Math 425 Pset 1

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1. By the theorem due to Hadamond, the radius of convergence of e^z is just

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}}} = \frac{1}{\lim_{N\to\infty} \sup_{k\geq N} \sqrt[k]{\frac{1}{k!}}}$$

$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n$ iff the limit exists. So it suffices to show that $\lim_{k\to\infty} 1/(k!)^{1/k}$ exists. This exists and equals 0 if $\lim_{k\to\infty}(k!)^{1/k}$ equals infinity, which is what I shall show. Since $\sqrt{2\pi k}(k/e)^k/k! \to 1$ (Stirling's approximation), for k sufficiently large $k! \leq 2\sqrt{2\pi k}(k/e)^k$. Taking k-th roots on both sides, noting that $c^{1/k}$ is a decreasing function, we get that $k!^{1/k} \geq (2\sqrt{2\pi})^{1/k} k^{1/2k}(k/e)$. Finally, $\lim_{k\to\infty}(2\sqrt{2\pi})^{1/k} = (2\sqrt{2\pi})^{\lim_{k\to\infty} 1/k} = (2\sqrt{2\pi})^0 = 1$. Note: $\lim_{k\to\infty}(k^{1/k})^{1/2} = (\lim_{k\to\infty} k^{1/k})^{1/2} = (e^{\lim_{k\to\infty}\log(k)/k})^{1/2} = (e^0)^{1/2} = 1$. We have used the well-known fact that $\log(k)/k \to 0$. Clearly $k/e \to \infty$ as $k \to \infty$. Since $k!^{1/k}$ was larger than this, $k!^{1/k}$ also tends to infinity, as claimed. So the radius of convergence is ∞, and this series converges on the entire complex plane.

Similarly, the radius of convergence of the series defining $\sin(z)$ will be

$$\frac{1}{\lim_{N\to\infty} \sup_{k\geq N} \frac{1}{(2k+1)!}}$$

and by the exact same reasoning as last time ($\lim_{N\to\infty} 1/(2N+1)! = 0$), we see this tends to ∞. So indeed $\sin(z)$ also converges on the entire complex plane.

One notes that, by the taylor expansion,

$$\begin{aligned}\frac{e^{iz}-e^{-iz}}{2i} &= \frac{1}{2i}\left(\sum_{k=0}^{\infty} \frac{(iz)^k}{k!} - \frac{(-1)^k(iz)^k}{k!}\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{i^{k-1}z^k}{k!} - \frac{(-1)^k i^{k-1} z^k}{k!}\right)\end{aligned}$$

If $k \equiv 2 \pmod 4$, $i^{k-1} = i^{2+4k-1} = i$, and clearly $(-1)^k = 1$. We see that in this case the terms cancel exactly, so the series vanishes when $k \equiv 2 \pmod 4$. Similarly, if $k \equiv 0 \pmod 4$,

$i^{k-1} = i^{-1} = -i$, and also $(-1)^k = 1$. Th terms cancel again. We are left with just odd terms in our series, so our series becomes:

$$\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{i^{2k+1-1}z^{2k+1}}{(2k+1)!} - \frac{(-1)^{2k+1}i^{2k+1-1}z^{2k+1}}{(2k+1)!}\right) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{i^{2k}z^{2k+1}}{(2k+1)!} - \frac{(-1)^{2k+1}i^{2k}z^{2k+1}}{(2k+1)!}\right)$$

If k is even, $i^{2k} = 1$, and if k is odd, $i^{2k} = -1$. So $i^{2k} = (-1)^k$. Also, $(-1)^{2k+1} = (-1)^{2k} \cdot -1 = -1$. We see that:

$$\begin{aligned}
\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{i^{2k}z^{2k+1}}{(2k+1)!} - \frac{(-1)^{2k+1}i^{2k}z^{2k+1}}{(2k+1)!}\right) &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} - \frac{-1\cdot(-1)^k z^{2k+1}}{(2k+1)!}\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{2(-1)^k z^{2k+1}}{(2k+1)!}\right) \\
&= \sin(z)
\end{aligned}$$

As claimed. For the final part, note that

$$|\sin(iR)| = \left|\frac{1}{2i}\right| \cdot |e^{i\cdot iR} - e^{-i\cdot iR}| = \frac{1}{2} \cdot |e^{-R} - e^{R}| \geq \frac{1}{2}(e^R - e^{-R})$$

As $R \geq 1$, $e^{-R} \leq 1$, so $-e^{-R} \geq -1$, and we see that

$$\frac{1}{2}(e^R - e^{-R}) \geq \frac{1}{2}(e^R - 1)$$

Finally, I claim that $\frac{1}{2}(e^R - 1) \geq \frac{1}{1000}e^{R/1000}$. For clearly, plugging in $R = 1$ to the LHS gives us $\frac{12}{(}e - 1) \geq \frac{1}{2} \geq e^{1/1000}/1000$. The LHS is clearly growing faster than the RHS, so we have proven this mini claim. Finally, $\max_{|z|\leq R} |\sin(z)| \geq |\sin(iR)|$, so we have proven the entire claim. It is very weird that $\sin(z)$ is growing faster than exponentially (in some sense).

