CSE 311 HW4

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2.1 (a) The "proof" is incorrect because they started with the conclusion and concluded the hypothesis.

(b) It is false, take $a = 1, b = 0, c = 2$. Clearly $ab = bc = 0$, while $a \neq c$. The statement would however be true if we restrict our domain to non-zero real numbers.

2.2 (a) The above proof is incorrect as $\sqrt{a^2} = |a|$, which is not necessarily a (it could be $-a$). If we impose the additional restriction that $a, b \geqslant 0$, then the statement is true, as $|x| = x$ for all $x \geqslant 0$.

(b) The above statement is false. Take $a = 1, b = -1$. Clearly $a^2 = b^2 = 1$, while $a \neq b$.

3. (a) $\mathrm{Mysterious}(x) = \exists k(x - 3 = 4k)$.

   (b)

let a be arbitrary

1.1. $\mathrm{Mysterious}(a)$	(Assumption)
1.2. $\exists k(a - 3 = 4k)$	(Definition of Mysterious(a))
1.3. $a - 3 = 4b$	(Elim $\exists$)
1.4. $a = 4b + 3$	(Add 3 to both sides)
1.5. $a = 2(2b + 1) + 1$	(Factor out a 2)
1.6. $\exists l(a = 2l + 1)$	(Intro $\exists$)
1.7. $\mathrm{Odd}(a)$	(Definition of Odd(a))
2. $\mathrm{Mysterious}(a) \to \mathrm{Odd}(a)$	(Direct proof rule)
3. $\forall x(\mathrm{Mysterious}(x) \to \mathrm{Odd}(x))$	(Intro $\forall$)

   (c) Let $x \in \mathbb{Z}$ be arbitrary. If $4 \mid (x - 3)$, then $x - 3 = 4k$ for some $k \in \mathbb{Z}$. Rearranging, we get that $x = 4k + 3 = 4k + 2 + 1 = 2(2k + 1) + 1$, and clearly $2k + 1 \in \mathbb{Z}$, so x is odd by the definition of an odd number. As x was arbitrary, we have proven our statement. $\square$

   (d) The first sentence corresponds to the first 3 lines of my proof, that is a being arbitrary and using the definition of Mysterious(x). The part before the second comma corresponds to the next 3 lines of algebra, and the part after the comma corresponds to translating that to $\mathrm{Odd}(a)$. The last sentence corresponds to the last 2 lines of the proof–reintroducing the $\forall$.

4. Let $a \in (A \cap B) \cup (A \cap C)$ be arbitrary. By the definition of union, we know that $a \in A \cap B$, or $a \in A \cap C$. In the first case, $a \in A$ and $a \in B$, so $a \in A$, and we are done. In the second case, $a \in A$, and $a \in C$, so $a \in A$, and we are also done. In any case, $a \in A$, and as a was arbitrary, we have the inclusion $(A \cap B) \cup (A \cap C) \subseteq A$. $\square$

5. (a) Given any $x \in (B \setminus A) \cap (C \setminus A)$, we see that $x \in B$ and $x \notin A$, and that $x \in C$ and $x \notin A$. It is now clear that we have both $x \in B$ and $x \in C$, which shows that $x \in B \cap C$. We also have $x \notin A$ (twice, in fact), so by the definition of set difference, we have that $x \in (B \cap C) \setminus A$. As x was arbitrary, we have that $(B \setminus A) \cap (C \setminus A) \subseteq (B \cap C) \setminus A$. Given $x \in (B \cap C) \setminus A$, we have that $x \in B \cap C$ and that $x \notin A$. The first condition is equivalent to $x \in B$ and $x \in C$, so we see see that $x \in B$ and that $x \notin A$, and at the same time we have that $x \in C$ and $x \notin A$ (this step is similar to intro $\wedge$ in a formal proof). This shows that $x \in (B \cap C) \setminus A$, by the definition of $\cap$ and $\setminus$. As x was arbitrary, we have the reverse inclusion $(B \setminus A) \cap (C \setminus A) \supseteq (B \cap C) \setminus A$, which proves that the two sets are indeed equal.

