# Math 338 - Homework 7

Due Friday 2/25

*Answer the following questions. You are encouraged to work with other students and to seek help from the instructor while working on these problems, but please write up your answers on your own.*

1. (Based on ) Consider the function $D_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

   $$D_0(x, y) = (2x, 2y).$$

   Prove that $D_0$ is not an isometry.

   **Solution:** Consider the points $(0, 0)$ and $(1, 0)$. Their distance is 1. But $D_0(0, 0) = (0, 0)$, and $D_0(1, 0) = (2, 0)$. These points are 2 units apart. Since isometries preserve distance, this means $D_0$ is not an isometry.

2. (Boyce 7.3) Suppose $F$ and $G$ are isometries and $A$,$B$,$C$ are non-collinear points in $\mathbb{R}^2$. Prove that if $F(A) = G(A)$, $F(B) = G(B)$, and $F(C) = G(C)$, then $F = G$.

   **Solution:** Suppose $F$ and $G$ are isometries, and $A, B, C$ are non-collinear points in $\mathbb{R}^2$ such that $F(A) = G(A)$, $F(B) = G(B)$, and $F(C) = G(C)$. Suppose by way of contradiction that $D$ is a point in $\mathbb{R}^2$ such that $F(D) \neq G(D)$.

   First, suppose $D$ is collinear with one pair of points (A and B or B and C or A and C). Without loss of generality, suppose $D$ lies on $\overleftrightarrow{AB}$. By our theorem from class, $F(D)$ must lie on the line $\overleftrightarrow{F(A)F(B)}$ and must also lie on the line $\overleftrightarrow{G(A)G(B)}$.

   Because isometries preserve distance, $F(D)F(A) = DA$ and $F(D)F(B) = DB$. There is exactly one point in the plane that satisfies both of these. Since it also must be true that $G(D)G(A) = DA$ and $G(D)G(B) = DB$, and we know $F(A) = G(A)$ and $G(A) = G(B)$, then $G(D) = F(D)$, contradicting our assumption.

   Then we must assume that $D$ lies on none of the lines formed by $A, B, C$. Since the points are non-collinear, the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ form a triangle. By our previous argument, any point $Z$ on one of those lines satisfies $F(Z) = G(Z)$.

   Then we can construct a line from D that intersects two of these lines in points, say $X$ and $Y$. But then we showed for points on the lines, $F(X) = G(X)$ and $F(Y) = G(Y)$. Further, we now know that since $D$ is on $\overleftrightarrow{XY}$, $F(D) = G(D)$. This is a contradiction. So $D$ does not exist, and $F = G$.

3. (Based on Boyce 7.3) Suppose $A$, $B$, and $C$ are collinear. Is it possible that there are two isometries $F$ and $G$ such that $F \neq G$ but $F(A) = G(A)$, $F(B) = G(B)$, and $F(C) = G(C)$? Why or why not?

   **Solution:** Yes! For example let $F = e$ the identity isometry, and $G$ be the reflection over the line $y = 0$ (the $x$-axis). Then both $F$ and $G$ fix the points $(0,0)$, $(1,0)$, and $(2,0)$. But $F(0,1) = (0,1)$ and $G(0,1) = (0,-1)$. So $F \neq G$.

4. (Boyce 7.10) What is the set of symmetries of a regular polygon with $n$ sides? Does that set form a group under the operation of composition? What is the size of that set.

   **Solution:** A regular polygon with $n$ sides has $2n$ symmetries.

   - The identity $e$.
   - The rotations $R_{2k\pi/n}$ for $k = 1, 2, \ldots, n-1$ around the center of the polygon.
   - If $n$ is odd: the $n$ reflections over the lines which bisect the $n$ internal angles of the polygon.
   - If $n$ is even: the $n/2$ uniqure angle bisectors of the internal angles of the polygons, and the $n/2$ perpendicular bisectors of the segments which form its sides.

   These form a group since:

   - Closed under composition: performing one isometry which superimposes the polygon, and then another, will leave the polygon in its original position. So it is a symmetry as well.
   - Associative: $(F \circ G) \circ H = F \circ (G \circ H)$ since both mean 'perform H, then G, then F'.
   - Identity: The identity symmetry $e$ leaves the polygon unmoved.
   - Inverses: The identity is its own inverse, and each reflection is also its own inverse since performing a reflection twice gets back to the original position of the polygon. Any rotation $R_{2k\pi/n}$ has inverse $R_{2(n-k)\pi/n}$ since performing one and then the other in either order rotates the polygon by $2\pi$ radians (back to its original position).