CSE 311 Template

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1. Suppose for the sake of contradiction that Σ is regular. Then there is some DFA M that recognizes L.

   let S be $\{0^n 1^n \mid n \geqslant 0\}$.

   Because the DFA is finite, and S is infinite, there are two strings $x \neq y \in S$ so that x, y go to the same state when read by M. So we see that $x = 0^a 1^a$ for some $a \geqslant 0$, and $y = 0^b 1^b$ for some $b \geqslant 0$. WLOG $a > b$.

   Consider the string $z = 0^a$. We notice that $xz = 0^a 1^a 0^a$, and that $xy = 0^b 1^b 0^a$, so xz is of the form in Σ, while xy is not of the form in Σ.

   Since $xz \in \Sigma$ while $yz \notin \Sigma$, M does not recognize L. But that's a contradiction! So Σ must be an irregular language.

6. Let $y \in f(A \cap B)$ be an arbitrary element in the image. By definition of the image, we see that there exists an $x \in A \cap B$ so that $f(x) = y$. Then clearly $x \in A$ and $x \in B$, which means that $f(x) \in f(A)$ by definition and at the same time $f(x) \in f(B)$. By the definition of intersection, $f(x) = y \in f(A) \cap f(B)$. As y was an arbitrary element of $f(A \cap B)$, we see that $f(A \cap B) \subseteq f(A) \cap f(B)$.

7. For a Husky Tree T, let $P(T) :=$ "if T has a purple root, then it has an even number of leaves $\wedge$ if T has a gold root, then it has an odd number of leaves."

   Base case: If T is a single gold node, then it clearly has an odd number of leaves (it has only 1 leaf, which is itself). As T is does not have a purple root, we see the other part of P holds vacuously, so $P(T)$ holds.

   Inductive Hypothesis: Let T_1, T_2 be arbitrary Husky Trees, and suppose $P(T_1)$ and $P(T_2)$.

   Inductive Step: There are three cases (without loss of generality): both T_1 and T_2 have gold roots, both T_1 and T_2 have purple roots, or T_1 has a gold root and T_2 has a purple root.

   In the first case, by the recursive rule the new husky tree has a purple root with T_1 and T_2 as it's children. By our inductive hypothesis, T_1 and T_2 both have an odd number of leaves, and as odd + odd = even, the new tree has an even number of leaves. As this new tree does not have a gold root, the second part of P holds vacuously, so we see $P(\text{newtree})$ holds.

In the second case, by the recursive rule the new tree would be one with a purple root and T_1, T_2 as its children. As in this case T_1 and T_2 have purple nodes, and as $P(T_1)$ and $P(T_2)$ hold true, we see that T_1 and T_2 both have an even number of leaves. The number of leaves of the new tree is equal to the sum of the number of leaves of T_1 and T_2, which would be even + even, which is of course even. As this new tree does not have a gold root, the second part of P holds vacuously, So $P(\text{newtree})$ holds in this case.

Finally, in the last case T_1 has a gold root and T_2 has a purple root. The new tree would have a gold root and T_1 and T_2 as it's children. By the inductive hypothesis, T_1 has an odd number of leaves and T_2 has an even number of leaves. So the new tree would have odd + even = odd number of leaves. As the new tree does not have a purple root, the second part of P holds vacuously, which shows $P(\text{newtree})$.

Therefore, $P(T)$ holds for every husky tree T by structural induction.

8. For a positive integer n, let $P(n) \coloneqq \sum_{k=1}^{n} 4k - 3 = n(2n-1)$. We show $P(n)$ for all $n \geqslant 1$ by induction.

   Base case: $\sum_{k=1}^{1} 4k - 3 = 4 \cdot 1 - 3 = 1 = 1(2 \cdot 1 - 1)$, $P(1)$ holds.

   Inductive Hypothesis: Suppose $P(w)$ for an arbitrary $w \geqslant 1$.

   Inductive Step: We notice that

$$\begin{aligned}
\sum_{k=1}^{w+1} 4k - 3 &= \sum_{k=1}^{w} 4k - 3 + 4(w+1) - 3 \\
&= w(2w-1) + 4w + 4 - 3 && \text{(Inductive Hypothesis)} \\
&= 2w^2 - w + 4w + 1 \\
&= 2w^2 + 3w + 1 \\
&= (w+1)(2w+1) \\
&= (w+1)(2(w+1)-1)
\end{aligned}$$

   Which shows $P(w+1)$. So we conclude that $P(n)$ holds for all $n \geqslant 1$.