Part I: Axiomatic Systems, Models, & Introduction to Logic

An **axiomatic system** is a system of statements (called *axioms* or *postulates*) that serves as a starting point from which other statements can be logically derived.

Consider the set of postulates below:

Existence of Lines and Circles (Euclid's first three postulates)

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

A **model** for an axiomatic system is a setting which assigns meaning for the undefined terms presented in the axiomatic system, in a manner that is logically consistent with the statements in the system. Our familiar models of planar geometry rely on our meanings for these terms, such as point, (straight) line, extended indefinitely, endpoint, line segment, circle, center and radius.

Discussion: In groups, discuss the following terms and describe their meanings.

point	radius
straight line	extended indefinitely
circle	center
endpoint	straight line segment

Discussion: What other meanings are important for understanding Euclid's first three postulates?

Some Basic Assumptions

The three postulates above each refer to actions (such as drawing or extending) upon a figure or diagram. The mathematician David Hilbert created an extensive set of axioms that codify just about every assumption that one might make about these terms, as well as Euclid's other Postulates, removing the focus on constructive actions like drawing (which instead become part of a particular *model* of a geometry in Hilbert's axiomatic system).

In this course, we will work from a smaller set of axioms than Hilbert. We will generally introduce axioms throughout this course at the moment we will need them. However, there are three basic assumptions about *order* and *incidence* that we will make without codifying them as axioms. The geometer David Clark calls these "foundational concepts."

1. Two points determine a line; every line contains at least two points, and there exist three non-collinear points.

2. If the point B is **between** points A and C, then A, B, and C are collinear. If the points A, B, and C are collinear, then exactly one of them is between the other two.

3. Points of intersection of lines and/or circles exist if, when constructed in the plane, they necessarily cross.

Note: we define collinear as follows: three points in the plane A, B, and C are **collinear** if one point lies on the line that connects the other two points.

Discussion: What is meant by "two points *determine* a line"?

Types of Logical Statements

Once we establish an axiomatic system, we can then build from it by developing **theorems**. Theorems are statements that are proven using axioms, previously-proven theorems, and deductive reasoning. We often call a mathematical statement a **conjecture**, until we have proved it.

A mathematical statement will often have the following structure: *If* X, *then* Y. In this case, X is the **hypothesis** of the statement and Y is the **conclusion**. Statements with this structure are called **conditional statements** or **implications**.

Each part of a statement (that is, both X and Y) has a **truth value**, which is independent of the other part, and the entire statement also has a truth value that depends on the hypothesis and conclusion. All the possible combinations of truth values for X, Y, and X→Y are listed below. We see that the only way for a conditional statement to be false is if its hypothesis (X) is true and its conclusion (Y) is false.

X	**Y**	**X→Y**
T	T	T
T	F	F
F	T	T
F	F	T

The opposite of X, called **not X** or the **negation of X**, is denoted ~X. If X is true, then ~X is false, and vice versa.

Two important logical connectors are **and** and **or**, as in *X and Y* or *X or Y*. Sometimes, it is helpful to consider that *or* can be inclusive or exclusive. The **inclusive or** allows for X and Y to be true at the same time, while the **exclusive or** only allows for one of them to be true. Unless otherwise stated, *or* will mean the inclusive or.

1.1	Create truth tables for *X and Y* and *X or Y*.

A conditional statement has several related statements to go with it. Let us consider the statement:

If X, then Y.

Then, we have the following definitions:

- The **negation** of the original conditional statement is
  If X, then ~Y.
- The **converse** of the original conditional statement is
  If Y, then X.
- The **contrapositive** of the original conditional statement is
  If ~Y, then ~X.
- Then **inverse** of the original conditional statement is
  If ~X, then ~Y.

Discussion: Come up a conditional statement that you know is true, and write down its converse, contrapositive, and inverse.

Two statements are considered **logically equivalent** if they have the same truth values.

A **biconditional** statement is a statement of the form *X if and only if Y*, sometimes denoted X↔Y , which means **both** *if X, then Y* and *if Y, then X*. A biconditional statement is true if both conditional statements are true, and it is false if either conditional statement is false.

Discussion: Write the definition of *collinear* as a biconditional statement.

1.2	Consider the following statement, for which *n* represents an arbitrary positive integer: If $n < 2$, then $n < 5$. a. Is the conditional statement true (for any value of *n*)? b. State the negation, converse, contrapositive, and inverse forms of this statement. i. Which of these forms of the conditional statement are true (for any value of *n*)? ii. Which of these forms of the conditional statement are false (for at least one value of *n*)? iii. Which of the forms of this conditional statement are logically equivalent?
1.3	Consider the following statement: If *m* and *n* are distinct lines, then *m* and *n* are non-collinear. a. Is the statement true or false? b. Write the contrapositive of the conditional statement. Is it true or false?
1.4	We defined "collinear" as follows: three points in a plane A, B, and C are **collinear** if one point lies on the line that connects the other two points. a. Create a definition for **non-collinear** points in a plane. b. According to your definition, can two points ever be non-collinear? Explain.

1.5	Consider the following statements, for which A, B, and C represent arbitrary distinct points in a plane. *X* is the statement "A and B are collinear." *Y* is the statement "B is between A and C." a. Is (X and Y) true or false? b. Is (X or Y) true or false? c. Is (If X, then Y) true or false? d. Is (If Y, then X) true or false?
1.6	Consider the four statements, and recall that a **chord** is a line segment that has its endpoints on a circle: I. A diameter is a line segment that intersects the center of a circle and has endpoints on the circle. II. A diameter is a chord. III. A chord is a diameter. IV. A diameter is a chord that intersects the center of a circle. a. Which pair or pairs of statements I, II, III, and IV are equivalent? Explain. b. There are two meanings for the word "is" in these four statements. Identify these two meanings and explain the differences in these meanings. c. Which of I, III, III, and IV would be your preferred definition of diameter? Explain your choice.

Part II: Transformations and Congruence

Segment Measure (Length)
Discussion: What is length? What ways do we have of measuring length?

Axioms of Length: Each segment $\overline{AB}$ can be assigned a positive number, called the **length** of $\overline{AB}$, denoted AB, such that the following properties hold:

- The length of the unit segment is 1.
- $AB = BA$.
- If A, B, and C are three points with B between A and C, then AC is equal to $AB + BC$.
- There exists a unique point *M* between A and B, called the *midpoint* of $\overline{AB}$, such that $AM = MB$.

Discussion: In the first length axiom, the length of a "unit segment" is defined as 1. What does this mean? Is this enough of a definition to allow us to actually measure a segment?

Discussion: Consider the set of all points in a plane that are equidistant (that is, the same distance) from a single point. What does this set look like?

Angle Measure
When two rays share an endpoint (called a **vertex**), two angles are created. What ways do we have of measuring angles?

