Math 334 HW 9

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1. As $f(x)$, $g(y)$ are differentiable on all of $\mathbb{R}$, given any point (x, y) there is some δ, δ' about x and y (resp.) so that $f(x+h) = f(x) + f'(x)h + E(h)$, and $g(y+h') = g(y) + g'(y)h' + E'(h')$ where $E(h), E'(h')$ shrink faster than h, h' (resp.). Then we see that, for $h = (h, h')$ with $\|h\| < \min \delta, \delta'$,

$$\begin{aligned} h(x+h, y+h') &= [f(x) + f'(x)h + E(h)][g(y) + g'(y)h' + E'(h')] \\ &= f(x)g(y) + f'(x)g(y)h + f(x)g'(y)h' \\ &+ \big(f'(x)g'(y)hh' + E(h)[g(y) + g'(y)h + E'(h')] \\ &+ E'(h')[f(x) + f'(x)h + E(h)]\big) \end{aligned}$$

We now see that everything in brackets goes to 0 faster than $\|h\|$, as for example $\lim_{(h,h')\to(0,0)} E(h)[g(y)+g'(y)h' + E'(h')]/\sqrt{h^2 + h'^2}$ is trivial as $\sqrt{h^2 + h'^2} \geq \sqrt{h^2} = |h|$, so $E(h)/\sqrt{h^2 + h'^2} \leq E(h)/|h|$, which tends to 0 as $h \to 0$, so both limits go to 0. The right side now is just a constant, namely $g(y)$, so the entire limit goes to 0. The last term is $f'(x)g'(y)hh'/\sqrt{h^2 + h'^2}$, and as before $hh'/\sqrt{h^2 + h'^2} \leq hh'/|h|$, which is always smaller in magnitude than h', which is tending towards 0. So all terms in the big parenthesis go to 0, and we see that $h(x, y)$ is differentiable at all $(x, y) \in \mathbb{R}^2$.

2. We construct a counterexample.

$$f(x) = -x + 1$$

The theorem states for all functions that satisfy the hypothesis and all $0 \leq x_0 \leq 1$. However, for our function, if we choose $x_0 = 1$, then we see that $x_1 = f(1) = 0, x_2 = f(0) = 1, \ldots$ and so on. Then our sequence is $(x_n)_{n=1}^{\infty} = \begin{cases} 1, & n \equiv 0 \mod 2 \\ 0, & n \equiv 1 \mod 2 \end{cases}$.

Clearly this sequence doesn't converge, as if it were to, there would be some $N > 0$ so that for all $m, n > N$, $|x_m - x_n| < 1/2$. But if we choose $m = n+1$, we get that $|x_m - x_n| = 1 < 1/2$, a contradiction.

3. (a) First off, consider the function defined by $G(s) = f(s + \frac{b-a}{2}) - f(s) - 1/2[f(b) - f(a)]$. We see that $G((a+b)/2) = 1/2f(b) - f((a+b)/2) + 1/2f(a) = -G(a)$. If one side is 0, $s = a$ is so that $\frac{f(a+(b-a)/2)-f(a)}{\frac{b-a}{2}} = \frac{f(b)-f(a)}{b-a}$. Else, one side is positive and one side is negative, so by the IVT ($G(s)$ is continuous because f differentiable $\implies$ continuity, and the composition of continuous functions is continuous) there is some $s \in [a, (a+b)/2]$ so that $G(s) = 0$. This makes s satisfy $\frac{f(s+(b-a)/2)-f(s)}{\frac{b-a}{2}} = \frac{f(b)-f(a)}{b-a}$ by rearranging and dividing both sides by $(b-a)/2$.

