Math 522 Hw1

Rohan Mukherjee

January 22, 2025

1. Let

$$\xi_n = \begin{cases} -1 & \text{w.p. } 1 - \frac{1}{n^2} \\ n^2 - 1 & \text{w.p. } \frac{1}{n^2} \end{cases}$$

be independent. Then clearly $\mathbb{E}[\xi_n] = 0$. With $X_n = \xi_1 + \cdots + \xi_n$, we know that X_n is a martingale. I claim that $\mathbb{P}\left(\limsup_{n\to\infty} X_n = -\infty\right) \geq 1/2$. This is because $\limsup_{n\to\infty} X_n = -\infty$ occurs certainly whenever all the ξ_n are -1. Thus, by independence:

$$\mathbb{P}\left(\limsup_{n\to\infty} X_n = -\infty\right) \geq \mathbb{P}\left(\bigcap_{n=1}^{\infty} \{\xi_n = -1\}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$$

Now the event $\{\limsup X_n = -\infty\}$ is tail measurable, since if we throw out the first m many terms, the limsup will still be $-\infty$. Thus, by Kolmogorov 0-1 law, we know that $\mathbb{P}(\limsup X_n = -\infty) = 0$ or 1. But we have shown that $\mathbb{P}(\limsup X_n = -\infty) \geq 1/2$, so we must have $\mathbb{P}(\limsup X_n = -\infty) = 1$.

2. We know that $X_n = \prod_{m\leq n} Y_n$ is a non-negative martingale, thus by Martingale convergence theorem, $X_n \to X$ a.s. with X finite a.s., as $\mathbb{E}[X] \leq \mathbb{E}[X_0] = 1$. In particular, X_n is Cauchy in the following sense:

$$\mathbb{P}(|X_{n+1} - X_n| > \eta) \to 0$$

This is beacuse $\eta \leq |X_{n+1} - X_n|$ implies $\eta \leq |X_n - X| + |X_{n+1} - X|$ which in turn implies $\eta/2 \leq |X_n - X|$ or $\eta/2 \leq |X_{n+1} - X|$. Since $X_n \to X$ almost surely, it converges in probability, so it is easy to see that those last 2 events converge to 0 as $n \to \infty$. Thus the first event must also converge to 0. Since $\mathbb{P}(Y_m = 1) < 1$, and $\mathbb{E}[Y_m] = 1$, we can find some $\delta > 0$ so that $\mathbb{P}(Y_m > 1 - \delta) > 0$. This is because otherwise $Y_m \geq 1$ almost surely, and $\mathbb{P}(Y_m > 1) > 0$, so $Y_m - 1$ is a non-negative r.v. with positive expectation, a contradiction.

3. First, we prove that if $y_n > -1$ for all n and $\sum |y_n| < \infty$, then $\prod_{m=1}^{\infty}(1 + y_m)$ exists. By Taylor's theorem for remainders, $\log(1 + x) = 0 + \frac{1}{1+\zeta}x$ for some $\zeta \in (x, 0)$ for $x < 0$ and $(0, x)$ for $x > 0$. Since $|y_n| \to 0$ since the series converges, we know that $|y_n| < 1/2$ eventually, and since $y_n > -1$, this means that $y_n > -1/2$ eventually. For these values, we then have $|\log(1 + y)| \leq 2|y|$ by the calculation above. Thus, for n sufficiently large:

$$\left|\sum_{m \geq n} \log(1 + y_m)\right| \leq 2 \sum_{m \geq n} |y_m| \to 0$$

So $\sum \log(1 + y_m)$ converges as it is cauchy, so $\exp(\sum \log(1 + y_m)) = \prod(1 + y_m)$ converges.

Now define a new r.v.

$$Z_n = \frac{X_n}{\prod_{m \leq n-1}(1 + Y_m)}$$

where the empty product has value 1. Then:

$$\mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) = \frac{1}{\prod_{m \leq n}(1 + Y_m)} \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \leq \frac{1}{\prod_{m \leq n}(1 + Y_m)}(1 + Y_n)X_n = \frac{X_n}{\prod_{m \leq n-1}(1 + Y_m)} = Z_n$$

so Z_n is a non-negative supermartingale, thus it converges almost surely. Since $\sum Y_n \leq \infty$ a.s., the product $\prod_{m \leq n}(1 + Y_m)$ converges a.s. so $Z_n \cdot \prod_{m \leq n}(1 + Y_m) = X_n$ converges a.s. which completes the proof.

4. Let X_n^1, X_n^2 be supermartingales adapted to $\mathcal{F}_n$ and let $N = \inf\left\{m : X_m^1 \geq X_m^2\right\}$. Then N is a stopping time, and let:

$$Y_n = X_n^1 1_{N > n} + X_n^2 1_{N \leq n}$$
$$Z_n = X_n^1 1_{N \geq n} + X_n^2 1_{N < n}$$

We claim that Y_n, Z_n are supermartingales. First, we show that $Y_n \leq Z_n$ everywhere. This is because when $N > n$, $Y_n = X_n^1$ and $Z_n = X_n^1$, when $N < n$, $Y_n = X_n^2$ and $Z_n = X_n^2$, and when $N = n$, $X_n^1 \geq X_n^2$ by definition of N, and $Y_n = X_n^2$ while $Z_n = X_n^1$, so $Y_n \leq Z_n$

in all cases.

