425 HW5

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1. Given any function $h(x) : [a,b] \to \mathbb{R}$, note that $|h(x)| \geq 0$. Therefore, $\sup_{x\in[a,b]} |h| \geq 0$. Therefore, $\|f - g\|_\infty \geq 0$ for any functions f, g. We conclude that $\|f' - g'\|_\infty \geq 0$ for any $f, g \in C^1([a,b], \mathbb{R})$, and therefore the sum of these two is positive too. if $d(f, f) = 0$, then $\|f - g\|_\infty + \|f' - g'\|_\infty = 0$, and since this is a sum of two positive things, it follows that $\|f - g\|_\infty = 0$. This is equivalent to saying that $f = g$. Finally, note that for any functions f, g, h, we have that $|f(x) - g(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$ by the regular triangle inequality. We therefore see that $\sup_{x\in[a,b]} |f(x) - g(x)| \leq \sup_{x\in[a,b]} |f(x) - g(x)| + |g(x) - h(x)| \leq \sup_{x\in[a,b]} |f(x) - g(x)| + \sup_{x\in[a,b]} |g(x) - h(x)|$, i.e. that $\|f - g\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty$. Since this holds for any functions $f, g, h \in C^1([a,b], \mathbb{R})$, it also holds for their derivatives (if they exist, which they do). We conclude that $\|f' - g'\|_\infty \leq \|f' - h'\|_\infty + \|h' - g'\|_\infty$. Adding these gives $\|f - g\|_\infty + \|f' - g'\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty + \|f' - h'\|_\infty + \|h' - g'\|_\infty$, i.e. that $d(f, g) \leq d(f, h) + d(g, h)$. Also, $\sup_{x\in[a,b]} |f - g| = \sup_{x\in[a,b]} |g - f|$, so it follows that $d(f, g) = d(g, f)$.

2. Let $\varepsilon > 0$. We can find a $\delta > 0$ so that for all x, y with $d(x, y) < \delta$, we have that $d(f(x), f(y)) < \varepsilon$. Since X is totally bounded, there exists a finite set $\{x_1, \ldots, x_n\} \subset X$ so that
$$X \subset \bigcup_{j=1}^{n} N_\delta(x_j)$$
Let $y \in f(X)$ be arbitrary. Then $y = f(x)$ for some $x \in X$. Since $x \in \bigcup_{j=1}^{n} N_\delta(x_j)$, $x \in N_\delta(x_j)$ for some $j \in \{1, \ldots, n\}$. Therefore, $d(x, x_j) < \delta$. It follows then that $d(f(x), f(x_j)) < \varepsilon$, i.e. that $x \in N_\varepsilon(f(x_j))$. It follows that $f(X) \subset \bigcup_{j=1}^{n} N_\varepsilon(f(x_j))$.

3. Since each E_n is nonempty, we can find an $x_n \in E_n$. For any $\varepsilon > 0$, since $\mathrm{diam}(E_n) \to 0$, we can find an $N > 0$ so that for all $n \geq N$, one has that $\mathrm{diam}(E_n) < \varepsilon$. Since $x_n \in E_n \subset E_N$ for all $n \geq N$, it follows that for any $m, n > N$, $d(x_m, x_n) < \varepsilon$, so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $x_n \to x$ for some $x \in X$. Since x_n is convergent, all subsequences converge. Then for any $l \in \mathbb{N}$, $\{x_{l+n}\}_{n=1}^{\infty} \to x$. Also, $\{x_{l+n}\}_{n=1}^{\infty} \subset E_l$ since $x_{l+n} \in E_{l+n} \subset E_l$ for each $n \in \mathbb{N}$. If $x = x_{l+n}$ for some $n \geq 1$, $x \in E_l$. Else, $x \neq x_{l+n}$ for every $n \geq 1$, and then for every $r > 0$, we can find an $x \neq x_{l+n} \in E_{l+n} \subset E_l$ so that $d(x_{l+n}, x) < r$ by the definition of convergence. Then x is a limit point of E_l, and therefore $x \in E_l$ since E_l is closed. Since l was arbitrary, it follows that $x \in \bigcap_{n=1}^{\infty} E_n$. Suppose there was a $x \neq y \in \bigcap_{n=1}^{\infty} E_n$. We can find N sufficiently large so that for all $n \geq N$, $\mathrm{diam}(E_n) < d(x, y)/2$. But then $x, y \in E_N$, so $d(x, y)/2 > \mathrm{diam}(E_N) \geq d(x, y)$, a contradiction.

