

HW 2

MTH 344-001 Winter 2022

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1.

- (a) $2, 5 \in H$, but $1 + 5 = 6 \notin H$
- (b) Any positive number say $a > 0 \in H$, but their inverse $-a$ does not exist in H .
- (c) Let $(a_1, a_2), (b_1, b_2) \in H$, then by the definition given $a_2 = \sqrt{a_1}$. So, $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (a_1 + b_1, \sqrt{a_1} + \sqrt{b_1}) \notin H$. Since a_1 and b_1 cannot be factored out of the $\sqrt{}$, it is not closed under coordinate-wise addition and therefore H is not a subgroup of G .

2.

(a)

Proof:

Let $H = \{3^m 4^n \mid m, n \in \mathbb{Z}\}$.

We want to show that $H \leq G = \mathbb{R}^*$.

- i. Since $0 \in \mathbb{Z}$ then for $n, m = 0$, $3^0 4^0 = 1 \cdot 1 = 1$, meaning the identity 1 is in H .
- ii. Let $a, b, c, d \in H$. Then by definition $(a, b) = 3^a 4^b$ and $(c, d) = 3^c 4^d$. So, $(a, b)(c, d) = (3^a 4^b)(3^c 4^d) = 3^a 4^b 3^c 4^d = 3^a 3^c 4^b 4^d = 3^{a+c} 4^{b+d} = 3^m 4^n$. $\Rightarrow H$ is closed under multiplication.
- iii. Let $a_1, a_2 \in H$. Then by definition $(a_1, a_2) = 3^{a_1} 4^{a_2}$. So, $-(a_1, a_2) = (-a_1, -a_2) = 3^{-a_1} 4^{-a_2} = \frac{1}{3^{a_1}} \frac{1}{4^{a_2}} \in H$. $\Rightarrow H$ is closed under inverses.

Completing the proof that $H \leq G$ \square .

(b)

Proof:

Let $H = \{(x, y) \mid y = -3x\}$.

We want to show that $H \leq G = \mathbb{R} \times \mathbb{R}$.

- i. Since $0 = -3 \cdot 0$, the identity $(0, 0)$ is in H .

- ii. Let $a_1, a_2, b_1, b_2 \in H$. Then by definition $a_2 = -3a_1$ and $b_2 = -3b_1$. So, $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (a_1 + b_1, -3a_1 - 3b_1) = (a_1 + b_1, -3(a_1 + b_1)) \in H$. $\Rightarrow H$ is closed under coordinate-wise addition.
- iii. Let $a_1, a_2 \in H$. Then by definition $a_2 = -3a_1$. So, $-(a_1, a_2) = (-a_1, -a_2) = (-a_1, -(-3a_1)) = (-a_1, 3a_1) \in H$. $\Rightarrow H$ is closed under inverses.

Completing the proof that $H \leq G$ \square .

3.

Let G be a group. We say that elements $a, x \in G$ *commute* if $ax = xa$. Prove that if a and x commute, then a^{-1} and x also commute, i.e. $a^{-1}x = xa^{-1}$.

Proof:

$$\begin{aligned}
 ax = xa &\Leftrightarrow axa^{-1} = xaa^{-1} \\
 &\Leftrightarrow axa^{-1} = x \\
 &\Leftrightarrow a^{-1}axa^{-1} = a^{-1}x \\
 &\Leftrightarrow xa^{-1} = a^{-1}x \\
 &\Leftrightarrow a^{-1}x = xa^{-1}
 \end{aligned} \tag{1}$$

Therefore $a, x \in G$ commutes \square .

4.

Let G be a group. The *center* of G is the set C of all elements of G that commute with every other element of G :

$$C = \{g \in G \mid xg = gx \forall x \in G\}.$$

Prove that C is a subgroup of G . (Hint: Use problem 3 when showing C is closed under inverses.)

Proof:

Let $C = \{g \in G \mid xg = gx \forall x \in G\}$.

We want to show that $C \leq G$.

- i. Since $g = g \cdot e = e \cdot g$, the identity $e = 1$ is in C .
- ii. Let $a, b \in C$. Then by definition $ab = ba$, $ax = xa$, and $bx = xb$. So, $abx = xba \in C$. $\Rightarrow C$ is closed under multiplication.
- iii. Let $a_1, a_2 \in C$. Then by definition $a_1a_2 = a_2a_1$. So, $-(a_1a_2) = -(a_2a_1)$. We also showed in 3 that if $ax = xa \Rightarrow a^{-1}x = xa^{-1}$. Therefore C is closed under inverses.

5.

Let H be a subgroup of G and let $a \in G$ be a constant. Show that

$$K = \{aha^{-1} | h \in H\}$$

is also a subgroup of G .

Proof:

- i. Since $H \leq G$, $e \in H$. Then $e = aea^{-1} \in K$.
- ii. Let $x, y \in K$. Then $x = axa^{-1}$ and $y = aya^{-1}$ for some $a \in G$ and $x, y \in H$. So,

$$\begin{aligned}
 (xy)^{-1} &= (x^{-1}y^{-1}) \\
 &= a(x^{-1}y^{-1})a^{-1} \\
 &= ax^{-1}y^{-1}a^{-1} \\
 &= ax^{-1}(1)y^{-1}a^{-1} \\
 &= ax^{-1}(a^{-1}a)y^{-1}a^{-1} \\
 &= ax^{-1}a^{-1}ay^{-1}a^{-1} \\
 &= (ax^{-1}a^{-1})(ay^{-1}a^{-1})
 \end{aligned} \tag{2}$$

$$\Rightarrow (xy)^{-1} \in K \Rightarrow K \leq G \quad \square.$$

6.

(a) Prove that f_a is a bijection.

- i. Suppose $x_1, x_2 \in G$, $f(x_1) = f(x_2)$. Then $f(x_1) = f(x_2) \Rightarrow ax_1 = ax_2 \Rightarrow x_1 = x_2$. So, f is one-to-one.
- ii. Choose any $x \in G$ such that $a^{-1}x \in G$. Then $f(a^{-1}x) = a(a^{-1}x) = (aa^{-1})x = x$. So, f is also onto.

(b) Show that $f_a \circ f_b = f_{ab}$.

Define $f : G \rightarrow G$ where $f_a(x) = ax$ and $f_b(x) = bx$. Then $f_a \circ f_b : (f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(bx) = abx$. Therefore $f_{ab} = abx$.

(c) Find a formula for f_a^{-1} .

$$\begin{aligned}
 y = f(x) &\Rightarrow y = ax \\
 &\Rightarrow a^{-1}y = x \\
 &\Rightarrow x = a^{-1}y \\
 &\Rightarrow f^{-1}(x) = a^{-1}y
 \end{aligned} \tag{3}$$