2. A nice one is $f(z) = 1$. A function $f(z)$ is diff'able at z if $\lim_{w\to 0} \frac{f(z+w)-f(z)}{w}$ exists, in our case,

$$\lim_{w\to 0} \frac{f(z+w) - f(z)}{w} = \lim_{w\to 0} \frac{1-1}{w} = \lim_{w\to 0} 0 = 0$$

As claimed. Also, clearly $|1| = 1$.

3. Interpreting $f : \mathbb{R}^2 \to \mathbb{R}^2$, as $f(x, y) = (u(x, y), v(x, y))$ we see that for any pair of curves $r_1(t), r_2(t)$, satisfying $\langle r_1'(t), r_2'(t)\rangle = 0$, for every $t \in \mathbb{R}$, we have that $\langle f'(r_1(t)), f'(r_2(t))\rangle = 0$. This is just putting the wording in the problem into something I can use. By the chain rule, $f'(r(t)) = Df \cdot r'(t)$, where the $\cdot$ is a matrix product. For convention we write $Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. The first pair of curves I shall use is $r_1(t) = (t, 0)$ and $r_2(t) = (0, t)$. It is obvious these curves are orthogonal, and note that $r_1'(t) = (1, 0)$, $r_2'(t) = (0, 1)$. Our inner product becomes

$$\begin{aligned}
\left\langle \begin{pmatrix} u_x(t,0) & u_y(t,0) \\ v_x(t,0) & v_y(t,0) \end{pmatrix} \cdot (1,0), \begin{pmatrix} u_x(0,t) & u_y(0,t) \\ v_x(0,t) & v_y(0,t) \end{pmatrix} \cdot (0,1) \right\rangle &= 0 \\
\langle (u_x(t,0), v_x(t,0)), (u_y(0,t), v_y(0,t))\rangle &= 0 \\
u_x(t,0)u_y(0,t) + v_x(t,0)v_y(0,t) &= 0 \\
u_x(0,0)u_y(0,0) + v_x(0,0)v_y(0,0) &= 0
\end{aligned}$$

The last line comes from seeing that this is indeed true for all nonzero t, then taking a limit as $t \to 0$ noting that $u, v \in C^\infty$. The next two curves we use are $r_1(t) = (t, t)$, and $r_2(t) = (t, -t)$. I won't write this one out, as you go through a very similar calculation, but in the end you get that $u_x^2 - u_y^2 + v_x^2 - v_y^2 = 0$. Note also that $2iu_x(0,0)u_y(0,0) = -2iv_x(0,0)v_y(0,0)$ (just multiply by $2i$ and rearrange). Note: from here forward all partials are evaluated at $(0,0)$. The first equation tells us that $u_x^2 + i^2u_y^2 = v_y^2 + i^2v_x^2$, and adding the new second equation gives us that $(u_x + iu_y)^2 = (v_y - iv_x)^2$. In the complex plane, $\sqrt{z^2} = z$ or $-z$. This gives us two cases (the other two cases collapse into these two): $u_x + iu_y = v_y - iv_x$, or $u_x + iu_y = -v_y + iv_x$. In the first case, $u_x = v_y$ and $u_y = -v_x$, so f satisfies the Cauchy-Riemann equations. In the second case, $\overline{f}$ satisfies the Cauchy-Riemann equations. For any $z_0 \in \mathbb{C}$, apply this exact same argument to $f(z + z_0)$, to get that at z_0 either f or $\overline{f}$ satisfy the Cauchy-Riemann equations. Suppose there were points z_0, w_0 so that at z_0 f satisfies the Cauchy-Riemann equations, and at w_0 $\overline{f}$ satisfies the Cauchy-Riemann equations. By our argument above, on the line connecting z_0 with w_0 either f or $\overline{f}$ satisfy the Cauchy-Riemann equations–in particular, we can walk on this line starting at z_0 until $\overline{f}$ satisfies the Cauchy-Riemann equations. Call this first point x_0 (Note: such a point exists, as w_0 at least satisfies this, but it could occur closer to z_0. Also, if it turns out that you can always find a point ε away with this property for every ε, you can just take a limit now like I have done below and see that the argument still works out–this could be done at every point). By construction, any point closer to z_0 than x_0 on this line is so that f satisfies Cauchy-Riemann. Because v_x, u_y are continuous, for $|w|$ sufficiently small, $|v_x(x_0 + w) - v_x(x_0)| < |v_x(x_0)|/10000$ and $|u_y(x_0 + w) - u_y(x_0)| < |v_x(x_0)|/1000$ (Note: $|v_x(x_0)| = |u_y(x_0)|$). Then for all w a sufficiently small (in magnitude) multiple of $z_0 - x_0$, we see that $f(x_0 + w)$ satisfies the Cauchy-Riemann equations. By our construction, $\overline{f(x_0)}$ satisfies the Cauchy-Riemann equations too. So $v_x(x_0 + w) = -u_y(x_0 + w)$, while $-v_x(x_0) = u_y(x_0)$. Letting $w \to 0$ shows us that $v_x(x_0) = -v_x(x_0)$, or that $v_x(x_0)$, so in this case f is Cauchy-Riemann at x_0 too. Continuing this argument until we have covered the entire line (Note: u_y and v_x are *uniformly* continuous on our line, so this approach works), we see that in our original construction, $f(w_0)$ was also Cauchy-Riemann. Therefore, in this construction, one of f or $\overline{f}$ satisfy the Cauchy-Riemann equations, depending on who satisfies it at the origin. **Q.E.F.D.**

