A

Theorem 1. *The amount of diagonals of an n sided regular polygon is* $\frac{n(n-1)}{2} - n$

Proof:

Since we are only considering regular polygons with n sides, we can consider this polygon graphically as isomorphic to the complete graph of n vertices K_n when you draw in all of the diagonals. And thus the amount of diagonals is $\frac{n(n-1)}{2} - n$. The n term is subtracted from the total amount of edges, because we don't want to count the edges which are drawn from any vertex to its adjacent vertices. ∎

So if we consider the 2022 sided n-regular polygon then the number of diagonals is $\frac{(2022)(2021)}{2} - 2022 = 2041209$

B

Theorem 2. *If* $A = \{1, ..., n\}$ *then the number of odd order subsets & even order subsets of* A *is* 2^{n-1}

Proof:

The number of odd order subsets = The numbers of subsets of order $1, ..2\lfloor \frac{n}{2} \rfloor + 1$. That is equal to the sum $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1}$. Similarly we have that the number of even subsets of A is $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}$. Now we just have to show that $\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1} = 2^{n-1}$. We can prove that by looking at the sum $\sum_{i=0}^{n} (-1)^i \binom{n}{i}$. It is 0 since it equal to $(1-1)^n$ by the binomial theorem. Also every term that is negative is of the form $-\binom{n}{i}$ where i is odd. And thus we have the equality $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}$.

And since trivially we have that $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1} + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = 2^n$ we have that $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} = 2^{n-1}$ ∎

C

Definition 0.0.1. *Given any graph* $G = (V, E)$ *with vertices and any permutation* $f \in S_V$ *then one can define* N_f *which inputs any vertex of* G *and outputs the set of vertices* u*, such that* $f(u)$ *is connected to* $f(v)$ *or* u *such that* $(f(u), f(v))$

2 *permutations are said to be equivalent,* $f \sim g \iff N_f = N_g$

Let $A(G)$ *be the subgroup of* S_V *such that* $(u, v) \iff (f(u), f(v))$

Lemma 1. $f \sim g \iff \exists_{h \in A(G)} f = h \circ g$

Proof:

If $f \sim g$ then $N_f = N_g$. So for any vertices u and v we have that $(f(u), f(v)) \iff (g(u), g(v))$. Thus if we choose u' and v' such that $g(u') = u, g(v') = v$ we have that $(u, v) \iff (g(u'), g(v')) \iff (f(u'), f(v')) \iff (f(g^{-1}(u)), f(g^{-1}(v)))$. Which implies that $fg^{-1} \in A(G)$ and thus we can choose h to be fg^{-1}. If there is an $h \in A(G)$ such that $f = h \circ g$ then we have that $(h(g(u)), h(g(v))) \iff (g(u), g(v))$ which implies that $N_{h \circ g} = N_g$ and thus $N_f = N_g \implies f \sim g$ ∎

Theorem 3. $|S_V/\sim| = [S_V : A(G)] = \frac{|V|!}{|A(G)|}$

Proof:
This is due to the fact that $S_V/\sim$ are precisely the left cosets of $A(G)$ in S_V by lemma 1. The theorem now trivially follows from lagrange's theorem ∎

Corollary 1. *Let G_n denote the cyclic graph and since $A(G_n) = D_n$, the symmetries of the regular n-gon, the set $S_n/\sim$ has $\frac{n!}{2n}$ elements*

Corollary 2. *Let n be divisible by 4 so $n = 4k$ for some $k \geq 2$ and let the graph $G_{n,k}$ be the graph of 4 isomorphic copies of a graph of k vertices which looks like a line segment with all of the vertices on it. Then $|A(G_{n,k})| = 4!(2^4) = 384$ and thus $|S_n/\sim| = \frac{n!}{384}$*

To summarize my results to compute the amount of different ways to permute the vertices of graphs such that the neighbors of every vertex you need to know the order of the automorphism group of the graph $A(G)$.

The key element behind this result was to show that the "neighbor preserving" relation is exactly the same as the relation used to define the quotient object $S_G/A(G)$ where S_G denotes the symmetric group over the vertices and $A(G)$ is the subgroup of automorphisms of the graph G. The reason I say quotient object is because the set $S_G/A(G)$ is usually not a group.

D

Definition 0.0.2. *For n even let a_n denote the amount of ways to draw line segments between pairs of the n points on the unit circle such that none of the line segments intersect.*

If $n = 0$ then $a_n = 1$ and if n is odd then a_n is 0

Let $A_n = a_{2n}$

The sequence a_n can be computed recursively via this simple observation.

