# Math 425 Pset 2

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**Lemma 0.1.** *Let $(X, d)$ be a metric space, let $A \subset X$, and suppose there is a sequence $A \supset \{a_n\} \to a \notin A$. Then $a$ is a limit point of $A$.*

*Proof.* Given any $r > 0$, we can find a $N$ so that if $n \geq N$, $d(a_n, a) < r$. Note also that since $a_n \neq a$, $a_n \in N_r(a) \setminus \{a\}$ by definition, and consequently since $a_n \in A$ for every $n$, we see that $(N_r(a) \setminus \{a\}) \cap A \neq \emptyset$, so $a$ is indeed a limit point of $A$, as claimed. □

I use this lemma countless times during this homework assignment, so I thought it would be a good idea to put it right here.

1. A clean example is $\mathfrak{G} = \{(1/n, 1-1/n)\}_{n=1}^{\infty}$. One notes clearly, that as each $(1/n, 1-1/n) \subset (0,1)$ for positive $n$, we have that the union is contained in $(0,1)$. Similarly, given any $x \in (0,1)$, WLOG $|x| \leq 1/2$, we can choose an integer $n > 1/x$ so that $x \in (1/n, 1-1/n)$, which completes double inclusion (I realize now that double inclusion is not necessary, but this doesn't affect the claim). Let $S_i = (1/i, 1-1/i)$, and suppose $\mathfrak{G}$ had a finite subcover, say $\{S_l, \ldots, S_k\}$ (where the indices are increasing). One notes clearly that $S_l \subset S_{l+1} \subset \cdots \subset S_k$, so it suffices to show that $S_k$ does not cover $(0,1)$ (i.e., $\bigcup_{i=l}^{k} S_l = S_k$). Clearly, $(2k)^{-1} \notin (1/k, 1-1/k)$.

2. $E$ is not countable. For if it was, there would be a bijection $f : \mathbb{N} \to E$. From this consider the number with its $i$-th digit a 4 if $f(i)$ has a 7 in the $i$-th place, or in the case where $f(i)$ has a 4 in the $i$-th place, make the new number have a 7 in the $i$-th place. Clearly this number is not on our list, so indeed $E$ is not countable. $E$ is also not dense in $[0,1]$, for the largest possible number in $E$ would be the number with all digits being a 7. Then we could just choose $x = 1$ and $\varepsilon = 0.2$, and we see that as every number in $E$ is less than 0.8, we cannot be within 0.2 of $x$, so $E$ is not dense by counterexample. Let $x$ be a point in $[0,1]$ with a non 4 or 7 digit, and call the first digit place that is not a 4 or 7 $n$, and for any number $z \in [0,1]$, let $z_l$ where $l \in \mathbb{N}$ denote the $l$th digit of $z$. Let $y \in E$, and let $m \leq n$ be the first digit place that $y$ and $x$ differ (Note: $y$ and $x$ certainly differ at the digit place $n$, but they might differ before that too). Then $|y - m| = |(y_m - x_m)/10^m + \sum_{k=m+1}^{\infty}(y_k - x_k)/10^k| \geq |(y_m - x_m)/10^m| - |\sum_{k=m+1}^{\infty}(y_k - x_k)/10^k|$ by the reverse triangle inequality. Since $y_k$ is either a 4 or a 7, and $x_k \in \{0, \cdots, 9\}$, we see that $|y_k - x_k| \leq 7$. Also, $|\sum_{k=m+1}^{\infty}(y_k - x_k)/10^k| \leq \sum_{k=m+1}^{\infty}|y_k - x_k|/10^k \leq \sum_{k=m+1}^{\infty} 7/10^k = 7 \cdot (1/10^{m+1})/(9/10) = 7/9 \cdot 1/10^m$, so $|(y_m - x_m)/10^m| - |\sum_{k=m+1}^{\infty}(y_k - x_k)/10^k| \geq |(y_m - x_m)/10^m| -$