- **Straight** angle: An angle formed by two rays with the same endpoint whose union coincides with a line.
- **Right** angle: An angle with measure exactly half the measure of a straight angle.
- **Acute** angle: An angle that is smaller than a right angle.
- **Obtuse** angle: An angle that is larger than a right angle but smaller than a straight angle.
- **Minor** angle: An angle that is smaller than a straight angle.
- **Major** angle: An angle that is larger than a straight angle.

Two angles are **supplementary** if their measures sum to a straight angle's measure. Two angles are **complementary** if their measures sum to a right angle's measure. Two angles are **adjacent** if they have a common vertex and share exactly one ray. Two angles are **vertical** if their intersection is only a vertex and the union of the four rays forming the two angles forms two lines.

Discussion: Can an angle be adjacent to itself?
Discussion: How many vertical angles are formed from one pair of intersecting lines?

A ray $\overrightarrow{OP}$ **bisects** an angle $\angle AOB$ if the resulting adjacent angles $\angle AOP$ and $\angle BOP$ have the same measure. If O is between A and B, a straight angle and two other angles are created.

A line $\overleftrightarrow{PQ}$ is a **perpendicular bisector** of a segment $\overline{AB}$ with midpoint *O* if $\overleftrightarrow{PQ}$ intersects $\overline{AB}$ at *O* and if the resulting adjacent angles $\angle AOP$ and $\angle BOP$ have the same measure.

Similarly to the axioms of length, we also establish axioms of angle measure.

Axioms of Angle Measure: Every minor angle $\angle ABC$ can be assigned a positive number between 0 and 180, called the degree measure of the angle, so that the following properties hold:

- The degree measure of a right angle is 90°.
- Measure of $\angle ABC$ = Measure of $\angle CBA$.
- If point D is in the interior of $\angle ABC$, then the measure of $\angle ABC$ is equal to the sum of the measures of $\angle ABD$ and $\angle DBC$.
- There exists a unique ray that is the angle bisector of $\angle ABC$.

2.1	**Extension of Segment Measure** Suppose A=B. What number must be assigned as the length of "segment" $\overline{AB}$ in order to be consistent with the axioms of length? Explain why assigning any other number would result in a contradiction to one or more of the axioms of length.
2.2	**Extensions of Angle Measure** a. Suppose A=C. What number must be assigned as the measure of "angle" $\angle ABC$ in order to be consistent with the axioms of angle measure? Explain why assigning any other number would result in a contradiction to one or more of the axioms of angle measure. b. Suppose B is between A and C. What number must be assigned as the measure of "angle" $\angle ABC$ in order to be consistent with the axioms of angle measure? Explain why assigning any other number would result in a contradiction to one or more of the axioms of angle measure.

Geometric Congruence

It is important to have some understanding of what it means for two things to be "the same" in the field of mathematics in which you are working. In geometry, there are multiple ways in which we can consider two things to be the same. Two points, lines, or rays are called equal if they coincide exactly. A weaker idea of sameness that we will explore is *congruence*. However, before we do that, we need to think about *transformations*. Transformations are mappings from the plane to itself. We often use the prime notation to indicate the image of a point under a transformation. For instance, we would denote P' as the image of the point P.

We call a transformation a **rigid motion** if it does not change the size or shape of an object in the plane. There are three basic rigid motions that are fundamental to geometry.

- A **translation** is a rigid motion that slides the plane along a fixed non-zero vector. Given a vector $\vec{v}$, the image of a point P under a translation by $\vec{v}$ is a point P' such that the vector $\overrightarrow{PP'} = \vec{v}$.
- A **rotation** about center O of angle θ is a rigid motion that maps a point $P \neq O$ to a point P' on the circle centered at O with radius OP, such that $\angle POP' = \theta$. The image of O is O.
- A **reflection** in a line b is a rigid motion that maps any point on b to itself, and any other point P to a point P' so that b is the perpendicular bisector of segment $\overline{PP'}$.

An **isometry** is a transformation of the plane that preserves distance and angle measure.

Axiom of Isometries: The three basic rigid motions are isometries.

Two geometric figures are **congruent** if they are equivalent or if there exists an isometry (or sequence of isometries) that superimposes one figure directly onto the other figure. We use the symbol ≅ to indicate congruence.

Axiom of Equivalence: Congruence is an *equivalence relation*.
This means that congruence is

1. *Reflexive*: $X \cong X$. A geometric figure must be congruent to itself.
2. *Symmetric*: If $X \cong Y$, then $Y \cong X$.
3. *Transitive:* If $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

2.3	**Congruence and Length Theorem** Two segments are congruent if and only if they have the same length.
2.4	**Congruence and Angle Measure Theorem** Two angles are congruent if and only if they have the same measure.
2.5	Suppose that B is between A and C. Prove $ray\ \overrightarrow{AB} \cong\ ray\ \overrightarrow{AC}$.
2.6	Suppose $\angle ABC$ and $\angle CBD$ are supplementary and $\angle ABC \cong \angle EBD$. Prove $\angle ABE \cong \angle CBD$.

Part III: Synthetic Geometry

Euclid's First Postulates

The axioms in this section are based on Euclid's Postulates. Since Euclid wrote his postulates 3000 years ago, mathematicians have identified flaws in the axiomatic structure of Euclidean geometry. Euclid's first three Postulates introduced in Part I are extended by the Clark's common notions to address some of those issues. Euclid's fourth Postulate states that all right angles are congruent, which we expanded upon in introducing Axioms of Angle Measure and by defining congruence in Part II. However, Euclid's axioms are still useful – in particular, the multiple forms of Euclid's Fifth Postulate, relating to the existence of parallel lines.

Lines & Euclid's Fifth Postulate

Two lines are **parallel** if they that lie in the same plane but do not intersect in the plane. A **transversal** is a line that intersects two other lines. Two lines are **perpendicular** if, when they intersect, they form right angles.

Discussion: Two lines are *coincident* or *coinciding* if they lie in the same position in the plane. Are coincident lines parallel?

We introduce the following definition and axiom to qualify the regions of the plane formed by a transversal.

We define a subset S of the plane to be **convex** if, for any points A and B in S, the line segment $\overline{AB}$ is also in S .

Plane Separation Axiom: For every line ℓ contained in a plane, the points not on ℓ form two disjoint and convex sets such that if point A is in one set and point B is in the other set, then $\overline{AB}$ and ℓ must intersect.

Suppose we have a transversal which cuts across two other lines, that is, it intersects the other lines in exactly one point each. We will call the intersection of the first line and the transversal A and the intersection of the second line and the transversal B. Then, we have the following definitions:

- Two angles which are in the same relative position to points A and B are called **corresponding angles**.
- Angles with vertices at either A or B that lie between the two non-transversal lines are called **interior angles**. Interior angles pairs made by each of the non-transversal lines and the transversal are **alternate interior angles** if they lie on opposite sides of the transversal.
- Angles with vertices at either A or B that are *not* between the two non-transversal lines are called **external angles**. Exterior angles pairs made by each of the non-transversal lines and the transversal are **alternate exterior angles** if they lie on opposite sides of the transversal.

