Name: Solutions

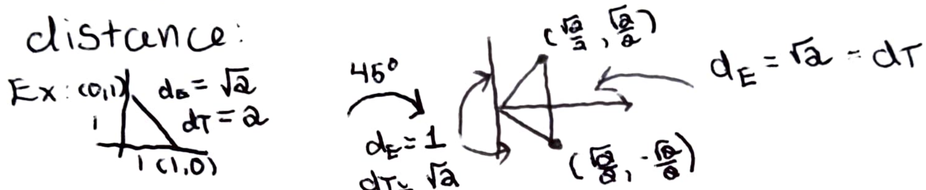
Math 338 Practice Final Exam

Answer each question in the space provided. To receive full credit, you must clearly present your solution and any necessary computations.

1. (15 points) Decide if each of the following statements is True or False. Explain your answer.

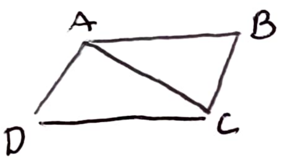
   (a) If two triangles are congruent in Euclidean geometry, then they're congruent in Taxicab geometry.

   False: not all isometries that preserve Euclidean distance also preserve Taxicab distance:



   (b) If $ABCD$ is a parallelogram, then $\Delta ABC \cong \Delta CDA$.

   True: Since opposite sides and angles are congruent in a parallelogram

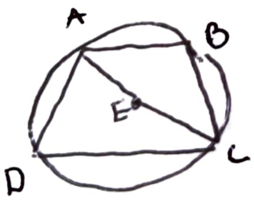


   - $\overline{AB} \cong \overline{CD}$
   - $\overline{BC} \cong \overline{DA}$
   - $\angle ABC \cong \angle CDA$

   So SAS says they're congruent.

   (c) If $ABCD$ is a quadrilateral inscribed in a circle, then $\angle ABC$ and $\angle CDA$ are supplementary.

   True: The Inscribed Angle Theorem says $m\angle AEC$ facing D is 2 times $m\angle ABC$, and $m\angle AEC$ facing B is 2 times $m\angle ADC$.



   Since $m\angle AEC$ (facing D) $+ m\angle AEC$ (facing B) =3

   $m\angle ABC + m\angle ADC = 180°$.

2. (10 points) Define the following terms.

(a) Euclidean metric (give the formula too!)

The distance between 2 points in the Euclidean plane. If $A = (x_A, y_A)$, $B = (x_B, y_B)$

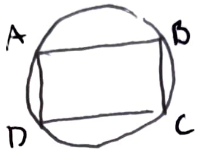
$d_E(A, B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$.

(b) Circle inversion

A transformation of the plane that swaps the inside and outside of the circle.
(If O is its center and P is a point,
P is sent to P' where P' on $\vec{OP}$,
$\vec{OP} \cdot \vec{OP'} = r^2$, r the radius of the circle)

3. (15 points) Prove that if $ABCD$ is a parallelogram inscribed in a circle, $ABCD$ is a rectangle.

Suppose ABCD is a parallelogram inscribed in a circle.



By the Parallelogram Theorem

- $\angle ABC \cong \angle CDA$
- $\angle DAB \cong \angle BCD$

By the Inscribed Angle Theorem, since the central angles corresponding to $\angle ABC$ and $\angle CDA$ add to $360°$, $\angle ABC$ and $\angle CDA$ are supplementary. Similarly, $\angle DAB$ and $\angle BCD$ are supplementary.

So: $m\angle ABC = m\angle CDA$

$m\angle ABC + m\angle CDA = 180°$

$\Rightarrow m\angle ABC = m\angle CDA = 90°$

Same argument shows $m\angle DAB = m\angle BCD = 90°$.

This means ABCD is a rectangle.

4. (10 points) Explain why it isn't possible to construct a square in the Poincare disk.

Imagine ABCD was a square in the Poincaré disk.

