let λ be the order of an entire function $f : \mathbb{C} \to \mathbb{C}$. Then

$$\begin{aligned}
&|f(z)| \leq Ae^{B|z|^{\lambda}} \\
&\implies \log|f(z)| \leq B|z|^{\lambda} \quad \text{for sufficiently large } |z| \text{ (Take } B \text{ bigger)} \\
&\implies \log\log|f(z)| \leq \lambda \log|z| \\
&\implies \frac{\log\log|f(z)|}{\log|z|} \leq \lambda \\
&\implies \frac{\log\log\sup_{|z|=r}|f(z)|}{\log r} \leq \lambda \\
&\implies \limsup_{r\to\infty} \frac{\log\log\sup_{|z|=r}|f(z)|}{\log r} \leq \lambda
\end{aligned}$$

If $\eta = \limsup_{r\to\infty} \frac{\log\log\sup_{|z|=r}|f(z)|}{\log r} < \lambda$, then we could do these steps backwards to get that $|f(z)| < Ae^{B|z|^\eta}$ for sufficiently large $|z|$, but then it would not be the case that λ was the order of f, a contradiction. This establishes that $\limsup_{r\to\infty} \frac{\log\log\sup_{|z|=r}|f(z)|}{\log r} = \lambda$.

Now, expanding f into a power series as $f(z) = \sum_{n=0}^{\infty} c_n z^n$, let

$$\beta = \limsup_{n\to\infty} \frac{n\log n}{\log\frac{1}{|c_n|}}$$

And let $\varepsilon > 0$. By how we defined order, we can take r sufficiently large so that

$$|f(z)| \leq Ae^{B|z|^{\lambda+\varepsilon/2}} = e^{B|z|^{\lambda+\varepsilon/2}+C}$$

For some constant $C = \log(A)$. Notice now that

$$\lim_{|z|\to\infty} \frac{|z|^{\lambda+\varepsilon}}{B|z|^{\lambda+\varepsilon/2}+C} = \lim_{|z|\to\infty} \frac{|z|^{\lambda+\varepsilon}}{B|z|^{\lambda+\varepsilon/2}} = \lim_{|z|\to\infty} \frac{1}{B}|z|^{\varepsilon/2} \to \infty$$

Hence we may choose $|z|$ sufficiently large so that $B|z|^{\lambda+\varepsilon/2} + C \leq |z|^{\lambda+\varepsilon}$. Thus for $|z|$ sufficiently large,

$$|f(z)| \leq e^{|z|^{\lambda+\varepsilon}} \implies \sup_{|z|=r}|f(z)| \leq e^{r^{\lambda+\varepsilon}}$$

We recall that $c_n = \frac{f^{(n)}(0)}{n!}$, and Cauchy's inequality, which says

$$|c_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leq \sup_{|z|=r}|f(z)|r^{-n}$$

Taking $r = n^{1/(\lambda+\varepsilon)}$ gives us (and n sufficiently large)

$$\begin{aligned} |c_n| &\leq \sup_{|z|=r}|f(z)|n^{-n/(\lambda+\varepsilon)} \leq n^{-n/(\lambda+\varepsilon)}e^n \\ \implies \log|c_n| &\leq -\frac{n}{\lambda+\varepsilon}\log(n) + n \end{aligned}$$

We recall that entire functions are holomorphic everywhere, that is, they radius of convergence ∞, which says $\limsup_{n\to\infty}|c_n|^{1/n} = 0$, and in particular $\limsup_{|c_n|} \to 0$. Then for any positive $\varepsilon > 0$, $|a_n|^{1/n} < \varepsilon$ for all large n, and taking logs on both sides gives $\frac{1}{n}\log|c_n| < \log(\varepsilon)$. Since ε will be made smaller than 1, $\log(\varepsilon)$ is negative, and hence $\frac{1}{\log(\varepsilon)} < \frac{n}{\log|c_n|}$. Letting $\varepsilon \to 0$ and taking a limsup on both sides shows that $\limsup_{n\to\infty} \frac{n}{\log|c_n|} = 0$ (since $n/\log|c_n|$ is negative, and bounded below by something going to 0). Taking quotients, noting that $\log|c_n| < 0$ eventually, and being very careful with the sign gives,

$$\frac{n\log n}{-\log|c_n|} + \frac{n}{\log|c_n|} \leq \lambda + \varepsilon$$

By what we just proved above, taking the limsup on both sides gives

$$\limsup_{n\to\infty} \frac{n\log n}{-\log|c_n|} + \frac{n}{\log|c_n|} = \limsup_{n\to\infty} \frac{n\log n}{-\log|c_n|} \leq \lambda + \varepsilon$$

Letting $\varepsilon \to 0$ gives us that $\beta \leq \lambda$. Now, let $\varepsilon > 0$ once again. By definition of β, we can find n sufficiently large so that (Note: we can also find n sufficiently large so that $|c_n| < 1/2$, so the log of it is negative, and hence $-\log|c_n| > 0$),

$$\begin{aligned}
&\frac{n\log n}{-\log|c_n|} < \beta + \varepsilon \\
&\implies n\log n < -(\beta+\varepsilon)\log|c_n| \\
&\implies (\beta+\varepsilon))\log|c_n| < -n\log n \\
&\implies \log|c_n| < -\frac{n}{\beta+\varepsilon}\log n \\
&\implies |c_n| < n^{-n/(\beta+\varepsilon)}
\end{aligned}$$

One also notices, by triangle inequality, that

$$\sup_{|z|=r}|f(z)| \leq \sum_{n=0}^{\infty}\sup_{|z|=r}|c_n||z|^n = \sum_{n=0}^{\infty}|c_n|r^n$$

