MTH 344-001 Winter 2022 HW 1

Randi Bolt

1/14/2022

1. Do the following define operations on the given set? Why or why not? (Your answers only need to be a sentance or two; if these do define operations, you do not need to check if they're commutative, associatve, ect. in this problem.)

(a) $a * b = 4a - b$ **on** $\mathbb{Z}$

No, because the operation * is neither commutative, or associative.

Commutative : Let $a = 1$, and $b = 2$,

$$a * b = 4a - b = 4(1) - 2 = 4 - 2 = 2$$

$$b * a = 4b - a = 4(2) - a = 8 - 1 = 7$$

$$2 \neq 7$$

□

Associative :

$$(a * b) * c = (4a - b) * c = 4(4a - b) - c = 16a - 4b - c$$

$$a * (b * c) = 4a - (4b - c) = 4a - 4b + c$$

$$16a - 4b - c \neq 4a - 4b + c$$

□

(b) $a * b = 3^b$ **on** $\mathbb{Q}$

No, because * is neither commutative, or associative.

Commutative : Let $a = 1$ and $b = 2$ then,

$$a * b = 3^2 = 9 \neq 3 = 3^1 = b * a$$

□

Associative :

$$a * (b * c) = (3^c)^{3^c} \neq 3^c = (a * b) * c$$

□

(c) $a * b = \frac{a}{ab+2}$ **on** $\mathbb{R}$

No, because the operation * is neither commutative, or associative.

Commutative : Let $a = 1$, and $b = 2$,

$$a * b = \frac{a}{ab+2} = \frac{1}{4} \neq \frac{1}{2} = \frac{b}{ba+2} = b * a$$

□

Associative :

$$a * (b * c) = a * \frac{b}{bc+2} = \frac{a}{a\frac{b}{bc+2} + 2} = \frac{\frac{a}{1}}{\frac{ab+2(bc+2)}{bc+2}} = \frac{a(bc+2)}{ab+2(bc+2)} = \frac{abc+2}{ab+2bc+4}$$

$$(a * b) * c = \frac{a}{ab+2} * c = \frac{\frac{a}{ab+2}}{\frac{a}{ab+2}c + 2} = \frac{\frac{a}{ab+2}}{\frac{ac+2(ab+2)}{ab+2}} = \frac{a(ab+2)}{(ac+2(ab+2))(ab+2)} = \frac{a^2b+2a}{(ac+2ab+4)(ab+2)}$$

$$a * (b * c) \neq (a * b) * c$$

□

(d) $a * b = \frac{1}{ab+2}$ **on** $\mathbb{R}^+$

No, because the operation * is not associative.

$$a * (b * c) = a * (\frac{1}{bc+2}) = \frac{1}{a(\frac{1}{bc+2}) + 2} = \frac{\frac{1}{1}}{\frac{a+2(bc+2)}{bc+2}} = \frac{bc+2}{a+2bc+4}$$

Let $a = 1$, $b = 2$, $c = 3$,

$$\frac{bc+2}{a+2bc+4} = \frac{2(3)+2}{1+2(2)(3)+4} = \frac{8}{17} \approx 0.470588$$

$$(a * b) * c = (\frac{1}{ab+2}) * c = \frac{1}{(\frac{1}{ab+2})c + 2} = \frac{\frac{1}{1}}{\frac{c+2(ab+2)}{ab+2}} = \frac{ab+2}{c+2ab+4}$$

Let $a = 1$, $b = 2$, $c = 3$,

$$\frac{ab+2}{c+2ab+4} = \frac{4}{3+4+4} = \frac{4}{11} \approx 0.3636$$

$$\frac{8}{17} \neq \frac{4}{11}$$

□

2. Define an operation * on $\mathbb{R}$ by $a * b = 3a - 4b + 2$

(a) Is * commutative?

No, let $a = 1$, and $b = 2$.

$$\begin{aligned} 3a - 4b + 2 &= 3b - 4a + 2 \\ 3(1) - 4(2) + 2 &= 3(2) - 4(1) + 2 \\ 3 - 8 + 2 &= 6 - 4 + 2 \\ -3 &\neq 4 \end{aligned} \tag{1}$$

Since $a * b \neq b * a$, then * is NOT commutative.

$\square$

(b) Is * associative?

