# Math 521 HW3

Rohan Mukherjee

November 13, 2024

1. We check the conditions of the Lindeberg-Feller theorem. First shift all the variables by their expectation so they have expectation 0. Notice that, with $p = 2/(2+\delta)$ and $q = \delta/(2+\delta)$, by Holder's inequality we have that:

$$\mathbb{E}\left[|X_m|\mathbb{1}_{|X_m|>\varepsilon\alpha_n}\right] \leq \mathbb{E}\left[|X|^{2+\delta}\right]^{2/(2+\delta)} \cdot \mathbb{E}\left[\mathbb{1}_{|X_m|>\varepsilon\alpha_n}\right]^{\delta/(2+\delta)}$$

Now, notice that, by Markov's inequality:

$$\mathbb{E}\left[\mathbb{1}_{|X_m|>\varepsilon\alpha_n}\right] = \mathbb{P}(|X_m| > \varepsilon\alpha_n) = \mathbb{P}\left(|X_m|^{(2+\delta)/2} > \varepsilon^{(2+\delta)/2}\alpha_n^{(2+\delta)/2}\right) \leq \frac{\mathbb{E}\left[|X_m|^{2+\delta}\right]}{\varepsilon^{2+\delta}\alpha_n^{2+\delta}}$$

And thus,

$$\mathbb{E}\left[\mathbb{1}_{|X_m|>\varepsilon\alpha_n}\right]^{\delta/(2+\delta)} \leq \frac{\mathbb{E}\left[|X_m|^{2+\delta}\right]^{\delta/(2+\delta)}}{\varepsilon^{\delta}\alpha_n^{\delta}}$$

We conclude that:

$$\begin{aligned}\mathbb{E}\left[|X_m|\mathbb{1}_{|X_m|>\varepsilon\alpha_n}\right] &\leq \frac{\mathbb{E}\left[|X|^{2+\delta}\right]^{2/(2+\delta)} \cdot \mathbb{E}\left[|X_m|^{2+\delta}\right]^{\delta/(2+\delta)}}{\varepsilon^{\delta}\alpha_n^{\delta}} \\ &= \mathbb{E}\left[|X_m|^{2+\delta}\right]\varepsilon^{-\delta}\alpha_n^{-\delta}\end{aligned}$$

Thus,

$$\alpha_n^{-2}\sum_{m=1}^{n}\mathbb{E}\left[|X_m|\mathbb{1}_{|X_m|>\varepsilon\alpha_n}\right] \leq \varepsilon^{-\delta}\alpha_n^{-2-\delta}\sum_{m=1}^{n}\mathbb{E}\left[|X_m|^{2+\delta}\right] \to 0$$

By hypothesis. Thus the Lindeberg-Feller theorem applies and we have that:

$$\frac{1}{\alpha_n^2}\sum_{m=1}^{n} X_m \Rightarrow \mathcal{N}(0,1)$$

2. Notice that, for $X$ uniformly distributed in $[-n, n]$, $\mathbb{E}[X] = 0$ and $\mathbb{E}\big[X^2\big] = n^2/3$. Thus,

$$\sigma_n^2 = \sum_{m=1}^{n} \mathbb{E}\big[X_m^2\big] = \sum_{m=1}^{n} \frac{n^2}{3} = \frac{1}{18} n(n+1)(2n+1)$$

Fix $\varepsilon > 0$. We want to show that:

$$\frac{1}{\sigma_n^2} \sum_{m=1}^{n} \mathbb{E}\big[|X_m|^2 \mathbb{1}_{|X_m| > \varepsilon\sigma_n}\big] \to 0$$

Note that $\sigma_n \sim \dfrac{1}{3} n^{3/2}$. Since $|X_m| \leq m \leq n$, and since $n/n^{3/2} \to 0$, we know that for large enough $n$, $n < \dfrac{\varepsilon}{2}\sigma_n$. But then, for every $1 \leq m \leq n$, $|X_m| \leq m \leq n < \dfrac{\varepsilon}{2}\sigma_n$, so the function $|X_m|^2 \mathbb{1}_{|X_m| > \varepsilon\sigma_n}$ is identically 0. Thus, this sum is eventually equal to 0, and hence its limit is 0 as well, so we are done.

This means that the Lindberg-Feller theorem applies, and we have that:

$$\frac{1}{\sigma_n} \sum_{m=1}^{n} X_m \Rightarrow \mathcal{N}(0,1)$$

And since $\sigma_n / \dfrac{1}{3} n^{3/2} \to 1$ in distribution (treated as a constant random variable), we can multiply by this to get that:

$$\frac{3}{n^{3/2}} \sum_{m=1}^{n} X_m \Rightarrow \mathcal{N}(0,1)$$

By dividing both sides by 3, noting that this will make the variance decrease by a factor of 9, we get that $\dfrac{1}{n^{3/2}} \displaystyle\sum_{m=1}^{n} X_m \Rightarrow \mathcal{N}(0, 1/9)$. Thus we can take $\alpha = 3/2$, $\mu = 0$ and $\sigma^2 = 1/9$ in the statement of the quesiton.

