Math 505 HW7

Anonymous

March 10, 2024

1. Recall that an (eventually the) identity morphism is defined as the following: $\mathrm{id}_A \in \mathrm{Mor}(A, A)$ satisfies $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$ for all $f \in \mathrm{Mor}(B, A)$ and $g \in \mathrm{Mor}(A, B)$ for every object B. Let i be another identity element. Then $i \circ \mathrm{id}_A = i = \mathrm{id}_A$, where in the first equality we used the right identity property for id_A and in the second we used the left identity property for i.

2. (a) We claim that $\varphi : M_2 \to M_1 \oplus M_3$ defined by $\varphi(m_2) = (\pi(m_2), g(m_2))$ is the desired isomorphism. By a similar argument from part (b) this is an R-module homomorphism, so we just need to show that φ is bijective. Assuming that $\pi(m_2) = 0$ and $g(m_2) = 0$, we would have $m_2 \in \operatorname{Im} f$, so write $m_2 = f(m_1)$. Then $m_1 = \pi(f(m_1)) = \pi(0) = 0$, so $m_1 = 0$ and thus $m_2 = f(m_1) = 0$ as well, which shows injectivity. Let $(m_1, m_3) \in M_1 \oplus M_3$. Since g is surjective, find m_2 so that $g(m_2) = m_3$. Restricting the codomain of f to $\operatorname{Im} f$ and the domain of π to $\operatorname{Im} f$ will still have the property that $\pi \circ f = \mathrm{id}_{M_1}$, so $\pi : \operatorname{Im} f \to M_1$ is still surjective. This means we can find $m_2' \in \operatorname{Im} f$ so that $\pi(m_2') = m_1 - \pi(m_2)$. Letting $m = m_2 + m_2'$, we see that $\pi(m) = \pi(m_2) + \pi(m_2') = m_1$, and, since $m_2' \in \operatorname{Im} f = \ker g$, we have $g(m) = g(m_2) + g(m_2') = m_3$, which shows that φ is surjective. The φ from this part is precisely the ψ^{-1} from part (b), so the exact same reasoning shows that the diagram commutes.

   (b) Suppose there exists an R-module homomorphism $\iota : M_3 \to M_2$ such that $g \circ \iota = \mathrm{id}_{M_3}$. We claim that

$$\begin{aligned} \psi &: M_1 \oplus M_3 \to M_2 \\ \psi(m_1, m_3) &= f(m_1) + \iota(m_3) \end{aligned}$$

Is the desired R-module isomorphism. First,

$$\begin{aligned}\psi((m_1,m_3)+(m_1',m_3')) &= f(m_1)+\iota(m_3)+f(m_1')+\iota(m_3') = f(m_1+m_1')+\iota(m_3+m_3') \\ &= \psi(m_1+m_1',m_3+m_3') \\ r\psi(m_1,m_3) &= rf(m_1)+r\iota(m_3)+f(rm_1)+\iota(rm_3) = \psi(rm_1,rm_3) = \psi(r(m_1,m_3))\end{aligned}$$

So ψ is indeed an R-module homomorphism. If $\psi(m_1,m_3)=0$, then $f(m_1) = -\iota(m_3)$. Applying g to both sides shows that $g(f(m_1)) = -m_3$. Since $\operatorname{Im} f = \ker g$, $g(f(m_1)) = 0$ and $m_3 = 0$. Thus $f(m_1) = -\iota(m_3) = -\iota(0) = 0$. Since f is injective, $m_1 = 0$, which shows that ψ is injective.

Now let $m_2 \in M_2$ and consider $k = m_2 - \iota(g(m_2))$. By applying g, we see that $g(k) = g(m_2) - (g\circ\iota)(g(m_2)) = g(m_2) - g(m_2) = 0$, since $g\circ\iota = \mathrm{id}_{M_3}$. Since $\ker g = \operatorname{Im} f$, $k \in \operatorname{Im} f$, so find m_1 so that $k = f(m_1)$. Then $m_2 = f(m_1)+\iota(g(m_2))$, so m_2 is the image of $(m_1, \iota(g(m_2)))$ which shows surjectivity. Now we claim the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "\mathrm{id}_{M_1}"] & M_2 \arrow[r, "g"] \arrow[d, "\psi^{-1}"] & M_3 \arrow[r] \arrow[d, "\mathrm{id}_{M_3}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_1\oplus M_3 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

We see that $\psi^{-1}(f(m_1)) = \psi^{-1}(f(m_1)+\iota(0)) = (m_1,0)$ as desired. Similarly, writing $m_2 = f(m_1)+\iota(m_3)$ uniquely as above, and letting $\pi : M_1\oplus M_3 \to M_3$ be the canonical projection, we see that $\pi\circ\psi^{-1}(m_2) = \pi((m_1,m_3)) = m_3 = g(m_2)$, as desired.

(c) We show that this part implies the last two parts, which will complete the equivalence. Let $\psi : M_2 \to M_1\oplus M_3$ be the isomorphism, let $\pi : M_1\oplus M_3 \to M_3$ be the natural projection, and consider the map $\iota : M_3 \to M_2$ by $\iota(m_3) = \psi^{-1}(0,m_3)$. It is clear that ψ is a R-module homomorphism, so we need only verify that $g\circ\iota(m_3) = m_3$. We know that $g(m_2) = \pi(\psi(m_2))$ holds for every m_2, so $g(\iota(m_3)) = \pi(\psi(\psi^{-1}(0,m_3))) = \pi(0,m_3) = m_3$, as desired.

