# Math 338 - Homework 9

Due Friday 3/11

*Answer the following questions. You are encouraged to work with other students and to seek help from the instructor while working on these problems, but please write up your answers on your own.*

1. (Boyce 9.6) Using the definitions of parallelogram and square, explain why parallelograms exist in hyperbolic geometry, but none of the parallelograms can be squares.

   **Solutions:** In hyperbolic space, it is easy to construct parallel lines. As an example, the points (0,0), (1/2,0), (0,1/2), and (1/2, 1/2) in the Poincare disk form a parallelogram, since the vertical and horizontal diameters don't cross the opposite arcs. However, if one were a square, then by connecting the opposite vertices (0,0) and (1/2,1/2) we'd form two triangles $T_1$ and $T_2$. If we add the interior angles of $T_1$ and $T_2$ all together they'd need to add to 360°. This would only be possible if one of $T_1$ and $T_2$ had interior angles sum to at least 180°, which is not possible in hyperbolic geometry.

2. (Boyce 10.1) Show that the equation for the unit sphere $S^2$ centered at the origin $O$ is the same as the collection of points $A$ in $\mathbb{R}^3$ such that $d_E(A, O) = 1$.

   $$S^2 = \{x^2 + y^2 + z^2 = 1\}.$$

   **Solutions:** The points in $S$ are the points $A = (x, y, x)$ that fit the equation $x^2 + y^2 + z^2 = 1$. So they also fit the equation $(x - 0)^2 + (y - 0)^2 + (z - 0)^2 = 1$. That means that $d_E(A, 0) = 1$.

3. (Boyce 10.3) Find the distance along the sphere between the points $S = (1/3, -1/3, -\sqrt{7}/3)$ and $T = (2/3, 2/3, -1/3)$.

   **Solutions:** The radius of the sphere containing these points is $(1/3)^2 + (-1/3)^2 + (-\sqrt{7}/3)^2 = 1$. So the distance between the two points is

   $$d(S, T) = \arccos(S \cdot T).$$

   In this case,

   $$S \cdot T = 2/9 - 2/9 + \sqrt{7}/9.$$

So the distance between the two points is $\arccos(\sqrt{7}/9)$.

4. (Boyce 10.4) Prove that rectangles do not exist on the sphere.

   **Solutions:** On the sphere, it is impossible for two lines to be parallel. Therefore, it is not possible to construct any parallelogram, let alone a rectangle.