Math Template

Rohan Mukherjee

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1. Writing

$$A = \begin{pmatrix} a_1^T \\ \vdots \\ a_n^T \end{pmatrix}$$

we can see that,

$$\left\| A \frac{a_i}{|a_i|} \right\| = \frac{1}{||a_i||} \sum_{j=1}^{n} (a_j^T a_i)^2 \geq |a_i|$$

Since this holds for every row, $||A||_{op} \geq \sup_{1 \leq i \leq m} \left(\sum_{j=1}^{n} |A_{ij}|^2 \right)$.

For $||x|| = 1$,

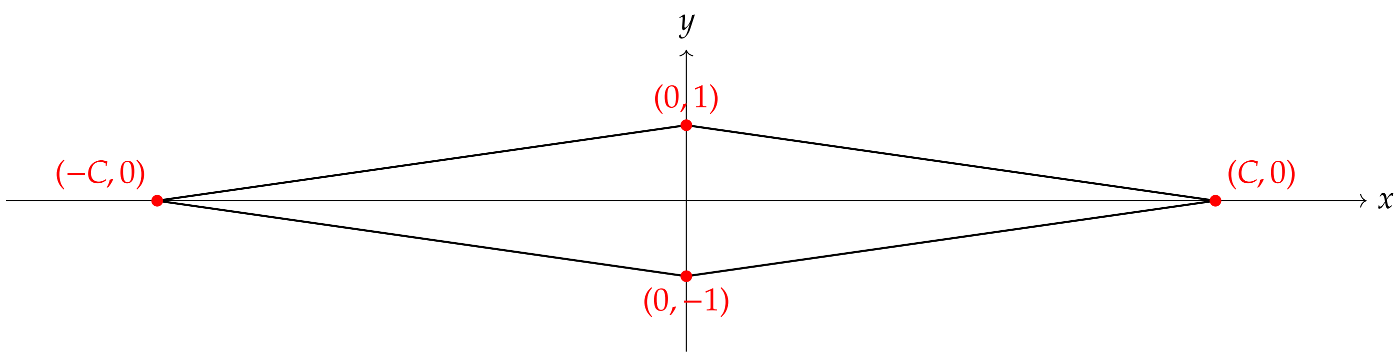
$$||ABx|| = ||Bx|| \cdot ||A(Bx/||Bx||)|| \leq ||Bx|| \cdot ||A||_{op} \leq ||A||_{op} ||B||_{op}$$

Notice from that last part,

$$||ABx|| \leq ||Bx|| \cdot ||A||_{op}$$

Now, $\sup_{||x||=1} ||Bx||^2 = \sup_{||x||=1} x^T B^T B x$ is the largest eigenvalue of $B^T B$, say λ. Recall that $||B||_F^2$ can be written as $\text{Tr}\left(B^T B\right)$. By properties of trace, $\text{Tr}\left(B^T B\right) = \sum_i \lambda_i$ where λ_i is the ith largest eigenvalue of $B^T B$ (breaking ties arbitrarily). Then clearly, since $B^T B$ is PSD, $\lambda_1 \leq \sum \lambda_i$. This shows that $||B||_{op} \leq ||B||_F$. Using that above completes the proof.

2. Consider any centrally symmetric convex set Ω, which is compact, and let B be the largest ellipse contained in Ω. Via an affine transformation, we can ensure that B is the unit ball. Let $C = \inf\{t > 0 \mid \Omega \subset tB\}$. Assume by contradiction that C were large. Then there is some point of Ω of magnitude at least bigger than C. By connecting this point to the origin, we can find a point in Ω with norm precisely equal to C. By symmetry, the opposite of this point is also in Ω, and after a rotation, we may assume that these two points are $(\pm C, 0)$. By connecting the top and bottom of the unit circle with these points, we get the following shape fully contained in Ω:



Now the idea is simply to show that there is a ellipse other than the unit ball that has larger volume contained in this smaller shape. Consider the ellipse:

$$\frac{x^2}{C^2} + y^2 = \frac{1}{2}$$

It just might be that these coefficients were specifically chosen to make this algebra work better. To show that this ellipse is fully contained in our shape, we need only consider $0 \le x \le C$, and then show that all the points the boundary of that ellipse are below the line $y = -1/Cx + 1$ in the first quadrant. Plugging in, we need to show that, if (x, y) is a point on the ellipse, then:

$$y \le (-1/C)x + 1$$

This is equivalent to (since all the values are positive):

$$y^2 \le \frac{x^2}{C^2} - \frac{2}{C}x + 1$$

plugging in the value for y,

$$\begin{aligned}
&\frac{1}{2} - \frac{x^2}{C^2} \le \frac{x^2}{C^2} - \frac{2}{C}x + 1 \\
\iff\ &0 \le \frac{2x^2}{C^2} - \frac{2}{C}x + \frac{1}{2} = \frac{(C-2x)^2}{2C^2}
\end{aligned}$$

This shows that the ellipse is fully contained in the set. Rewriting the ellipse in a familiar form:

$$\frac{x^2}{(C/\sqrt{2})^2} + \frac{y^2}{(1/\sqrt{2})^2} = 1$$

The area of this ellipse becomes $\pi C/2$. So we have a larger ellipse contained in Ω whenever $C > 2$, a contradiction. Thus $C = 2$ suffices.

3. Consider $g(x) = \frac{1}{2}x^T Ax + b^T x$. By compactness g has a maxmium on the sphere S^{n-1}, say y. The only way that $g(y)$ could be maximal is if the gradient of g were perpendicular to the sphere, because otherwise we could walk in a projected direction and increase our value. The perpendicular to the sphere is precisely y, so we need $\nabla g(y) = \lambda y$ for some y. Notice that $\nabla g(x) = \frac{1}{2}(A + A^T)x + b$, and since A is symmetric, we get $\nabla g(x) = Ax + b$. Thus we have found a vector y with $Ay + b = \lambda y$ as desired.

4. We prove by Rolle's theorem. Since $p(x)$ is d dimensional and d distinct roots, we know that $p'(x)$ has $\le d-1$ distinct roots, since it has dimension $d-1$. Labeling the roots of $p(x)$ as $x_1, \dots, x_d$, we know by Rolle's theorem that since $p(x_i) = p(x_{i+1})$, there is some y_i so that $p'(y_i) = 0$. Then $x_1 < y_1 < \cdots < y_{d-1} < x_d$. This gives $d-1$ distinct roots for $p'(x)$, so these must be all the roots, and we are done.