Math 506 HW4

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1. If we consider the group $G = D_3$, we see that $srsr^{-1} = srrs = sr^2s = r^{-2} = r$, so $\langle r \rangle \subset [G, G]$. Since $D_3/\langle r \rangle \cong \mathbb{Z}_2$, we have that $[G, G] = \langle r \rangle$ since $[G, G]$ is the smallest subgroup with abelian quotient. The one-dimensional irreducible representations of D_3 are thus the irreducible representations of $\mathbb{Z}/2$, which are just going to be the trivial representation and the representation sending $r \to 1$, $s \to -1$. Finally, we know the last irreducible representation of D_3 to be the representation sending r to the matrix that rotates by $\frac{2\pi}{3}$ radians about the origin, and s to the matrix that flips across the x-axis. This is the representation that sends

$$r \to \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad s \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Since the first matrix is orthogonal, it has determinant ± 1, and since it is orientation preserving it has determinant 1. The second matrix can easily be seen to have matrix -1. The three conjugacy classes of D_3 are $\{e\}, \{r, r^2\}, \{s, sr, sr^2\}$. Clearly, for a 1-dimensinoal representation, the determinant equals the trace. By our above calculations, if we call the only 1-dimensional non-trivial character $\pi_2 = \chi_2$, we know that $\chi_2(e) = 1$, $\chi_2(r) = 1$ and $\chi_2(s) = -1$. If we call the 2-dimensional representation π_3, we also see that $\det(\pi_3(e)) = 1$, $\det(\pi_3(r)) = 1$ and $\det(\pi_3(s)) = -1$. Since π_2, π_3 are inequivalent irreducible representations, we have disproven the claim.

2. Recall that if $\pi : G \to \mathrm{GL}_n(\mathbb{C})$, then $\pi(g)$ is diagonalizable for any g, and that every eigenvalue of $\pi(g)$ is a root of unity. Suppose that the elements on the diagonal are $\lambda_1, \ldots, \lambda_n$ (where repretition is possible). If ever $\lambda \neq 1$, then λ has a non-zero imaginary part. If we were to have $\chi(g) = \chi(1) = n$, we would need the sum of the eigenvalues to be n. The only way to have a sum of n norm 1 numbers to be n is if all

the numbers are themselves 1, otherwise the weight of the imaginary part would take from the real part and we couldn't possibly have n as the sum. This shows that if $\chi(g) = \chi(1)$, then $\pi(g)$ is similar to the identity matrix and hence just the identity matrix, so $g \in \ker \pi$. Now, if $g \neq 1$, then g is clearly not conjugate to the identity, so by the orthogonality relations we have that

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(1)} = 0.$$

If somehow $\chi_i(g) = \chi_i(1)$ for all irreducible characters χ_i, then the above sum would evaluate to

$$\sum_{\chi_i} \chi_i(1)\overline{\chi_i(1)} = |C_G(e)| = |G|$$

which is certainly nonzero. This completes the proof.

3. Recall that if $g_1, \ldots, g_m$ are distinct representatives from each of the m left cosets of H,

$$c_G(g) = \mathrm{Ind}_H^G(c_H)(g) := \sum_{g_i^{-1}gg_i \in H} c_H(g_i^{-1}gg_i).$$

Notice first that if $xh_1x^{-1} = yh_2y^{-1}$, then we have $y^{-1}xh_1x^{-1}y = h_2$, so either $y^{-1}x \in H$ or $h_1 = h_2 = 1$. This means that $xHx^{-1} \cap yHy^{-1} = 1$ if x and y are in different left cosets of H, and if $x \in yH$, then $x^{-1}Hx = y^{-1}Hy$ so there are precisely $|G : H|$ distinct conjugacy classes of H. We proceed with the following calculation:

$$\begin{aligned}
\langle c_G, c_G \rangle &= \frac{1}{|G|}\sum_{g\in G} c_G(g)\overline{c_G(g)} = \frac{1}{|G|}\sum_{g\in G}\left(\sum_{g_i^{-1}gg_i\in H} c_H(g_i^{-1}gg_i)\overline{\sum_{g_j^{-1}gg_j\in H} c_H(g_j^{-1}gg_j)}\right) \\
&= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{g_i^{-1}gg_i\in H} c_H(g_i^{-1}gg_i)\sum_{g_j^{-1}gg_j\in H} \overline{c_H(g_j^{-1}gg_j)}\right) \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{g_igg_i^{-1}\in H \\ g_jgg_j^{-1}\in H}} c_H(g_i^{-1}gg_i)\overline{c_H(g_j^{-1}gg_j)}
\end{aligned}$$

As before, if $g_i \neq g_j$ and $g_igg_i^{-1}, g_jgg_j^{-1} \in H$ we have $g \in g_i^{-1}Hg_i \cap g_j^{-1}Hg_j = 1$ so $g = 1$ and since $c_H(1) = 0$, the only terms that contribute to the sum are those where $g_i = g_j$.

