Math 334 HW 4

Rohan Mukherjee

November 2, 2022

1. $\boxed{\Longrightarrow}$

   First note that every ball centered about x is of the form $B(x, \varepsilon)$ where $\varepsilon > 0$. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^n$ that converges to $x \in \mathbb{R}^n$. Then given any $\varepsilon > 0$, there is some $N > 0$ such that for all $n > N$, $\|x_n - x\| < \varepsilon \iff x_n \in B(x, \varepsilon)$. Clearly there are infinitely many $n > N$, so there are also infinitely many $x_n \in B(x, \varepsilon)$. $\square$

   $\boxed{\Longleftarrow}$

   Suppose that every ball centered at x contains infinitely many elements of the sequence and that $(x_n)_{n=1}^{\infty}$ does not converge to x. Then there is some $\varepsilon > 0$ such that for all $N > 0$ there is some $n > N$ such that $\|x_n - x\| \geqslant \varepsilon$.

   Then given any $\varepsilon > 0$, we have that

2. Because $(x_n)_{n=1}^{\infty} \to x$ for some $x \in \mathbb{R}$, we know that for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that if $n > N$, we have that $|x_n - x| < \varepsilon$. Now let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that if $n > N$, $|x_n - x| < \varepsilon/2$. We see that because $m(n) > n > N$, $|x_{m(n)} - x| < \varepsilon/2$, therefore $|x_{m(n)} - x_n| = |x_{m(n)} - x + x - x_n| \leqslant |x_{m(n)} - x| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. This proves that $x_{m(n)} - x_n \to 0$. $\square$

3. Suppose that there was no such c. Then $f(x)$ gets arbitrarily small, that is for every $n \in \mathbb{N}$, there is some $0 \leqslant x_n \leqslant 1$ such that $f(x_n) < 1/n$. Then $(x_n)_{n=1}^{\infty}$ is a bounded sequence, because every point is contained in $[0, 1]$. Then it contains a convergent subsequence, $(x_{n_k}) \to x$, and because $[0, 1]$ is compact, $x \in [0, 1]$. We see that $f(x_{n_k}) \to 0$, because given $\varepsilon > 0$, choose $N > 1/\varepsilon$. Then for all $k > N$, $n_k \geqslant k > N$, therefore $|x_n - x| < 1/n_k < 1/N < \varepsilon$. But $f(x)$ is continuous, so $f(x_{n_k}) \to f(x)$, so therefore $f(x) = 0$, a contradiction. So some c must exist. $\square$

4. Let $f(x) = 4$ for all $x \in \mathbb{R}$, and let $(x_n)_{n=1}^{\infty} = 4$ for all $n \in \mathbb{N}$. We see that $f(x_n) = 4$ for every x_n. Then $y_n = \displaystyle\sum_{k=1}^{n} f(x_k) = \sum_{k=1}^{n} 4 = 4n$. Given any $M > 0$, choose $N > M/4$. Then for all $n > N$, $y_n \geqslant y_N = 4N > M$. So y_n diverges, which means that we have disproven this statement. $\square$