Math 336 HW3

Rohan Mukherjee

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1. Write $f(x+iy) = u(x,y) + iv(x,y)$, and write $\gamma(t) = x(t) + iy(t)$. Then, by definition of the complex line integral, we get that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 (u(x(t),y(t)) + iv(x(t),y(t))(x'(t) + iy'(t))dt \\ &= \int_0^1 [u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)]dt + i\int_0^1 [v(x(t),y(t))x'(t) + u(x(t),y(t))y'(t)]dt \\ &= \int_\gamma udx - vdy + i\int_\gamma vdx + udy\end{aligned}$$

In our case, since $\gamma = \partial\mathbb{D}$, we may apply Green's theorem to see that

$$\begin{aligned}\int_\gamma udx - vdy &= \int_{\mathbb{D}} -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}dA \\ &= \int_{\mathbb{D}} 0dA \\ &= 0\end{aligned}$$

Where we used the Cauchy-Riemann equations in the second equality. Also,

$$\begin{aligned}i\int_\gamma vdx + udy &= i\int_{\mathbb{D}} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}dA \\ &= i\int_{\mathbb{D}} 0dA \\ &= 0\end{aligned}$$

where once again we used the Cauchy-Riemann equations. A sum of zeros is another zero, so indeed the integral over the boundary of the unit disk of any holomorphic function is 0 (Also, this argument clearly generalizes, since all we used was that γ is closed).

2. Notice that, for any polynomial $p(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{-x^2} p(x) = -e^{-x^2}(p'(x) - 2xp(x))$$

We know that $\frac{\mathrm{d}^0}{\mathrm{d}e^0}^{-x^2} = e^{-x^2}$. Suppose that $\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} = e^{-x^2} p(x)$ for some polynomial $p(x)$ of degree n with leading coefficient $(-1)^n 2^n$. By what we did above, we see that $\frac{\mathrm{d}^{n+1}}{\mathrm{d}e^{n+1}}^{-x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^2} p(x)\right) = -e^{-x^2}(p'(x) - 2xp(x))$. $p'(x)$ has degree $n-1$, while $2xp(x)$ has degree $n+1$, so this new polynomial has degree $n+1$, and picked up a leading coefficient of -2, so the new polynomial has leading coefficient $-2 \cdot (-1)^n \cdot 2^n = (-1)^{n+1} 2^{n+1}$, as claimed. We can paramameterize the rectangle described by $\gamma_1(x) = x$, $-R \leqslant x \leqslant R$, $\gamma_2(x) = R + ix$, $0 \leqslant x \leqslant t$, $\gamma_3(x) = x + it$, $R \leqslant x \leqslant -R$ (again, makes sense in the integral), and finally $\gamma_4(x) = -R + ix$, $t \leqslant x \leqslant 0$ (makes sense in the integral!). Since the rectangle is a closed loop,

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)dz = 0$$

I claim that the integral over γ_2 and γ_4 equal 0. In fact, one will imply the other. We start with γ_2:

$$\int_{\gamma_2} f(z)dz = \int_0^t e^{(R+i(x-t))^2/2} e^{-(R+ix)^2} p(R + ix) \cdot i dx$$

Since $\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} = (-1)^n e^{-x^2} H_n(x)$ where $H_n(x)$ is a polynomial of degree n with leading coefficient 2^n, so here we are just calling $(-1)^n H_n(x) = p(x)$. This is smaller in magnitude than (by the triangle inequality)

$$\begin{aligned}\int_0^t |e^{R^2/2}| \cdot |e^{-Ri(x-t)}| \cdot |e^{-(x-t)^2/2}| \cdot |e^{-R^2}| \cdot |e^{-2iRx}| \cdot |e^{x^2}| \cdot |p(R + ix)| dx \\ \leqslant \int_0^t e^{-R^2/2} \cdot 1 \cdot 1 \cdot 1 \cdot e^{x^2} \cdot |p(R + ix)| dx\end{aligned}$$

