Math 425 HW8

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1. Suppose that f has a local maximum at $\mathbf{x} \in E$ and that $f'(\mathbf{x}) \neq 0$. Then, $\left|f'(\mathbf{x})\right| > 0$. Now let $\varepsilon > 0$. We expand f as:

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + E(h)$$

So that $|E(\mathbf{h})/\mathbf{h}| \to 0$. Now let $\varepsilon > 0$. We shall consider $\mathbf{h} = \varepsilon f'(\mathbf{x}) \neq 0$ since $f'(\mathbf{x}) \neq 0$. Indeed, we see that

$$f(x + \varepsilon f'(\mathbf{x})) = f(\mathbf{x}) + f'(\mathbf{x}) \cdot f'(\mathbf{x})\varepsilon + E(f'(\mathbf{x}) \cdot \varepsilon) = f(x) + \varepsilon\left|f(x)\right|^2 + E(f'(\mathbf{x}) \cdot \varepsilon)$$

Also, since $|E(\mathbf{h})/\mathbf{h}| \to 0$, we can find a $\delta > 0$ so that $|E(\mathbf{h})/\mathbf{h}| < \left|f'(\mathbf{x})\right|/2$. It follows that $|E(\mathbf{h})| < \frac{1}{2}\left|f'(\mathbf{x})\mathbf{h}\right|$ for all $|\mathbf{h}| < \delta$. Now, for all $\varepsilon < \delta/\left|\nabla f(\mathbf{x})\right|$, we have that $\left|E(\varepsilon f'(\mathbf{x}))\right| < \frac{\varepsilon}{2}\left|f(\mathbf{x})\right|^2$, and in particular, $E(\varepsilon \nabla f(\mathbf{x})) > -\varepsilon/2|f(\mathbf{x})|^2$. It follows then that

$$f(x + \varepsilon f'(\mathbf{x})) > f(\mathbf{x}) + \frac{\varepsilon}{2}|f(\mathbf{x})|^2$$

So it cannot be the case that $\mathbf{x}$ was a local minimum of f, a contradiction. Thus, $f'(\mathbf{x}) = 0$.

2. Fix $x \in E$, and let $A = f^{-1}(\{f(x)\})$, and $B = A^c$, i.e. $B = \{b \in E \mid f(b) \neq f(x)\}$. Suppose that f is non-constant, i.e. that B is nonempty. We shall show that A is a clopen subset of E, which will immediately show that E cannot be connected. First, A is obviously closed since it is the preimage of a closed set (singletons are closed sets). B is therefore open. Suppose that B were not closed. Then there would be some sequence $y_n \to y$ so that $y_n \in B$ and $y \notin B$, i.e. $y \in A$. Now find N sufficiently large so that $|y_n - y| < 1$ for all $n \geq N$. Now we shall consider $N_1(y)$. First, we prove the following lemma:

Lemma 1. *Let $r > 0$ and $x \in \mathbb{R}^n$. Then $N_r(x)$ is convex.*

Proof. Letting $x, y \in N_r(x)$, we see for any $0 \leq t \leq 1$,

$$||tx + (1-t)y|| \leq t||x|| + (1-t)||y|| \leq tr + (1-t)r = r$$

Which completes the proof. $\square$

So now let $z \in N_1(y)$. Since $N_1(y)$ is convex, we may apply the mean value theorem to see that, for all $i \in \{1, \dots, n\}$,

$$e_i \cdot (f(z) - f(y)) = e_i \cdot [f'(\xi)(b-a)]$$

for some $\xi \in \mathbb{R}^n$. Now,

$$||e_i \cdot [f'(\xi)(b-a)]|| \leq 1 \cdot ||f'(\xi)(b-a)|| \leq ||f'(\xi)|| \cdot ||b-a|| = 0$$

This shows that the ith coordinate of $f(z) - f(y)$ is 0. Since this holds for all $i \in \{1, \dots, n\}$, $f(z) = f(y)$. But now, since $|y_N - y| < 1$, we have that $f(y_N) = f(y) = f(x)$, a contradiction, since $y_N \notin A$. So, looking back at what we were contradicting, we see that f is constant.

