Math Template

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Problem 3.10

Suppose $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions. Show that if S and T are stopping times and X is a bounded $\mathcal{F}_\infty$ measurable random variable, then

$$E[E[X \mid \mathcal{F}_S]\mathcal{F}_T] = E[X \mid \mathcal{F}_{S\wedge T}]$$

Let $Y_t = E[X \mid \mathcal{F}_t]$ and $Z_t = Y_{t\wedge S}$. We have two short things to verify. First, I claim that for any stopping time T, $Y_T = E[X \mid \mathcal{F}_T]$ where $F_t = \{A \subset \Omega : A \cap \{T \leq t\} \in \mathcal{F}_T\}$. Let $T_n \downarrow T$. By taking a right-continuous version of Y_t, and noting that the filtration $\mathcal{F}_t$ satisfies the usual conditions, Proposition 3.10 shows that Y_{T_n} is $\mathcal{F}_{T_n} \subset \mathcal{F}_{T+\frac{1}{2^n}}$ measurable. Let $A \in \mathcal{F}_T$. Then, as X is bounded, the dominated convergence theorem lets us move the sum out of the expectation, so

$$\begin{aligned}
E(Y_{T_n}1_A) &= \sum_{k=0}^{\infty} E\Big(Y_{(k+1)/2^n}\ ;\ A, \{T_n = (k+1)/2^n\}\Big) \\
&= \sum_{k=0}^{\infty} E(E(X \mid \mathcal{F}_{(k+1)/2^n})\ ;\ A, \{T_n = (k+1)/2^n\}) \\
&= \sum_{k=0}^{\infty} E(X\ ;\ A, \{T_n = (k+1)/2^n\}) = E(X\ ;\ A)
\end{aligned}$$

Again the dominated convergence theorem shows that $E(Y_T\ ;\ A) = \lim_{n\to\infty} E(Y_{T_n}\ ;\ A) = E(X\ ;\ A)$, so $Y_T = E(X \mid \mathcal{F}_T)$ as desired.

Now take a look at $E(Y_S \mid \mathcal{F}_t)$. By what we just proved, this is precisely equal to $E(E(X \mid \mathcal{F}_S) \mid \mathcal{F}_t)$.

We try something similar. Let $S_n \downarrow S$. Then:

$$E(Y_{S_n} \mid \mathcal{F}_t) = \sum_{k=0}^{\infty} E(Y_{(k+1)/2^n} \mid \mathcal{F}_t) 1_{\{S_n=(k+1)/2^n\}}$$

Since if $\mathcal{E} \subset \mathcal{F}$, $E(E(X \mid \mathcal{E}) \mid \mathcal{F}) = E(E(X \mid \mathcal{F}) \mid \mathcal{E}) = E(X \mid \mathcal{E})$, we can write this as:

$$\sum_{k=0}^{\infty} E(Y_{(k+1)/2^n} \mid \mathcal{F}_t) 1_{\{S_n=(k+1)/2^n\}} = \sum_{k=0}^{\infty} E(Y_{(k+1)/2^n \wedge t}) 1_{\{S_n=(k+1)/2^n\}} = Y_{S_n \wedge t}$$

Another application of dominated convergence theorem using that Y_t has right-continuous paths yields $E(Y_S \mid \mathcal{F}_t) = Y_{S \wedge t} = Z_t$. And now, we are done. Using the aboove, the first argument again, plugging in $t = T$ yields $Z_T = E(Y_S \mid \mathcal{F}_T) = E(E(X \mid \mathcal{F}_S) \mid \mathcal{F}_T)$.

Problem 7.1

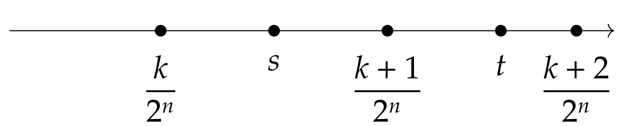
Prove that

$$\lim_{\delta \to 0} \sup_{s,t \in [0,1], 0<|t-s|<\delta} \frac{|W_t| - W_s|}{\sqrt{\delta \log(1/\delta)}} < \infty \text{ a.s.}$$

Let W be a brownian motion and set $A_n = \left\{ \exists k \leq 2^n - 1 \;\middle|\; \sup_{k/2^n \leq t \leq (k+1)/2^n} |W_t| \geq \sqrt{2n} 2^{-n/2} \right\}$. Then by the Markov property of Brownian motion and union bound, we have:

$$P(A_n) \leq 2^n P(\sup_{t \leq 1/2^n} |W_t| \geq \sqrt{2n} 2^{n/2}) \leq 2^n \exp(-2n 2^n / 2(2^{-n})) = 2^n e^{-n}$$

where the second inequality follows by a form of Doob's maximal inequality found in Proposition 3.15. $\sum P(A_n) < \infty$, so $P(A_n \text{ i.o}) = 0$, meaning that for almost every ω, there exists N so that if $n \geq N$, $\omega \notin A_n$. Let $s \leq t$ be two points in $[0,1]$, and choose n so that $2^{-n} \leq t - s \leq 2^{-n+1}$. We have something like the following picture:



Then applying that $\omega \notin A_n$, we have that:

$$\begin{aligned}|W_t - W_s| &\le |W_t - W_{t\wedge(k+1)/2^n}| + |W_{t\wedge(k+1)/2^n} - W_{k/2^n}| + |W_s - W_{k/2^n}| \\ &\le 3 \cdot \sqrt{2n}2^{-n/2} \le 3\sqrt{2|t-s|\log(1/|t-s|)}\end{aligned}$$

The function $\sqrt{x\log(1/x)}$ is increasing in a neighborhood of 0 , so as long as $|t-s| \le 2^{-(N+2)}$ and δ is below that increasing threshold, we are good. This shows that as long as $\delta \le 2^{-(N+2)}$, we have:

$$\sup_{s,t\in[0,1],0<|t-s|<\delta} \frac{|W_t - W_s|}{\sqrt{\delta\log(1/\delta)}} \le 3\sqrt{2}$$

Which completes the proof.

Problem 7.4

Supose that $\alpha > 1/2$ and W_t is a brownian motion. Show that the event

$$A = \{\exists t \in [0,1] : W \text{ is Holder continuous of order } \alpha \text{ at } t\}$$

has probability 0.

Fix N be so large that $N(\alpha - 1/2) > 1$, and define:

$$\begin{aligned}A_{M,h} &= \{\exists s \in [0,1] : |W_t - W_s| \le M|t-s|^\alpha, |t-s| \le h\} \\ B_n &= \left\{\exists k \le 2n : \bigwedge_{i=1}^{N} |W_{(k+i)/n} - W_{(k+i-1)/n}| \le \frac{2N^\alpha M}{n^\alpha}\right\}\end{aligned}$$

I show that for $n \ge \dfrac{N}{h}$, $A_{M,h} \subset B_n$. This is becaus for k the largest integer with $k/n \le s$, $\left|\dfrac{k+i}{n} - s\right| \le \dfrac{N}{n} \le h$. Then,

$$\begin{aligned}\left|W_{(k+i)/n} - W_{(k+i-1)/n}\right| &\le |W_{(k+i)/n} - W_s| + |W_s - W_{(k+i-1)/n}| \\ &\le \frac{N^\alpha M}{n^\alpha} + \frac{N^\alpha M}{n^\alpha} = \frac{2N^\alpha M}{n^\alpha}\end{aligned}$$

Now, using independent increments, and that $P(|Z| \le r) \le 2r$ for standard normal Z,

$$P(B_n) \le 2nP\left(|W_{1/n}| \le \frac{2N^\alpha M}{n^\alpha}\right)^N = 2nP\left(|Z| \le \frac{2N^\alpha M}{n^{\alpha-1/2}}\right)^N \le 2n\frac{4N^{\alpha N}M^N}{n^{N(\alpha-1/2)}} = Cn^{-\beta}$$

for some $\beta > 0$ by our choice of N (recall $N(\alpha - 1/2) > 1$). This shows that

$$P(A_{M,h}) \le \limsup_{n\to\infty} P(B_n) = 0.$$

This implies that the probability that there exists $s \le 1$ such that

$$\limsup_{h\to 0} \frac{|W_{s+h} - W_s|}{|h|^\alpha} \le M$$

is zero, which shows that in fact, W_t is not Holder continuous of order α at any $s \in [0, 1]$.

Problem 7.8

Let $H_\gamma(A)$ be the Hausdorff measure of order γ. Many books (see, for example, Stein & Shakarchi Book 3) have shown that $H_\gamma(A)$ is a true measure on the borel subsets of $\mathbb{R}$. Using this, we know that if $C_n \downarrow C$, then $\lim_{n\to\infty} H_\gamma(C_n) = H_\gamma(C)$. Let C_n be the nth level of the cantor set. C_n is a union of 2^n intervals of length 3^{-n}. So letting δ be arbitrary, choose n so that $3^{-n} < \delta$. The order $\gamma = \log 2/\log 3$ length of C_n is $2^n \cdot 3^{-\gamma n} = 1$. This shows that

$$\lim_{\delta\to 0}\left[\inf\left\{\sum_{i=1}^{\infty}[b_i - a_i]^\gamma : A \subset \bigcup_{i=1}^{\infty}[a_i, b_i], \sup_i |b_i - a_i| \le \delta\right\}\right] \le 1$$

