CSE 311 Template

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2. We prove that $\sqrt{19}$ is irrational by contradiction. Suppose on the contrary that it was rational. Then there exists $p, q \in \mathbb{Z}$ so that $q \neq 0$, and $\sqrt{19} = \frac{p}{q}$. We then see that as $\sqrt{19} > 0$, either p and q are both positive or both negative, if they are both negative replace them with their positive counterparts. Along with this, choose $s = \frac{p}{\gcd(p,q)}$ and $t = \frac{q}{\gcd(p,q)}$. By the fundamental theorem of arithmetic, we have divided out all common factors of p and q, so $\gcd(s,t) = 1$. At the same time that $\sqrt{19} = \frac{p}{q} = \frac{\frac{p}{\gcd(p,q)}}{\frac{q}{\gcd(p,q)}} = \frac{s}{t}$. Squaring both sides we see that $19 = \frac{s^2}{t^2}$, which shows that $t^2 \cdot 19 = s^2 = s \cdot s$. Therefore, $19 \mid s^2 = s \cdot s$. As 19 is prime, either $19 \mid s$ (case 1) or $19 \mid s$ (case 2) (by the fact given in the problem). In case 1, $19 \mid s$, and in case 2, $19 \mid s$, so in any case, $19 \mid s$. Then there exists a $k \in \mathbb{Z}$ so that $s = 19k$ (this line is equivalent to letting $P = 19 \mid s$, we have concluded $P \vee P$, and as $P \vee P \equiv P$, we may conclude P). Plugging this back into the equation for above, we see that $19t^2 = (19k)^2 = 19^2k^2$. Canceling the 19 from both sides, we see that $t^2 = 19k^2$. So we see that $19 \mid t^2$, which, in the same manner as above, means that $19 \mid t$. But then $\gcd(s,t) \geq 19$, in particular it is not 1, which is a contradiction.

3. Let $P(n) = T(n) \leq 20n$. We prove $P(n)\ \forall n \in \mathbb{N}$ with $n \geq 1$ by strong induction.

   Base cases: $T(1) = 5 \cdot 1 \leq 20 \cdot 1$, so $P(1)$ holds. $T(2) = 5 \cdot 2 = 10 \leq 20 \cdot 2 = 40$, so $P(2)$ holds. $T(3) = 5 \cdot 3 = 15 \leq 20 \cdot 3 = 60$, so $P(3)$ holds. $T(4) = 5 \cdot 4 = 20 \leq 20 \cdot 4 = 80$, so $P(4)$ holds.

   Inductive Hypothesis: Suppose $P(1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 4$.

   Inductive Step: Noting that $k + 1 \geq 5 > 4$, we see that $T(k+1) = T(\lfloor (k+1)/2 \rfloor) + T(\lfloor (k+1)/4 \rfloor) + 5(k+1)$ by the definition of T. We see that as $1 \leq k$, $1 + k \leq 2k$ (add k to both sides), and finally $(1+k)/2 \leq k$. We then notice that $1 \leq \lfloor (k+1)/2 \rfloor \leq (k+1)/2 \leq k$ by our discussion above, and from the second hint: $1 \leq \lfloor x \rfloor \leq x$. Similarly, because $0 \leq k$, we may multiply both sides by the positive number 2 to get that $0 \leq 2k$. We then use the property that $1 \leq k$ to get that $1 \leq 3k$. Finally we add k to both sides to get that $1 + k \leq 4k$, which means that $(1+k)/4 \leq k$. So we see that $1 \leq \lfloor (1+k)/4 \rfloor \leq k$ (by our discussion, and the second hint). Finally, note that both $\lfloor (1+k)/4 \rfloor, \lfloor (1+k)/2 \rfloor$ are integers in the correct range, as we showed above, so we may apply our inductive hypothesis to $T(\lfloor (1+k)/4 \rfloor)$ and $T(\lfloor (1+k)/2 \rfloor)$. We conclude that

