Math 336 HW7

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1. I believe I have found a solution that doesn't use the hint, but instead **Theorem 3.4** in the book. I like it, so I will be using it instead. Since $f(z)$ is holomorphic everywhere, by definition $f(1/z)$ is a meromorphic function in the extended complex plane. By **Theorem 3.4**, $f(1/z)$ is a rational function. Thus,

$$F(z) = f(1/z) = \frac{p(z)}{q(z)}$$

Since $F(z)$ is rational, $F(1/z)$ is also a rational function (we could clear denominators). So, outside of 0,

$$f(z) = \sum_{i=1}^{n} \frac{p_i(z)}{(z-\alpha_i)^{m_i}}$$

where the α_i's are distinct (Note: we used implicitly that rational functions may be expanded into partial fractions) and all p_i's are not equivalently zero. I claim that $(z-\alpha_i)^{m_i} \mid p_i(z)$. We shall show that this is true for $i = 1$ and argue that the general case holds immediately. Note that

$$f(z) = \sum_{i=1}^{n} \frac{p_i(z)}{(z-\alpha_i)^{m_i}} = \frac{p_1(z)}{(z-\alpha_1)^{m_1}} + \sum_{i=2}^{n} \frac{p_i(z)}{(z-\alpha_i)^{m_i}}$$

Since we assumed that the α_i's are distinct, the only place the remaining sum blows up is (potentially) at $\{\alpha_2, \ldots, \alpha_n\}$. Note that since these are all positive distance from α_1, we may bound the remaining sum by some constant $C > 0$. Suppose that $(z-\alpha_1)^{m_1} \nmid p_1(z)$. Then we may cancel all multiples of $(z-\alpha_1)$ from $p_1(z)$, noting that at least one must be remaining to rewrite this as

$$\frac{p_1(z)}{(z-\alpha_1)^{m_1}} = \frac{q(z)}{(z-\alpha_1)^{n}}$$

where $(z-\alpha) \nmid q(z)$ and $1 \leq n \leq m_1$. We now notice that

$$\left| \frac{p_1(z)}{(z-\alpha_1)^{m_1}} + \sum_{i=2}^{n} \frac{p_i(z)}{(z-\alpha_i)^{m_i}} \right| \geq \left| \frac{q(z)}{(z-\alpha_1)^{n}} \right| - C \to \infty$$

It cannot be that f is holomorphic at α_1 since f is unbounded in a neighborhood of α_1, a contradiction. We could've done this argument with any of the factors, so it follows that $(z-\alpha_i)^{m_i} \mid p_i(z)$ for all $1 \leq i \leq n$. But then $f(z)$ is just a polynomial. If $\deg(f) \geq 2$, then f cannot be injective since we have two cases, either $f(z) = (z-\alpha)^m$ where $m \geq 2$, or f has at least two distinct roots. In the ladder case f already isn't injective, and in the first case we could consider $f(z) - 1 = 0$ and see that this has m distinct solutions, so $f(z)$ is again not injective. So, $f(z) = az + b$. We already showed that f cannot be constant, so it follows that $a \neq 0$, completing the proof. **Q.E.D.**

2. Let $u : \Omega \to \mathbb{R}^2$ be the harmonic function in question, and extend u to a holomorphic function $f : \Omega \to \mathbb{C}$ (we talked about this before, it is sufficient for Ω to be simply connected) so that $f(x+iy) = u(x,y) + iv(x,y)$. Let $z \in \Omega$ be arbitrary. Since Ω is open, there is some ball of radius $r > 0$ $B_r(z) \subset \Omega$. f is assumed to be nonconstant, so the open mapping theorem applies, and so by the open mapping theorem $f(B_r(z))$ is open. Thus, there is some $\eta > 0$ so that $B_\eta(f(z)) \subset f(B_r(z))$. This inclusion tells us there is some point $z_1 \in \Omega$ so that $f(z_1) = f(z) + \eta/2$ (which is clearly in $B_\eta(f(z))$), which has real part $u(z) + \eta/2 > u(z)$, so z is not a maximum of u. Since z was arbitrary, u has no maximum on Ω, which completes the proof.