3. Let $f, g : \mathbb{R}^n \to \mathbb{R}$. We notice, by the regular product rule, that

$$\frac{\partial}{\partial x_i} fg = g\frac{\partial}{\partial x_i} f + f\frac{\partial}{\partial x_i} g$$

We therefore see that

$$\nabla(fg) = \begin{pmatrix} \frac{\partial}{\partial x_1} fg \\ \dots \\ \frac{\partial}{\partial x_n} fg \end{pmatrix} = \begin{pmatrix} g\frac{\partial}{\partial x_1} f + f\frac{\partial}{\partial x_1} g \\ \dots \\ g\frac{\partial}{\partial x_n} f + f\frac{\partial}{\partial x_n} g \end{pmatrix} = g\begin{pmatrix} \frac{\partial}{\partial x_1} f \\ \dots \\ \frac{\partial}{\partial x_n} f \end{pmatrix} + f\begin{pmatrix} \frac{\partial}{\partial x_1} g \\ \dots \\ \frac{\partial}{\partial x_n} g \end{pmatrix} = g\nabla f + f\nabla g$$

Similarly, if $f \neq 0$, $\frac{\mathrm{d}}{\mathrm{d}x_i}\frac{1}{f} = \frac{-1}{f^2} \cdot \frac{\partial f}{\partial x_i}$ by the regular quotient rule. It follows then that

$$\nabla(1/f) = \begin{pmatrix} \frac{-1}{f^2}\frac{\partial f}{\partial x_1} \\ \dots \\ \frac{-1}{f^2}\frac{\partial f}{\partial x_n} \end{pmatrix} = \frac{-1}{f^2}\nabla f$$

4.

$$f'(0) = \lim_{h\to 0} \frac{h + 2h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = 1 + \lim_{h\to 0} 2h \sin\left(\frac{1}{h}\right)$$

Now,

$$\left|2h \sin\left(\frac{1}{h}\right)\right| \leq 2h$$

And so $\left|2h\sin\left(\frac{1}{h}\right)\right| \to 0$, which shows that $f'(0) = 1$. For $t \neq 0$, we can calculate f' using regular derivative rules as

$$f'(t) = 1 - 2 \cdot \cos\left(\frac{1}{t}\right) + 4t \sin\left(\frac{1}{t}\right)$$

By the triangle inequality, this is bounded above by $1 + 2 + 4 = 7$ on $(-1, 1) \setminus 0$, and we already showed it equals 1 at 0. First we notice that $f'\left(\frac{1}{2\pi n}\right) = -1$ for all $n \in \mathbb{N}$. Fix a neighborhood of 0, say $(-t, t)$. First find $n > 0$ so that $\frac{1}{2\pi n} < t$. We also notice that $f\left(\frac{1}{2\pi n}\right) = \frac{1}{2\pi n}$ at all $n \in \mathbb{N}$. We know that f is continuously differentiable outside of 0, in particular at $x = \frac{1}{2\pi(n+1)} < \frac{1}{2\pi n}$. So find a $\delta > 0$ so that $y \in (x - \delta, x)$ means $f'(y) - f'(x) < 1/2$ and $f(y) - f(x) < f\left(\frac{1}{2\pi n}\right) - f(x)$ (which is obviously positive), in particular, $f(y) < f(\frac{1}{2\pi n})$, and also $f'(y) < -1/2$. Now choosing $z = x - \delta/2$, since $f'(y) < -1/2$ on all of $(x - \delta, x)$, it follows by the mean value theorem that $f(x) - f(z) = f'(\xi)(z - x)$ for some $\xi \in (x - \delta, x)$, and this tells us that $f(x) - f(z) \leq -\frac{1}{2}(z - x) < 0$, so in particular $f(z) > f(x)$. One also notes that $f(z) < f(\frac{1}{2\pi n})$ by how we defined δ. Then $f(z) \in (f(x), f(\frac{1}{2\pi n}))$, and by the intermediate value theorem there is some $\eta \in (x, \frac{1}{2\pi n})$ so that $f(z) = f(x)$. This tells us that f is not injective, which completes the proof.