Math 582 HW1

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1. Let $A = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}$. For $n \geq 3$, consider the quantity $(a_1 + a_2)^T(a_2 + a_3) = a_2^T a_2 = n$ by orthogonality. However,

$$(a_1 + a_2)^T(a_2 + a_3) = \sum_i (a_{1i} + a_{2i})(a_{2i} + a_{3i})$$

$a_{1i} + a_{2i}$ and $a_{2i} + a_{3i}$ are both divisible by 2 (being either 0, 2 or -2 each), so this sum is divisible by 4. But this means that $4 \mid n$.

2. The matrix is as follows:

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$

Since every vector $x \in \{\pm 1\}^4$ will have a prefix of one of those rows (up to a possible sign change), we can always get a value of $\frac{3}{\sqrt{3}} = \sqrt{3}$. An important paper by Steinerberger discussing the bad science problem shows that when we normalize the rows, the Kolmos constant cannot be finite. There is even an explicit construction giving $\geq \sqrt{\log_2(n+1)}$, yet, it alludes me on how to achieve 17% better $\sqrt{2\ln(n+1)}$.

For when the columns are normalized, consider the following 5×5 matrix:

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & -1/2 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Every vector $x \in \{\pm 1\}^5$ will have a prefix of one of the top 4 rows (up to a possible sign change). This will then give an inner product of $\frac{3}{2}$. The last row is just a throwaway. This shows the Kolmos constant with rows is at least $\frac{3}{2} > \sqrt{2}$. After readinng Kunisky's paper, I have been enlightened even further. Let $A_0 \in \mathbb{R}^{n \times n}$ be any starting matrix with normalized columns. Define

$$A_{n+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} A_n & I \\ -A_n & I \end{pmatrix}$$

Further define $K(A) = \inf_{x \in \{\pm 1\}^n} \|Ax\|_\infty$. Then I claim that $K(A_n) = C \cdot 2^{-n/2} + 2/(\sqrt{2} - 1)$. This follows by letting $x_n = K(a_n)$, then noticing that $x_n = \frac{1}{\sqrt{2}} x_{n-1} + \frac{1}{\sqrt{2}}$. This is inhomogeneous linear recurrence, which can be solved easily. The constant part satisfies $c = c/\sqrt{2} + \frac{1}{\sqrt{2}}$, which gives $c = 2/(\sqrt{2} - 1)$. The exponential part satisfies $b_n = b_{n-1}/\sqrt{2}$, which gives $b_n = C \cdot 2^{-n/2}$. Their sum $K(A_n)$ is then $C \cdot 2^{-n/2} + 2/(\sqrt{2} - 1)$, which is the desired result. Taking $n \to \infty$ gives a sequence of matrices whose K value goes upto $\frac{2}{\sqrt{2}-1}$, which is around 4.8, much bigger than 2.

3. First I prove the following lemma. Let W, V be subspaces, and consider the map $f : V \to (V + W)/W$ sending v to $\overline{v}$. As every vector in $V + W$ can be written as $v + w$, passing to a quotient gives just $\overline{v}$, so this map is surjective. By the first isomorphism theorem, $\dim V = \dim \ker f + \dim \operatorname{Im} f$. So, $\dim V \geq \dim \operatorname{Im} f = \dim(V + W)/W = \dim(V + W) - \dim W$. Thus, $\dim(V + W) \leq \dim V + \dim W$. In fact we prove the stronger result that equality holds iff $V \cap W = \{0\}$. This is because equality holds when $\ker f = \{0\}$, and the kernel can be described by the set of v so that $v \in W$, which is just $V \cap W$.

   Now, let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. Clearly,

$$\operatorname{span}\{a_1, \ldots, a_n\} \subset \operatorname{span}\{a_1 + b_1, \ldots, a_n + b_n, b_1, \ldots, b_n\}$$

The first subset has dimension $\operatorname{rank} A$ and the last has dimension $\leq \operatorname{rank}(A+B)+\operatorname{rank} B$. This shows that $\operatorname{rank} A - \operatorname{rank} B \leq \operatorname{rank}(A+B)$, and similarly the reverse holds, so $|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A+B)$.

