Math 521 HW3

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1. We prove that if for every subsequence of x_{n_k} of x_n, if $\liminf_{k\to\infty} x_{n_k} \geq x$ then also $\liminf_{n\to\infty} x_n \geq x$. Otherwise, $\inf_{k\geq m} x_m < x$ infinitely often. Taking an increasing sequence of m making this happen, we get a subsequence with $\liminf_{k\to\infty} x_{n_k} < x$, and clearly no subsequence of this can have a limit inferior $\geq x$.

   Now, because X_n converges in probability, for every subsequence n_k, there is a further subsequence n_{k_j} so that $X_{n_{k_j}}$ converges almost surely. In this case, by Fatou's lemma,

$$\mathbb{E}[X] \leq \liminf_{j\to\infty} \mathbb{E}\left[X_{n_{k_j}}\right]$$

   Since this holds for every subsequence of the numbers $x_n = \mathbb{E}[X_n]$, it must hold for the whole sequence by the lemma we proved above. This shows that $\liminf_{n\to\infty} \mathbb{E}[X_n] \geq \mathbb{E}[X]$.

   Similarly, find a subsequence n_{k_j} so that $X_{n_{k_j}} \to X$ almost surely. Then by the dominated convergence theorem,

$$\mathbb{E}\left[X_{n_{k_j}}\right] \to \mathbb{E}[X]$$

   Since for every subsequence of the numbers $x_n = \mathbb{E}[X_n]$, there is a further subsequence converging to $\mathbb{E}[X]$, this shows that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as well.

2. Divide $[0,1]$ into $m = \varepsilon^{-1}$ parts: $[0,\varepsilon]$, $[\varepsilon, 2\varepsilon]$, $\ldots$, $[1-\varepsilon, 1]$. Since F is continuous, and the imagee of a connected set is connected, there are $-\infty = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = \infty$ so that $F(x_i) = i\varepsilon^{-1}$. Choose a global n so that $|F_n(x_i) - F(x_i)| < \varepsilon$ for each i (we may do this because there are only finitely many is). Then for each x, there is an i so that $x \in [x_i, x_{i+1})$. Then,

$$|F_n(x) - F(x)| \leq |F_n(x) - F_n(x_{i+1})| + |F_n(x_{i+1}) - F(x_{i+1})| + |F(x_{i+1}) - F(x)|$$

First, $|F_n(x_{i+1}) - F_n(x)| = F_n(x_{i+1}) - F_n(x)$. By our choice of n, $F_n(x_{i+1}) \leq F(x_{i+1}) + \varepsilon$, and $F_n(x) \geq F_n(x_i) \geq F(x_i) - \varepsilon$. Thus, $|F_n(x_{i+1}) - F_n(x)| \leq 2\varepsilon$. Similarly, $|F(x_{i+1}) - F(x)| \leq \varepsilon$, and the middle term is precisely equal to ε. Thus,

$$|F_n(x) - F(x)| \leq 4\varepsilon$$

Which completes the proof.

3. Let $C > 0$ be arbitrary. Then,

$$\sum_{n=2}^{\infty} \mathbb{P}(X_n \geq Cn\log n) \geq \int_3^{\infty} \frac{1}{Cx\log x} dx = \int_{\log 3}^{\infty} \frac{1}{Cx} dx = \infty$$

Thus $\mathbb{P}(X_n \geq Cn\log n \text{ i.o.}) = 1$. Suppose that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{n\log n} < \infty\right) > 0.$$

Then since,

$$\left\{\limsup_{n\to\infty} \frac{X_n}{n\log n} < k\right\} \uparrow \left\{\limsup_{n\to\infty} \frac{X_n}{n\log n} < \infty\right\}$$

we have that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{n\log n} < k\right) > 0 \text{ for some } k.$$

But this is a contradiction: since $\mathbb{P}(X_n \geq kn\log n \text{ i.o.}) = 1$, $\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{n\log n} \geq k\right) = 1$. Thus $\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{n\log n} < \infty\right) = 0$. In particular, since $X_n \geq 1$ a.s., we have that $\limsup_{n\to\infty} \frac{X_n}{n\log n} \leq \limsup_{n\to\infty} \frac{S_n}{n\log n}$ a.s. and hence the latter limit is ∞ a.s. as well.