Discussion: Draw two lines and a transversal, and label enough points on those lines to denote each one of the angles formed. Identify all pairs of corresponding, alternate interior, and alternate exterior angles. Do this again, but this time, make the original two lines parallel. What do you notice? Can you make any conjectures about these pairs of angles?

Axiom of Parallelism (Playfair's Axiom, equivalent to Euclid's Fifth Postulate): For every line ℓ and every point P not on ℓ, there is at most one line containing P that is parallel to ℓ.

Theorems about Parallel Lines and Angles

In order to prove some of the theorems below, it may be helpful also to think about or use Euclid's Fifth Postulate, as it was originally formulated below.

If a transversal that cuts two other lines makes the interior angles on the same side add up to less than a straight angle, the two (non-transversal) lines meet on that side of the transversal.

3.1	**Vertical Angle Theorem** Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines intersecting at the point O. Assume that O is between A and B, and that O is between C and D. Then $\angle AOC \cong \angle BOD$.
3.2	**Same-side Interior Angle Theorem** Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines, and that $\overleftrightarrow{AC}$ is a transversal such that B and D lie on the same side of $\overleftrightarrow{AC}$. Then angles $\angle BAC$ and $\angle DCA$ are supplementary if and only if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.
3.3	**Alternate Interior Angle Theorem** Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines, and that $\overleftrightarrow{AC}$ is a transversal such that B and D lie on opposite sides of $\overleftrightarrow{AC}$. Then $\angle BAC \cong \angle DCA$ if and only if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.
3.4	**Corresponding Angle Theorem** Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines, and that $\overleftrightarrow{AC}$ is a transversal such that B and D lie on the same side of $\overleftrightarrow{AC}$. Let E be a point on $\overleftrightarrow{AC}$ such that C is between E and A. Then $\angle BAE \cong \angle DCE$ if and only if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.

Triangles and Triangle Congruence

Suppose A, B, and C are non-collinear points in the plane. Then the figure ABC consisting of the line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ is called a **triangle**. The line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are called the **sides** of the triangle. The angles $\angle CAB$, $\angle ABC$, and $\angle BCA$ are called the **interior angles** of the triangle.

- A triangle that contains a right angle is a **right triangle**.
- A triangle that contains only acute angles is an **acute triangle.**
- A triangle that contains an obtuse angle is an **obtuse triangle**.
- A triangle with three congruent sides is an **equilateral triangle**.
- A triangle with at least two congruent sides is an **isosceles triangle**.
- A triangle with no two sides congruent is a **scalene triangle**.

Let ΔABC be a triangle, and suppose we extend $\overline{AB}$ beyond B to D, without loss of generality. The angle $\angle DBC$ is an **exterior angle** of ΔABC. The interior angles $\angle BAC$ and $\angle BCA$ are called the **remote interior angles** of $\angle DBC$.

3.5	**Interior Angles of a Triangle Theorem** The sum of the measures of the three interior angles of a triangle is 180°.
3.6	**Transitivity of Parallelness Theorem** Suppose that $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ are lines such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$. Then $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$ are parallel (or they are coincident).
3.7	**Exterior Angles of a Triangle Theorem** The sum of the measures of the three exterior angles of a triangle is 360°.
3.8	**Exterior and Remote Interior Angles of a Triangle Theorem** The measure of an exterior angle of a triangle is equal to the sum of the measures of its remote interior angles.

The triangle congruence conditions (3.9-3.11, 3.13) are often proved by considering one of them axiomatic and proving the rest as resulting from it. Usually, geometers consider the SAS condition to be a "forgotten" postulate of Euclid's, or they integrate it into their axiomatic systems for geometry. In this course (and in the standards documents for K-12 mathematics), it is important to understand the congruence of two geometric figures as an invariant property of applying sequence of isometries. Therefore, establishing the validity of (at least one of) the triangle congruence theorems listed below will depend on transformational reasoning (see Part II).

3.9	**SAS Congruence Theorem** If two sides and the angle between them in one triangle are congruent to the corresponding two sides and the angle between them in another triangle, then the triangles themselves are congruent.

Proving this theorem can lead us to the following important principle: ***Corresponding parts of congruent triangles are congruent (CPCTC)***. This means that knowing that triangles are congruent implies knowing that three pairs of corresponding angles are congruent and three pairs of corresponding sides are congruent.

3.10	**ASA Congruence Theorem** If two angles and the side between them in one triangle are congruent to the corresponding two angles and the side between them in another triangle, then the triangles themselves are congruent.
3.11	**AAS Congruence Theorem** If two angles and a side not between them in one triangle are congruent to the corresponding two angles and side not between them in another triangle, then the triangles themselves are congruent.
3.12	**Isosceles Triangle Theorem** Let ΔABC be a triangle. Then the following statements are equivalent: 1. $AB = AC$. 2. $\angle ABC \cong \angle ACB$.

3.13	**SSS Congruence Theorem** If the three sides of one triangle are each congruent to the corresponding three sides of another triangle, then the triangles themselves are congruent.

Geometric Inequalities

Inequalities are rare in Euclidean geometry, but the ones listed below are important and used often.

3.14	**Triangle Inequality Theorem** Let ΔABC be a triangle. Then $BC < AB + AC$, without loss of generality.
3.15	**Side/Angle Inequality Theorem** Let ΔABC be a triangle. Then $AB < AC$ if and only if $\text{m}\angle ACB < \text{m}\angle ABC$.
3.16	**Hinge Theorem** Let ΔABC and ΔDEF be triangles such that $AB = DE$, $AC = DF$, and $\text{m}\angle BAC < \text{m}\angle EDF$. Then $BC < EF$.

Quadrilaterals

A geometric figure enclosed by a finite number of straight sides is called a **polygon**. A polygon with four sides is a **quadrilateral**. Quadrilaterals can be classified into particular classes. It is important to note that these classes are not mutually exclusive.

A quadrilateral $ABCD$ is

- a **parallelogram** if $\overline{AB} \parallel \overline{DC}$ and $\overline{BC} \parallel \overline{AD}$,
- a **rectangle** if each interior angle of $ABCD$ is a right angle,
- a **rhombus** if $AB = BC = CD = DA$, and
- a **trapezoid** if *at least* one pair of sides of $ABCD$ are parallel.
- a **square** if it is both a rectangle and a rhombus.

3.17	Using the definitions given, create a hierarchy of set inclusions for the types of quadrilaterals. How would this hierarchy change if trapezoids were defined as having exactly one pair of parallel sides instead of at least one pair of parallel sides?
3.18	**Parallelogram Theorem** Let $ABCD$ be a quadrilateral. Then the following four statements are equivalent: 1. $ABCD$ is a parallelogram; that is, $\overline{AB} \parallel \overline{DC}$ and $\overline{BC} \parallel \overline{AD}$. 2. $\angle DAB \cong \angle BCD$ and $\angle ABC \cong \angle CDA$. 3. $AB = CD$ and $BC = DA$. 4. The diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.
3.19	**Rectangle Diagonals Theorem** Let $ABCD$ be a parallelogram. Then $ABCD$ is a rectangle if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.