Now, also $\operatorname{span}\{a_1+b_1, \dots, a_n+b_n\} \subset \operatorname{span}\{a_1, \dots, a_n, b_1, \dots, b_n\}$. The first subset has dimension $\operatorname{rank}(A+B)$ and the last has dimension $\leq \operatorname{rank} A + \operatorname{rank} B$. This shows that $\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$, which completes the proof.

4. Let $B_r(x) \subset \mathbb{R}^n$ be the ℓ^1 ball of radius r around the point x. I claim that $\operatorname{Vol}(B_r(x)) = \frac{2^n}{n!} r^n$. We prove this by induction. Suppose that it is true for $n-1$. Then the volume of the 1-ball in n dimensions is just:

$$\int_{-1}^{1} \frac{2^n}{n!} r^n dr = \frac{2^n}{n!} \frac{1}{n+1} r^{n+1} \Big|_{-1}^{1} = \frac{2^{n+1}}{(n+1)!}$$

By simple scaling map and translational invariance of volume, we see that the volume of the r-ball around any point is just $\frac{2^{n+1}}{(n+1)!} r^{n+1}$.

I give second proof for fun. We know that there are 2^n quadrants in in n dimensions, so we only look at the first quadrant $x_i \geq 0$ for all i. Then we just have to show that the volume of $\sum x_i \leq 1$ is $\frac{1}{n!}$ when $x_i \geq 0$ for all i.

Since this is linear algebra class, we use linear algebra. Consider linear map sending $(x_1, \dots, x_n)$ to $(x_1, x_1+x_2, \dots, x_1+\dots+x_n)$. The matrix A is just:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Clearly, the image of $S = \{x_1 + \dots + x_n \leq 1\}$ under A is just $T = \{0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq 1\}$. Intuitively, for almost every point $x \in [0,1]^n$, there is a unique $\pi \in S_n$ so that $x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(n)}$ (just look at the numbers and order them in increasing order; points where there are multiple correspond to where two of the coordinates are equal, and these have measure 0). Since the $n!$ permutations of the indices of T all have the same volume, whose volume is a (almost) disjoint union of $n!$ copies summing to the volume of $[0,1]^n$ which is 1, we see that the volume of T is just $\frac{1}{n!}$. Since $\det A = 1$, this means the volume of S is $(\det A)^{-1} \frac{1}{n!} = \frac{1}{n!}$ too. Incorporating the other quadrants, the volume of $|x_1| + \dots + |x_n| \leq 1$ is just $\frac{2^n}{n!}$.

Assume that S is countably infinite (otherwise throw away some points).

We now use a very standard volume argument. Consider the set $U = B_{3/2}(x_1)$. I claim that $\bigcup_i B_{1/2}(x_i) \subset U$. This is because if $y \in B_{1/2}(x_i)$, then $|y - x_1| \leq |y - x_i| + |x_i - x_1| < 3/2$. The open balls $B_{1/2}(x_1), B_{1/2}(x_2), \ldots$ are disjoint (if $y \in B_{1/2}(x_1) \cap B_{1/2}(x_2)$, then $|x_1 - x_2| \leq |y - x_1| + |y - x_2| < 1$, a contradiction), and so the volume of their union is the sum of the volumes. This sum is:

$$\sum_{i=1}^{\infty} \mathrm{Vol}(B_{1/2}(x_i)) = \sum_{i=1}^{\infty} \frac{2^n}{n!} 2^{-n} = \sum_{i=1}^{\infty} \frac{1}{n!} = \infty$$

But S is contained in $B_3/2(x_1)$ which has finite volume $\leq 2^n/n! \cdot (3/2)^n = 3^n/n!$, a contradiction. Further, this shows that $|S| \leq 3^n$.