Refinement on spectral Turán's theorem*

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April 27, 2022

Abstract

A well-known result in extremal spectral graph theory, known as Nosal's theorem, states that if G is a triangle-free graph on n vertices, then $\lambda(G) \le \lambda(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$, equality holds if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Nikiforov [Linear Algebra Appl. 427 (2007)] extended Nosal's theorem to K_{r+1}-free graphs for every integer $r \ge 2$. This is known as the spectral Turán theorem. Recently, Lin, Ning and Wu [Combin. Probab. Comput. 30 (2021)] proved a refinement on Nosal's theorem for non-bipartite triangle-free graphs. In this paper, we provide alternative proofs for the result of Nikiforov and the result of Lin, Ning and Wu. Our proof can allow us to extend the later result to non-r-partite K_{r+1}-free graphs. Our result refines the theorem of Nikiforov and it also can be viewed as a spectral version of a theorem of Brouwer.

Key words: Mantel theorem; Turán theorem; Spectral radius; Nikiforov theorem; Zykov symmetrization.

2010 Mathematics Subject Classification. 05C50, 05C35.

1 Introduction

Extremal graph theory is becoming one of the significant branches of discrete mathematics nowadays, and it has experienced an impressive growth during the last few decades. With the rapid development of combinatorial number theory and combinatorial geometry, extremal graph theory has a large number of applications to these areas of mathematics. Problems in extremal graph theory deal usually with the question of determining or estimating the maximum or minimum possible size of graphs satisfying some certain requirements, and further characterize the extremal

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graphs attaining the bound. For example, one of the most well-studied problems is the Turán-type problem, which asks to determine the maximum number of edges in a graph which forbids the occurence of some specific substructures. Such problems are related to other areas including theoretical computer science, discrete geometry, information theory and number theory.

1.1 The classical extremal graph problems

Given a graph F, we say that a graph G is F-free if it does not contain an isomorphic copy of F as a subgraph. For example, every bipartite graph is C_3-free, where C_3 is a triangle. The *Turán number* of a graph F is the maximum number of edges in an F-free n-vertex graph, and it is usually denoted by $\mathrm{ex}(n,F)$. An F-free graph on n vertices with $\mathrm{ex}(n,F)$ edges is called an *extremal graph* for F. Over a century old, a well-known theorem of Mantel [35] states that every n-vertex graph with more than $\lfloor \frac{n^2}{4} \rfloor$ edges must contain a triangle as a subgraph. We denote by $K_{s,t}$ the complete bipartite graph with parts of sizes s and t.

Theorem 1.1 (Mantel, 1907)**.** *Let G be an n-vertex graph. If G is triangle-free, then $e(G) \le e(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = \lfloor \frac{n^2}{4} \rfloor$, equality holds if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Let K_{r+1} be the complete graph on $r+1$ vertices. In 1941, Turán [48] proposed the natural question of determining $\mathrm{ex}(n,K_{r+1})$ for every integer $r \ge 2$. Let $T_r(n)$ denote the complete r-partite graph on n vertices whose part sizes are as equal as possible. That is, $T_r(n) = K_{t_1,t_2,\ldots,t_r}$ with $\sum_{i=1}^r t_i = n$ and $|t_i - t_j| \le 1$ for $i \ne j$. This implies that each vertex part of $T_r(n)$ has size either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. The graph $T_r(n)$ is usually called the Turán graph. In particular, we have $T_2(n) = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Importantly, Turán [48] extended Mantel's theorem and proved the following result.

Theorem 1.2 (Turán, 1941)**.** *Let G be a graph on n vertices. If G is K_{r+1}-free, then*

$$e(G) \le e(T_r(n)),$$

equality holds if and only if G is the r-partite Turán graph $T_r(n)$.

Many different proofs of Turán's theorem have been found in the literature; see [1, pp. 269–273] and [4, pp. 294–301] for more details. Furthermore, there are various extensions and generalizations on Turán's result; see, e.g., [5, 7]. In the language of extremal number, Turán's theorem can be stated as

$$\mathrm{ex}(n,K_{r+1}) = e(T_r(n)).$$

Moreover, we can easily see that $(1-\frac{1}{r})\frac{n^2}{2} - \frac{r}{8} \le e(T_r(n)) \le (1-\frac{1}{r})\frac{n^2}{2}$ and $e(T_r(n)) = \lfloor (1-\frac{1}{r})\frac{n^2}{2} \rfloor$ for every integer $r \le 7$. Thus Theorem 1.2 implies the explicit numerical bound $e(G) \le (1-\frac{1}{r})\frac{n^2}{2}$ for every n-vertex K_{r+1}-free graph G. This bound is more concise and called the weak version of Turán's theorem. The problem of determining $\mathrm{ex}(n,F)$ is usually referred to as the Turán-type extremal graph problem. It is a cornerstone of extremal graph theory to understand $\mathrm{ex}(n,F)$ for various graphs F; see [18, 44] for comprehensive surveys.

1.2 The spectral extremal graph problems

Let G be a simple graph on n vertices. The *adjacency matrix* of G is defined as $A(G) = [a_{ij}] \in \mathbb{R}^{n\times n}$ where $a_{ij} = 1$ if two vertices v_i and v_j are adjacent in G, and $a_{ij} = 0$ otherwise. We say that G has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if these values are eigenvalues of the adjacency matrix $A(G)$. Since $A(G)$ is a symmetric real matrix, we write $\lambda_1, \lambda_2, \ldots, \lambda_n$ for the eigenvalues of G in decreasing order. Let $\lambda(G)$ be the maximum value in absolute among all eigenvalues of G, which is known as the *spectral radius* of graph G. Since the adjacency matrix $A(G)$ is nonnegative, the Perron–Frobenius theorem (see, e.g., [56, p. 120–126]) implies that the spectral radius of a graph G is actually the largest eigenvalue of G and it corresponds to a nonnegative eigenvector. Moreover, if G is further connected, then $A(G)$ is an irreducible nonnegative matrix, $\lambda(G)$ is an eigenvalue with multiplicity one and there exists an entry-wise positive eigenvector corresponding to $\lambda(G)$.

The classical extremal graph problems usually study the maximum or minimum number of edges that the extremal graphs can have. Correspondingly, the extremal spectral problems are well-studied in the literature. The spectral Turán function $\mathrm{ex}_{\lambda}(n, F)$ is define to be the largest spectral radius (eigenvalue) of the adjacency matrix in an F-free n-vertex graph, that is,

$$\mathrm{ex}_{\lambda}(n, F) := \max\big\{\lambda(G) : |G| = n \text{ and } F \nsubseteq G\big\}.$$

In 1970, Nosal [42] determined the largest spectral radius of a triangle-free graph in terms of the number of edges, which also provided the spectral version of Mantel's theorem. Note that when we consider a graph with given number of edges, we shall ignore the possible isolated vertices if there are no confusions.

Theorem 1.3 (Nosal, 1970)**.** *Let G be a graph on n vertices with m edges. If G is triangle-free, then*

$$\lambda(G) \le \sqrt{m}, \tag{1}$$

equality holds if and only if G is a complete bipartite graph. Moreover, we have

$$\lambda(G) \le \lambda(K_{\lfloor \frac{n}{2}\rfloor, \lceil \frac{n}{2}\rceil}), \tag{2}$$

equality holds if and only if G is a balanced complete bipartite graph $K_{\lfloor \frac{n}{2}\rfloor, \lceil \frac{n}{2}\rceil}$

Nosal's theorem implies that if G is a bipartite graph, then $\lambda(G) \le \sqrt{m}$, equality holds if and only if G is a complete bipartite graph. On the one hand, Nosal's theorem implies the classical Mantel theorem. Indeed, applying the Rayleigh inequality, we have $\frac{2m}{n} \le \lambda(G) \le \sqrt{m}$, which yields $m \le \lfloor \frac{n^2}{4} \rfloor$. On the other hand, applying the Mantel theorem to (1), we obtain that $\lambda(G) \le \sqrt{m} \le \sqrt{\lfloor n^2/4 \rfloor} = \lambda(K_{\lfloor \frac{n}{2}\rfloor, \lceil \frac{n}{2}\rceil})$. So inequality (1) in Nosal's theorem can imply inequality (2), which is called the spectral Mantel theorem.

Over the past few years, various extensions and generalizations on Nosal's theorem have been obtained in the literature; see, e.g., [36, 37, 38, 51] for extensions of K_{r+1}-free graphs, [33, 54, 55, 31] for extensions of graphs with given size. In addition, many spectral extremal problems are also obtained recently; see [11, 12] for the friendship graph and the odd wheel, [29, 13] for intersecting odd cycles and cliques, [50] for a recent conjecture. We recommend the surveys [39, 10, 28] for interested readers. The eigenvalues of the adjacency matrix sometimes can give some information about the structure of a graph. There is a rich history on the study of bounding the eigenvalues of a graph in terms of various parameters; see [6] for spectral radius and cliques, [46, 32] for eigenvalues of outerplanar and planar graphs, and [47] for the Colin de Verdière parameter, excluded minors, and the spectral radius.

In 1986, Wilf [51] provided the first result regarding the spectral version of Turán's theorem and proved that for every n-vertex K_{r+1}-free graph G, we have

$$\lambda(G) \le \left(1 - \frac{1}{r}\right) n. \tag{3}$$

In 2002, Nikiforov [36] proved that for every m-edge K_{r+1}-free graph G,

$$\lambda(G) \le \sqrt{2m\left(1 - \frac{1}{r}\right)}. \tag{4}$$

It is worth mentioning that both (3) and (4) are direct consequences of the celebrated Motzkin–Straus theorem. On the one hand, combining with $\frac{2m}{n} \le \lambda(G)$, we see that both Wilf's result and Nikiforov's result can imply the weak version of Turán theorem: $m \le (1 - \frac{1}{r})\frac{n^2}{2}$. On the other hand, using the weak Turán theorem $m \le (1 - \frac{1}{r})\frac{n^2}{2}$, we know that Nikiforov's result (4) implies Wilf's result (3).

In 2007, Nikiforov [37] showed a spectral version of the Turán theorem.

Theorem 1.4 (Nikiforov, 2007)**.** *Let G be a graph on n vertices. If G is K_{r+1}-free, then*

$$\lambda(G) \le \lambda(T_r(n)),$$

equality holds if and only if G is the r-partite Turán graph $T_r(n)$.

In other words, Nikiforov's result implies that

$$\mathrm{ex}_{\lambda}(n, K_{r+1}) = \lambda(T_r(n)).$$

By calculation, we can obtain that $(1 - \frac{1}{r})n - \frac{r}{4n} \le \lambda(T_r(n)) \le (1 - \frac{1}{r})n$. Thus, Theorem 1.4 implies Wilf's result (3). It should be mentioned that the spectral version of Turán's theorem was early studied independently by Guiduli in his PH.D. dissertation [21, pp. 58–61] dating back to 1996 under the guidance of László Babai.

A natural question is the following: what is the relation between the spectral Turán theorem and the edge Turán theorem? Does the spectral bound imply the edge bound of Turán's theorem? This question was also proposed and answered in [21, 38]. The answer is positive. It is well-known that $e(G) \le \frac{n}{2}\lambda(G)$, with equality if and only if G is regular. Although the Turán graph $T_r(n)$ is sometimes not regular, but it is nearly regular. Upon calculation, it follows that $e(T_r(n)) = \left\lfloor \frac{n}{2}\lambda(T_r(n)) \right\rfloor$. With the help of this observation, the spectral Turán theorem implies that

$$e(G) \le \left\lfloor \frac{n}{2}\lambda(G) \right\rfloor \le \left\lfloor \frac{n}{2}\lambda(T_r(n)) \right\rfloor = e(T_r(n)).$$

Thus the spectral Turán Theorem 1.4 implies the classical Turán Theorem 1.2. To some extent, this interesting implication has shed new lights on the study of spectral extremal graph theory.