Less than or equal to since $-(x-t)^2/2 \leqslant 0$. Next, since p is a polynomial of degree n with leading coefficient 2^n, $|p(z)/2^n z^n| \to 1$, so we have the bound $|p(z)| \leqslant 2^{n+1} z^n = Cz^n$ for some C. So,

$$\begin{aligned}\int_0^t e^{-R^2/2} e^{x^2} \cdot |p(R + ix)| dx &\leqslant Ce^{-R^2/2} \int_0^t e^{x^2} |(R + ix)^n| dx \\ &\leqslant Ce^{-R^2/2} \int_0^t e^{x^2} \sum_{k=0}^n \binom{n}{k} R^{n-k} |ix|^k dx \\ &= C \sum_{k=0}^n \binom{n}{k} e^{-R^2/2} R^{n-k} \int_0^t e^{x^2} x^k dx\end{aligned}$$

As t is fixed, for any k, $\int_0^t e^{x^2} x^k dx$ is finite, and bounded in absolute value by some constant D_k. Letting $D = \max_k D_k$, we see that

$$C \sum_{k=0}^{n} \binom{n}{k} e^{-R^2/2} R^{n-k} \int_0^t e^{x^2} x^k dx \leqslant C \sum_{k=0}^{n} \binom{n}{k} e^{-R^2/2} R^{n-k} D$$

As $R \to \infty$, $e^{-R^2/2} \cdot R^{n-k} \to 0$ for any $n - k \geqslant 0$. A sum of zeros is another 0, so we see our integral tends towards 0, as claimed. If we recall back to the first step I did, either R was squared, or it was in the power of $e^{i \cdot \text{real}}$, so replacing $R \to -R$ doesn't change anything, as the sign of R never mattered. The only change is that the bounds of integration are backwards, which would change the sign of the integral, but as $-0 = 0$, we see the integral over γ_4 is also zero. We are left to evaluate the integral over γ_3. Note that

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} e^{-z^2} \bigg|_{z=x+it} = (-1)^n e^{z^2} H_n(z) \bigg|_{x+it} = (-1)^n e^{-(x+it)^2} H_n(x+it)$$

We prove that this equals $\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{(x+it)^2}$ by induction. The base case is clear:

$$\frac{\mathrm{d}^0}{\mathrm{d}x^0} e^{-(x+it)^2} = e^{-(x+it)^2} = \frac{\mathrm{d}^0}{\mathrm{d}z^0} e^{-z^2} \bigg|_{z=x+it}$$

Since $H_0(x+it) \equiv 1$. Suppose it is true for some $n \geqslant 1$. We see that

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} e^{-(x+it)^2} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-(x+it)^2} &= \frac{\mathrm{d}}{\mathrm{d}x} (-1)^n e^{-(x+it)^2} H_n(x+it) \\
&= (-1)^n e^{-(x+it)^2} \left[-2(x+it) H_n(x+it) + H_n'(x+it) \right] \\
&= (-1)^n e^{-(x+it)^2} \cdot (-1) \cdot H_{n+1}(x+it) && \text{(See question 3)} \\
&= (-1)^{n+1} e^{-(x+it)^2} H_{n+1}(x+it)
\end{aligned}$$

As claimed. Finally, notice that

$$\begin{aligned}
-\int_{-R}^{R} e^{x^2/2} \cdot \frac{\mathrm{d}^n}{\mathrm{d}z^n} e^{-z^2} \bigg|_{z=x+it} dx &= -\int_{-R}^{R} e^{x^2/2} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-(x+it)^2} dx \\
&= -\int_{-R}^{R} e^{x^2/2} \cdot \frac{\mathrm{d}^n}{\mathrm{d}t^n} (-i)^n e^{-(x+it)^2} dx \\
&= -(-i)^n \int_{-R}^{R} e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}t^n} e^{-(x+it)^2} dx \\
&= -(-i)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-R}^{R} e^{x^2/2} \cdot e^{-x^2-2itx+t^2} dx \\
&= -(-i)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} e^{t^2} \int_{-R}^{R} e^{-x^2/2} e^{2itx} dx
\end{aligned}$$

