

## Basic Results

**Definition 0.0.1.** The arithmetic derivative,  $D$ , is defined recursively in the following manner for all natural numbers  $n \in \mathbb{N}$

1.  $D(1) = D(0) = 0$
2.  $D(p) = 1$  for any prime  $p$
3. For every pair of natural numbers  $n$  &  $m$ ,  $D(mn) = mD(n) + nD(m)$

The arithmetic derivative is much like the normal derivative operator if one sees the numbers 0 & 1 as the "constant" functions, prime numbers as "lines" with slope 1 and if one required the arithmetic derivative to satisfy the Leibniz rule.

Since this definition doesn't allow one to directly compute the derivative of a number quickly, the following lemma shall help us greatly.

**Lemma 1.** If

$$x = \prod_{i=1}^n m_i$$

then

$$D(x) = x \sum_{i=1}^n \frac{D(m_i)}{m_i}$$

*Proof:*

We shall proceed by induction on the amount of elements in the product expansion of  $x$ . If  $n = 1$  then  $x = m_1$  and thus

$$x \sum_{i=1}^n \frac{D(m_i)}{m_i} = (m_1) \frac{D(m_1)}{m_1} = D(m_1) = D(x)$$

Now let's assume that for any  $x$  with  $n$  factors in its product expansion the formula

$$D(x) = x \sum_{i=1}^n \frac{D(m_i)}{m_i}$$

applies. Let  $x'$  be equal to  $xm_{n+1}$  and thus  $D(x') = D(x)m_{n+1} + xD(m_{n+1})$ . We can rewrite  $D(x)m_{n+1} + xD(m_{n+1})$  as

$$xm_{n+1} \sum_{i=1}^n \frac{D(m_i)}{m_i} + \frac{x'D(m_{n+1})}{m_{n+1}} = x' \sum_{i=1}^n \frac{D(m_i)}{m_i} + x' \frac{D(m_{n+1})}{m_{n+1}}$$

.

We can then factor out the  $x'$  and our final expression for  $D(x')$  is  $x' \sum_{i=1}^{n+1} \frac{D(m_i)}{m_i}$  which is exactly what we wanted to show ■

With lemma 1 in hand we can easily write down an expression for  $D(x)$  in terms of its prime factors.

$$\text{Let } x = \prod_{i=1}^n p_i^{k_i}$$

be  $x$ 's prime factorization.

$$D(x) = x \sum_{i=1}^n \frac{D(p_i^{k_i})}{p_i^{k_i}}$$

Obviously,  $D(p^k)$  is  $kp^{k-1}$ , since due to lemma 1 we have

$$D(p^k) = p^k \sum_{i=1}^n \frac{D(p)}{p} = p^k \sum_{i=1}^n \frac{1}{p}$$

and thus

$$D(p^k) = \sum_{i=1}^n p^{k-1} = kp^{k-1}$$

Finally, we have a fully simplified expression for the arithmetic derivative of  $x$ .

$$D(x) = x \sum_{i=1}^n \frac{D(p_i^{k_i})}{p_i^{k_i}} = x \sum_{i=1}^n \frac{k_i p_i^{k_i-1}}{p_i^{k_i}} = x \sum_{i=1}^n \frac{k_i}{p_i}$$

By using the notation  $v_p(x)$  to denote the exponent of a prime  $p$  in the prime factorization of  $x$  the above formula can be rewritten as the following.

$$D(x) = x \sum_{p|x} \frac{v_p(x)}{p}$$

## Differential Equations

For this section,  $x'$  will be used as shorthand for  $D(x)$ .