Recently, Lin, Ning and Wu [33, Theorem 1.4] proved a generalization of Nosal's theorem for non-bipartite triangle-free graphs (Theorem 3.2). In this paper, we shall extend the result of Lin, Ning and Wu to non-r-partite K_{r+1}-free graphs. Our result is also a refinement on Theorem 1.4 in the sense of stability result (Theorem 4.3). The motivation is inspired by the works [21, 37, 23, 24], and it uses mainly the spectral extension of the Zykov symmetrization [57]. This article is organized as follows. In Section 2, we shall give an alternative proof of the spectral Turán Theorem 1.4. To make the proof of Theorem 4.3 more transparent, we will present a different proof of the result of Lin, Ning and Wu [33] in Section 3. In Section 4, we shall show the detailed proof of our main result (Theorem 4.3). In Section 5, we shall discuss the spectral extremal problem in terms of the p-spectral radius. Section 6 contains some possible open problems, including the spectral extremal problems for F-free graphs with the chromatic number $\chi(G) \ge t$, the problems in terms of the signless Laplacian spectral radius, and the A_α-spectral radius of a graph.

2 Alternative proof of Theorem 1.4

The proof of Nikiforov [37] for Theorem 1.4 is more algebraic and based on the characteristic polynomial of the complete r-partite graph. Moreover, his proof also relies on a theorem relating the spectral radius and the number of cliques [36], as well as an old theorem of Zykov [57] (also proved independently by Erdős [15]), which is now known as the clique version of Turán's theorem.

In addition, the proof of Guiduli [21, pp. 58–61] for the spectral Turán theorem is completely different from that of Nikiforov. The main idea of Guiduli's proof reduces the problem of bounding the largest spectral radius among K_{r+1}-free graphs to complete r-partite graphs by applying a spectral extension of Erdős' degree majorization algorithm [16]. Then one can show further that the balanced complete r-partite graph attains the maximum spectral radius among all complete r-partite graphs; see, e.g., [23, 24] for more spectral applications, and [19, 3] for related topics.

In this section, we shall provide an alternative proof of Theorem 1.4. The proof is motivated by the papers [21, 23, 24], and it is based on a spectral extension of the Zykov symmetrization [57], which is becoming a powerful tool for extremal graph problems; see, e.g., [20] for a recent application on the minimum number of triangular edges.

For $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, we denote $\|\boldsymbol{x}\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. Since the adjacency matrix $A(G)$ is real and symmetric, the Rayleigh formula gives

$$\lambda(G) = \max_{\|\boldsymbol{x}\|_2=1} \boldsymbol{x}^T A(G)\boldsymbol{x} = \max_{\|\boldsymbol{x}\|_2=1} 2 \sum_{\{i,j\}\in E(G)} x_i x_j.$$

We denote $|\boldsymbol{x}| = (|x_1|, |x_2|, \dots, |x_n|)^T$. Suppose that $\boldsymbol{x} \in \mathbb{R}^n$ is an optimal vector, i.e., $\|\boldsymbol{x}\|_2 = 1$ and $\lambda(G) = \boldsymbol{x}^T A(G)\boldsymbol{x}$, then so is $|\boldsymbol{x}|$. Thus there is always a non-negative unit vector $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$ such that $\lambda(G) = \boldsymbol{x}^T A(G)\boldsymbol{x}$. Given a vector $\boldsymbol{x}$, we know from Rayleigh's formula (or Lagrange's multiplier method) that $\boldsymbol{x}$ is an optimal vector if and only if $\boldsymbol{x}$ is a unit eigenvector corresponding to $\lambda(G)$. Namely, for every $v \in V(G)$, we have

$$\lambda(G)x_v = (A(G)\boldsymbol{x})_v = \sum_{u\in V(G)} a_{vu}x_u = \sum_{u\in N_G(v)} x_u.$$

This equation implies that if G is connected, then every nonnegative optimal vector is entry-wise positive. Indeed, otherwise, if $x_v = 0$ for some $v \in V(G)$, then we get $x_u = 0$ for every $u \in N(v)$. Similarly, we get $x_w = 0$ for every $w \in N(u)$. The connectivity of G leads to $x_w = 0$ for every $w \in V(G)$, and so $\boldsymbol{x}$ is a zero vector, which is a contradiction. Thus there exists an entry-wise positive eigenvector $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ corresponding to $\lambda(G)$ whenever G is a connected graph. This fact will be frequently used throughout the paper.

The following Lemma was proved by Feng, Li and Zhang in [17, Theorem 2.1] by using the characteristic polynomial of a complete multipartite graph; see, e.g., [25, Theorem 2] for an alternative proof and more extensions.

Lemma 2.1 (Feng–Li–Zhang, 2007)**.** *If G is an r-partite graph on n vertices, then*

$$\lambda(G) \leq \lambda(T_r(n)),$$

equality holds if and only if G is the r-partite Turán graph $T_r(n)$.

Now, we present our alternative proof of Theorem 1.4.

Proof of Theorem 1.4. Let G be a K_{r+1}-free graph on n vertices with maximum value of the spectral radius and $V(G) = \{1, 2, \dots, n\}$. Firstly, we show that G is a connected graph. Otherwise, adding a new edge between a component attaining the spectral radius of G and any other component will strictly increase the spectral radius of G, and it does not create a copy of K_{r+1}. Since G is connected, we take $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ as a unit positive eigenvector of $\lambda(G)$. Hence, we have

$$\lambda(G) = 2 \sum_{\{i,j\}\in E(G)} x_i x_j.$$

Our goal is to show that G is the Turán graph $T_r(n)$. First of all, we will prove that G must be a complete t-partite graph for some integer t. Since G is K_{r+1}-free, this implies $2 \le t \le r$. Observe that G attains the maximum spectral radius, Lemma 2.1 implies that G is further balanced, i.e., G is the t-partite Turán graph $T_t(n)$. Note that $\lambda(T_t(n)) \le \lambda(T_r(n))$. The maximality gives $t = r$ and $G = T_r(n)$.

We assume on the contrary that G is not complete t-partite for every $t \in [2, r]$, so there are three vertices $u, v, w \in V(G)$ such that $vu \notin E(G)$ and $uw \notin E(G)$ while $vw \in E(G)$. (This reveals that the non-edge relation between vertices is not an equivalent binary relation, as it does not satisfy the transitivity.) Throughout the paper, we denote by $s_G(v, \boldsymbol{x})$ the sum of weights of vertices in $N_G(v)$. Namely,

$$\boxed{s_G(v, \boldsymbol{x}) := \sum_{i \in N_G(v)} x_i.}$$

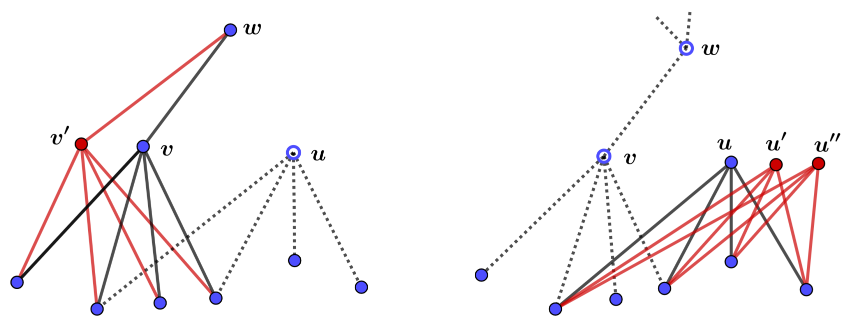

Figure 1: The spectral Zykov symmetrization.

Case 1. $s_G(u, \boldsymbol{x}) < s_G(v, \boldsymbol{x})$ or $s_G(u, \boldsymbol{x}) < s_G(w, \boldsymbol{x})$.

We may assume that $s_G(u, \boldsymbol{x}) < s_G(v, \boldsymbol{x})$. Then we duplicate the vertex v, that is, we create a new vertex v' which has exactly the same neighbors as v, but vv' is not an edge, and we delete the vertex u and its incident edges; see the left graph in Figure 1. Moreover, we distribute the value x_u to the new vertex v', and keep the other coordinates of $\boldsymbol{x}$ unchanged. It is not hard to verify that the new graph G' has still no copy of K_{r+1} and

$$\begin{aligned}\lambda(G') \ge 2 \sum_{\{i,j\} \in E(G')} x_i x_j &= 2 \sum_{\{i,j\} \in E(G)} x_i x_j - 2x_u s_G(u, \boldsymbol{x}) + 2x_u s_G(v, \boldsymbol{x}) \\ &> 2 \sum_{\{i,j\} \in E(G)} x_i x_j = \lambda(G),\end{aligned}$$

where we used the positivity of vector $\boldsymbol{x}$. This contradicts with the choice of G.

Case 2. $s_G(u, \boldsymbol{x}) \ge s_G(v, \boldsymbol{x})$ and $s_G(u, \boldsymbol{x}) \ge s_G(w, \boldsymbol{x})$.

We copy the vertex u twice and delete vertices v and w with their incident edges; see the right graph in Figure 1. Similarly, we distribute the value x_v to the new vertex

u', and x_w to the new vertex u'', and keep the other coordinates of $\boldsymbol{x}$ unchanged. Moreover, the new graph G'' contains no copy of K_{r+1} and

$$\begin{aligned}\lambda(G'') \geq 2 \sum_{\{i,j\}\in E(G'')} x_i x_j = 2 \sum_{\{i,j\}\in E(G)} x_i x_j - 2x_v s_G(v,\boldsymbol{x}) - 2x_w s_G(w,\boldsymbol{x}) \\ + 2x_v x_w + 2x_v s_G(u,\boldsymbol{x}) + 2x_w s_G(u,\boldsymbol{x}) \\ > \sum_{i=1}^{n} x_i s_G(i,\boldsymbol{x}) = \lambda(G).\end{aligned}$$

So we get a contradiction again. □

We conclude here that the spectral version of Zykov's symmetrization starts with a K_{r+1}-free graph G with vertex set $V = \{1, 2, \ldots, n\}$, and at each step takes two *non-adjacent* vertices v_i and v_j such that $s_G(v_i,\boldsymbol{x}) > s_G(v_j,\boldsymbol{x})$, and deleting all edges incident to v_j, and adding new edges between vertex v_j and the neighborhood $N(v_i)$. We do the same if $s_G(v_i,\boldsymbol{x}) = s_G(v_j,\boldsymbol{x})$ and $N(v_i) \neq N(v_j)$ for $i < j$. The spectral version of Zykov's symmetrization does not increase the size of the largest clique and does not decrease the spectral radius[1]. When the process terminates, it yields a complete multipartite graph with at most r vertex parts. Otherwise, there are three vertices $u, v, w \in V(G)$ such that $vu \notin E(G)$ and $uw \notin E(G)$ but $vw \in E(G)$. Applying the same case analysis as in Theorem 1.4, we will get a new graph with larger spectral radius, which is a contradiction.

