Math 504 HW7

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1. We claim that if $\mathcal{C}$ is a conjugacy class of G, and $H \trianglelefteq G$, then either $\mathcal{C} \subset H$ or $\mathcal{C} \cap H = \emptyset$. If there is an $x \in \mathcal{C} \cap H$, then $gxg^{-1} \in H$ as well, and since every $y \in \mathcal{C}$ is of the form gxg^{-1}, this shows that $\mathcal{C} \subset H$. Since $G = Z(G) \cup \mathcal{C}_{g_1} \cup \cdots \cup \mathcal{C}_{g_r}$ where $g_1, \ldots, g_r$ are the representatives of the distinct noncentral conjugacy classes of G, we have that $H = (Z(G) \cap H) \cup \mathcal{C}_{g_1} \cup \cdots \cup \mathcal{C}_{g_s}$, where we have potentially re-ordered the g_i to be so that $\mathcal{C}_{g_i} \subset H$ for all $1 \leq i \leq s$ and $\mathcal{C}_{g_i} \cap H = \emptyset$ for all $s+1 \leq i \leq r$. The above is a disjoint union, since the $\mathcal{C}_{g_i}$ are disjoint and each were disjoint from $Z(G)$, thus they must be disjoint from $Z(G) \cap H$. We have deduced that

$$|H| = |Z(G) \cap H| + \sum_{i=1}^{s} |\mathcal{C}_{g_i}|$$

We recall that the orbit of the conjugacy class containing g_i is just $|G : C_G(g_i)|$, and since $g_i \notin Z(G)$ by hypothesis we must have $|G : C_G(g_i)| > 1$. Since G is a p-group, $p \mid |G : C_G(g_i)|$, and since $p \mid |H|$ as well, p divides $|H| - \sum_{i=1}^{s} |\mathcal{C}_{g_i}| = |Z(G) \cap H|$, which shows that $Z(G) \cap H \neq \langle 1 \rangle$.

2. We claim that $\langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle \cong Z_4 \rtimes Z_4$. Clearly $\langle a^4 \rangle = \{ e, a, a^2, a^3 \}$, since free groups are defined on words, and similarly $\langle b^4 \rangle = \{ e, b, b^2, b^3 \}$. Since $a \neq b$, it follows immediately that $\langle a \rangle \cap \langle b \rangle = \langle 1 \rangle$. We also claim that $\langle a \rangle \trianglelefteq \langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$. We need only show that every element is in the normalizer of a, since then we could show it by induction. Elements of the form a^α commute with a, and $b^\beta a b^{-\beta} = a^{-\beta}$, so every word in our free group will normalize $\langle a \rangle$. Since $\langle a \rangle \cong Z_4$ and $\langle b \rangle \cong Z_4$ we may use Theorem 12 in chapter 4 to conclude the above claim.