No,

$$\begin{aligned} (a * b) * c &= (3a - 4b + 2) * c \\ &= 3(3a - 4b + 2) - 4c + 2 \\ &= 9a - 12b + 6 - 4c + 2 \\ &= 9a - 12b - 4c + 8 \end{aligned} \tag{2}$$

$$\begin{aligned} a * (b * c) &= a * (3b - 4c + 2) \\ &= 3a - 4(3b - 4c + 2) + 2 \\ &= 3a - 12b + 16c - 8 + 2 \\ &= 3a - 12b + 16c - 6 \end{aligned} \tag{3}$$

Since $(a * b) * c \neq a * (b * c)$, then * isn't associative.

$\square$

(c) Is there an identity element $e \in \mathbb{R}$ w.r.t. *?

No, let $e \in \mathbb{R}$ such that $e * a = a \forall a \in \mathbb{R}$.

$$\begin{aligned} e * a = a \Rightarrow 3a - 4e + 2 &= a \\ &\Rightarrow 4e = 2a + 2 \\ &\Rightarrow e = \frac{1}{4}(2a + 2) = \frac{1}{2}a + \frac{1}{2} \end{aligned} \tag{4}$$

Checking the other order,

$$
\begin{aligned}
e * a &= 3(\frac{1}{2}a + \frac{1}{2}) - 4a + 2 \\
&= \frac{3a}{2} + \frac{3}{2} - \frac{8a}{2} + \frac{4}{2} \\
&= \frac{5a}{2} + \frac{7}{2} \\
&\neq a
\end{aligned} \tag{5}
$$

Since $e * a \neq a$ there is no identity in $\mathbb{R}$ w.r.t. *.

$\square$

(d) Does every element $a \in \mathbb{R}$ have an inverse w.r.t. *?

No, suppose $b = a^{-1}$, then $a * b = 0$.

$$
\begin{aligned}
a * b = 0 \Rightarrow 3a - 4b + 2 = 0 \\
&\Rightarrow 3a + 2 = 4b \\
&\Rightarrow b = \frac{3a + 2}{4} \\
&\Rightarrow a^{-1} = \frac{3a + 2}{4}
\end{aligned} \tag{6}
$$

Checking the other order,

$$
\begin{aligned}
b * a = 0 &\Rightarrow 3b - 4a + 2 = 0 \\
&\Rightarrow 3(\frac{3a + 2}{4}) = 4a - 2 \\
&\Rightarrow 9a + 6 = a - \frac{1}{2} \\
&\Rightarrow 8a = -5.5 \\
&\rightarrow a = \frac{-5.5}{8} = 0.6875 \neq 0
\end{aligned} \tag{7}
$$

Since $b * a \neq 0$ there is no inverse in $\mathbb{R}$ w.r.t. *.

$\square$

3. Define an operation * on $\mathbb{R}$ by $a * b = a + b - ab$.

(a) Is * commutative?

Yes,

$$
\begin{aligned}
a * b &= a + b - ab \\
&= b + a - ab \quad \text{since } + \text{ is commutative} \\
&= b + a - ba \quad \text{since } \cdot \text{ is commutative} \\
&= b * a
\end{aligned} \tag{8}
$$

Since $a * b = b * a$ then * is commutative.

$\square$

(b) Is * associative?

Yes,

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - ab - ac - bc + abc \end{aligned} \tag{9}$$

$$\begin{aligned} (a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - c(a + b - ab) \\ &= a + b - ab + c - ca + cb - abc \\ &= a + b + c - ab - ac - bc + abc \end{aligned} \tag{10}$$

Since $a * (b * c) = (a * b) * c$ then * is associative.

$\square$

(c) Is there an identity element $e \in \mathbb{R}$ w.r.t. *?

Yes, since $0 \in \mathbb{R}$ $a * 0 = 0 * a = a + 0 - a(0) = a$. Therefore there is an identity element $e = 0$ w.r.t. *.

$\square$

(d) Does every element $a \in \mathbb{R}$ have an inverse w.r.t. *?

Yes, suppose $b = a^{-1}$, then $a * b = b * a = 0$ (Identity found if part c).

$$\begin{aligned} a * b = 0 &\Rightarrow a + b - ab = 0 \\ &\Rightarrow b(1 - a) = -a \\ &\Rightarrow b = \frac{-a}{1 - a} \end{aligned} \tag{11}$$

Therefore $\forall a \in \mathbb{R}$ except for 1 has an inverse given by: $a^{-1} = \frac{-a}{1-a}$.

$\square$

4. Define an operation * on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$ by

$$(a, b) * (c, d) = (ac, ad + bc).$$

Prove that $\langle G, * \rangle$ is an abelian group. (You do not need to check associativity. We will show this in class.)