in particular it is finite. So the Hausdorff dimension of C is at most $\log 2/\log 3$. Now, suppose that it were less than $\log 2/\log 3$. Then $H_\gamma(C_n) \le M$ for some $\gamma = \log 2/\log 3 - \varepsilon$ and all $n \ge N$, by the limit argument. Since if $[a, b]$ is an interval covered by $\bigcup_i [a_i, b_i]$, all of which have length $\le |b - a|$, we must have

$$\sum_i |b_i - a_i| \ge |b - a|$$

by union bound and the regular lebesgue measure. But then I claim that:

$$\sum_i |b_i - a_i|^\gamma \ge |b - a|^\gamma$$

This follows since:

$$\sum_i \left(\frac{|b_i - a_i|}{|b-a|}\right)^\gamma \geq \sum_i \frac{|b_i - a_i|}{|b-a|} \geq 1$$

The first inequality is true as $\frac{|b_i - a_i|}{|b-a|} \leq 1$ and $x^\gamma \geq x$ for $x \leq 1$. So in particular, $H_\gamma([a,b]) \geq |a-b|^\gamma$. By disjointness, $H_\gamma(C_n) \geq 2^n \cdot 3^{-n\gamma} = 3^{n\varepsilon}$. Sending $n \to \infty$ yields a contradiction.

With that done, we prove exercise 7.7.

Problem 7.7.

Let W be a brownian motion and let Z be the zero set $Z = \{t \in [0,1] : W_t = 0\}$.

1. Show there exists a constant c not depending on x or δ such that:

$$P(\exists s \leq \delta : W_s = -x) \leq ce^{-x^2/2\delta}$$

2. Use the Markov Property of brownian motion to show that there exists a constant c not depending on s or t such that:

$$P(Z \cap [s,t] \neq \emptyset) \leq c\left(1 \wedge \sqrt{\frac{t-s}{t}}\right)$$

The first is a simple application of Doob's maximal inequality. Indeed, $P(\exists s \leq \delta : W_s = -x) \leq P(\sup_{s\leq\delta} |W_s| \geq |x|) \leq 2e^{-x^2/2\delta}$ per prop 3.15 in the book.

For the second, notice that $P(Z \cap [s,t] \neq \emptyset) = P(\exists \delta \leq t-s : W_{s+\delta} - W_s = -W_s)$. By what we proved above, and using that $W_{s+\delta} - W_s$ is a brownian motion independent of $\mathcal{F}_s$, we have:

$$P(\exists \delta \leq t-s : W_{s+\delta} - W_s = -W_s) = E(P(\exists \delta \leq t-s : W_{s+\delta} - W_s = -W_s \mid \mathcal{F}_s)) \leq E(2e^{-W_s^2/2(t-s)})$$

This equals:

$$\frac{2}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-x^2/2(t-s)} \cdot e^{-x^2/2s} dx = 2\sqrt{\frac{t-s}{t}}$$

Since probabilities are less than 1, we immediately get $P(Z \cap [s,t] \neq \emptyset) \leq 2\left(1 \wedge \sqrt{\frac{t-s}{t}}\right)$.

Let C_n be the (random) collection of intervals $[i/2^n, (i+1)/2^n]$ that intersect Z. Then $\#C_n$ is a real-valued random variable of our brownian motion. By 7.7 and lineararity of expectation, we have:

$$E[\#C_n] = \sum_{i=0}^{2^n-1} P(Z \cap [i/2^n, (i+1)/2^n] \neq \emptyset) \leq 2\sum_{i=0}^{2^n-1} \frac{1}{\sqrt{i}} \leq \int_0^{2^n} \frac{2}{\sqrt{x}} = 4 \cdot 2^{n/2}$$

The cover C_n of Z has Hausdorff γ measure:

$$\sum_{[i/2^n,(i+1)/2^n] \in C_n} |2^{-n}|^{\gamma} = 2^{-n\gamma}\#C_n$$

Thus, for each δ you can find an n with $2^{-n} < \delta$, also with:

$$E(2^{-n\gamma}\#C_n) \leq 4 \cdot 2^{-n\gamma} \cdot 2^{n/2}$$

We have found a sequence of covers C_n of Z with diameter shrinking to 0 so that its 1/2 Hausdorff measure is finite almost surely. Taking a limit, with probability 1, the Hausdorff 1/2 measure of all the C_n's bounded by 4 almost surely. As the quantity $\inf\left\{\sum_{i=1}^{\infty}[b_i - a_i]^{\gamma} : A \subset \cup_{i=1}^{\infty}[a_i, b_i],\ \sup_i [b_i - a_i] \leq \delta\right\}$ is increasing in δ, the limit as $\delta \to 0$ exists (possibly infinite), and equals the same among any sequence of δ's going to 0. This concludes the proof that $H_{1/2}(Z)$ is finite a.s., meaning that the Hausdorff dimension of Z is at most 1/2.