Math 461 HW7

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1. (a) We construct a bijection between the partitions of n where the largest part occurs at least twice and partitions of $n-2$ with smallest part at least 2. The bijection is to take the transpose then remove the last

   (b) We use the same trick as before. We shall construct a bijection between partitions of n where the largest part appears exactly 2 times and partitions of $n-2$ where the smallest part is at least 2. The bijection this time is to take a transpose then remove the bottom row, and the inverse being add 2 blocks in a new row on the bottom and take a transpose. The condition of having the smallest part at least 2 this operation indeed yields a young diagram. In any case, the generating function for the number of partitions of n with smallest part at least 2 is just

   $$\prod_{k=2}^{\infty}(1+x^k+x^{2k}+\cdots) = P(x)/(1+x+x^2+\cdots) = (1-x)P(x) = p(0)+\sum_{n=1}^{\infty}(p(n)-p(n-1))x^n$$

   So the number of partitions of n in which the smallest part is at least 2 is just $p(n)-p(n-1)$ for $n \geq 1$ and $p(0)=1$ for $n=0$. So the number of partitions of n where the largest part appears exactly 2 times is just $p(n-2)-p(n-3)$ for $n \geq 3$. We conclude that the number of ways to have the largest part occur at least 3 times is just the number of ways to have it occur at least 2 times minus the number of ways to have it occur exactly 2 times, which is just $p(n)-p(n-1)-(p(n-2)-p(n-3)) = p(n)-p(n-1)-p(n-2)+p(n-3)$.

2. Let A be the ways Alice is sitting next to Bob, C Charlie next to Danielle, and E ed next to Francine. We use complimentary counting to find $|A|$, so put Alice and Bob in a block, pick an ordering of the block Alice and Bob and everyone else, then choose if Alice is to the left or right of Bob. This can be done in $5! \cdot 2$ ways, and the same logic applies to C and E to get they also equal $5! \cdot 2$. If we want Alice next to Bob and Charlie next to Danielle, i.e. $|A \cap C|$, we would now have 2 blocks of people (Alice and Bob and Charlie and Danielle), with 2 remaining people, which using the same logic as above gives a total of $4! \cdot 2^2$, since we need to choose if Alice is left/right of Bob and if Charlie is left/right of Danielle. Finally, using the same logic we would have $|A \cap C \cap E| = 3! \cdot 2^3$. Ignoring the ordering, the total number of seatings is just $6!$, so using complimentary counting and inclusion/exclusion gives us: $|(A^c \cap C^c \cap D^c)^c| = 6! - |A \cup C \cup D| = 6! - (|A|+|C|+|E|-|A \cap C|-|A \cap E|-|E \cap C|+|A \cap C \cap |E|) = 6! - (3 \cdot 5! \cdot 2 - 3 \cdot 4! \cdot 2^2 + 3! \cdot 2^3) = 240$.

3. This is just a question of derangements. First, with no conditions, there are 7! ways to return their hats back to them. Now we need to find the number of ways less than 2 of them get their own hats back. This is just the number of ways that precisely 1 gets his hat back or precisely 0 gets his hat back. This is just $\binom{7}{0}D_7 + \binom{7}{1}D_6$. From the notes we have $D_5 = 44$, so $D_6 = 6 \cdot 44 + (-1)^6 = 265$, and $D_7 = 7 \cdot 265 + (-1)^7 = 1854$. So our final answer is just

$$7! - (1 \cdot 1854 + 7 \cdot 265) = 1131$$

Which is around 1/7th of the possibilities. This sort of says most (6/7) of the possibilities return either 1 or no hats. Also, in expectation you would expect 1 hat back, so this makes sense, but is tighter than I would've expected.

4. This is just exclusion exclusion. Let A denote the ways the first shelf gets exactly 6 books, B, the number of ways the second gets exactly books, C the third, and D the fourth. We want to find $|(A^c \cap B^c \cap C^c \cap D^c)|$, and this is of course just an inclusion-exclusion problem. To find $|A|$, put 6 books in the first shelf, then stars and bars the remaining books into the other shelves, which can be done in $\binom{18+3-1}{3-1} = \binom{20}{2}$ ways. The same logic can be applied to the other shelves, so $|A| = |B| = |C| = |D| = \binom{20}{2} = 190$. To find $|A \cap B|$, we put 6 books in A, 6 in B, then we stars and bars the remaining 12 books among the last 2 shelves to get $\binom{12+2-1}{2-1} = \binom{13}{1} = 13$ ways. So all $\binom{4}{2}$ ways of picking 2 bookshelves with exactly 6 books in both will yield 13 ways. We see that if we have exactly 6 books in the first 3 shelves, then we also have exactly 6 books in the last one, which would yield 1 way. This the $\binom{4}{3} = 4$ ways of picking 3 shelves to have exactly 6 books all give one way. Finally, there is 1 way to have exactly 6 books in all the shelves. This gives us an answer of

$$4 \cdot \binom{20}{2} - \binom{4}{2} \cdot 13 + \binom{4}{3} \cdot 1 - 1 = 685$$

5. Let $m = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, and $A_i = \{ \ell \mid 1 \leq i \leq m, p_i \mid \ell \}$. Notice that for any integer $1 \leq x \leq m$, $\gcd(x, m) \neq 1$ iff $p_i \mid x$ for some p_i (Both directions are clear). We see that $|A_i| = \frac{m}{p_i}$, that $|A_i \cap A_j| = \frac{m}{p_i p_j}$, and in general, $|A_{i_1} \cap \cdots \cap A_{i_\ell}| = \frac{m}{p_{i_1} \cdots p_{i_\ell}}$. By inclusion-exclusion, we have that

$$|A_1 \cup \cdots \cup A_k| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_k|$$

We now claim that

$$-\prod_{i=1}^{n}(1 - x_i) = -1 + \sum_i x_i - \sum_{i<j} x_i x_j + \sum_{i<j<k} x_i x_j x_k + \cdots + (-1)^{n+1} x_1 \cdots x_n$$

First because there is only one way to pick all 1s, and the coefficient is 1 but we flip it. Then notice that to get x_i, we need to pick $-x_i$ from the ith factor and 1 from all the other ones, and finally we flip the sign to get a coefficient of 1. To get $x_i x_j$, we need to choose $-x_i$ from the ith factor and $-x_j$ from the jth factor and 1 from all the other ones, and finally flip the sign, for a coefficient of -1. In general to get $x_{i_1} \cdots x_{i_k}$ we need to pick $-x_{i_1}$ from the i_1th factor and so on and 1s for every other factor, finally flipping the sign, for a coefficient of $(-1)^{k+1}$, which proves the above claim. We may now conclude that,

$$|A_1 \cup \cdots \cup A_k| = n\left(\sum_i \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i p_j} + \cdots + (-1)^{n+1} \frac{1}{p_1 \cdots p_n}\right) = -n\left(\prod_{i=1}^{n}\left(1 - \frac{1}{p_i}\right) + 1\right)$$

But we wanted to find the number which have gcd 1 with n. This is just the complement of the above set, who's cardinality is

$$n - \left(n \prod_{i=1}^{n}\left(1 - \frac{1}{p_i}\right) + n\right) = n \prod_{i=1}^{n}\left(1 - \frac{1}{p_i}\right)$$

Which completes the proof.