We illustrate the difference of the spectral extension between the Erdős degree majorization algorithm and the Zykov symmetrization. Recall that the spectral version of the Erdős degree majorization algorithm asks us to choose a vertex $v \in V(G)$ with maximum value of $s_G(v,\boldsymbol{x})$ among all vertices of G, and we remove all edges incident to vertices of $V(G) \setminus (N_G(v) \cup \{v\})$, and then add all edges between $N_G(v)$ and $V(G)\setminus N_G(v)$. We observe that this operation makes each vertex of $V(G)\setminus(N_G(v)\cup\{v\}$ being a copy of the vertex v. Since G is K_{r+1}-free, we see that the subgraph of G induced by $N_G(v)$ is K_r-free. We denote by $V_1 = V(G) \setminus N_G(v)$. Next, we do the same operation on vertex set $V_1^c = N_G(v)$. More precisely, we further choose a vertex $u \in V_1^c$ with maximum value of $s_G(u,\boldsymbol{x})$ over all vertices of V_1^c, and we remove all edges incident to vertices $V_1^c \setminus (N_{V_1^c}(u) \cup \{u\})$, and then add all edges between $N_{V_1^c}(u)$ and $V_1^c \setminus N_{V_1^c}(u)$. By using this operation repeatedly, we get a complete r-partite graph H on the same vertex set $V(G)$. Furthermore, one can verify that the majorization inequality $s_G(v,\boldsymbol{x}) \leq s_H(v,\boldsymbol{x})$ holds for every vertex $v \in V(G)$; see, e.g., [21, 23, 24].

The spectral extensions of the Erdős majorization algorithm and the Zykov symmetrization share some similarities. For example, these two operations ask us to compare the sum of weights of neighbors, and turn a K_{r+1}-free graph into a complete

[1]Combining Rayleigh's formula or Lagrange's multiplier method, one can show further that the spectral radius will increase strictly whenever all coordinates of the vector $\boldsymbol{x}$ are positive.

r-partite. Importantly, these two operations do not create a copy of K_{r+1} and do not decrease the value of spectral radius. The only difference between them is that one step of the Erdős operation will change many vertices and its incident edges, while one step of the Zykov operation will only change two vertices and its incident edges. This subtle difference will bring great convenience in later Sections 3 and 4. As a matter of fact, at each step of the Erdős operation, there are many times of actions of the Zykov operation. In other words, each step of the Erdős operation can be decomposed as a series of the Zykov operation.

3 Refinement for triangle-free graphs

Mantel's theorem has many interesting applications and miscellaneous generalizations in the literature; see, e.g., [4, 5, 7, 44] and references therein. In particular, Mantel's Theorem 1.1 was refined in the sense of the following stability form.

Theorem 3.1 (Erdős). *Let G be an n-vertex graph containing no triangle. If G is not bipartite, then $e(G) \le \lfloor \frac{(n-1)^2}{4} \rfloor + 1$.*

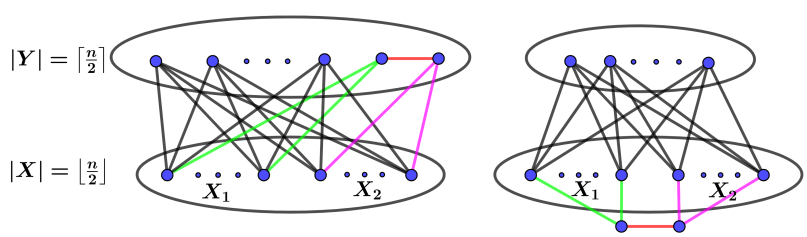

Figure 2: Two drawings of extremal graphs in Theorem 3.1.

It is said that this stability result attributes to Erdős; see [8, Page 306, Exercise 12.2.7]. The bound in Theorem 3.1 is best possible and the extremal graph is not unique. To show that the bound is sharp for all integers n, we take two vertex sets X and Y with $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. We take two vertices $u, v \in Y$ and join them, then we put every edge between X and $Y \setminus \{u, v\}$. We partition X into two parts X_1 and X_2 *arbitrarily* (this shows that the extremal graph is not unique), then we connect u to every vertex in X_1, and v to every vertex in X_2; see Figure 2. This yields a graph G which contains no triangle and $e(G) = \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$. Note that G has a 5-cycle, so G is not bipartite.

In 2021, Lin, Ning and Wu [33, Theorem 1.4] proved a generalization on (2) in Nosal's Theorem 1.3 for non-bipartite graphs. Let $SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ denote the subdivision of the complete bipartite graph $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ on one edge; see Figure 3. Clearly, $SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ is one of the extremal graphs in Theorem 3.1 by setting $|X_1| = \lfloor \frac{n}{2} \rfloor - 1$ and $|X_2| = 1$ in Figure 2.

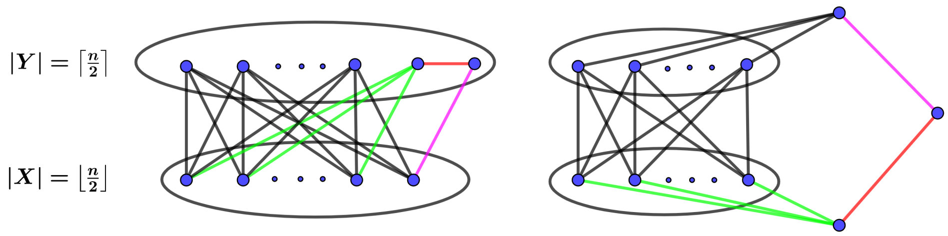

Figure 3: Two drawings of the graph $SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

Theorem 3.2 (Lin–Ning–Wu, 2021)**.** *Let G be an n-vertex graph. If G is triangle-free and non-bipartite, then*

$$\lambda(G) \le \lambda(SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}),$$

equality holds if and only if $G = SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

Theorem 3.2 is also a corresponding spectral version of Erdős' stability Theorem 3.1, while the extremal graph in spectral problem is uniquely $SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$; see [34, 27] for a recent extension on graphs without short odd cycles, and [30] for more stability theorems on spectral graph problems.

In this section, we shall provide an alternative proof of Theorem 3.2. One of the key ideas in the proof is to use the spectral Zykov symmetrization, which provides great convenience to yield a clearly approximate structure of the required extremal graph. Moreover, the ideas in this proof can benefit us to extend Theorem 3.2 to K_{r+1}-free non-r-partite graphs, which will be discussed in Section 4. Before starting the proof, we include the following lemma, which is a direct consequence by computations; see, e.g., [33, Appendix A].

Lemma 3.3. *If G is a graph on $n = a+b+1$ vertices obtained from $K_{a,b}$ by subdividing an edge arbitrarily, then*

$$\lambda(G) \le \lambda(SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}),$$

equality holds if and only if $G = SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

Proof. We denote by $SK_{a,b}$ the graph obtained from $K_{a,b}$ by subdividing an edge. Let s, t be two positive integers with $t \ge s \ge 1$. It suffices to show that

$$\lambda(SK_{s+1,t+3}) < \lambda(SK_{s+2,t+2}).$$

By computation, the spectral radius of $SK_{a,b}$ is the largest root of

$$F_{a,b}(x) := x^5 - (ab+1)x^3 + (3ab - 2a - 2b + 1)x - 2ab + 2a + 2b - 2.$$

Hence $\lambda(SK_{s+2,t+2})$ is the largest root of

$$F_{s+2,t+2}(x) = x^5 - (2s + 2t + st + 5)x^3 + (4s + 4t + 3st + 5)x - 2s - 2t - 2st - 2.$$

Similarly, $\lambda(SK_{s+1,t+3})$ is the largest root of $F_{s+1,t+3}(x)$. Note that

$$F_{s+2,t+2}(x) - F_{s+1,t+3}(x) = -(x-1)^2(x+2)(t-s+1).$$

This implies $F_{s+2,t+2}(x) < F_{s+1,t+3}(x)$ for every $x > 1$. Since $K_{2,3}$ is a subgraph of $SK_{s+1,t+3}$, we know that $\lambda(SK_{s+1,t+3}) \ge \lambda(K_{2,3}) = \sqrt{6}$. Thus, we have

$$F_{s+2,t+2}(\lambda(SK_{s+1,t+3})) < F_{s+1,t+3}(\lambda(SK_{s+1,t+3})) = 0.$$

Therefore, we obtain $\lambda(SK_{s+1,t+3}) < \lambda(SK_{s+2,t+2})$. $\square$

Now we are ready to show our proof of Theorem 3.2. For two *non-adjacent* vertices $u, v \in V(G)$, we denote the Zykov symmetrization $Z_{u,v}(G)$ to be the graph obtained from G by replacing u with a twin of v, that is, deleting all edges incident to vertex u, and then adding new edges from u to $N_G(v)$. We can verify that the Zykov symmetrization does not increase both the clique number $\omega(G)$ and the chromatic number $\chi(G)$. More precisely, we have $\omega(Z_{u,v}(G)) = \omega(G \setminus \{u\})$ and $\chi(Z_{u,v}(G)) = \chi(G \setminus \{u\})$. Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ be a *positive unit eigenvector* corresponding to $\lambda(G)$. Recall that $s_G(v, \boldsymbol{x}) = \sum_{i \in N_G(v)} x_i$ is denoted by the sum of weights of all neighbors of v in G. For two *non-adjacent* vertices u, v, if $s_G(u, \boldsymbol{x}) < s_G(v, \boldsymbol{x})$, then we replace G with $Z_{u,v}(G)$. Apparently, the spectral Zykov symmetrization does not make triangles. More importantly, it will increase strictly the spectral radius, since

$$\begin{aligned}\lambda(Z_{u,v}(G)) \ge 2 \sum_{\{i,j\} \in E(Z_{u,v}(G))} x_i x_j &= 2 \sum_{\{i,j\} \in E(G)} x_i x_j - 2x_u s_G(u, \boldsymbol{x}) + 2x_u s_G(v, \boldsymbol{x}) \\ &> 2 \sum_{\{i,j\} \in E(G)} x_i x_j = \lambda(G).\end{aligned}$$

If $s_G(u, \boldsymbol{x}) = s_G(v, \boldsymbol{x})$ and $N_G(u) \neq N_G(v)$, then we can apply either $Z_{u,v}$ or $Z_{v,u}$, which leads to $N(u) = N(v)$ after making the spectral Zykov symmetrization, and while the operation will keep the spectral radius $\lambda(G)$ increasing strictly. Indeed, we can easily see that

$$\lambda(Z_{u,v}(G)) \ge 2 \sum_{\{i,j\} \in E(Z_{u,v}(G))} x_i x_j = 2 \sum_{\{i,j\} \in E(G)} x_i x_j = \lambda(G).$$

We claim that $\lambda(Z_{u,v}(G)) > \lambda(G)$. Assume on the contrary that $\lambda(Z_{u,v}(G)) = \lambda(G)$, then the inequality in above become an equality, thus $\boldsymbol{x}$ is an eigenvector of $\lambda(Z_{u,v}(G))$. Namely, $A(Z_{u,v}(G))\boldsymbol{x} = \lambda(Z_{u,v}(G))\boldsymbol{x} = \lambda(G)\boldsymbol{x}$. Taking any vertex $z \in N_G(v) \setminus N_G(u)$, we observe that $\lambda(Z_{u,v}(G))x_z = \sum_{t \in N_G(z) \cup \{u\}} x_t > \sum_{t \in N_G(z)} x_t = \lambda(G)x_z$. Consequently, we get $\lambda(Z_{u,v}(G)) > \lambda(G)$, which contradicts with our assumption. It is worth emphasizing that the positivity of $\boldsymbol{x}$ is necessary in above discussions. Roughly speaking, applying the spectral Zykov symmetrization will make the K_{r+1}-free graph more regular in some sense according to the weights of the eigenvector.