4. (a) Let $F(x) = 1 - x^{-\alpha}$. Then,

$$\mathbb{P}\big(M_n/n^{1/\alpha} \le y\big) = \mathbb{P}\big(M_n \le n^{1/\alpha}y\big) = F^n(n^{1/\alpha}y) = \left(1 - \frac{y^{-\alpha}}{n}\right)^n \quad \text{for } n^{1/\alpha}y \ge 1$$

Using the fact that for any $a, b \in \mathbb{R}$, $\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$, and that for large enough n, $n^{1/\alpha}y \ge 1$ since $y > 0$, we see that this tends to $e^{-y^{-\alpha}}$.

(b) This one follow similarly:

$$\mathbb{P}\big(M_n \le yn^{-1/\beta}\big) = \big(1 - |yn^{-1/\beta}|^\beta\big)^n = \left(1 - \frac{|y|^\beta}{n}\right)^n \to e^{-|y|^\beta}$$

(c) Again,

$$\mathbb{P}(M_n \le y + \log n) = \big(1 - e^{-y-\log n}\big)^n = \left(1 - \frac{1}{n}e^{-y}\right)^n \to e^{-e^{-y}}$$

5. Let $g \ge 0$ be continous and $X_n \implies X$. Let F_n be the distribution function of F_n and F be the distribution function of F. Then we can find a probability space $(\Omega, \mathcal{F}, \mathbb{P}())$ and random variables Y_n, Y so that Y_n has distribution function F_n, Y has distribution function Y, and $Y_n \to Y$ almost surely. Then since g is continuous $g(Y_n) \to g(Y)$ almost surely. Since $g \ge 0$ we have that $g(Y_n) \ge 0$ and hence Fatou's lemma applies. This gives us:

$$\liminf_{n\to\infty} \mathbb{E}[g(Y_n)] \ge \mathbb{E}[g(Y)]$$

Notice that:

$$\mathbb{E}[g(Y_n)] = \int_{\mathbb{R}} g(x)\mu_{Y_n}(dx) = \mathbb{E}[g(X_n)]$$

and,

$$\mathbb{E}[g(Y)] = \int_{\mathbb{R}} g(x)\mu_Y(dx) = \mathbb{E}[g(X)]$$

This shows that $\liminf_{n\to\infty} \mathbb{E}[g(X_n)] \ge \mathbb{E}[g(X)]$ as desired.

Let $g(x) = x$, $X_n = n\mathbb{1}_{(0,1/n)}$, and $X = 0$. Then clearly $X_n \to X$ almost surely, and hence in distribution. But,

$$\mathbb{E}[X_n] = 1 \quad \forall n$$

And hence $\liminf_{n\to\infty} \mathbb{E}[X_n] = 1 \geq \mathbb{E}[X] = 0$.

6. Let $\sigma^2 = \mathbb{E}\big[X_i^2\big]$. If $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ were to converge in probability, to say Y, then we know that Y is $N(0, \sigma^2)$ distributed. Consider $\sqrt{2}S_{2n} - S_n = \frac{1}{\sqrt{n}} \sum_{i=n+1}^{2n} X_i$. This is a sum of n i.i.d. random variables divided by $\sqrt{n}$ with mean 0p, so again by the central limit theorem it converges in distribution to $N(0, \sigma^2)$. But also, it converges in probability to $(\sqrt{2}-1)Y$. This then means that $\mathrm{Var}(\sqrt{2}Y - Y) = (\sqrt{2}-1)^2\sigma^2 = \sigma^2$, so $\sigma^2 = 0$ a contradiction.

7. First we prove the "coming together theorem". Let $X_n \to X$ in distribution. Then if $Y_n \to c$ in probability where c is a constant, then $X_n + Y_n \to X + c$ in distribution.

   Notice first that:

$$\mathbb{P}(X_n + Y_n \leq x) = \mathbb{P}(X_n + Y_n \leq x, |Y_n - c| \leq \delta) + \mathbb{P}(X_n + Y_n \leq x, |Y_n - c| > \delta)$$

   The second term goes to 0 as $\delta \to 0$, so we don't have to worry about it. We need only show that the first term goes to $\mathbb{P}(X + c \leq x)$. Notice that if $c - \delta \leq Y_n \leq c + \delta$ and $X_n + Y_n \leq x$, then $X_n + c - \delta \leq X_n + Y_n \leq x$, so $\mathbb{P}(X_n + Y_n \leq x, |Y_n - c| \leq \delta) \leq \mathbb{P}(X_n + c - \delta \leq x)$. Thus,

$$\mathbb{P}(X_n + Y_n \leq x) \leq \mathbb{P}(X_n + c - \delta \leq x) + \mathbb{P}(|Y_n - c| \geq \delta)$$

   if $x - c + \delta$ is a continuity point of F, then:

$$\limsup_{n\to\infty} \mathbb{P}(X_n + Y_n \leq x) \leq \mathbb{P}(X + c - \delta \leq x)$$

