

# MTH 344-001 Winter 2022 HW 1

Randi Bolt

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**1. Do the following define operations on the given set? Why or why not? (Your answers only need to be a sentence or two; if these do define operations, you do not need to check if they're commutative, associative, ect. in this problem.)**

**(a)  $a * b = 4a - b$  on  $\mathbb{Z}$**

No, because the operation  $*$  is neither commutative, or associative.

Commutative : Let  $a = 1$ , and  $b = 2$ ,

$$a * b = 4a - b = 4(1) - 2 = 4 - 2 = 2$$

$$b * a = 4b - a = 4(2) - a = 8 - 1 = 7$$

$$2 \neq 7$$

□

Associative :

$$(a * b) * c = (4a - b) * c = 4(4a - b) - c = 16a - 4b - c$$

$$a * (b * c) = 4a - (4b - c) = 4a - 4b + c$$

$$16a - 4b - c \neq 4a - 4b + c$$

□

**(b)  $a * b = 3^b$  on  $\mathbb{Q}$**

No, because  $*$  is neither commutative, or associative.

Commutative : Let  $a = 1$  and  $b = 2$  then,

$$a * b = 3^2 = 9 \neq 3 = 3^1 = b * a$$

□

Associative :

$$a * (b * c) = (3^c)^{3^c} \neq 3^c = (a * b) * c$$

□

(c)  $a * b = \frac{a}{ab+2}$  on  $\mathbb{R}$

No, because the operation  $*$  is neither commutative, or associative.

Commutative : Let  $a = 1$ , and  $b = 2$  ,

$$a * b = \frac{a}{ab+2} = \frac{1}{4} \neq \frac{1}{2} = \frac{b}{ba+2} = b * a$$

□

Associative :

$$\begin{aligned} a * (b * c) &= a * \frac{b}{bc+2} = \frac{a}{a \frac{b}{bc+2} + 2} = \frac{\frac{a}{1}}{\frac{ab+2(bc+2)}{bc+2}} = \frac{a(bc+2)}{ab+2(bc+2)} = \frac{abc+2}{ab+2bc+4} \\ (a * b) * c &= \frac{a}{ab+2} * c = \frac{\frac{a}{ab+2}}{\frac{a}{ab+2}c + 2} = \frac{\frac{a}{ab+2}}{\frac{ac+2(ab+2)}{ab+2}} = \frac{a(ab+2)}{(ac+2(ab+2))(ab+2)} = \frac{a^2b+2a}{(ac+2ab+4)(ab+2)} \end{aligned}$$

$$a * (b * c) \neq (a * b) * c$$

□

(d)  $a * b = \frac{1}{ab+2}$  on  $\mathbb{R}^+$

No, because the operation  $*$  is not associative.

$$a * (b * c) = a * \left( \frac{1}{bc+2} \right) = \frac{1}{a \left( \frac{1}{bc+2} \right) + 2} = \frac{\frac{1}{1}}{\frac{a+2(bc+2)}{bc+2}} = \frac{bc+2}{a+2bc+4}$$

Let  $a = 1$ ,  $b = 2$ ,  $c = 3$ ,

$$\frac{bc+2}{a+2bc+4} = \frac{2(3)+2}{1+2(2)(3)+4} = \frac{8}{17} \approx 0.470588$$

$$(a * b) * c = \left( \frac{1}{ab+2} \right) * c = \frac{1}{\left( \frac{1}{ab+2} \right) c + 2} = \frac{\frac{1}{1}}{\frac{c+2(ab+2)}{ab+2}} = \frac{ab+2}{c+2ab+4}$$

Let  $a = 1$ ,  $b = 2$ ,  $c = 3$ ,

$$\frac{ab+2}{c+2ab+4} = \frac{4}{3+4+4} = \frac{4}{11} \approx 0.3636$$

$$\frac{8}{17} \neq \frac{4}{11}$$

□

**2. Define an operation  $*$  on  $\mathbb{R}$  by  $a * b = 3a - 4b + 2$**

**(a) Is  $*$  commutative?**

No, let  $a = 1$ , and  $b = 2$ .

$$\begin{aligned} 3a - 4b + 2 &= 3b - 4a + 2 \\ 3(1) - 4(2) + 2 &= 3(2) - 4(1) + 2 \\ 3 - 8 + 2 &= 6 - 4 + 2 \\ -3 &\neq 4 \end{aligned} \tag{1}$$

Since  $a * b \neq b * a$ , then  $*$  is NOT commutative.

