Sample Size & Power Calculations

Sample size calculations

$$n = \frac{\$ \text{ available}}{\$ \text{ per sample}}$$
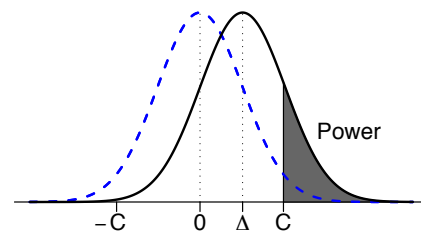
Power

$X_1, \ldots, X_n$ iid Normal(μ_A, σ_A) $Y_1, \ldots, Y_m$ iid Normal(μ_B, σ_B)

Test $H_0 : \mu_A = \mu_B$ vs $H_a : \mu_A \neq \mu_B$ at α = 0.05.

Test statistic: $T = \dfrac{\overline{X} - \overline{Y}}{\widehat{SD}(\overline{X} - \overline{Y})}$.

$\longrightarrow$ Critical value: C such that $\Pr(|T| > C \mid \mu_A = \mu_B) = \alpha$.

Power: $\Pr(|T| > C \mid \mu_A \neq \mu_B)$

Power depends on...

- The design of your experiment

- What test you're doing

- Chosen significance level, α

- Sample size

- True difference, $\mu_A - \mu_B$

- Population SD's, σ_A and σ_B.

The case of known population SDs

Suppose σ_A and σ_B are known.

Then $\overline{X} - \overline{Y} \sim \text{Normal}(\,\mu_A - \mu_B,\, \sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}\,)$

Test statistic: $\tilde{Z} = \dfrac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}}$

If H_0 is true (i.e. $\mu_A = \mu_B$), we have $\tilde{Z} \sim \text{Normal}(0,1)$.

$\longrightarrow$ $C = z_{\alpha/2}$ so that $\Pr(|\tilde{Z}| > C \mid \mu_A = \mu_B) = \alpha$.

For example, for $\alpha = 0.05$, $C = \texttt{qnorm(0.975)} = 1.96$.

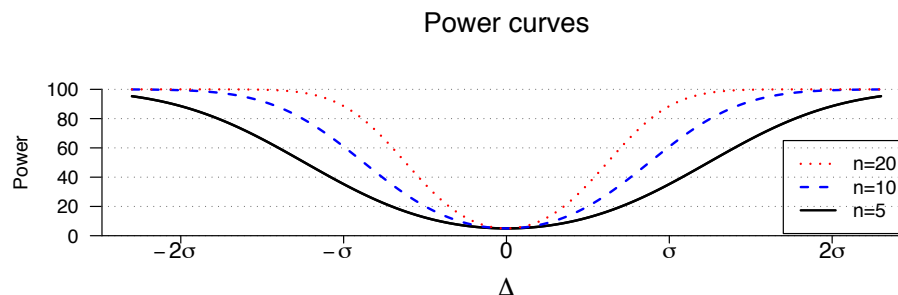# Power when the population SDs are known

If $\mu_A - \mu_B = \Delta$, then $Z = \dfrac{(\overline{X} - \overline{Y}) - \Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \sim \text{Normal}(0,1)$

$$\Pr\left(\frac{|\overline{X} - \overline{Y}|}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} > 1.96 \right) = \Pr\left(\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} > 1.96 \right) + \Pr\left(\frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} < -1.96 \right)$$

$$= \Pr\left(\frac{\overline{X} - \overline{Y} - \Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} > 1.96 - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right) + \Pr\left(\frac{\overline{X} - \overline{Y} - \Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} < -1.96 - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right)$$

$$= \Pr\left(Z > 1.96 - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right) + \Pr\left(Z < -1.96 - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right)$$

Calculations in R

$$\text{Power} = \Pr\left(Z > 1.96 - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right) + \Pr\left(Z < -1.96 - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right)$$

```
C <- qnorm(0.975)
se <- sqrt( sigmaA^2/n + sigmaB^2/m )
power <- 1-pnorm(C-delta/se) + pnorm(-C-delta/se)
```


Power curves

Power depends on ...

$$\text{Power} = \Pr\left(Z > C - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right) + \Pr\left(Z < -C - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right)$$

- Choice of α (which affects C)

  Larger $\alpha \rightarrow$ less stringent $\rightarrow$ greater power.