   Here is the chain of equivalences:

$$
\begin{aligned}
& x \in (B \setminus A) \cap (C \setminus A) \\
& \iff x \in B \setminus A \wedge x \in C \setminus A && \text{(Definition of } \cap\text{)} \\
& \iff (x \in B \wedge x \notin A) \wedge (x \in C \wedge x \notin A) && \text{(Definition of } \setminus\text{)} \\
& \iff (x \in B \wedge x \in C) \wedge (x \notin A \wedge x \notin A) && \text{(Commutativty, and associativity twice)} \\
& \iff (x \in B \wedge x \in C) \wedge x \notin A && (p \wedge p \equiv p) \\
& \iff x \in (B \cap C) \wedge x \notin A && \text{(Definition of } \cap\text{)} \\
& \iff x \in (B \cap C) \setminus A && \text{(Definition of } \setminus\text{)}
\end{aligned}
$$

   (b) Take $\{1, 2, 3\} = A = B = C$. Note that $A \setminus B = B \setminus C = \emptyset$. Therefore, the LHS is $A \setminus \emptyset = A = \{1, 2, 3\}$, while the RHS is $\emptyset \setminus C = \emptyset \neq \{1, 2, 3\}$, so we have found a counterexample. This is not a surprising result, as subtraction of real numbers is also not associative.

6. (a) The first problem is that $X \subseteq S \cup T \rightarrow X \subseteq S \vee X \subseteq T$. This is false, take $X = \{1, 2, 3\}$, $S = \{1, 2\}$, and $T = \{3\}$. Clearly the first statement is true while the second is false. The proof strategy error is that all they attempted to do was show that $P(S \cup T) \subseteq P(S) \cup P(T)$, but they also had to show the other direction, which is that $P(S) \cup P(T) \subseteq P(S \cup T)$.

   (b) This statement is also false, take $S = \{1\}, T = \{2\}$. $P(S) = \{\emptyset, \{1\}\}$, $P(T) = \{\emptyset, \{2\}\}$, and as $S \cap T = \emptyset$, $P(S \cap T) = \{\emptyset\}$. The union of all of these powersets is $\{\emptyset, \{1\}, \{2\}\}$, which clearly does not contain $S \cup T = \{1, 2\}$, which is certainly in the LHS. So these aren't equal, and we are done.

7. (a) Given two arbitrary positive integers x, y, we know that $x \geqslant 1$, and $y \geqslant 1$. Then $x + y \geqslant 2$, so at the very least $x + y \neq 1$, which proves that it is not an arbitrary positive integer (as *arbitrary* would mean that it can take on the value of all positive integers!)

   (b) Let $(x, y) \in (S \cup T) \times V$ be arbitrary.

   ...

   Therefore, $(x, y) \in (S \times V) \cup (T \times V)$. As (x, y) was arbitrary, this shows that $(S \cup T) \times V \subseteq (S \times V) \cup (T \times V)$.

   Next, let $(x, y) \in (S \times V) \cup (T \times V)$ be arbitrary.

   ...

   Therefore, $(x, y) \in (S \cup T) \times V$. As (x, y) was arbitrary, this shows that $(S \cup T) \times V \supseteq (S \times V) \cup (T \times V)$. Because both sets are contained in each other, they are equal. $\square$

8. This assignment was particularly short. I have been really excited to get to English proofs, as it is something that I have practiced a LOT. I love math, and started learning English proofs this summer when I was reading, ”Abstract Algebra: An Introduction” by T.W. Hungerford, which is where i was introduced to a lot of the number theory that we are now learning in class. I hope to take graduate algebra next year, so although I know a lot of proofs already, I find it helpful that I finally understand the underlying logic behind what I was doing (or at least, I have a better understanding of the sort). This problem set took me ≈ 3 hours (I take a while to review stuff). I spent the most time on problem 3, as I had to write an inference proof, which takes long to type up (it was also fairly confusing).