3.20	**Rhombus Diagonals Theorem** Let $ABCD$ be a parallelogram. Then $ABCD$ is a rhombus if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular.

Part IV: Area and Perimeter

Discussion: What is area? What are we measuring when we measure area? What is a *unit* of area? How do we use units of area to measure area?

Axioms of Area
1. To every polygonal region there corresponds a unique positive number called the **area** of the polygonal region. 2. If two triangles are congruent, then the triangular regions have the same area. 3. Suppose that the region R is the union of two regions R_1 and R_2. Suppose also that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2. 4. The area of a rectangle is equal to the product of the length of its base and the length of its altitude.

Many other area formulas can be built from the formulas for the areas of rectangles, parallelograms and triangles. Justify the following area formulas.

4.1	**Area of a Parallelogram** Let $ABCD$ be a parallelogram, and let X be a point on $\overleftrightarrow{AB}$ such that $\overline{CX} \perp \overleftrightarrow{AB}$. Then the area of $ABCD$ is $A_{ABCD} = AB \cdot CX$.
4.2	**Area of Triangle** Let ΔABC be a triangle, and let X be a point on $\overleftrightarrow{AB}$ such that $\overline{CX} \perp \overleftrightarrow{AB}$. Then the area of ΔABC is $A_{\Delta ABC} = \frac{1}{2}(AB \cdot CX)$.
4.3	**Area of Trapezoid** The area of a trapezoid is the product of its height and the arithmetic mean of its bases.
4.4	**Area of Rhombus** The area of a rhombus is one-half the product of the lengths of its diagonals.
4.5	**Parallel Line Triangle Area Theorem** Suppose that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel. Then the area of ΔABC is equal to the area of ΔABD.
4.6	Let $ABCD$ be a parallelogram. Prove that the diagonal $\overline{BD}$ divides the parallelogram into two triangles of equal area.
4.7	**The Pythagorean Theorem**: Let ΔABC be a triangle with right angle $\angle ABC$. Then, we have the following identity: $AC^2 = AB^2 + BC^2$.

Distance Around a Geometric Figure

The **perimeter** of a polygon is the sum of the lengths of its sides.

Discussion: Prove or disprove: As the perimeter of a polygon increases, so does its area.

A **circle** the set of all points (in a plane) that are equidistant from a given point, which is called the **center**. The **radius** of a circle is the distance for any point on the circle to the center. The **diameter** of a circle is the length of a segment that connects two points on the circle and goes through the center of the circle.

Discussion: How might we use the lengths of line segments (which we can measure easily) to approximate the length of a curved segment (for example, an **arc**)? How might we use this idea to come up with an approximation to the circumference (distance around a circle? How can we make this approximation as accurate as possible?

Archimedes' Method

Archimedes approximated the circumference of a circle with radius 1 by inscribing and circumscribing hexagons in and around it. What is the relationship between the perimeter of the inscribed hexagon (call this h), the perimeter of the outside hexagon (call this H), and the circumference of this circle?

He then developed better and better approximations by doubling the number of sides of the inscribed and circumscribed polygons. Why might those approximations get better? From this method, Archimedes approximated the circumference of a circle with radius 1 to be somewhere between $\frac{22}{7}$ and $\frac{223}{71}$.

In fact, we can assume that for a circle of radius 1, the ratio of the circumference to the radius is 2π.

4.8	**Circumference of a Circle** The circumference of a circle is $C = 2\pi r$.
4.9	**Area of a Circle** The area of a circle is $A = \pi r^2$, where r is the radius of the circle.

Part V: Similarity

5.1	**Parallel Lines in a Triangle Theorem I** Let ΔABC be a triangle, and let P and Q be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{PQ} \parallel \overline{BC}$. Then each of the following are true: 1. The triangles ΔAPC and ΔAQB have the same area. 2. $\frac{AP}{AB} = \frac{AQ}{AC}$. 3. $\frac{PB}{AP} = \frac{QC}{AQ}$.
5.2	**Lemma on Shortest Distance** Let m and l be distinct lines and let l contain distinct points P and Q that are not on m and on the same side of *m*. If the shortest distance between P and m is the same as the shortest distance between Q and m, then m and l are parallel.
5.3	**Parallel Lines in Triangle Theorem II** Let ΔABC be a triangle, and let P and Q be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\frac{AP}{AB} = \frac{AQ}{AC}$. Then, $\overline{PQ} \parallel \overline{BC}$.

A **dilation** is a transformation that associates each point in the plane P with a point P' by identifying a point O, called the **center** of the dilation, and a positive number k, called the **scaling factor** of the dilation, such that:

For all k, and *P* distinct from O, $OP' = k \cdot OP$
For k>1, P is between O and P'
For k<1, P' is between O and P.
For k= 1, $P = P'$.

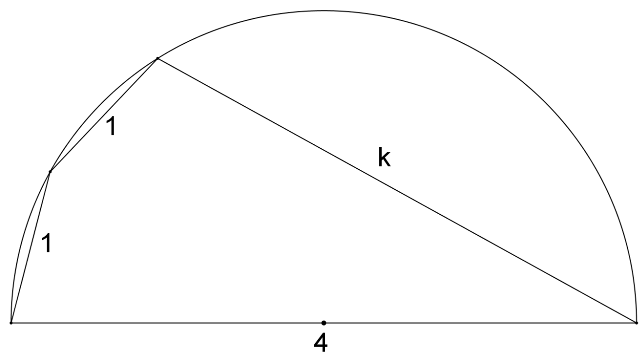
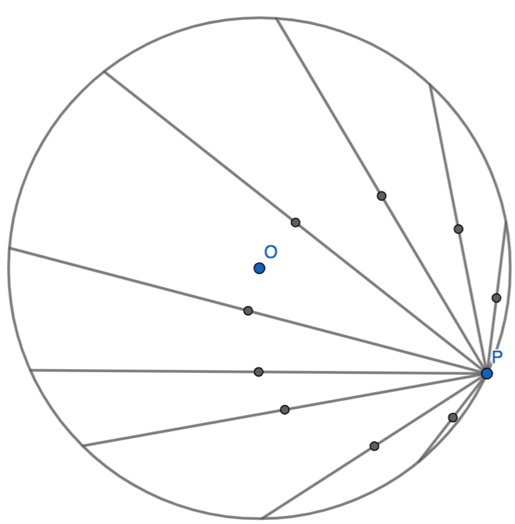
Two geometric figures are **similar** if composing a dilation and a sequence of isometries superimposes one figure onto the other.

