Math HW5

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1. Consider $b_n = \frac{a_n}{n} - a_1$. Now,

$$b_{n+1} \leq \frac{a_n + a_1 - (n+1)a_1}{n+1} = \frac{a_n - na_1}{n+1} \leq b_n$$

Also, since a_n was assumed to be non-negative, $b_n \geq -a_1$, so it is bounded below. Now the monotone convergence theorem implies that b_n converges so a_n does too.

Further, if a_n is not bounded below, we know that it converges to $-\infty$ by the previous argument.

Now, if $\|\cdot\|$ is a matrix norm, then by submultiplicativity $\|AB\| \leq \|A\|\|B\|$. Thus,

$$\|A^{m+n}\| \leq \|A^m\|\|A^n\|$$

So by the previous theorem applied to $a_n = \log(\|A^n\|)$, we know that a_n/n converges to some a (possibly $-\infty$ as it is monotonically decreasing), and hence $e^a = \lim_{n\to\infty} \|A^n\|^{1/n}$ exists as well.

2. For convienience I will work with $Ax > 0$ instead (consider $x' = -x$). First, if there is some $x \in \mathbb{R}^n$ with $Ax > 0$ and also some $y \geq 0$ with $y \neq 0$ with $A^T y = 0$, then $x^T A^T y = (Ax)^T y = 0$, however this must be strictly positive. Now, by considering $\eta = \min(Ax)_i$, $Ax > 0$ is equivalent to there being some η with $Ax - \eta 1 \geq 0$. Writing $x' = (x, \eta)$, this is the same as $[Ax \ \ -1]x' \geq 0$ with $\langle -e_{n+1}, x' \rangle < 0$ (the last coordiante must be positive).

By Farkas lemma, if this doesn't happen, then there is some $y \geq 0$ so that $\begin{pmatrix} A^T \\ -1^T \end{pmatrix} y = -e_{n+1}$. This implies that $A^T y = 0$ with $\sum y_i = 1$, in particular it is not 0, completing the proof.

3. Consider the probability simplex $\Delta = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0} : x_1 + x_2 + x_3 = 1\right\}$. Give it the natural triangulation, and consider a function $f : \Delta \to \Delta$ that is continuous. Since Δ is compact, f is uniformly continuous. Now suppose that f does not have a fixed point–by

compactness, there is some $c > 0$ so that $|f(x) - x| \geq c$ for all $x \in \Delta$. Now, by uniform continuity, there is some $\delta > 0$ so that $|f(x) - f(y)| < c/100$ whenever $|x - y| < \delta$. Now label each vertex of the triangulation via $c(v) = \inf\{i : (f(v) - v)_i < 0\}$. This always exists, as $f(x)$ goes to Δ, we know that its entries sum to 1, and so does $x \in \Delta$, so $f(x) - x$ has entries summing to 0, and by hypothesis this is not the zero vector.

By Sperner's triangle theorem, there is a 1-2-3 triangle. We assume the grid has been chosen so small so that the diameter of this triangle is $< \delta$. Call the vertices of this triangle x_1, x_2, x_3 with corresponding $v_i = f(x_i)$. After possibly permuting the v_i, we know these are of the form:

$$v_1 = \begin{pmatrix} - \\ \\ \end{pmatrix}, \; v_2 = \begin{pmatrix} + \\ - \\ \end{pmatrix}. \; v_3 = \begin{pmatrix} + \\ + \\ - \end{pmatrix}$$

We shall now argue that since these are all within $c/100$ of each other, they cannot possibly exhibit this behavior, given that $|v_i| \geq c$. Since $|v_i| \geq c$, there must be a coordinate i so that $|v_i| \geq c/3$ (otherwise the norm is $< c/3 < c$). We now do a grand case distinction. I will say a coordinate is big if it is $> c/3$ in absolute value. If v_{11} is big, since v_{21} has opposite sign, $|v_1 - v_2| > c/3$, a contradiction as it should be $< c/100$. Similar logic shows that if v_{22} is big, then $|v_2 - v_3| > c/3$. The same logic works if any of the plusses are big ($v_{21} \to v_{21}$, $v_{31} \to v_{11}$, and $v_{32} \to v_{22}$). Now I will show that v_2 cannot have both top coordinates small. If v_{21}, v_{22} are both smaller than $c/3$ in absolute value, we know that v_{23} is bigger than $c/3$ in absolute value. But, $|v_{23} + v_{22} + v_{21}| \geq |v_{23}| - |v_{22} + v_{21}| > 0$, since $v_{22} - v_{21}$ is smaller than $c/3$ in absolute value since they point in different directions, and v_{23} is bigger than $c/3$. So one of the previous cases applies, and we are done.

4. Let $z_1, \ldots, z_n$ be arbitrary points on the the circle. Consider the vandermonde matrix:

$$V^T = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{n-1} \end{pmatrix}$$

We discussed at great length that $\det(V) = \prod_{i<j}(z_j - z_i)$. By Hadamard's determinant inequality,

as this matrix is $n \times n$ with all entries bounded by 1 in norm,

$$|\det(V)| = \prod_{i<j} |z_j - z_i| \leq n^{n/2}$$

Now,

$$\lim_{n\to\infty} n^{n/2 \cdot \frac{1}{\binom{n}{2}}} = \lim_{n\to\infty} n^{1/(n-1)} = 1$$

This shows that it goes to 1.

I present a second proof that gets $\leq \sqrt{3}$. Recall the circumradius formula for triangles, once again, that $4RA = abc$. In our case, since all the points are on a circle, $R = 1$, so $4A = abc$. Now, by doing the same angle trick, and using Jensens inequality we get that $A \leq \frac{3\sqrt{3}}{4}$ (equilateral is the maximizer). So $abc \leq 3^{3/2}$.

We now instead consider the product

$$\prod_{i<j<k} |z_i - z_j| \cdot |z_i - z_k| \cdot |z_j - z_k| \leq 3^{\frac{3}{2} \cdot \binom{n}{3}}$$

By what we just proved, since z_i, z_j, z_k make a triangle. Now we wonder how many times each term $|z_i - z_j|$ appears in this product. Once we have fixed vertices i, j, we just have to pick the last vertex k, which can be done in $n - 2$ ways, so we have the equality:

$$\prod_{i<j<k} |z_i - z_j| \cdot |z_i - z_k| \cdot |z_j - z_k| = \prod_{i<j} |z_i - z_j|^{n-2}$$

Taking the $n - 2$th root on both sides, we get:

$$\prod_{i<j} |z_i - z_j| \leq \sqrt{3}^{\binom{n}{2}}$$

However it alludes me on how to get $\sqrt{2}$, this approach fails immediately when you try a square since there are non-cyclic quadrilaterals on 4 vertices. The vandermonde determinant approach is honestly shocking in that regard.