Groups of order $n \leq 10$

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Groups of order 1

This is just the trivial group $\langle 1 \rangle$.

Groups of order 2, 3, 5, 7

For prime p there is only one group up to isomorphism, namely $\mathbb{Z}/p$. So for $n = 2, 3, 5, 7$ we get $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/5$ and $\mathbb{Z}/7$ respectively.

Groups of order 4, 9

Every group of order p^2 is abelian. From homework 7, we showed that if $H \trianglelefteq G$, and $|G| = p^n$, then $G \cap Z(G) \neq \langle 1 \rangle$. Taking $H = G$ shows that $G \cap Z(G) = Z(G) \neq \langle 1 \rangle$, so $Z(G) = p$ or p^2. In the second case we are done, in the first $G/Z(G) \cong \mathbb{Z}/p$ which is cyclic so G is abelian.
By fundamental theorem, $G \cong \mathbb{Z}/p^2$ or $\mathbb{Z}/p \times \mathbb{Z}/p$. So for $n = 4$ we have $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, and for $n = 9$ we get $\mathbb{Z}/9$ and $\mathbb{Z}/3 \times \mathbb{Z}/3$.

Groups of order 6, 10

Homework 5 presentation problem P.2 shows that for a group of order pq with $p < q$ and $p \mid q - 1$, the only two groups of order pq up to isomorphism are $\mathbb{Z}/pq$ and $\mathbb{Z}/q \rtimes \mathbb{Z}/p$ where the automorphism is nontrivial. So $n = 6$ gives us $\mathbb{Z}/6$ and $\mathbb{Z}/3 \rtimes \mathbb{Z}/2 \cong D_3 \cong S_3$ and $n = 10$ gives us $\mathbb{Z}/10$ and $\mathbb{Z}/5 \rtimes \mathbb{Z}/2 \cong D_5$.

Groups of order 8

First, the abelian ones: $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, and $(\mathbb{Z}/2)^3$. Let G be a nonabelian group of order 8. Then G has an element of order 4: otherwise all elements have order dividing 2, so $[x, y] = xyx^{-1}y^{-1} = xyxy = (xy)^2 = e$. Let x be an element of order 4, and notice that $\langle x \rangle \trianglelefteq G$ since it has index 2. If $G \setminus \langle x \rangle$ has an element of order 2, say y, then,

$$\langle x \rangle < \langle x \rangle \langle y \rangle \leq G$$

So $|\langle x \rangle \langle y \rangle| \mid 8$ and $|\langle x \rangle \langle y \rangle| > 4$, which shows that $\langle x \rangle \langle y \rangle = G$. This shows that $\langle x \rangle \cap \langle y \rangle = \langle 1 \rangle$, so $G \cong \langle x \rangle \rtimes \langle y \rangle \cong \mathbb{Z}/4 \rtimes \mathbb{Z}/2$. Since G is nonabelian we have $G \cong D_4$.

Groups of order 8 contd.

Otherwise, every element in $G \setminus \langle x \rangle$ has order 4. Let y be such an element. Again $\langle x \rangle \langle y \rangle = G$. Now, y^2 has order 2 and is not in $G \setminus \langle x \rangle$, so is in $\langle x \rangle$. The only element of order 2 in $\langle x \rangle$ is x^2, so $y^2 = x^2$. Also, since $G / \langle y \rangle$ is abelian, notice that $xyx \langle y \rangle = x^2 y \langle y \rangle = y^3 \langle y \rangle = \langle y \rangle$, so $xyx \in \langle y \rangle$. Next, $(xyx)^2 = xyx^2yx = xy^4x = x^2 = y^2$, so $xyx = y$ or y^3. In the second case, $xyx = y^3 = x^2 y$, so $yx = xy$, which shows that G is abelian, a contradiction. So $xyx = y$, and $G = \left\langle x, y \mid x^4 = e, x^2 = y^2, xyx = y \right\rangle \cong Q_8$.