Since $i^{2n} = (-1)^n$, we used the fact in the problem, and we simplified the denominator. We recall that the Gaussian is it's own Fourier transform:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{2\pi i x \xi} dx = e^{-\pi \xi^2}$$

Finally, by applying the substitution $x = \sqrt{2\pi}u$, with $\sqrt{2\pi}\mathrm{d}u = \mathrm{d}x$, we get that

$$\int_{-R}^{R} e^{-x^2/2}e^{2itx}dx \overset{R\to\infty}{=} \sqrt{2\pi}\int_{-\infty}^{\infty} e^{-\pi u^2}\cdot e^{2\pi iu(\sqrt{2/\pi}t)}\mathrm{d}u$$

$$= \sqrt{2\pi}e^{-\pi 2/\pi\cdot t^2} = \sqrt{2\pi}e^{-2t^2}$$

Therefore,

$$-(-i)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}e^{t^2}\int_{-R}^{R} e^{-x^2/2}e^{2itx}dx \to -(-i)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}e^{t^2}\sqrt{2\pi}e^{-2t^2}$$

$$= -(-i)^n\sqrt{2\pi}\frac{\mathrm{d}^n}{\mathrm{d}t^n}e^{-t^2}$$

$$= -(-i)^n\sqrt{2\pi}(-1)^n e^{-t^2}H_n(t)$$

Which is nice. Since $\int_{\gamma_2} f(z)dz = \int_{\gamma_4} f(z)dz = 0$, we have that $\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = 0$, which tells us that $\int_{\gamma_1} f(z)dz = -\int_{\gamma_3} f(z)dz$. Plugging the parameterization in, this tells us that

$$\int_{-\infty}^{\infty} e^{(x-it)^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}dx = (-i)^n\sqrt{2\pi}(-1)^n e^{-t^2}H_n(t)$$

Finally, note that $e^{(x+it)^2/2} = e^{x^2/2}\cdot e^{itx}\cdot e^{-t^2/2}$, so the LHS equals

$$e^{-t^2/2}\int_{-\infty}^{\infty} e^{x^2/2}e^{itx}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}dx$$

Once again, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2} = (-1)^n e^{-x^2}H_n(x)$, so this equals

$$e^{-t^2/2}(-1)^n\int_{-\infty}^{\infty} e^{x^2/2}e^{itx}e^{-x^2}H_n(x)dx = e^{-t^2/2}(-1)^n\int_{-\infty}^{\infty} e^{itx}e^{-x^2/2}H_n(x)dx$$

$$= e^{-t^2/2}(-1)^n\int_{-\infty}^{\infty}\phi_n(x)e^{itx}dx$$

We conclude that

$$e^{-t^2/2}(-1)^n\int_{-\infty}^{\infty}\phi_n(x)e^{itx}dx = (-i)^n\sqrt{2\pi}(-1)^n e^{-t^2}H_n(t)$$

$$\implies \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_n(x)e^{itx}dx = (-i)^n e^{-t^2/2}H_n(t) = (-i)^n\phi_n(t)$$

And we are done. Q.E.MF'n.D.

3. By definition, $(-1)^n H_n(x)e^{-x^2} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}$ (note: $n \equiv -n \bmod 2$). We see that

$$\begin{aligned}
H_{n+1}(x) &= -(-1)^n e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2} \\
&= -(-1)^n e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(-1)^n H_n(x)e^{-x^2} \\
&= -e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}H_n(x)e^{-x^2} \\
&= -e^{x^2}e^{-x^2}(H_n'(x) - 2xH_n(x)) \\
&= -H_n'(x) + 2xH_n(x)
\end{aligned}$$

Also, by the generalized Liebnitz rule, we see that

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}e^{-x^2} = \frac{\mathrm{d}^n}{\mathrm{d}x^n} - 2xe^{-x^2} &= -2\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}e^{-x^2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}x \\
&= -2x\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2} - 2n\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}e^{-x^2}
\end{aligned}$$