$$\begin{aligned}
T(\lfloor (k+1)/2 \rfloor) + T(\lfloor (k+1)/4 \rfloor) + 5(k+1) &\leq 20 \cdot \lfloor (k+1)/2 \rfloor + 20 \cdot \lfloor (k+1)/4 \rfloor + 5(k+1) && \text{(I.H.)} \\
&\leq 20 \cdot (k+1)/2 + 20 \cdot (k+1)/4 + 5(k+1) && (\lfloor x \rfloor \leq x \\
&= 10(k+1) + 5(k+1) + 5(k+1) \\
&= 20(k+1)
\end{aligned}$$

   which shows $P(k+1)$. So we conclude that $P(n)$ holds for all $n \in \mathbb{N}$ with $n \geq 1$ by the principle of induction.

4. Let $P(n) = 1^n \in S$. We prove $P(n)\ \forall n \geq 20$ by strong induction.

   Base cases: $1^{20} = 1^{11} \cdot 1^3 \cdot 1^3 \cdot 1^3$. Clearly $1^3 \in S$ (by S's construction), so $1^3 \cdot 1^3 \in S$, so therefore $1^3 \cdot (1^3 \cdot 1^3) \in S$ (S is closed under string concatenation), and as $1^{11} \in S$, we see that $1^{20} \in S$, so $P(20)$ holds. Similarly, $1^{21} = \underbrace{1^3 \cdot \cdots \cdot 1^3}_{7 \text{ times}}$, so $1^{21} \in S$ (Once again S is closed under concatenation, we showed above that $1^3 \cdot 1^3 \cdot 1^3 \in S$, we could simply multiply this by 1^3 to get $1^{3 \cdot 4}$, and then continue until we get all 7 1^3s). Finally, as $1^{11} \in S$, $1^{22} = 1^{11} \cdot 1^{11} \in S$ (closed under string concatenation), so $P(22)$ holds.

   Inductive Hypothesis: Suppose $P(20) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 22$.

   Inductive Step: We notice that $1^{k+1} = 1^3 \cdot 1^{k-2}$ (a string of $(k+1)$ 1's is just a string of 3 1's followed by a string of $k-2$ 1's). We see that as $20 \leq k-2 \leq k$, the inductive hypothesis applies, so we conclude $1^{k-2} \in S$. As $1^3 \in S$, we see that $1^3 \cdot 1^{k-2} = 1^{k+1} \in S$ (because S is closed under string concatenation–see the recursive step), which is what $P(k+1)$ asserts. By the principle of induction, we conclude that $P(n)$ holds for all $n \geq 20$.

5. For a JTree X, let $P(X) =$ "if X has $c-1$ copies of $data$, then X has c copies of nil." We prove $P(X)$ for all JTrees X by structural induction on X.

   Base Case ($X = nil$): nil has 0 copies of data, and 1 copy of nil, so we conclude that $P(nil)$ is true.

   Inductive Hypothesis: Suppose $P(X)$ and $P(Y)$ hold for some arbitrary JTrees X, Y, and let $c-1$ be the number of copies of data X has, and $d-1$ be the number of copies of data Y has.

   Inductive Step: $\boxed{\text{Goal: Show that } P(data, X, Y) \text{ holds.}}$

   We notice that $(data, X, Y)$ has $(c-1)+(d-1)+1 = c+d-1$ copies of $data$, because it would have all the copies of data that X has, plus all the copies of data that Y has, plus 1 because we are adding 1 piece of data to this tree. By the inductive hypothesis, X has c copies of nil, and Y has d copies of nil. So, $(data, X, Y)$ has $c+d$ copies of nil as $(data, X, Y)$ would have all of X's nil copies (look at the left half of the tree), and all of Y's nil copies (look at the right half of the tree). This proves $P(data, X, Y)$.

   Conclusion: Thus, $P(X)$ holds for all JTrees X by structural induction.

6. This problem set took me around 3 hours to complete. I spent the most time on problem 5, as the argument was pretty hard to think about. I don't have any other feedback.