- $\Delta = \mu_A - \mu_B =$ the true "effect."

  Larger $\Delta \rightarrow$ greater power.

- Population SDs, σ_A and σ_B

  Smaller σ's $\rightarrow$ greater power.

- Sample sizes, n and m

  Larger n, m $\rightarrow$ greater power.

Choice of sample size

We mostly influence power via n and m.

Power is greatest when $\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}$ is as small as possible.

Suppose the total sample size N = n + m is fixed.

$\longrightarrow$ $\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}$ is minimized when n = $\frac{\sigma_A}{\sigma_A + \sigma_B} \times$ N and m = $\frac{\sigma_B}{\sigma_A + \sigma_B} \times$ N

For example:

- If $\sigma_A = \sigma_B$, we should choose n = m.

- If $\sigma_A = 2\,\sigma_B$, we should choose n = 2 m.

  That means, if σ_A = 4 and σ_B = 2, we might use n=20 and m=10.

Calculating the sample size

Suppose we seek 80% power to detect a particular value of $\mu_A - \mu_B = \Delta$, in the case that σ_A and σ_B are known.

(For convenience here, let's pretend that $\sigma_A = \sigma_B$ and that we plan to have equal sample sizes for the two groups.)

$$\text{Power} \approx \Pr\left(Z > C - \frac{\Delta}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{m}}} \right) = \Pr\left(Z > 1.96 - \frac{\Delta\sqrt{n}}{\sigma\sqrt{2}} \right)$$

$\longrightarrow$ Find n such that $\Pr\left(Z > 1.96 - \frac{\Delta\sqrt{n}}{\sigma\sqrt{2}} \right) = 80\%$.

Thus $1.96 - \frac{\Delta\sqrt{n}}{\sigma\sqrt{2}} = \texttt{qnorm(0.2)} = -0.842$.

$\longrightarrow$ $\sqrt{n} = \frac{\sigma}{\Delta}\{1.96 - (-0.842)\}\sqrt{2}$ $\qquad \longrightarrow$ n = $15.7 \times \left(\frac{\sigma}{\Delta}\right)^2$

Equal but unknown population SDs

$X_1, \ldots, X_n$ iid Normal(μ_A, σ) $\qquad$ $Y_1, \ldots, Y_m$ iid Normal(μ_B, σ)

Test $H_0 : \mu_A = \mu_B$ vs $H_a : \mu_A \neq \mu_B$ at $\alpha = 0.05$.

$$\hat{\sigma}_p = \sqrt{\frac{s_A^2(n-1) + s_B^2(m-1)}{n+m-2}} \qquad \widehat{SD}(\overline{X} - \overline{Y}) = \hat{\sigma}_p \sqrt{\frac{1}{n} + \frac{1}{m}}$$

Test statistic: $T = \dfrac{\overline{X} - \overline{Y}}{\widehat{SD}(\overline{X} - \overline{Y})}$.

In the case $\mu_A = \mu_B$, T follows a t distribution with $n + m - 2$ d.f.

$\longrightarrow$ Critical value: C = `qt(0.975,n+m-2)`

Power: equal but unknown pop'n SDs

Power $= \Pr\left(\dfrac{|\overline{X} - \overline{Y}|}{\hat{\sigma}_p \sqrt{\frac{1}{n} + \frac{1}{m}}} > C \right)$