Then ΔABC and ΔACD are both right triangles. But in the Poincaré disk

- $m\angle ABC + m\angle BCA + m\angle CAB < 180°$
- $m\angle ACD + m\angle CDA + m\angle DAC < 180°$

The sum

$$m\angle ABC + m\angle BCA + m\angle CAB + m\angle ACD + m\angle CDA + m\angle DAC < 360°$$

This is equal to

$$m\angle ABC + \underbrace{m\angle BCD}_{m\angle BCA + m\angle ACD} + m\angle CDA + \underbrace{m\angle DAB}_{m\angle DAC + m\angle CAB}$$

But then ABCD can't be a square since its angles don't add to 360°.

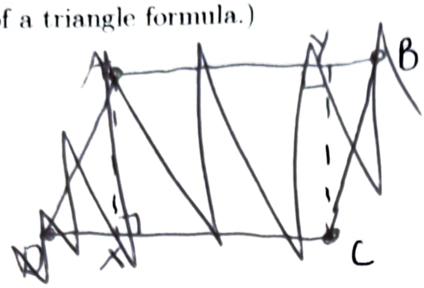
5. (25 points) Fill in the missing steps in the following proof.

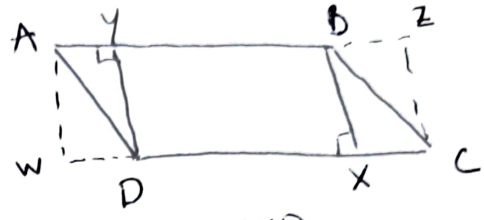
Claim: Suppose ΔABC and ΔDEF are triangles such that $\angle ABC \cong \angle DEF$, and $\angle BCA \cong \angle EFD$, then $\Delta ABC \sim \Delta DEF$.

Proof:

- Apply a dilation to ΔABC with center B and scaling factor $k =$ DE/AB.
- After the dilation, call the triangle $\Delta A'B'C'$ where $B = B'$ and A moved to A', C moved to C'.
- Then $A'B' = DE$ by our choice of k.
- We also know that $\angle A'B'C' \cong \angle DEF$ because dilations preserve angles and ∠A'B' ≅ ∠AB
- Further, $\angle B'C'A' \cong \angle EFD$ because dilations preserve angles and ∠B'C'A' ≅ ∠BCA
- So by the AAS Triangle Congruence Theorem, $\Delta A'B'C' \cong \Delta DEF$.
- Therefore ΔABC ~ ΔDEF.

6. (25 points) Suppose $ABCD$ is a parallelogram. Prove that if X is a point on $\overline{CD}$ such that $\overline{BX}$ is perpendicular to $\overline{CD}$, then the area of $ABCD$ is $BX \cdot CD$. (You can use the axioms for area, including the area of a rectangle formula, but **not** the area of a triangle formula.)





By Area Axiom 3, Area(ABCD) = Area(YBXD) + Area(BXC) + Area(AYD)

• By Area Axiom 4, Area(YBXD) = BX · XD

• By Area Axiom 3, Area(AYDW) = Area(AYD) + Area(AWD)

By Area Axiom 4, Area(AYDW) = AY · YD

Since ∠AYD ≅ ∠DWA, $\overline{AD} \cong \overline{AD}$, ∠YAD ≅ ADW (parallel lines)

by AAS, △AYD ≅ △DWA.

By Area Axiom 2, Area(AYD) = Area(AWD)

So Area(AYD) = $\frac{1}{2}$ Area(AYDW) = $\frac{1}{2}$ AY · YD

• Same argument (sorry all!!!) →

Area(BXC) = $\frac{1}{2}$ Area(BXCZ)

= $\frac{1}{2}$ · BX · CX

• ∠YAD ≅ ∠XCB, $\overline{BX} \cong \overline{YD}$, $\overline{AD} \cong \overline{BC}$ so SAS shows △AYD ≅ △CXB

so CX = AY

• All together,

Area(ABCD) = BX · XD + $\frac{1}{2}$ AY · YD + $\frac{1}{2}$ BX · CX

= BX · XD + $\frac{1}{2}$ AY · BX + $\frac{1}{2}$ BX · CX

= BX · (XD + $\frac{1}{2}$ AY + $\frac{1}{2}$ CX)

= BX · (XD + $\frac{1}{2}$ CX + $\frac{1}{2}$ CX)

= BX · CD.