Proof of Theorem 3.2. Let G be a non-bipartite triangle-free graph on n vertices with the largest spectral radius. Our goal is to show that $G = SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$. Clearly, we know that G is connected. Otherwise, any addition of an edge between a component with the maximum spectral radius and any other component will strictly increase the spectral radius. Since G is connected, there exists a positive unit eigenvector corresponding to $\lambda(G)$, and then we denote such a vector by $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, where $x_i > 0$ for every i. Since G is triangle-free, we apply repeatedly the spectral Zykov symmetrization for every pair of non-adjacent vertices until it becomes a bipartite graph. Without loss of generality, we may assume that G is triangle-free and non-bipartite, while $Z_{u,v}(G)$ is bipartite. We next are going to show that $\lambda(G) \le \lambda(SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil})$, equality holds if and only if $G = SK_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

Since $Z_{u,v}(G)$ is bipartite, we know that $G \setminus \{u\}$ is bipartite. We denote $V(G) \setminus \{u\} = V_1 \cup V_2$, where V_1, V_2 are disjoint and $|V_1| + |V_2| = n - 1$. Assume that $C = N(u) \cap V_1$ and $D = N(u) \cap V_2$. We denote $A = V_1 \setminus C$ and $B = V_2 \setminus D$. Since G is triangle-free, there are no edges between parts C and D. As G attains the largest spectral radius, we know that the pair of parts (A, B), (A, D) and (B, C) are complete bipartite subgraphs; see Figure 4.

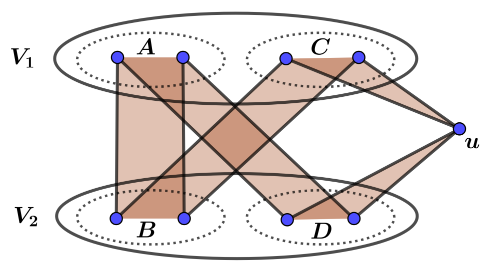

Figure 4: An approximate structure of G.

Note that each vertex in A has the same neighborhood, we know that the coordinates $\{x_v : v \in A\}$ are all equal. This property holds similarly for vertices in B, C and D respectively. Thus, we write x_a for the value of the entries of $\boldsymbol{x}$ in vertex set A. And x_b, x_c and x_d are defined similarly.

The remaining steps of our proof are outlined as follows.

☆ If $|A|x_a \ge |B|x_b$, then we delete $|C| - 1$ vertices in C with its incident edges, and add $|C| - 1$ vertices to D and connect these vertices to $A \cup \{u\}$. We keep the weight of these new vertices being x_c and denote the new graph by G'. We can verify that $\lambda(G') \ge 2\sum_{\{i,j\} \in E(G)} x_i x_j - 2(|C| - 1)|B|x_c x_b + 2(|C| - 1)|A|x_c x_a \ge 2\sum_{\{i,j\} \in E(G)} x_i x_j = \lambda(G)$. In fact, we can further prove that $\lambda(G') > \lambda(G)$. Otherwise, if $\lambda(G') = \lambda(G)$, then $\boldsymbol{x}$ is the Perron vector of G', that is, $A(G') = \lambda(G')\boldsymbol{x} = \lambda(G)\boldsymbol{x}$. Taking any vertex $z \in A$, we observe that $\lambda(G')x_z = \sum_{v \in N_{G'}(z)} x_v = \sum_{v \in N_G(z)} x_v + (|C| - 1)x_c > \sum_{v \in N_G(z)} x_v = \lambda(G)x_z$, and then $\lambda(G') > \lambda(G)$, which is a contradiction.

- ✯ If $|A|x_a < |B|x_b$, then we can delete $|D|-1$ vertices from D with its incident edges, and add $|D|-1$ vertices to C and join these vertices to $B\cup\{u\}$. Similarly, we can show that this process will increase the spectral radius strictly. From the above discussion, we can always remove the vertices to force either $|C|=1$ or $|D|=1$. Without loss of generality, we may assume that $|C|=1$ and $C=\{c\}$.

- ✭ If $x_u \ge x_c$, then we remove $|B|-1$ vertices from B with its incident edges, and add $|B|-1$ vertices to D and join these vertices to $A\cup\{u\}$. We keep the weight of these new vertices being x_b and denote the new graph by G^*. Then $\lambda(G^*) \ge 2\sum_{\{i,j\}\in E(G)} x_ix_j - 2(|B|-1)x_bx_c + 2(|B|-1)x_bx_u \ge 2\sum_{\{i,j\}\in E(G)} x_ix_j = \lambda(G)$. Furthermore, by Rayleigh's formula, we know that the first inequality holds strictly. Thus we conclude in the new graph G^* that B is a single vertex, say $B=\{b\}$. We observe that the graph G^* is a subdivision of a complete bipartite graph on $(A\cup\{u\},\{b\}\cup D)$ by subdividing the edge $\{b,u\}$.

- ★ If $x_u < x_c$, then we delete $|D|-1$ vertices from D with its incident edges, and add $|D|-1$ vertices to B and join these new vertices to $A\cup\{c\}$. Keeping the weight of vertices unchanged, we denote the new graph by $G^\star$. Then we can similarly get $\lambda(G^\star) > \lambda(G)$. In the graph $G^\star$, we have $|D|=1$ and write $D=\{d\}$. Thus $G^\star$ is a subdivision of a complete bipartite graph on $(A\cup\{c\}, B\cup\{d\})$ by subdividing the edge $\{c,d\}$.

From our discussion above, we know that if G is an n-vertex triangle-free non-bipartite graph and attains the maximum spectral radius, then G is a subdivision of a complete bipartite by subdividing exactly one edge. Lemma 3.3 implies that G is a subdivision of a balanced complete bipartite graph on $n-1$ vertices. □

4 Refinement of spectral Turán theorem

In 1981, Brouwer [9] proved the following improvement on Turán's Theorem 1.2.

Theorem 4.1 (Brouwer, 1981)**.** *Let $n\ge 2r+1$ be an integer and G be an n-vertex graph. If G is K_{r+1}-free and G is not r-partite, then*

$$e(G) \le e(T_r(n)) - \left\lfloor \frac{n}{r} \right\rfloor + 1.$$

Theorem 4.1 was also independently studied in many references, e.g., [2, 22, 26, 49]. Similar with that of Theorem 3.1, the bound of Theorem 4.1 is sharp and there are many extremal graphs attaining this bound.

We would like to illustrate the reason why we are interested in the study of the family of non-r-partite graphs. On the one hand, the Erdős degree majorization algorithm [16] or [4, pp. 295–296] implies that if G is an n-vertex K_{r+1}-free graph, then there exists an r-partite graph H on the same vertex set $V(G)$ such that $d_G(v) \le d_H(v)$ for every vertex v. Consequently, we get $e(G)\le e(H)\le e(T_r(n))$. Hence it is meaningful to determine the family of graphs attaining the second largest value of the extremal function. This problem is also called the stability problem. On the other hand, there are various ways to study the extremal graph problems under some reasonable constraints. For example, the condition of non-r-partite graph is equivalent to saying the chromatic number $\chi(G) \geq r+1$. Moreover, we can also consider the extremal problem under the restriction $\alpha(G) \leq f(n)$ for a given function $f(n)$, where $\alpha(G)$ is the independence number of G. This is the well-known Ramsey–Turán problem; see [45] for a comprehensive survey.

The proof of Theorem 3.2 stated in Section 3 can bring us more effective treatment for the extremal spectral problem when K_{r+1} is a forbidden subgraph. In what follows, we shall extend Lin–Ning–Wu's Theorem 3.2. Our result is also a spectral version of Brouwer's Theorem 4.1.

Recall that $T_r(n)$ is the n-vertex r-partite Turán graph in which the parts T_1, T_2, $\ldots, T_r$ have sizes $t_1, t_2, \ldots, t_r$ respectively. We may assume that $\lfloor \frac{n}{r} \rfloor = t_1 \leq t_2 \leq \cdots \leq t_r = \lceil \frac{n}{r} \rceil$. Now, we are going to define a new graph obtained from $T_r(n)$. Firstly, we choose two vertex parts T_1 and T_r. Secondly, we add a new edge into T_r, denote by uw, and then remove all edges between T_1 and $\{u, w\}$. Finally, we connect u to a vertex $v \in T_1$, and connect w to the remaining vertices of T_1. The resulting graph is denoted by $Y_r(n)$; see Figure 5. Clearly, $Y_r(n)$ is one of the extremal graphs of Brouwer's theorem.

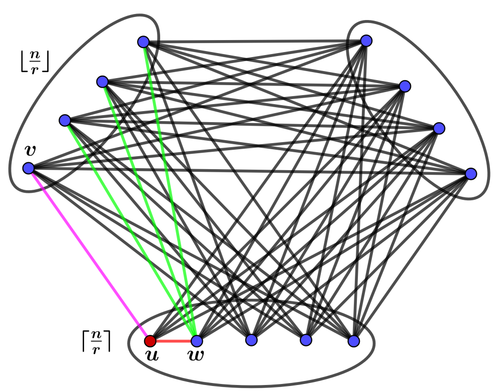

Figure 5: The graph $Y_r(n)$ for $n = 13$ and $r = 3$.

Lemma 4.2. *Let $K_{b_1, b_2, \ldots, b_r}$ be the complete r-partite graph with parts $B_1, B_2, \ldots, B_r$ satisfying $|B_i| = b_i$ for every $i \in [r]$ and $\sum_{i=1}^r b_i = n-1$. Let G be an n-vertex graph obtained from $K_{b_1, b_2, \ldots, b_r}$ by adding a new vertex u and choosing $v \in B_1, w \in B_2$, and removing the edge vw, and adding the edges uv, uw and ut for every $t \in \cup_{i=3}^r B_i$. Then*

$$\lambda(G) \leq \lambda(Y_r(n)).$$

Moreover, the equality holds if and only if $G = Y_r(n)$.

Next, we illustrate the construction of $Y_r(n)$ in another way. Let $T_r(n-1)$ be the r-partite Turán graph on $n-1$ vertices whose parts S_1, $S_2, \ldots,$ S_r have sizes $s_1, s_2, \ldots, s_r$ such that $\lfloor \frac{n-1}{r} \rfloor = s_1 \le s_2 \le \cdots \le s_r = \lceil \frac{n-1}{r} \rceil$. Note that $Y_r(n)$ can also be obtained from $T_r(n-1)$ by adding a new vertex u, and choosing two vertices $v \in S_1$ and $w \in S_2$, and deleting the edge vw, and adding the edges uv, uw and ut for every vertex $t \in \cup_{i=3}^r S_i$. Hence Lemma 4.2 states that G attains the maximum spectral radius when its part sizes $b_1, b_2, \ldots, b_r$ are as equal as possible and the two special vertices v, w are located in the smallest two parts respectively. We know that $\lambda(G)$ is the largest root of the characteristic polynomial $P_G(x) = \det(xI_n - A(G))$. It is operable to compute $\lambda(G)$ exactly for some small integers r by using computers, while it seems complicated for large r.

Proof of Lemma 4.2. Let G be a graph satisfying the requirement of Lemma 4.2 and G has the maximum spectral radius. We will show that $G = Y_r(n)$. Since G is connected, there exists a positive unit eigenvector $\boldsymbol{x} \in \mathbb{R}^n$ corresponding to $\lambda(G)$. Then $A(G)\boldsymbol{x} = \lambda(G)\boldsymbol{x}$ and

$$\lambda(G) = \boldsymbol{x}^T A(G) \boldsymbol{x} = 2 \sum_{\{i,j\} \in E(G)} x_i x_j.$$

Moreover, the eigen-equation gives that $\lambda(G) x_v = \sum_{u \in N(v)} x_u$ for every $v \in V(G)$. It follows that if two non-adjacent vertices have the same neighborhood, then they have the same value in the corresponding coordinate of $\boldsymbol{x}$. Thus all coordinates of $\boldsymbol{x}$ corresponding to the vertices of B_i are equal for every $i \in \{3, 4, \ldots, r\}$. We write x_i for the value of the entries of $\boldsymbol{x}$ corresponding to vertices of B_i for each $i \in \{3, \ldots, r\}$. We denote $B_1^- = B_1 \setminus \{v\}$ and $B_2^- = B_2 \setminus \{w\}$. Similarly, all coordinates of $\boldsymbol{x}$ corresponding to the vertices of B_i^- are equal for $i \in \{1, 2\}$.

