HW 3

MTH 344-001 Winter 2022

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1. Consider the following permutations in S_8:

$$\alpha = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 3 & 8 & 2 & 6 & 4 & 7 \end{smallmatrix}\right) \qquad \beta = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 2 & 1 & 6 & 5 & 3 & 8 & 4 \end{smallmatrix}\right).$$

(a) Compute $\alpha \circ \beta$ and β^{-1}

$\alpha \circ \beta = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 4 & 2 & 3 & 6 & 8 \end{smallmatrix}\right)$

$\beta^{-1} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 6 & 8 & 5 & 4 & 1 & 7 \end{smallmatrix}\right)$

(b) Write α as a composition of disjoint cycles. Then show that $\alpha^3 = \epsilon$.

Permutation:

$\alpha = (152)(3)(487)(6)$

Show $\alpha^3 = \epsilon$..

$\alpha = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 3 & 8 & 2 & 6 & 4 & 7 \end{smallmatrix}\right)$

$\alpha^2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 3 & 7 & 1 & 6 & 8 & 4 \end{smallmatrix}\right)$

$\alpha^3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{smallmatrix}\right)$

$\Rightarrow \alpha^3 = \epsilon.$

(c) Write β as a composition of transpositions. Is β an even or odd permutation?

$$\begin{aligned} \beta &= (178463)(2)(5) \\ &= (178463) \\ &= (631784) \\ &= (63)(61)(67)(68)(64) \end{aligned} \tag{1}$$

Since the number of compositions of transpositions of β is odd, the permutation β is odd.

2. Let G be a group. Prove that the function: $f : G \to G$ given by $f(x) = x^{-1}$ is a permutation of G.

Define $f : G \to G$ by $f(x) = x^{-1}$.

Since a permutation is a bijective we can show that $f(x)$ has an inverse $f^{-1}(x) = \frac{1}{x}$ by,

$f^{-1}(f(x)) = f^{-1}(x^{-1}) = f^{-1}(\frac{1}{x}) = x$

and

$f(f^{-1}(x)) = f(\frac{1}{x}) = (\frac{1}{x})^{-1} = x$

Therefore, since $f(x)$ is bijective then $f : G \to G$. $\square$

3. Let G be a group. Prove G is abelian if and only if the function $f(x) = x^{-1}$ is an isomorphism from G to G. (Note: By #2 we already know that f is a bijection.)

i) G is abelian $\Rightarrow f(x) = x^{-1}$ is an isomorphism from G to G

Take any $a, b \in G$.

Since G is abelian $f(ab) = f(ba)$.

We showed in 2 that f is bijective, so now we need to show that it is closed under multiplication.

$f(ab) = (ab)^{-1} = a^{-1}b^{-1} = f(a)f(b)$.

Therefore, since G is abelian $\Rightarrow f(x) = x^{-1}$ is an isomorphism from G to G.

ii) $f(x) = x^{-1}$ is an isomorphism from G to G $\Rightarrow$ G is abelian.

Since $G \cong G$, then there is an isomorphism $f : G \to G$. Take any $a, b \in G$.

$$\begin{aligned} f(ab) &= f(a)f(b) \qquad \text{, since f is an isomorphism} \\ &= a^{-1}b^{-1} \\ &= aa^{-1}b^{-1}a^{-1} \\ &= b^{-1}a^{-1} \\ &= f(b)f(a) \\ &= f(ba) \end{aligned} \tag{2}$$

$\Rightarrow f(ab) = f(ba)$, meaning f is commutative.

Since we already prove in 2 that f is bijective, then f is abelian.

Therefore, since $f(x) = x^{-1}$ is an isomorphism from G to G $\Rightarrow$ G is abelian.

4. Let G_1 and G_2 be groups and let $f : G_1 \to G_2$ be an isomorphism. Prove that if H is a subgroup of G_1, then

$$f(H) = \{f(h) | h \in H\}$$

is a subgroup of G_2.

5. Determine whether each of the following groups of size 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. (Recall that $D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ is the group of symmetriese of a square, where r denotes a $90°$ rotation clockwise and s denotes a reflection about a vertical axis.)

(a) $G_1 = \{\epsilon, (12), (34), (12)(34)\} \leq S_4$

G_1 is cyclic so $G_1 \cong \mathbb{Z}_4$.

(b) $G_2 = \{\epsilon, (1234), (13)(24), (1432)\} \leq S_4$

G_2 is cyclic so $G_2 \cong \mathbb{Z}_4$.

(c) $G_3 = \{1, r, r^2, r^3\} \leq D_4$

Any element of G_3 times itself is the identity, so $G_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) $G_4 = \{1, s, r^2, sr^2\} \leq D_4$

Any element of G_4 times itself is the identity, so $G_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.