To generate all pairings of n points via line segments on the unit circle such that none of the lines intersect I will first designate a base point n, O and then enumerate the rest of the points from 1 to $n - 1$. Then for any i in between 1 and $n - 1$ then points that are not connected to anything will then by partitioned into two sets. One of $i - 1$ points and the other with $(n-2)-(i-1)$ points. We can then choose some connection of points in the first set and then another in the second set. So there are $a_i a_{(n-2)-(i-1)}$ possible types of connects. Also the two choices I made are independent of one another since any point in one set must be connected to another in the same set since line segments can't intersect. If you had two points from the 2 sets then the line segment between them would intersect the line segment from O our base point and point i.

Finally we can sum over all indices i and we get the following recurrence relation. Notice that since a_0 is 1 if we choose the left most point or right most point in the way I described above we would only have 1 choice instead of 2. And since a_n is 0 when n is odd we aren't summing anything undefined

$$a_n = \sum_{i=1}^{n-1} a_{i-1}a_{(n-2)-(i-1)}, a_0 = 1$$

Now we can reindex

$$a_n = \sum_{i=0}^{n-2} a_i a_{(n-2)-i}, a_0 = 1$$

Now this recurrence relation may seem quite similar to the one for catalan numbers which is below. There are two key differences.

$$C_n = \sum_{i=0}^{n-1} C_i C_{(n-1)-i}, C_0 = 1$$

For n odd a_n is 0 so we must use A_n to account for that AND our last index is $n-2$ not $n-1$

We can fix this by using A_n which is just equal to a_{2n}. First we have to reindex the sum once again based on the fact that $a_n = 0$ if n is odd. Thus we get the following relation.

$$a_n = \sum_{i=0}^{\frac{n}{2}-1} a_{2i}a_{(n-2)-2i}$$

And now we can change the n to $2n$ and substitute in A_n

$$A_n = \sum_{i=0}^{n-1} A_i A_{(n-1)-i}, A_0 = 1$$

And thus we have that A_n is just equal to C_n and for n even we have that $a_n = C_{\frac{n}{2}}$

E

Given set A of order L let's say there is some partition of it into k sets, $C_1, ...C_k$, each of order g. And let's say there was some function $c : A \to X$, such that when c is restricted to C_i it injectively maps it to the entirety of X. The questions asks how many ways are there to pick n elements, $a_1, ...a_n$ of A such that there is some $0 \le z \le k-n$ such that $a_1 \in C_{z+1}, ...a_n \in C_{z+n}$ AND $c(a_1), ...c(a_n)$ are all distinct

So to count this we first need to have a choice of z. The amount of z we can choose is obviously $k-n+1$. Next we need to choose the actual elements. So the first element there are g choices since c is a bijection from C_i to X, where $|C_i| = g$. And then the second there are $g-1$ and so on and so. So there are $g...(g-(n-1))$ possibilities. We can then multiply $k-n+1$ and $x..(x-(n-1))$ or rather $\frac{g!}{(g-n)!}$ to get $(k-n+1)\frac{g!}{(g-n)!}$. But we aren't done yet since we picked everything *accounting for order*. So our final step is to divide the entire thing by $n!$. And thus our final result is $(k-n+1)\binom{g}{n}$

So to summarize given the context, if you have L people with k different birthday dates, g people in each separate category such that everyone in each category wears different colored clothing, and a number n then the amount of "royalty tables" with n people with consecutive birthdays one could make is $(k-n+1)\binom{g}{n}$

F

The solution to the problem at hand is quite simple. Given 28 friends the amount of non-empty subsets of these friends, which corresponds to dinner parties is $2^{28} - 1$. And if we wanted to only invite a certain amount of people like 3 or 4 to each dinner party our answer would be $\binom{28}{3}$ or $\binom{28}{4}$

Now what if were to tweak the question a bit? Let's say that we want dinner parties which have *totally distinct* people in them. Additionally what if we only wanted to invite k people to every dinner party for some k?

Definition 0.0.3. *Let b_n denote the amount of partitions of a set of order n thus $b_0 = b_1 = 1$*

Let $b_{n,k}$ denote the amount of partitions such that each set in the partition is of equal order and all have order k

Let's first start off with $b_{n,k}$. Obviously for this to be a well defined expression we need $k|n$.

But other than that the expression for this number is quite straight-forward. To make a partition of a set of n elements in the manner I described above we first need to choose k elements. The amount of ways to do so is $\binom{n}{k}$. Now we just need to choose k more elements FROM the set of order $n - k$ since we are creating a partition. And we keep on doing this until we reach the moment we have $n - k(\frac{n}{k} - 1))$. While the number $k(\frac{n}{k} - 1)$ may look complicated and out of place it is the way I think about it is that if we were to do this process again, choose a subset of k elements, we would ONLY have 1 choice since we only have $n - k(\frac{n}{k} - 1) = n - (n - k) = k$ elements left.