By the maximum modulus princple, $\sup_{|z|\leq r}|f(z)| = \sup_{|z|=r}|f(z)|$, so it follows that the maximum on the disk is also less than or equal to this sum. Now, we see that

$$\sum_{n=0}^{\infty}|c_n|r^n = \sum_{n=0}^{\lfloor(2r)^{\mu+\varepsilon}\rfloor}|c_n|r^n + \sum_{\lfloor(2r)^{\mu+\varepsilon}\rfloor}^{\infty}|c_n|r^n$$

Examining the first sum,

$$\sum_{n=0}^{\lfloor(2r)^{\mu+\varepsilon}\rfloor}|c_n|r^n \leq r^{(2r)^{\mu+\varepsilon}}\sum_{n=0}^{\lfloor(2r)^{\mu+\varepsilon}\rfloor}|c_n|$$

Since r is positive and eventually greater than 1. Since for sufficiently large n $|c_n| < n^{-n/(\beta+\varepsilon)}$, comparing the RHS of

$$\sum_{n=0}^{\lfloor(2r)^{\mu+\varepsilon}\rfloor}|c_n| \leq \sum_{n=0}^{\infty}|c_n|$$

to

$$\sum_{n=0}^{\infty}n^{-n/(\beta+\varepsilon)}$$

will give that the first sum converges (Note: $\lim_{n\to\infty} \sqrt[n]{n^{-n/(\beta+\varepsilon)}} = n^{-1/(\beta+\varepsilon)} \to 0 < 1$, so the last sum I wrote does indeed converge by the root test). Hence,

$$\sum_{n=0}^{\lfloor (2r)^{\mu+\varepsilon} \rfloor} |c_n| r^n \leq C \cdot r^{(2r)^{\mu+\varepsilon}}$$

for some constant $C > 0$. Next, notice that for the second sum, since $n \geq (2r)^{\mu+\varepsilon}$, $n^{n/(\beta+\varepsilon)} \geq (2r)^n$, and hence $n^{-n/(\beta+\varepsilon)} \leq (2r)^{-n}$. Therefore, for r sufficiently large,

$$\sum_{\lfloor (2r)^{\beta+\varepsilon} \rfloor}^{\infty} |c_n| r^n \leq \sum_{\lfloor (2r)^{\beta+\varepsilon} \rfloor}^{\infty} n^{-n/(\beta+\varepsilon)} r^n \leq \sum_{n=\lfloor (2r)^{\beta+\varepsilon} \rfloor}^{\infty} (2r)^{-n} \cdot r^n \leq \sum_{n=0}^{\infty} 2^{-n} = 1$$

And hence, letting $M(r) = \sup_{|z|=r} |f(z)|$,

$$\begin{aligned}
&M(r) \leq r^{(2r)^{\beta+\varepsilon}} + 1 \leq 2r^{(2r)^{\beta+\varepsilon}} \\
&\implies \log M(r) \leq (2r)^{\beta+\varepsilon} \log(r) + \log 2 \\
&\implies \log\log M(r) \leq \log((2r)^{\mu+\varepsilon} \log(r) + \log 2) \leq \log\Big(2(2r)^{\beta+\varepsilon} \log(r)\Big) \\
&\implies \log\log M(r) \leq \log 2 + (\beta+\varepsilon)\log(2r) + \log\log r = (1+\beta+\varepsilon)\log 2 + \log\log r + (\beta+\varepsilon)\log(r) \\
&\implies \frac{\log\log M(r)}{\log(r)} \leq \frac{(1+\beta+\varepsilon)\log 2 + \log\log r}{\log r} + \beta + \varepsilon
\end{aligned}$$

Finally, letting $u = \log r$, noting that $u \to \infty$ as $r \to \infty$,

$$\lim_{r\to\infty} \frac{\log\log r}{\log r} = \lim_{u\to\infty} \frac{\log u}{u} = 0$$

Hence taking a limit as $r \to \infty$ on both sides gives

$$\lambda \leq \beta + \varepsilon$$

Letting $\varepsilon \to 0$ gives $\lambda \leq \beta$. Hence, $\lambda = \beta$. Finally, the order of the series we wanted is

$$\sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\alpha}}$$

Using the formula we just proved gives

$$\lambda = \lim_{n\to\infty} \frac{n\log n}{\alpha \log n!} = \left(\lim_{n\to\infty} \frac{\alpha \log n!}{n \log n}\right)^{-1}$$

Using that $\log(n!) = n\log n - n + O(\log n)$, we get that inside limit equals

$$\alpha \lim_{n\to\infty} \frac{n\log n - n + O(\log n)}{n\log n} = \alpha + \alpha \lim_{n\to\infty} \frac{n + O(\log n)}{n \log n} = \alpha$$

Since clearly $1/\log n \to 0$, and $O(\log n)/(n\log n) = O(1)/n \to 0$. Thus, $\lambda = \alpha^{-1}$, and we are done.

6.

$$\lim_{k\to\infty} \frac{2\sqrt{k+1}}{\sqrt{2k+1}} = \lim_{k\to\infty} \frac{2\sqrt{1+\frac{1}{k}}}{\sqrt{2+\frac{1}{k}}} = \frac{2\lim_{k\to\infty} \sqrt{1+\frac{1}{k}}}{\lim_{k\to\infty} \sqrt{2+\frac{1}{k}}}$$

$$= \frac{2\sqrt{\lim_{k\to\infty} 1+\frac{1}{k}}}{\sqrt{\lim_{k\to\infty} 2+\frac{1}{k}}} = \frac{2\sqrt{1+0}}{\sqrt{2+0}} = \sqrt{2}$$