# Math 425 HW7

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**Lemma 1.** *Let $A : \mathbb{R}^n \to \mathbb{R}^m$. If for every $x, y \in \mathbb{R}^n$, and $c \in \mathbb{R}$ we have that*

$$A(cx + y) = cAx + Ay$$

*then $A$ is linear.*

*Proof.* Choosing $y = 0$, we see that for every $x \in \mathbb{R}^n$, and $c \in \mathbb{R}$, we have that $A(cx) = A(cx + 0) = cAx + A0 = cAx$. Choosing $c = 1$, we see that for every $x, y \in \mathbb{R}^n$, we have that $A(x + y) = A(1 \cdot x + y) = 1 \cdot Ax + Ay = Ax + Ay$. $\square$

Since I used this lemma countless times without mentioning it, I thought I should add (and prove) it here.

1. Denote $S = \{x_1, \ldots, x_n\}$. We recall that

$$\text{span}(S) = \left\{ \sum_{i=1}^n a_i x_i \;\middle|\; a_i \in \mathbb{R}, x_i \in X \right\}$$

Given $\sum_{i=1}^n a_i x_i, \sum_{i=1}^n b_i x_i \in S$, we know that

$$\sum_{i=1}^n a_i x_i + \sum_{i=1}^n b_i x_i = \sum_{i=1}^n a_i x_i + b_i x_i = \sum_{i=1}^n (a_i + b_i) x_i \in \text{span}(S)$$

Also, given $c \in \mathbb{R}$,

$$c \sum_{i=1}^n a_i x_i = \sum_{i=1}^n (c a_i) x_i \in \text{span}(S)$$

which shows that span$(S)$ is a vector space.

2. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then,

$$BA(cx + y) = B(cAx + Ay) = cBAx + BAy$$

So indeed, $BA$ is linear. Next, by definition $A^{-1}$ is defined as the left inverse of $A$. Since $A$ is onto, it has a right inverse, say $B$ (We are thinking of $A, A^{-1}, B$ as functions). Now,

$$Bx = A^{-1}ABx = A^{-1}x$$

Since this holds for all $x \in \mathbb{R}^n$, it follows that $B = A^{-1}$, and hence $A^{-1}$ is also a right inverse of $A$. Note now that, for any $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$, $x = Az_1$ for some $z_1 \in \mathbb{R}^n$ and $y = Az_2$ for some $z_2 \in \mathbb{R}^n$ (since $A$ is onto), and so

$$A^{-1}(cx + y) = A^{-1}(cAz_1 + Az_2) = A^{-1}(A(cz_1) + Az_2) = A^{-1}(A(cz_1 + z_2)) = cz_1 + z_2 = cA^{-1}x + A^{-1}y$$

Since $A^{-1}x = A^{-1}Az_1 = z_1$, and similarly $A^{-1}y = z_2$.

3. Let $x, y \in \mathbb{R}^n$, and suppose that $Ax = Ay$. By lineararity, $A(x - y) = 0$. Then $x - y = 0$, which says that $x = y$, so $A$ is indeed 1–1.

4. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation, null($A$) be its nullspace, and range($A$) be its range. First, let $x, y \in \text{null}(A)$ and $c \in \mathbb{R}$. Then,

$$A(cx + y) = cAx + Ay = c \cdot 0 + 0 = 0 + 0 = 0$$

So $cx + y \in \text{null}(A)$, which shows that null($A$) is a vector space. Similarly, let $x, y \in \text{range}(A)$, and $c \in \mathbb{R}$. Then $x = Az_1$ and $y = Az_2$ for some $z_1, z_2 \in \mathbb{R}^n$. Now,

$$cx + y = cAz_1 + Az_2 = A(cz_1 + z_2)$$

So, $cx + y$ is the image of $cz_1 + z_2$ which shows that it is in the range.

5. Let $a_i = Ae_i$ for $i \in \{1, \dots, n\}$ where $e_i \in \mathbb{R}^n$ is the vector with a 1 in position $i$ and a 0 everywhere else. We notice that for any vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, we have that $x = \sum_{i=1}^n x_i e_i$, and since $A$ is linear,

$$Ax = A\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i A(e_i) = \sum_{i=1}^n x_i a_i$$

If we let $y = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, we know that

$$x \cdot y = \sum_{i=1}^n x_i a_i$$

which shows that $Ax = x \cdot y$ for every $x$. Note also that $y$ is unique, since $A$ is a function (i.e., $Ae_i$ can only have 1 value). If $||y|| = 0$, then $y = 0$ and so $A = 0$ since it maps everything to 0, and then given any vector $x \in \mathbb{R}^n$ with $||x|| = 1$, we have that $||Ax|| = ||0|| = 0$, and since $||A|| \geq 0$, it follows that $||A|| = 0$. Indeed, $||A|| = 0 = ||y||$. So now suppose that $||y|| \neq 0$.

We also notice that (by Cauchy-Schwartz),

$$||Ax|| = ||x \cdot y|| \leq ||x|| \cdot ||y||$$

Taking the sup over all $x \in \mathbb{R}^n$ with $||x|| = 1$ yields $||A|| \leq ||y||$. We also note that since $||y|| \neq 0$, $y/||y||$ is a vector of norm 1, and

$$\frac{1}{||y||}||Ay|| = ||A\frac{y}{||y||}|| = \frac{y}{||y||} \cdot y = \frac{||y||^2}{||y||} = ||y||$$

So $||A|| \geq ||y||$. Therefore, $||A|| = ||y||$.

6. Suppose that $A \not\equiv 0$. Then there is some $x \in \mathbb{R}^n$ so that $Ax \neq 0$. It is clear that $x \neq 0$. Letting $z = x/||x||$, we see that $||z|| = 1$ and that $Az = Ax/||x|| = 1/||x||Ax \neq 0$, since $\frac{1}{||x||} \neq 0$. Since $||\cdot||$ is a norm, we see that $||Az|| > 0$. Since

   $$||Ax - Ay|| < C||x - y||^r$$

   holds for all $x, y \in \mathbb{R}^n$. We may choose $y = 0$ to see that

   $$||Ax|| = ||Ax - A0|| < C||x - 0||^r = C||x||^r$$

   holds for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$. In particular, for all $\varepsilon > 0$ we have that

   $$\varepsilon||Az|| = ||A(\varepsilon z)|| < C||\varepsilon z||^r = C\varepsilon^r \cdot ||z||^r = C\varepsilon^r$$

   Which tells us that

   $$||Az|| < C\varepsilon^{r-1}$$

   Choosing $\varepsilon < \left(\frac{||Az||}{C}\right)^{\frac{1}{r-1}}$ yields a contradiction (Notice: since the power $(r-1)^{-1} > 0$, it follows that $\varepsilon^{r-1} < \frac{||Az||}{C}$ since $x^{\frac{1}{r-1}}$ is increasing on $x > 0$).