# Math Template

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## Exercise 11.2

Let $0 < a < b$ and let $f$ be the function such that $f(0) = 0$, $f'(0) = 0$, and

$$f'(x) = \int_0^x 1_{[a,b]}(x)\, dx$$

Show that Ito's formula holds for $f$.

Change this function slightly by keeping all the same conditions except we require the second derivative to also have the lines connecting $(a - 1/n, 0)$ with $(a, 1)$ and $(b, 1)$ with $(b + 1/n, 0)$. Call those line segments $a_n''$ and $b_n''$. This function is now $C^2$ so Ito's formula holds. We first seek to show that:

$$\int_0^t a_n''(B_s)\, ds \to 0$$

This will follow by:

$$\mathbb{E}\left[\int_0^t a_n''(B_s)\, ds\right] = \int_0^t \mathbb{E}[a_n''(B_s)]\, ds \le \int_0^t \mathbb{E}[1_{[a-1/n,a]}(B_s)]\, ds$$

This last is simply bound by

$$\frac{1}{\sqrt{2\pi(a - 1/n)}} \cdot \frac{1}{n} \to 0$$

Similarly,

$$\int_0^t b_n''(B_s)\, ds \to 0.$$

Now, notice that $a_n' \leq \frac{1}{2} \cdot \frac{1}{n}$ (the width of the triangle is $1/n$ and its height is 1). So, we have, using Ito's isometry:

$$\mathbb{E}\left(\int_0^t a_n'(B_s)\,dB_s\right)^2 = \mathbb{E}\left(\int_0^t a_n'(B_s)^2\,ds\right) \leq \mathbb{E}\left(\frac{1}{4n^2}\right) \to 0$$

The exact same argument holds for $b_n'$. Now, $g_n = f + a_n + b_n$ converges to $f''$ as $n \to \infty$ to $f$. Putting it all together,

$$g_n(B_t) = \int_0^t f'(B_s) + a_n'(B_s) + b_n'(B_s)\,dB_s + \int_0^t f''(B_s) + a_n''(B_s) + b_n''(B_s)\,ds$$

Using everything we just proved and linearity of all the integrals above, the Ito's formula holds for $f$ as well.

## Exercise 11.6

Suppose $M$ is a bounded continuous martingale, $A$ is a continuous process whose paths have total variation bounded by $N > 0$ a.s., and $X_t = M_t + A_t$.

1. Prove that for each $t$,

$$\sum_{i=1}^{2^n t} (X_{(i+1)/2^n} - X_{i/2^n})^2 \to \langle X \rangle_t$$

2. Prove that if $f$ is a $C^2$ function whose second derivative is bounded, then

$$\sum_{i=1}^{2^n t} f''(X_{i/2^n})(X_{(i+1)/2^n} - X_{i/2^n})^2 \to \int_0^t f''(X_s)\,d\langle X \rangle_s.$$

Write $\Delta X_i = X_{(i+1)/2^n} - X_{i/2^n}$ and $\Delta M_i = M_{(i+1)/2^n} - M_{i/2^n}$, and $\Delta\langle X \rangle_i = \langle X \rangle_{(i+1)/2^n} - \langle X \rangle_{i/2^n}$. First,

$$\sum_i (\Delta X_i)^2 = \sum_i (\Delta M_i)^2 + 2\Delta M_i \Delta A_i + (\Delta A_i)^2$$

With $\eta_n = \sup_i |\Delta A_i|$, for each fixed $\omega$, on the compact interval $[0, t]$ $A_t(\omega)$ is uniformly continuous, so $\eta_n \to 0$ a.s. Also, $\eta_n \leq \|A\|_\infty$, recalling that $A$ is bounded on a compact interval. By hypothesis,

$$\mathbb{E}\left[\sum_i \Delta A_i^2\right] \le \mathbb{E}\left[\sum_i \eta_n(\Delta A_i)^2\right] \le N\mathbb{E}[\eta_n] \to 0$$

by the DCT.

Similarly, with $\xi_n = \sup_i |\Delta M_i|$, we have

$$\mathbb{E}\left[\sum_i \Delta M_i \Delta A_i\right] \le \mathbb{E}\left[\xi_n \sum_i |\Delta A_i|\right] \le N\mathbb{E}[\xi_n] \to 0$$

By the same argument as above.

