Math 335 HW6

Rohan Mukherjee

November 14, 2023

1. (a) For the first one, we apply the transformation $u = 1 - x^2$. We therefore see that

$$\int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{1}{2} \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{\sqrt{u}} du$$
$$= \lim_{\varepsilon \to 0} \sqrt{u} \bigg|_0^1 = 1$$

So it converges, with value 1.

(b) For the second one, note that $x^2 + x \geqslant x$ on $[0, 1]$. Then we may say that

$$0 \leqslant \frac{1}{\sqrt{x}\sqrt[3]{x^2+x}} \leqslant \frac{1}{\sqrt{x}\sqrt[3]{x}} = \frac{1}{x^{5/6}}$$

By taking cuberoot (preserves order), inverting both sides (reserves inequality), and multiplying by a positive number $(1/\sqrt{x})$. So we may say that

$$0 \leqslant \int_0^1 \frac{1}{\sqrt{x}\sqrt[3]{x^2+x}} dx \leqslant \int_0^1 \frac{1}{x^{5/6}} dx = \lim_{\varepsilon \to 0} 6x^{1/6} \bigg|_\varepsilon^1 = 6$$

(c) First, we apply the transformation $u = x^{-1}$, and get that

$$\int_1^\infty \tan\left(\frac{1}{x}\right) dx = \int_0^1 \frac{\tan(u)}{u^2} du = \int_0^1 \frac{\sin(u)}{\cos(u)u^2} du$$

We notice that on $[0, 1]$, $\cos(u) \leqslant 1$, so we can say that this integral is bigger than

$$\int_0^1 \frac{\sin(u)}{u^2} du$$

So suffices to show that the integrand is larger than $1/2u^{-1}$ around 0. Notice that

$$\lim_{u \to 0} \frac{\frac{\sin(u)}{u^2}}{\frac{1}{u}} = \lim_{u \to 0} \frac{\sin(u)}{u} = 1$$

So we can certainly find a $\delta > 0$ so that for every $u \in \mathbb{R}_{>0}$ with $u < \delta$, we have that $|\frac{\frac{\sin(u)}{u^2}}{\frac{1}{u}} - 1| < 1/2$. Using that $-x \leqslant |x|$, we get that $1 - \frac{\frac{\sin(u)}{u^2}}{\frac{1}{u}} < 1/2$, and rearranging this gives us that $1/2 < \frac{\frac{\sin(u)}{u^2}}{\frac{1}{u}}$. Multiplying both sides by the positive quantity (we chose u to be positive) u^{-1}, we get that $\frac{\sin(u)}{u^2} > 1/2u^{-1}$ for a δ-neighborhood around 0. Now applying **Corollary 4.60**, we see that $\int_0^1 \frac{\sin(u)}{u^2}$ diverges, and finally, we deduce that our original integral, $\int_1^\infty \tan(x^{-1})$ also diverges (it was bigger than this integral).

2. We recognize that

$$\int_0^\infty x^{-1/5} \sin\left(\frac{1}{x}\right) dx = \int_0^{\frac{1}{\pi}} x^{-1/5} \sin\left(\frac{1}{x}\right) dx + \int_{\frac{1}{\pi}}^\infty x^{-1/5} \sin\left(\frac{1}{x}\right) dx$$

So, it suffices to show that the latter two integrals converge. For the first one note that $|\sin\left(\frac{1}{x}\right)| \leqslant 1$, so we can say that

$$\left|\int_0^{\frac{1}{\pi}} x^{-1/5} \sin\left(\frac{1}{x}\right) dx\right| \leqslant \int_0^{\frac{1}{\pi}} x^{-1/5} dx$$

By the triangle inequality and noting that $x^{-1/5}$ is strictly positive on $(0, \frac{1}{\pi})$. Next note that by comparing areas, we could instead think of this integral as the area under the curve of the inverse function, which is a function of y. $x^{-1/5}$ maps $(0, \frac{1}{\pi})$ bijectively to $(\pi^{1/5}, \infty)$, so we may say that our integral is equal to

$$\int_{\pi^{1/5}}^\infty \frac{1}{y^5} dy$$

Which clearly converges ($p = 5 > 1$). For the second integral, we first apply the change of variables $u = x^{-1}$, and get that

$$\int_{\frac{1}{\pi}}^\infty x^{-1/5} \sin\left(\frac{1}{x}\right) dx = \int_0^\pi \frac{\sin(u)}{u^{9/5}} du$$

Using the sharp bound that $|\sin(x)| \leqslant x$ on $[0, \pi]$, we see that

$$\left|\int_0^\pi \frac{\sin(u)}{u^{9/5}} du\right| \leqslant \int_0^\pi u^{-4/5} du$$

By doing the same idea from before, we can say this integral equals

$$\int_{\pi^{-4/5}}^\infty \frac{1}{u^{5/4}} du$$

Which clearly converges as again $p = 5/4 > 1$. So the entire integral converges.

