Fun facts about rotation matrices

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Here is a fun proof of the trig addition identities. We know that rotating the vector (x, y) by θ degrees is given by

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

This can be found by drawing the points $(1, 0)$ and $(0, 1)$ and then sliding these points on the unit circle counterclockwise by θ degrees, and then describing the new x, y coordinates with $\sin(\theta)$ and $\cos(\theta)$.

We know that if we want to rotate by $\theta + \gamma$ degrees we could use the formula above, or we could first rotate by θ degrees and then by γ degrees, whose matrix is given by multiplying the matrix that rotates by θ degrees by the matrix that rotates by γ degrees, given by the formula above. So, putting it all together:

$$\begin{aligned} \begin{pmatrix} \cos(\theta+\gamma) & -\sin(\theta+\gamma) \\ \sin(\theta+\gamma) & \cos(\theta+\gamma) \end{pmatrix} &= \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta)\cos(\gamma) - \sin(\theta)\sin(\gamma) & -\cos(\theta)\sin(\gamma) - \sin(\theta)\cos(\gamma) \\ \sin(\theta)\cos(\gamma) + \cos(\theta)\sin(\gamma) & -\sin(\theta)\sin(\gamma) + \cos(\theta)\cos(\gamma) \end{pmatrix} \end{aligned}$$

Finally, equating the $(1, 1)$ coordinate and the $(1, 2)$ coordinate yields the familiar formulae: $\sin(\theta + \gamma) = \sin(\theta)\cos(\gamma) + \cos(\theta)\sin(\gamma)$, and $\cos(\theta + \gamma) = \cos(\theta)\cos(\gamma) - \sin(\theta)\sin(\gamma)$. A fun corollary of this geometric interpretation is that the special orthogonal group of 2x2 matrices, namely, SO(2) must be abelian, as there is no difference between rotating first by θ degrees and then γ, versus rotating by γ degrees and then θ.

Theorem 0.1. *$p\mathbb{Z}/p^m\mathbb{Z}$ is a nilpotent ideal in $\mathbb{Z}/p^m\mathbb{Z}$, where p is prime.*

Proof. For simplicity let $p\mathbb{Z}/p^m\mathbb{Z} = M$. We see that, everything in M^2 is of the form $\sum a_i b_i$. Then everything in M^3 is of the form $\sum c_j \sum a_i b_i = \sum\sum a_i b_i c_j$, and in general everything in M^n is of the form $\sum\sum \dots \sum a_1 a_2 \cdots a_n$, where there are $n - 1$ sums. Because p is prime, the only elements of $p\mathbb{Z}/p^m\mathbb{Z}$ are $0, p, 2p, \dots, p^{m-1}$. So there are p^{m-1} elements of M. Note also that as every element of M is of the form pn, where $n \in \mathbb{Z}/p^m\mathbb{Z}$, we see that $(pn)^m = p^m n^m = 0$, as $p^m = 0$. Finally, note that every element of $M^{p^{m-1}\cdot m}$ is the sum of $p^{m-1} \cdot m$ factors, and by the pigeonhole principle there must be $p^{m-1} \cdot m/p^{m-1}$ elements that are the same in this factorization, as there are only p^{m-1} elements of M. Then there is necessarily a 0 in this factorization, and we are simply summing up a bunch of 0's. So 0 is the only element in $M^{p^{m-1}m}$, and we are done. $\square$