Similarly, let $\iota : M_1 \to M_1\oplus M_3$ be the canonical embedding, $\eta : M_1\oplus M_3 \to M_1$ the canonical projection, and define $\pi(m_2) = \eta(\psi(m_2))$. Most importantly notice that $(\eta\circ\iota)(m_1) = m_1$. Then $\pi\circ f(m_1) = \eta(\psi(f(m_1))) = \eta(\iota(m_1)) = m_1$, as desired.

3. (a) We shall show that projective $\implies$ (1) $\implies$ (2) $\implies$ projective. Suppose that P

is projective, and let

$$0 \longrightarrow N \xrightarrow{g} M \xrightarrow{f} P \longrightarrow 0$$

be an exact sequence. Then,

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\exists\iota"'] \arrow[d, "\mathrm{id}_P"] & \\ 0 \arrow[r] & N \arrow[r, "g"] & M \arrow[r, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

By the projective property. This precisely says that $f \circ \iota = \mathrm{id}_P$, so the exact sequence splits by question 2.

Now suppose that any exact sequence

$$0 \longrightarrow N \xrightarrow{g} M \xrightarrow{f} P \longrightarrow 0$$

splits. Let $F = \bigoplus_{p \in P} R$ be the free module indexed by P, and let $\left\{ x_p \right\}_{p \in P}$ be a basis. Let ψ be the unique R-module homomorphism sending x_p to p, as per the universal property. ψ is clearly surjective, so the following sequence is exact:

$$0 \longrightarrow \ker\psi \longrightarrow F \xrightarrow{\psi} P \longrightarrow 0$$

By hypothesis this sequence is split, so $P \oplus \ker\psi \cong F$, as desired.

Suppose we are given maps $\varphi : P \to N$ and $f : M \twoheadrightarrow N$. Let Q be so that $P \oplus Q$ is free, let $\{ x_i \}_{i \in I}$ be a basis, and define $\psi(p, q) = (\varphi \circ \pi)(p, q) = \varphi(p)$. Then the following diagram commutes:

$$\begin{tikzcd} & P \arrow[r, hook] \arrow[d, "\varphi"'] & P \oplus Q \arrow[dl, "\psi"] \\ M \arrow[r, two heads, "f"] & N & \end{tikzcd}$$

Since f is surjective, for each x_i let m_i be any element of M satisfying $f(m_i) = \psi(x_i)$. In particular, m_i need not be unique. Extend the map sending each x_i to m_i to an R-module homomorphism $\overline{\psi} : P \oplus Q \to M$, as per the universal property of free modules. We now have the following commutative diagram:

$$
\begin{tikzcd}
& & P \oplus Q \arrow[dll, bend right, "\overline{\psi}"'] \arrow[ddl, bend left, "\psi"] \\
M \arrow[dr, "f"'] & P \arrow[ur, hook] \arrow[d, "\varphi"] & \\
& N &
\end{tikzcd}
$$

This diagram commutes precisely because if $m = \sum_{i \in I} r_i x_i$, with only finitely many $r_i \neq 0$, then

$$f(\overline{\psi}(m)) = f\left(\overline{\psi}\left(\sum_{i \in I} r_i x_i\right)\right) = \sum_{i \in I} r_i f(\overline{\psi}(x_i)) = \sum_{i \in I} r_i f(m_i) = \sum_{i \in I} r_i \psi(x_i) = \psi(m)$$

By construction, and if we let $\iota : P \to P \oplus Q$ be the canonical inclusion, $f \circ \overline{\psi} \circ \iota(p) = f \circ \overline{\psi}(p, 0) = \psi(p, 0) = \varphi(p)$ as desired, which completes the proof.

(b) Added this part before the deadline but forgot to submit... Oops.

Let R be a ring and $I \subset R$ an ideal. We claim there is a natural bijection

$$\{\, R/I\text{-modules} \,\} \leftrightarrow \{\, R\text{-modules } M \text{ annihilated by } I \,\}$$

Let M be an R-module annihilated by I, i.e. $i \cdot m = 0$ for every $m \in M$ and $i \in I$. We seek to define $\bar{r} \cdot m = r \cdot m$. We need to show that this is well-defined: suppose that $\bar{r}_1 = \bar{r}_2$. Then $r_2 = r_1 + i$ for some $i \in I$, and we see that $r_2 \cdot m = (r_1 + i) \cdot m = r_1 \cdot m + 0$. Thus M can be made into an R/I-module. Similarly, if M is an R/I-module, simply define $r \cdot m := \bar{r} \cdot m$. Since M is an R/I module this shows that this action makes M into an R-module. Lastly notice that under this definition, I annihilates M, so we have verified the above correspondence.

This in particular shows that the $\mathbb{Z}$-module $\mathbb{Z}/2$ can be made into a $\mathbb{Z}/6$ module since (6) annihilates $\mathbb{Z}/2$. Similarly $\mathbb{Z}/3$ is a $\mathbb{Z}/6$ module. The inclusion map $\iota : \mathbb{Z}/3 \to \mathbb{Z}/6$ by $\iota(1) = 2$ and extending linearly, and the map $f : \mathbb{Z}/6 \to \mathbb{Z}/2$ by $1 \mapsto 3$ form the following exact sequence:

$$0 \longrightarrow \mathbb{Z}/3 \xrightarrow{\iota} \mathbb{Z}/6 \xrightarrow{f} \mathbb{Z}/2 \longrightarrow 0$$

The ring isomorphism $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$ forgets to a module isomorphism, so $\mathbb{Z}/2$ is projective by the previous part. However, $\mathbb{Z}/2$ cannot be free since if it was, since it is obviously finitely generated it would be of the form $(\mathbb{Z}/6)^{\oplus n}$, but the latter has order $6^n \neq 2$ for any n.