5.4	**AA Triangle Similarity Theorem** Let ΔABC and ΔDEF be triangles such that $\angle CAB \cong \angle FDE$ and $\angle ABC \cong \angle DEF$. Then $\Delta ABC \sim \Delta DEF$.
5.5	**SAS Similarity Theorem** Let ΔABC and ΔDEF be triangles such that $\angle CAB \cong \angle FDE$ and $\frac{AB}{DE} = \frac{AC}{DF}$. Then $\Delta ABC \sim \Delta DEF$.
5.6	**SSS Similarity Theorem** Let ΔABC and ΔDEF be triangles such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$. Then $\Delta ABC \sim \Delta DEF$.

Part VI: Circles

- A **unit circle** is a circle with radius of length 1 unit.
- An **arc** of a circle is a connected subset of the points that make up the circle.
- A **chord** is a segment that connects any two points on the circle.
- A **sector** is the area enclosed by the edge of the circle and two rays that extend from the center of the circle.
- A **central angle** subtends two different arcs. We call the smaller arc the **minor arc**. We call the larger arc the **major arc**.
- A central angle that has measure one **radian** is an angle that cuts an arc of length one on a unit circle. This is equivalent to a measure of $180/\pi$ degrees.
- A geometric figure is **inscribed** in a circle if all its vertices are points on the circle.
- An **inscribed angle** has its one vertex on the circle.
- An **inscribed polygon** has all of its vertices on the circle.
- A line is **tangent** to a circle if it intersects the circle in exactly one point.

6.1	**Inscribed Angles Theorem** Let P, A, and B be points on a circle with center O, and let Q be a point on the circle that is not on arc APB. Then $\angle APB$ is one half of the measure of the arc AQB.
6.2	**Corollary to the Inscribed Angles Theorem** Any two inscribed angles in a circle that subtend the same arc are congruent.
6.3	**Inscribed Quadrilateral Theorem** Let ABCD be a quadrilateral inscribed in a circle. Then $\angle ABC$ and $\angle CDA$ are supplementary, and $\angle BCD$ and $\angle DAB$ are supplementary.
6.4	**Inscribed Right Angle Theorem** Let P, A, and B be distinct points on a circle. Then $\overline{AB}$ is a diameter of the circle if and only if $\angle APB$ is a right angle.
6.5	**Lemma** Let A, B, and C be points on a circle. Given a tangent line $\overleftrightarrow{AD}$ going through point A and another line $\overleftrightarrow{BC}$, the inscribed angle $\angle ACB$ has the same measure as $\angle DAB$.
6.6	**Power of a Point I Theorem** Let $\overline{AB}$ and $\overline{CD}$ be chords of a circle meeting at a point X inside the circle. Then, $AX \cdot XB = CX \cdot XD$.
6.7	**Power of a Point II Theorem** Let P be a point outside a given circle. Suppose we draw two rays from the point P: one ray intersects the circle at the points A and B (in that order), and the other intersects the circle at the points C and D (in that order). Then $PA \cdot PB = PC \cdot PD$.

6.8	**Tangent-Radius Theorem** Let P be a point on a circle centered at point O, and let $\overleftrightarrow{PQ}$ be a line through P that is tangent to the circle. Then $\overline{OP} \perp \overleftrightarrow{PQ}$.
6.9	**Circles Tangent to Each Other Theorem** Suppose that two circles, centered at points O and P respectively, are tangent at the point Q. Then the points O, P, and Q are collinear.
6.10	**Equidistant Tangents Theorem** Let P be a point outside a given circle. Suppose we draw the two lines from P that are tangent to the circle at points Q and R. Then $PQ = PR$.
6.11	Below is a semicircle with diameter 4 units and marked chords of length 1 unit each. Find the value of k. 
6.12	Consider the circle below, in which point P is on the circle centered at O and the points marked are the midpoints of chords. a. Form a conjecture about the midpoints of these chords. b. Prove your conjecture is true. 

Part VII: Geometry of Position

Isometries: An Analytic Approach
We discussed isometries at the beginning of the course, and we defined them as transformations that preserve distance and angle measure. We talked about three transformations, which we called rigid motions, but we did not define what a transformation was. Here, we will define transformations analytically.

First, we should remember what we know about functions. A function is a map from one set to another in which each member of the set of inputs (the **domain**) corresponds uniquely to a member in the set of outputs (the **range**); that is, each element of the domain is mapped to only one element of the range. A function is **one-to-one** if it takes each point in the range can only be mapped from a unique point in the range. A function is **onto** means that every element in the set that is being mapped into (the **co-domain**) is the image of an element in the domain.

In order to discuss the position of geometric objects, we need to have some way of locating them. In 1637, Rene Descartes introduced the idea of using a number line (or **axis**) to denote location; in 1649, Frans von Schooten introduced the idea of using a pair of axes to precisely denote two-dimensional planes. This is what we now call the **Cartesian coordinate system**.

We will be defining the three rigid motions by using the language of functions. In this case, the functions we are talking about map every point in the Cartesian coordinate plane to another point in the Cartesian coordinate plane, and we can say that a **transformation** is a function

$$F: \mathbb{R}^2 \to \mathbb{R}^2$$

is a *bijective* transformation; a mapping of the entire plane to the entire plane.

We also need to recall/introduce the notion of algebraic **group**.

A **group** is a set of objects, G, together with an operation, $*$, such that the following rules are satisfied for all the elements of the set:

- Closure: $a * b \in G$, for all $a, b \in G$
- Associativity: $(a * b) * c = a * (b * c)$, for all $a, b, c \in G$
- Identity: there exists fixed e in G, such that $a * e = e * a = a$, for all a in G.
- Inverse: for all a in G, there exists some element a^{-1} in G such that
$$a^{-1} * a = a * a^{-1} = e.$$

The isometries of the plane form a **group,** where the operation * is function composition**.**
That is, for any pair of isometries F and G and point P in the plane, $F * G(P) = F(G(P))$

The following transformations are all isometries:

- A **reflection across line $\boldsymbol{m}$** is the transformation r_m that takes a point P in the Euclidean plane to a point P' so that line m equidistant from both points.

- A **translation** is a transformation which moves each point a fixed distance in a single direction along a line (a **vector**). We call τ_{AB} the translation that takes point A to point B.

- A **rotation** is an isometry defined by a point and an angle. A point is rotated about a point C by angle α, denoted $R_{C,\alpha}$. We call C the **center of rotation**.

We define the **trivial isometry** as the transformation that maps every point in the plane to itself. It is also known as the **identity** transformation.

Isometries that result in a "mirror image" of the original object are called **orientation-reversing**, or **opposite**, isometries. Isometries that preserve orientation are called **direct**.

Discussion: For each transformation function, $F: \mathbb{R}^2 \to \mathbb{R}^2$, decide whether or not it is an isometry of the Euclidean plane and classify it.

$$F(x, y) = (-x + 6,\ y)$$
$$F(x, y) = (2x,\ y)$$

Discussion: Explain why, for any line m, the composition of r_m with itself is the identity mapping.