Let $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$. Define * on G by

$$(a, b) * (c, d) = (ac, ad + bc)$$

For $\langle G, * \rangle$ is an abelian group it must be commutative, associative, have an identity element, and each element has an inverse.

(i) Commutative

Let $a = 1$, $b = 2$, $c = 3$, $d = 4$,

$$\begin{aligned}(a, b) * (c, d) &= (ac, ad + bc) \\ &= (1(3), 1(4) + 2(3)) \\ &= (3, 10)\end{aligned} \tag{12}$$

$$\begin{aligned}(c, d) * (a, b) &= (ca, cb + da) \\ &= (3(1), 3(2) + 4(1) \\ &= (3, 10)\end{aligned} \tag{13}$$

Since $(a, b) * (c, d) = (c, d) * (a, b)$ then $\langle G, * \rangle$ is commutative.

(ii) Associative

done in class.

(iii) Identity

Let $(a, b) * (e_1, e_2) = (a, b)$ for $e_1, e_2 \in \mathbb{R}$.

$$\begin{aligned}(a, b) * (e_1, e_2) = (a, b) &\Rightarrow (ae_1, ae_2 + be_1) = (a, b) \\ &\Rightarrow (e_1, e_2) = (1, 0)\end{aligned} \tag{14}$$

Checking the other order,

$$\begin{aligned}(e_1, e_2) * (a, b) = (1, 0) * (a, b) &= (1a, 1b + a0) \\ &= (a, b)\end{aligned} \tag{15}$$

Since $(a, b) * (e_1, e_2) = (a, b)$ there is an identity in $\mathbb{R}$ w.r.t. $\langle G, * \rangle$.

□

(iv) Inverse

Suppose $(c, d) = (a, b)^{-1}$, then $(a, b) * (c, d) = (1, 0)$ (identity found in part iii).

$$\begin{aligned}(a, b) * (c, d) = (1, 0) &\Rightarrow (ac, ad + bc) = (1, 0) \\ &\Rightarrow (c, d) = (\frac{1}{a}, -b) \\ &\Rightarrow (a, b)^{-1} = (\frac{1}{a}, -b)\end{aligned} \tag{16}$$

Therefore $\forall(a,b) \in \mathbb{R}$, $\langle G, * \rangle$ has an inverse given by $(\frac{1}{a}, -b)$.

$\square$

By parts i - iv, G is commutative, associative, has an identity element, and every element has an inverse. Therefore $\langle G, * \rangle$ is an abelian group.

$\square$

5. Let a,b,c,x be elements of a group G. Solve the following system of equations for x :

(a) $bx^2 = ax^{-1}$ **and** $x^4 = c$

$x^4 = c \Rightarrow x = c^{1/4}$

$bx^2 = ax^{-1} \Rightarrow b(c^{1/4})^2 = a(c^{1/4})^{-1} \Rightarrow bc^{2/4} = ac^{-1/4}$

$\Rightarrow b = a\frac{c^{-1/4}}{c^{2/4}} = ac^{-3/4}$

$\Rightarrow a = b\frac{c^{2/4}}{c^{-1/4}} = ac^{3/4}$

(b) $x^2c = bxa^{-1}$ **and** $xca = cax$

$$\begin{aligned}
x^2b &= bxa^{-1} \\
\Rightarrow x^{-1}x^2c &= x^{-1}bxa \\
\Rightarrow xc &= x^{-1}bxa^{-1} \\
\Rightarrow xca &= x^{-1}bx \\
\Rightarrow cax &= x^{-1}bx \\
\Rightarrow caxx^{-1} &= x^{-1}bxx^{-1} \\
\Rightarrow ca &= x^{-1}b \\
\Rightarrow xca &= xx^{-1}b \\
\Rightarrow xca &= b
\end{aligned} \tag{17}$$

$\Rightarrow b = xca$

$\Rightarrow a = cx^{-1}b$

$\Rightarrow c = x^{-1}ba^{-1}$

6. This problem asks you to consider the importance of using proper notation for inverses in groups. Your answer only needs to be a sentence or two.

Suppose a,b, and x are elements of a *nonabelian* group G, and that we want to solve thte equation $ax = b$ for x. Why would it be incorrect and unclear to say that the solution is $x = \frac{b}{a}$

It would be incorrect and unclear to say that $x = \frac{b}{a}$ for a *nonabelian* group G, because a *nonabelian* group wouldn't be commutative. Meaning that $(ax = b) \Rightarrow (x = \frac{b}{a}) \neq (\frac{a}{b} = x) \Leftarrow (bx = a)$.