Basic Results

Definition 0.0.1. *The arithmetic derivative, D, is defined recursively in the following manner for all natural numbers* $n \in \mathbb{N}$

1. $D(1) = D(0) = 0$
2. $D(p) = 1$ *for any prime p*
3. *For every pair of natural numbers n & m,* $D(mn) = mD(n) + nD(m)$

The arithmetic derivative is much like the normal derivative operator if one sees the numbers 0 & 1 as the "constant" functions, prime numbers as "lines" with slope 1 and if one required the arithmetic derivative to satisfy the Lebiniz rule.

Since this definition doesn't allow one to directly compute the derivative of a number quickly, the following lemma shall help us greatly.

Lemma 1. *If*

$$x = \prod_{i=1}^{n} m_i$$

then

$$D(x) = x \sum_{i=1}^{n} \frac{D(m_i)}{m_i}$$

Proof:

We shall proceed by induction on the amount of elements in the product expansion of x. If $n = 1$ then $x = m_1$ and thus

$$x \sum_{i=1}^{n} \frac{D(m_i)}{m_i} = (m_1)\frac{D(m_1)}{m_1} = D(m_1) = D(x)$$

Now let's assume that for any x with n factors in its product expansion the formula

$$D(x) = x \sum_{i=1}^{n} \frac{D(m_i)}{m_i}$$

applies. Let x' be equal to xm_{n+1} and thus $D(x') = D(x)m_{n+1} + xD(m_{n+1})$. We can rewrite $D(x)m_{n+1} + xD(m_{n+1})$ as

$$xm_{n+1} \sum_{i=1}^{n} \frac{D(m_i)}{m_i} + \frac{x' D(m_{n+1})}{m_{n+1}} = x' \sum_{i=1}^{n} \frac{D(m_i)}{m_i} + x' \frac{D(m_{n+1})}{m_{n+1}}$$

.

We can then factor out the x' and our final expression for $D(x')$ is $x' \sum_{i=1}^{n+1} \frac{D(m_i)}{m_i}$ which is exactly what we wanted to show ∎

With lemma 1 in hand we can easily write down an expression for $D(x)$ in terms of its prime factors.

$$\text{Let } x = \prod_{i=1}^{n} {p_i}^{k_i}$$

be x's prime factorization.

$$D(x) = x \sum_{i=1}^{n} \frac{D({p_i}^{k_i})}{{p_i}^{k_i}}$$

Obviously, $D(p^k)$ is kp^{k-1}, since due to lemma 1 we have

$$D(p^k) = p^k \sum_{i=1}^{n} \frac{D(p)}{p} = p^k \sum_{i=1}^{n} \frac{1}{p}$$

and thus

$$D(p^k) = \sum_{i=1}^{n} p^{k-1} = kp^{k-1}$$

Finally, we have a fully simplified expression for the arithmetic derivative of x.

$$D(x) = x \sum_{i=1}^{n} \frac{D({p_i}^{k_i})}{{p_i}^{k_i}} = x \sum_{i=1}^{n} \frac{k_i {p_i}^{k_i - 1}}{{p_i}^{k_i}} = x \sum_{i=1}^{n} \frac{k_i}{p_i}$$

By using the notation $\upsilon_p(x)$ to denote the exponent of a prime p in the prime factorization of x the above formula can be rewritten as the following.

Theorem 1.

$$D(x) = x \sum_{p|x} \frac{\upsilon_p(x)}{p}$$

Cool Results

Differential Equations

Let x' be shorthand for $D(x)$ and of course x'' will be $D(D(x))$ and so on.

In calculus, the most basic differential equation is $x' = 0$ and the solutions of such an equation are the constant functions, so we would expect the same to be true in this context, and it indeed is.

Theorem 2. *If* $x' = 0$ *then* $x \in \{0, 1\}$

Proof:

By our definition of $D(x)$, $0'$ and $1'$ are both 0. Now let $x > 1$ and x' is equal to $x \sum_{p|x} \frac{\upsilon_p(x)}{p}$. The only way it can be 0 is if $\sum_{p|x} \frac{\upsilon_p(x)}{p}$ is 0 and that's only true if and only $\upsilon_p(x)$ is 0 for all $p \mid x$ which is obviously false and thus x' is never 0 if $x > 1$ ∎

Another simple differential equation is $x' = 1$ which characterizes all linear functions with slope 1. Similarly, in this context we should see that if x' is 1 then x is a prime number. Before I proceed with the proof, I will present two useful corollaries, whose proofs are obvious, and one lemma.

Corollary 1. $x > 1 \iff x' \geq 1$

Corollary 2. *If* $x = p^k m$ *then* $x' = p^{k-1}(km + pm')$

Lemma 2. *If* $p^k||x$ *and* $1 \leq k \leq p-1$ *then* $p^{k-1}||x'$

Theorem 3. *If* $x' = 1$ *then* x *is prime*

Proof:
By corollary 1 we have that $x > 1$ and let's say that x is not prime. Thus we can rewrite x as $p^k m$ where $p \nmid m$ and $k > 0$. Now let's assume that $k \geq p$ then by theorem 1 we have that $x' = x \sum_{p|x} \frac{v_p(x)}{p}$ and the value of $\sum_{p|x} \frac{v_p(x)}{p}$ is larger than 1, since we know that there is at least one $\frac{v_p(x)}{p}$ term which is larger than or equal to 1 and thus $x' \geq x$ and since $x > 1$ and along with the fact that $x' = 1$ we have a contradiction and thus $k \leq p-1$. Now, since we assumed that $0 < k$ and we can apply corollary 2 and we can see quite easily that $p^{k-1}||x' = 1$ and thus $k-1 = 0$ and thus $k = 1$. So x is of the form pm where p is a prime that doesn't divide m. Finally, let's assume that $m > 1$ and now we symbolically differentiate x, getting $pm' + m$, and since $m \geq 2$ (and thus $m' \geq 1$) and $p \geq 2$ we have that $x = pm' + m \geq 2(1) + 2 = 4$ which is greater than 4 and thus $m = 1$ and we have that $x = p$ where p is some prime ∎

The final result of this section will solve the next classical differential equation which is $x' = x$

Theorem 4. *If* $x' = x$ *then* $x = p^p$ *where* p *is prime*

Proof:
Let's assume that $p \mid x$ and let $x = p^k m$ so we have that $k \geq 1$ and let's also assume that $m \geq 2$. We also choose p such that $k \geq p$ since by corollary 2, if $k < p$ we have that $p^{k-1}||x' = x$ and thus k must be greater than or equal to p. By corollary 2, we have that $x' = \frac{x}{pm}(pm' + km) \geq \frac{x}{pm}(p + pm) = \frac{m+1}{m}x$. And since $x' = x$, m must be 1 so x is of the form p^k. Finally, let's assume that $k > p$ and then we have that $x' = kp^{k-1} > p(p^{k-1}) = p^k = x$ and thus $k = p$, so we have that $x = p^p$ where p is prime ∎