CSE 421 Last Homework

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Problem 1.

The LP in standard form is the following:

$$\begin{aligned}\min \quad & 3x_1 - x_2 \\ \text{s.t.,} \quad & x_1 + x_2 + z_1 - z_2 \leq 1 \\ & -x_1 - x_2 - z_1 + z_2 \leq -1 \\ & -2x_1 + x_2 + z_1 - z_2 \leq 2 \\ & x_1, x_2, z_1, z_2 \geq 0\end{aligned}$$

Problem 2.

The standard form of the LP is just:

$$\begin{aligned}\max & -\sum_{i=1}^{m} c_i x_i \\ \text{s.t.,} \quad & -\sum_{i:j\in S_i} x_i \leq -1 \quad \forall j \in \{1, \dots, n\} \\ & x_i \geq 0 \quad \forall i \in \{1, \dots, m\}\end{aligned}$$

We see that the LP in question is a relaxation of the weighted set cover problem because given a minimum cost family $F \subset \{S_1, \dots, S_m\}$, let $x_i = 1$ if $S_i \in F$ and $x_i = 0$ otherwise. The condition that every element in $\{1, \dots, n\}$ is in at least one set is thus equivalent to the following:

$$\sum_{i:j\in S_i} x_i \geq 1$$

Since this sum counts the number of sets in our family with j in it for each j. Clearly the $x_i \geq 0$, so the answer to the weighted set cover problem is a feasible solution to the LP and hence the LP is a relaxation of the weighted set cover problem. If you let y_j be the variable for the constraint $\sum_{i:j-\in S_i} x_i \leq -1$, then we need the coefficient of x_i to be $\geq -c_i$. Notice that x_i shows up in the above sum iff $j \in S_i$ for each j. Thus the coefficient of x_i in terms of the y_i is just $-\sum_{j\in S_i} x_i$. This gives the following dual LP:

$$\begin{aligned} \min \quad & -\sum_{j=1}^{n} y_j \\ \text{s.t.,} \quad & -\sum_{j\in S_i} y_j \geq -c_i \quad \forall i \in \{1,\ldots,m\} \\ & y_j \geq 0 \quad \forall j \in \{1,\ldots,n\} \end{aligned}$$

Putting this in standard forms yields:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} y_j \\ \text{s.t.,} \quad & \sum_{j\in S_i} y_j \leq c_i \quad \forall i \in \{1,\ldots,m\} \\ & y_j \geq 0 \quad \forall j \in \{1,\ldots,n\} \end{aligned}$$

Problem 3.

Define a free variable x_e for each directed edge $e \in E$. This will represent the flow passing along edge e. By the flow constraint that the incoming and outgoing flow have to be equal for each vertex, we can see that $f(v) = \sum_{e \text{ into } v} x_e$ and clearly $f(e) = x_e$. From these observations, we can see that we want the following constraints:

$$\begin{aligned} x_e \leq c_e \quad & \forall e \in E \\ \sum_{e \text{ into } v} x_e \leq c_v \quad & \forall v \in V - \{s,t\} \end{aligned}$$

With the above observations about the flow values, we can see that the old payment value:

$$\sum_v f(v)p_v + \sum_e f(e)p_e = \sum_v \sum_{e \text{ into } v} x_e p_v + \sum_e x_e p_e$$

Since each edge $e = (u, v)$ is going into precisely one vertex v, we can rewrite the first sum on the right hand side as:

$$\sum_{e=(u,v)} p_v x_e$$

We also want the constraints that the incoming flow to a vertex is the same as the outcoming flow, and finally that the flow leaving S is equal to D, by the demand constraint. Since we want to minimize the total payment, the LP becomes the following:

$$\begin{aligned}
\min \quad & \sum_{e=(u,v)} p_v x_e \\
\text{s.t.,} \quad & x_e \leq c_e \quad \forall e \in E \\
& \sum_{e \text{ into } v} x_e \leq c_v \quad \forall v \in V - \{s, t\} \\
& \sum_{e \text{ into } v} x_e = \sum_{e \text{ out of } v} x_e \quad \forall v \in V - \{s, t\} \\
& \sum_{e \text{ out of } s} x_e = D \\
& x_e \geq 0 \quad \forall e \in E
\end{aligned}$$

Problem 4.

Construct an undirected graph G' so that for each vertex $v \in G$, we add 3 new vertices to G': v_s, v, v_e with two new edges $v_s \to v$ and $v \to v_e$. For each edge $u \to v$ in the original directed graph G, add an edge $u_e \to v_s$ in G. We claim that G has a hamiltonian path from a to b iff G' has a hamiltonian path from a_s to b_e. If G has a hamiltonian path from a to b, say $P = a = v_1, \ldots, a_n = b$, then we can construct a hamiltonian path for G' by setting $P' = a_s, a, a_e, \ldots, b_s, b, b_e$. Since the edge from $v_{i,e} \to v_{i+1,s}$ always exists, and this path runs through all vertices of G' by looking at our construction, this gives a Hamiltonian path for G'. Similarly, suppose that P is a Hamiltonian path for G' starting at a_s and ending at b_e. We seek to show that if v_s is in this path for some s, the next two elements of the path must be v and v_e. Suppose otherwise. The vertex v of G' has precisely two edges: $v_s \to v$ and $v_e \to v$. Since this is a Hamiltonian path, and v did not come right after v_s, we must have used the edge $v_e \to v$. But then there would be no way to escape v: v must be the last element in the path. But this is clearly nonsensical, a contradiction. Thus after every v_s, the next two elements of the Hamiltonian path are v, v_e. The only edges out of v_e are those of the form $v_e \to u_s$ for some u. By stringing these together, we see that every path has the aforementioned form, that being,

$$a_s, a, a_e, v_{1,s}, v_1, v_{1,e}, \ldots, b_s, b, b_e$$

This shows that we can take the path $P = a, v_1, \ldots, b$ in G to get a Hamiltonian path of the original graph. The function transforming G to G' is clearly polynomial in the input size: we simply add 3 vertices for each vertex of G and add 2 edges for vertex of G. Thus we see that

$$\text{Directed Hamiltonian Path} \leq_p \text{Undirected Hamiltonian Path}$$