3. We shall write $\gamma(t) = f(re^{it}) = \Re(f(re^{it})) + i \cdot \Im(f(re^{it}))$ for $0 \leq t \leq 2\pi$. We notice that

$$\Re(f(re^{it})) = \frac{1}{2}\left(f(re^{it}) + \overline{f(re^{it})}\right) = \frac{1}{2}\left(\frac{1}{r}e^{-it} + \sum_{n=0}^{\infty} a_n r^n e^{int} + \frac{1}{r}e^{it} + \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int}\right)$$

and similarly,

$$\Im(f(re^{it})) = \frac{1}{2i}\left(\frac{1}{r}e^{-it} + \sum_{n=0}^{\infty} a_n r^n e^{int} - \frac{1}{r}e^{it} - \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int}\right)$$

(We used multiple times that the complex conjugate is linear). We now wish to evaluate

$$\int_\gamma x dy = \int_0^{2\pi} \Re(f(re^{it})) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Im(f(re^{it}))dt$$

We see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Im(f(re^{it})) &= \frac{1}{2i}\left(\frac{-i}{r}e^{-it} + i\sum_{n=0}^{\infty} na_n r^n e^{int} - \frac{i}{r}e^{it} + i\sum_{n=0}^{\infty} n\overline{a_n} r^n e^{-int}\right) \\ &= \frac{1}{2}\left(-\frac{1}{r}e^{-it} + \sum_{n=0}^{\infty} na_n r^n e^{int} - \frac{1}{r}e^{it} + \sum_{n=0}^{\infty} n\overline{a_n} r^n e^{-int}\right)\end{aligned}$$

Hence,

$$\int_\gamma x dy = \frac{1}{4}\int_0^{2\pi}\left(\frac{1}{r}e^{-it} + \sum_{n=0}^{\infty} a_n r^n e^{int} + \frac{1}{r}e^{it} + \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int}\right) \cdot \left(-\frac{1}{r}e^{-it} + \sum_{n=0}^{\infty} na_n r^n e^{int} - \frac{1}{r}e^{it} + \sum_{n=0}^{\infty} n\overline{a_n} r^n e^{-int}\right)dt$$

We notice that

$$\int_0^{2\pi} e^{int}dt = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases}$$

When multiplying out all the $e^{\pm it}$ terms with each other, we apply this and see that we are left with $\frac{-4\pi}{r^2}$, which handles the first 4 terms. Next, notice that

$$\int_0^{2\pi} e^{\pm it} \sum_{n=0}^{\infty} a_n r^n e^{\pm int} dt = 0$$

Since we could interchange sum and integral and use the identity from above, and noting that the sign of the $\pm$ actually matters. This handles the next 4 terms. 8 to go! Next note that

$$\int_0^{2\pi} e^{it} \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int} dt = \int_0^{2\pi} \overline{a_n} r dt = 2\pi \overline{a_n} r$$

Since all terms cancel unless $n = 1$, and similarly that $\int_0^{2\pi} e^{-it} \sum_{n=0}^{\infty} a_n r^n e^{int} dt = 2\pi a_n r$. By literally the exact same reasoning, which I don't want to write out again,

$$\int_0^{2\pi} e^{it} \sum_{n=0}^{\infty} n\overline{a_n} e^{-int} dt = 1 \cdot 2\pi \overline{a_n} r$$
$$\int_0^{2\pi} e^{-it} \sum_{n=0}^{\infty} n a_n e^{int} dt = 2\pi a_n r$$

A careful observations of the signs of the $\pm\frac{1}{r}e^{\pm it}$ shows that summing up these 4 terms cancels. We are left with evaluating the 4 terms generated by multiplying out the sums. We see that

$$\int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{int} \cdot \sum_{n=0}^{\infty} n a_n r^n e^{int} = \int_0^{2\pi} \sum_{m,n=0}^{\infty} a_n m a_m r^{n+m} e^{i(n+m)t} = 0$$

Since this sum always vanishes (from the e's if $n, m \neq 0$, else its from $m = 0$). Similarly,

$$\int_0^{2\pi} \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int} \cdot \sum_{n=0}^{\infty} n\overline{a_n} r^n e^{-int} dt = 0$$

Finally, we are left to evaluate the two good sums. One notices that the remaining 2 terms are equal, so we shall only evaluate one of them and double the final answer.