Discussion: Let v represent the *y*-axis and let h represent the *x*-axis.
- Find the image of $(2,3)$ under the reflection r_h.
- Find the image of $(2, -3)$ under the reflection r_v.
- Find the image of $(2,3)$ under the composition mapping $r_v \circ r_h$.
- What is the image of an arbitrary point (x, y) under $r_v \circ r_h$?
- Give a simple geometric description of the composition $r_v \circ r_h$.

Discussion: $R_{C,\alpha}(x, y) = (-y, x)$ is a rotation about the origin $C = (0,0)$ by some angle α. Find the angle α.

7.1	**Fixed Points Theorem I** If an isometry F fixes two distinct points, A and B, then F fixes all the points on line $\overleftrightarrow{AB}$.
7.2	**Fixed Points Theorem II** If an isometry F fixes three non-collinear points, then F must fix every point in the plane and thus be the trivial isometry.
7.3	**Equivalence of Isometries Theorem** Suppose F and G are isometries and A, B, and C are non-collinear points in $\mathbb{R}^2$. If $F(A) = G(A); F(B) = G(B); F(C) = G(C)$ then $F = G$.
7.4	Translations can be defined the composition of reflections over two lines. What must be true about these lines? Given two lines with the required characteristics, how can you determine the direction and magnitude of the translation?
7.5	Rotations can be defined as the composition of reflections over two lines. What must be true about these lines? Given two lines with the required characteristics, how can you determine the angle and center of the rotation?
7.6	Under what conditions is the composition of two rotations a rotation?
7.7	**Theorem: Isometries map lines to lines** If F is an isometry, and C is on line $\overleftrightarrow{AB}$, then $F(C)$ is on line $\overleftrightarrow{F(A)F(B)}$

Discussion: Find the parameters h and k for the translation $F(x, y) = (x + h, y + k)$ if $F = r_w \circ r_m$, where m is the line $x = 0$ and w is the line $x = 2$. Give a short description of this translation.

A **symmetry** is an isometry, F, that maps a geometric object, T, onto itself without a change in position. That is, an isometry such that $F(T) = \{A \in \mathbb{R}^2 | A \in T\}$.

The trivial isometry is a symmetry. We call it the **trivial symmetry**.

Let F be a symmetry of geometric object T:

- If $F = r_m$, T is said to be **symmetric with respect to line $\boldsymbol{m}$**, where m is a **line of symmetry** of T.
- If $F = R_{C,\alpha}$, T is said to have **rotation symmetry about the point $\boldsymbol{C}$**.
- If $F = \tau_{AB}$, T is said to have **translational symmetry**.

7.8	**Symmetries of the Triangle Theorem** The set of symmetries of an equilateral triangle form a group under the operation of composition.
7.9	**Symmetries of the Square Theorem** The set of symmetries of a square form a group under the operation of composition.
7.10	What is the set of symmetries of a regular polygon with n sides? Does that set form a group under the operation of composition? What is the size of that set?
7.11	Let *P* be a polygon with *n* vertices. Prove that *P* cannot have a (non-trivial) translation as a symmetry.. If a geometric object *T* has a (non-trivial) translation as a symmetry, what must be true about *T* ?

Part VIII: Introduction to TaxiCab Geometry

Measuring Distance

Given two points on the Euclidean plane, how can we measure the distance between them? In fact, the distance formula is pretty familiar, but in the next proof, it helps to think about where it might come from.

8.1	**Euclidean Metric** For any points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in $\mathbb{R}^2$, show that the Euclidean distance between them is given by the distance function d_E, defined as follows $$d_E(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

Ideal City (Krause, 1986)

Ideal City is a city that runs along grid lines. The streets all run vertically and horizontally. In Ideal City, *we can also instantaneously build new streets wherever we want and whenever we want, assuming they are either horizontal or vertical.* When thinking about the distance between two locations in Ideal City, we have to consider the distance we travel along horizontal and vertical streets; that is, we cannot travel diagonally.

Discussion: Jason is a Lyft driver in Ideal City. Like all Lyfts in Ideal City, Jason's car is equipped with the ability to create streets anywhere he wants, but they can be only horizontal or vertical (adapted from Kemp (2018))

- Jason is located at point $(3, 2)$ and needs to pick up a passenger at point $(6, -10)$. What is the shortest distance between Jason and the passenger?
- Jason is now located at the point $(5.5, -2.1)$ and has to pick up a passenger at the point $(2.7, 6.3)$. What is the shortest distance between Jason and the passenger?
- Write a formula for distance between any two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in Ideal City, assuming that Jason is driving his Lyft between them.
- Jason was wondering how many different routes could be taken and still have them be the shortest routes. How many different routes would result in the shortest distance between Jason and the passenger based on what you know of Ideal City?

Discussion: Ideal City recently received a grant to build a library. The city planner wants to build the library so that the two schools, A and B, are located at an equal distance to the library. School A is at point $(-3, 2)$ and school B is located at point $(6, 5)$.

- What is the distance between the two schools?
- Where can the library be located?
- Ideal City has money left over to also build a new school, C. The city manager wants school C to be exactly as close to the new library as schools A and B. Where should school C be located?
- Good news! Ideal City had enough money left over for 100 more schools. Where can these new schools be located so that they are *all* equidistant from the library?

Taxicab Geometry

The distance formula in Ideal City is actually a valid metric[1] on $\mathbb{R}^2$, called the **Taxicab metric**. For any points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in $\mathbb{R}^2$, the Taxicab distance, $d_T(A, B)$, between them is given by

$$d_T(A, B) = |x_A - x_B| + |y_A - y_B|.$$

8.2	**Euclidean/Taxicab Metric Inequality Theorem** For any points A and B in $\mathbb{R}^2$, $d_E(A, B) \leq d_T(A, B)$.

Discussion: A line of reflection in Euclidean geometry forms a perpendicular bisector between a point and its image. This is also true in TaxiCab Geometry; this operation is well-defined. Prove that if M is the midpoint of a line segment AB in Euclidean Geometry, then M is the midpoint of the line segment AB in Taxicab Geometry.

Discussion: Graph the set of all points P that are a Taxicab distance 3 from a point A:

$$\{P | d_T(P, A) = 3\}.$$

What should we call the geometric object that the graph of this set of points makes?

