Math 334 HW 4

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February 26, 2024

1. Let $\varepsilon > 0$, and let $a \in \mathbb{R}^m$. We know that $B(f(a), \varepsilon)$ is certainly open, so therefore $f^{-1}(B(f(a), \varepsilon))$ is open. Clearly a is in this preimage, and because every point of this set is an interior point, there is some δ so that $B(a, \delta) \subset f^{-1}(B(f(a), \varepsilon))$. Because of this, we know that everything in the ball also maps to something in $B(f(a), \varepsilon)$, and so f is continuous. $\square$

2. Let $\varepsilon > 0$. Because $(a_n)_{n=1}^\infty \to a$, it is bounded, so choose $R > 0$ so that $|a_n - a| < R$ for all $n \geqslant 1$. Now choose N_1 so that for all $n > N_1$, we have that $|a_n - a| < \varepsilon/2$. Finally, if $R > \varepsilon/2$, choose N_2 so that $N_1(R - \varepsilon/2) < N_2 \cdot \varepsilon/2$, else choose $N_2 = 0$. Now choose $N = \max\{N_1, N_2\}$. Then for all $n > N$, we have that $|b_n - a| = \left|\frac{1}{n}\sum_{k=1}^n (a_k - a)\right| \leqslant \frac{1}{n}\sum_{k=1}^n |a_k - a| = \frac{1}{n}\left(\sum_{k=1}^{N_1} |a_k - a| + \sum_{k=N_1+1}^n |a_k - a|\right) < \frac{1}{n}\left(\sum_{k=1}^{N_1} R + \sum_{k=N_1+1}^n \varepsilon/2\right) = \frac{N_1 \cdot R}{n} + \varepsilon/2\frac{n - N_1}{n} = \frac{N_1(R - \varepsilon/2)}{n} + \varepsilon/2$. Now note if $R \leqslant \varepsilon/2$, this quantity is $\leqslant \frac{N_1(\varepsilon/2 - \varepsilon/2)}{n} + \varepsilon/2$, which is clearly $< \varepsilon$. If this is not the case, by our choice of N_2 we see that $\frac{N_1(R - \varepsilon/2)}{n} + \varepsilon/2 < \frac{N_2 \cdot \varepsilon/2}{n} + \varepsilon/2$, and because $N_2 < n$, this quantity is $< \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

3. **Part 2:** Suppose that $(x_n)_{n=1}^\infty \to x$ is a sequence of reals that converges to x. Then there is some $N > 0$ such that $\forall n > N$, $|x_n - x| < 1$. Clearly there are only finitely many elements in $\{x_n \mid 1 \leqslant n \leqslant N\}$. But every finite set is bounded, therefore there is some $R > 0$ such that $\{x_n \mid 1 \leqslant n \leqslant N\} \subset B(x, R)$. But clearly $\{x_n \mid n > N\} \subset B(x, 1)$, and so if we pick $R^* = \max\{1, R\}$, we have that $(x_n)_{n=1}^\infty \subset B(x, R^*)$, i.e. that $(x_n)_{n=1}^\infty$ is bounded. $\square$

   Part 3: Let $(x_{n_k})_{k=1}^\infty$ be a subsequence of $(x_n)_{n=1}^\infty \to x$, and let $\varepsilon > 0$. Then, because $(x_n)_{n=1}^\infty$ is convergent, we know that there is some $N > 0$ such that $|x_n - x| < \varepsilon$ for all $n > N$. Suppose that $n_k < k$. It is clear that $n_1 \geqslant 1$, by the definition of a subsequence. Then we have that $1 \leqslant n_k < k$. If we choose $k = 1$, we see that $1 \leqslant n_k < 1$, which is impossible. So $n_k \geqslant k$. Then for all $k > N$, we see that $n_k \geqslant n$ so $|x_{n_k} - x| < \varepsilon$, and so $(x_{n_k})_{k=1}^\infty \to x$. $\square$

Part 1: We shall construct a subsequence of $(x_{2n})_{n=1}^{\infty}$, that is, $(x_{6n})_{n=1}^{\infty}$. Because this sequence is a subsequence of $(x_{2n})_{n=1}^{\infty}$, its limit must also go to α_1. But we notice that each $6n$ is divisible by 3, and so we see that $(x_{6n})_{n=1}^{\infty}$ is also a subsequence of $(x_{3n})_{n=1}^{\infty}$. As all subsequences must go to the same limit as the main sequence by part 2 (if the main sequence converges), we see that $(x_{6n})_{n=1}^{\infty} \to \alpha_2$. Because the limit is unique, we see that $\alpha_1 = \alpha_2$. Similarly, we define $(x_{3^n})_{n=1}^{\infty}$. Clearly this is a subsequence of $(x_{3n})_{n=1}^{\infty}$, as each index is certainly divisible by 3, and it is also a subsequence of $(x_{2n+1})_{n=1}^{\infty}$ because each 3^n is odd. By the same reasoning, we see that $\alpha_2 = \alpha_3$, so we conclude that $\alpha_1 = \alpha_2 = \alpha_3$. $\square$

4. The base case is clear: $\sqrt{2} < 2$. (I.H.) Suppose that $x_n < 2$ for some $n \geqslant 1$. Then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{2+2} = 2$. $\square$

   So we know that $x_n < 2$ for all $n \geqslant 1$. For the other case (base), we know that $x_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1$. (I.H.) Suppose that $x_n < x_{n+1}$ for some $n > 1$. Then $x_{n+2} = \sqrt{2 + x_{n+1}} > \sqrt{2 + x_n} = x_{n+1}$. $\square$

   Now we take the original equation, apply the limit to both sides, and see that both x_n and $x_{n+1} \to x$ for some limiting value x (all subsequences go to the same value as the original sequence). We also know that, because $f(x) = \sqrt{2+x}$ is continuous, if $x_n \to x$, then $\sqrt{2 + x_n} \to \sqrt{2+x}$. Thus $x = \sqrt{2+x}$, and we see that $(x-2)(x+1) = 0$. Clearly $x \neq -1$, because $x_1 = \sqrt{2} > 0$, and our sequence is strictly increasing. So $\lim\limits_{n\to\infty} x_n = 2$. $\square$