Assume on the contrary that G is not isomorphic to $Y_r(n)$. In other words, there are two parts B_i and B_j such that $|b_i - b_j| \ge 2$, or $b_i \le b_j - 1$ for some $i \in \{3, 4, \ldots, n\}$ and $j \in \{1, 2\}$. By the symmetry, there are four cases listed below.

(i) $b_i \le b_j - 2$ for some $i, j \in \{3, \ldots, r\}$;

(ii) $b_1 \le b_2 - 2$;

(iii) $b_1 \le b_i - 2$ for some $i \in \{3, \ldots, r\}$;

(iv) $b_i \le b_1 - 1$ for some $i \in \{3, \ldots, r\}$.

Case 1. First and foremost, we shall consider case (i) that $b_i \le b_j - 2$ for some $i, j \in \{3, \ldots, r\}$. The treatment for this case has its root in [25]. If $b_i + b_j = 2b$ for some integer b, then we will balance the number of vertices of parts B_i and B_j. Namely, we define a new graph G' obtained from G by deleting all edges between B_i and B_j, and then we move some vertices from B_j to B_i such that the resulting sets, say B_i', B_j', have size b, and then we add all edges between B_i' and B_j'. In this process, we keep the other edges unchanged. We define a vector $\boldsymbol{y} \in \mathbb{R}^n$ such that $y_s = (\frac{b_i x_i^2 + b_j x_j^2}{2b})^{1/2}$ for every vertex $s \in B_i' \cup B_j'$, and $y_t = x_t$ for every $t \in V(G') \setminus (B_i' \cup B_j')$. Clearly, $\sum_{v \in V(G')} y_v^2 = 1$. Furthermore, we have

$$\boldsymbol{y}^T A(G')\boldsymbol{y} - \boldsymbol{x}^T A(G)\boldsymbol{x} = 2((by_s)^2 - b_i x_i b_j x_j) + 2(2by_s - (b_i x_i + b_j x_j)) \sum_{t \notin B_i' \cup B_j'} x_t.$$

Note that $b = \frac{b_i + b_j}{2} > \sqrt{b_i b_j}$ and

$$(by_s)^2 = b^2 \cdot \frac{b_i x_i^2 + b_j x_j^2}{2b} \geq b\sqrt{b_i x_i^2 b_j x_j^2} > b_i x_i b_j x_j.$$

Moreover, the weighted power-mean inequality gives

$$2by_s = 2b \left(\frac{b_i x_i^2 + b_j x_j^2}{b_i + b_j} \right)^{1/2} \geq 2b \frac{b_i x_i + b_j x_j}{b_i + b_j} = b_i x_i + b_j x_j.$$

Thus we get $\boldsymbol{y}^T A(G')\boldsymbol{y} > \boldsymbol{x}^T A(G)\boldsymbol{x}$. Rayleigh's formula gives

$$\lambda(G') \geq \boldsymbol{y}^T A(G')\boldsymbol{y} > \boldsymbol{x}^T A(G)\boldsymbol{x} = \lambda(G),$$

which contradicts with the choice of G. If $b_i + b_j = 2b + 1$ for some integer b, then we move similarly some vertices from B_j to B_i such that the resulting sets B_i', B_j' satisfying $|B_i'| = b$ and $|B_j'| = b + 1$. We construct a vector $\boldsymbol{y} \in \mathbb{R}^n$ satisfying $y_s = (\frac{b_i x_i^2 + b_j x_j^2}{2b+1})^{1/2}$ for every vertex $s \in B_i' \cup B_j'$, and $y_t = x_t$ for every $t \in V(G') \setminus (B_i' \cup B_j')$. Similarly, we get

$$\begin{aligned}\boldsymbol{y}^T A(G')\boldsymbol{y} - \boldsymbol{x}^T A(G)\boldsymbol{x} &= 2(b(b+1)y_s^2 - b_i x_i b_j x_j) \\ &\quad + 2((2b+1)y_s - (b_i x_i + b_j x_j)) \sum_{t \notin B_i' \cup B_j'} x_t.\end{aligned}$$

We are going to show that

$$b(b+1)y_s^2 - b_i x_i b_j x_j > 0, \quad \text{and} \quad (2b+1)y_s - (b_i x_i + b_j x_j) \geq 0.$$

For the first inequality, by applying AM-GM inequality, we get

$$b(b+1)y_s^2 = b(b+1)\frac{b_i x_i^2 + b_j x_j^2}{b_i + b_j} \geq \frac{2b(b+1)}{b_i + b_j}\sqrt{b_i b_j} x_i x_j.$$

Then it is sufficient to prove that $2b(b+1) > (b_i + b_j)\sqrt{b_i b_j}$. Note that $b_i \leq b_j - 2$ and $b_i + b_j = 2b + 1$ is odd. Then $b_i \leq b - 1$ and $b_j \geq b + 2$. The desired inequality holds. For the second one, the weighted power-mean inequality yields

$$(2b+1)y_s = (2b+1)\left(\frac{b_i x_i^2 + b_j x_j^2}{b_i + b_j} \right)^{1/2} \geq (2b+1)\frac{b_i x_i + b_j x_j}{b_i + b_j} = b_i x_i + b_j x_j.$$

This case also contradicts with the choice of G.

For the remaining three cases, we will show our proof by considering the characteristic polynomial of G and then applying induction on integer r.

Case 2. Now, we consider case (ii) that $b_1 \leq b_2 - 2$. Recall that $B_1^- = B_1 \setminus \{v\}$ and $B_2^- = B_2 \setminus \{w\}$. We define a graph G' obtained from G by deleting a vertex of B_2^-, and adding a copy of a vertex of B_1^-. This makes the two parts B_1^-, B_2^- more balanced. Our goal is to prove that $\lambda(G) < \lambda(G')$, which contradicts with the maximality of G. Let x_v, x_w and x_u be the weights of vertices v, w and u respectively. We denote by x_1^- and x_2^- the weights of vertices of B_1^- and B_2^- respectively. The eigen-equation $A(G)\boldsymbol{x} = \lambda(G)\boldsymbol{x}$ gives $\sum_{j\in N(i)} x_j = \lambda(G)x_i$ for every $i \in [n]$. Then

$$
\begin{cases}
x_u + (b_2-1)x_2^- + b_3x_3 + \cdots + b_rx_r = \lambda(G)x_v, \\
x_u + (b_1-1)x_1^- + b_3x_3 + \cdots + b_rx_r = \lambda(G)x_w, \\
x_v + x_w + b_3x_3 + \cdots + b_rx_r = \lambda(G)x_u, \\
x_w + (b_2-1)x_2^- + b_3x_3 + \cdots + b_rx_r = \lambda(G)x_1^-, \\
x_v + (b_1-1)x_1^- + b_3x_3 + \cdots + b_rx_r = \lambda(G)x_2^-, \\
x_v + x_w + x_u + (b_1-1)x_1^- + (b_2-1)x_2^- + b_4x_4 + \cdots + b_rx_r = \lambda(G)x_3, \\
\quad \vdots \\
x_v + x_w + x_u + (b_1-1)x_1^- + (b_2-1)x_2^- + b_3x_3 + \cdots + b_{r-1}x_{r-1} = \lambda(G)x_r.
\end{cases}
$$

Thus $\lambda(G)$ is the largest eigenvalue of the matrix A_r corresponding to eigenvector $(x_v, x_w, x_u, x_1^-, x_2^-, x_3, \ldots, x_r)$, where $A_r (r \geq 3)$ is defined as the following.

$$
A_r := \left[\begin{array}{ccccc:ccc}
0 & 0 & 1 & 0 & b_2-1 & b_3 & \cdots & b_r \\
0 & 0 & 1 & b_1-1 & 0 & b_3 & \cdots & b_r \\
1 & 1 & 0 & 0 & 0 & b_3 & \cdots & b_r \\
0 & 1 & 0 & 0 & b_2-1 & b_3 & \cdots & b_r \\
1 & 0 & 0 & b_1-1 & 0 & b_3 & \cdots & b_r \\
\hdashline
1 & 1 & 1 & b_1-1 & b_2-1 & 0 & \cdots & b_r \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & 1 & 1 & b_1-1 & b_2-1 & b_3 & \cdots & 0
\end{array}\right].
$$

For notational convenience, we denote

$$
A_2 := \begin{bmatrix}
0 & 0 & 1 & 0 & b_2-1 \\
0 & 0 & 1 & b_1-1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & b_2-1 \\
1 & 0 & 0 & b_1-1 & 0
\end{bmatrix},
$$

and

$$R_{b_1,b_2}(x) := \det \begin{bmatrix} x+1 & 1 & 0 & b_1-1 & 0 \\ 1 & x+1 & 0 & 0 & b_2-1 \\ 0 & 0 & x+1 & b_1-1 & b_2-1 \\ 1 & 0 & 1 & x+b_1-1 & 0 \\ 0 & 1 & 1 & 0 & x+b_2-1 \end{bmatrix}.$$

For every $r \geq 2$, the characteristic polynomial of A_r is denoted by

$$F_{b_1,b_2,\dots,b_r}(x) = \det(xI_{r+3} - A_r).$$

In particular, the polynomial $F_{b_1,b_2}(x)$ is the same as that in Lemma 3.3. By expanding the last column of $\det(xI_{r+3} - A_r)$, we get the following recurrence relations:

$$F_{b_1,b_2,b_3}(x) = (x+b_3)F_{b_1,b_2}(x) - b_3 R_{b_1,b_2}(x), \tag{5}$$

and for every integer $r \geq 4$,

$$F_{b_1,b_2,\dots,b_r}(x) = (x+b_r)F_{b_1,b_2,\dots,b_{r-1}}(x) - b_r \prod_{i=3}^{r-1}(x+b_i)R_{b_1,b_2}(x), \tag{6}$$

where $F_{b_1,b_2}(x)$ and $R_{b_1,b_2}(x)$ are computed as below:

$$\begin{aligned} F_{b_1,b_2}(x) &= x^5 - (b_1b_2+1)x^3 + (3b_1b_2 - 2b_1 - 2b_2 + 1)x - 2b_1b_2 + 2b_1 + 2b_2 - 2, \\ R_{b_1,b_2}(x) &= x^5 + (b_1+b_2+1)x^4 + (b_1b_2+1)x^3 - (b_1b_2 + b_1 + b_2 - 3)x^2 \\ &\quad + (2b_1 + 2b_2 - 3b_1b_2 - 1)x + 3(b_1-1)(b_2-1). \end{aligned}$$

Note that $b_1 \leq b_2 - 2$. Upon computations, we obtain

$$F_{b_1+1,b_2-1}(x) - F_{b_1,b_2}(x) = (b_1 - b_2 + 1)(x-1)^2(x+2) < 0,$$

and

$$R_{b_1+1,b_2-1}(x) - R_{b_1,b_2}(x) = -(b_1 - b_2 + 1)(x-1)(x^2-3) > 0.$$

Note that $b_1 - b_2 + 1 \leq -1$. Combining with equation (5), we obtain

$$\begin{aligned} &F_{b_1+1,b_2-1,b_3}(x) - F_{b_1,b_2,b_3}(x) \\ &= (x+b_3)(F_{b_1+1,b_2-1}(x) - F_{b_1,b_2}(x)) - b_3(R_{b_1+1,b_2-1}(x) - R_{b_1,b_2}(x)) \\ &= (b_1-b_2+1)(x-1)^2(x+2)(x+b_3) + b_3(b_1-b_2+1)(x-1)(x^2-3) \\ &= (b_1-b_2+1)(x-1)\big((x-1)(x+2)(x+b_3) + b_3(x^2-3)\big) < 0. \end{aligned}$$

Next we prove by induction that for every $r \geq 3$ and $x \geq 2$,

$$F_{b_1+1,b_2-1,b_3,\dots,b_r}(x) - F_{b_1,b_2,b_3,\dots,b_r}(x) < 0. \tag{7}$$

Firstly, the base case $r = 3$ was verified in the above. For $r \geq 4$, we get from (6) that

$$F_{b_1+1,b_2-1,b_3,\dots,b_r}(x) - F_{b_1,b_2,b_3,\dots,b_r}(x)$$

$$= (x + b_r)\big(F_{b_1+1,b_2-1,b_3,\dots,b_{r-1}}(x) - F_{b_1,b_2,b_3,\dots,b_{r-1}}(x)\big)$$
$$- b_r \prod_{i=3}^{r-1}(x + b_i)\big(R_{b_1+1,b_2-1}(x) - R_{b_1,b_2}(x)\big) < 0,$$

where the last inequality holds by applying inductive hypothesis on the case $r - 1$ and invoking the fact $R_{b_1+1,b_2-1}(x) - R_{b_1,b_2}(x) > 0$. From inequality (7), we know that $F_{b_1+1,b_2-1,b_3,\dots,b_r}(\lambda(G)) < F_{b_1,b_2,b_3,\dots,b_r}(\lambda(G)) = 0$. Since $\lambda(G')$ is the largest root of $F_{b_1+1,b_2-1,b_3,\dots,b_r}(x)$, this implies $\lambda(G) < \lambda(G')$.

