Math 582 HW3

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1. Look at the n-simplex $\{e_1, \ldots, e_n\}$ in n-dimensional space. This convex hull is an $n-1$ dimensional shape, which we can draw the $n-1$ dimensional sphere around. In this new affine subspace, call the vectors $\{v_1, \ldots, v_n\}$. For the function $\sum_i \langle v_i, x\rangle^2 = x^T \sum_i v_i v_i^T x$. This projects x onto the v_i's, takes a sum, and then calculates an inner product. Since the shape is symmetrical, and by rotational symmetry of the sphere, any rotation of the v_i's will yield the same value of the function for all x's. Again by symmetry, the function on $f(v_i)$ for each i will be the same. Finally, it is proportional to $|x|^2$, if we call the common value of $f(v_i) = C$, then $f(x_i) = C|x_i|^2$. This is a tight frame by lots of symmetry. I am unfortunately not sure how to get the exact coordinates of these v_i, but many symmetries shows it is a tight frame.

   I didn't realize that this was only for 2-dimensions inintially, but now I have. Let's use the vectors $(\cos(2\pi k/n), \sin(2\pi k/n))$ for $k = 1, \ldots, n$. Then the function becomes $\sum x_1^2 \cos^2(2\pi k/n) + x_2^2 \sin(2\pi k/n) + x_1 x_2 \cos(2\pi k/n)\sin(2\pi k/n)$.

   Now, notice that:

$$\sum_{i=1}^{n} \cos^2(2\pi k/n) = \frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n} \cos(4\pi i/n)$$

   Let $\zeta = e^{4\pi i/n}$. For odd n, ζ generates the group of nth roots of unity, so the sum will certainly be 0 (by the factorization $x^n - 1 = (x-1)(x^{n-1} + \cdots + 1)$). For even n, $\zeta, \ldots, \zeta^n = 1$ goes twice through the $n/2$ roots of unity, so by applying this fact about the sum of the odd ones, we get that this eventually equals 0 too. In any case, the sum on the rightmost side is just 0, so $\sum_{i=1}^{n} \cos^2(2\pi k/n) = n/2$. Similarly, since $\cos^2(x) + \sin^2(x) = 1$, we get that $\sum_{i=1}^{n} \sin^2(2\pi k/n) = n/2$. Finally, $\sum_{i=1}^{n} \cos(2\pi k/n)\sin(2\pi k/n) = \frac{1}{2}\sum_{i=1}^{n} \sin(4\pi k/n) = 0$ by the same reason. So the function is just $n|x|^2/2$.

2. If $\|\cdot\|_1$ and $\|\cdot\|_2$ have the same unit ball, this means that $\|\cdot\|_1 = 1$ iff $\|\cdot\|_2 = 1$. For arbitrary $x \in \mathbb{R}^n$, we know that $\|x/\|x\|_1\|_1 = 1$, so $\|x/\|x\|_1\|_2 = 1$, which says that $\|x\|_1 = \|x\|_2$ for all $x \in \mathbb{R}^n$.

   With the assumption that $\|x\| = 0$ iff $x = 0$ (I believe you would need some more conditions on B to ensure this is true always. Like, you could take just the set $\{0\}$ or a line through the origin and this definition would break down as the norm would be infinite). Then, $\|\eta x\| = \inf\{\lambda > 0 \mid \eta x \in \lambda B\} = \{\lambda > 0 \mid x \in \lambda/\eta B\} = \{\eta\lambda \mid x \in \lambda B\} = \eta\|x\|$. Since B is convex, $\|x\|B \subset B$, so $x \in B$. On the other hand, if $x \in B$, then $\inf\{\lambda > 0 \mid x \in \lambda B\} \leq 1$ (since 1 works), which shows the other inclusion.

   I now claim that $B = \{x \mid \| x\| \leq 1\}$. Clearly, $\{x \mid \| x\| < 1\} \subset B$, since if $\|x\| < 1$, then $\|x\| < 1 - \eta$ for some small η, and then $x \in (1-\eta)B \subset B$. Since the interior of $\{x \mid \| x\| \leq 1\}$ is just $\{x \mid \| x\| < 1\}$, (this follows since all norms are equivalent to ℓ^2), and we shown that the second set is contained in B, we know that the closure of it is contained in B (assuming that B is closed, which we can just replace it with if not), so B contains its closure too. The reverse inclusion is clear.

   Finally, let $x, y \in \mathbb{R}^n$ and consider $u = x/\|x\|$ and $v = y/\|y\|$. Then since B is convex, and $u, v \in B$, we know that:

$$\frac{\|x\|u}{\|x\| + \|y\|} + \frac{\|y\|v}{\|x\| + \|y\|} \in B$$

   This says that $\|\frac{\|x\|u}{\|x\| + \|y\|} + \frac{\|y\|v}{\|x\| + \|y\|}\| \leq 1$, so $\|x + y\| \leq \|x\| + \|y\|$, which shows that it is a norm.

3. Plugging in $y = x/\|x\|$, we get that $\sup_{\|y\|=1}\langle x, y\rangle \geq \langle x, x/\|x\|\rangle = \|x\|$. On the other hand, if $\|x\| = 1$, by the simple inequality $2ab \leq a^2 + b^2$,

$$\langle x, y\rangle = \sum_i x_i y_i \leq \frac{1}{2}\sum_i x_i^2 + y_i^2 = 1$$

   And otherwise, $\langle x, y\rangle = \|x\|\langle x/\|x\|, y\rangle \leq \|x\|$ by above. This completes the proof.

4. We need only show that B° is centrally symmetric and convex. It is clearly centrally

symmetric, since $\sup_{x\in B} |\langle x, y\rangle| \le 1$ is invariant under $y \mapsto -y$. Let $x, y \in B^\circ$ and $t \in [0,1]$. Then $\sup_{z\in B} |\langle tx+(1-t)y, z\rangle| = \sup_{z\in B} |t\langle x, y\rangle+(1-t)\langle y, z\rangle| \le t \sup_{z\in B} |\langle x, z\rangle|+(1-t) \sup_{z\in B} |\langle y, z\rangle| \le 1$. So $tx + (1-t)y \in B^\circ$, showing that B° is centrally symmetric and convex, so it induces a norm.

5. Let $y = \|y\|_{B^\circ}\zeta$ for $\zeta \in B^\circ$ (taking $\zeta = y/\|y\|_{B^\circ}$, we by a previous question that $\zeta \in B^\circ$). Then by definition of B°, we have $\sup_{a\in B} |\langle a, \zeta\rangle| \le 1$. Then again $x/\|x\|_B \in B$ by the previous question again, so $|\langle x/\|x\|_B, \zeta\rangle| \le 1$, which shows after multiplying both sides by $\|x\|_B\|y\|_{B^\circ}$ that $|\langle x, y\rangle| \le \|x\|_B\|y\|_{B^\circ}$.