3. We see that:

$$\mathbb{E}\big[|X^k|\big] = 2\int_0^\infty x^k e^{-x^2/2} dx$$

Since $x^k e^{-x^2/4} \to 0$ as $x \to \infty$, there is some $T$ so if $x > T$, $x^k < e^{x^2/4}$. Also as $x^k e^{-x^2/2}$ is continous, it is bounded. Thus we get that:

$$\mathbb{E}\big[|X^k|\big] \leq 2\int_0^T x^k e^{-x^2/2} dx + 2\int_T^\infty e^{x^2/4} dx$$

The first integral is bounded by $T \cdot \sup_{0 \le x \le T} x^k e^{-x^2/2}$ and the second integral is obviously finite. Thus we know that $\varphi^{(k)}(0) = \mathbb{E}\big[X^k\big]$. Now recall that the characteristic function of the standard normal distribution is just:

$$\varphi(t) = e^{-t^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!}$$

Since this has no odd power terms, $\mathbb{E}\big[X^k\big] = 0$ for every $k$ odd. On the other hand,

$$\mathbb{E}\big[(iX)^{2n}\big] = \frac{(-1)^n (2n)!}{2^n n!}$$

Now using that $i^{2n} = (-1)^n$ we get that $\mathbb{E}\big[X^{2n}\big] = (2n-1)!!$, and we are done.

4. First we prove the following lemma. If $\lim_{t \to 0^+} \frac{\varphi(t) - 1}{t^2} = c > -\infty$, then $\mathbb{E}()X = 0$ and $\mathbb{E}\big[X^2\big] = -2c < \infty$.

   First,

$$\begin{aligned} c &= \mathrm{Re}\left( \lim_{t \to 0^+} \frac{\varphi(t) - 1}{t^2} \right) \\ &= \lim_{t \to 0^+} \frac{\mathrm{Re}\,(\varphi(t)) - 1}{t^2} \end{aligned}$$

   Since $\mathrm{Re}(z)$ is continous at $c$. Now notice that:

$$\lim_{h \to 0^+} \frac{\varphi(h) - 2 + \varphi(-h)}{h^2} = 2 \lim_{h \to 0^+} \frac{\mathrm{Re}(\varphi(h)) - 1}{h^2} = 2c > -\infty$$

   Thus $\mathbb{E}\big[X^2\big] < \infty$ by Theorem 3.3.21 in the book. As $\varphi''(0) = 2c$, we know that $i^2 \mathbb{E}\big[X^2\big] = 2c$, and hence $\mathbb{E}\big[X^2\big] = -2c$.

   Now, suppose that $X+Y, X$ have the same distribution. Then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = \varphi_X(t)$. Since $\varphi_X(0) = 1$, there is a neighborhood around 0 so that $\varphi_X(t) > 1/2$ for all $t$. In this neighborhood, dividing both sides shows that $\varphi_Y(t) = 1$. But then obviously, $\lim_{t \to 0^+} \frac{\varphi_Y(t) - 1}{t^2} = 0$, which means that $\mathbb{E}\big[Y^2\big] = 0$ or that $Y = 0$ almost surely.

5. Let $Y$ be an independent copy of $X$. Then,

$$\mathbb{E}\big[e^{it(X-Y)}\big] = \mathbb{E}\big[e^{itX}\big]\mathbb{E}\big[e^{i(-t)Y}\big] = \varphi_X(t)\varphi_X(-t) = |\varphi_X(t)|^2$$

   So $|\varphi_X(t)|^2$ is a chf as well. There is a theorem in the book that says if $a_i \ge 0$ and

$\sum_i a_i = 1$, and $\varphi_i$ is a chf. then $\sum_i a_i \varphi_i$ is one as well (They say it follows from $\int f d(\mu+\nu)) = \int f d\mu + \int f \nu$.)

Recall that $\varphi_X(-t) = \overline{\varphi}_X(t)$, and that $\varphi_X(-t)$ is the chf of $-X$. Thus $\overline{\varphi}_X(t)$ is a chf as well. Thus,

$$\mathrm{Re}(\varphi) = \frac{\varphi + \overline{\varphi}}{2}$$

is one as well.

6. Since $X_n, Y_n$ are independent:

$$\varphi_{X_n+Y_n}(t) = \mathbb{E}\left[e^{it(X_n+Y_n)}\right] = \mathbb{E}\left[e^{itX_n}\right]\mathbb{E}\left[e^{itY_n}\right] = \varphi_{X_n}(t)\varphi_{Y_n}(t)$$

Since $X_n \to X_\infty$ and $Y_n \to Y_\infty$, $\varphi_{X_n} \to \varphi_{X_\infty}$ pointwise and the same for $Y_n$. Thus,

$$\varphi_{X_n+Y_n}(t) \to \varphi_{X_\infty}(t)\varphi_{Y_\infty}(t) = \varphi_{X_infty+Y_\infty}(t)$$

pointwise, which means that $X_n + Y_n \to X_\infty + Y_\infty$ in distribution.