3. We showed that the arclength of a curve $\gamma : [a, b] \to \mathbb{R}^n$ is $\int_\gamma d\sigma = \int_a^b \|\gamma'\| dt$.

(a) $\gamma' = (-a\sin(t), a\cos(t), b)$. So we see that $\|\gamma'\| = \sqrt{a^2+b^2}$. Our integral is therefore

$$\int_0^{2\pi} \sqrt{a^2+b^2}dt = 2\pi\sqrt{a^2+b^2}$$

(b) For the second one, notice that $\|\gamma'(t)\| = \sqrt{(t^2-1)^2+4t^2} = \sqrt{t^4-2t^2+1+4t^2} = \sqrt{t^4+2t^2+1} = \sqrt{(t^2+1)^2} = t^2+1$. Our integral therefore becomes

$$\int_0^2 t^2+1dt = \frac{4}{3}+2$$

(c) We see that $\gamma'(t) = \sqrt{\frac{1}{t^2}+4+4t^2} = \sqrt{(2t+t^{-1})^2} = 2t+t^{-1}$ As the inside is positive the whole time. Our integral becomes:

$$\int_0^{2\pi} 2t+t^{-1}dt$$

Which sadly doesn't converge! I suppose then the arclength would be infinite.

4. We parameterize the curve using $\gamma(t) = (t, t^2)$ where $-1 \leqslant t \leqslant 1$. We see that $\gamma'(t) = (1, 2t)$, so $\|\gamma'(t)\| = \sqrt{1+4t^2}$. The x-coordinate (times the length of gamma) becomes

$$\int_{-1}^1 t\sqrt{1+4t^2}dt = 0$$

Where we have noticed that the integrand is odd. So the x-coordinate center of mass of our curve is 0. Next we find the length.

$$\int_{-1}^1 \sqrt{1+4t^2}dt$$

Applying the change of variables $2t = \tan(\theta)$, and noting that our function is even, we get the integral to be

$$\int_0^{\arctan(2)} \sec^3(\theta)d\theta$$

Now I prove the reduction formula for $\sec^n(\theta)$. By integration by parts ($u = \sec^{n-2}(\theta)$, $dv = \sec^2(\theta)$, we may say that

$$\begin{aligned}\int \sec^n(\theta)d\theta &= \sec^{n-2}(\theta)\tan(\theta) - (n-2)\int \sec^{n-2}(\theta)\tan^2(\theta)d\theta \\ &= \sec^{n-2}(\theta)\tan(\theta) - (n-2)\int \sec^{n-2}(\theta) + \sec^n(\theta)d\theta\end{aligned}$$

Adding $n-2$ copies of our goal integral to both sides, we get that

$$(n-1)\int \sec^n(\theta)d\theta = \sec^{n-2}(\theta)\tan(\theta) - (n-2)\int \sec^{n-2}(\theta)d\theta$$

Dividing both sides by $n-1$ yields the familiar result

$$\int \sec^n(\theta) = \frac{\sec^{n-2}(\theta)\tan(\theta)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(\theta)d\theta$$

So now we can find $\int_0^{\arctan(2)} \sec^3(\theta)d\theta$. We have:

$$\int_0^{\arctan(2)} \sec^3(\theta)d\theta = \left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|\right)\Bigg|_0^{\arctan(2)}$$

We notice that $\sec(\arctan(2)) = \sqrt{1+\tan^2(\arctan(2))} = \sqrt{5}$, and clearly $\tan(\arctan(2)) = 2$. So our integral is therefore $\frac{1}{2}(2\sqrt{5}+\ln(\sqrt{5}+2))$, which was the length of our curve. Finally we must find

$$\int_{-1}^{1} t^2\sqrt{1+4t^2}dt = \frac{2}{4}\int_0^1 4t^2\sqrt{1+4t^2}dt$$

Once again applying the substitution $2t = \tan(\theta)$, we get that our integral is:

$$\frac{1}{4}\int_0^{\arctan(2)} \tan^2(\theta)\sec^3(\theta)d\theta = \frac{1}{4}\int_0^{\arctan(2)} \sec^3(\theta)+\sec^5(\theta)d\theta$$