Case 3. Thirdly, we consider case (iii) that $b_1 \le b_i - 2$ for some $i \in \{3, \dots, r\}$. We may assume by symmetry that $b_1 \le b_3 - 2$. Our treatment in this case is similar with that of case (ii). Let G^* be the graph obtained from G by deleting a vertex of B_3 with its incident edges, and add a new vertex to B_1^- and connect this new vertex to all remaining vertices of B_3 and all vertices of $B_2 \cup B_4 \cup \dots \cup B_r$. We will prove that $\lambda(G) < \lambda(G^*)$. By case (ii), we may assume that $|b_1 - b_2| \le 1$. Clearly, $\lambda(G^*)$ is the largest root of $F_{b_1+1,b_2,b_3-1,b_4,\dots,b_r}(x)$. First of all, we will show that

$$F_{b_1+1,b_2,b_3-1}(x) - F_{b_1,b_2,b_3}(x) < 0, \tag{8}$$

and then by applying induction, we will prove that for each $r \ge 4$,

$$F_{b_1+1,b_2,b_3-1,b_4,\dots,b_r}(x) - F_{b_1,b_2,b_3,b_4,\dots,b_r}(x) < 0. \tag{9}$$

Indeed, we next verify inequalities (8) and (9) for the case $r = 4$ only, since the inductive steps are the same as that of case (ii) with slight differences. By computation, we obtain

$$\begin{aligned}
&(x + b_3 - 1)F_{b_1+1,b_2}(x) - (x + b_3)F_{b_1,b_2}(x)\\
&= -x^5 - b_2x^4 + (b_2(b_1 - b_3 + 1) + 1)x^3 + (3b_2 - 2)x^2\\
&\quad + (3b_2b_3 - 3b_1b_2 + 2b_1 - 3b_2 - 2b_3 + 3)x + 2b_1b_2 - 2b_1 - 2b_2b_3 + 2b_3,
\end{aligned}$$

and

$$\begin{aligned}
&- (b_3 - 1)R_{b_1+1,b_2}(x) + b_3R_{b_1,b_2}(x)\\
&= x^5 + (b_2 + b_1 - b_3 + 2)x^4 + (b_2(b_1 - b_3 + 1) + 1)x^3\\
&\quad + (-b_1b_2 - b_1 + b_2b_3 - 2b_2 + b_3 + 2)x^2\\
&\quad + (3b_2b_3 - 3b_1b_2 + 2b_1 - b_2 - 2b_3 + 1)x\\
&\quad + 3b_1b_2 - 3b_1 - 3b_2b_3 + 3b_3.
\end{aligned}$$

Combining these two equations with (5), we get

$$\begin{aligned}
&F_{b_1+1,b_2,b_3-1}(x) - F_{b_1,b_2,b_3}(x)\\
&= (x + b_3 - 1)F_{b_1+1,b_2}(x) - (x + b_3)F_{b_1,b_2}(x) - (b_3 - 1)R_{b_1+1,b_2}(x) + b_3R_{b_1,b_2}(x)\\
&= (b_1 - b_3 + 2)x^4 + 2(b_2(b_1 - b_3 + 1) + 1)x^3 + (b_2b_3 - b_1b_2 - b_1 + b_2 + b_3)x^2
\end{aligned}$$

$$+ (6b_2b_3 - 6b_1b_2 + 4b_1 - 4b_2 - 4b_3 + 4)x + 5b_1b_2 - 5b_1 - 5b_2b_3 + 5b_3.$$

Combining $|b_1 - b_2| \le 1$ and $b_1 - b_3 \le -2$, one can verify that $F_{b_1+1,b_2,b_3-1}(x) < F_{b_1,b_2,b_3}(x)$ for every $x \ge 2(b_1 - 2)$. This completes the proof of (8). We now consider (9) in the case $r = 4$. Note that $b_1 - b_3 + 2 \le 0$ and

$$\begin{aligned}
&- (x + b_3 - 1)R_{b_1+1,b_2}(x) + (x + b_3)R_{b_1,b_2}(x) \\
&= (b_1 - b_3 + 2)x^4 + (b_2(b_1 - b_3 + 2) + 2)x^3 + (b_2(b_3 - b_1 + 1) + b_3 - b_1)x^2 \\
&\quad + (3b_2b_3 - 3b_1b_2 + 2b_1 - 4b_2 - 2b_3 + 4)x + 3b_1b_2 - 3b_1 - 3b_2b_3 + 3b_3 < 0,
\end{aligned}$$

which together with (6) and the case $r = 3$ yields

$$\begin{aligned}
&F_{b_1+1,b_2,b_3-1,b_4}(x) - F_{b_1,b_2,b_3,b_4}(x) \\
&= (x + b_4)(F_{b_1+1,b_2,b_3-1}(x) - F_{b_1,b_2,b_3}(x)) \\
&\quad - b_4(x + b_3 - 1)R_{b_1+1,b_2}(x) + b_4(x + b_3)R_{b_1,b_2}(x) < 0.
\end{aligned}$$

Let $t = \min\{b_i : 1 \le i \le r\} - 1$. Since the complete r-partite $K_{t,t,\dots,t}$ is a subgraph of G, we know that $\lambda(G) \ge \lambda(K_{t,t,\dots,t}) = (r - 1)t$. Thus, we can similarly get that $F_{b_1+1,b_2,b_3-1,b_4,\dots,b_r}(\lambda(G)) < F_{b_1,b_2,b_3,b_4,\dots,b_r}(\lambda(G)) = 0$, which yields $\lambda(G) < \lambda(G^*)$, which contradicts with the choice of G.

Case 4. Finally, we consider the case (iv) that $b_i \le b_1 - 1$ for some $i \ge 3$. We may assume that $b_3 \le b_1 - 1$. This case can similarly be completed by applying a similar argument of case (iii). Let G^* be the graph obtained from G by removing a vertex of B_1^- with its incident edges, and adding a copy of a vertex of B_3. In what follows, we will show that

$$F_{b_1-1,b_2,b_3+1}(x) - F_{b_1,b_2,b_3}(x) < 0, \tag{10}$$

and then we prove by induction that for every $r \ge 4$,

$$F_{b_1-1,b_2,b_3+1,b_4,\dots,b_r}(x) - F_{b_1,b_2,b_3,b_4,\dots,b_r}(x) < 0. \tag{11}$$

By computation, we obtain that

$$\begin{aligned}
&(x + b_3 + 1)F_{b_1-1,b_2}(x) - (x + b_3)F_{b_1,b_2}(x) \\
&= x^5 + b_2x^4 + (b_2(b_3 - b_1 + 1) - 1)x^3 + (-3b_2 + 2)x^2 \\
&\quad + (3b_1b_2 - 2b_1 - 3b_2b_3 - 3b_2 + 2b_3 + 1)x - 2b_1b_2 + 2b_1 + 2b_2b_3 + 4b_2 - 2b_3 - 4,
\end{aligned}$$

and

$$\begin{aligned}
&- (b_3 + 1)R_{b_1-1,b_2}(x) + b_3R_{b_1,b_2}(x) \\
&= -x^5 + (b_3 - b_1 - b_2)x^4 + (b_2(b_3 - b_1 + 1) - 1)x^3 \\
&\quad + (b_1b_2 - b_2b_3 + b_1 - b_3 - 4)x^2 \\
&\quad + (3b_1b_2 - 2b_1 - 3b_2b_3 - 5b_2 + 2b_3 + 3)x
\end{aligned}$$

$$+ 3b_2b_3 - 3b_1b_2 + 3b_1 + 6b_2 - 3b_3 - 6.$$

Combining with the recurrence equation (5), we get

$$\begin{aligned}
&F_{b_1-1,b_2,b_3+1}(x) - F_{b_1,b_2,b_3}(x)\\
&= (x+b_3+1)F_{b_1-1,b_2}(x) - (x+b_3)F_{b_1,b_2}(x) - (b_3+1)R_{b_1-1,b_2}(x) + b_3R_{b_1,b_2}(x)\\
&= (b_3-b_1)x^4 + (2b_2(b_3-b_1+1)-2)x^3 + (b_1b_2 - b_2b_3 + b_1 - b_3 - 3b_2 - 2)x^2\\
&\quad + (6b_1b_2 - 6b_2b_3 - 4b_1 - 8b_2 + 4b_3 + 4)x - 5b_1b_2 + 5b_1 + 5b_2b_3 + 10b_2 - 5b_3 - 10.
\end{aligned}$$

Since $b_3-b_1 \leq -1$ and $|b_1-b_2| \leq 1$, one can verify that $F_{b_1-1,b_2,b_3+1}(x)-F_{b_1,b_2,b_3}(x) < 0$ for every $x \geq 2(b_3-1)$. This completes the proof of (10). Next we will prove (11) for the case $r=4$ only, since the inductive steps are similar with that of case (ii) and (iii). By computation, we have

$$\begin{aligned}
&-(x+b_3+1)R_{b_1-1,b_2}(x) + (x+b_3)R_{b_1,b_2}(x)\\
&= (b_3-b_1)x^4 + (b_2(b_3-b_1)-2)x^3 + (b_1b_2 + b_1 - b_2b_3 - 3b_2 - b_3 - 2)x^2\\
&\quad + (3b_1b_2 - 2b_1 - 3b_2b_3 - 2b_2 + 2b_3)x - 3b_1b_2 + 3b_1 + 3b_2b_3 + 6b_2 - 3b_3 - 6 < 0,
\end{aligned}$$

which together with (6) and the case $r=3$ gives

$$\begin{aligned}
&F_{b_1-1,b_2,b_3+1,b_4}(x) - F_{b_1,b_2,b_3,b_4}(x)\\
&= (x+b_4)(F_{b_1-1,b_2,b_3+1}(x) - F_{b_1,b_2,b_3}(x))\\
&\quad - b_4(x+b_3+1)R_{b_1-1,b_2}(x) + b_4(x+b_3)R_{b_1,b_2}(x) < 0.
\end{aligned}$$

Since $F_{b_1-1,b_2,b_3+1,b_4,\ldots,b_r}(\lambda(G)) < F_{b_1,b_2,b_3,b_4,\ldots,b_r}(\lambda(G)) = 0$ and $\lambda(G^*)$ is the largest root of $F_{b_1-1,b_2,b_3+1,b_4,\ldots,b_r}(x)$, we know that $\lambda(G) < \lambda(G^*)$, which contradicts with the choice of G. In summary, we complete the proof of all possible cases. $\square$

Remark. It seems possible to prove the last three cases by using a weight-balanced argument similar with that of the first case. Nevertheless, it is inevitable that a great deal of tedious calculations are required in the proof of these cases. Moreover, applying the recursive technique of determinants in the proof of Lemma 4.2, one can compute the characteristic polynomial of the adjacency matrix and signless Laplacian matrix of the n-vertex complete r-partite graph $K_{t_1,\ldots,t_r}$. More precisely,

$$\det(xI_n - A(K_{t_1,\ldots,t_r})) = x^{n-r}\left(1 - \sum_{i=1}^{r}\frac{t_i}{x+t_i}\right)\prod_{i=1}^{r}(x+t_i),$$

and

$$\det(xI_n - Q(K_{t_1,\ldots,t_r})) = \prod_{i=1}^{r}(x-n+t_i)^{t_i-1}(x-n+2t_i)\left(1 - \sum_{i=1}^{r}\frac{t_i}{x-n+2t_i}\right).$$

It has its own interests to compute the eigenvalues of complete multipartite graphs; see, e.g., [14, 53, 43, 52] for different proofs and related results.