8.3	Students sometimes have a conception that π is 3.14, or 22/7**,** despite that they are taught that π is irrational. But is π always irrational? • What *quantitative relationship* relating to circles' circumference does π represent? • Justify that the value of π is invariant for any circle in Euclidean geometry. • What is the value of π in taxicab geometry?
8.4	A definition of *radian* angle measure is the following: the radian unit is the angle at the center of a circle whose arc is equal in length to the circle's radius. Compare an angle of radian measure π/2 in Euclidean geometry and an angle of radian measure π/2 in taxicab geometry.
8.5	Come up with examples of each, or explain why such an example is not possible: 1. 2 triangles that are congruent in both Euclidean geometry and Taxicab geometry 2. 2 triangles that are congruent in Euclidean geometry but **not** in Taxicab geometry

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[1] A **metric** is a function d that takes a pair of points in a space X and assigns to them a real number such that for any points A, B and C in X, the following three statements are true:

- $d(A, B) \geq 0$, and $d(A, B) = 0$ if and only if $A = B$,
- $d(A, B) = d(B, A)$, and
- $d(A, C) \leq d(A, B) + d(B, C)$.

The Euclidean distance formula is famously a metric, as is the formula for Taxicab distance.

	3. 2 triangles that are congruent in Taxicab geometry but **not** in Euclidean geometry
8.6	Identify all the Taxicab isometries.

Discussion: What are the conditions for two segments to be congruent in Taxicab geometry? Would you say that two segments of the same Taxicab length are congruent? Why or why not?

Part IX: Circle Inversions and Hyperbolic Geometry

We have investigated one non-Euclidean geometry that we arrived at by imposing a different distance function, d_T, on $\mathbb{R}^2$. This alternate distance function, which we called the Taxicab metric, when paired with $\mathbb{R}^2$ gave us a metric space. We have already seen some of the consequences of doing geometry in this new metric space. For example, the isometries in Taxicab geometry are different from the isometries of Euclidean geometry. Similarly, we do not have the Pythagorean Theorem for Taxicab right triangles. In fact, there are many more consequences to doing work in the metric space $(\mathbb{R}^2, d_T)$. And all these changes came only from changing the distance function on our usual canvas of $\mathbb{R}^2$!

Constructing Other Geometries

Now, think of altering something even more fundamental than the distance function. In fact, we are going to go all the way back to some of our establishing axioms.

Recall Playfair's Postulate:

> *Given a point that does not lie on a given line, there exists exactly one line through that point that is parallel to the given line.*

In fact, this leads to a question that was eventually asked (separately) by mathematicians Bolyai and Lobachevsky: what if there was more than, or less than, one parallel line through a given point?

By reconsidering this postulate, we can create multiple geometries.

Euclidean	Exactly one parallel may be drawn through a point not on a given line.
Elliptic (Spherical)	No parallel may be drawn through a point not on a given line.
Hyperbolic	More than one parallel may be drawn through a point not on a given line.

We are going to spend more time discussing elliptical (spherical) geometry soon, but first we are going to set up the foundations for hyperbolic geometry. Specifically, how can we talk about congruence and similarity? For this purpose, we begin with discussion of a transformation of the plane that is *not* an isometry or dilation, what is known as an **inversion** (about a circle).

9.1	**Inversion about a circle** Let *C* be a circle centered at O with radius *r*. Let P be a point in the Euclidean plane that is distinct from O. Then the inversion about circle *C*, denoted σ_C, maps P to be the point on $\overrightarrow{OP}$ that is distance $\frac{r^2}{OP}$ from *O*.

Discussion:

- Why isn't a circle inversion an isometry of the Euclidean plane?
- Suppose an equilateral triangle is inscribed in a circle with center *O*. What is the image of the triangle under inversion in circle *O* ?

9.2	**Conformal map** A mapping is said to be *conformal* if it preserves angle measure and orientation. A mapping is said to be *anticonformal* if it preserves angle measure and reverses orientation.

Discussion: Which of the following transformations (translations, rotations, reflections, glide reflections, identity, dilation, inversion) are conformal mappings from $\mathbb{R}^2 \to \mathbb{R}^2$?

9.3	**Cross ratio** Given four distinct points (A,B,C, D) in the plane , the *cross ratio* is defined $$(A,\ B;\ C,\ D) = \frac{AC \cdot BD}{BC \cdot AD}$$

Discussion:

- Is the cross-ratio invariant under permutations of the points A, B, C, and D?
- Verify the cross-ratio is invariant under circle inversion so long as none of the four points is the center of the inversion. That is, if A', B', C', D' are the images of the A, B, C, and D under the inversion, then $(A, B; C, D) = (A', B'; C', D')$.
- How might the cross-ratio be used to define an isometry involving a circle inversion?

The Poincaré Disk

♦ The Poincaré disk D consists of the *interior* of the unit circle. It is a model of hyperbolic geometry that is particularly useful in complex analysis.

  - The unit circle itself is called the "circle at infinity" and is often labeled ∂D.
  - Geodesics are paths on arcs of circles (or diameters) that are *orthogonal* (perpendicular) to ∂D.
  - Angles formed at intersection points of two geodesics are measured by finding the angle between the Euclidean lines tangent (coincident in the case of a diameter) to each geodesic at the point of intersection.

9.4	**Hyperbolic distance formula:**

	Let A, B be two points in the Poincaré disk D. Let P and Q on ∂D be the endpoints of the geodesic through A and B, such that A is between P and B and B is between A and Q. Then the hyperbolic distance between A and B, denoted $d_H(A, B)$, is defined as $d_H(A, B) = \ln\lvert(A,\ B; P,\ Q)\rvert$
9.5	Use the tool available at https://www.geogebra.org/classic/aDCnWYfQ to explore and verify the following properties of hyperbolic geometry in the Poincaré disk model. 1. Infinitely many parallel lines to a given line through a point 2. Distance between parallel lines is not preserved 3. Triangles with three corresponding angles of same measure must have three corresponding sides of same measure 4. Triangle angle sum is strictly less than 180.
9.6	Using the definitions of parallelogram and square, explain why parallelograms exist in hyperbolic geometry, but none of the parallelograms can be squares.

Part X: Spherical Geometry

We will now investigate another non-Euclidean geometry. This is the geometry of the sphere. Why might we want to investigate how geometry works on a sphere?

Discussion[2]: Imagine yourself to be a bug crawling around on a sphere. (This bug can neither fly nor burrow into the sphere.) The bug's universe is just the surface; it never leaves it. What is "straight" for this bug? What will the bug see or experience as straight? How can you convince yourself of this? What might straight lines on a sphere look like?

Spherical geometry is a special case of the larger class of geometries that are collectively called elliptic geometry. This is because the elliptic parallel postulate gives rise to geometries on a surface with positive curvature. If the curvature of the surface is 1, then we're doing geometry on a sphere.

For our purposes, let us restrict ourselves to the unit sphere centered at the origin, that is, the sphere with radius 1. We can think of this sphere as embedded in $\mathbb{R}^3$, that is, space.

The surface of this sphere, S^2, is the collection of points given by

$$S^2 = \{A = (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

In spherical geometry, we have to assume that the entire universe we are working in is only the surface of the sphere, which is two-dimensional. That means, any lines we draw between points must follow the surface of the sphere. However, note that we can also use some of our understandings about $\mathbb{R}^3$, which is the space the sphere lives in, to talk about points, lines and distances on the sphere.