Recalling that $\langle X\rangle_t = \langle M\rangle_t$, we need only show that:

$$\sum_i (\Delta M_i)^2 - \Delta\langle M\rangle_i \to 0 \text{ in probability.}$$

We will show that it converges in $L^2$. Observe:

$$\mathbb{E}\left[\left(\sum_i (\Delta M_i)^2 - \Delta\langle M\rangle_i\right)^2\right] = \sum_i \mathbb{E}\Big[(\Delta M_i)^2 - \Delta\langle M\rangle_i\Big]^2 \tag{1}$$

$$+ \sum_{i<j} \mathbb{E}\Big[((\Delta M_i)^2 - \Delta\langle M\rangle_i)((\Delta M_j)^2 - \Delta\langle M\rangle_j)\Big] \tag{2}$$

Now notice that:

$$\mathbb{E}\Big[((\Delta M_i)^2 - \Delta\langle M\rangle_i)((\Delta M_j^2) - \Delta\langle M\rangle_j)\Big] = \mathbb{E}\Big[((\Delta M_i)^2 - \Delta\langle M\rangle_i)\mathbb{E}\Big[((\Delta M_j)^2 - \Delta\langle M\rangle_j) \mid \mathcal{F}_{j/2^n}\Big]\Big]$$

Since $M$ is a martingale,

$$\mathbb{E}\Big[(M_{(j+1)/2^n} - M_{j/2^n})^2 \mid \mathcal{F}_{j/2^n}\Big] = \mathbb{E}\Big[M^2_{(j+1)/2^n} - M^2_{j/2^n} \mid \mathcal{F}_{j/2^n}\Big]$$

Meaning the diagonal terms, (2), all vanish.

To deal with the first term,

$$\sum_i \mathbb{E}\Big[(\Delta M_i)^2 - \Delta\langle M\rangle_i\Big]^2 = \sum_i \mathbb{E}\Big[(\Delta M_i)^4 - 2(\Delta M_i)^2\Delta\langle M\rangle_i + \Delta\langle M\rangle_i^2\Big]$$

We go term by term. First,

$$\mathbb{E}\left[\sum_i (\Delta M_i)^4\right] \le \mathbb{E}\left[\sum_i \xi_n^2 (\Delta M_i)^2\right] \le \mathbb{E}\left[\xi_n^4\right]^{1/2} \mathbb{E}\left[\left(\sum_i (\Delta M_i)^2\right)^2\right]^{1/2}$$

Define

$$V_n = \sum_i \Delta M_i^2$$

We will prove that $\mathbb{E}\left[V_n^2\right] \le 4\|M\|_\infty^2 \mathbb{E}[V_n]$. We will then prove that $\mathbb{E}[V_n]$ is bounded, hence the term will go to 0. By our calculation above:

$$\mathbb{E}\left[V_n^2\right] = \sum_i \mathbb{E}\left[(\Delta M_i)^4\right] + \sum_{i<j} \mathbb{E}\left[(\Delta M_i)^2 (\Delta M_j)^2\right]$$

Clearly,

$$\sum_{i<j} \mathbb{E}\left[(\Delta M_i)^2 (\Delta M_j)^2\right] = \sum_{i<j} \mathbb{E}\left[(\Delta M_i)^2 \Delta M_j^2\right] = \sum_i \mathbb{E}\left[(\Delta M_i)^2 (M_t^2 - M_{(i+1)/2^n}^2)\right] \le 2\|M\|_\infty^2 \mathbb{E}[V_n]$$

The easier part is this one:

$$\mathbb{E}\left[\sum_i (\Delta M_i)^4\right] \le \sum_i 4\|M\|_\infty^2 \mathbb{E}[V_n]$$

By the same conditioning trick,

$$\mathbb{E}[V_n] = \mathbb{E}\left[M_t^2 - M_0^2\right] < \infty$$

DCT applied to the bounded random variable $\xi_n^4$ shows that this term goes to 0. The other are much easier. With $\chi_n = \sup_i |\Delta\langle M\rangle_i|$, once again using it is bounded, and goes to 0:

$$\sum_i \mathbb{E}\left[(\Delta\langle M\rangle_i)^2\right] \le \mathbb{E}\left[\chi_n \sum_i \Delta\langle M\rangle_i\right] = \mathbb{E}[\chi_n \langle M\rangle_t] \le \|\langle M\rangle\|_\infty \mathbb{E}[\chi_n] \to 0$$

by DCT again.

Lastly,

$$\sum_i \mathbb{E}\Big[(\Delta M_i)^2 \Delta\langle M\rangle_i\Big] \le \mathbb{E}[\xi_n V_n] \le \mathbb{E}\Big[\chi_n^2\Big]^{1/2} \mathbb{E}\Big[V_n^2\Big]^{1/2} \to 0$$

This completes the proof.

For the second part, if $f$ is $C^2$ with bounded second derivative,

$$\left|\sum_i f''(X_{i/2^n})[(\Delta X_i)^2 - \Delta\langle X\rangle_i]\right| \le \sum_i \|f''\|_\infty \left|\sum_i (\Delta X_i)^2 - \Delta\langle X\rangle_i\right| \to 0 \text{ in probability}$$

by what we just proved. This completes the problem.