Now one may see why I did the reduction formula! Forgetting about the $\frac{1}{4}$, we know that

$$\begin{aligned}
\int_0^{\arctan(2)} -\sec^3(\theta)+\sec^5(\theta)d\theta &= -\int_0^{\arctan(2)} \sec^3(\theta)d\theta + \frac{1}{4}\sec^3(\theta)\tan(\theta)\Bigg|_0^{\arctan(2)} + \frac{3}{4}\int_0^{\arctan(2)} \sec^3(\theta)d\theta \\
&= -\frac{1}{4}\int_0^{\arctan(2)} \sec^3(\theta)d\theta + \frac{1}{2}5^{\frac{3}{2}} \\
&= -\frac{1}{8}\left(2\sqrt{5}+\ln\left(\sqrt{5}+2\right)\right) + \frac{5^{\frac{3}{2}}}{2}
\end{aligned}$$

Remembering the $\frac{1}{4}$, we see that

$$\int_{-1}^{1} t^2\sqrt{1+4t^2}dt = -\frac{1}{32}\left(2\sqrt{5}+\ln\left(\sqrt{5}+2\right)\right) + \frac{5^{\frac{3}{2}}}{8}$$

So therefore our final answer, the center of mass of the curve, is

$$\left(0, \frac{-2\sqrt{5}-\ln\left(\sqrt{5}+2\right)+4\cdot 5^{3/2}}{32\sqrt{5}+16\ln\left(\sqrt{5}+2\right)}\right)$$

5. First, we prove a mean value theorem.

   Theorem 0.1. *(I don't end up using this theorem). Suppose that* $f:\mathbb{R}^2\to\mathbb{R}, \gamma(t):[0,1]\to\mathbb{R}^2$ *are differentiable everywhere. Then*

   $$\int_\gamma f(x)d\sigma/\text{Length}(\gamma) = f(\gamma(d))$$

   for some $d\in[0,1]$.

Proof. Noting that the line integral preserves inequalities, we may apply the extreme value theorem to f to see that

$$\begin{aligned}
\mathrm{Length}(\gamma)\cdot\min f(\gamma(t)) &= \int_0^1 \min_{t\in[0,1]} f(\gamma(t))\cdot\|\gamma'(t)\|dt \\
&\leqslant \int_0^1 f(\gamma(t))\cdot\|\gamma'(t)\|dt \\
&\leqslant \int_0^1 \max_{t\in[0,1]} f(\gamma(t))\cdot\|\gamma'(t)\|dt \\
&= \max_{t\in[0,1]} f(\gamma(t))\cdot\mathrm{Length}(\gamma(t))
\end{aligned}$$

So $\int_0^1 f(\gamma(t))\cdot\|\gamma'(t)\|dt/\mathrm{Length}(\gamma(t))$ lies between $\min f(\gamma(t))$ and $\max_{t\in[0,1]} f(\gamma(t))$. By the intermediate value theorem, we may say there exists a $d\in[0,1]$ so that $f(\gamma(d)) = \int_0^1 f(\gamma(t))\cdot\|\gamma'(t)\|dt/\mathrm{Length}(\gamma(t))$. $\square$

Theorem 0.2. *(Convex hull!) If $\Omega\subseteq\mathbb{R}^2$ is convex, then given any n points $x_1,\dots,x_n\in\Omega$, and given any positive numbers $\alpha_1,\dots,\alpha_n$ so that $\sum_i\alpha_i=1$, $\alpha_1x_1+\dots+\alpha_nx_n\in\Omega$.*

Proof. We proceed inductively. Given any two points $x,y\in\Omega$, we see that if we are given positive scalars $\alpha_1,\alpha_2\in\Omega$, with $\alpha_2+\alpha_1=1$, i.e. $\alpha_2=1-\alpha_1$, we may write $\alpha_1x+\alpha_2y=\alpha_1x+(1-\alpha_1)y$ which is in our set by the definition of convexity. Now suppose that given any k points $x_1,\dots,x_k\in\Omega$, and any scalars $\alpha_1,\dots,\alpha_k\in\mathbb{R}_{\geqslant 0}$ with $\sum_1^k\alpha_i=1$, we have that $\sum\alpha_ix_i\in\Omega$.

