Basic Results

Definition 0.0.1. *The arithmetic derivative, D, is defined recursively in the following manner for all natural numbers* $n \in \mathbb{N}$

1. $D(1) = D(0) = 0$
2. $D(p) = 1$ *for any prime* p
3. *For every pair of natural numbers* n *&* m*,* $D(mn) = mD(n) + nD(m)$

The arithmetic derivative is much like the normal derivative operator if one sees the numbers 0 & 1 as the "constant" functions, prime numbers as "lines" with slope 1 and if one required the arithmetic derivative to satisfy the Lebiniz rule.

Since this definition doesn't allow one to directly compute the derivative of a number quickly, the following lemma shall help us greatly.

Lemma 1. *If*

$$x = \prod_{i=1}^{n} m_i$$

then

$$D(x) = x \sum_{i=1}^{n} \frac{D(m_i)}{m_i}$$

Proof:

We shall proceed by induction on the amount of elements in the product expansion of x. If $n = 1$ then $x = m_1$ and thus

$$x \sum_{i=1}^{n} \frac{D(m_i)}{m_i} = (m_1)\frac{D(m_1)}{m_1} = D(m_1) = D(x)$$

Now let's assume that for any x with n factors in its product expansion the formula

$$D(x) = x \sum_{i=1}^{n} \frac{D(m_i)}{m_i}$$

applies. Let x' be equal to xm_{n+1} and thus $D(x') = D(x)m_{n+1} + xD(m_{n+1})$. We can rewrite $D(x)m_{n+1} + xD(m_{n+1})$ as

$$xm_{n+1} \sum_{i=1}^{n} \frac{D(m_i)}{m_i} + \frac{x' D(m_{n+1})}{m_{n+1}} = x' \sum_{i=1}^{n} \frac{D(m_i)}{m_i} + x' \frac{D(m_{n+1})}{m_{n+1}}$$

.

We can then factor out the x' and our final expression for $D(x')$ is $x' \sum_{i=1}^{n+1} \frac{D(m_i)}{m_i}$ which is exactly what we wanted to show ∎

With lemma 1 in hand we can easily write down an expression for $D(x)$ in terms of its prime factors.

$$\text{Let } x = \prod_{i=1}^{n} {p_i}^{k_i}$$

be x's prime factorization.

$$D(x) = x \sum_{i=1}^{n} \frac{D({p_i}^{k_i})}{{p_i}^{k_i}}$$

Obviously, $D(p^k)$ is kp^{k-1}, since due to lemma 1 we have

$$D(p^k) = p^k \sum_{i=1}^{n} \frac{D(p)}{p} = p^k \sum_{i=1}^{n} \frac{1}{p}$$

and thus

$$D(p^k) = \sum_{i=1}^{n} p^{k-1} = kp^{k-1}$$

Finally, we have a fully simplified expression for the arithmetic derivative of x.

$$D(x) = x \sum_{i=1}^{n} \frac{D({p_i}^{k_i})}{{p_i}^{k_i}} = x \sum_{i=1}^{n} \frac{k_i {p_i}^{k_i - 1}}{{p_i}^{k_i}} = x \sum_{i=1}^{n} \frac{k_i}{p_i}$$

By using the notation $\upsilon_p(x)$ to denote the exponent of a prime p in the prime factorization of x the above formula can be rewritten as the following.

$$D(x) = x \sum_{p|x} \frac{\upsilon_p(x)}{p}$$

Upper & Lower Bounds