And thus $b_{n,k}$ is $\prod_{i=0}^{\frac{n}{k}-1} \binom{n-ik}{k} = \binom{n}{k} .... \binom{n-k(\frac{n}{k}-1)}{k}$. Well not quite since we could've done this process in any certain order. Choosing one subset first or last is equivalent in our framework so we would have to then divide by $(\frac{n}{k})!$ to account for all of the ways I could choose these subsets in an ordered fashion

So the true expression for $b_{n,k}$ is $\frac{\prod_{i=0}^{\frac{n}{k}-1} \binom{n-ik}{k}}{(\frac{n}{k})!} = \frac{\binom{n}{k} .... \binom{n-k(\frac{n}{k}-1)}{k}}{(\frac{n}{k})!}$

Now to tackle b_n we will need to think recursively. Supposed we had a set of order $n+1$ and let's say we chose some subset of order between 0 and n and let's call it k. Then we can partition said subset into b_k ways and make a partition of our original set by just adding in the set which we get when removing our set of order k. And since any partition can be thought of our original set can be thought of in this manner we have the recurrence relation $b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} b_i$.

I will end this solution here since a closed form formula is out of the scope of this problem

G

This problem is quite trivial since we are only considering distinct people. The # of ways the passengers can get off is precisely the amount of functions from a set of order 2022 to a set of order 100. The amount of these functions is precisely 100^{2020}

Below is the proof of a very cool theorem in algebraic topology

Theorem 4. $\pi_1(S^1) \cong \mathbb{Z}$

In layman terms this theorem basically says that every loop on the circle *basically* has 3 forms. Either we are looking at the loop which is constant, i.e stays on one point. The loop goes clockwise around the circle some amount of time, until ending where it started. Or the loop goes counter-clockwise some amount of times, until ending where it started.

Now while that may be a self-evident fact, in the framework of algebraic topology it is very non-trivial and to prove it we must rely on the fact that the real line, $\mathbb{R}$ is a covering space of the circle S^1

Definition 0.0.4. *A space C is the covering space of a space $X \iff$ there exists a continuous map $p : C \to X$ such that for every $x \in X$ there is an open neighborhood of it, U, such that the pre-image $p^{-1}(U)$ is the disjoint union of open sets of C such that each of them are homomorphically mapped onto U by p*

Now before we get to that we still need to define the homotopy group and even before that I must define what I mean by a path homotopy and more generally homotopy

Also during this entire section when I use the term "map" I am referring to continuous functions, since the study of topology also includes the study of continuous functions which are the "special" maps of topological spaces. Formally, they are the arrows used in the definition of the category **Top**.

Definition 0.0.5. *Given 2 maps between 2 spaces X, Y a set of maps $\{f_t\}_{t \in I}$, where $I = [0, 1]$ are a homotopy between f and $g \iff H : I \times X \to Y$ defined by $H(t, x) = f_t(x)$ is continuous and $H(0, x) = f(x), H(1, x) = g$*

While the definition may seem tricky the next definition about path homotopy is much more intuitive

Definition 0.0.6. *Given 2 paths in the space X, i.e $f, g : I \to X$ which have the same endpoints, $f(0) = g(0), f(1) = g(1)$ a path homotopy is a homotopy such that the endpoints of each function in the homotopy are the same.*

So if we had H as defined before a path homotopy requires that $H(t, 0)$ & $H(t, 1)$ are constant for all inputs t

Now to get an intuitive of this look at the following image and consider the paths f_0 and f_1. The paths in between the functions are to visualize the path homotopy between them, and what a path homotopy let's us do is basically consider the 2 maps "equivalent" since one can be deformed into the other.

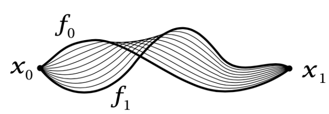

Figure 1: Stolen from a book

Now why is this useful? Well because in other areas of "geometry" which is quite a loaded term, to classify certain objects it is REALLY important to know how many "holes" it has. I use quotes since holes in arbitrary spaces may not have a useful visualization. And different dimensional holes are quite odd, since we live in a 3 dimensional world. But despite that in the study of 2 dimensional surfaces the amount of holes, or "genus" of an object is crucial in studying it so ways to mimic the genus have been created. Mathematicians have done this through the concept of homotopy!

The intuition is that if we look at loops, paths who's endpoints are the same, on a space X and considering 2 maps to be "the same" if they are path homotopic we can encounter holes. Consider a simple object such as the torus and notice the giant gaping hole it has in the middle of it. The way we can "encounter" it is by looking at loops which have to go through it. So any loop which has to go through the hole it has, can't be homotopic or be continuously deformed into a loop which never goes around the hole, since that would require us to split apart the loop if we would want to continuously deform it into the second loop.