Which, through a similar calculation as above, shows that

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

These together show that $H_n'(x) = 2nH_{n-1}(x)$. We also recall that the definition of $\phi_n(x) = e^{-x^2/2}H_n(x)$. Algebra shows that

$$\begin{aligned}
\phi_n''(x) &= e^{-x^2/2}((-1+x^2)H_n(x) - 2xH_n'(x) + H_n''(x)) \\
&= e^{-x^2/2}((-1+x^2)H_n(x) + (-2xH_n(x) + H_n'(x))' + 2H_n(x)) \\
&= e^{-x^2/2}((1+x^2)H_n(x) - H_{n+1}'(x)) \\
&= (1+x^2)\phi_n(x) - e^{-x^2/2}H_{n+1}'(x) \\
&= (1+x^2)\phi_n(x) - 2(n+1)e^{-x^2/2}H_n(x) \\
&= (1+x^2)\phi_n(x) - 2(n+1)\phi_n(x) \\
&= x^2\phi_n(x) - (2n+1)\phi_n(x)
\end{aligned}$$

So indeed, $\phi_n(x)$ satisfies $y'' - x^2y + (2n+1)y = 0$ for every $n \geqslant 0$. Also, note that $\phi_n(x) \to 0$ as $|x| \to \infty$, since e^{-x^2} shrinks faster than any polynomial, and of course $H_n(x)$ is a polynomial of degree n. Similarly, $\phi_n'(x) = e^{-x^2}(H_n'(x) - 2xH_n(x))$, and as $H_n'(x) - 2xH_n(x)$ is just another polynomial, as $|x| \to \infty$, $\phi_n'(x) \to 0$ for any n. We see that

$$\begin{aligned}
\int_{\mathbb{R}} \phi_n''(x)\phi_m(x)dx &= \phi_m(x)\phi_n(x)\Big|_{-\infty}^{\infty} - \phi_m'(x)\phi_n(x)\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} \phi_m''(x)\phi_n(x)dx \\
&= \int_{\mathbb{R}} \phi_m''(x)\phi_n(x)dx
\end{aligned}$$

Since we showed that $\phi'_m(x) \to 0$ and $\phi_n(x) \to 0$, so their product also tends towards 0, and the argument is similar for the other term going to 0. By the DE that ϕ satisfies, we see that

$$\begin{aligned}\int_{\mathbb{R}} \phi''_n(x)\phi_m(x)dx &= \int_{\mathbb{R}} (x^2\phi_n(x) - (2n+1)\phi_n(x))\phi_m(x)dx \\ &= -(2n+1)\int_{\mathbb{R}} \phi_n(x)\phi_m(x)dx + \int_{\mathbb{R}} x^2\phi_n(x)\phi_m(x)dx\end{aligned}$$

And similarly,

$$\int_{\mathbb{R}} \phi''_m(x)\phi_n(x)dx = -(2m+1)\int_{\mathbb{R}} \phi_n(x)\phi_m(x)dx + \int_{\mathbb{R}} x^2\phi_n(x)\phi_m(x)dx$$

Since these are equal, we can subtract the integral with x^2 to see that

$$(2n - 2m)\int_{\mathbb{R}} \phi_n(x)\phi_m(x)dx = 0$$

Since $n \neq m$, dividing by $(2n - 2m)$ on both sides gives us our desired result.

4. Let Γ be the semicircle oriented counter-clockwise around the origin of radius 1. Then $\int_\Gamma f(z)dz = 0$, since Γ is a closed loop. We decompose Γ into two parts, $\gamma_1(t) = t$, $-1 \leqslant t \leqslant 1$, and $\gamma_2(t) = e^{it}$, $0 \leqslant t \leqslant \pi$, and by our formula above we get that

$$0 = \int_\Gamma f^2(z)dz = \int_{-1}^{1} f^2(x)dx + \int_0^\pi f^2(e^{it}) \cdot ie^{it}dt$$

Similarly, let Ξ be the contour that starts at 1, moves to -1 in a straight line, and then moves counter-clockwise in a circular form to close the loop. Ξ can be decomposed into $\xi_1(t) = t$, $1 \leqslant t \leqslant -1$ (as formal symbols, think of this as ξ_1 being oriented backwards, in the integral it makes sense), and similarly $\xi_2(t) = e^{it}$, $\pi \leqslant t \leqslant 2\pi$. We get that

$$0 = \int_\Xi f^2(z)dz = \int_1^{-1} f^2(x)dx + \int_\pi^{2\pi} f^2(e^{it})ie^{it}dt$$