$\longrightarrow$ In the case $\mu_A - \mu_B = \Delta$, the statistic $\dfrac{\overline{X} - \overline{Y}}{\hat{\sigma}_p \sqrt{\frac{1}{n} + \frac{1}{m}}}$ follows a non-central t distribution.
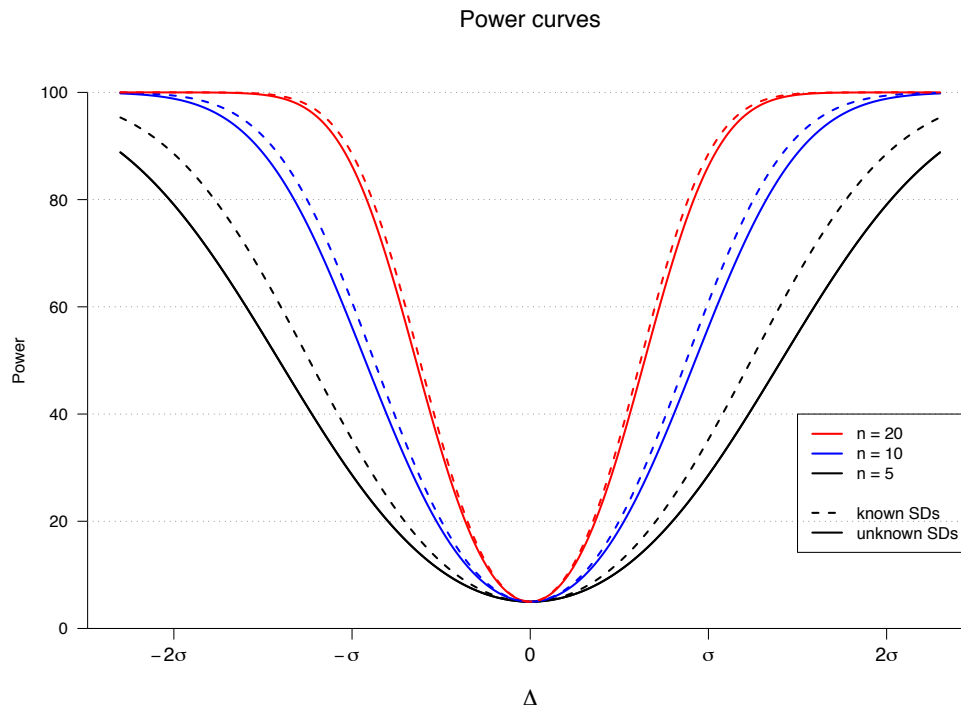
This distribution has two parameters:

$\longrightarrow$ The degrees of freedom (as before)

$\longrightarrow$ The non-centrality parameter, $\dfrac{\Delta}{\sigma \sqrt{\frac{1}{n} + \frac{1}{m}}}$

```
C <- qt(0.975, n + m - 2)
se <- sigma * sqrt( 1/n + 1/m )
power <- 1 - pt(C, n+m-2, ncp=delta/se) +
            pt(-C, n+m-2, ncp=delta/se)
```

Power: equal population SDs

Power curves

A built-in function: power.t.test()

Calculate power (or determine the sample size) for the t-test when:

- Sample sizes equal
- Population SDs equal

Arguments:

- `n` = sample size
- `delta` = Δ = $\mu_2 - \mu_1$
- `sd` = σ = population SD
- `sig.level` = α = significance level
- `power` = the power
- `type` = type of data (two-sample, one-sample, paired)
- `alternative` = two-sided or one-sided test

Examples

A. n = 10 for each group; effect = $\Delta = 5$; pop'n SD = $\sigma = 10$

```
power.t.test(n=10, delta=5, sd=10)
```

$\longrightarrow$ 18%

B. power = 80%; effect = $\Delta = 5$; pop'n SD = $\sigma = 10$

```
power.t.test(delta=5, sd=10, power=0.8)
```

$\longrightarrow$ n = 63.8 $\longrightarrow$ 64 for each group

C. power = 80%; effect = $\Delta = 5$; pop'n SD = $\sigma = 10$; one-sided

```
power.t.test(delta=5, sd=10, power=0.8,
             alternative="one.sided")
```

$\longrightarrow$ n = 50.2 $\longrightarrow$ 51 for each group

Unknown and different pop'n SDs

$X_1, \ldots, X_n$ iid Normal(μ_A, σ_A) $Y_1, \ldots, Y_m$ iid Normal(μ_B, σ_B)

Test $H_0 : \mu_A = \mu_B$ vs $H_a : \mu_A \neq \mu_B$ at $\alpha = 0.05$.

Test statistic: $T = \dfrac{\overline{X} - \overline{Y}}{\sqrt{\frac{s_A^2}{n} + \frac{s_B^2}{m}}}$

To calculate the critical value for the test, we need the null distribution of T (that is, the distribution of T if $\mu_A = \mu_B$).

To calculate the power, we need the distribution of T given the value of $\Delta = \mu_A - \mu_B$.