   On the other hand, if $X_n + c + \delta \leq x$ and $c - \delta \leq Y_n \leq c + \delta$, then $X_n + Y_n \leq x$, so $\mathbb{P}(X_n + c + \delta \leq x, |Y_n - c| \leq \delta) \leq \mathbb{P}(X_n + Y_n \leq x, |Y_n - c| \leq \delta)$. This implies that:

$$\mathbb{P}(X_n + c - \delta \leq x) \leq \mathbb{P}(X_n + Y_n \leq x) + \mathbb{P}(|Y_n - c| \geq \delta)$$

This shows that, if $x - c - \delta$ is a continuous point of F:

$$\liminf_{n\to\infty} \mathbb{P}(X_n + Y_n \le x) \ge \mathbb{P}(X + c - \delta \le x)$$

If $x-c$ is a continous point of F, then we can find a sequence of decreasing $\delta \to 0$ so that $x-c-\delta \to x-c$, where $x+c-\delta$ and $x+c+\delta$ are continuous points of F (since F has only countably many discontinuities). This then shows that $\mathbb{P}(X_n + Y_n \le x) \to \mathbb{P}(X + c \le x)$ as desired.

Next we prove that if $X_n \to X$ in distribution and $Y_n \to c \in \mathbb{R}$ in probability, then $X_n Y_n \to Xc$ in distribution.

Notice that, since $c - \delta \le Y_n \le c + \delta$ iff $\dfrac{1}{c+\delta} \le \dfrac{1}{Y_n} \le \dfrac{1}{c-\delta}$, we have that:

$$\begin{aligned}
\mathbb{P}(X_n Y_n \le x) &\le \mathbb{P}\left(X_n \le \frac{x}{Y_n}, |Y_n - c| \le \delta \right) + \mathbb{P}(|Y_n - c| \ge \delta) \\
&\le \mathbb{P}\left(X_n \le \frac{x}{c-\delta} \right) + \mathbb{P}(|Y_n - c| \ge \delta)
\end{aligned}$$

Similarly,

$$\mathbb{P}\left(X_n \le \frac{x}{c+\delta} \right) \le \mathbb{P}(X_n Y_n \le x) + \mathbb{P}(|Y_n - c| \ge \delta)$$

Now, Thus,

$$\limsup_{n\to\infty} \mathbb{P}(X_n Y_n \le x) \le \limsup_{n\to\infty} \mathbb{P}\left(X_n \le \frac{x}{c-\delta} \right)$$

If $x/(c-\delta)$ is a continuous point of F, then the right side equals $\mathbb{P}\left(X \le \dfrac{x}{c-\delta} \right)$, and similarly,

$$\liminf_{n\to\infty} \mathbb{P}\left(X_n \le \frac{x}{c+\delta} \right) \le \liminf_{n\to\infty} \mathbb{P}\left(X_n Y_n \le \frac{x}{c+\delta} \right)$$

And if $x/(c+\delta)$ is a continuous point of F then the left side equals $\mathbb{P}\left(X \le \dfrac{x}{c+\delta} \right)$. If c is a continous point of F, then once again since there are only countably many discontinuities of F, we can find a sequence of δ decreasing to 0 so that $x/(c+\delta)$, $x/(c-\delta)$ are always continuous points of F. Sending $\delta \to 0$ shows that $\mathbb{P}(X_n Y_n \le x) \to \mathbb{P}(Xc \le x)$ as desired.

Now we are ready to defeat the beast. Let $\mathbb{E}\big[X_1^2\big] = \sigma^2$. Then by the weak law of large numbers, noticing that the X_i^2 are i.i.d., $\dfrac{1}{\sigma^2 n}\displaystyle\sum_{i=1}^n X_i^2 \to 1$ in probability. Since $f(x) = \dfrac{1}{\sqrt{x}}$ is continous at 1, we must have that $\dfrac{\sigma\sqrt{n}}{\left(\sum_{i=1}^n X_i^2\right)^{1/2}} \to 1$ in probability. By the central limit theorem, $\dfrac{1}{\sigma\sqrt{n}}\displaystyle\sum_{i=1}^n X_i \to N(0,1)$ in distribution. Thus by the theorem we just proved we must have:

$$\frac{\sum_{i=1}^n X_i}{\left(\sum_{i=1}^n X_i^2\right)^{1/2}} = \frac{\frac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}}}{\frac{(\sum_{i=1}^n X_i^2)^{1/2}}{\sigma\sqrt{n}}} \to N(0,1)$$

in distribution, which completes the proof.