Math 425 HW4

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1. (a) Suppose that $G \setminus F_1$ was empty. Then $F_1 \supset G$. Take any $g \in G$. Since G is open, there is a neighborhood $N(g) \subset G \subset F$, so the interior of F_1 isn't empty, a contradiction. Since $G \setminus F_1$ is not empty, we can take x in it. Next, since F_1 is closed, $d(\{x\}, F_1) = d > 0$ because if it was 0 then $x \in \overline{F_1} = F_1$, which is impossible. Since $x \in G$, there is some $r_1 > 0$ so that $N_{r_1}(x) \subset G$. Take $r_2 = \min\{r_1, d/2\}$. If $N_{r_2}(x) \cap F_1 \neq \emptyset$, there would be some $f \in F_1$ so that $d(x, f) < d/2$. But for any $f \in F$, $d(x, f) \geq d$, a which is a contradiction. So $N_{r_2}(x) \cap F_1 = \emptyset$, and therfore $N_{r_2}(x) \subset G \setminus F_1$. Finally, let $r = r_2/2$. Since we are in $\mathbb{R}^d$, $\overline{N_r(x)} \subset N_{r_2}(x)$, so indeed, there is a closed ball about a point in G.

   (b) I claim that if A, B are closed and have empty interiors, then $A \cup B$ is also closed and has empty interior. It is closed since the finite union of closed sets is also closed. Suppose it had an interior point–say $u \in (A \cup B)^\circ$. Then there is some $r_1 > 0$ so that $N_{r_1}(u) \subset A \cup B$. Since A, B have empty interiors, we have that $N_{r_1}(u) \not\subset A$, and that $N_{r_1}(u) \not\subset B$. Since it is contained in the union, and not entirely contained in A, we can find a point $b \in B$ so that $b \in N_{r_1}(u) \setminus A$. From last time, $d(\{b\}, A) = \delta > 0$, since if it was 0 then $b \in \overline{A} = A$, which isn't true. Let $r = \min\{\delta/2, (r_1 - d(u, b))/2\}$. If there was some $x \in N_r(b) \cap A$, then $\delta \leq d(x, b) < \delta/2$, a contradiction. So $N_r(b) \cap A = \emptyset$. Since $N_r(b) \subset N_{r_1}(u) \subset A \cup B$, we see by the last two facts that $N_r(b) \subset B$, which shows that B doesn't have empty interior, a contradiction. It follows clearly by induction that if $F_1, \ldots, F_n$ are closed sets with empty interior, then $\cup_{k=1}^n F_k$ is also a closed set with empty interior. Continuing on from part (a), since $N_r(x) \subset G$, we see that $N_r(x) \cap G = N_r(x)$. Now we need to find a sub-neighborhood of this neighborhood that is contained in $N_r(x) \setminus (F_1 \cup F_2)$. Since $F_1 \cup F_2$ has empty interior (by the lemma above), we see that $F_1 \cup F_2 \not\supset N_r(x)$. Then there is a point $x_2 \in N_r(x)$ so that $x_2 \notin F_1 \cup F_2$. We see that $d(\{x_2\}, F_1 \cup F_2) = \delta > 0$, since it can't be 0 by the same reasoning above. Since x_2 is an interior point of $N_r(x)$ (clearly open) we can find an r_2^* so that $N_{r_2^*}(x) \subset N_r(x)$. Clearly $\overline{N_{r_2*/2}(x)} \subset N_r(x)$.Taking $r_2 = \min\{r_2^*/2, (r - d(x_2, x)/10)\}$ will have $\overline{N_{r_2}(x_2)} \subset N_r(x) \setminus (F_1 \cup F_2)$ since $\overline{N_{r_2}(x_2)} \subset N_{r_2*}(x)$ which is disjoint from $F_1 \cup F_2$ by the reasoning above (if it wasn't, it would contradict the minimality of d). We have therefore generated a point x_2 and a radius r_2 so that $\overline{N_{r_2}(x_2)} \subset N_r(x) = G \cap N_r(x) \subset G \setminus G \setminus (F_1)$ while also having that $N_{r_2}(x_2) \subset N_r(x) \setminus (F_1 \cup F_2) \subset G \setminus (F_1 \cup F_2)$. We can now clearly continue this process for all $n \in \mathbb{N}$ (the important part is that the union of closed sets with empty interiors is also a closed set with empty interior), so we can indeed generate a sequence of points $\{x_n\}_{n=1}^{\infty}$ and radii $\{r_n\}_{n=1}^{\infty}$ satisfying the conditions in this part.

(c) We see that $\overline{N_{r_n}(x_n)}$ is a sequence of nested, nonempty, closed sets, and therefore,

$$\bigcap_{n=1}^{\infty} \overline{N_{r_n}(x_n)} \neq \emptyset$$

Since this intersection is in $G \setminus (\bigcup_{k=1}^{\infty} F_k)$ by construction, we see that $\bigcup_{k=1}^{\infty} F_k \not\supset G$, and therefore can't be all of $\mathbb{R}^d$.

