CSE 521 HW6 (Midterm)

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1. (a) Consider the following greedy algorithm:

Algorithm 1 1/4-net

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N ← { (1, 0, ..., 0) }
while there is some x ∈ B̄(0, 1) with d(x, N) > 1/4 do
    N ← N ∪ { x }
end while
return N
```

We can see that at the end, every point in the ball is within $1/4$ of N. We also see that every point in N is at least $1/4$ away from all other points. This means that the balls centered at each $x \in N$ with radius $1/8$ are disjoint, and the union of all these balls is contained in $\overline{B}(1+1/8, 0)$, which has volume $c_n \cdot (1+1/8)^n = c_n \cdot (9/8)^n = 9^n \cdot c_n \cdot 1/8^n$. Each ball has volume $c_n \cdot (1/8)^n$, so we must have no more than $9^n = 2^{\log_2(9)n} = 2^{O(n)}$ balls.

(b) Let $x \in \overline{B}(0, 1)$ be the maximizer of $|x^T Mx|$ and decompose $x = y + z$ where $y \in N$ and $|z| < 1/4$. Then

$$\begin{aligned}\sigma_1 = |x^T Mx| &= |(y+z)^T M(y+z)| = |(y+z)^T (My + Mz)| \\ &= |y^T My + z^T Mz + z^T My + y^T Mz| \\ &\le |y^T My| + |z^T Mz| + 2|z^T My| \\ &\le |y^T My| + \frac{1}{4}\sigma_1 + 2 \cdot \frac{1}{4}\sigma_1 = |y^T My| + \frac{3}{4}\sigma_1\end{aligned}$$

Hence $\sigma_1 \le 4|y^T My|$, thus $\sigma_1 \le 4 \max_{y \in N} |y^T My|$.

(c) Notice that

$$\mathbb{E}\Big[\sum r_i a_i\Big] = \sum a_i \mathbb{E}[r_i] = 0$$

Since $\mathbb{E}[r_i] = 0$. Therefore, by Hoeffding's inequality,

$$\Pr\Big[|\sum r_i a_i - 0| \ge t\Big] \le 2\exp\left(\frac{-2t^2}{\sum(2a_i)^2}\right) = 2\exp\left(-\frac{t^2}{2\sum a_i^2}\right)$$

Since $-1 \le r_i \le 1$ we have $-|a_i| \le a_i r_i \le |a_i|$.

(d) Fix $y \in N$. Then,

$$y^T(A - \mathbb{E}[A])y = \sum_{i,j} (A - \mathbb{E}[A])_{ij} y_i y_j$$

Now notice that A_{ij} is a Bernoulli random variable with $p = \frac{1}{2}$ for $i \neq j$. Then $\mathbb{E}[A_{ij}] = \frac{1}{2}$, and $A_{ij} = \mathbb{E}[A_{ij}] = 0$ for $i = j$. Then for $i \neq j$,

$$(A - \mathbb{E}[A])_{ij} = A_{ij} - \mathbb{E}[A_{ij}] = \begin{cases} \frac{1}{2}, \text{ w.p } \frac{1}{2} \\ -\frac{1}{2}, \text{ o.w.} \end{cases}$$

That is, $(A - \mathbb{E}[A])$ is $1/2$ times a Radamacher random variable. Thus we write $A_{ij} = \sigma_{ij}$ for $i \neq j$ and $A_{ij} = 0$ otherwise. Now,

$$y^T(A - \mathbb{E}[A])y = \sum_{i \neq j} \frac{y_i y_j}{2} \sigma_{ij}$$

By part c) we can conclude that

$$\Pr\left[\left|\sum_{i \neq j} \frac{y_i y_j}{2} \sigma_{ij}\right| \geq C\sqrt{n}\right] \leq 2 \exp\left(-\frac{C^2 n}{\frac{2}{4} \sum_{i \neq j} y_i^2 y_j^2}\right)$$

Notice now that,

$$\sum_{i \neq j} y_i^2 y_j^2 \leq \sum_{i,j} y_i^2 y_j^2 = \sum_i y_i^2 \cdot \sum_j y_j^2 = \|y\|^4 \leq 1$$

So,

$$\Pr\left[\left|\sum_{i \neq j} \frac{y_i y_j}{2} \sigma_{ij}\right| \geq C\sqrt{n}\right] \leq 2e^{-2C^2 n}$$

By the union bound we have that

$$\Pr\left[\max_{y \in N} |y^T(A - \mathbb{E}[A])y| \geq C\sqrt{n}\right] \leq 2 \cdot 9^n e^{-C^2 n}$$

Choosing $C = 4\sqrt{\ln(18)}$ yields

$$\Pr\left[4 \max_{y \in N} |y^T(A - \mathbb{E}[A])y| \geq C\sqrt{n}\right] \leq 2^{-n}$$

Now, by part b), since $\sigma_1(A - \mathbb{E}[A]) = \max_{x \in \mathbb{S}^n} |x^T(A - \mathbb{E}[A])x| \leq 4 \max_{y \in N} |y^T(A - \mathbb{E}[A])y|$, we have that

$$\begin{aligned} \Pr\left[\|A - \mathbb{E}[A]\| \leq C\sqrt{n}\right] &\geq \Pr\left[4 \max_{y \in N} |y^T(A - \mathbb{E}[A])y| \geq C\sqrt{n}\right] \\ &\geq 1 - 2^{-n} \end{aligned}$$