$7/9 \cdot 1/10^m \geq 1/10^m - 7/9 \cdot 1/10^m = 2/9 \cdot 1/10^m \geq 2/9 \cdot 1/10^n$. Since $y$ was arbitrary, we see that taking $r < 2/9 \cdot 1/10^n$ will be so that $(N_r(x) \setminus x) \cap E = \emptyset$, which shows that $x$ is not a limit point of $E$. Since everything not in $E$ is not a limit point of $E$, we see that all limit points of $E$ lie in $E$, i.e. that $E$ is closed. Finally, $E$ is also perfect since $E$ is closed (we showed this above), and given any $x \in E$, and given any $\varepsilon > 0$, by the archimedian property we can find $n$ so that $1/10^n < \varepsilon$, and just flip the $n$th digit of $x$ to get something in $E$, not equal to $x$, which is within $1/10^n < \varepsilon$ of $x$. So indeed every point in $E$ is a limit point of $E$, as claimed.

3. Let $x$ be a limit point of $A + B \subset \mathbb{R}^d$. Then for $i \in \mathbb{N}$, there is some $a_i + b_i \in A + B$ so that $|a_i + b_i - x| < 1/i$. So, there is a sequence $\{a_i + b_i\}_{i=1}^{\infty} \to x$. Because $\{a_i\} \subset A$, and $A$ is compact, there is some $R > 0$ so that $|a_i| < R$. Similarly, since there is some $N$ sufficiently large so that if $n \geq N$, $|a_n + b_n - x| < \varepsilon$, we see that $|b_n| \leq |a_n + b_n| + |a_n| \leq \varepsilon + x + R$, and as $\{b_1, \ldots, b_{n-1}\}$ is clearly finite, the $b_i$'s are bounded. Therefore, $b_i$ has a convergent subsequence, so pass to a convergent subsequence say $b_k \to b$. Suppose $c_k \to c$, $\{c_k\} \subset C$, then if $C$ is closed, $c \in C$. For if $c$ was not in $C$, then given any $\varepsilon > 0$ there is some $N$ so that if $n \geq N$, $|c_n - c| < \varepsilon$, i.e. that $(N_\varepsilon(c) \setminus c) \cap C \neq \emptyset$ (clearly $c_n \neq c$ for every $c_n$). So $c$ is a limit point of $C$ that $C$ does not contain, a contradiction. So $b \in B$. Given any $\varepsilon > 0$, since $a_i + b_i \to x$, there is $N_1$ so that for every $n \geq K_1$, $|a_k - b_k - x| < \varepsilon/2$ (we passed to the same indices as the new convergent subsequence of $b$). Similarly, since $b_k \to b$, there is some $K_2$ so that if $k \geq K_2$, $|b_k - b| < \varepsilon/2$. So for any $k \geq \max\{K_1, K_2\}$, and any $m, n \geq K$, $|a_m - a_n| - \varepsilon/2 \leq |a_m - a_n| - |b_m - b_n| \leq |a_m + b_m - a_n - b_n| < \varepsilon/2$, i.e. that $|a_m - a_n| < \varepsilon$, so $a_k$ is Cauchy and therefore convergent, call its limit $a$. Since $a_k \subset A$ and $A$ is closed (By Heine-Borel, compact $\implies$ closed + bounded), $a \in A$. Clearly $a_k + b_k$ is a subsequence of the original $a_i + b_i$, so $a_k + b_k \to x$ as well, but this time we know that we can split up the limits to get that $a_k + b_k \to a + b$ as well. Since $x = a + b \in A + B$, $A + B$ contains all it's limit points, and is therefore closed, as claimed. Q.E.D.