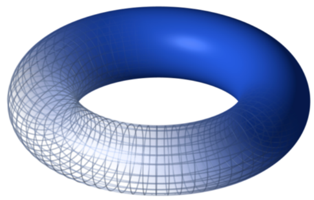

Figure 2: Stolen from wikipedia

Okay so we got the intuition down but how exactly do we formally construct an algebraic object based on the concept of path homotopy. Well first of all we need to show that the relation "is path homotopic to" or usually notated "$\simeq$" holds the same properties as the usual equality relation $=$. This is so when we want to create an algebraic object, or specifically a group we can formally define the elements of said group, and have them be well behaved.

Lemma 2. $\simeq$ *is an equivalence relation*

Proof:
Given any path f in any space X it is trivally path homotopic to itself by considering the constant homotopy $f_t = f$. If $f \simeq g$ and $H(t,x)$ is our path homotopy then we can consider the homotopy $H(1-t,x)$ from g to f. This is trivally also a path homotopy between g & f.

Finally if $f \simeq g$ and $g \simeq z$ then we have 2 path homotopies H and H'. We can define a third homotopy between f and z as $H''(t,x) = f_{2t}(x)$ for $0 \le t \le \frac{1}{2}$ and for $\frac{1}{2} \le t \le 1$ $H(t,x) = g_{2t-1}$. Which informally means that for each map in the homotopy you go through f twice as fast then go through g twice as fast. Obviously we have that $H''(0,x) = H(0,x)$ and $H''(1,x) = H'(1,x)$. To prove continuity we use the fact that a function is continuous $\iff$ It is continuous when each time it is restricted to a finite amount of closed sets who's union is the domain. The closed sets which when restricted too H'' is obviously continuous are $[0, \frac{1}{2}] \times I$, $[\frac{1}{2} \times I]$, and $\{\frac{1}{2}\} \times I$. And since they are the entirety of $I \times I$ we are done ∎

Now since $\simeq$ is an equivalence relation we can now consider the set $\pi_1(X,x)$ of loops *based at* x. The reason we have to identify some point of X is because all of our loops have to start somewhere. Now while this is terrible for most topological spaces, for path-connected ones i.e for every 2 points there is a path between them, the base point is irrelevant. I will prove this fact rigorously later.

Now to make $\pi_1(X,x)$ worth this amount of build-up we must define an operation on its elements. Which I will denote "$\cdot$" and is defined in the following way.

Given any 2 loops based at x $f \cdot g$ is defined to be the loop z which in the interval $t \in [0, \frac{1}{2}]$ is just $f(2t)$, and in the interval $[\frac{1}{2}, 1]$ is defined to be $g(2t-1)$. So given 2 loops this operation $\cdot$ creates a loop which traces through the first loop at double speed, comes back to based point, then goes through the other loop at double speed and then stops at the base point.

Now this operation is highly unstable in the set $\pi_1(X,x)$ we tweak the definition of $\pi_1(X,x)$ and define it to be the set of loops at base point x, modulo the path homotopy relation. We define the operation between equivalency classes the same may and instead taking the equivalence class of the new path created

This still is a well defined definition of our operation due to the following lemma

Lemma 3. *If $f_0 \simeq f_1$ and $g_0 \simeq g_1 \implies f_0 \cdot f_1 \simeq g_0 \cdot g_1$*

Proof:
Since $f_0 \simeq f_1$ and $g_0 \simeq g_1$ we have path homotopies f_t and g_t. We can then define a new path homotopy by $H(t,x) = (f_t(x) \cdot g_t)(x)$. It is constant on the endpoints since $H(t,0) = (f_t \cdot g_t)(0) = f_t(0)$ and $H(1,t) = (f_t \cdot g_t)(1) = g_t(1)$.

The reason the map H is continuous is because it is continuous on $[0, \frac{1}{2}] \times I$, and $[\frac{1}{2}, 1] \times I$ ∎

We can now consider $\pi_1(X,x)$ as an algebraic object by endowing it with the operation $\cdot$. The following lemma shows that $\pi_1(X,x)$ is indeed a group.

Lemma 4. *$\pi_1(X,x)$ is a group with operation $\cdot$ and identity $[c(s)]$ where $c(s) = 1$ for all $s \in [0,1]$*

Proof:
Before we dive into the proof let me define what a reparametrization of a loop f. It is the composite map defined as $f\chi$ where $\chi : I \to I$ such that $\chi(0) = 0$ and $\chi(1) = 1$. We have that $f\chi \simeq f$ by the path homotopy $f\chi_t$, $\chi_t(s) = (1-t)\chi(s) + ts$

Now we can see trivially that $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$ since they are just reparametrizations of each other since they traverse the same path but just as different speeds.

