Math 441 Final HW

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1. We recall that $\mathbb{S}^n := \{ x \in \mathbb{R}^n \mid |x| = 1 \}$. I claim then that $\mathbb{S}^n \subset [-1,1]^n$. Indeed, if $x \notin [-1,1]^n$, then x has a coordinate who's value is strictly greater in absolute value than 1. Thus its square is strictly greater than 1, contradicting the fact that $|x| = 1$. $[-1,1] \cong [0,1]$ via the homeomorphism $x \to x/2 + 1/2$, which is a homeomorphism since it is a linear bijection (and linear maps are continuous and have continuous inverses). Since the product of compact sets are compact, $[-1,1]^n$ is compact. I claim that $\mathbb{S}^n$ is closed. Indeed, if $\mathbb{S}^n \ni x_n \to x \in \mathbb{R}^n$, we have that

$$\|x_n\| = 1 \text{ for all } n \in \mathbb{N}$$

Thus, since $x \mapsto \|x\|$ is continuous, we have that

$$1 = \lim_{n\to\infty} \|x_n\| = \|\lim_{n\to\infty} x_n\| = \|x\|$$

Which shows that $\mathbb{S}^n$ is closed, thus $\mathbb{S}^n$ is compact as it is a closed subset of a compact space. Similarly, we recall that $\mathbb{RP}^n := \mathbb{S}^n / \sim$, where $x \sim -x$. The quotient topology is defined so that the projection is a quotient map–in particular, it is continuous (and onto). Thus $\mathbb{RP}^n$ is the continuous image of a compact set, and hence compact.

2. We shall construct the topology in the most obvious way: consider the topology generated by the base containing elements of the form

$$B = \left\{ (a_{ij})_{1 \le i \le m,\ 1 \le j \le n} \;\middle|\; a_{ij} \in U_{ij} \subset \mathbb{R},\ U_{ij} \text{ open } \right\}$$

First we need to show that this is indeed a base. It is clear that if we take each $U_{ij} = \mathbb{R}$, then we would get the entire space, thus the basis elements cover $\mathrm{Mat}(m \times n;\ \mathbb{R})$. Next notice that given $B_1 = \left\{ (a_{ij}) \;\middle|\; a_{ij} \in U_{ij} \subset \mathbb{R},\ U_{ij} \text{ open } \right\}$ and $B_2 = \left\{ (a_{ij}) \;\middle|\; a_{ij} \in B_{ij} \subset \mathbb{R},\ B_{ij} \text{ open } \right\}$, we have that

$$B_1 \cap B_2 = \left\{ (a_{ij}) \;\middle|\; a_{ij} \in U_{ij} \cap V_{ij} \right\}$$

And since $U_{ij} \cap V_{ij}$ is always the finite intersection of open sets, it too is open. Thus $B_1 \cap B_2$ is another element in the base, and we are done with that part. Now consider the "identity" map $(a_{ij}) \mapsto (a_{11}, a_{21}, \ldots, a_{mn})$. This map is obviously a bijection. If $\prod_{i=1}^{mn} U_i$ is a subbasis element for $\mathbb{R}^{mn}$, pulling back would give us the set

$$\{ (a_{ij}) \mid a_{ij} \in U_{ij} \}$$

(where the ordering and column/row jumping is clear) which is of course open since it is a basis element. Now, if $B = \left\{ (a_{ij}) \,\middle|\, a_{ij} \in U_{ij} \subset \mathbb{R},\ U_{ij} \text{ open } \right\}$ is a subbasis element for the topology we just constructed, it would get mapped to $\prod_{1 \le i \le m, 1 \le j \le n} U_{ij}$, which is indeed open in $\mathbb{R}^{mn}$. A bjiective open map is a homeomorphism, thus we are done.

3. (a) We use the fact from linear algebra that the determinant is a continuous map from $\mathrm{Mat}(n \times n;\ \mathbb{R})$ to $\mathbb{R}$. Indeed, the determinant is a polynomial in the entries of the matrix, and since we equipped the matrix with the topology of (effectively) $\mathbb{R}^{n^2}$, this map is going to be continuous because the same polynomial from $\mathbb{R}^{n^2}$ to $\mathbb{R}$ is continuous. Each matrix in $\mathrm{GL_n}(\mathbb{R})$ has nonzero determinant, and matrices with 0 determinant are not invertible. Thus $\mathrm{GL_n}(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus 0)$, and since $\mathbb{R} \setminus 0 = (-\infty, 0) \cup (0, \infty)$ is the union of two open sets, it is open thus $\mathrm{GL_n}(\mathbb{R})$ is open.

   (b) Furthermore, restricting domain preserves continuity, thus $\det : \mathrm{GL_n}(\mathbb{R}) \to \mathbb{R}$ is continuous. If $\mathrm{GL_n}(\mathbb{R})$ were connected, then it's image under a continuous map would be connected. However, its image is $\mathbb{R} \setminus 0$, as for any $x \neq 0$, the matrix with 1's along the main diagonal with the first 1 replaced by x has determinant equal to x, and 0 is not the determinant of any invertible matrix, by definition, and since $\mathbb{R} \setminus 0$ is clearly not connected, we have reached a contradiction.

4. (a) Given the Euclidean topology on $\mathbb{C}$, we recall that $\mathbb{R}^2 \setminus K$ where K is countable is path-connected from in class. Call ι the prescribed map. Since $\{ \iota(\alpha_1), \ldots, \iota(\alpha_k) \}$ is countable, we have that $\mathbb{R}^2 \setminus \{ \iota(\alpha_1), \ldots, \iota(\alpha_k) \}$ is path-connected. Since we defined the topology on $\mathbb{C}$ to be so that ι is a homeomorphism, and since the restriction of a homeomorphism is a homeomorphism, $\iota(\mathbb{R}^2 \setminus \{ \iota(\alpha_1), \ldots, \iota(\alpha_k) \}) = \mathbb{C} \setminus \{ \alpha_1, \ldots, \alpha_k \}$ is path-connected. We remark this can also be used to show that $\mathbb{C} \setminus K$ where $K \subset \mathbb{C}$ is countable is path-connected.

   (b) $\det(A + z(I - A))$ is a polynomial in z, that is not equivalently 0 (since we showed it is nonzero somewhere). By the fundamental theorem of algebra it has finitely many roots, say $\{ \alpha_1, \ldots, \alpha_k \}$. Now, since $\mathbb{C} \setminus \{ \alpha_1, \ldots, \alpha_k \}$ is path-connected, we can find a path $\gamma : [0, 1] \to \mathbb{C} \setminus \{ \alpha_1, \ldots, \alpha_k \}$ with $\gamma(0) = 0$, $\gamma(1) = 1$, and $P(\gamma(t)) \neq 0$ for all $t \in [0, 1]$, since $0, 1 \in \mathbb{C} \setminus \{ \alpha_1, \ldots, \alpha_k \}$ (we showed they were not roots), and since γ maps to the subset of $\mathbb{C}$ precisely defined as such so that it excludes the roots of P.

   (c) Since $A + z(I - A)$ is clearly a continuous map, and since $\gamma(t)$ is continuous, $A + \gamma(t)(I - A)$ is continuous. We see immediately that $A + \gamma(0)(I - A) = A$, and $A + \gamma(1)(I - A) = I$. By the gluing lemma, given any matrix A, B, we could glue together two of these paths to get a path from A to I then to B, thus $\mathrm{GL_n}(\mathbb{C})$ is path-connected, and we are done.