## Problem 24.10

Let $W$ be a one-dimensional Brownian motion and let $X_t^x$ be the solution to

$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt, X_0 = x$$

suppose $\sigma$ and $b$ are $C^\infty$ functions and all their derivatives are bounded. Show that for each $t$ the map $x \to X_t^x$ is continuous in $x$ a.s. and further thnat it is differentiable.

Using that $(x+y+z)^4 \le 27(x^4+y^4+z^4)$, Doob's $L^p$ inequality, and the Lipschitz continuity of differentiable functions, we have that:

$$\begin{aligned}\mathbb{E}\left[\sup_{r\le t} |X_r^x - X_r^y|^4\right] &\le 27|x-y|^4 + 27\mathbb{E}\left[\sup_{r\le t}\left(\int_0^r (\sigma(X_s^x) - \sigma(X_s^y))\, dW_t\right)^4\right] \\ &+ 27\mathbb{E}\left[\sup_{r\le t}\left(\int_0^r (b(X_s^x) - b(X_s^y))\, ds\right)^4\right]\end{aligned}$$

The first term is more more complicated, first we use inequality due to Burkholder-Davis-Gundy and Cauchy-Schwarz:

$$\begin{aligned}\mathbb{E}\left[\sup_{r\le t}\left(\int_0^r (\sigma(X_s^x) - \sigma(X_s^y))\, dW_t\right)^4\right] &\le C\mathbb{E}\left[\left(\int_0^t (\sigma(X_s^x) - \sigma(X_s^y))^2\, ds\right)^2\right] \\ &\le t^2 C\mathbb{E}\left[\int_0^t (\sigma(X_s)^x - \sigma(X_s^y))^4\, ds\right]\end{aligned}$$

The other one is easier:

$$\mathbb{E}\left[\sup_{r\leq t}\left(\int_0^r (b(X_s^x) - b(X_s^y))\,ds\right)^4\right] \leq t^4\mathbb{E}\left[\int_0^t (b(X_s)^x - b(X_s^y))^4\,ds\right]$$

Putting it all together and using that $\sigma, b$ are Lipschitz continuous, we get (for a constant independent of $x, y, \omega$):

$$\mathbb{E}\left[\sup_{r\leq t}|X_r^x - X_r^y|^4\right] \leq 27|x-y|^4 + C\int_0^t \mathbb{E}\left[\sup_{r\leq s}|X_r^x - X_r^y|^4\right]ds$$

Letting $g(t) = \mathbb{E}\left[\sup_{r\leq t}|X_r^x - X_r^y|^4\right]$, we have:

$$g(t) \leq 27|x-y|^4 + C\int_0^t g(s)\,ds$$

Using Gronwall's lemma, we get $g(t) \leq 27|x-y|^4 e^{Ct}$. In particular, as $t$ is fixed,

$$\mathbb{E}\left[|X_t^x - X_t^y|^4\right] \leq D|x-y|^4$$

Applying Kolmogorov's continuity theorem, $X_t$ has a continuous verison.

Now we prove the existence of the derivative. Consider the sequence of stochastic processes defined by $Y_0^x(t) = 1$, and:

$$Y_{i+1}^x(t) = 1 + \int_0^t \sigma'(X_s^x)Y_i^x(s)\,dW_s + \int_0^t b'(X_s^x)Y_i^x(s)\,ds$$

Notice that:

$$\mathbb{E}\left[\sup_{r\leq t}|Y_{i+1}^x(r) - Y_i^x(r)|^2\right] \leq 8\mathbb{E}\left[\int_0^t (\sigma'(X_s^x)(Y_i^x(s) - Y_{i-1}^x(s)))^2\,ds\right] + 8\mathbb{E}\left[\int_0^t (b'(X_s^x)(Y_i^x(s) - Y_{i-1}^x(s)))^2\,ds\right]$$

Using that the derivatives are bounded, and defining $f(t) = \mathbb{E}\left[\sup_{r\leq t}|Y_{i+1}^x(r) - Y_i^x(r)|^2\right]$ yields:

$$f(t) \leq A\int_0^t f(s)\,ds$$

As in the book this gives $f(t) \le A^i t^{i-1}/(i-1)!$. Then

$$\mathbb{E}\left[\sup_{r\le t} |Y^x_m(r) - Y^x_n(r)|^2\right]^{1/2} \le \sum_{i=m}^{n} \mathbb{E}\left[\sup_{r\le t} |Y^x_{i+1}(r) - Y^x_i(r)|^2\right]^{1/2} \le \sum_{m}^{n} \sqrt{\frac{A^i t^{i-1}}{(i-1)!}}$$

This norm is complete (as in the book) so converges to a limit $Y^x$. Taking a limit in the integral equations above yields that $Y^x$ is a solution to the SDE $dY^x_t = \sigma'(X^x_t)Y^x_t\, dW_t + b'(X^x_t)Y^x_t\, dt$.