Let f be some loop and let $c(s) = f(1)$. Then $f \cdot c \simeq f$ since $f \cdot c$ is the map which goes through f twice as fast and stays on the endpoint in the interval $[\frac{1}{2}, 1]$. Thus $f \cdot c = f\chi$ where $\chi(s) = 2s$ for $0 \le s \le \frac{1}{2}$ and is just $f(1)$ on $[\frac{1}{2}, 1]$. Similarly $c \cdot f = f\psi$ where $\psi(s) = f(0) = f(1)$ on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$ is $2s$.

Now given any path f let $\bar{f}$ denote the inverse path $f(1-t)$ which starts at the endpoint of f, traces through f and then ends at the startpoint of f. Given a loop f consider the homotopy, $f_t \cdot g_t$ from $c(s) = f(0)$ to $f \cdot \bar{f}$. Where f_t is f on $[0,t]$ and $[t,1]$ is just equal to $f(t)$. And $g(t)$ is just the inverse path of f_t. Then this is a valid path homotopy since $(f_t \cdot g_t)(0)$ is always just $f(0)$ and $(f_t \cdot g_t)(1)$ is also just $f(0)$. Also the map $H(t,x) = (f_t \cdot g_t)(x)$ is continuous trivially. So we have that $f \cdot \bar{f} \simeq c(s)$ and analgously we have that $\bar{f} \cdot f \simeq c(s)$ ∎

Okay so now we know that $\pi_1(X,x,)$ is a valid group, but remember that I said that if we require a space to be path-connected, i.e any 2 points have a path which connects them, we don't have to worry about our base point. Well now I can rigorously formulate this property

Lemma 5. *Let h be a path from x to y then $b_h : \pi_1(X,x,\to)\pi_1(X,y,)$ is $b_h([f]) = [h \cdot f \cdot \bar{h}]$ is an isomorphism*

Proof:
It is well defined since if f_t is a homotopy from a loop f to g then $h \cdot f_t \cdot \bar{h}$ is a homotopy from $h \cdot f \cdot \bar{h}$ and $h \cdot g \cdot \bar{h}$

It is a homomorphism since $b_h[f \cdot g] = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}] = b_h[f]b_h[g]$

It is bijective and its functional inverse is $b_{\bar{h}}$ since $b_h(b_{\bar{h}}[f]) = b_h[\bar{h} \cdot f \cdot h] = [f]$ and similarly $b_{\bar{h}}(b_h[f]) = b_{\bar{h}}[h \cdot f \cdot \bar{h}] = [f]$ ∎

Now we can actually get to proving that $\pi_1(S^1) \cong \mathbb{Z}$

The main part of the proof is the map $p : \mathbb{R} \to S^1$ defined by the rule $p(s) = (\cos 2\pi s, \sin 2\pi s)$

One can imagine this map as the composition of 2 maps. The first map being the embedding of $\mathbb{R}$ into $\mathbb{R}^3$ as a helix via $f(x) = (\cos 2\pi s, \sin 2\pi s, s)$ and secondly a projection of this helix onto S^1 by just removing the z-coordinate. Here's an image showing the map from the the helix to the circle.

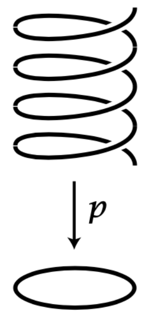

Figure 3: Stolen from Hatcher

Now one can somewhat see the intuition behind the definition of a covering space which I gave at the start of this section, but I will just put it here again. Every pre-image of a neighborhood of a point on the circle maps to countably many neighborhoods on the helix

Definition 0.0.7. *A space C is the covering space of a space $X \iff$ there exists a continuous map $p : C \to X$ such that for every $x \in X$ there is an open neighborhood of it, U, such that the pre-image $p^{-1}(U)$ is the disjoint union of open sets of C such that each of them are homomorphically mapped onto U by p*

Now finally I can start with the first definition on our journey of proving that the fundamental group, $\pi_1(X)$ of the circle is $\mathbb{Z}$.