4. Consider $A = \{1 + 1/10, 2 + 1/10^2, \cdots\}$ and $B = \{-1, -2, \cdots\}$. Both these sets are closed because both of their complements is a countable union of open intervals (For example, $B^c = (-1, \infty) \cup \bigcup_{k=-\infty}^{-1}(k-1, k)$). Also, if $0 \in A + B$, then $0 = a + b$ for some $a \in A, b \in B$. Everything in $A$ is of the form $a + 1/10^a$ for a natural $a$, and everything in $B$ is of the form $-b$ for a natural $b$. This says that $a + 1/10^a = b$, i.e. that $1/10^a = b - a$, which is impossible since the LHS is not an integer. Finally, given any $\varepsilon > 0$, $N_\varepsilon(0) \setminus 0 \cap A + B$ is nonempty because we can find a natural number $i$ so that $10^i > \varepsilon$ (by the Archimedian property), i.e. that $1/10^i < \varepsilon$, which tells us that $|i + 1/10^i - i - 0| = 1/10^i < \varepsilon$, and clearly $i + 1/10^i \in A$ and $-i \in B$, so 0 is a limit point of $A + B$ that is not in $A + B$, so $A + B$ is not closed.

5. Suppose instead that $d(A, B) \leq 0$ for some compact set $A$ and closed set $B$. Since $d(x, y) \geq 0$ for every $x, y$, it follows that $d(A, B)$ (the inf of many distances) is also nonnegative, so we see that $d(A, B) = 0$. By the definition of the inf, given any $n \in \mathbb{N}$, we can find $a_n \in A$, $b_n \in B$, so that $d(a_n, b_n) < 1/n$. Now clearly $\{a_n\} \subset A$, and since compact $\iff$ sequentially compact, $a_n$ has a convergent subsequence. So, pass to a convergent subsequence $a_k \to a \in A$. I claim that $b_k \to a$ as well. Given any $\varepsilon > 0$, choose $K_1$ large enough so that $d(a_k, a) < \varepsilon/2$, and also choose $K_2 > 2/\varepsilon$, and choose $K = \max\{K_1, K_2\}$. Then $d(b_k, a) \leq d(b_k, a_k) + d(a_k, a) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, so indeed $b_k \to a$ (Note: by construction, for

any $m > K_2$, $d(a_m, b_m) < 1/m < 1/K_2 < \varepsilon$). Since $A \cap B = \emptyset$, we know that $a \notin B$, and by Lemma 0.1 $a$ is a limit point of $B$. But then $B$ doesn't contain one of it's limit points, so $B$ is not closed, a contradiction. So $d(A, B) > 0$, as claimed.

6. (a) Suppose that one of $A \cap \overline{B} \neq \emptyset$ or $\overline{A} \cap B \neq \emptyset$ were true. Since both $A, B$ are closed, $\overline{A} = A$ and similarly $\overline{B} = B$. Then this would tell us that $A \cap B \neq \emptyset$ is true, a contradiction.

   (b) WLOG, suppose that $A \cap \overline{B} \neq \emptyset$. Then let $y \in A \cap \overline{B}$. If $y \in B$, then $y \in A \cap B$, a contradiction, so $y \in B'$. Then since $y$ is a limit point of $B$, $\forall r > 0\, N_r(y) \setminus y \cap B \neq \emptyset$. Also, since $y \in A$, and $A$ is open, $\exists l > 0, N_r(y) \subset A$. One therefore sees that $N_l(y) \setminus y \cap B \neq \emptyset$, and similarly that $N_l(y) \subset A$, which combined tells us that $A \cap B \neq \emptyset$, a contradiction. The other case follows in exactly the same way, so we are done.

   (c) Theorem 2.19 tells us that every neighborhood is an open set, so $N_\delta(p)$ is open. Similarly, let $q \in \{q \in X \mid d(p, q) > \delta\}$. Since $d(p, q) > \delta$, $d(p, q) = \delta + a$ for some $a > 0$. Now consider $N_{a/2}(q)$. I claim that everything in this neighborhood also has distance at least $\delta$ from $p$. Certainly, given $b \in N_{a/2}(q)$, $\delta + a = d(p, q) \leq d(q, b) + d(b, p) < a/2 + d(b, p)$, which says that $\delta + a/2 < d(b, p)$, or more importantly that $\delta < d(b, p)$, as claimed. So $q$ is an interior point of $\{q \in X \mid d(p, q) > \delta\}$, and since $q$ was arbitrary $\{q \in X \mid d(p, q) > \delta\}$ is open. Finally, suppose there was an $x \in N_\delta(p) \cap \{q \in X \mid d(p, q) > \delta\}$. Then $d(x, p) < \delta$, and $d(x, p) > \delta$, a contradiction, so these sets are disjoint. We proved in the last part that two disjoint open sets are separated, so these two sets are separated, as claimed.

