Math 504 HW4

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1. Let G be a group of order 8. If G is abelian, then $G \cong \mathbb{Z}/8$, or $\mathbb{Z}/4 \times \mathbb{Z}/2$, or $(\mathbb{Z}/2)^3$ by the fundamental theorem of finite abelian groups. Else, G has an element of order 4, and none of order 8. This is because a group where each element has order dividing 2 is abelian. Indeed, for any x, y $[x, y] = x^{-1}y^{-1}xy = xyxy = (xy)^2 = e$ since every element has order dividing 2. So take x to be an element of order 4. If $G \setminus \langle x \rangle$ has an element of order 2, say y, then $\langle x \rangle < \langle x, y \rangle \leq G$, so $\langle x, y \rangle = G$ by order considerations. This tells us (also by order considerations), that $\langle y \rangle \cap \langle x \rangle = 1$, so $G \cong \langle x \rangle \rtimes \langle y \rangle$ since $\langle x \rangle$ is normal in G since it has index 2. This yields $\mathbb{Z}/4 \times \mathbb{Z}/2$ and D_4. Otherwise, we can find an element $y \in G \setminus \langle x \rangle$ of order 4, and every element of order 2 is in $\langle x \rangle$. Thus, $y^2 \in \langle x \rangle$, and hence $y^2 = x^2$ since the only element of order 2 in $\langle x \rangle$ is x^2. Notice now that, since $\langle y \rangle \trianglelefteq G$ and since $G/\langle y \rangle \cong \mathbb{Z}/2$, in particular it is abelian, so $xyx\langle y \rangle = x^2y\langle y \rangle = y^2\langle y \rangle = \langle y \rangle$, so $xyx \in \langle y \rangle$. It also has order 4: notice that $(xyx)^2 = xyy^2yx = xy^4x = x^2$, which has order 2. Thus $xyx = y$ or $xyx = y^3$. In the second case, $xyx = x^2y$ meaning $yx = xy$. Then since $G = \langle x, y \rangle$ we have that G is abelian whose case we must've already covered above. Thus $xyx = y$, and $G \cong \left\langle x, y \mid x^4 = e, x^2 = y^2, xyx = y \right\rangle \cong Q_8$.

2. Let G be nilpotent and $\varphi : G \to H$ be a surjective homomorphism. We see that $\varphi(Z_0(G)) = 1 \leq Z_0(H)$, so suppose $\varphi(Z_i(G)) \leq Z_i(H)$ for some $i > 0$. Define $\psi : G/Z_i(G) \to H/Z_i(H)$ by $\psi(xZ_i(G)) = \varphi(x)Z_i(H)$. This map is well-defined since if $xZ_i(G) = yZ_i(G)$, then $xy^{-1} \in Z_i(G)$, so $\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \in Z_i(H)$. We have the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, two heads, "\varphi"] \arrow[d, two heads, bend right, "\pi_1"'] & H \arrow[d, two heads, bend right, "\pi_2"'] \\
G/Z_i(G) \arrow[r, dashed, two heads, "\psi"'] \arrow[u, bend right, "\pi_1^{-1}"'] & H/Z_i(H) \arrow[u, bend right, "\pi_2^{-1}"']
\end{tikzcd}$$

Where $\pi : H \to H/Z_i(H)$ is the natural projection.

Lemma 1. *If $\varphi : G \to H$ is a surjective homomorphism then $\varphi(Z(G)) \leq Z(H)$.*

Proof. Let $g \in Z(G)$ and $h \in H$. Since φ is surjective, find $x \in G$ so that $\varphi(x) = h$. Now, $\varphi(g)h = \varphi(gx) = \varphi(xg) = h\varphi(g)$, so $\varphi(g) \in Z(H)$, which completes the proof. $\square$

Using this lemma with the above commutative diagram shows that $\psi(Z(G/Z_i(G))) \leq Z(H/Z_i(H))$. Thus,

$$\pi_2^{-1} \circ \pi_2 \circ \varphi \circ \pi_1^{-1}(Z(G/Z_i(G)) = \pi_2^{-1} \circ \psi(Z(G/Z_i(G))) \leq \pi_2^{-1}(Z(H/Z_i(H))) = Z_{i+1}(H)$$
$$\implies \varphi(Z_{i+1}(G)) \leq Z_{i+1}(H)$$

In particular, if $Z_n(G) = G$, we have that $Z_n(H) \geq \varphi(Z_n(G)) = H$, which completes the proof. In particular, if G is nilpotent then $G/Z(G)$ is.

Now we shall show that if $G/Z(G)$ is nilpotent then G is. Notice that the ascending central series for $G/Z(G)$ starts $1, Z(G/Z(G)) = Z_2(G)/Z(G)$. By the third isomorphism theorem, we have the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] & G/Z_2(G) \arrow[d, two heads, bend right, "\varphi"'] \\ G/Z(G) \arrow[r, two heads, "\rho"'] & \frac{G/Z(G)}{Z_2(G)/Z(G)} \arrow[u, two heads, bend right, "\varphi^{-1}"'] \end{tikzcd}$$

Recall that isomorphic groups have isomorphic centers. Thus the center of $\frac{G/Z(G)}{Z_2(G)/Z(G)}$ is just $\varphi(Z(G/Z_2(G))) = \varphi(Z_3(G)/Z_2(G))$. By commutativity of the diagram,

$$\varphi(Z_3(G)/Z_2(G)) = \rho \circ \pi_1 \circ \pi_2^{-1}(Z_3(G)/Z_2(G)) = \rho \circ \pi_1(Z_3(G)) = \rho(Z_3(G)/Z(G))$$

Simple induction on i will show that $Z_i(G/Z(G)) = Z_{i+1}(G)/Z(G)$. So, if there is an $n \geq 0$ so that $Z_n(G/Z(G)) = G/Z(G)$, then we'd have $Z_{n+1}(G)/Z(G) = G/Z(G)$, so $Z_{n+1}(G) = G$ which completes the proof.

3. Let α have minimum polynomial of odd degree. Write

$$\sum_{i=0}^{n} c_i \alpha^i = 0$$

Rearrange this sum as,

$$a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} = \alpha(a_1 + \cdots + a_n\alpha^{n-1})$$

If $a_1 + \cdots + a_n\alpha^{n-1} = 0$, then since $a_n \neq 0$, there would be a relation on $1, \alpha, \cdots, \alpha^{n-1}$, a contradiction, since n is minimal. Thus $\alpha = (a_0 + \cdots + a_{n-1}\alpha^{n-1})(a_1 + \cdots + a_n\alpha^{n-1})^{-1}$, and we are done.