Also the reason I don't specify a base point is because S^1 is path-connected

Definition 0.0.8. *$\psi : \mathbb{Z} \to \pi_1(S^1)$ is defined as $\psi(n) = [\omega_n(s)]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$*

Notice that $\omega_n(s) = (p \circ f_n)(s)$ where $f_n(s) = ns$ which is a path from 0 to n

Lemma 6. *$\psi(n) = [pf]$ where f is any path from 0 to n*

Proof:
Trivial since we already have that $pf_n = \omega_n(s)$ and since any 2 paths in $\mathbb{R}$ are path homotopic the lemma is true ∎

Lemma 7. *ψ is a homorphism*

Proof:
Let t_m denote the translation $t_m(x) = x + m$. Then we have that $\psi(m+n) = [p(f_m \cdot (t_m \circ f_n))]$ due to lemma 6. But $p(f_m \cdot (t_m \circ f_n)) = \omega_m \cdot \omega_n$ and thus $\psi(m+n) = \psi(m) \cdot \psi(n)$ ∎

Now we just need to show that ψ is bijective. We do this by first introducing the concept of a lift. Given any path f in S^1 a lift is a path in $\mathbb{R}$, f' such that $pf' = f$.

Now I can state 2 statements which prove that ψ is bijective

1. Given any path f in S^1 that starts at x then for any $y \in p^{-1}(x)$ there is a unique lift f' of f which starts at y
2. Given any path homotopy f_t starting at x then for any $y \in p^{-1}(x)$ there is a unique homotopy of lifts of each of the f_t, ${f'}_t$ which start at y

Now let's show that these imply that ψ is bijective

To show that ψ is surjective let us first fix $(1, 0)$ as the base point we are working with. And let f be a path starting at $(1, 0)$ then there is unique lift by (1) f' which is a path from 0 to some integer n since $f'(1)$ must be such that $pf'(1) = f(1) = (1, 0)$ and we have that $p^{-1}(1, 0) = \mathbb{Z}$. So by lemma 6 $\psi(n) = [f]$

To show that ψ is injective let n and m be such that $w_n \simeq w_m$. Then by (2) there is unique lifted path homotopy from f_n to f_m which starts from 0. Since this is a path homotopy the end points must be constant for any map in the homotopy, which I will denote as f_t. So $f_1(1) = f_m(1) = m$ and since $f_0(1) = w_n(1) = n$ we have that $n = m$.

Now we just need to prove those 2 statements but due to the following lemma we just need to prove one general statement

Lemma 8. *If given any map $F : Y \times I \to S^1$ and a map $F' : Y \times \{0\} \to \mathbb{R}$ which lifts F restricted to $Y \times \{0\}$ then there is a unique extension of F', $F'' : Y \times I \to \mathbb{R}$ such that $pF' = F$ THEN the 2 statements which I stated above are true*

Proof:
For the first statement given any path in S^1 by our hypothesis if Y is just a point and we make f our F, $f(0)$ represent F and any point y such that $p(y) = f(0)$ represent our F' then we have a map $F'' : I \to \mathbb{R}$, or rather a path in $\mathbb{R}$ which lifts F or rather our original path f

For the second statement if we have any homotopy of paths in S^1 denoted f_t, i.e a map $F : I \times I \to S^1$ then our F' would just be a lift of $F : I \times \{0\} \to S^1$ or rather $f_0(x)$ which we knows should exist due to the above proof. Then by our hypothesis F' can be uniquely extended to a map $F'' : I \times I \to \mathbb{R}$. And when we restrict F'' to $I \times \{0\}$ and $I \times \{1\}$ we get lifts of the constant path at $x = f_t(0)$ and by uniqueness we have that F'' is a homotopy of paths ∎

Now we're almost done

Theorem 5. *If given any map $F : Y \times I \to S^1$ and a map $F' : Y \times \{0\} \to \mathbb{R}$ which lifts F restricted to $Y \times \{0\}$ then there is a unique extension of F', $F'' : Y \times I \to \mathbb{R}$ such that $pF' = F$*

Proof:
Before I get to the meat of the proof let me just state what we already know. F is a map from $Y \times I \to S^1$, F' is a map from $Y \times \{0\} \to \mathbb{R}$ which is a lift of F when it is restricted to $Y \times \{0\}$.

Now we construct F'' for any neighborhood of a point of y. So let $y \in Y$ and let $y \in U$ be open. Since F is continuous any point $(y, t) \in Y \times I$ has a neighborhood $U_t \times (a_t, b_t)$ such that $F(U_t \times (a_t, b_t)) \subset V_\alpha$ for some evenly covered neighborhood of $F(y_0, t_0)$. Which is a neighborhood such that there exist a disjoint collection of neighborhoods in $\mathbb{R}$ which are mapped homeomorphically onto said neighborhood or rather evenly covered

Since $I \times \{y\}$ is compact there are finitely many $V_t \times (a_t, b_t)$ which cover $\{y\} \times I$. So given our neighborhood U of y we can choose t_i such that $0 = t_0 < ... < t_n = 1$ and such that $F(U \times [t_i, t_{i+1}]) \subset V_i$ where V_i is some evenly covered neighborhood.

