Math 338 - Homework

Randi Bolt

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Answer the following questions. You are encouraged to work with other students and to seek help from the instructor while working on these problems, but please write up your answers on your own.

1. (Boyce 4.1) Let $\triangle ABC$ be a triangle, and let X be a point on $\overline{AB}$ such that $\overline{CX}$ is perendicular to $\overline{AB}$. Prove that the area of $\triangle ABC$ is $\frac{1}{2}(AB \times CX)$.

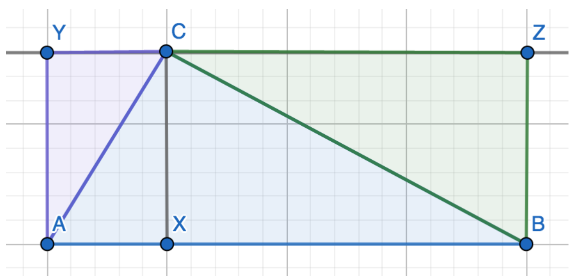

Figure 1: Triangle ABC with an added linesegment YZ, and points X,Y,Z.

Consider a line $\overline{YZ}$ parallel to $\overline{AB}$ such that C is a point on $\overline{YZ}$, and point Y is perpendicular to A and Z is perpendicular to B.

By our axioms of area we know that the area of a rectangle is base $\times$ height. Therefore,

$Area(AYCX) = AX \cdot CX$

$Area(BZCX) = BX \cdot CX$

We can see that $\triangle AXC \cong CYA$ because

- $\angle AXC \cong \angle CYA$ by defining them both as perpendicular, $90°$.
- $\angle YCA \cong \angle XAC$ by alternate interior angle theorem
- $\overline{AC} = \overline{CA}$ by length axioms.

Similarly we can say that $\triangle BXC \cong \triangle CYB$.

Then

$Area(BZCX) = 2Area(\triangle BXC) \Rightarrow Area(\triangle BXC) = \frac{1}{2}(AC \cdot CX)$

and

$Area(AYCX) = 2Area(\triangle AXC) \Rightarrow Area(\triangle BXC) = \frac{1}{2}(BX \cdot CX)$

Therefore,

$$
\begin{aligned}
Area(\triangle ABC) &= Area(\triangle AXC) + Area(\triangle BXC) \\
&= \frac{1}{2}(AX \cdot CX) + \frac{1}{2}(BX \cdot CX) \\
&= \frac{1}{2}CX(AX + BX) \\
&= \frac{1}{2}CX(AB)
\end{aligned} \tag{1}
$$

Therefore the area of $\triangle ABC$ is $\frac{1}{2}(AB \times CX)$. $\square$

2. (Boyce 4.4) Prove that the area of a rhombus is one half the product of the lengths of the diagonals.

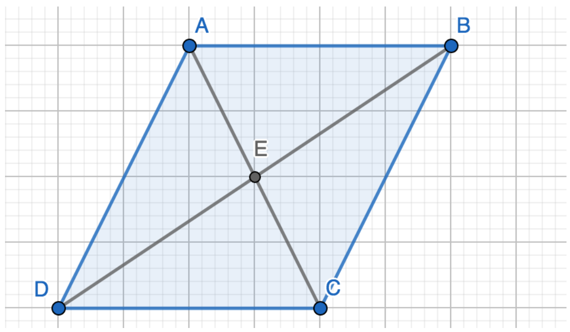

Figure 2: Rhombus ABCD with point E where the diagnols intersect.

Define a rhombus ABCD, and let E be the point that AC and BD intersect.

$$
\begin{aligned}
Area(ABCD) &= Area(\triangle ABC) + Area(\triangle CDA) \\
&= \frac{1}{2}AC(DE) + \frac{1}{2}AC(BE) \qquad \text{, proved in 1} \\
&= \frac{1}{2}AC(BE + DE) \\
&= \frac{1}{2}AC(BD)
\end{aligned} \tag{2}
$$

Therefore the area a rhombus is one half the product of the lengths of the diagonals. $\square$

3. (Boyce 4.7) Suppose $\overline{AB}$ and $\overline{CD}$ are parallel. Prove the area of $\triangle ABC$ is equal to the area of $\triangle ABD$.

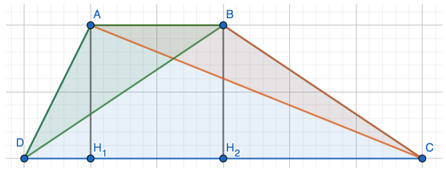

Figure 3: Two parallel points with H_1 and H_2

We proved the area of a triangle is $\frac{1}{2}$ base $\times$ height in problem 1, so define two points to represent the height of both triangles H_1 and H_2 on $\overline{CD}$ that are perpendicular to A and B respectively. See **Figure 3** for reference.

Since $\overline{AB}$ is parallel to $\overline{CD}$, and H_1 and H_2 are perpendicular to A and B respectively then $H_1 = H_2$.

Therefore,

$$\begin{aligned} &Area(\triangle ABC) = Area(\triangle ABD) \\ &\Rightarrow \frac{1}{2}AB \cdot H_1 = \frac{1}{2}AB \cdot H_2 \\ &\Rightarrow \frac{1}{2}AB \cdot H_1 = \frac{1}{2}AB \cdot H_1 \end{aligned} \tag{3}$$

Therefore the area of $\triangle ABC$ is equal to the area of $\triangle ABD$. $\square$

4. (Boyce 4.8) Let $ABCD$ be a paralellogram. Prove that the diagnal of $\overline{BD}$ divides the parallelogram into two triangles of equal area.

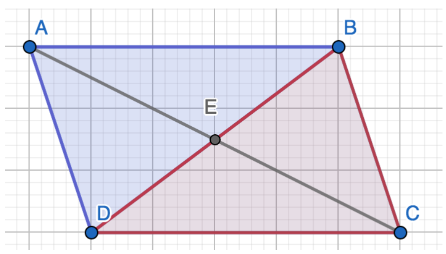

Figure 4: A parallelogram ABCD split into two triangles.

From the first problem we know the area of $\triangle DAB$ and $\triangle BCD$ are:

$Area(\triangle DAB) = \frac{1}{2}DB \cdot AE$

$Area(\triangle BCD) = \frac{1}{2}DB \cdot CE$

Since it is given that ABCD is a parallelogram, we know by parallelogram theorem 4 $\overline{AC}$ and $\overline{DB}$ bisect each other. Meaning that $AE = CE$.

Therefore,

$$\begin{aligned} &Area(\triangle DAB) = Area(\triangle BCD) \\ &\Rightarrow \frac{1}{2}DB \cdot AE = \frac{1}{2}DB \cdot CE \\ &\Rightarrow \frac{1}{2}DB \cdot AE = \frac{1}{2}DB \cdot AE \end{aligned} \tag{4}$$

Therefore the diagnal of $\overline{BD}$ divides the parallelogram into two triangles of equal area. $\square$