HW 5

MTH 344-001 Winter 2022

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1. Define a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f(x, y) = 3x - y$.

(a) Prove that f is a surjective homomorphism.

Let's show f is surjective. Take any $y \in \mathbb{Z}$. Then $(0, -y) \in \mathbb{Z} \times \mathbb{Z}$ and

$$f(0, -y) = 3(0) - (-y) = y.$$

Hence, f is surjective.

Now let's show f is a homomorphism.

$$\begin{aligned} f(3(a_1, a_2) - (b_1, b_2)) &= f(3a_1 - b_1, 3a_2 - b_2) \\ &= 3(3a_1 - b_1) - (3a_2 - b_2) \\ &= 9a_1 - 3b_1 - 3a_2 + b_2 \\ &= 9a_1 - 3a_2 - 3b_1 + b_2 \\ &= 3(3a_1 - a_2) - (3b_1 - b_2) \\ &= 3f(a_1, a_2) - f(b_1, b_2) \end{aligned} \tag{1}$$

Therefore f is a homomorphism.

(b) Find the kernel of f.

Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x, y) = 3x - y$.

$$\begin{aligned} \text{f} &= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | f(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | 3x - y = 0\} \\ &= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | 3x = y\} \\ &= \{...(-2, -6), (-1, 3), (0, 0), (1, 3), (2, 6)...\} \end{aligned} \tag{2}$$

2. Let G be a group. Recall that the *center* of G is the set

$$C = \{C = \{g \in G | xg = gx \forall x \in G\}.$$

We showed in HW 2 that C is a subgroup of G. Prove that C is a normal subgroup of G.

Let $g \in G$ and $h \in C$. Then since C is abelian for all x in G,

$$ghg^{-1} = g^{-1}hg = g^{-1}gh = h \in C.$$

3. Let $G = \{\mathcal{E}, (12), (34), (1324), (1423), (12)(34), (13)(24), (14)(23)\}$. Prove that $H = \{\mathcal{E}, (12)(34)\}$ is a normal subgroup.

Right Cosets

$$\begin{aligned} H\mathcal{E} &= H = \{\mathcal{E}, (12)(34)\} \\ H(12) &= \{\mathcal{E}(12), (12)(34)(12)\} = \{(12), (34)\} \\ H(1324) &= \{\mathcal{E}(1324), (12)(34)(1324)\} = \{(1324), (1423)\} \\ H(13)(24) &= \{\mathcal{E}(13)(24), (12)(34)(13)(24)\} = \{(13)(24), (14)(23)\} \end{aligned} \tag{3}$$

Left Cosets

$$\begin{aligned} \mathcal{E}H &= H = \{\mathcal{E}, (12)(34)\} \\ (12)H &= \{(12)\mathcal{E}, (12)(12)(34)\} = \{(12), (34)\} \\ (1324)H &= \{(1324)\mathcal{E}, (1324)(12)(34)\} = \{(1324), (1423)\} \\ (13)(24)H &= \{(13)(24)\mathcal{E}, (13)(24)(12)(34)\} = \{(13)(24), (14)(23)\} \end{aligned} \tag{4}$$

Since $aH = Ha \forall a \in G$, H is a normal subgroup of G. □

Note: $\frac{\text{8 elements in G}}{\text{2 elements in H}} = 4$ cosets

4. Let $G = \mathbb{Z}_3 \times \mathbb{Z}_4$ and let $H = \langle(1,0)\rangle = \{(0,0),(1,0)\}$.

(a) Explain why H is a normal subgroup of G.

Every subgroup of an abelian group is normal since if G is abelian, then for all $g \in G$ and $h \in H$

$$ghg^{-1} = gg^{-1}h \in H.$$

(b) List the elements of the quotient group G/H.

$$\begin{aligned} H + (0,0) &= \{(0,0),(1,0)\} \\ H + (1,0) &= \{(1,0),(2,0)\} \\ H + (0,1) &= \{(0,1),(1,1)\} \\ H + (1,1) &= \{(1,1),(2,1)\} \end{aligned} \tag{5}$$

(c) The quotient group G/H is a group of size 4. Is it isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$? Justify your answer.

$G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ since neither group is cyclic, and the any element in G/H times itself is the identity.

5. Let G be a group and H be a normal subgroup of G. Suppose that for every $a \in G$ there is a positive integer n such that $a^n \in H$. Prove that every element of the quotient group G/H has finite order.

Let G be a group and H be a normal subgroup of G.

We need to show $\forall a \in G \exists n > 0$ such that $a^n \in H$. Therefore,

$$\begin{aligned} \operatorname{ord}(Ha \in G/H) = n &\Leftrightarrow (Ha)^n = H \\ &\Leftrightarrow Ha^n = H \\ &\Leftrightarrow He = H \\ &\Leftrightarrow H = H \end{aligned} \tag{6}$$

$\Leftrightarrow a^n \in H$ by Theorem 5 chapter 15. Since if $Ha = Hb$, then $ab^{-1} \in H$ where $a = e$ and $b = a^n$

6. Use the Fundamental Homomorphism Theorem to prove that

$$(\mathbb{Z} \times \mathbb{Z})/K \cong \mathbb{Z}$$

where $K = \langle(0,1)\rangle = \{..., (0,-2), (0,-1), (0,0), (0,1), (0,2), (0,3), ...\}$

We need a surjective homomorphism $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ with kernel

$$\begin{aligned} ker(f) &= \langle(0,1)\rangle \\ &= \{..., (0,-2), (0,-1), (0,0), (0,1), (0,2), (0,3), ...\} \\ &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} | x\} \end{aligned} \tag{7}$$

Let $f(x,y) = x$. We've shown the $\ker(f) = \langle(0,1)\rangle$.

Now lets show f is surjective. Take any $n \in \mathbb{Z}$. Then $f(n,0) \in \mathbb{Z} \times \mathbb{Z}$ and

$$f(n,0) = (n,0) = (0,0).$$

So f is surjective. Last, let's check f is a homomorphism:

$$\begin{aligned} f((a_1,a_2) + (b_1,b_2)) &= f(a_1 + b_1, a_2 + b_2) \\ &= a_1 + b_1 \\ &= f(a_1,a_2) + f(b_1,b_2) \end{aligned} \tag{8}$$

Hence by the FHT, $(\mathbb{Z} \times \mathbb{Z})/K \cong \mathbb{Z}$