Math 461 HW5

Rohan Mukherjee

November 7, 2023

1. (a) I give two proofs for fun. First, let $g(x) = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i)x^n$. We see that $xg(x) = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i)x^{n+1} = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} a_i\right)x^n$. Thus $g(x) - xg(x) = \sum_{n=0}^{\infty} a_n x^n = f(x)$, hence $g(x) = \frac{f(x)}{1-x}$ is the generating function for $f(x)$. Similarly,

$$(a_0 + a_1 x + a_2 x^2 + \ldots)(1 + x + x^2 + \ldots) = a_0 + x(a_0 + a_1) + x^2(a_0 + a_1 + a_2) + \ldots$$

So $f(x)(1 + x + x^2 + \ldots) = f(x)/(1-x)$ is the generating function for $(\sum_{i=0}^{n} a_i)_n$.

(b) I claim that $a_0 + a_2x^2 + a_4x^4 + \ldots = \frac{1}{2}(f(x) + f(-x))$. This is clear because

$$f(x) + f(-x) = \sum_{n=0}^{\infty} a_n x^n + (-1)^n a_n x^n = \sum_{n=0}^{\infty} (a_n + (-1)^n a_n)x^n = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} 2a_n x^n$$

Since if n is odd, then $(-1)^n = -1$, hence $a_n + (-1)^n a_n = a_n - a_n = 0$, and for n even, $a_n + (-1)^n a_n = a_n + a_n = 2a_n$. We also notice that $a_1 + a_3x^3 + \ldots = \frac{1}{2}(f(x) - f(-x))$, but we do not prove this.

(c) First, I claim that the generating function for $(na_n)_n$ is $xf'(x)$. Notice that

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

Thus $xf'(x) = \sum_{n=0}^{\infty} n a_n x^n$. Finally, notice that

$$xf'(x) + f(x) = \sum_{n=0}^{\infty} (na_n + a_n)x^n = \sum_{n=0}^{\infty} (n+1)a_n x^n$$

2. The generating function for a_n, the answer to the question, is just

$$(1 + x)(1 + x + x^2)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)$$

The first factor corresponding to picking 0 or 1 pears, the second 0 or 1 or 2 oranges, the third an even number of apples, and the fourth a multiple of 3 bananas. More formally, to get the coefficient of x^n, we would have to pick an x^a where $a = 0, 1$ from the first factor, an

x^b from the second factor where $b = 0, 1, 2$, an x^c from the third factor where c is even, and an x^d from the fourth factor where d is a multiple of 3 such that $x^a \cdot x^b \cdot x^c \cdot x^d = x^{a+b+c+d}$. If we let a be the number of pears, b the number of oranges, c the number of apples, and d the number of bananas, we see that this is exactly the quantity we are looking for. We conclude that

$$\begin{aligned}(1+x)(1+x+x^2)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots) &= \frac{1+x}{(1+x)(1-x)} \cdot \frac{1+x+x^2}{(1-x)(1+x+x^2)} \\ &= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n\end{aligned}$$

Thus $a_n = n + 1$.

3. We notice that

$$\sum_{n=1}^{\infty} h_n x^n = \sum_{n=1}^{\infty} 3h_{n-1}x^n - 4\sum_{n=1}^{\infty} nx^n = \sum_{n=0}^{\infty} 3h_n x^{n+1} - 4\sum_{n=0}^{\infty} nx^n = 3x\sum_{n=0}^{\infty} h_n x^n - 4\frac{x}{(1-x)^2}$$

Since $\sum_{n=0}^{\infty} nx^n$ is just $x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{x}{(1-x)^2}$. Thus,

$$f(x) - 2 = \sum_{n=0}^{\infty} h_n x^n - h_0 = 3xf(x) - 4\frac{x}{(1-x)^2}$$

We see that

$$f(x)(1-3x) = 2 - \frac{4x}{(1-x)^2}$$

So,

$$f(x) = \frac{2}{1-3x} - \frac{4x}{(1-x)^2(1-3x)}$$

We write

$$\frac{4x}{(1-x)^2(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}$$

Plugging in $x = 1/3$ and covering up that factor yields $C = 4/3/(2/3)^2 = 3$. Plugging in $x = 0$ and $x = -1$ and solving gives $A = -1$ and $B = -2$. Thus,

$$f(x) = \sum_{n=0}^{\infty}(2\cdot 3^n - (-1 - 2(n+1) + 3\cdot 3^n))x^n = \sum_{n=0}^{\infty}(-3^n + 2n + 3)x^n$$

Thus $h_n = 3^n + 2n - 1$.

4. We claim that the generating function for a_n is $(1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)$. This is true because the coefficient of x^n is precisely the number of ways to choose an x^a from the first factor, a x^{2b} from the second factor, and a x^{3c} from the third factor such that $x^n = x^a \cdot x^{2b} \cdot x^{3c} + x^{a+2b+3c}$, so this is equivalent to just finding a solution (a, b, c) to $a + 2b + 3c = n$. If we let the generating function of a_n be $f(x)$, we have

$$f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3}$$

So $f(x)$ is a rational polynomial with denominator $(1-x)(1-x^2)(1-x^3) = 1-x-x^2+x^4+x^5-x^6$. Thus a_n must satisfy the the recurrence

$$a_n - a_{n-1} - a_{n-2} + a_{n-4} + a_{n-5} - a_{n-6} = 0$$

5. Notice that, by HW3 #5,

$$\left(\frac{1}{\sqrt{1-4x}}\right)^2 = \sum_{k=0}^{\infty} \binom{2k}{k} x^k \sum_{j=0}^{\infty} \binom{2j}{j} x^k = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{2k}{k} \cdot \binom{2(n-k)}{n-k} \right) x^k$$

Thus it suffices to find the coefficient of x^n in $1/(1-4x)$. But this is just (obviously) 4^n. Thus,

$$\sum_{k=0}^{n} \binom{2k}{k} \cdot \binom{2(n-k)}{n-k} = 4^n$$