So now given any $k+1$ points $x_1,\dots,x_{k+1}$, and arbitrary scalars $\alpha_1,\dots,\alpha_{k+1}$ that sum to 1, we see that $1-\alpha_{k+1}=\sum_i^k\alpha_k$ as they sum to 1. Then $\sum_1^{k+1}\alpha_ix_i=\alpha_{k+1}x_{k+1}+(1-\alpha_{k+1})\sum_1^k\frac{\alpha_i}{1-\alpha_{k+1}}x_i$. We see that $\sum_1^k\frac{\alpha_i}{1-\alpha_{k+1}}=1$ as the bottom is the constant equal to $\sum_i^k\alpha_k$ (we may pull this number out of the sum), so by the inductive hypothesis $\sum_1^k\frac{\alpha_i}{1-\alpha_{k+1}}x_i$ is also in Ω. Then by the definition of a convex set, we see that $\sum_1^{k+1}\alpha_ix_i\in\Omega$, which finishes the proof. $\square$

This is the proof for center of mass of a convex set, not its boundary: (it felt unfitting to remove, although I haven't checked it's correctness) Now given any $n\in\mathbb{N}$, there exists a partition $P=\{X_1,\dots,X_n\}$ so that $\frac{1}{\mathrm{Vol}(\Omega)}\sum_{i=1}^n x_i\mathsf{Vol}(X_i)$ (where x_i is the bottom left corner of X_i, is within $\frac{1}{n}$ of $\frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega x dx$ (vector-valued integral). We see that $\frac{1}{\mathrm{Vol}(\Omega)}\sum_{i=1}^n x_i\mathsf{Vol}(X_i)$ is of the form described in the second lemma, as $\sum\mathsf{Vol}(X_i)/\mathsf{Vol}(\Omega)=1$, so we see that $\frac{1}{\mathrm{Vol}(\Omega)}\sum_{i=1}^n x_i\mathsf{Vol}(X_i)\in\Omega$. This generates a sequence (y_n), which converges by our construction, and as each y is in Ω, $\lim_{n\to\infty}y_n\in\bar{\Omega}$ by bolzano-weirstrass. As Ω contains its boundary, we may conclude that $\lim_{n\to\infty}y_n\in\Omega$ as well, which is by our construction its center-of-mass.

This is where the real proof starts:

We may say that there exists a partition of $[0,1]$ so that

$\left|\frac{1}{\mathrm{Length}(\gamma)}\sum_{i=1}^{n}\gamma(t_k)\cdot\|\gamma(t_{k+1})-\gamma(t_k)\|-\frac{1}{\mathrm{Length}(\gamma)}\int_\gamma x d\sigma\right| < \frac{1}{n}$ by how we defined the line integral (and simply dividing both sides by a constant, also note that these are vector-valued). Let $\varepsilon_n = \sum_1^n \|\gamma(t_{k+1})-\gamma(t_k)\| - \mathrm{Length}(\gamma)$. Once should notice that $\varepsilon_n \to 0$ in the limit. We see that the point $\sum_{i=1}^{n}\gamma(t_k)\cdot\frac{\|\gamma(t_{k+1})-\gamma(t_k)\|}{\mathrm{Length}(\gamma)+\varepsilon_n} \in \Omega$ by Theorem 0.2. Given any sequence $a_n \to a$, and $b_n \to 0$, and any scalar $c \neq 0$, I claim that $a_n/(c+b_n) \to a/c$. It is clear that $a_n/c \to 0$. Given any $\varepsilon > 0$, note that $a_n/c - a_n/(c+b_n) = a_n b_n/(c+b_n)$. For sufficiently large n, $|b_n| < \varepsilon$, and $a_n < \varepsilon + |a|$. So, $a_n b_n \leqslant (|a|+\varepsilon)\varepsilon$. We can also make N sufficiently large so that $|b_n| < \varepsilon$, so $b_n > -\varepsilon$, and therefore $c + b_n > c - \varepsilon$. Finally, we see that for sufficiently large n $a_n b_n/(c+b_n) < (\varepsilon+|a|)\varepsilon/(c+\varepsilon)$, which is arbitrarily small, completing the mini proof. So we can generate a sequence $(x_n) = \frac{\|\gamma(t_{k+1})-\gamma(t_k)\|}{\mathrm{Length}(\gamma)+\varepsilon_n}$ that also converges to the integral. Note that all $x_n \in \Omega$, and as Ω is closed, we may conclude that its limit is also in Ω. Finally, as its limit is indeed going to be the center-of-mass, we have shown that the center in mass lies in Ω.

Note that the line segment $\{(t,0) \mid 0 \leqslant t \leqslant 1\}$ is a convex set, so if Ω didn't contain its boundary, then it wouldn't contain anything at all, which would be an easy disprove. But obviously this problem wouldn't be that easy (this is my reasoning to conclude Ω is closed).