MTH 344-001 Winter 2022 HW 1

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1. Do the following define operations on the given set? Why or why not? (Your answers only need to be a sentence or two; if these do define operations, you do not need to check if they're commutative, associatve, ect. in this problem.)

(a) $a * b = 4a - b$ **on** $\mathbb{Z}$

Yes, a*b should work for any two elements in $\mathbb{Z}$, and produce a unique output that exists in $\mathbb{Z}$.

(b) $a * b = 3^b$ **on** $\mathbb{Q}$

No, because an operation needs two elements a and b, and a*b only performs an operation on one element.

(c) $a * b = \frac{a}{ab+2}$ **on** $\mathbb{R}$

No, because it is possible for $ab = -2$ which would make a*b undefined.

(d) $a * b = \frac{1}{ab+2}$ **on** $\mathbb{R}^+$

Yes, a*b should work for any two elements in $\mathbb{R}^+$, and product a unique output that also exists in $\mathbb{R}^+$.

2. Define an operation * on $\mathbb{R}$ by $a * b = 3a - 4b + 2$

(a) Is * commutative?

No, it is given that $a * b = 3a - 4b + 2$, but $b * a = 3b - 4a + 2$, which is clearly not the same.

To show by this by example let $a = 1$, and $b = 2$. We see that $a * b = 3a - 4b + 2 = 3(1) - 4(2) + 2 = -3$ and $b * a = 3b - 4a + 2 = 3(2) - 4(1) + 2 = 4$.

Since $a * b \neq b * a$, then * is NOT commutative. $\square$

(b) Is * associative?

No,

$$
\begin{aligned}
(a * b) * c &= (3a - 4b + 2) * c \\
&= 3(3a - 4b + 2) - 4c + 2 \\
&= 9a - 12b + 6 - 4c + 2 \\
&= 9a - 12b - 4c + 8
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
a * (b * c) &= a * (3b - 4c + 2) \\
&= 3a - 4(3b - 4c + 2) + 2 \\
&= 3a - 12b + 16c - 8 + 2 \\
&= 3a - 12b + 16c - 6
\end{aligned} \tag{2}
$$

Since $(a * b) * c \neq a * (b * c)$, then * isn't associative. □

(c) Is there an identity element $e \in \mathbb{R}$ w.r.t. *?

No, let $e \in \mathbb{R}$ such that $e * a = a \forall a \in \mathbb{R}$.

$$
\begin{aligned}
e * a = a \Rightarrow 3a - 4e + 2 &= a \\
&\Rightarrow 4e = 2a + 2 \\
&\Rightarrow e = \frac{1}{4}(2a + 2) = \frac{1}{2}a + \frac{1}{2}
\end{aligned} \tag{3}
$$

Since the identity element must be constant then there is no identity in $\mathbb{R}$ w.r.t. *. □

(d) Does every element $a \in \mathbb{R}$ have an inverse w.r.t. *?

No, since there is no identity then are no inverse in $\mathbb{R}$ w.r.t. *. □

3. Define an operation * on $\mathbb{R}$ by $a * b = a + b - ab$.

(a) Is * commutative?

Yes,

$$
\begin{aligned}
a * b &= a + b - ab \\
&= b + a - ab \quad \text{since } + \text{ is commutative} \\
&= b + a - ba \quad \text{since } \cdot \text{ is commutative} \\
&= b * a
\end{aligned} \tag{4}
$$

Since $a * b = b * a$ then * is commutative. □

(b) Is * associative?

Yes,

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - ab - ac - bc + abc \end{aligned} \tag{5}$$

$$\begin{aligned} (a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - c(a + b - ab) \\ &= a + b - ab + c - ca + cb - abc \\ &= a + b + c - ab - ac - bc + abc \end{aligned} \tag{6}$$

Since $a * (b * c) = (a * b) * c$ then * is associative. $\square$

(c) Is there an identity element $e \in \mathbb{R}$ w.r.t. *?

Yes, since $0 \in \mathbb{R}$ $a * 0 = 0 * a = a + 0 - a(0) = a$. Therefore there is an identity element $e = 0$ w.r.t. *. $\square$

(d) Does every element $a \in \mathbb{R}$ have an inverse w.r.t. *?

Yes, suppose $b = a^{-1}$, then $a * b = b * a = 0$ (Identity found if part c).

$$\begin{aligned} a * b = 0 &\Rightarrow a + b - ab = 0 \\ &\Rightarrow b(1 - a) = -a \\ &\Rightarrow b = \frac{-a}{1 - a} \end{aligned} \tag{7}$$

Therefore $\forall a \in \mathbb{R}$ except for 1 has an inverse given by: $a^{-1} = \frac{-a}{1-a}$. $\square$

4. Define an operation * on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$ by

$$(a, b) * (c, d) = (ac, ad + bc).$$

Prove that $\langle G, * \rangle$ is an abelian group. (You do not need to check associativity. We will show this in class.)

Let $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$. Define * on G by

$$(a, b) * (c, d) = (ac, ad + bc)$$

For $\langle G, * \rangle$ is an abelian group it must be commutative, associative, have an identity element, and each element must have an inverse.

(i) Commutative

For $(a, b) * (c, d)$ to be commutative then $(a, b) * (c, d) = (d, a) * (b, c) = (c, d) * (a, b) = (b, c) * (d, a)$.

