Math 334 HW 7

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1. Given any $a < b$, we see that because $f'(x) \leq g'(x)$ for every $x \in \mathbb{R}$, we have that

$$f(b) - f(a) = \int_a^b f'(x)dx \leq \int_a^b g'(x)dx = g(b) - g(a)$$

, because the integral (on an increasing interval) preserves inequalities.

2. Given any $a < b$, because g is differentiable at a and b, and because f is differentiable on all of $\mathbb{R}^2$, the composition function $f \circ g : \mathbb{R} \to \mathbb{R}$ is differentiable on all of $\mathbb{R}$ by the chain rule. Because differentiable $\implies$ continuity, we see that the mean value theorem applies, and there is some $c \in (a, b)$ so that

$$f(g(b)) - f(g(a)) = (f(g(c)))'(b - a)$$

Again by the chain rule, we see that $(f(g(a)))' = \langle \boldsymbol{\nabla} f(g(c)), g'(a) \rangle$, which completes the proof.

3. If f' is continuous: let $(x_n)_{n=1}^{\infty}$ be the described sequence. Because $|x_{n+1}| = |f(x_n)| = |f'(c)||x_n| \leq |x_n|$ for some c, $(|x_n|)$ is bounded and monotonically decreasing, so it converges. Suppose it doesn't converge to 0. Then $\inf |x_n| = r > 0$. Because $1 - f'(x) > 0$ for every $x \in I = [-|x_0|, -\inf |x_n|] \cup [\inf |x_n|, |x_0|]$, there is some $c > 0$ so that $1 - f'(x) \geq c$ on I. Then $f'(x) \leq 1 - c$, so we see f is a contraction mapping, and I is a complete metric space, therefore f has a unique fixed point by Banach, which $(x_n)_{n=1}^{\infty}$ converges to. But the only fixed point of f is $x = 0$, and we already determined that x_n does not go to 0, a contradiction.

Alternatively, let $x_0 \in \mathbb{R}$. Note that $|x_{n+1}| = |f(x_n)| \leq |x_n|$ and so $|x_n| \to r$ for some $r \geq 0$ (monotone convergence theorem). Note also that $|f(x_n)| = |x_{n+1}| \to r$. Note x_n must take the same sign infinitely many times, so choose a subsequence $x_{n_k} \to ar$ with $a \in \{\pm 1\}$. This is justified because if x_{n_k} always has the same sign, this means $|x_{n_k}|$ is either always x_{n_k} or $-x_{n_k}$ in the first case, we see that $|x_{n_k}| = x_{n_k}$, so by taking the limit x_{n_k} converges to r. In the second case, $|x_{n_k}| = -x_{n_k}$, so by taking the limit again we see that $r = -\lim x_{n_k}$. Then $r = \lim_{n \to \infty} |f(x_{n_k})| = |f(ar)|$. If $r \neq 0$, $|r| = |f(ar)| < r$ (see uniqueness), a contradiction. Then $|x_n| \to 0$, so given any $\varepsilon > 0$, there is some $N > 0$ so that if $n > N$, $|x_n| = ||x_n|| < \varepsilon$, which means that $x_n \to 0$. $\square$

Uniqueness: Suppose $f(y) = y$, for some $y \neq 0$. Then by the mean value theorem, there is some $c \in (\min(0, y), \max(0, y))$ so that $|y - 0| = |f(y) - f(0)| = |f'(c)||(y - 0)| < |y - 0|$, a contradiction.

Consider the function

$$f(x) = \begin{cases} x - \dfrac{x^3}{8N}, & |x| < \sqrt{8N}/3, \\ \sqrt{8N}/3 - (\sqrt{8N}/3)^3/(8N), & |x| \geq \sqrt{8N}/3 \end{cases}$$

For clarity I will call $C = 8N$. We see that $f(2) = 2 - 2^3/C$. Then $f(f(2)) = (2 - 2^3/C) - 1/C(2 - 2^3/C)^3$. Note that $2 - 8/C$ is positive by construction, because the second term will never exceed 1 (so it is in fact always greater than 1). Then because $-x^3$ is decreasing, if we plug a larger value into it we will get a smaller value, so this quantity is greater than $(2 - 2^3/C) - 1/C \cdot 2^3 = 2 - 2 \cdot 2^3/C$. Note now that $f(x)$ is increasing on $|x| < C$, because the derivative is 0 for the first time at $|x| = C$, and positive before that. So each value we are plugging in is going to be less than the actual value after k compositions. We continue this process and see after N steps we get that $f(f(...(2))) > 2 - 8N/8N = 1$. Note also that every composition before this would be $>$ than this value, by construction. □

4. $Tf_0 = 1, Tf_1 = 1 + 2x, Tf_2 = 1 + 2x + 2x^2, Tf_3 = 1 + 2x + 2x^2 + 4x^3/3 = (2x)^0/0! + (2x)^1/1! + (2x^2)/2! + (2x^3)/3!$. Then I claim that

$$f_n = \sum_{k=0}^{n} \frac{2x^k}{k!} \tag{1}$$

Proof. For $n = 0$, we see that $f_0 = 1$ which certainly equals $\sum_{k=0}^{0}(2x^k)/k!$. Then suppose for some $n \geq 0$ we have that $f_n = \sum_{k=0}^{n} \frac{2x^k}{k!}$. We see that $f_{n+1} = T(f_n) = 1 + \int_0^x 2f_n(t)dt = 1 + 2\sum_{k=0}^{n} \frac{1}{k!} \int (2t)^k dt = 1 + 2\sum_{k=0}^{n} \frac{1}{k!} \int (2t^k) dt = 1 + 2\sum_{k=0}^{n} \frac{1}{k!}(2x)^{k+1}/(2(k+1)) = 1 + \sum_{k=0}^{n}(2x)^{k+1}/(k+1)!$ Re-indexing, and seeing that $(2x)^0/0! = 1$, we get that $f_{n+1} = \sum_{k=0}^{n+1}(2x)^k/k!$
□

Then the solution to the integral equation is probably e^{2x}. I actually did this in the reverse order, by seeing that if we plug in $x = 0$ to the integral equation, we get that $f(0) = 1$, and differentiating both sides we get that $f' = 2f$. Now it is very clear what function this should be, and also that the iterations are just going to be the taylor approximations of this function.