   (b) The base case, $n = 1$, has been proven above. Suppose that for some $n \geq 1$, we have that there is some $s_n \in [a, b]$ so that $\frac{f\left(s_n+\frac{b-a}{2^n}\right)-f(s_n)}{\frac{b-a}{2^n}} = \frac{f(b)-f(a)}{b-a}$. Now I define $G(s) = f\left(s + \frac{b-a}{2^{n+1}}\right) - f(s) - \frac{f(b)-f(a)}{2^{n+1}}$. We see that

$$
\begin{aligned}
G(s_n) &= f\left(s_n + \frac{b-a}{2^{n+1}}\right) - f(s_n) + f\left(s_n + \frac{b-a}{2^n}\right) - f\left(s_n + \frac{b-a}{2^n}\right) - \frac{f(b)-f(a)}{2^n} \\
&= -f\left(s_n + \frac{b-a}{2^n}\right) + f\left(s_n + \frac{b-a}{2^{n+1}}\right) + \frac{f(b)-f(a)}{2^{n+1}}
\end{aligned}
$$

   As $-f(s_n) + f\left(s_n + \frac{b-a}{2^n}\right) = \frac{f(b)-f(a)}{2^n}$ by hypothesis. Note also that

$$
G\left(s_n + \frac{b-a}{2^{n+1}}\right) = f\left(s_n + \frac{b-a}{2^n}\right) - f\left(s_n + \frac{b-a}{2^{n+1}}\right) - \frac{f(b)-f(a)}{2^{n+1}}
$$

   That is, $G(s_n + \frac{b-a}{2^{n+1}}) = -G(s_n)$. Then if we look back to the 3 cases in the base case, they also hold here, and we see that in any case, there is some s_{n+1} so that $G(s_{n+1}) = 0 \iff \frac{f\left(s_{n+1}+\frac{b-a}{2^{n+1}}\right)-f(s_{n+1})}{\frac{b-a}{2^{n+1}}} = \frac{f(b)-f(a)}{b-a}$.

   (c) Then we have generated a sequence $(s_n)_{n=1}^{\infty}$ entirely contained in $[a, b]$. Therefore it has a convergent subsequence, say $(s_{n_k})_{k=1}^{\infty} \to s$. Now,

$$
\lim_{k\to\infty} \frac{f\left(s_{n_k} + \frac{b-a}{2^{n_k}}\right) - f(s_{n_k})}{\frac{b-a}{2^{n_k}}} = \lim_{k\to\infty} \frac{f(b) - f(a)}{b-a}
$$

   Because $f(x)$ is differentiable,

$$
\frac{f\left(s_{n_k} + \frac{b-a}{2^{n_k}}\right) - f(s_{n_k})}{\frac{b-a}{2^{n_k}}} = f'(s_{n_k}) - \frac{E(\frac{b-a}{2^{n_k}})}{\frac{b-a}{2^{n_k}}}
$$

   where $E(\frac{b-a}{2^{n_k}})/\frac{b-a}{2^{n_k}} \to 0$. Thus the entire LHS tends to $f'(s)$, where $s \in [a, b]$. Equating the LHS to the RHS gives the desired result. $\square$

4. By the chain rule, the derivative of $f(g(x))$ is $Df(g(x)) \cdot Dg(x)$. Because $f(g(x)) = x$, its jacobian is also just the identity matrix (2x2). So all that we have to do is find what $g(0,0)$ can be. We know that $f(g(0,0)) = (0,0)$. So we see that, by the definition of $f(x)$, $x + \sin(y) = 0$, and $y - x^2 = 0$. $g(0,0)$ will be whatever (x, y) we get. The second equation tells us that $y = x^2$, so we get that $-x = \sin(x^2)$. Notice that if $|x| > 1$, this situation is impossible, so $|x| \leq 1$.