Each can be defined:

$(a, b) * (c, d) = (ac, ad + bc)$

$(d, a) * (b, c) = (db, dc + ab)$

$(c, d) * (a, b) = (ca, cb + da)$

$(b, c) * (d, a) = (bd, ba + cd)$

Since multiplication and addition are communicative on $\mathbb{R}$, we can also show that

(1) $(a, b) * (c, d) = (ac, ad + bc) = (ca, da + cb) = (ca, cb + da) = (c, d) * (a, b)$

and

(2) $(d, a) * (b, c) = (db, dc + ab) = (db, cd + ba) = (db, ba + cd) = (b, c) * (d, a)$

However (1) $\neq$ (2).

To show this by example let $a = 1$, $b = 2$, $c = 3$, and $d = 4$. Then,

(1) $(a, b) * (c, d) = (ac, ad + bc) = (1(3), 1(4) + 2(3)) = (3, 10)$

(2) $(d, a) * (b, c) = (db, dc + ab) = (4(2), 4(3) + 1(2)) = (8, 14)$

Since $(3, 10) \neq (8, 14)$ and $(a, b) * (c, d) = (c, d) * (a, b) \neq (d, a) * (b, c) = (b, c) * (d, a)$ then $\langle G, * \rangle$ is not commutative. $\square$

(ii) Associative

done in class.

(iii) Identity

Let $(a, b) * (e_1, e_2) = (a, b)$ for $e_1, e_2 \in \mathbb{R}$.

$$\begin{aligned} (a, b) * (e_1, e_2) = (a, b) &\Rightarrow (ae_1, ae_2 + be_1) = (a, b) \\ &\Rightarrow (e_1, e_2) = (1, 0) \end{aligned} \tag{8}$$

Checking the other order,

$$\begin{aligned} (e_1, e_2) * (a, b) = (1, 0) * (a, b) &= (1a, 1b + a0) \\ &= (a, b) \end{aligned} \tag{9}$$

Since $(a, b) * (e_1, e_2) = (a, b)$ there is an identity in $\mathbb{R}$ w.r.t. $\langle G, * \rangle$. $\square$

(iv) Inverse

Suppose $(c, d) = (a, b)^{-1}$, then $(a, b) * (c, d) = (1, 0)$ (identity found in part iii).

$$
\begin{aligned}
(a, b) * (c, d) = (1, 0) &\Rightarrow (ac, ad + bc) = (1, 0) \\
&\Rightarrow (c, ad) = (\frac{1}{a}, -bc) \\
&\Rightarrow (c, a^{-1}ad) = (\frac{1}{a}, a^{-1}(-b)c) \\
&\Rightarrow (c, d) = (\frac{1}{a}, -ba^{-1}c) \\
&\Rightarrow (c, d) = (\frac{1}{a}, -ba^{-1}a^{-1}) \\
&\Rightarrow (a, b)^{-1} = (\frac{1}{a}, -ba^{-2})
\end{aligned} \tag{10}
$$

Therefore $\forall (a, b) \in \mathbb{R}$, $\langle G, * \rangle$ has an inverse given by $(\frac{1}{a}, -ba^{-2})$. $\square$

By parts ii - iv, G is associative, has an identity element, and every element has an inverse. Therefore $\langle G, * \rangle$ is an group, but because it isn't commutative, then $\langle G, * \rangle$ is not an abelian group. $\square$

5. Let a,b,c,x be elements of a group G. Solve the following system of equations for x :

(a) $bx^2 = ax^{-1}$ **and** $x^4 = c$

$bx^2 = ax^{-1} \Rightarrow b = ax^{-3} \Rightarrow a^{-1}b = x^{-3}$

$x^4 = c \Rightarrow x = cx^{-3} = ca^{-1}b = a^{-1}bc$

Therefore $x = a^{-1}bc$. $\square$

(b) $x^2c = bxa^{-1}$ **and** $xca = cax$

$$
\begin{aligned}
x^2c &= bxa^{-1} \\
\Rightarrow xxc &= bxa^{-1} \\
\Rightarrow xxca &= bxa^{-1}a \\
\Rightarrow x(xca) &= bx(a^{-1}a) \\
\Rightarrow x(cax) &= bx(e) \\
\Rightarrow xcaxx^{-1} &= bxx^{-1} \\
\Rightarrow xca(xx^{-1}) &= b(xx^{-1}) \\
\Rightarrow xca(e) &= b(e) \\
\Rightarrow xc(aa^{-1}) &= ba^{-1} \\
\Rightarrow x(cc^{-1}) &= ba^{-1}c^{-1} \\
\Rightarrow x &= ba^{-1}c^{-1}
\end{aligned} \tag{11}
$$

Therefore $x = ba^{-1}c^{-1}$. $\square$.

6. This problem asks you to consider the importance of using proper notation for inverses in groups. Your answer only needs to be a sentence or two.

Suppose a,b, and x are elements of a *nonabelian* group G, and that we want to solve thte equation $ax = b$ for x. Why would it be incorrect and unclear to say that the solution is $x = \frac{b}{a}$

It would be incorrect and unclear to say that $x = \frac{b}{a}$ for a *nonabelian* group G, because a *nonabelian* group wouldn't be commutative. This is important because the side an operation happens on matters. Meaning that $(ax = b) \Rightarrow (x = \frac{b}{a}) \neq (\frac{a}{b} = x) \Leftarrow (bx = a)$.