We next show our main result in this paper.

Theorem 4.3. *Let G be an n-vertex graph. If G is K_{r+1}-free and G is not r-partite, then*

$$\lambda(G) \le \lambda(Y_r(n)).$$

Moreover, the equality holds if and only if $G = Y_r(n)$.

Theorem 4.3 is a refinement of the spectral Turán Theorem 1.4 and also it is an extension of Theorem 3.2. Our proof is mainly based on Zykov's symmetrization.

Proof. First of all, we assume that G is a K_{r+1}-free non-r-partite graph with maximum value of spectral radius. Our goal is to prove that $G = Y_r(n)$. Clearly, we know that G must be a connected graph. Let $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ be a positive unit eigenvector of $\lambda(G)$.

Claim 4.1. *There exists a vertex $u \in V(G)$ such that $G \setminus \{u\}$ is r-partite.*

Proof of Claim 4.1. Recall that for two non-adjacent vertices $u, v \in V(G)$, the spectral Zykov symmetrization $Z_{u,v}(G)$ is defined as the graph obtained from G by removing all edges incident to vertex u and then adding new edges from u to $N_G(v)$. We can verify that the spectral Zykov symmetrization does not increase the clique number and the chromatic number. Recall that $s_G(v, \boldsymbol{x}) = \sum_{i \in N_G(v)} x_i$ is the sum of weights of all neighbors of v in G. For two non-adjacent vertices u, v, if $s_G(u, \boldsymbol{x}) < s_G(v, \boldsymbol{x})$, then we replace G with $Z_{u,v}(G)$. If $s_G(u, \boldsymbol{x}) = s_G(v, \boldsymbol{x})$, then we can apply either $Z_{u,v}$ or $Z_{v,u}$, which leads to $N(u) = N(v)$ after making the spectral Zykov symmetrization. Obviously, the spectral Zykov symmetrization does not create a copy of K_{r+1}. More significantly, it will increase the spectral radius strictly, since $\boldsymbol{x}$ is entry-wise positive.

The proof of Claim 4.1 is based on the spectral Zykov symmetrization stated in above. Since G is K_{r+1}-free, we can repeatedly apply the Zykov symmetrization on every pair of non-adjacent vertices until G becomes an r-partite graph. Without loss of generality, we may assume that G is K_{r+1}-free and G is not r-partite, while $Z_{u,v}(G)$ is r-partite. Thus $G \setminus \{u\}$ is r-partite, and we assume that $V(G) \setminus \{u\} = V_1 \cup V_2 \cup \cdots \cup V_r$, where $V_1, V_2, \ldots, V_r$ are pairwise disjoint and $\sum_{i=1}^r |V_i| = n - 1$. $\square$

We denote $A_i = N(u) \cap V_i$ for every $i \in [r] := \{1, \ldots, r\}$. Note that G has maximum spectral radius among all K_{r+1}-free non-r-partite graphs. Then for each $i \in [r]$, every vertex of $V_i \setminus A_i$ is adjacent to every vertex of V_j for every $j \in [r]$ and $j \ne i$. We remark here that the difference between the K_{r+1}-free case (Theorem 4.3) and the triangle-free case (Theorem 3.2) is that there may exist some edges between the pair of sets A_i and A_j, which makes the problem seems more difficult.

Claim 4.2. *There exists a pair $\{i, j\} \subseteq [r]$ such that $G[A_i, A_j]$ forms an empty graph, and for other pairs $\{s, t\} \ne \{i, j\}$, $G[A_s, A_t]$ is a complete bipartite subgraph in G.*

Proof of Claim 4.2. Let $G[A_1, A_2, \ldots, A_r]$ be the subgraph of G induced by the vertex sets $A_1, A_2, \ldots, A_r$. Claim 4.2 is equivalent to say that $G[A_1 \cup A_2, A_3, \ldots, A_r]$ forms a complete $(r-1)$-partite subgraph in G. Since G is K_{r+1}-free, we know that the subgraph $G[A_1, A_2, \ldots, A_r]$ is a K_r-free subgraph in G.

First of all, we choose a vertex $v_1 \in A_1$ such that $s_G(v_1, \boldsymbol{x})$ is maximum among all vertices of A_1, then we apply the Zykov operation Z_{u,v_1} on G for every $u \in A_1 \setminus \{v_1\}$. These operations will make all vertices of A_1 being equivalent, that is, every pair of vertices in A_1 has the same neighbors. Secondly, we choose a vertex $v_2 \in A_2$ such that $s_G(v_2, \boldsymbol{x})$ is maximum over all vertices of A_2, and then we apply similarly the Zykov operation Z_{u,v_2} on G for every $u \in A_2 \setminus \{v_2\}$. *Note that all vertices in A_1 have the same neighbors.* After doing Zykov's operations on vertices of A_2, we claim that the induced subgraph $G[A_1, A_2]$ is either a complete bipartite graph or an empty graph. Indeed, if $v_2 \in \cap_{v\in A_1} N(v)$, then the operations Z_{u,v_2} for all $u \in A_2 \setminus \{v_2\}$ will lead to a complete bipartite graph between A_1 and A_2. If $v_2 \in \cap_{v\in A_1} N(v)$, then v_2 is not adjacent to all vertices of A_1, and so is u for every $u \in A_2 \setminus \{v_2\}$, which yields that $G[A_1, A_2]$ is an empty graph. Moreover, by applying the similar operations on $A_3, A_4, \ldots, A_r$, we can obtain that for every $i, j \in [r]$ with $i \neq j$, the induced bipartite subgraph $G[A_i, A_j]$ is either complete bipartite or empty. Since $G[A_1, A_2, \ldots, A_r]$ is K_r-free and G attains the maximum spectral radius, we know that there is exactly one pair $\{i, j\} \subseteq [r]$ such that $G[A_i, A_j]$ is an empty graph. $\square$

We may assume that $\{i, j\} = \{1, 2\}$ for convenience. In what follows, we intend to enlarge A_i to the whole set V_i for every $i \in \{3, 4, \ldots, r\}$. Observe that every vertex of $V_i \setminus A_i$ is adjacent to every vertex of V_j for every $j \in [r]$ with $j \neq i$, and adding all edges between $\{u\}$ to $V_i \setminus A_i$ does not create a copy of K_{r+1} in G, and does not decrease the spectral radius of G. From this observation, we know that u is adjacent to every vertex of V_i for each $i \in \{3, 4, \ldots, r\}$; see (a) in Figure 6.

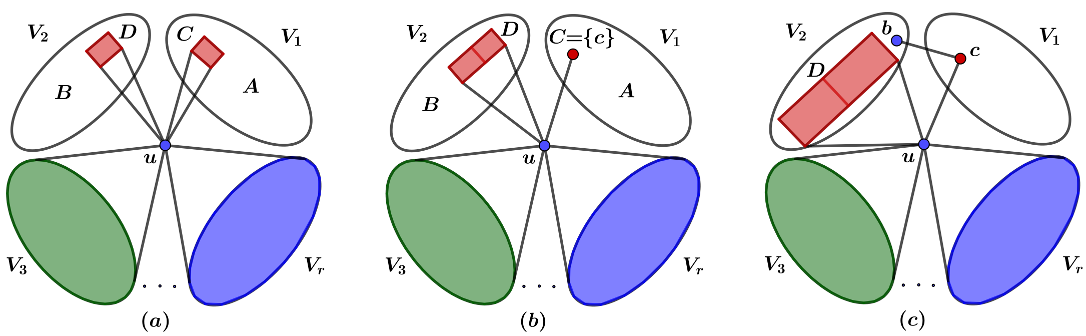

Figure 6: Further movement step.

Assume that $C := N(u) \cap V_1$ and $D := N(u) \cap V_2$. We denote $A := V_1 \setminus C$ and $B := V_2 \setminus D$; see (a) in Figure 6. Note that there is no edge between C and D, since G does not contain K_{r+1} as a subgraph. In the remaining of our proof, we will prove by two steps that both C and D are single vertex sets.

Claim 4.3. *The set C is a single vertex, i.e., $|C| = 1$.*

Proof of Claim 4.3. The treatment is similar with that of our proof of Theorem 3.2. If $\sum_{v\in A} x_v \geq \sum_{v\in B} x_v$, then we choose $|C| - 1$ vertices of C and delete its incident edges only in B, then we move these $|C|-1$ vertices into D and connect these vertices to A. In this process, the edges between these $|C|-1$ vertices and $V_3 \cup \cdots \cup V_r$ are unchanged. We write G' for the resulting graph. Using the similar computation as in Section 3, we can verify that $\lambda(G') > \lambda(G)$.

If $\sum_{v\in A} x_v < \sum_{v\in B} x_v$, then we can choose $|D|-1$ vertices of D and delete its incident edges only in A, and then move these $|D|-1$ vertices into C and join these vertices to B. This process will increase strictly the spectral radius.

From the above case analysis, we can always remove the vertices of G to force either $|C| = 1$ or $|D| = 1$. Without loss of generality, we may assume that $|C| = 1$ and denote $C = \{c\}$; see (b) in Figure 6. $\square$

Claim 4.4. *The set D is a single vertex, i.e., $|D| = 1$.*

Proof of Claim 4.4. If $x_u < x_c$, then we choose $|D|-1$ vertices of D and delete its incident edges to vertex u, then we move these $|D|-1$ vertices into B and join these these vertices to c, and keeping the other edges unchanged, we denote the new graph by $G^\star$. Then we can similarly get $\lambda(G^\star) > \lambda(G)$. In the graph $G^\star$, we have $|D| = 1$ and write $D = \{d\}$. Thus $G^\star$ is the graph obtained from a complete r-partite graph $K_{t_1,t_2,\ldots,t_r}$, where $\sum_{i=1}^r t_i = n-1$, by adding a new vertex u and then joining u to a vertex $c \in V_1$, and joining u to a vertex $d \in V_2$, and joining u to all vertices of $V_3 \cup \cdots \cup V_r$, and finally removing the edge $cd \in E(K_{t_1,t_2,\ldots,t_r})$.

