Hypercube Vertices

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Our main result today is to show that in the $1 \times n$ case, picking vertices of the hypercube is the "worst possible" in the sense that the beta value tends to 0. We make the following result precise in the following sense:

Theorem 1 (Hypercube Vertices Scale Poorly)**.** *Let $a \in \{-1, 1\}^n$ be any hypercube vertex. Then,*

$$\sqrt{n}\beta(a) = \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} |a^t x| \ll n^2 (0.91)^n.$$

We claim that we can assume without loss of generality that $a = \mathbf{1}$. Indeed, If $a \in \{-1, 1\}^n$, then we can write

$$a^t x = \operatorname{diag}(a)\mathbf{1}^t x = x^t \operatorname{diag}(a)\mathbf{1}$$

Since x will run through all ± 1 combinations, multiplying x^t on the right by a ± 1 diagonal matrix will just permute those vectors among themselves. Thus, we can assume that $a = \mathbf{1}$.

Notice that in this case, $a^t x$ = # of 1s in x - # of -1s in x. We divide the x's into classes based on the number of 1s in x. Let S_{n-k} be the set of x's with $n-k$ 1s and k -1s. Clearly, $|S_{n-k}| = \binom{n}{k}$. We see then that (after some simple approximations)

$$\begin{aligned}
\sqrt{n}\beta(\mathbf{1}) &= \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} |a^t x| = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} |n - 2k| = \frac{1}{2^n} \sum_{k=0}^{n/2} \binom{n}{k} (n - 2k) \\
&= \sum_{k=0}^{n/2} \frac{n^k}{k!} (n - 2k) = n^2 \frac{n^{n/4}}{(n/4)!} = n^2 \frac{(4e)^{n/4}}{2^n} = n^2 (0.91)^n.
\end{aligned}$$

We give a second proof. Let $S_n = X_1 + \cdots + X_n$ be the sum of n i.i.d. Radamacher random variables. By Hoeffding's Inequality,

$$\mathbb{P}(S_n \neq 0) = \mathbb{P}\left(|S_n| \geq \frac{1}{2}\right) \leq e^{-4/n}$$

Recall that if a has norm 1 and $x \in \{\pm 1\}^n$, then by Cauchy-Schwartz $|a^T x| \leq \sqrt{n}$. Putting these together tells us that,

$$\beta\left(\frac{\mathbf{1}}{\sqrt{n}}\right) = \mathbb{E}_x\left[\left|\left(\frac{\mathbf{1}}{\sqrt{n}}\right)^T x\right|\right] \leq \sqrt{n}e^{-4/n} \to 0$$