2. Let $\{r_n\}$ be an enumeration of the rationals. Suppose there were open sets $\{U_n\}_{n=1}^{\infty}$ so that $\mathbb{Q} = \cap_{n=1}^{\infty} U_n$. Since $\mathbb{Q}$ is dense, each U_n is dense (i.e., $\mathbb{Q} \subset U_n \implies \mathbb{R} = \overline{\mathbb{Q}} \subset \overline{U_n}$). First, take any point $u_1 \in U_1$, since u_1 is an interior point there is an $\eta > 0$ so that $N_\eta(u_1) \subset U$. Let $\eta_1 = \min\{r, |u - r_1\}/2$. Clearly, $a_1 \notin \overline{N_{\eta_1}(u_1)} \subset U$. Next, choose any point $u_2 \in \overline{N_{\eta_1}(u_1)} \cap U_2$ (such a point exists since U_2 is dense). Repeat the same process, choose η_2 so that $\eta_2 < |u_2 - u_1|/10$, $\eta_2 < |u_2 - r_2|/10$ and $\overline{N_{\eta_2}(u_2)} \subset U_2$. Repeating this process, we get a sequence of closed intervals $\overline{N_{\eta_n}(u_n)} \subset U_n$ all nested, and nonempty. Then there is some point in the intersection $x \in \bigcap_{k=1}^{\infty} N_{\eta_k}(u_k)$. Since $x \neq r_k$ for every $k \in \mathbb{N}$, it cannot be that x is rational (we excluded the nth rational in I_n). Since $\bigcap_{n=1}^{\infty} I_n \subset \bigcap_{n=1}^{\infty} U_n$, we have shown that U_n contains an irrational point, a contradiction.

3. $\mathbb{R}$ is closed since it is the entire space, and bounded since $\mathbb{R} \subset N_2(0)$. I claim that $\{a_n\}_{n=1}^{\infty}$ where $a_n = n$ has no convergent subsequence. Suppose it did, say $\{a_{n_k}\}_{k=1}^{\infty}$. Then this would also be cauchy, so there would be some $K > 0$ so that if $k > K$, $d(n_k, n_{k+1}) = d(a_{n_k}, a_{n_{k+1}}) < 1/2$. Since n_k is an injection from $\mathbb{Z}_+$ to $\mathbb{Z}_+$, it is also strictly increasing, so $|n_{k+1} - n_k| \geq 1$. But then $d(n_k, n_{k+1}) \geq 1$, a contradiction. So $\mathbb{R}$ is not compact with this metric.

4. Suppose instead there was some $\emptyset \neq A \subset X$ that is both closed and open. Then A^c is nonempty, since $A \neq X$, closed (since A is open) and open (since A is closed). Clearly $X = A \cup A^c$. Also, $A \cap \overline{A^c} = A \cap A^c = \emptyset$, and $\overline{A} \cap A^c = A \cap A^c = \emptyset$. But then X is disconnected, a contradiction.

5. $\boxed{\implies}$

If $E \subset X$ is closed and bounded, since its bounded there is some $R > 0$ and $x \in E$ so that if $y \in E$, $d(y, x) < R$. Then $E \subset N_r(x) \subset \overline{N_r(x)}$, and since E is closed and a subset of a compact set, it is also compact.

$\boxed{\impliedby}$

If E is compact, we know that $E \subset \cup_{x \in E} N_1(x)$, so it admits a finite subcover $\cup_{k=1}^{n} N_1(x_k)$. Let $d = \max_{1 \leq i,j \leq n} d(x_i, x_j)$. For any $q \in E$, we know that $q \in N_1(x_j)$ for some j. Then $d(x_1, q) \leq d(x_1, x_j) + d(q, x_j) \leq d + 1$, so E is bounded. Since E is compact, it is limit point compact. If E wasn't closed, we could take $x \in E' \setminus E$. Then at step n, choose $x_n \in E$ so that $d(x_n, x) < 1/n$. Clearly x is a limit point of $\{x_n\}_{n=1}^{\infty}$. Suppose it had another limit point, say y. At step 1, find an $x_{n_1} \in N_{1/1}(y) \setminus y$. At step k, since $N_{1/n}(y) \setminus y$ intersects E infinitely many times, we can find an x_n in this intersection with index greater than all previous x_{n_l}'s, so call this new one x_{n_k}. This gives a sequence $x_{n_k} \to y$. Since x_{n_k} is a subsequence of x_n, it also converges to x, so $x = y$. Then $\{x_n\}_{n=1}^{\infty}$ is an infinite subset of E with no limit points in E, which is a contradiction. So indeed, E is closed and bounded.

6. Let

$$f_n(x) = \begin{cases} 1, & 0 \le x \le 1/n \\ -n/2(x - (1/2 + 1/n)), & 1/2 - 1/n \le x \le 1/2 + 1/n \\ 0, & x > 1/2 + 1/n \end{cases}$$

For $n \ge 2$. A quick calculation shows that $\lim_{x \to 1/2 - 1/n} f_n(x) = 1$, and that $\lim_{x \to 1/2 + 1/n} f_n(x) = 0$, so f is continuous for every $n \ge 1$. Suppose that f_n had some convergent subsequence, say $f_{n_k} \to f$. Since $f_{n_k} \to f$ uniformly (given the metric), f is also continuous. Let $\delta > 0$. We can find $K > 100/\delta$ so that $d_\infty(f_{n_K}, f) < 1/3$. In particular, $|f_{n_K}(1/2 - 1/K) - f(1/2 - 1/K)| = |1 - f(1/2 - 1/K)| < 1/3$, and that $|f_{n_K}(1/2 + 1/K) - f(1/2 + 1/K)| = |f(1/2 + 1/N)| < 1/3$. Clearly $2/K < \delta$. The first part says, by the reverse triangle inequality, that $|f(1/2 - 1/K)| > 1 - 1/3 = 2/3$, and the second says that $|f(1/2 + 1/K)| < 1/3$. Then, $|f(1/2 - 1/K) - f(1/2 + 1/K)| \ge |f(1/2 - 1/K)| - |f(1/2 + 1/K)| > 2/3 - 1/3 = 1/3$. Since this is true for every δ, we see that f is not continuous. But then f_{n_k} didn't converge in this metric space, a contradiction.