These together tell us that

$$\begin{aligned}2\int_{-1}^{1} f^2(x)dx = 2\left|\int_{-1}^{1} f^2(x)dx\right| &= \left|\int_0^\pi f^2(e^{it})ie^{it}dt\right| + \left|\int_\pi^{2\pi} f^2(e^{it})ie^{it}dt\right| \\ &\leqslant \int_0^\pi |f^2(e^{it})| \cdot 1dt + \int_\pi^{2\pi} |f^2(e^{it})| \cdot 1dt = \int_0^{2\pi} |f(e^{it})|^2 dt\end{aligned}$$

Next,

$$
\begin{aligned}
|f(e^{it})|^2 = f(e^{it}) \cdot \overline{f(e^{it})} = \sum_{k=0}^{n} a_k e^{ikt} \cdot \overline{\sum_{l=0}^{n} a_l e^{ilt}} &= \sum_{k=0}^{n} a_k e^{ikt} \cdot \sum_{l=0}^{n} \overline{a_l e^{ilt}} \\
&= \sum_{k=0}^{n} a_k e^{ikt} \cdot \sum_{l=0}^{n} a_l e^{-ilt} \\
&= \sum_{l,k=0}^{n} a_k a_l e^{it(k-l)} \\
&= \sum_{l \neq k}^{n} a_k a_l e^{it(k-l)} + \sum_{k=0}^{n} a_k^2 \cdot e^0
\end{aligned}
$$

Next, note that, for $k \neq l$,

$$
\begin{aligned}
\int_0^{2\pi} a_k a_l e^{it(k-l)} &= a_k a_l \int_0^{2\pi} \cos((k-l)t) + i\sin((k-l)t)dt \\
&= \frac{a_k a_l}{k-l}\Big[\sin((k-l)t) - i\cos((k-l)t)\Big]_0^{2\pi} \\
&= 0
\end{aligned}
$$

We see that

$$
\begin{aligned}
\int_0^{2\pi} |f(e^{it})|^2 dt &= \int_0^{2\pi} \left[\sum_{l \neq k}^{n} a_k a_l e^{it(k-l)} + \sum_{k=0}^{n} a_k^2\right] dt \\
&= 2\pi \sum_{k=0}^{n} a_k^2 + \sum_{l \neq k}^{n} \int_0^{2\pi} a_k a_l e^{it(k-l)} dt \\
&= 2\pi \sum_{k=0}^{n} a_k^2
\end{aligned}
$$

Looking at where we started, we see that we in fact have derived that

$$
\int_{-1}^{1} f^2(x)dx \leqslant \pi \sum_{k=0}^{n} a_k^2
$$

for any polynomial f. That is quite nice!

5. Let Γ be the curve described in the question. We can decompose Γ into three parts: $\gamma_1(t) = Rt$, $0 \leqslant t \leqslant 1$, $\gamma_2(t) = Re^{it}, 0 \leqslant t \leqslant \pi/4$, and finally $\gamma_3(t) = Rt + Rit$, $0 \leqslant t \leqslant \sqrt{2}/2$, where $\gamma_3(t)$ is oriented backwards (we will correct this by adding a minus sign). We see that

$$
\int_\Gamma e^{iz^2} dz = \int_{\gamma_1} e^{iz^2} dz + \int_{\gamma_2} e^{iz^2} dz - \int_{\gamma_3} e^{iz^2} dz = 0
$$

since Γ is a closed loop in the complex plane. We want to find $\lim_{R\to\infty}\int_{\gamma_1} e^{iz^2}dz$, so it suffices to find $\lim_{R\to\infty}\int_{\gamma_3} e^{iz^2}dz - \int_{\gamma_2} e^{iz^2}dz$ (since these quantities are equal). Notice that

$$\begin{aligned}\int_{\gamma_2} e^{iz^2} &= \int_0^{\pi/4} e^{iR^2e^{i2t}} \cdot ie^{it}dt \\ &= \int_0^{\pi/4} \left[e^{iR^2\cos(2t)-R^2\sin(2t)}\right] \cdot ie^{it}dt \\ &= \int_0^{\pi/4} \left[e^{iR^2\cos(t)dt} \cdot e^{-R^2\sin(t)}\right] ie^{it}dt\end{aligned}$$