   (d) Suppose there was a connected metric space that is countable. Fix $p \in X$. I claim that there is a $\delta \in \mathbb{R}_{>0}$ so that given any $y \in X$, $d(p, y) \neq \delta$. If for all $\delta > 0$ there was a point $y_\delta \in X$ so that $d(y_\delta, p) = \delta$, we could define an onto map to the reals by $f : X \to \mathbb{R}_{>0}$ by $f(y) = d(y, p)$. But then $X$ would be uncountable, a contradiction. So, there is at least one $\delta > 0$ so that for every $y \in X$, $d(y, p) \neq \delta$. Since given any $x \in X$ $d(x, p)$ is either $\geq \delta$, or $< \delta$, and we have shown that it cannot equal $\delta$, $N_\delta(p)$ and $\{q \in X \mid d(p, q) > \delta\}$ partition $X$. But by part c), these two sets are separated, so $X$ is not connected, a contradiction.

7. Let $k \in K$. Since $K \subset U$, there is a $r_k > 0$ so that $N_{r_k}(k) \subset U$. Also, note that $\overline{N_{r_k/2}(k)} \subset N_{r_k}(k)$, since if $x \in \overline{N_{r_k/2}(k)}$, then either $x \in N_{r_k/2}(k)$ or $x \in N_{r_k/2}(k)'$. In the first case, $d(x, k) < r_k/2 < r_k$, so clearly $x \in N_{r_k}(k)$. In the second case, we know that since $x$ is a limit point of $N_{r_k/2}(k)$, $(N_{r_k/2}(x) \setminus x) \cap N_{r_k/2}(k) \neq \emptyset$, so we can find a $y \in (N_{r_k/2}(x) \setminus x) \cap N_{r_k/2}(k)$. Finally, one notes that $d(x, k) \leq d(x, y) + d(y, k) < r_k/2 + r_k/2 = r_k$, so indeed $x \in N_{r_k}(k)$, as claimed. Next, one notes that since $k \in N_{r_k/2}(k)$, $K \subset \bigcup_{k \in K} N_{r_k/2}(k)$. Since $K$ is compact, this open cover has a finite subcover, say $\bigcup_{i=1}^{l} N_{r_{k_i}/2}(k)$, where $l < \infty$. Because this is an open cover of $K$, $K \subset \bigcup_{i=1}^{l} N_{r_{k_i}/2}(k)$. Next, $\overline{\bigcup_{i=1}^{l} N_{r_{k_i}/2}(k)} = \bigcup_{i=1}^{l} \overline{N_{r_{k_i}/2}(k)}$, since this is a *finite* union. Finally, since $\overline{N_{r_{k_i}/2}(k)} \subset N_{r_{k_i}}(k)$, we see that $\overline{\bigcup_{i=1}^{l} N_{r_{k_i}/2}(k_i)} \subset \bigcup_{i=1}^{l} N_{r_{k_i}}(k_i) \subset U$ (again, since each $N_{r_{k_i}}(k_i) \subset U$ by construction, and since this is a finite union). In the end, we get the chain $K \subset \bigcup_{i=1}^{l} N_{r_{k_i}/2}(k) \subset \overline{\bigcup_{i=1}^{l} N_{r_{k_i}/2}(k)} = \bigcup_{i=1}^{l} \overline{N_{r_{k_i}/2}(k)} \subset \bigcup_{i=1}^{l} N_{r_{k_i}}(k_i) \subset U$, which completes the proof. Q.E.D.