Answer to P2

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9. Define the function f by $f(x) = e^{-1/x^2}$ if $x \neq 0$, and $f(0) = 0$.

   (a) Show that $\lim_{x\to 0} f(x)/x^n = 0$ for all $n > 0$.

   $$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{y\to\infty} \frac{y^{n/2}}{e^y} = 0$$

   (b) Show that f is differentiable at 0 and that $f'(0) = 0$.

   $$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \frac{e^{-1/h^2}}{h} = 0 \quad \text{by part (a)}$$

   (c) Show by induction on k that for $x \neq 0$, $f^{(k)}(x) = P(1/x)e^{-1/x^2}$, where P is a polynomial of degree $3k$.

   We see that, for $x \neq 0$, $f'(x) = -\frac{2}{x^3}e^{-1/x^2}$, and that $-2/x^3$ is a polynomial in $1/x$ of degree 3.

   Next, suppose that $f^{(k)}(x) = P(1/x)e^{-1/x^2}$ for some polynomial $P(x)$ of degree $3k$ and all $x \neq 0$. We see that:

   $$f^{(k+1)}(x) = e^{-1/x^2}\left(P(1/x) \cdot \frac{-2}{x^3} + P'(1/x) \cdot \frac{-1}{x^2}\right)$$

   $P'(x)$ is a polynomial of degree $3k-1$ and thus $P'(1/x) \cdot -1/x^2$ is a polynomial in $1/x$ of degree $3k+1$. $P(1/x) \cdot -2/x^3$ is a polynomial of $3k+3$, thus $P(1/x) \cdot -2/x^3 + P'(1/x) \cdot -1/x^2$ is a polynomial of degree $3k+3$, completing the inductive step.

   (d) We recall that $f'(0) = 0$. Suppose that $f^{(k)}(0) = 0$ for some $k \geq 2$. Then

   $$f^{(k+1)}(0) = \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h\to 0} \frac{P(1/h) \cdot e^{-1/h^2}}{h}$$

   for some polynomial $P(x)$. Then $1/hP(1/h)$ is a polynomial in $1/h$ with no constant term, and thus may be written in the form $\sum_{k=1}^N a_k h^{-k}$. Now,

   $$\lim_{h\to 0} \frac{P(1/h) \cdot e^{-1/h^2}}{h} = \lim_{h\to 0} e^{-1/h^2} \cdot \sum_{k=1}^N a_k h^{-k} = \sum_{k=1}^N a_k \lim_{h\to 0} e^{-1/h^2} \cdot h^{-k}$$

   Part (a) shows that the limit inside of the sum is always 0, thus this quantity is 0, which completes the inductive step.