This means that the above sum is just

$$\frac{1}{|G|}\sum_{g\in G}\sum_{g_i g g_i^{-1}\in H}|c_H(g_i^{-1}gg_i)|^2$$

We want this to equal

$$\frac{1}{|H|}\sum_{h\in H}|c_H(h)|^2=\frac{1}{|H|}\sum_{h\in H}|c_H(h)|^2=\langle c_H,c_H\rangle.$$

Thus, we count the number of times $|c_H(h)|^2$ for $h\neq 1$ shows up in the first sum. We showed above that there are precisely $|G:H|$ distinct conjugacy classes of H. The above also shows that h has precisely $|G:H|$ distinct conjugates in G, namely, $g_i^{-1}hg_i$ for the g_i as above. Thus the term $|c_H(h)|^2$ shows up precisely $|G:H|$ times for each h. This shows that

$$\frac{1}{|G|}\sum_{g\in G}\sum_{g_i g g_i^{-1}\in H}|c_H(g_i^{-1}gg_i)|^2=\frac{|G:H|}{|G|}\sum_{h\in H}|c_H(h)|^2=\frac{1}{|H|}\sum_{h\in H}|c_H(h)|^2=\langle c_H,c_H\rangle.$$

We first seek to show that if c_H is any class function on H, then $\mathrm{Ind}_H^G(c_H)$ restricts to itself on H. This is true because the condition $ghg^{-1}\in H$ is equivalent to $h\in g^{-1}Hg$, and by hypothesis this is only true if $g\in H$. Thus,

$$\mathrm{Ind}_H^G(c_H)(h)=\frac{1}{|H|}\sum_{h'\in H}c_H((h')^{-1}hh')=\frac{1}{|H|}\sum_{h'\in H}c_H(h)=c_H(h).$$

Notice that if χ_1,χ_2 are two characters of H, then,

$$\begin{aligned}\mathrm{Ind}_H^G(a\chi_1+b\chi_2)&=\frac{1}{|H|}\sum_{x\,:\,xgx^{-1}\in H}(a\chi_1(xgx^{-1})+b\chi_2(xgx^{-1}))\\&=a\frac{1}{|H|}\sum_{x\,:\,xgx^{-1}\in H}\chi_1(xgx^{-1})+b\frac{1}{|H|}\sum_{x\,:\,xgx^{-1}\in H}\chi_2(xgx^{-1})=a\mathrm{Ind}_H^G(\chi_1)+b\mathrm{Ind}_H^G(\chi_2).\end{aligned}$$

Holding for any complex numbers $a,b\in\mathbb{C}$. In particular, $c_\chi=\mathrm{Ind}_H^G(\chi-d\chi_1)$ is an integer linear combination of two characters. This tells us that $c_\chi=c_1\chi_1+\sum_{i=2}^m c_i\chi_i$ for some $c_i\in\mathbb{Z}$ with the χ_i being the distinct irreducible characters of G. In the setting that χ is not the trivial character on H, and noting that restricting the trivial character clearly gives the trivial character, by Frobenius reciprocity we see that

$$\langle c_\chi, \chi_1\rangle_G = \langle \chi - d, \chi_1\rangle_H = \langle \chi, \chi_1\rangle_H - d\langle \chi_1, \chi_1\rangle_H = -d.$$

