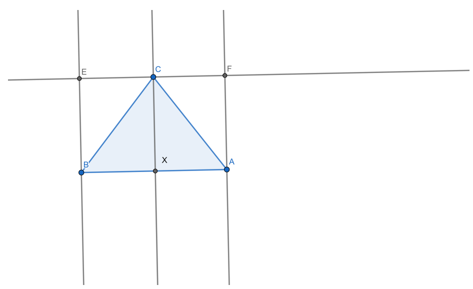
Math 338 - Homework 4

Due Friday 2/4

Answer the following questions. You are encouraged to work with other students and to seek help from the instructor while working on these problems, but please write up your answers on your own.

1. (Boyce 4.1) Let ΔABC be a triangle, and let X be a point on $\overline{AB}$ such that $\overline{CX}$ is perendicular to $\overline{AB}$. Prove that the area of ΔABC is $(1/2)(AB \times CX)$.

Solution: We'll use the following picture for reference.



By Axiom 3 of Area, the area of ΔABC is the sum of the areas of ΔAXC and ΔBXC, so we'll calculate each one separately.

Let $\overline{EF}$ be parallel to $\overline{AB}$ so that:

- $\overline{EF}$ passes through C.
- $\overline{AF}$ is perpendicular to $\overline{AB}$ (and $\overline{EF}$ by Euclid's 5th axiom).
- $\overline{BE}$ is perpendicular to $\overline{AB}$ (and $\overline{EF}$ by Euclid's 5th axiom).

Then since $\overline{CX}$ is also perpendicular to $\overline{EF}$, it is parallel to both $\overline{BE}$ and $\overline{AF}$. The quadrilaterals $BECX$ and $AFCX$ are then rectangles. This tells us that:

- The area of $BECX$ is $BX \cdot CX$ by Axiom 4 of Area.
- The area of $AFCX$ is $AX \cdot CX$ by Axiom 4 of Area.

Let's look at each rectangle separately. The rectangle $BECX$ is composed of the triangle ΔBXC and the triangle CEB. Since the two triangles overlap only on a line segment, Axiom 2 says the area of $BECX$ is the sum of the areas of the two triangles. This is a rectangle, the Parallelogram Theorem says $BE = CX$. They also share a side BC. Since $\overline{BE}$ and $\overline{CX}$ are parallel, alternate angles $\angle EBC$ and BCX are congruent. Then by SAS, the two triangles are congruent, and so by Axiom 2 of area they also have the same area. Therefore the area of ΔBCX is $(1/2) \cdot BX \cdot CX$.

I'm going to cheat now and say that as the instructor, I don't want to write the same exact proof for $AFCX$. But you can replace the letters in the last paragraph appropriately to show that the area of ΔACX is $(1/2) \cdot AX \cdot CX$.

All together, the area of ΔABC is $(1/2) \cdot BX \cdot CX + (1/2) \cdot AX \cdot CX$. Factoring, this becomes $(1/2) \cdot (BX + AX) \cdot CX$.Since $AB = AX + BX$ by Axiom 3 of Line Segment Measure, we get that the area of ΔABC is $(1/2) \cdot AB \cdot CX$.

2. (Boyce 4.4) Prove that the area of a rhombus is one half the product of the lengths of the diagonals.

   Solution: Suppose ABCD is a rhombus. Imagine the triangles ΔABD and ΔBCD. They form the top and bottom halves of the rhombus, respectively. By Axiom 3 of Area, the area of ABCD is the area of ΔABD plus the area of ΔBCD. So we will calculate each one separately, then add the results.

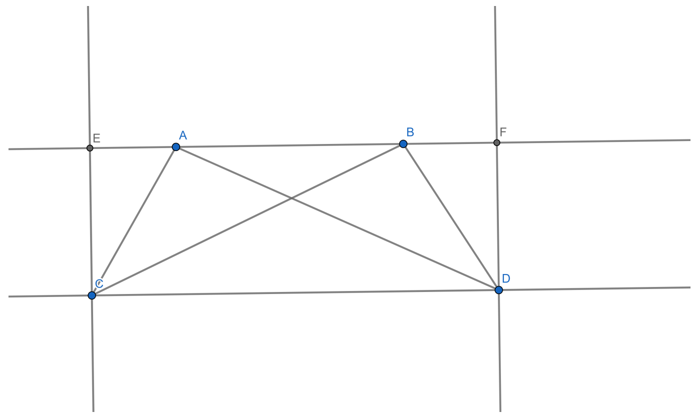
   First, for ΔABD, we'll use BD as the base. The 'height' would then the length of a line connecting A and $\overline{BD}$ that is perpendicular to $\overline{BD}$. By the Rhombus Diagonals Theorem (shown in class), the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. And so if we let E be the point of intersection of $\overline{AC}$ and $\overline{BD}$, then the area of ΔABD is $(1/2) \cdot BD \cdot AE$.

   We can use the same argument on ΔBCD. If we use BD for its base then its height would be CE, the distance from the point C and the diagonal $\overline{BD}$, since we know $\overline{AC}$ and $\overline{BD}$ are perpendicular.

   Putting this together, the area of $ABCD$ is $(1/2) \cdot BD \cdot AE + (1/2) \cdot BD \cdot CE$. By factoring, we get that this is $(1/2) \cdot BD \cdot (AE + CE)$. By Axiom 3 of Line Segment Measure, $AE + CE = AC$, so we end up with the desired formula, $(1/2) \cdot BD \cdot AC$.

3. (Boyce 4.7) Suppose $\overline{AB}$ and $\overline{CD}$ are parallel. Prove the area of ΔABC is equal to the area of ΔABD.

   Solution: Since $\overline{AB}$ and $\overline{CD}$ are parallel, they form two sides of a trapezoid. I'll use the following picture, but note that the argument does not rely on the positions of drawn points.



By our work in exercise 1, we know the area of ΔABC is $(1/2) \cdot AB \cdot CE$. The same reasoning shows that the area of ΔABD is $(1/2) \cdot AB \cdot DF$. But since CE and DF are both perpendicular to $\overline{AB}$ and $\overline{CD}$, they form two sides of a rectangle. The Parallelogram Theorem tells us that $DF = CE$. Then the area of $\Delta ABC = (1/2) \cdot AB \cdot CE = (1/2) \cdot AB \cdot DF =$ the area of ΔABD.

4. (Boyce 4.8) Let ABCD be a parallelogram. Prove that the diagonal $\overline{BD}$ divides the parallelogram into two triangles of equal area.

   Solution: Suppose ABCD is a parallelogram. By the Parallelogram Theorem, AB=DC and AD = BC. So since BD is congruent to itself, by the SSS Triangle Congruence Theorem, $\Delta ABD \cong \Delta CDB$. Then Axiom 2 of Area tells us that the area of ΔABD and the area of ΔCDB are equal.