Math 336 HW4

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1. Let $u(x, y)$ be harmonic everywhere. Since $u(x, y)$ is a harmonic function on a simply connected domain (i.e., a domain without holes), it can be extended to a holomorphic function $f(x, y) = u(x, y) + iv(x, y)$ where $f : \mathbb{C} \to \mathbb{C}$ is holomorphic everywhere. Let $z \in \mathbb{C}$ be arbitrary, and $R > 0$. By the Cauchy Integral Formula, if we let $C =$ the circle of radius R about the point z, we get

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Evaluating this integral directly by parameterizing the circle as $\gamma(t) = z + Re^{it}$, $0 \leq t \leq 2\pi$, and noting that $\gamma'(t) = iRe^{it}$, we see that

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + Re^{it})}{z + Re^{it} - z} iRe^{it} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{it}) dt \end{aligned}$$

Writing $f(z) = u(x, y) + iv(x, y)$, and noting that $z + Re^{it} = (x + r\cos(t), y + r\sin(t))$,

$$u(x, y) + iv(x, y) = \frac{1}{2\pi} \int_0^{2\pi} (x + r\cos(t), y + r\sin(t)) dt + \frac{i}{2\pi} \int_0^{2\pi} v(x + r\cos(t), y + r\sin(t)) dt$$

Matching real parts gives us $u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos(t), y + r\sin(t)) dt$, which completes the proof.

2. The cauchy integral formula states

$$\frac{2\pi i f^{(n)}(z)}{n!} = \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

For the first integral, since $\cosh(z)$ is holomorphic, we see that this integral is just the 2nd derivative of $\cosh(z)$ evaluated at the origin, times $2\pi i/3!$. Noting the crucial identity that $\frac{\mathrm{d}}{\mathrm{dz}}\cosh(z) = \sinh(z)$, and that $\frac{\mathrm{d}}{\mathrm{dz}}\sinh(z) = \cosh(z)$, we get that this integral equals $\frac{\pi i}{3}\cosh(0) = \frac{\pi i}{3}$, which is nice. For the second integral, note that

$$\int_{\gamma_2} \frac{1}{z(z^2-4)}dz = \int_{\gamma_2} \frac{-\frac{1}{4}}{z} + \frac{\frac{1}{8}}{z-2} + \frac{\frac{1}{8}}{z+2}dz$$

$\frac{1}{z+2}$ is holomorphic on a neighborhood of the inside of γ_2, so integrating it along a circle is 0 by Goursat's theorem. By the Cauchy integral theorem, $\int_{\gamma_2} \frac{1}{z}dz = 2\pi i$, and similarly, $\int_{\gamma_2} \frac{1}{z+2}dz = 2\pi i$ (the function in both cases is just $f(z) \equiv 1$). Therefore, our original integral equals $-\frac{1}{4}2\pi i + \frac{1}{8}2\pi i = -\frac{1}{4}\pi i$.

3. For any $\theta \in [0, 2\pi)$, $f(z) = e^{i\theta}z^n$ satisfies the hypothesis, since $|f(z)| = |e^{i\theta}z^n| = |z^n| \leq |z^n|$, and similarly, $|f(z)/z^n| = |e^{i\theta}| = 1 \overset{|z|\to\infty}{\to} 1$, and f is just z^n times a constant, so it is holomorphic and therefore analytic. Clearly $[0, 2\pi)$ is uncountable, so the set of all functions satisfying these conditions is too.

4. We prove that H is holomorphic by giving a formula for the derivative. By definition,

$$\lim_{w\to 0} \frac{1}{w}(H(z+w) - H(z)) = \lim_{w\to 0} \frac{1}{w}\left(\int_a^b h(t)\left[e^{-it(z+w)} - e^{-itz}\right]dt\right)$$

Next, since $g(z) = e^{-itz}$ is a composition of holomorphic functions, it too is holomorphic. We therefore see that for all sufficiently small w, $g(z+w) - g(z) = g'(z) \cdot w + E(w)$ where $|E(w)/w| \to 0$. This says that $e^{-it(z+w)} - e^{-itz} = w(-it)e^{-itz} + E(w)$. Plugging this in we get:

$$\begin{aligned}\lim_{w\to 0} \frac{1}{w}\left(\int_a^b h(t)\left[e^{-it(z+w)} - e^{-itz}\right]\right)dt &= \lim_{w\to 0} \int_a^b (-it)e^{-itz}dt + \frac{1}{w}\int_a^b h(t)E(w)dt \\ &= \int_a^b (-it)e^{-itz}dt + \lim_{w\to 0}\int_a^b h(t)\frac{E(w)}{w}dt\end{aligned}$$

Finally, one notes that given any $\varepsilon > 0$, there is some $\delta > 0$ so that if $|w| < \delta$, $|E(w)/w| < \varepsilon$. Then

$$\left|\int_a^b h(t)E(w)/wdt\right| \leq \varepsilon \int_a^b |h(t)|dt$$

The integral on the right exists because $|h|$ is just another continuous function. Since the LHS is less than every positive number, we see it equals 0, so our function is indeed holomorphic everywhere. By the triangle inequality,

$$\begin{aligned}|H(x+iy)| \leq \int_a^b |h(t)||e^{-it(x+iy)}|dt &= \int_a^b |h(t)|e^{ty}dt \\ &\leq \int_a^b |h(t)|e^{b|y|}dt = \int_a^b |h(t)|dt \cdot e^{b|y|}\end{aligned}$$

So indeed, $H(z)$ is entire of finite type.

5. We are going to show that the Riemann zeta function, $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z$ is holomorphic on $\Re(z) \geq 2$. This is clearly not obvious, as the Riemann zeta function is actually very complicated. We recall that if $f_n \to f$ uniformly, then

$$\lim_{n\to\infty} \oint f_n(z)dz = \oint \lim_{n\to\infty} f_n(z)dz$$

So I claim that $\zeta_n(z) = 1/n^z$ converges uniformly. Notice that, for any z with real part at least 2,

$$\left|\frac{1}{n^z}\right| \leq |\frac{1}{n^{x+iy}}| = \frac{1}{n^x} \leq \frac{1}{n^2}$$

And clearly $\sum_{n=1}^{\infty} \frac{1}{n^2} \to \pi^2/6$, so by the Weistrass M-test the Riemann zeta function converges uniformly on $\Re(z) \geq 2$. Therefore, for any triangle $T \subset \Re(z) > 2$, we see that

$$\oint_T \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^z} dz = \lim_{N\to\infty} \oint_T \sum_{n=1}^{N} \frac{1}{n^z} dz = \lim_{N\to\infty} \sum_{n=1}^{N} \oint_T \frac{1}{n^z} dz = \sum_{n=1}^{\infty} 0 = 0$$

Since n^z is clearly holomorphic on $\Re(z) \geq 2$ for any natural number n (indeed, $n^z = \exp(-z\ln(n))$, and as $\ln(n)$ is just a number, $\exp(z\ln(n))$ is a composition of holomorphic functions and therefore also holomorphic). By Moreras Theorem, we see that the Riemann zeta function is holomorphic on $\Re(z) \geq 2$.