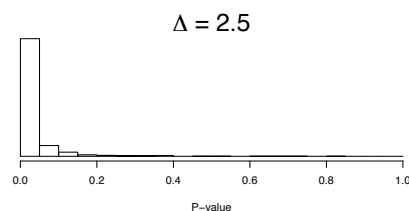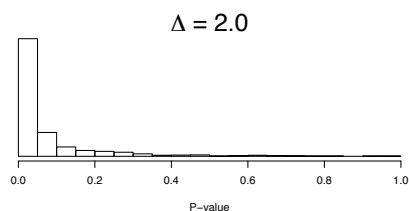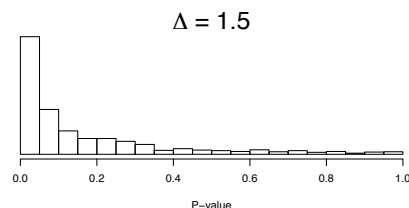We don't really know either of these.
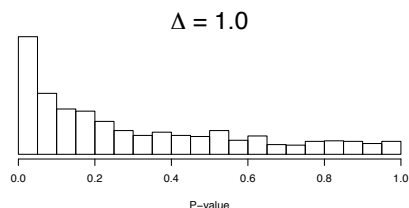
Power by computer simulation

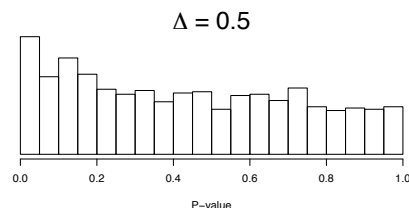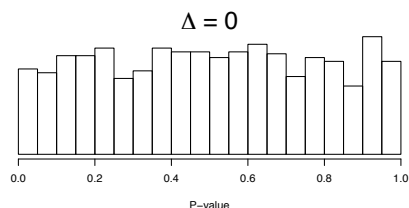- Specify n, m, σ_A, σ_B, $\Delta = \mu_A - \mu_B$, and the significance level, α.

- Simulate data under the model.

- Perform the proposed test and calculate the P-value.

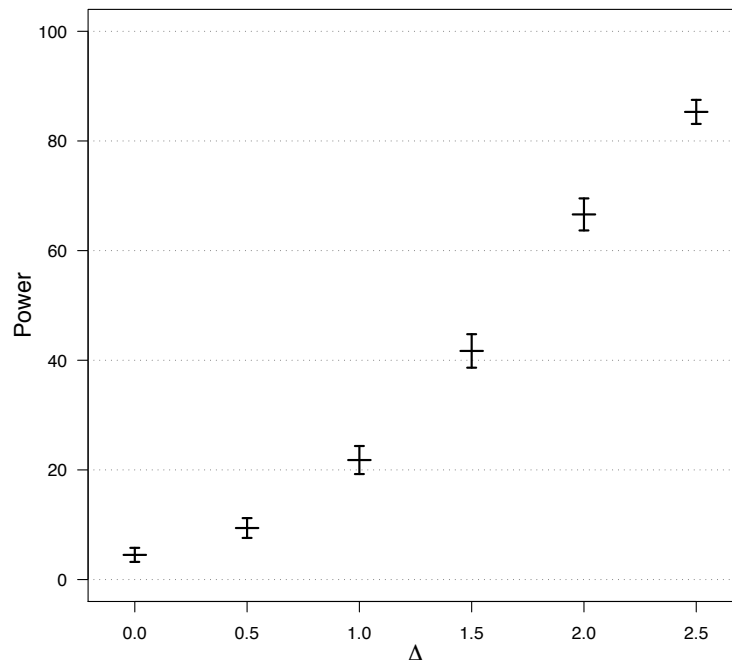- Repeat many times.

$\longrightarrow$ Example:

  n = 5, m = 10, σ_A = 1, σ_B = 2,

  Δ = 0.0, 0.5, 1.0, 1.5, 2.0 or 2.5.

Example

Example

Determining sample size

The things you need to know:

- Structure of the experiment

- Method for analysis

- Chosen significance level, α (usually 5%)

- Desired power (usually 80%)

- Variability in the measurements
  $\rightarrow$ If necessary, perform a pilot study, or use data from prior experiments or publications.

- The smallest meaningful effect

Reducing sample size

- Reduce the number of treatment groups being compared.

- Find a more precise measurement (e.g., average survival time rather than proportion dead).

- Decrease the variability in the measurements.
  - Make subjects more homogenous.
  - Use stratification.
  - Control for other variables (e.g., weight).
  - Average multiple measurements on each subject.