Discussion: (Visualizing Spherical geometry) Take a moment to think about by the elliptical parallel postulate would result in geometry being done on a sphere. Suppose you have a line m and a point P not on the line. In order for there to be no lines through P that are parallel to m, every line through P needs to intersect with m at some point. What might that result in? Draw a picture to reflect your thinking. What would straight lines look like?

Points on the Sphere

There are two ways to talk about points on the unit sphere. First, we can use the Cartesian coordinate system from $\mathbb{R}^3$ to list points. However, we need to make sure these points are on the sphere (that is, that the

[2] Adapted from Course Notes written by Nathanial Miller for the *Journal of Inquiry-Based Learning in Mathematics*.

Euclidean distance in three dimensions between the given point and the origin is exactly 1). Euclidean distance in space is given by the following formula:

$$d_E(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

10.1	Show that the equation for the unit sphere centered at the origin, O, is the same as the collection of points A in $\mathbb{R}^3$ such that $d_E(A, O) = 1$.

It's easy to see that the following points will be on the unit sphere, S^2: $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$, and $(0,0,-1)$.

Discussion: Which of these other points are also on the unit sphere?

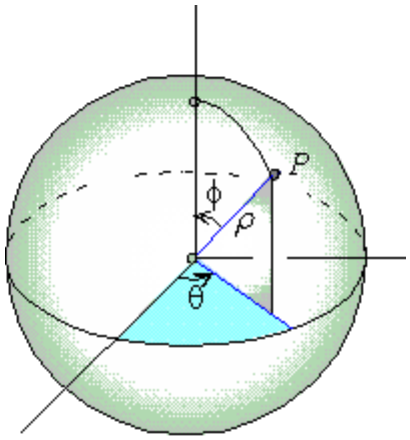
$$P = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$$

$$Q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$R = \left(\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$$

10.2	Find an analytic process for determining whether a point is on a unit sphere centered at the origin.

It should be emphasized that this method only tells us if a point is on the sphere – the Euclidean distance formula is not the appropriate way to measure distance on the surface of the sphere. This is because Euclidean distance will calculate the distance between two points in space by measuring the length of the straight line segment that connects them in space. This is akin to measuring the distance between Portland and San Antonio by boring a straight tunnel into the Earth to connect the two cities and measuring it. In actuality, we measure the distance between San Antonio and Portland by measuring along the surface of the Earth (which, you may remember, is approximately spherical). Similarly, we'll measure distance *in* spherical geometry by measuring along the sphere, but in order to define the unit sphere in the first place, we use Euclidean distance. In fact, you can probably see a clear connection between the Euclidean distance formula in space and the formula for the unit sphere.



The second way we can talk about points on S^2 is by using spherical coordinates. Spherical coordinates are much like polar coordinates, but generalized to three dimensions. In fact, when projected onto the xy-plane, spherical coordinates pretty much are polar coordinates. Spherical coordinates (ρ, θ, ϕ) are defined as follows: ρ represents the Euclidean distance between the point and the origin (note that for points on the unit sphere, $\rho = 1$), θ represents the angle that the point makes with the positive x-axis, and ϕ represents the angle the point makes with the positive z-axis.

Imagine placing the point on the unit sphere like this: rotate around the equator of the sphere (starting from $(1,0,0)$) until you reach the right "longitude" for your point, then rotate directly upward toward the "North Pole" or directly downward to the "South Pole" to reach the desired point. The first rotation will give you θ, measuring the angle between the z-axis and where you ended up after the second rotation will give you ϕ.

The relationship between spherical coordinates (which are of the form (ρ, θ, ϕ)) and Euclidean coordinates (which are of the form (x, y, z)), can be expressed as

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\phi$$

$$z = \rho \cos\theta$$

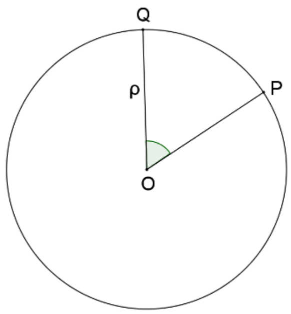
Note that $\rho = 1$ on the unit sphere.

Lines on the Sphere

Lines in spherical geometry are defined in the same way they are defined in Euclidean geometry, that is, as the most "direct" way to connect two points. As in Euclidean geometry, two points will define a line. However, remember that in spherical geometry, we need to travel along the surface of a sphere, so we must determine the shortest path on the sphere that connects two points. This path is along the great circle on which both points lie. (A **great circle** of a sphere is the circle of largest diameter that lies on the sphere. For example, the equator is a great circle of the Earth.) Given two points P and Q, there is only one great circle containing both of them, except if P and Q are diametrically opposed. If P and Q are **diametrically opposed** (that is, they are distinct points on the same diameter of the sphere), how many different spherical lines (i.e. great circles) will pass through them?

Calculating Distance on the Sphere

Since the shortest path between two points P and Q is along a spherical line (a great circle), we calculate the distance between those two points as the length of the minor arc on the great circle that connects them. Let the picture at the left be a representation of the great circle that connects P and Q. Remember that this circle is on the sphere of radius ρ.



The arclength of the minor arc between P and Q is easy to express:

$$d_s(P, Q) = \rho \cdot \mathrm{m}\angle POQ,$$

where $\mathrm{m}\angle POQ$ is in radians.

However, how do we find $\mathrm{m}\angle POQ$? For an arbitrary choice of P and Q on the sphere, the angles used to define their spherical coordinates do not directly tell us the angle formed by the two points and the origin. However, by using Euclidean coordinates, we can use a useful fact from linear algebra to help us develop the formula for spherical distance.

The **dot product** of two vectors $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$ is defined as

$$P \cdot Q = x_P x_Q + y_P y_Q + z_P z_Q.$$

As we can think of points in space as vectors from the origin, we can see that the angle between the two vectors P and Q is actually the angle created by the points P and Q and the origin. Using the useful fact that $P \cdot Q = |P||Q|\cos(\angle POQ)$, and noting that the magnitudes of the vectors P and Q are $|P| = |Q| = \rho$, we get that

$$\mathrm{m}\angle POQ = \cos^{-1}\left(\frac{P \cdot Q}{\rho^2}\right).$$

Note that $\mathrm{m}\angle POQ$ is still being measured in radians. Substituting this into the formula for arclength (remember, the angle is being given in radians), we get the spherical distance formula

$$d_s(P, Q) = \rho \cdot \cos^{-1}\left(\frac{P \cdot Q}{\rho^2}\right).$$

Note that we always measure the minor arc between the points P and Q, meaning that the longest the arc could be is $\rho\pi$, or half the circumference of the great circle.

10.3	Find the distance along the sphere between the points $S = \left(\frac{1}{3}, -\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$ and $T = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.
10.4	Prove that rectangles do not exist on the sphere.