Now,

$$\mathbb{E}\left[\sup_{h\le 2^{-n}} \left|\frac{X^{x+h}_t - X^x_t}{h} - Y^x_t\right|^2\right] \le 2\mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} \left|\frac{\sigma(X^{x+h}_s) - \sigma(X^x_s)}{h} - \sigma(X^x_s)Y^x_s\right|^2 ds\right] + 2\mathbb{E}\left[\sup_{h\le 2^{-n}} \left(\int_0^t \left|\frac{b'(X^{x+h}_s) - b(X^x_s)}{h} - b(X^x_s)Y^x_s\right| ds\right)^2\right]$$

With $\sigma(X^{x+h}_s) - \sigma(X^x_s) = \sigma'(C^h_s)(X^{x+h}_s - X^x_s)$ for some $C^h_s$ between $X^x_s$ and $X^{x+h}_s$, we have:

$$\mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} \left|\frac{\sigma(X^{x+h}_s) - \sigma(X^x_s)}{h} - \sigma(X^x_s)Y^x_s\right|^2 ds\right] \le L\mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} \left|\frac{X^{x+h}_s - X^x_s}{h} - Y^x_s\right|^2 ds\right] + \mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} \left|\sigma'(C^h_s) - \sigma'(X^x_s)\right|^2 \left|Y^x_s\right|^2 ds\right]$$

This second term, by Cauchy-Schwarz and using that $\sigma'$ is lipschitz,

$$\mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} \left|\sigma'(C^h_s) - \sigma'(X^x_s)\right|^2 \left|Y^x_s\right|^2 ds\right] \le \int_0^t L\mathbb{E}\left[\sup_{h\le 2^{-n}} |C^h_s - X^x_s|^4\right]^{1/2} \mathbb{E}\left[|Y^x_s|^4\right]^{1/2} ds$$

We deal with the second expectation first:

$$\mathbb{E}\left[|Y^x_t|^4\right] \le 27 + 27t^2\mathbb{E}\left[\int_0^t |Y^x_s|^4\, ds\right] + 27t^4\mathbb{E}\left[\int_0^t |Y^x_s|^4\, ds\right]$$

This tells us that, for some $A$ independent of $x, y, \omega$:

$$\mathbb{E}\left[|Y^x_t|^4\right] \le 27e^{At}$$

Therefore, it is bounded. The last term:

$$\mathbb{E}\left[\sup_{h\le 2^{-n}} |X^{x+h}_t - X^x_t|^4\right] \le 27\cdot 2^{-4n} + 27t^2\mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} |X^{x+h}_s - X^x_s|^4\, ds\right] + 27t^4\mathbb{E}\left[\int_0^t \sup_{h\le 2^{-n}} |X^{x+h}_s - X^x_s|^4\, ds\right]$$

This tells us that, for a different constant $B$ independent of $x, y, \omega$:

$$\mathbb{E}\left[\sup_{h \leq 2^{-n}} |X_t^{x+h} - X_t^x|^4\right] \leq 27e^{Bt} \cdot 2^{-4n}$$

Encompass $27e^{Bt} = E^2$, then we have:

$$\mathbb{E}\left[\sup_{h \leq 2^{-n}} \left|\frac{X_t^{x+h} - X_t^x}{h} - Y_t^x\right|^2\right] \leq L \int_0^t \mathbb{E}\left[\sup_{h \leq 2^{-n}} \left|\frac{X_s^{x+h} - X_s^x}{h} - Y_s^x\right|^2\right] ds + E \cdot 2^{-2n}$$

Meaning that:

$$\mathbb{E}\left[\sup_{h \leq 2^{-n}} \left|\frac{X_t^{x+h} - X_t^x}{h} - Y_t^x\right|^2\right] \leq E \cdot 2^{-2n} e^{Lt}$$

Then for any $0 < \alpha < 1$, by Borel Cantelli lemma,

$$\mathbb{P}\left(\sup_{h \leq 2^{-n}} \left|\frac{X_t^{x+h} - X_t^x}{h} - Y_t^x\right|^2 \geq 2^{-2n\alpha} \text{ i.o.}\right) = 0$$

This shows that

$$\frac{X_t^{x+h} - X_t^x}{h} \to Y_t^x$$

completing the proof.