HW 2

MTH 344-001 Winter 2022

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1/28/2022

1.

(a) $2, 5 \in H$, but $1 + 5 = 6 \notin H$

(b) Any positive number say $a > 0 \in H$, but their inverse $-a$ does not exists in H.

(c) Let $(a_1, a_2), (b_1, b_2) \in H$, then by the definition given $a_2 = \sqrt{a_1}$. So, $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (a_1 + b_1, \sqrt{a_1} + \sqrt{b_1}) \notin H$. Since a_1 and b_1 cannot be factored out of the $\sqrt{}$, it is not closed under coordinate-wise addition and therefore H is not a subgroup of G.

2.

(a)

Proof:

Let $H = \{3^m 4^n | m, n \in \mathbb{Z}\}$.

We want to show that $H \leq G = \mathbb{R}^*$.

i. Since $0 \in \mathbb{Z}$ then for $n, m = 0$, $3^0 4^0 = 1 \cdot 1 = 1$, meaning the identity 1 is in H.

ii. Let $a, b, c, d \in H$. Then by definition $(a, b) = 3^a 4^b$ and $(c, d) = 3^c 4^d$. So, $(a, b)(c, d) = (3^a 4^b)(3^c 4^d) = 3^a 4^b 3^c 4^d = 3^a 3^c 4^b 4^d = 3^{a+c} 4^{b+d} = 3^m 4^n$. $\Rightarrow$ H is closed under multiplication.

iii. Let $a_1, a_2 \in H$. Then by definition $(a_1, a_2) = 3^{a_1} 4^{a_2}$. So, $-(a_1, a_2) = (-a_1, -a_2) = 3^{-a_1} 4^{-a_2} = \frac{1}{3^{a_1}} \frac{1}{4^{a_2}} \in H$. $\Rightarrow$ H is closed under inverses.

Completing the proof that $H \leq G$ $\square$.

(b)

Proof:

Let $H = \{(x, y) | y = -3x\}$.

We want to show that $H \leq G = \mathbb{R} \times \mathbb{R}$.

i. Since $0 = -3 \cdot 0$, the identity $(0, 0)$ is in H.

ii. Let $a_1, a_2, b_1, b_2 \in H$. Then by definition $a_2 = -3a_1$ and $b_2 = -3b_1$. So, $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (a_1 + b_1, -3a_1 - 3b_1) = (a_1 + b_1, -3(a_1 + b_1)) \in H$. $\Rightarrow$ H is closed under coordinate-wise addition.

iii. Let $a_1, a_2 \in H$. Then by definition $a_2 = -3a_1$. So, $-(a_1, a_2) = (-a_1, -a_2) = (-a_1, -(-3a_1)) = (-a_1, -3(-a_1)) \in H$. $\Rightarrow$ H is closed under inverses.

Completing the proof that $H \leq G$ $\square$.

3.

Let G be a group. We say that elements $a, x \in G$ *commute* if $ax = xa$. Prove that if a and x commute, then a^{-1} and x also commute, i.e. $a^{-1}x = xa^{-1}$.

Proof:

$$\begin{aligned} ax = xa &\Leftrightarrow axa^{-1} = xaa^{-1} \\ &\Leftrightarrow axa^{-1} = x \\ &\Leftrightarrow a^{-1}axa^{-1} = a^{-1}x \\ &\Leftrightarrow xa^{-1} = a^{-1}x \\ &\Leftrightarrow a^{-1}x = xa^{-1} \end{aligned} \tag{1}$$

Therefore $a, x \in G$ commutes $\square$.

4.

Let G be a group. The *center* of G is the set C of all elements of G that commute with every other element of G:

$$C = \{g \in G | xg = gx \forall x \in G\}.$$

Prove that C is a subgroup of G. (Hint: Use problem 3 when showing C is closed under inverses.)

Proof:

Let $C = \{g \in G | xg = gx \forall x \in G\}$.

We want to show that $C \leq G$.

i. Since $g = g \cdot e = e \cdot g$, the identity $e = 1$ is in C.

ii. Let $a, b \in C$. Then by definition $ab = ba$, $ax = xa$, and $bx = xb$. So, $abx = xba \in C$. $\Rightarrow$ C is closed under multiplication.

iii. Let $a_1, a_2 \in C$. Then by definition $a_1 a_2 = a_2 a_1$. So, $-(a_1 a_2) = -(a_2 a_1)$. We also showed in 3 that if $ax = xa \Rightarrow a^{-1}x = xa^{-1}$. Therefore C is closed underinverses.

5.

Let H be a subgroup of G and let $a \in G$ be a constant. Show that

$$K = \{aha^{-1} | h \in H\}$$

is also a subgroup of G.

Proof:

i. Since $H \leq G$, $e \in H$. Then $e = aea^{-1} \in K$.

ii. Let $x, y \in K$. Then $x = axa^{-1}$ and $y = aya^{-1}$ for some $a \in G$ and $x, y \in H$. So,

$$\begin{aligned}
(xy)^{-1} &= (x^{-1}y^{-1}) \\
&= a(x^{-1}y^{-1})a^{-1} \\
&= ax^{-1}y^{-1}a^{-1} \\
&= ax^{-1}(1)y^{-1}a^{-1} \\
&= ax^{-1}(a^{-1}a)y^{-1}a^{-1} \\
&= ax^{-1}a^{-1}ay^{-1}a^{-1} \\
&= (ax^{-1}a^{-1})(ay^{-1}a^{-1})
\end{aligned} \tag{2}$$

$\Rightarrow (xy)^{-1} \in K \Rightarrow K \leq G \qquad \square.$

6.

(a) Prove that f_a is a bijection.

i. Suppose $x_1, x_2 \in G$, $f(x_1) = f(x_2)$. Then $f(x_1) = f(x_2) \Rightarrow ax_1 = ax_2 \Rightarrow x_1 = x_2$. So, f is one-to-one.

ii. Choose any $x \in G$ such that $a^{-1}x \in G$. Then $f(a^{-1}x) = a(a^{-1}x) = (aa^{-1})x = x$. So, f is also onto.

(b) Show that $f_a \circ f_b = f_{ab}$.

Define $f : G \to G$ where $f_a(x) = ax$ and $f_b(x) = bx$. Then $f_a \circ f_b$: $(f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(bx) = abx$. Therefore $f_{ab} = abx$.

(c) Find a formula for f_a^{-1}.

$$\begin{aligned}
y = f(x) &\Rightarrow y = ax \\
&\Rightarrow a^{-1}y = x \\
&\Rightarrow x = a^{-1}y \\
&\Rightarrow f^{-1}(x) = a^{-1}y
\end{aligned} \tag{3}$$