Now,

$$\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) \leq \mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) = \mathbb{E}\Big(X^1_{n+1} 1_{N \geq n+1} + X^2_{n+1} 1_{N < n+1} \mid \mathcal{F}_n\Big)$$

From here, $1_{N \geq n+1}$ and $1_{N<n+1}$ are $\mathcal{F}_n$ measurable, since the first event is the complement of $N \leq n$ and the second is equal to $N \leq n$. Thus we can take them out of the conditional expectation, getting:

$$\mathbb{E}\Big(X^1_{n+1} \mid \mathcal{F}_n\Big) 1_{N \geq n+1} + \mathbb{E}\Big(X^2_{n+1} \mid \mathcal{F}_n\Big) 1_{N<n+1} \leq X^1_n 1_{N>n} + X^2_n 1_{N \leq n} = Y_n \leq Z_n$$

where we used that X^1_n and X^2_n are supermartingales in the middle step, and just rewrote the indices of the indicator functions. By then re-using that $Y_n \leq Z_n$, we get that both Y_n, Z_n are supermartingales when all the equations are put together.

5. We can obviously write:

$$\begin{aligned} Z^{2j-1}_n &= 1_{N_1>n} + (X_n/a) 1_{N_1 \leq n < N_2} \\ &+ (b/a) 1_{N_2 \leq n < N_3} + (b/a)(X_n/a) 1_{N_3 \leq n < N_4} \\ &\vdots \\ &+ (b/a)^{j-1} 1_{N_{2j-2} \leq n < N_{2j-1}} + (b/a)^{j-1} (X_n/a) 1_{N_{2j-1} \leq n} \end{aligned}$$

and

$$\begin{aligned} Z^{2j}_n &= 1_{N_1>n} + (X_n/a) 1_{N_1 \leq n < N_2} \\ &+ (b/a) 1_{N_2 \leq n < N_3} + (b/a)(X_n/a) 1_{N_3 \leq n < N_4} \\ &\vdots \\ &+ (b/a)^{j-1} 1_{N_{2j-2} \leq n < N_{2j-1}} + (b/a)^{j-1} (X_n/a) 1_{N_{2j-1} \leq n < N_{2j}} \\ &+ (b/a)^j 1_{N_{2j} \leq n} \end{aligned}$$

We prove that Z^j_n is a supermartingale by induction. First, $Z^1_n = 1_{N_1>n} + (X_n/a) 1_{N_1 \leq n}$. Let X^1_n be the supermartingale (constant) 1, and $X^2_n = X_n/a$. Then the stopping time N that is $1 \geq X_n/a$ is just the first time $X_n \leq a$, which is precisely N_1. Thus by switching principle, Z^1_n is a supermartingale. Then clearly, $Z^{2j}_n = 1_{N_{2j}>n} Z^{2j-1}_n + (b/a)^j 1_{N_{2j} \leq n}$.

We now investigate the first time when $Z^{2j-1}_n \geq (b/a)^j$. If $n < N_{2j-1}$, then precisely one term of the form $(b/a)^\ell$ or $(b/a)^{\ell-1} X/a$ is non-zero for $\ell \leq j-1$. Then the condition would become either $(b/a)^\ell \geq (b/a)^j$, obviously impossible, or $(b/a)^{\ell-1} X_n/a \geq (b/a)^j$, which is also impossible since in this case $N_{2\ell-1} \leq n < N_{2\ell}$, and the condition would become $X_n \geq a(b/a)^j (a/b)^{\ell-1} \geq b$, but this would mean that we would already be over b, which contradicts the definition of $N_{2\ell}$ since we should be in between an upcrossing. Thus the only way this can happen is if $N_{2j-1} \leq n$. The term would becomme $(b/a)^{j-1} X_n/a \geq (b/a)^j$, which is equivalent to $X_n \geq b$, which is precisely how N_{2j} is defiend. Thus N_{2j} is the stopping time in the switching theorem with $X_n^1 = Z_n^{2j-1}$ and $X_n^2 = (b/a)^j$, so Z_n^{2j} is a supermartingale as well.

Similarly, $Z_n^{2j+1} = 1_{N_{2j+1} > n} Z_n^{2j} + (b/a)^j (X/a) 1_{N_{2j+1} \leq n}$, and the same logic applies and the only way $Z_n^{2j} \geq (b/a)^j (X/a)$ is when $N_{2j} \leq n$ and in this case the only term that shows up is $(b/a)^j$, so the condition becomes equivalent to $(b/a)^j \geq (b/a)^j (X/a)$, which is thus the first time after N_{2j} that $X_n \leq a$, which is how N_{2j+1} is defined. Thus Z_n^{2j+1} is a supermartingale as well.

From martingale convergence theorem, we know that:

$$\mathbb{E}(Y_{n \wedge N_{2k}}) \leq \mathbb{E}(Y_0)$$

We claim that $Y_0 = X_0/a \wedge 1$. This is because if $X_0 \leq a$ then N_1 is 0, and Y_0 will be X_0/a, otherwise $X_0 > a$ and $X_0 = 1$. In conclusion,

$$\left(\frac{b}{a}\right)^{2k} \mathbb{P}(N_{2k} \leq n) = \mathbb{E}(Y_{n \wedge N_{2k}} 1_{N_{2k} \leq n}) \leq \mathbb{E}(Y_{n \wedge N_{2k}}) \leq \mathbb{E}(X_0/a \wedge 1)$$

Sending $n \to \infty$ shows that:

$$\mathbb{P}(N_{2k} < \infty) \leq \left(\frac{a}{b}\right)^{2k} \mathbb{E}(1 \wedge X_0/a)$$

Since $N_{2k} < \infty$ iff the number of upcrossings is at least k, we conclude Dubins that:

$$\mathbb{P}(U \geq k) \leq \left(\frac{a}{b}\right)^{2k} \mathbb{E}(1 \wedge X_0/a)$$

which completes the proof. $\blacksquare$