CSE 525 UW

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Problem 1.

Fix $p > 0$. Let S be our independent set. First, for each vertex v, indepentently add it to S with probability p. Then for each edge appearing in the subgraph with vertices in S, remove one of the vertices uniformly at random. This clearly gives an independent set as we have made all edges disappear. Now, $E[\#\text{ added}] = np$, and $E[\#\text{ edges}] = p^3 m$. As we remove at most one vertex for each edge in the subgraph with vertices in S, $E[|S|] = E[\#\text{ added} - \#\text{ removed}] \geq E[\#\text{ added} - \#\text{ edges}] = np - mp^3$. Setting $p = \sqrt{\frac{n}{3m}}$ yields an independent set of size

$$E[|S|] \geq \frac{n^{3/2}}{\sqrt{3m}} - \frac{mn^{3/2}}{3^{3/2}m^{3/2}} = O\left(\frac{n^{3/2}}{\sqrt{m}}\right)$$

Problem 2.

Let E be the event that G has an isolated vertex. Enumerate the vertices of G as $V = \{v_1, \ldots, v_n\}$. Then let $X_i = 1[v_i \text{ is isolated}]$. Since each edge appears with probability p, $E[X_i] = (1-p)^{n-1}$. Now, $E(X) = n(1-p)^{n-1}$. Then, $\mathrm{Cov}(X_i, X_j) = E(X_iX_j) - E(X_i)^2$. The first is $P(i, j \text{ isolated}) = (1-p)^{2(n-2)+1}$. On the other hand, by our previous calculation $E(X_i)^2 = (1-p)^{2(n-1)} = (1-p)^{2(n-2)+2}$. Then $\mathrm{Cov}(X_i, X_j) = (1-p)^{2(n-2)}((1-p) - (1-p)^2) \leq p(1-p)^{2(n-2)}$. This shows that for $X = \sum_i X_i$, we have that:

$$\mathrm{Var}(X) \leq n(1-p)^{n-1} + n^2 p(1-p)^{2(n-2)}.$$

Since $E(X)^2 = n^2(1-p)^{2(n-1)}$, we have that,

$$P(X = 0) \leq \frac{\mathrm{Var}(X)}{E(X)^2} \leq \frac{n(1-p)^n + n^2 p(1-p)^{2n}}{n^2(1-p)^{2n}}$$

Using that for small p, $1 - x \approx e^{-x}$, and letting $p = \frac{\log n}{2n}$, we have that:

$$P(X = 0) \leq \frac{n(1-p)^n + n^2 p(1-p)^{2n}}{n^2(1-p)^{2n}}$$
$$= \frac{ne^{-n\log n/2n} + n^2 \frac{\log n}{2n} e^{-2n\log n/2n}}{n^2 e^{-2n\log n/2n}} = \frac{\sqrt{n} + 1/2\log n}{n} = O\left(\frac{1}{\sqrt{n}}\right).$$

Thus $P(G \text{ disconnected}) \geq P(X \geq 1) = 1 - O\left(\frac{1}{\sqrt{n}}\right)$.

For the second part of the problem, recall that a graph is connected iff every cut has an edge. Let (S, S^c) be an arbitrary cut with $|S| = k$. The number of possible edges between S and S^c is $k(n-k)$. Thus $P(E(S) = \emptyset) = (1-p)^{k(n-k)}$. Now,

$$P(G \text{ disconnected}) = P(\exists S \text{ s.t. } E(S) = \emptyset) \leq \sum_{k=1}^{n/2} \binom{n}{k}(1-p)^{k(n-k)} \leq \sum_k \left(\frac{ne}{k}\right)^k e^{-pk(n-k)}$$

Now let $p = \frac{3\log n}{n}$. We get:

$$\sum_k \left(\frac{ne}{k}\right)^k e^{-pk(n-k)} = \sum_k \left(\frac{ne}{k}\right)^k n^{-3k(1-\frac{k}{n})} = \sum_k \left(\frac{e}{k}\right)^k n^{-2k+\frac{3k^2}{n}}$$

Now, n^x is maximized when x is maximized. As $-2k + 3k^2/n$ is a convex quadratic, it is maximized at the bounadry, so either $k = 1$ or $n/2$. A simple check shows that eventually $k = 1$ yields the bigger value of $-2 + 3/n$. Thus,

$$P(G \text{ disconnected}) \leq \sum_k \left(\frac{e}{k}\right)^k \cdot n^{3/n} \cdot n^{-2}$$

Now, $n^{3/n} \to 1$ so it is bounded and $\sum_{k=1}^{n/2} \left(\frac{e}{k}\right)^k \leq \sum_{k=1}^{\infty} \left(\frac{e}{k}\right)^k < \infty$. Thus, we have that $P(G \text{ disconnected}) = O(n^{-2})$. So G is connected with probability, $1 - O(n^{-2})$.