Where we recall that $\langle \chi_1, \chi_2\rangle = 0$ if χ_1, χ_2 are distinct irreducible characters. Thus $c_\chi = -d\chi_1 + \sum_{i=2}^m c_i\chi_i$. By the previous result, we know that $\|c_\chi\|_G^2 = \|\chi - d\|_H^2$. Now, since χ and the trivial character are distinct irreducibles, we clearly have that $\|\chi - d\chi_1\|_H^2 = d^2 + 1$. It follows then that $\|c_\chi\|_G^2 = d^2 + 1$. Since distinct characters are orthogonal, this tells us that $\| \sum_{i=2}^m c_i\chi_i\|_G^2 = 1$, so precisely one of the $c_i = \pm 1$ for $i \geq 2$, and the rest are 0. Recalling that the restrictiion of $\mathrm{Ind}_H^G(c_H) = c_H$ for any class function c_H on H,

$$c_\chi(1) = \chi_H(1) - d = 0$$

This shows that $c_i = 1$, since for any character χ_G of G, $\chi_G(1)$ equals the dimension of the representation, and is importantly positive. Thus we can write $c_\chi = \chi_G - \chi_G(1)$ for some irreducible representation χ_G of G. In the case where χ is the trivial character on H, we are inducing the 0 class function, so taking the trivial character on G will suffice for χ_G. Notably, we have induced a degree d irreducible character and ended up with another degree d irreducible character of G. Now define

$$N = \left\{g \in G \mid xgx^{-1} \notin H \text{ for all } x \in G\right\} \cup \{1\}.$$

Let $n \in N$. Then n is not conjugate to anything in H. In particular, if c_H is any class function, $\mathrm{Ind}_H^G(c_H)(n) = \frac{1}{|G|}\sum_{xgx^{-1}\in H} c_H(xgx^{-1}) = 0$ because the sum is empty. Conversely, if $g \neq 1 \in G$ is so that $xgx^{-1} = h \in H$ for some $x \in G$, we see first that $h \neq 1$. Thus by question 2 there is an irreducible character χ_H of H so that $\chi_H(h) \neq \chi_H(1)$. In particular, if we define $c_H = \chi_H - \chi_H(1)$, $\chi_H(h) \neq 0$. Write $c_G = \mathrm{Ind}_H^G\chi_H = \chi_G - \chi_G(1)$ for some irreducible character χ_G of G. Then,

$$c_G(h) = \sum_{g_i h g_i^{-1}} c_H(g_i h g_i^{-1})$$

For the final time, $g_i h g_i^{-1} \in H$ means $h \in g_i^{-1}Hg_i$ which is only true if $g_i = 1$. Thus the above sum evaluates to $c_H(h) \neq 0$. Since a class function on G is constant on conjugacy classes, we see that $c_G(xgx^{-1}) = c_G(g) \neq 0$. If we write π_G as the representation of χ_G, this means that $g \notin \pi_G$. Thus $N = \bigcap_{\pi_G} \ker \pi_G$, which shows that N is the intersection of normal subgroups of G and hence itself normal.

4. Let $\lambda = (2, 1)$. If we let $T_1 = \begin{array}{c|c} 1 & 2 \\ \hline 3 & \end{array}$, $T_2 = \begin{array}{c|c} 1 & 3 \\ \hline 2 & \end{array}$, we can see that the Tabloids of λ are

$$a = \{T_1\} = \left\{ \begin{array}{c|c} 1 & 2 \\ \hline 3 & \end{array}, \begin{array}{c|c} 2 & 1 \\ \hline 3 & \end{array} \right\}, c = \{T_2\} = \left\{ \begin{array}{c|c} 1 & 3 \\ \hline 2 & \end{array}, \begin{array}{c|c} 3 & 1 \\ \hline 2 & \end{array} \right\}, b = \left\{ \begin{array}{c|c} 2 & 3 \\ \hline 1 & \end{array}, \begin{array}{c|c} 3 & 2 \\ \hline 1 & \end{array} \right\}.$$

We see, noting that the only nontrivial element of the column group is (13), that $V_{T_1} = a - b$ and by similar reasoning $V_{T_2} = c - b$. We claim that span $\{V_{T_1} + V_{T_2}\}$ is a submodule of S^λ. Note that $(12)V_{T_1} = a - c$, $(12)V_{T_2} = b - c$. Thus $(12)(V_{T_1} + V_{T_2}) = a - c + b - c = a + b - 2c = a + c - 2b = V_{T_1} + V_{T_2}$, since $1 = -2$ in characteristic 3. Similarly, one calculates $(123)V_{T_1} = b - c$ and $(123)V_{T_2} = a - c$, so $(123)(V_{T_1} + V_{T_2}) = b - c + a - c = a + b - 2c = a + c - 2b = V_{T_1} + V_{T_2}$. Thus span $\{V_{T_1} + V_{T_2}\}$ is a submodule of S^λ, which shows that for this choice of λ S^λ is not irreducible. Notably, this representation collapses into the identity representation of S_3, since every element of S_3 acts as the identity on the basis elements of S^λ.