Also note that

$$\begin{aligned}\left|\int_0^{\pi/4} \left[e^{iR^2\cos(2t)dt} \cdot e^{-R^2\sin(2t)}\right] ie^{it}dt\right| &\leqslant \int_0^{\pi/4} \left|e^{iR^2\cos(2t)dt} \cdot e^{-R^2\sin(2t)}\right| dt \\ &= \int_0^{\pi/4} e^{-R^2\sin(2t)}dt \\ &= \frac{1}{2}\int_0^{\pi/2} e^{-R^2\sin(t)}dt\end{aligned}$$

Given any $d > 0$,

$$\lim_{R\to\infty} \frac{R^2}{e^{R^2 d}} = \lim_{R\to\infty} \frac{2R}{e^{R^2 d}\cdot 2Rd} = \lim_{R\to\infty} \frac{1}{e^{R^2 d}d} = 0$$

Let $\varepsilon > 0$. On $\varepsilon \leqslant t \leqslant \pi/2$, $\sin(t) \geqslant \xi > 0$ for some $\xi > 0$. By our limit calculation above, we can find R sufficiently large so that

$$R^2/e^{R^2\xi} = \left|R^2/e^{R^2\xi}\right| < \varepsilon \text{ and } 1/R^2 < \varepsilon/(\pi/2 - \varepsilon)$$

i.e. that $R^2 < \varepsilon \cdot e^{R^2\xi}$. Hence, $e^{-R^2\xi} < \varepsilon/R^2$. Therefore,

$$\begin{aligned}\int_0^{\pi/2} e^{-R^2\sin(t)}dt = \int_0^{\varepsilon} e^{-R^2\sin(t)}dt + \int_{\varepsilon}^{\pi/2} e^{-R^2\sin(t)}dt &\leqslant \int_0^{\varepsilon} 1dt + \int_{\varepsilon}^{\pi/2} e^{-R^2\xi}dt \\ &\leqslant \varepsilon + \int_{\varepsilon}^{\pi/2} 1/R^2 dt \\ &\leqslant \varepsilon + (\pi/2 - \varepsilon)/R^2 \\ &< 2\varepsilon\end{aligned}$$

So indeed, our original integral tends towards zero in the limit. Next, notice that

$$\begin{aligned}
\int_{\gamma_3} e^{iz^2} dz &= \int_0^{\sqrt{2}/2} e^{i(Rt+Rit)^2} \cdot (R + Ri) dt \\
&= \int_0^{\sqrt{2}/2 \cdot R} e^{-2u^2} (1 + i) dt \\
&= \int_0^{\sqrt{2}/2 \cdot R} e^{-2u^2} dt + i \int_0^{\sqrt{2}/2 \cdot R} e^{-2u^2} dt \\
&\overset{R\to\infty}{=} \int_0^{\infty} e^{-2u^2} du + i \int_0^{\infty} e^{-2u^2} du
\end{aligned}$$

Finally, $\int_0^\infty e^{-2u^2} du = \frac{1}{\sqrt{2}} \int_0^\infty e^{-w^2} dw = \sqrt{2}/2 \cdot \sqrt{\pi}/2 = \sqrt{2\pi}/4$. We get that

$$\int_0^\infty e^{it^2} dt = \int_0^\infty \cos\left(t^2\right) dt + i \int_0^\infty \sin\left(t^2\right) dt = \frac{\sqrt{2\pi}}{4} + i \frac{\sqrt{2\pi}}{4}$$

Finally, matching real and imaginary parts shows that

$$\int_0^\infty \cos\left(t^2\right) dt = \int_0^\infty \sin\left(t^2\right) dt = \frac{\sqrt{2\pi}}{4}.$$