If $x_u \ge x_c$, then we choose $|B|-1$ vertices of B and delete its incident edges to vertex c, then we move these $|B|-1$ vertices into D and join these vertices to vertex u. We denote the new graph by G^*. Then $\lambda(G^*) > \lambda(G)$. Thus we conclude in the new graph G^* that B is a single vertex, say $B = \{b\}$; see (c) in Figure 6. In what follows, we will exchange the position of u and c. Note that $c \in V_1$ is adjacent to a vertex $b \in V_1$ and all vertices of $V_3 \cup \cdots \cup V_r$. Now, we move vertex c outside of V_1 and put vertex u into V_1. Thus the new center c is adjacent to a vertex $u \in V_1$, a vertex $b \in V_2$ and all vertices of $V_3 \cup \cdots \cup V_r$. Note that $bu \notin E(G^*)$. Hence G^* has the same structure as the previous case, and then we may assume that $|D| = 1$. $\square$

From the above discussion, we know that G is isomorphic to the graph defined as in Lemma 4.2. By applying Lemma 4.2, we know that $\lambda(G) \le \lambda(Y_r(n))$. Moreover, the equality holds if and only if $G = Y_r(n)$. This completes the proof. $\square$

5 Unified extension to the p-spectral radius

Recall that the spectral radius of a graph is defined as the largest eigenvalue of its adjacency matrix. By Rayleigh's theorem, we know that it is also equal to the maximum value of $\boldsymbol{x}^T A(G)\boldsymbol{x} = 2\sum_{\{i,j\}\in E(G)} x_i x_j$ over all $\boldsymbol{x} \in \mathbb{R}^n$ with $|x_1|^2 + \cdots + |x_n|^2 = 1$. The definition of the spectral radius was recently extended to *the p-spectral radius*. We denote the p-norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$. For every real number $p \geq 1$, the p-spectral radius of G is defined as

$$\lambda^{(p)}(G) := 2 \max_{\|\boldsymbol{x}\|_p = 1} \sum_{\{i,j\} \in E(G)} x_i x_j. \tag{12}$$

We remark that $\lambda^{(p)}(G)$ is a versatile parameter. Indeed, $\lambda^{(1)}(G)$ is known as the Lagrangian function of G, $\lambda^{(2)}(G)$ is the spectral radius of its adjacency matrix, and

$$\lim_{p \to +\infty} \lambda^{(p)}(G) = 2e(G), \tag{13}$$

which can be guaranteed by the following inequality

$$2e(G) n^{-2/p} \leq \lambda^{(p)}(G) \leq (2e(G))^{1-1/p}.$$

To some extent, the p-spectral radius can be viewed as a unified extension of the classical spectral radius and the size of a graph. In addition, it is worth mentioning that if $1 \leq q \leq p$, then $\lambda^{(p)}(G) n^{2/p} \leq \lambda^{(q)}(G) n^{2/q}$ and $(\lambda^{(p)}(G)/2e(G))^p \leq (\lambda^{(q)}(G)/2e(G))^q$; see [40, Propositions 2.13 and 2.14] for more details.

As commented by Kang and Nikiforov in [24, p. 3], linear-algebraic methods are irrelevant for the study of $\lambda^{(p)}(G)$ in general, and in fact no efficient methods are known for it. Thus the study of $\lambda^{(p)}(G)$ for $p \neq 2$ is far more complicated than the classical spectral radius.

The extremal function for p-spectral radius is given as

$$\mathrm{ex}_{\lambda}^{(p)}(n, F) := \max\{\lambda^{(p)}(G) : |G| = n \text{ and } G \text{ is } F\text{-free}\}.$$

To some extent, the proof of results on the p-spectral radius shares some similarities with the usual spectral radius when $p > 1$; see [40, 24, 25] for extremal problems for the p-spectral radius. In 2014, Kang and Nikiforov [24] extended the Turán theorem to the p-spectral version for $p > 1$. They proved that

$$\mathrm{ex}_{\lambda}^{(p)}(n, K_{r+1}) = \lambda^{(p)}(T_r(n)).$$

Theorem 5.1 (Kang–Nikiforov, 2014). *If G is a K_{r+1}-free graph on n vertices, then for every $p > 1$,*

$$\lambda^{(p)}(G) \leq \lambda^{(p)}(T_r(n)),$$

equality holds if and only if G is the n-vertex Turán graph $T_r(n)$.

Remark. We remark that a theorem of Motzkin and Straus states that Theorem 5.1 is also valid for the case $p = 1$ except for the extremal graphs attaining the equality.

Keeping (13) in mind, we can see that Theorem 5.1 is a unified extension of both Turán's Theorem 1.2 and Spectral Turán's Theorem 1.4 by taking $p \to +\infty$ and $p = 2$ respectively. We can obtain by detailed computation that $\lambda^{(p)}(T_r(n)) = (1 + O(\frac{1}{n^2})) 2e(T_r(n)) n^{-2/p}$ and $\lambda^{(p)}(T_r(n)) = (1 - O(\frac{1}{n^2})) \left(1 - \frac{1}{r}\right) n^{2-2/p}$, where $O(\frac{1}{n^2})$

stands for a positive error term. This theorem implies $\lambda^{(p)}(G) \le \left(1-\frac{1}{r}\right) n^{2-(2/p)}$, equality holds if and only if r divides n and $G = T_r(n)$.

Recall that the proof of Theorem 4.3 relies on the Rayleigh representation of $\lambda(G)$ and the existence of a positive eigenvector of $\lambda(G)$. For the p-spectral radius, there is also a positive vector corresponding to $\lambda^{(p)}(G)$. Indeed, we choose G as a K_{r+1}-free graph on n vertices with maximum value of the p-spectral radius, where $p > 1$. Clearly, we can assume further that G is connected. A vector $\boldsymbol{x} \in \mathbb{R}^n$ is called a unit (optimal) eigenvector corresponding to $\lambda^{(p)}(G)$ if it satisfies $\sum_{i=1}^n |x_i|^p = 1$ and $\lambda^{(p)}(G) = 2\sum_{\{i,j\}\in E(G)} x_i x_j$. From the definition (12), we know that there is always a non-negative eigenvector of $\lambda^{(p)}(G)$. Moreover, since $p > 1$, Lagrange's multiplier method gives that for every $v \in V(G)$, we have

$$\lambda^{(p)}(G) x_v^{p-1} = \sum_{u \in N_G(v)} x_u. \tag{14}$$

Therefore, if G is connected and $p > 1$, applying (14), then a non-negative eigenvector of $\lambda^{(p)}(G)$ must be entry-wise positive. Hence there exists a positive unit optimal vector $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ corresponding to $\lambda^{(p)}(G)$ such that

$$\lambda^{(p)}(G) = 2 \sum_{\{i,j\}\in E(G)} x_i x_j.$$

By applying a similar line of the proof of Theorem 4.3, one can extend Theorem 4.3 to the p-spectral radius. We leave the details for interested readers.

Theorem 5.2. *Let G be an n-vertex graph. If G does not contain K_{r+1} and G is not r-partite, then for every $p > 1$, we have*

$$\lambda^{(p)}(G) \le \lambda^{(p)}(Y_r(n)).$$

Moreover, the equality holds if and only if $G = Y_r(n)$.

6 Concluding remarks

In this paper, we studied the spectral extremal graph problems for graphs with given number of vertices. By extending the Mantel theorem and Nosal theorem, we presented an alternative proof of an extension of Nikiforov for K_{r+1}-free graphs, and provided a different proof of a refinement of Lin, Ning and Wu for non-bipartite K_3-free graphs. Furthermore, we generalized these two results to non-r-partite K_{r+1}-free graphs. Our result is not only a refinement on the spectral Turán theorem, but it is also a spectral version of Brouwer's theorem. In a forthcoming paper [31], we shall present some extensions and generalizations on Nosal's theorem for graphs with given number of edges.

At the end of this paper, we shall conclude with some possible problems for interested readers. To begin with, we define an extremal function as

$$\psi(n, F, t) := \max\{e(G) : F \nsubseteq G, \chi(G) \geq t\}.$$

Brouwer's theorem says that $\psi(n, K_{r+1}, r+1) = e(T_r(n)) - \lfloor \frac{n}{r} \rfloor + 1$. Similarly, we can define the spectral extremal function as

$$\psi_\lambda(n, F, t) := \max\{\lambda(G) : F \nsubseteq G, \chi(G) \geq t\}.$$

In Theorem 4.3, we proved that $\psi_\lambda(n, K_{r+1}, r+1) = \lambda(Y_r(n))$. Note that the extremal graph $Y_r(n)$ has chromatic number $\chi(Y_r(n)) = r+1$. It is possible to determine the function $\psi_\lambda(n, K_{r+1}, r+2)$. *More generally, it would be interesting to determine the functions $\psi(n, F, t)$ and $\psi_\lambda(n, F, t)$ for a general graph F and an integer t. For instance, it is possible to study these extremal functions by setting F as the odd cycle C_{2k+1}, the book graph $B_k = K_2 \vee kK_1$, the fan graph $F_k = K_1 \vee kK_2$, the wheel graph $W_k = K_1 \vee C_k$, and a color-critical graph F.*

We write $q(G)$ for the signless Laplacian spectral radius, i.e., the largest eigenvalue of the *signless Laplacian matrix* $Q(G) = D(G) + A(G)$, where $D(G) = \mathrm{diag}(d_1, \ldots, d_n)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix. In 2013, He, Jin and Zhang [23, Theorem 1.3] showed some bounds for the signless Laplacian spectral radius in terms of the clique number. As a consequence, they proved the signless Laplacian spectral version of Theorem 1.2, which states that if G is a K_{r+1}-free graph on n vertices, then $q(G) \leq q(T_r(n))$, equality holds if and only if $r = 2$ and $G = K_{t,n-t}$ for some t, or $r \geq 3$ and $G = T_r(n)$. This spectral extension also implies the classical edge Turán theorem. *It is possible to establish analogues of the results of our paper in terms of the signless Laplacian spectral radius. For example, whether $Y_r(n)$ is the extremal graph attaining the maximum signless Laplacian spectral radius among all non-r-partite K_{r+1}-free graphs.*

In 2017, Nikiforov [41] provided a unified extension of both the adjacency spectral radius and the signless Laplacian spectral radius. It was proposed by Nikiforov [41] to study the family of matrices A_α defined for any real $\alpha \in [0, 1]$ as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G).$$

In particular, we can see that $A_0(G) = A(G)$ and $2A_{1/2}(G) = Q(G)$. Nikiforov [41, Theorem 27] presented some extremal spectral results in terms of the spectral radius of A_α. It was proved that for every $r \geq 2$ and every K_{r+1}-free graph G, if $0 \leq \alpha < 1 - \frac{1}{r}$, then $\lambda(A_\alpha(G)) < \lambda(A_\alpha(T_r(n)))$, unless $G = T_r(n)$; if $\alpha = 1 - \frac{1}{r}$, then $\lambda(A_\alpha(G)) < (1 - \frac{1}{r})n$, unless G is a complete r-partite graph; if $1 - \frac{1}{r} < \lambda < 1$, then $\lambda(A_\alpha(G)) < \lambda(A_\alpha(S_{n,r-1}))$, unless $G = S_{n,r-1}$, where $S_{n,k} = K_k \vee I_{n-k}$ is the graph consisting of a clique on k vertices and an independent set on $n-k$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. *From this evidence, it is possible to extend the results of our paper into the A_α-spectral radius in the range $\alpha \in [0, 1 - \frac{1}{r})$ for non-r-partite K_{r+1}-free graphs.*

Acknowledgements

This work was supported by NSFC (Grant No. 11931002). This article was completed during a quarantine period due to the COVID-19 pandemic. The authors would like to express their sincere gratitude to all of the volunteers and medical staffs for their kind help and support, which makes our daily life more and more secure.

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