Let's then proceed by induction.

We'll be doing induction of the following statement

F'' is a well-defined map on $U \times [0, t_i]$ and lifts $F \equiv P(i)$

The statement $P(0)$ is already given since we can just define it as F' when restricted to $U \times \{0\} \subset Y \times \{0\}$

Then inductively that F'' is defined on $U \times [0, t_i]$. Then by our definition of covering space there should exist open sets of $\mathbb{R}$ such that ${V'}_i$ which are mapped homemorphically onto V_i by p which also contains the point $F''(y, t_i)$ we also can assume that $F''(U \times \{t_i\}) \subset {V'}_i$ by replacing U with a smaller neighborhood by say $U \bigcap F''|(U \times t_i)^{-1}({V'}_i)$. Now we can define F'' on $U \times [t_i, t_{i+1}]$ by $p^{-1}F$ by considering p^{-1} to be the homemorphism of V_i onto ${V'}_i$. We then finally get F'' to be a fully defined map from $U \times I \to \mathbb{R}$

Now we need to prove the uniqueness of F'' when Y is a point

Given any 2 lifts, z, g, of a path on S^1, f such that $z(0) = g(0)$ we can attempt to the thing we did before. Choose some partition of $[0, 1]$, t_i such that $F([t_i, t_{i+1}]) \subset V_i$.

Assume inductively that $z = g$ on $[0, t_i]$.Since $[t_i, t_{i+1}]$ is connected $z([t_i, t_{i+1}])$ is connected and thus must lie in exactly 1 ${V'}_i$. And thus the same is for g since $z(t_i) = g(t_i)$. And since z and g are lifts of f even when restricted to $z[t_i, t_{i+1}]$ and $g[t_i, t_{i+1}]$ we have that $p(z) = p(g)$ when we restrict them to ${V'}_i$. Since p is injective when restricted to ${V'}_i$ we have that $z = g$

Now the final step is to show that in general we can lift F'' to a map defined on $Y \times I$. By the preceding proof for any open set U we can construct F'' on $U \times I$. Whenever we have a $(y, t) \in U \bigcap U' \times I$ $F''(y, t)$ is uniquely determined since any lift of this manner on a single segment $\{y\} \times I$ is automatically unique, so constructing F'' globally via open sets will be unique on each intersection.

Finally F'' is continuous on $Y \times I$ since it is continuous when restricted to any set of the form $U \times I$ where $U \subset S^1$
∎

Now FINALLY I can state the following

Corollary 3. $\pi_1(S^1) \cong \mathbb{Z}$

This last section is just a nice application of what we just proved

Theorem 6. *Every non-constant polynomial with complex coefficients has a complex root*

Proof:
For the sake of contradiction let p be a non-constant polynomial that has no root. WLOG $p(z) = z^n + a_{n-1}z^{n-1} + ...a_0$

Since $p(z)$ has no roots we can define $f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$ for any $r \geq 0$ and this defines a loop on the unit circle S^1 when it is considered to be a part of the complex plane. If $r = 0$ then we get the constant map and since as r varies we have a path homotopy between all of the f_r we have that $[f_r] = [\omega_0]$ where ω_0 denotes the constant map for the start/end point of f_r.

Now consider a real number r such that it is larger than $1 + |a_0| + .. + |a_{n-1}|$ then we have the following inequalities if $|z| = r$

$$|z^n| = r^n = rr^{n-1} > (|a_0| + .. + |a_{n-1}|)|z|^{n-1} \geq |p(z) - z^n|$$

The inequalities above imply that for $t \in I$ the polynomial $p_t(z) = z^n + t(p(z) - z^n)$ must have no roots on the circle $|z| = r$.

Since if it did for some $|z| = r$ then we would have that $|z^n| = t|p(z) - z^n| \leq |p(z) - z^n|$ which would contradict our above inequality.

Now let r be as above and define $F_t(s) = \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}$. Now consider the homotopy from $t = 1$ to $t = 0$. It is a homotopy from F_1 which remember is still homotopic to the constant map. And $F_0(s) = \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|} = \frac{e^{2\pi ins}}{|e^{2\pi ins}|}$ which is equal to $e^{2\pi ins} = \omega_n(s)$ and thus we have that $\omega_n(s) \simeq \omega_0(s)$ which implies that $n = 0$ ∎

H

The problem can be restated & generalized in the following way

How many lattice paths are there from the point $(0, , , 0)$ and $(n_1 - 1, ..., n_k - 1)$ where n_i are the dimensions of the hyperrectangle we are considering and we can only move from one center of a unit-hypercube to one adjacent to it

The reason we can restate our problem like this is because it asks the amount of paths from one corner, or rather $(0, , 0)$ to the opposite corner or $(n_1 - 1, ..., n_k - 1)$.