Suppose that there was a solution other than $x = 0$, as $x = 0$ obviously works. Then by the mean value theorem, treating x^2 as our "x", we get that there is some $c \in (0, x^2)$ ($|x| > 0$ so $x^2 > 0$) so that $|\sin(x^2)| = |\cos(c)||x^2|$. On $c \in (0, 1)$, the possible values for x^2, $|\cos(c)| < 1$, so we see that $|\sin(x^2)| < |x^2| < |x|$, so we don't have a solution. Therefore $x = 0$, and $y = 0$. Then $g(0, 0) = (0, 0)$. $Df = \begin{pmatrix} 1 & \cos(y) \\ -2x & 1 \end{pmatrix}$, So $Df(g(0, 0)) = Df(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Now solving this linear system, we see that $Dg(0, 0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

5. I shall look at each function separately. We know that $\langle b, x\rangle = \sum_{i=1}^{n} b_i x_i$. If we apply $\frac{\partial}{\partial x_i}$, we get that $\boldsymbol{\nabla}\langle b, x\rangle = b$ (only the b_i stays per partial derivative). For the second part, let $g(x) = \langle x, Ax\rangle$. Notice that $A = (a_1 \ldots a_n)$ for some $a_1, \ldots, a_n \in \mathbb{R}^n$. Notice that $g(x + h) - g(x) = \langle x + h, A(x + h)\rangle = \langle h, Ah\rangle + \langle x, Ah\rangle + \langle Ax, h\rangle$, but because A is symmetric, the last two terms are the same, so $g(x + h) - g(x) = \langle h, Ah\rangle + 2\langle Ax, h\rangle = \langle \boldsymbol{\nabla} g(x), h\rangle + E(h)$, where $E(h)/\|h\| \to 0$. So therefore I claim that $\langle h, Ah\rangle/\|h\| \to 0$. Notice that $|\langle h, Ah\rangle/\|h\|| \leq \|h\|/\|h\|\|Ah\| = \|Ah\|$. Now, $h = h_1e_1 + \ldots + h_ne_n$ for some $h_i \in \mathbb{R}$, $i \in 1, \ldots, n$, where e_i has a 1 in the i-th column and 0 everywhere else. Then for any $\varepsilon > 0$, choose $\delta = \varepsilon/(n \max_{1\leq i\leq n} \|a_i\|)$. Then we see that for any $h \in \mathbb{R}^n$ with $\|h\| < \delta$, $\max_{1\leq i\leq n} |h_i| < \delta$, $\|Ah\| = \|h_1a_1 + \cdots + h_na_n\| \leq \sum_{i=1}^{n} |h_i|\|a_i\| \leq \sum_{i=1}^{n} \max_{1\leq i\leq n} |h_i| \max_{1\leq i\leq n} \|a_i\| < \varepsilon$. So therefore $\boldsymbol{\nabla} g(x) = 2Ax$. So $\boldsymbol{\nabla} f(x) = 2Ax + b$. By langrange multipliers, the gradient of $\|x\| = 1$ is the same as the gradient of $\langle x, x\rangle = 1$, i.e. $2x$. So we see that $2Ax + b = \lambda 2x$ for some λ. If $b = 0$, this certainly implies the existence of eigenvalues / vectors.

Note that $|x_n|$ is monotonically decreasing, and bounded below, so it converges to some $x \geq 0$. Then, as (x_n) is bounded by $|x_0|$, it has a convergent subsequence, say (x_{n_k}). $x = \lim_{k\to\infty} |x_{n_k}| = |\lim_{k\to\infty} x_{n_k}|$, so $(x_{n_k}) \to \pm x$ (one of them). Finally, $|\sin(x_{n_k})| = |x_{n_k+1}| \to x$, and also to $|\sin(\pm x)|$. In any case, $|\sin(\pm x)| = |\sin(x)|$, as $\sin(x)$ is odd. Note that $|x_1| \leq 1$, so $\inf |x_n| \leq 1$ as well (either 1 is the smallest element, or there is something smaller). If $x \in [-1, 1] - 0$, $|\cos(x)| < 1$, so by the mean value theorem $|\sin(x)| < |x|$ for x in that set. So x must be 0. Finally, as $|x_n| \to 0$, we see that $x_n \to 0$.