4. First, algebra shows that $g(z) = (dz - b)/(a - cz)$ is a two sided inverse for the abitrary mobius transform $f(z) = (az + b)/(cz + d)$, and as for sets two sided inverse iff bijective, we see that all mobius transforms are bijective (and, that the inverse function is also a mobius transform–similar to a homeomorphism). Given any two mobius transforms, $f(z) = (az + b)/(cz + d)$, and $g(z) = (tz + u)/(sz + r)$, simple algebra shows that $f \circ g(z) = \frac{z(at+bs)+au+br}{z(ct+ds)+cu+dr}$, so the composition of two mobius transforms is again another mobius transform. We wish to show that these operations preserve the circle / straight line structure (which is actually stronger than what the question asks). Every straight line is of the form $\{z \in \mathbb{C} \mid z = w + tz_0\}$ for fixed w, z_0 (Note: in this entire problem, $t \in \mathbb{R}$). In the case of dilations, if we take an arbitrary circle $C = \{z \in \mathbb{C} \mid |z - z_0| = r\}$ and apply this transformation (assuming $\lambda \neq 0$), we get $\{z \in \mathbb{C} \mid |\lambda z - z_0| = r\} = \{z \in \mathbb{C} \mid |z - z_0/\lambda| = r/\lambda\}$. Doing this transformation

to the straight line would give us $\{z \in \mathbb{C} \mid \lambda z = w + tz_0\} = \{z \in \mathbb{C} \mid z = w/\lambda + tz_0/\lambda\}$. Clearly $w/\lambda, z_0/\lambda$ are just other fixed complex numbers, so this indeed preserves straight lines. Doing $z \mapsto z + a$ for a regular circle gives us $\{z \in \mathbb{C} \mid |(z + a) - z_0| = r\} = \{z \in \mathbb{C} \mid |z - (z_0 - a)| = r\}$, which is clearly another circle. For straight lines, doing $z \to z + a$ gives us $\{z \in \mathbb{C} \mid z + a = w + tz_0\} = \{z \in \mathbb{C} \mid z = (w - a) + tz_0\}$, and as $w - a$ is just another complex number, this is indeed still a line. Finally, the inversion $z \mapsto z^{-1}$ is really just $z \to \overline{z}/|z|^2$, which we derived from the identity that $z\overline{z} = |z|^2$. For circles, this becomes $\{z \in \mathbb{C} \mid |\overline{z}/|z|^2 - z_0| = r\} = \{z \in \mathbb{C} \mid 1/|z|^2|\overline{z} - z_0| = r\} = \{z \in \mathbb{C} \mid |\overline{z - \overline{z_0}}| = |z|^2 r\} = \{z \in \mathbb{C} \mid |z - \overline{z_0}| = |z|^2 r\}$, where we used that $|z| = |\overline{z}|$. Clearly this is now in the form of a circle, as claimed. For the line, write $w = w_0 + iw_1$, and $z_0 = z_1 + iz_2$. Our original line becomes $\{z \in \mathbb{C} \mid z = w_0 + iw_1 + t(z_1 + iz_2)\} = \{a + bi \mid a + bi = w_0 + tz_1 + i(w_1 + tz_2)\}$, where we have used that every complex number is of the form $a + bi$ many times. In any case, doing $z \mapsto \overline{z}/|z|^2$ would give us $\{a + bi \mid (a - bi)/\sqrt{a^2 + b^2} = w_0 + tz_1 + i(w_1 + tz_2)\}$. Matching real and imaginary parts tells us that $a = \sqrt{a^2 + b^2}(w_0 + tz_1)$, and that $-b = \sqrt{a^2 + b^2}(w_1 + tz_2)$. So $b = -\sqrt{a^2 + b^2}(w + tz_2)$, which tells us that $a + bi = \sqrt{a^2 + b^2}(w_0 + tz_1) + i(-\sqrt{a^2 + b^2}(w + tz_2))$. Rearranging this gives us $a + bi = \sqrt{a^2 + b^2}(w_0 - iw_1) + t\sqrt{a^2 + b^2}(z_1 - iz_2) = \sqrt{a^2 + b^2}\overline{w} + t\sqrt{a^2 + b^2}\overline{z}$. Clearly this is in the form of the line definition that I gave above, so thusly inversion preserves the line structure. As a Mobius transformation is simply a finite composition that preserve the circle/line structure, a Mobius transformation also preserves the circle/line structure. **Q.E.D.**

