Math 461 HW6

Rohan Mukherjee

November 15, 2023

1. We see that

$$(a_0 + a_1 x + a_2 x^2 + \ldots)(1 - 2x) = a_0 + \sum_{n=1}^{\infty}(a_n - 2a_{n-1})x^n$$

Since $a_n - 2a_{n-1} = \begin{cases} 1, & a_n \text{ odd} \\ 0, & \text{o.w.} \end{cases}$, this evaluates to

$$0 + \sum_{n \text{ odd}} x^n$$

I noted on the previous homework that the generating function for $a_1, a_3, a_5, \ldots$ was $\frac{1}{2}(f(x) - f(-x))$, so this geometric series equals

$$\frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)$$

We conclude that $2A(x) = \frac{1}{(1-2x)(1-x)} - \frac{1}{(1-2x)(1+x)}$. Repeated applications of the cover-up method shows that these fractions equal

$$2A(x) = \frac{4/3}{1-2x} - \frac{1}{1-x} - \frac{1/3}{1+x} = \sum_{n=0}^{\infty}\left(\frac{4}{3}\cdot 2^n - 1 - \frac{(-1)^n}{3}\right)x^n$$

Thus $a_n = \frac{1}{6}(2^{n+2} - 3 + (-1)^{n+1})$.

2. I claim that the sum of abc over all $(a, b, c) \in S$ where S is the set of all positive integers such that $a + b + c = n$, which I shall denote a_n, has generating function

$$(0 + 1x + 2x^2 + \ldots)^3$$

This is because when we expand this product, to get x^n we would need to pick an ax^a from the first factor, a bx^b from the second, and a cx^c from the third, giving a term of $abcx^{a+b+c}$, and sum over all possible (a, b, c) so that $a + b + c = n$. Thus the coefficient is exactly what we are looking for. The above function now equals

$$\left(\frac{x}{(1-x)^2}\right)^3 = x^3 \cdot \frac{1}{(1-x)^6} = x^3 \cdot \sum_{n=0}^{\infty}\binom{n+5}{5}x^n = \sum_{n=0}^{\infty}\binom{n+5}{5}x^{n+3} = \sum_{n=3}^{\infty}\binom{n+2}{5}x^n$$

We conclude that $a_n = \binom{n+2}{5}$, and in particular $a_{25} = \binom{27}{5} = 80,730$.

3. First we find a generating function for the first row. Let a_n be the number of ways to have n identical coins, each lined up in a row, with an odd number of heads. We can evaluate a_n by summing over the number of heads. Let $1 \leq k \leq n$ be odd and the number of heads in our row. There are $\binom{n}{k}$ ways to do this. Summing over k yields

$$a_n = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$$

The generating function for a_n is thus $\sum_{n=1}^{\infty} 2^{n-1}x^n$ (We note for completeness that $a_0 = 0$, since in that case we can't have an odd number of heads). Similarly, if we let b_n be the number of ways we can have the second row with n identical coins, we would have $b_0 = 1$ (since 0 is even), and $b_n = 2^{n-1}$ for $n \geq 1$ (there are also 2^{n-1} even sized subsets). If we let c_n be the answer to the question, it's generating function is just the product of these two, since this asks what happens if we split n coins over the two different rows:

$$\begin{aligned} A(x) = \left(\sum_{n=1}^{\infty} 2^{n-1}x^n\right)\left(1 + \sum_{n=1}^{\infty} 2^{n-1}x^n\right) &= \sum_{n=1}^{\infty} 2^{n-1}x^n + \left(\sum_{n=1}^{\infty} 2^{n-1}x^n\right)^2 = \frac{x}{1-2x} + \frac{x^2}{(1-2x)^2} \\ &= \sum_{n=1}^{\infty} \left(2^{n-1} + 2^{n-2}(n-1)\right)x^n \end{aligned}$$

We conclude that $c_n = 2^{n-1} + 2^{n-2}(n-1)$ for $n > 0$ and $c_0 = 0$.

4. This is extremely similar to the problem we had in class. We proceed first by fixing the number of parts then summing over it. Let k be the number of parts. We shall find the generating function for the sequence a_n which counts the number of compositions of n with k parts, who are all odd. This is just,

$$\prod_{i=1}^{k}(x + x^3 + x^5 + \ldots) = \prod_{i=1}^{k} \frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right) = \left(\frac{x}{1-x^2}\right)^k$$

Now we can just sum over the number of parts. This yields, noting we need at least 1 part,

$$\sum_{k=1}^{\infty} \left(\frac{x}{1-x^2}\right)^k = \frac{x/(1-x^2)}{1-x/(1-x^2)} = \frac{x}{1-x-x^2}$$

The answer to our question is just the coefficient of x^n in the above generating function. At this time the curious reader is reminded of the Fibonacci sequence. Indeed, if we let $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$, and if we denote $F(x)$ as the generating function for F_n, we have that

$$F(x)(1-x-x^2) = F_0 + x(F_1 - F_0) + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})x^n$$

So

$$F(x) = \frac{x}{1-x-x^2}$$

We conclude that

$$a_n = F_n = \frac{1}{\sqrt{5}}\left(\left(1+\frac{\sqrt{5}}{2}\right)^n + \left(1-\frac{\sqrt{5}}{2}\right)^n\right)$$

5. The generating function where no part occurs more than 3 times is just

$$(1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^9)\cdots = \frac{1-x^4}{1-x}\cdot\frac{1-x^8}{1-x^2}\cdot\frac{1-x^{12}}{1-x^3}\cdots$$

We can see that all terms on the top will cancel with terms on the bottom of the form $1-x^{4k}$ for integers k. This leaves

$$\prod_{\substack{i\geq 1\\ i\not\equiv_4 0}} \frac{1}{1-x^i} = \prod_{\substack{i\geq 1\\ i\not\equiv_4 0}} (1+x^i+x^{2i}+x^{3i}+\cdots)$$

This is just the generating function for the number of partitions with no part a multiple of 4, which completes the proof.