4. We see that, for any $m, n \geq 1$,

$$\begin{aligned} & d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \\ \iff & d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n) \end{aligned}$$

and

$$\begin{aligned} & d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m) \\ \iff & d(p_m, q_m) - d(p_n, q_n) \leq d(p_m, p_n) + d(q_n, q_m) \end{aligned}$$

Which combined tells us that $|d(p_n, q_n) - d(p_m, q_m)| < d(p_n, p_m) + d(q_m, q_n)$. For any $\varepsilon > 0$, since both $\{p_n\}_{n=1}^{\infty}$, and $\{q_n\}_{n=1}^{\infty}$, we can find N_1, N_2 so that for any $m, n \geq N_1$, and any $r, s \geq N_2$, $d(p_n, p_m) < \varepsilon/2$, and $d(q_r, q_s) < \varepsilon/2$. Taking $N = \max\{N_1, N_1\}$ gives us that for any $m, n \geq N$, we have that both $d(p_n, p_m) < \varepsilon/2$, and $d(q_n, q_m) < \varepsilon/2$. Using the inequality from above, we conclude that $|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$, which establishes that $\{d(p_n, q_n)\}_{n=1}^{\infty}$ is Cauchy. Since $\mathbb{R}$ is complete, it follows that $\{d(p_n, q_n)\}_{n=1}^{\infty}$ converges.

5. We know from elementary set theory that $f(E \cup E') = f(E) \cup f(E')$. Clearly, $f(E) \subset \overline{f(E)}$. We are left to show that $f(E') \subset \overline{f(E)}$. Given any $y \in f(E')$, there is some $x \in E'$ so that $f(x) = y$. Since $x \in E'$, there is a sequence of points $\{x_n\}_{n=1}^{\infty} \to x$. Since f is continuous, $f(x_n) \to f(x)$. If at any point $f(x_n) = f(x)$, $f(x) \in f(E)$. If $f(x_n) \neq f(x)$ for all $n \in \mathbb{N}$, for any $r > 0$ we can find $N > 0$ so that for all $n \geq N$, $d(f(x_n), f(x)) < r$. Since $f(x_N) \neq f(x)$, $f(x_N) \in N_r(f(x)) \setminus f(x) \cap f(E)$, so $f(x) \in f(E)'$, completing the proof. A counterexample would be choosing $X = (0, 1)$ (as a metric space), and $Y = \mathbb{R}$, and letting $f : X \to Y$ be defined as $f(x) = x$. Since $\overline{X} \subset X$, $X = \overline{X}$, and so $f(\overline{X}) = f(X) = (0, 1)$. However, $\overline{f(X)} = \overline{(0, 1)} = [0, 1]$, so the inclusion can be strict.

6. First we show the forward direction, so suppose $f : X \to Y$ is uniformly continuous. Then for any $\varepsilon > 0$, there is some $\delta > 0$ so that for all $x, y \in X$, $d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon/2$. Now let E be an arbitrary set with $\operatorname{diam}(E) < \delta$. Taking any $x, y \in E$, since $d(x, y) < \operatorname{diam}(E) < \delta$, we see that $d(f(x), f(y)) < \varepsilon/2$. Since this is true for every $x, y \in E$, it follows that $\sup_{x,y \in E} d(f(x), f(y)) \leq \varepsilon/2 < \varepsilon$, as claimed. For the reverse direction, suppose to every $\varepsilon > 0$ there exists $\delta > 0$ so that if $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$. Let $x, y \in X$ be arbitrary, with $d(x, y) < \delta/2$. Clearly $\operatorname{diam}(N_{\delta/2})(x) = \delta/2 < \delta$, and note that $y \in N_{\delta/2}(x)$. Since $f(x), f(y) \in f(E)$, and $\operatorname{diam}(f(E)) < \varepsilon$, it follows that $d(f(x), f(y)) < \varepsilon$, which completes the proof.