The reason I subtract 1 is because we're only moving through the center of each unit-hypercube.

Now since we've successfully transported our problem into a much more combinatorics like setting we can easily see that each path must involve $n_i - 1$ movements in each dimension, or rather "direction". So we can think about it in terms of choices. First you choose $n_k - 1$ directions from a set of order $n_1 - 1 + ... + n_k - 1$ of possible directions. Then you choose $n_{k-1} - 1$ directions from a set of order $n_1 - 1 + ... + n_{k-1} - 1$ and so one until you have to choose $n_1 - 1$ directions from a set of $n_1 - 1$ directions, for which there is only 1 choice.

$$\binom{n_1 - 1 + ... + n_k - 1}{n_k - 1} \cdots \binom{n_1 - 1}{n_1 - 1}$$

And in fact there is another way of notating this and its called the multinomial coefficient

$$\binom{n_1 - 1 + ... + n_k - 1}{n_1 - 1, ..., n_k - 1} = \binom{n_1 - 1 + ... + n_k - 1}{n_k - 1} \cdots \binom{n_1 - 1}{n_1 - 1}$$

J

Theorem 7. *Given n points such that no* 3 *are collinear then the amount of* k*-sided polygons that can be drawn is* $\binom{n}{k}$

Proof:

Since no 3 points are collinear that means that any set of k points we choose constitutes a k-sided polygon. Formally let A be a set of points on the plane, and let N_k be the set of k-sided polygons. Then we have a natural bijection between k order subsets of A, A_k and N_k. Given $B \in A_k$ let $f(B)$ denote the polygon w/ points from that set. This map is obviously well defined due to the fact that no 3 points of A are collinear, and since a polygon in the plane is distinguished by its points if $f(B) = f(C)$ then $B = C$ since the polygon $f(A)$ is completely characterized by its points. And the map f is surjective since every polygon with k-points is just $f(B)$ where B is the set of its k points ∎

K

Definition 0.0.9. *Let I be some finite set, and let $M \subseteq I$*

Then I define the following set $I^k(M) = \{(a_1, ..., a_k) | \forall_{a_i \in M} a_{i+1} \notin M\}$

Lemma 9. *Let I and M be as above and let $|I| = A$ and let $|M| = m$ then $|I^k(M)| = pA^k$ where p is probability of picking $0's$ & $1's$ and getting a sequence of k numbers such that if $a_i = 1$ then $a_{i+1} = 0$ where the chance of getting a* 1 *is $\frac{m}{A}$ and the chance of getting a 0 is $\frac{A-m}{A}$*

Proof:

One can construct a tree with 2 possibilities each time you perform an experiment. The first one being that you pick an element randomly out of the set $I \setminus M$ and the other being that you pick an element from the set M randomly. The probability of getting a sequence of k elements such that $a_i \in M \implies a_{i+1} \in A \setminus M$ is obviously the same as the probability of the experiment mentioned in the lemma when you add up probabilities and when you think of getting a 1 as the same as getting an element in M ∎

Now the above lemma was that a cool result here's the lemma which shows how to actually compute the order of $|I^k(M)|$

Lemma 10. *If $|I| = A$ and $M \subseteq I$, $|M| = m$ then let $p_k = |I^k(M)|$ so $p_0 = 1$ and $p_1 = A$ and we have the following recurrence relation; $p_k = (A-m)p_{k-1} + m(A-m)p_{k-2}$*

Proof:

This relation comes from the fact that if $k \geq 2$ then $I^{k+1}(M) = (I^k(M) \times I \setminus M) \bigcup (I^{k-1}(M) \times I \setminus M \times M)$. Which basically mean that any element in I^{k+1} either is an element of $I^k(M)$ ending with an element of $I \setminus M$ OR it ends with an element from M and thus its second to last element is in $I \setminus M$ and the rest of the $k-1$ tuple is an element of $I^{k-1}(M)$ ∎

Corollary 4. *Let $I = \{0, 1\}$ and $M = \{1\}$ then if $|I^k(M)| = p_k$ we have the following recurrence relation*

$$p_k = p_{k-1} + p_{k-2}, p_0 = 1, p_1 = 2$$

And thus $p_k = F_{k+3} = \frac{\phi^{k+3} - (1-\phi)^{n+3}}{2\phi - 1}$ where F_k denotes the fibonacci sequence and $\phi = \frac{1+\sqrt{5}}{2}$

This is the end of this pc since I'm too lazy to do the other problems AND I'm too lazy to follow up on the full general version of lemma 10 since I don't want to type up 20 steps to diagonalize a 2×2 matrix