CSE Template

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Problem 1.

Let $e = \{v_1, \ldots, v_r\}$ be any hyperedge and color the vertices of the graph one of the 4 colors uniformly and independently. Then we shall look at $\mathbb{P}(\leq 3 \text{ colors in } e)$. $P(\text{exactly 1 color}) = 4/4^r$. $P(\text{exactly 2 colors}) = \binom{4}{2} \cdot (2^r - 2)/4^r$. This is because we need to pick a subset who is non-empty and whose complement is non-empty to get this splitting. Finally, $P(\text{exactly 3 colors}) = \binom{4}{3} \cdot (3^r - 3 - \binom{3}{2}(2^r - 2))/4^r$. This is because we first need to pick the 3 colors. Label them A,B,C in some order. Then the way to just put the r colors in those 3 boxes is 3^r. We then need to subtract the number of colors using exactly 2 of the 3 and using exactly 1 of the 3. This is $\binom{3}{2}(2^r - 2) + 3$ cases. So we get:

$$\begin{aligned} P(\leq 3 \text{ colors in } e) &= P(\text{exactly 1 color}) + P(\text{exactly 2 colors}) + P(\text{exactly 3 colors}) \\ &= \frac{4}{4^r} + \binom{4}{2} \cdot \frac{(2^r - 2)}{4^r} + \binom{4}{3} \cdot \frac{(3^r - 3 - \binom{3}{2}(2^r - 2))}{4^r} \\ &= \frac{1}{4^r}\left(4 - 6 \cdot 2^r + 4 \cdot 3^r\right) \end{aligned}$$

Since there are $4^{r-1}/3^r$ edges, a union bound tells us that:

$$\mathbb{P}(\exists e \text{ with} \leq 3 \text{ colors}) \leq \frac{4^{r-1}}{3^r} \cdot \mathbb{P}(\leq 3 \text{ colors in } e) = 1 - \frac{2^{r-1}}{3^{r-1}} + \frac{1}{3^r}$$

It is then easy to see that this quantity is < 1 for $r \geq 1$, so in particular, $P(\text{every edge has all 4 colors}) > 0$. So there exists a coloring where every edge has all 4 colors represented.

Problem 2.

Let $C = \bigcup_i S(v)$ be the set of all the colors. Since G is bipartite, we can write $G = (V_1, V_2, E)$ where V_1 and V_2 are the two parts of the bipartite graph. For each color $c \in C$, assign it to a side 1 or 2 uniformly and independently. Let C_1 be the colors with label 1 and C_2 similarly defined. Fixing $v \in V_i$, we show that:

$$\mathbb{P}(S(v) \cap C_i = \emptyset) = 2^{-\log_2(n+1)} = \frac{1}{n+1}$$

Since there are n vertices,

$$\mathbb{P}(\text{some vertex } v \text{ has no colors available}) \leq \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

So $\mathbb{P}(\text{all vertices } v \text{ have colors available}) \geq \frac{1}{n+1}$. If we run $k = (n+1)\log n$ independent trials of this coloring process, the probability that all of them fail is:

$$\mathbb{P}(\text{all trials fail}) \leq \left(1 - \frac{1}{n+1}\right)^{(n+1)\log n} \leq \exp(-(n+1)\log n/(n+1)) = \frac{1}{n}$$

Since given a color assignment, we can check if the algorithm succeeds in $O(n)$ time, by running $O(n \log n)$ independent trials we have a polynomial time algorithm to list color G. This completes the algorithm.