5. We see

$$\limsup_{n\to\infty} \sqrt[n]{na_n} = \limsup_{n\to\infty} \sqrt[n]{n} \cdot \limsup_{n\to\infty} \sqrt[n]{a_n}$$

This is true because both limsup's exist (I shall show this).

$$\limsup_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{1/n \log(n)} = e^0 = 1$$

So indeed, $\limsup_{n\to\infty} \sqrt[n]{na_n} = 1 \cdot \limsup_{n\to\infty} \sqrt[n]{a_n} = 1/R$. Therefore,

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{na_n}} = R$$

So the series does indeed have the same radius of convergence. Given $|z| < R$, we want to show that

$$\lim_{w\to 0} \frac{1}{w}\left(\sum_{n=0}^{\infty} a_n(z + w)^n - \sum_{n=0}^{\infty} a_n z^n\right)$$

exists and equals $g(z)$. For w sufficiently small, $|z + w| < R$, so we can take the sum outside to get

$$\lim_{w\to 0} \frac{1}{w}\left(\sum_{n=0}^{\infty} a_n(z + w)^n - a_n z^n\right)$$

An infinite sum is really just a limit of partial sums, so this equals

$$\lim_{w\to 0}\frac{1}{w}\left(\lim_{k\to\infty}\sum_{n=0}^{k}a_n((z+w)^n-z^n)\right)$$

Depending on n, we may find a radius R_n sufficiently small so that if $|w|<R_n$, $(z+w)^n-z^n = nz^{n-1}w+E(w)$ where $|E(w)|\leq \varepsilon|w|/(2^n|a_n|)$ (Note: $|z|^n$ is fixed). Taking $R=\min\{R_i\}\cup\{1\}$ to make sure all these inequalities are true at the same time, we find that

$$\begin{aligned}\lim_{w\to 0}\frac{1}{w}\left(\lim_{k\to\infty}\sum_{n=0}^{k}a_n((z+w)^n-z^n)\right) &= \lim_{w\to 0}\frac{1}{w}\left(\lim_{k\to\infty}\sum_{n=0}^{k}a_n(nz^{n-1}+E(w))\right)\\ &= \lim_{w\to 0}\frac{1}{w}\left(\lim_{k\to\infty}\sum_{n=0}^{k}a_n nz^{n-1}w+\sum_{n=0}^{k}a_nE(w)\right)\end{aligned}$$

Notice that

$$\left|\sum_{n=0}^{k}a_nE(w)\right|\leq\sum_{n=0}^{k}|a_n||E(w)|\leq\sum_{n=0}^{k}|w|\varepsilon/2^n\leq\varepsilon|w|$$

In particular, $|\left|\sum_{n=0}^{k}a_nE(w)\right|/|w|\leq\varepsilon$ for every positive ε, so as $k\to\infty$, it's limit is indeed 0. Finally, $\lim_{w\to 0}1/w\lim_{k\to\infty}\sum_{n=0}^{k}a_nnz^{n-1}w=\lim_{w\to 0}\lim_{k\to\infty}\sum_{n=0}^{k}a_nnz^{n-1}=\lim_{k\to\infty}\sum_{n=0}^{k}a_nnz^{n-1}$, which exists by our claim above. This completes the proof that both f is differentiable, and where it is differentiable it's derivative equals g.