Math 338 - Homework 3

Due Friday 1/28

Answer the following questions. You are encouraged to work with other students and to seek help from the instructor while working on these problems, but please write up your answers on your own.

1. (Boyce 3.4) Prove the Corresponding Angle Theorem: Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines, and that $\overleftrightarrow{AC}$ is a transversal such that B and D lie on the same side of $\overleftrightarrow{AC}$. Let E be a point on $\overleftrightarrow{AC}$ such that C is between E and A. Then $\angle BAE \cong \angle DCE$.

   Solution: Angles $\angle DCE$ and $\angle DCA$ are supplementary, so $m\angle DCE + m\angle DCA = 180°$. Place a point F on $\overline{AB}$ so that A is between F and B. By the Alternate Interior Angles Theorem, $\angle DCA \cong \angle FAC$. So $m\angle FCA + m\angle DCE = 180°$. But since $\angle FAC$ and $\angle BAE$ are supplementary, $m\angle BAE = m\angle DCE$. By our previous theorem, $\angle BAE \cong \angle DCE$.

2. (Boyce 3.5) Prove or disprove the following statements.

   (a) If the ray $\overrightarrow{OP}$ bisects the angle $\angle AOB$, and angles $\angle AOP$ and $\angle BOP$ are acute, then $\angle AOB$ is obtuse.

   (b) If the ray $\overrightarrow{OP}$ bisects the angle $\angle AOB$, and the angle $\angle AOB$ is obtuse, then angles $\angle AOP$ and $\angle BOP$ are acute.

   Solution:

   (a) This is false. If, for example, $m\angle AOP = 30°$, then $m\angle AOB = 60°$ so $\angle AOB$ is acute too.

   (b) This is true. Let $m\angle AOB = x$. We know $90° < x < 180°$. Then $\angle AOP$ and $\angle BOP$ have measures $x/2$, and $45° < x < 90°$. That means these angles are acute.

3. (Boyce 3.13) Prove the Isosceles Triange Theorem: Let ΔABC be a triangle. Then the following statements are equivalent:

   (a) $AB = AC$.

(b) $\angle ABC \cong \angle ACB$.

Solution: We will use Triangle Congruence Theorems for both implications. First suppose first that ΔABC is a triangle and $AB = AC$. Construct the unique bisector of $\angle BAC$ and label its intersection point with $\overline{BC}$ as D. Consider the triangles ΔBAD and CAD. They share side $\overline{AD}$ and we know $\overline{AC} \cong \overline{AB}$. By definition, $\angle BAD \cong \angle CAD$. So by SAS $\Delta BAD \cong \Delta CAD$. This means $\angle ABC \cong \angle ACB$.

Now suppose $\angle ABC \cong \angle ACB$. Again, construct the bisector $\overline{AD}$ of $\angle BAC$ with D between B and C. Then by construction, $\angle BAD \cong \angle CAD$. And since interior angles of a triangle add to $180°$, $\angle ADB \cong \angle ADC$. Since both triangle share $\overline{AD}$, the ASA Congruence Theorem shows $\Delta BAD \cong \Delta CAD$, and so $AB = AC$.

4. (Boyce 3.17) Prove the Rectangle Diagonals Theorem: Let $ABCD$ be a parallelogram. Then $ABCD$ is a rectangle if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent.

   Solution: Suppose $ABCD$ is a parallelogram. Now suppose $\overline{AC} \cong \overline{BD}$. Then since $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$, triangles ABC and BAD are congruent. That means $\angle ABC$ is congruent to $\angle BAD$.

   Because $ABCD$ is a parellelogram, we also know $\angle ABC \cong \angle ADC$ and $\angle BAD \cong \angle BCD$. Together, this means all 4 angles in $ABCD$ have the same measure. Since the total of all angles in $ABCD$ is $360°$, each angle has size $90°$. This makes $ABCD$ a rectangle.

   Now suppose $ABCD$ is a rectangle. Then all interior angles are right angles. Consider the triangles ΔABC and ΔBAD. They share side $\overline{AB}$ and $AD = BC$. And $\angle BAD \cong \angle ABC$ since they are both right angles. By SAS, $\Delta ABC \cong \Delta BAD$. This means $\overline{AC} \cong \overline{B}D$.