$$\int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{int} \cdot \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int} dt = \int_0^{2\pi} \sum_{m,n=0}^{\infty} m a_n \overline{a_m} r^{n+m} e^{i(n-m)t} dt = \int_0^{2\pi} \sum_{n=0}^{\infty} n r^{2n} |a_n|^2 dt = 2\pi \sum_{n=0}^{\infty} n r^{2n} |a_n|^2$$

Since, for the ten thousandths time, the only terms that don't vanish in the double sum is when $n = m$. We conclude that

$$\int_\gamma xdy = \frac{1}{4}\left(\frac{-4\pi}{r^2} + 2 \cdot 2\pi \sum_{n=0}^{\infty} nr^{2n}|a_n|^2\right) = \pi \sum_{n=-1}^{\infty} nr^{2n}|a_n|^2$$

(Note: $a_{-1} = 1$). Quite a spectacular result if I do say so myself. Since we actually wanted $-\int_\gamma xdy$, we have derived the identity in the problem. Finally, using that areas are nonnegative yields:

$$0 \le -\pi \sum_{n=-1}^{\infty} nr^{2n}|a_n|^2 \iff \sum_{n=-1}^{\infty} nr^{2n}|a_n|^2 \le 0$$

This says that

$$\sum_{n=0}^{\infty} nr^{2n}|a_n|^2 \le r^{-2}$$

Finally, since the limit preserves inequalities, we conclude that

$$\begin{aligned} &\lim_{r\to1^-} \sum_{n=0}^{\infty} nr^{2n}|a_n|^2 \le \lim_{r\to1^-} r^{-2} \\ \implies &\sum_{n=0}^{\infty} \lim_{r\to1^-} nr^{2n}|a_n|^2 \le 1 \\ \implies &\sum_{n=0}^{\infty} n|a_n|^2 \le 1 \end{aligned}$$

4. On $|z| = 2$, $|z^5 + 8z^3 + 2z + 1| \le 101 < |z|^9 = 512$, by Rouche's Theorem $z^9 + z^5 + 8z^3 + 2z + 1$ has 9 roots in $|z| \le 2$. Also, on $|z| = 1$, $|z^9 + z^5 + 2z + 1| \le 5 < 8 = |8z^3|$, so $z^9 + z^5 + 8z^3 + 2z + 1$ has 3 roots inside $|z| \le 1$. Therefore, $z^9 + z^5 + 8z^3 + 2z + 1$ has 6 roots in $1 \le |z| \le 2$. Given that $m < n$, we notice that on $|z| = 1$,

$$\left|\sum_{i=0}^{m} \frac{z^i}{i!}\right| = \sum_{i=0}^{m} \frac{1}{i!} \le \sum_{i=0}^{\infty} \frac{1}{i!} = e < 3 = |3z^n|$$

It follows that $\sum_{i=0}^{m} \frac{z^i}{i!} + 3z^n$ and $3z^n$ have the same number of solutions in $|z| \le 1$, which is n.

5. Since

$$|zf(z)| \le |z|^\varepsilon / c$$

$zf(z)$ is bounded and hence has a removable singularity in 0. We now have two cases: either $f(z)$ has a removable singularity in 0, in which case we are done, or $f(z)$ has a pole of order 1 in 0. In the ladder case, we would have

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$$

Where $a_{-1} \neq 0$. $\sum_{k=0}^{\infty} a_k z^k$ is a holomorphic function, and hence bounded above by some $C > 0$ in a neighborhood of 0. We see then that

$$\frac{|a_{-1}|}{|z|} - C \leq |f(z)| \leq \frac{c}{|z|^{1-\varepsilon}}$$

We shall now show that there exists a $z \in \mathbb{C}$ sufficiently small so that

$$\frac{c}{|z|^{1-\varepsilon}} < \frac{|a_{-1}|}{|z|} - C$$

We may take $|z|$ sufficiently small so that the RHS is positive. This claim is equivalent to showing there is some z so that

$$c|z|^{\varepsilon} < |a_{-1}| - C|z|$$

Which is equivalent to finding a z so that

$$c|z|^{\varepsilon} + C|z| < |a_{-1}|$$

Which obviously exists since a_{-1} was assumed to be nonzero and the LHS tends to 0. But that's a contradiction, which completes the proof.