Math HW5

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1. We need to show that:

$$\max_{\pi \in S_n} \sum_{i=1}^n \lambda_{\pi(i)}^2 + A_{ii}^2 - 2\lambda_{\pi(i)} A_{ii} \geq \frac{1}{n-1} \sum_{i=1}^n R_i^2$$

Then notice that

$$\frac{1}{n-1} \sum_{i=1}^n R_i^2 = \frac{1}{n-1} \sum_{i=1}^n \left(\sum_{j \neq i} |A_{ij}| \right)^2 \leq \sum_{i=1}^n \sum_{j \neq i} |A_{ij}|^2 = \|A\|_F^2 - \sum_{i=1}^n A_{ii}^2 = \sum_{i=1}^n \lambda_i^2 - \sum_{i=1}^n A_{ii}^2$$

So a strictly stronger inequality is to show that:

$$\sum_{i=1}^n \lambda_i^2 - \sum_{i=1}^n A_{ii}^2 \leq \max_{\pi \in S_n} \sum_{i=1}^n \lambda_{\pi(i)}^2 + A_{ii}^2 - 2\lambda_{\pi(i)} A_{ii}$$

Or equivalently,

$$0 \leq \max_{\pi \in S_n} \sum_{i=1}^n 2A_{ii}^2 - 2\lambda_{\pi(i)} A_{ii}$$

Put uniform measure on S_n. Then,

$$E_\pi \left(\sum_{i=1}^n \lambda_{\pi(i)} A_{ii} \right) = \sum_{i=1}^n A_{ii} E_\pi(\lambda_{\pi(i)})$$

For a fixed i, $\pi(i)$ is uniformly distributed in $[n]$. Thus this equals:

$$\sum_{i=1}^n A_{ii} \frac{1}{n} \left(\sum_{j=1}^n \lambda_j \right) = \frac{1}{n} \operatorname{Tr}(A)^2 = \frac{1}{n} \left(\sum_{i=1}^n A_{ii} \right)^2 \leq \sum_{i=1}^n A_{ii}^2$$

Since it happens on average, there exists permutation $\pi \in S_n$ that makes it happen. This completes the proof.

2. This is false. For any starting matrix A, consider $B = -A$. Then $1/2A + 1/2B = 0$, whose largest eigenvalue 0 is not simple as long as $n \geq 2$.

3. Since λ_{n-1} is closer to λ_n than λ_i for $i \leq n-1$, we have that:

$$\lambda_n' \leq \frac{n-1}{\lambda_n - \lambda_{n-1}}$$

By dropping all the positive terms, we also have that:

$$\lambda_{n-1}' \geq \frac{1}{\lambda_{n-1} - \lambda_n}$$

Thus,

$$\lambda_n' - \lambda_{n-1}' \leq \frac{n}{\lambda_n - \lambda_{n-1}}$$

Let $v = \lambda_n - \lambda_{n-1}$. The problem tells us that $v > 0$ always. So,

$$vv' \leq n$$

Integrating both sides,

$$v^2(t)/2 - v^2(0)/2 = \int_0^t vv'dx \leq \int_0^t ndx = nt$$

For $t \geq T$ for some T depending only on $v(0)$, we have $\lambda_n - \lambda_{n-1} = v(t) \leq 2\sqrt{nt}$ (More specifically, for $t \geq v^2(0)/2n = T$, $v^2(0)/2 \leq nt$). We seek to repeat this process for all the other gaps. By dropping the negative term, and using that λ_{n-2} is closer to λ_{n-1} than λ_i for $i \leq n-2$, we have that:

$$\lambda_{n-1}' \leq \frac{n-2}{\lambda_{n-1} - \lambda_{n-2}}$$

By a similar logic, we have that (recall that by making the denominator smaller, we are pushing the fraction far in the negative direction, giving a better lower bound):

$$\lambda_{n-2}' \geq \frac{2}{\lambda_{n-2} - \lambda_{n-1}}$$

As before this means that $\lambda_{n-1} - \lambda_{n-2} \leq 2\sqrt{nt}$ for T sufficiently large depending only on the initial conditions.

Then, (up to some constant depending only on $\lambda_i(0)$ for each i):

$$\lambda_n - \lambda_1 = \sum_{i=1}^{n} \lambda_i - \lambda_{i-1} \leq 2n^{3/2}\sqrt{t}$$

Now,

$$\lambda_n' \geq \frac{1}{\lambda_n - \lambda_1} \geq \frac{1}{2n^{3/2}\sqrt{t}}$$

Simutlaenously,

$$\lambda_1' \leq \frac{1}{\lambda_1 - \lambda_n} \leq \frac{-1}{2n^{3/2}\sqrt{t}}$$

Integrating these inequalities shows that (up to another constant depending on $\lambda_i(0)$ and $\lambda_n(0)$):

$$\lambda_1 \leq \frac{-C}{n^{3/2}}\sqrt{t} \leq \frac{C}{n^{3/2}}\sqrt{t} \leq \lambda_n \quad \text{for } t \geq T$$

where T depends only on the $\lambda_i(0)$. Indeed, in particular, the solutions are unbounded (also this is better than $\log t$).

4. There are two answers here, one that is more fun than the other. The first is to just consider $X = 0$. Since A has an eigenvalue 1 with multiplicity 2 it must not be 0. Then $A + \varepsilon X = A$ also has eigenvalue 1 with multiplicity 2 always. The other answer is to consider the Jordan normal form of the matrix A:

$$A = P \begin{pmatrix} 1 & 1 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix} P^{-1}$$

From here, taking

$$X = P \begin{pmatrix} 0 & 1 & \mathbf{0} \\ 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix} P^{-1}$$

Works for all $\varepsilon > 0$. You can just read the eigenvalues off the main diagonal for the 2×2

upper triangular block in the top left.