CSE 311 Quiz 7

Rohan Mukherjee

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0. Let $P(n) = f(n) = n$. I prove $P(n)\ \forall n \in \mathbb{N}$ by strong induction.

   Base cases: $f(0) = 0$, and $f(1) = 1$, so we may conclude $P(0),\ P(1)$.

   Inductive Hypothesis: Suppose $P(0) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geqslant 1$.

   Inductive Step: We see that as $k + 1 \geqslant 2$, we have to use the last definition for $f(k + 1)$. So we see that $f(k + 1) = 2f(k) - f(k - 1) = 2k - (k - 1)$ by the inductive hypothesis. We conclude that $f(k + 1) = k + 1$ by algebra. As this is what $P(k + 1)$ asserts, we see that $P(n)$ holds for all integer $n \geqslant 1$.

1. Let $P(n) = 6n + 6 < 2^n$. We prove $P(n) \forall n \geqslant 6$ by induction.

   Base case: $6 \cdot 6 + 6 = 42 < 2^6 = 64$ so we see that $P(6)$ holds.

   Inductive Hypothesis: Suppose $P(k)$ for an arbitrary $k \geqslant 6$.

   Inductive Step: We notice that $6(k + 1) + 6 = 6k + 6 + 6 < 2^k + 6$ by the inductive hypothesis. We also note that $6 < 2^6 \leqslant 2^k$, because $k \geqslant 6$, and 2^z is an increasing function. So $6(k + 1) + 6 < 2^k + 6 \leqslant 2^k + 2^k = 2^{k+1}$, which is what $P(k + 1)$ asserts. We conclude that $P(n)$ holds for all integer $n \geqslant 6$ by induction.

3 b) Let $P(x) = \mathit{leaves}(x) \geqslant \mathit{size}(x)/2 + 1/2$. We prove $P(x)$ for all trees x by structural induction. **Base case:** $\mathit{size}(\bullet)/2 + 1/2 = 1/2 + 1/2 = 1 \leqslant \mathit{leaves}(\bullet) = 1$, so we see that $P(\bullet)$ is true.

Inductive Hypothesis: Suppose L, R are trees, and suppose $P(L), P(R)$. We notice that

$$\begin{aligned}
\mathit{leaves}((L, R, \bullet)) &= \mathit{leaves}(L) + \mathit{leaves}(R) \\
&\geqslant \mathit{size}(L)/2 + 1/2 + \mathit{size}(R)/2 + 1/2 && \text{((I.H.))} \\
&= 1/2(\mathit{size}(L) + \mathit{size}(R) + 1) + 1/2 \\
&= 1/2\mathit{size}(\mathtt{Tree}(\bullet, L, R)) + 1/2
\end{aligned}$$

Which is precisely what $P(\mathtt{Tree}(\bullet, L, R))$ states. So we may conclude that $P(x)$ holds for all trees x by structural induction.