CSE 521 Last Homework

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1. Recall that $x \geq |y|$ iff $x \geq y$ and $x \geq -y$. This motivates the following: $A \in \mathbb{R}^{5\times 7}$ shall be the matrix with $|\varepsilon_{ij}|$ in entry ij, $a \in \mathbb{R}^5$ shall be the aptitude of the people, and $e \in \mathbb{R}^7$ shall be the easiness of the classes. Next, we shall set the entries of G to be 0 if they are undefined. So, we do not want to include the error in those spots. Therefore, define $F_{ij} = \mathbb{I}_{G_{ij}>0}$. Notice that $\text{tr}(AF^T) = \sum_{i,j} |\varepsilon_{ij}| \cdot \mathbb{I}_{G_{ij}>0}$ which is precisely the quantity we want. Finally, our other constraints shall be $G - a\mathbf{1}^T - \mathbf{1}e^T \leq A$ and $-G + a\mathbf{1}^T + \mathbf{1}e^T \leq A$, since the ij coordinate the first is just $G_{ij} - a_i - e_j \leq A_{ij}$ and the second is $-(G_{ij} + a_i + e_j) \leq A_{ij}$, which describes exactly what we want. Note also that since the only entries of A that affect the sum are the ones where G_{ij} is defined, the extra constraints on the other coordinates do not matter.

   So, we have the following linear program:

$$\begin{aligned} \text{minimize} \quad & \text{tr}\left(AF^T\right) \\ \text{s.t.} \quad & G - a\mathbf{1}^T - \mathbf{1}e^T \leq A \\ & -(G - a\mathbf{1}^T - \mathbf{1}e^T) \leq A \end{aligned}$$

   Here is my code:

```
G = [0 2.66 3 3.33 3.66 2 0; 2.66 3.66 0 0 4.33 1.66 3
2.66 0 3.33 0 3.66 3 3.33; 4.33 0 2.66 4 0 3.66 0;
0 2.66 1.33 3.33 0 3 2.33];
Filter = transpose(G > 0);
cvx_begin
  variables T(5,7) a(5) e(1,7)
  minimize trace(T*Filter)
  subject to
    G - a*ones(1, 7) - ones(5, 1)*e <= T;
    -G + a*ones(1, 7) + ones(5, 1)*e <= T;
    T >= 0;
cvx_end
```

   I got the calculated easiness to be: 0.960037046829499

   1.63003704798536

1.03457764835811

2.30003704582580

2.63003704559624

1.17342758396096

1.30003704812862

and the aptitudes of the students to be: 1.02996295373911

1.69996295314997

1.82657241603184

2.17422512162083

1.02996295245312

2. We construct a convex program. let $|E| = m$, and define to edge k, where k connects vertex i with vertex j,

$$a_l^k = \begin{cases} 1, \ l = i \\ -1, \ l = j \\ 0, \ \text{o.w} \end{cases}$$

Finally define $L(w) = \sum_{k=1}^m w_k a^k (a^k)^T = A\text{diag}(w)A^T$, where $A = [a_1 \dots a_m]$. We claim the function $f(w) = \lambda_2(L(w)) = \min_{\langle x, \mathbf{1} \rangle = 0} \frac{x^T L x}{x^T x}$ is concave. Notice that $\text{diag}((w+v)/2) = \frac{1}{2}(\text{diag}(w) + \text{diag}(v))$, so,

$$\begin{aligned} f\left(\frac{w+v}{2}\right) &= \min_{\langle x, \mathbf{1} \rangle = 0} \frac{1}{2} \frac{x^T(\text{diag}(w) + \text{diag}(v))x}{x^T x} \\ &= \frac{1}{2} \min_{\langle x, \mathbf{1} \rangle = 0} \left(\frac{x^T \text{diag}(w) x}{x^T x} + \frac{x^T \text{diag}(v) x}{x^T x} \right) \geq \frac{1}{2} \left(\min_{\langle x, \mathbf{1} \rangle = 0} \frac{x^T \text{diag}(w) x}{x^T x} + \min_{\langle x, \mathbf{1} \rangle = 0} \frac{x^T \text{diag}(v) x}{x^T x} \right) \\ &= \frac{1}{2}(f(v) + f(w)) \end{aligned}$$

We can now formulate the following linear program.

$$\begin{aligned} \text{maximize} \quad & f(w) \\ \text{s.t.} \quad & \text{diag}(L) = \mathbf{1} \end{aligned}$$

Since $\text{diag}(L)$ is a linear function in w as we showed above, the constraint is simply a linear system of equalities, where we are trying to maximize a convex function. The above is a convex program, and hence can be solved using the ellipsoid method in polynomial time. We now prove the correctness of this algorithm. We claim that $I - L$ is the adjacency matrix of the weighted graph. Notice first that the diagonal entry is going to be the sum of the weights of the edges connecting to vertex i, and by our constraint this diagonal entry is always 1. So, $I - L$ has diagonal entries all 0s. The ij entry of this matrix is 0 if there is no edge connecting i to j by construction, and $(-w_l \cdot a^k (a^k)^T)_{ij}$ if there is an edge connecting i to j, and this quantity is just w_l since a^k will be 1 in the ith coordinate and -1 in the jth coordinate. Now, L is symmetric, which proves the previous claim. Finally, the second largest eigenvalue of $I - L$ is the second smallest eigenvalue of L, and it will be minimized when the second smallest eigenvalue of L is maximized, which is what the above program does.