Math 334 HW 7

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1. First, given any $x \neq 0$, we know that $f'(x) = 2x\sin(1/x) - \sin(1/x)$ which exists for all $x \neq 0$ (just calculated using regular derivative rules). Then we consider the case where $x = 0$, that is,

$$\lim_{h \to 0} \frac{h^2 \sin(1/h) - f(0)}{h} = \lim_{h \to 0} h \sin(1/h)$$

Note that, because $-1 \leq \sin(x) \leq 1$ for every $x \in \mathbb{R}$,

$$\begin{aligned} -h \leq h\sin(1/h) &\leq h \\ \lim_{h \to 0} -h \leq \lim_{h \to 0} h\sin(1/h) &\leq \lim_{h \to 0} h \\ 0 \leq \lim_{h \to 0} h\sin(1/h) &\leq 0 \end{aligned}$$

So we conclude that the limit is in fact 0, that is, f is differentiable at $x = 0$.

2. Given any $x \in \mathbb{R}$, because f is differentiable at x, there is some $\delta* > 0$ so that if $|x - y| < \delta*, |\frac{f(x)-f(y)}{x-y} - f'(x)| < 1$. Then $|f(x) - f(y)| < (1 + |f'(x)|)|x - y|$. Then given any $\varepsilon > 0$, choose $\delta = \min(\delta*, \varepsilon/(1 + |f'(x)|))$. Then $|f(x) - f(y)| < (1 + |f'(x)|)|x - y| < (1 + |f'(x)|)\delta \leq \varepsilon$. $\square$.

We know from the above that because f is differentiable everywhere, it is continuous everywhere, so the mean value theorem applies. So we know that for any $x, y \in \mathbb{R}$, $|f(x) - f(y)| = |x - y||f'(c)|$ for some $c \in \mathbb{R}$. But the derivative is bounded, so this is $\leq C|x - y|$. If $C = 0$, $|f(x) - f(0)| = 0 \cdot |x - y| = 0$, so we see that $f(x) = f(0)$, that is, $f(x)$ is constant. Constant functions are uniformly continuous, so let $C \neq 0$. Then given any $\varepsilon > 0$, choose $\delta = \varepsilon/C$. Then for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have that $|f(x) - f(y)| < C|x - y| < \varepsilon$. $\square$

3. Let $x \in \mathbb{R}$. Because the first derivative is differentiable, there is some $c \in \mathbb{R}$ so that $f'(x) - f'(0) = (x - 0)f''(c) = 0$ because $f''(x) = 0$ for every x. So we see that $f'(x)$ is constant. Then $f'(x) = d$ for some d, and every x. Again, $f(x) - f(0) = (x - 0)f'(C)$ for some C, and we see that $f(x) - f(0) = xd$. By rearranging, $f(x) = xd + f(0)$, so $f(x)$ is linear.

4. Note that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{f(\varepsilon a,\varepsilon b)-f(0,0)}{\varepsilon} &= \lim_{\varepsilon\to 0}\frac{\varepsilon^2 a^2 \varepsilon b}{\varepsilon^3(a^2+b^2)}\\ &= \lim_{\varepsilon\to 0}\frac{a^2 b}{a^2+b^2} = \lim_{\varepsilon\to 0}\varepsilon\cdot a^2 b = a^2 b\end{aligned}$$

So all directional derivatives exist. In particular, $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$, so if f was differentiable at $(0,0)$, $\nabla f = (0,0)$. Then for all h sufficiently small, $f(0+h)-f(0) = \langle \nabla f, h\rangle + E(h)$, with $|E(h)|/\|h\| \to 0$. Then we see that $f(h) = E(h)$, so therefore $|f(h)|/\|h\| \to 0$. But if we take $h = (x,x)$, we see that $\lim_{x\to 0}|f(x,x)|/\sqrt{x^2+x^2} = x^3/(\sqrt{2}x\cdot 2x^2) = 1/\sqrt{8}$ which certainly does not tend to 0. So $f(x)$ is not differentiable at $(0,0)$. $\square$

5. Because all partial derivatives exist and are bounded, there is some number $C > 0$ that is bounds all of them. Now let $\varepsilon > 0$, and choose $\delta = \varepsilon/(nC)$. Then for every $y \in S$ with $\sup_{1\le i\le n}|y_i - x_i| < \delta$,

$$\begin{aligned}|f(y)-f(x)| \le\ & |f(y_1,\dots,y_n) - {\color{red}f(x_1,y_2,\dots,y_n)}|\\ & + |{\color{red}f(x_1,y_2,\dots,y_n)} - f(x_1,x_2,\dots,y_n)|\\ & \vdots\\ & + |f(x_1,\dots,x_{n-1},y_n) - f(x_1,\dots,x_n)|\end{aligned}$$

We consider only the first term: if we let $g(x) = f(x,y_2,\dots,y_n)$, we see that the first term is $|g(y_1)-g(x_1)|$. Note that because the partial derivative in the x_1 direction exists at every point of S, we have that $g(x)$ is continuous on all of $\{x_1 \in \mathbb{R} \mid (x_1,\dots,x_n)\in S\}$ (because differentiable on $(a,b) \implies$ continuous on (a,b)), which, $[\min\{x_1,y_1\},\max\{x_1,y_1\}]$ is a closed subset of. We therefore see that the mean value theorem applies, and that this argument would work for every other coordinate. Then by the mean value theorem, the first term on the right side is $|y_1-x_1||f_i(c)| < C|y_1-x_1|$ for some $c\in\mathbb{R}^n$. This is true for every term, all that's changing is the point at which you evaluate the derivative (which is not important because all the derivatives are bounded above by C), so all terms are $< C\cdot\sup_{1\le i\le n}|y_i-x_i|$. Then we see that $|f(y)-f(x)| < nC\cdot\sup_{1\le i\le n}|y_i-x_i| < \varepsilon$. $\square$