□

**(b) Is  $*$  associative?**

No,

$$\begin{aligned} (a * b) * c &= (3a - 4b + 2) * c \\ &= 3(3a - 4b + 2) - 4c + 2 \\ &= 9a - 12b + 6 - 4c + 2 \\ &= 9a - 12b - 4c + 8 \end{aligned} \tag{2}$$

$$\begin{aligned} a * (b * c) &= a * (3b - 4c + 2) \\ &= 3a - 4(3b - 4c + 2) + 2 \\ &= 3a - 12b + 16c - 8 + 2 \\ &= 3a - 12b + 16c - 6 \end{aligned} \tag{3}$$

Since  $(a * b) * c \neq a * (b * c)$ , then  $*$  isn't associative.

□

**(c) Is there an identity element  $e \in \mathbb{R}$  w.r.t.  $*$ ?**

No, let  $e \in \mathbb{R}$  such that  $e * a = a \forall a \in \mathbb{R}$ .

$$\begin{aligned} e * a = a &\Rightarrow 3a - 4e + 2 = a \\ &\Rightarrow 4e = 2a + 2 \\ &\Rightarrow e = \frac{1}{4}(2a + 2) = \frac{1}{2}a + \frac{1}{2} \end{aligned} \tag{4}$$

Checking the other order,

$$\begin{aligned}
e * a &= 3\left(\frac{1}{2}a + \frac{1}{2}\right) - 4a + 2 \\
&= \frac{3a}{2} + \frac{3}{2} - \frac{8a}{2} + \frac{4}{2} \\
&= \frac{5a}{2} + \frac{7}{2} \\
&\neq a
\end{aligned} \tag{5}$$

Since  $e * a \neq a$  there is no identity in  $\mathbb{R}$  w.r.t.  $*$ .

□

**(d) Does every element  $a \in \mathbb{R}$  have an inverse w.r.t.  $*$ ?**

No, suppose  $b = a^{-1}$ , then  $a * b = 0$ .

$$\begin{aligned}
a * b = 0 &\Rightarrow 3a - 4b + 2 = 0 \\
&\Rightarrow 3a + 2 = 4b \\
&\Rightarrow b = \frac{3a + 2}{4} \\
&\Rightarrow a^{-1} = \frac{3a + 2}{4}
\end{aligned} \tag{6}$$

Checking the other order,

$$\begin{aligned}
b * a = 0 &\Rightarrow 3b - 4a + 2 = 0 \\
&\Rightarrow 3\left(\frac{3a + 2}{4}\right) = 4a - 2 \\
&\Rightarrow 9a + 6 = 4a - 2 \\
&\Rightarrow 5a = -8 \\
&\Rightarrow a = -\frac{8}{5} = -1.6
\end{aligned} \tag{7}$$

Since  $b * a \neq 0$  there is no inverse in  $\mathbb{R}$  w.r.t.  $*$ .

□

**3. Define an operation  $*$  on  $\mathbb{R}$  by  $a * b = a + b - ab$ .**

**(a) Is  $*$  commutative?**

Yes,

$$\begin{aligned}
a * b &= a + b - ab \\
&= b + a - ab \quad \text{since } + \text{ is commutative} \\
&= b + a - ba \quad \text{since } \cdot \text{ is commutative} \\
&= b * a
\end{aligned} \tag{8}$$

Since  $a * b = b * a$  then  $*$  is commutative.

□

**(b) Is  $*$  associative?**

Yes,

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b + c - ab - ac - bc + abc \end{aligned} \tag{9}$$

$$\begin{aligned} (a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - c(a + b - ab) \\ &= a + b - ab + c - ca + cb - abc \\ &= a + b + c - ab - ac - bc + abc \end{aligned} \tag{10}$$

Since  $a * (b * c) = (a * b) * c$  then  $*$  is associative.

□

**(c) Is there an identity element  $e \in \mathbb{R}$  w.r.t.  $*$ ?**

Yes, since  $0 \in \mathbb{R}$   $a * 0 = 0 * a = a + 0 - a(0) = a$ . Therefore there is an identity element  $e = 0$  w.r.t.  $*$ .

□

**(d) Does every element  $a \in \mathbb{R}$  have an inverse w.r.t.  $*$ ?**

Yes, suppose  $b = a^{-1}$ , then  $a * b = b * a = 0$  (Identity found if part c).

$$\begin{aligned} a * b = 0 &\Rightarrow a + b - ab = 0 \\ &\Rightarrow b(1 - a) = -a \\ &\Rightarrow b = \frac{-a}{1 - a} \end{aligned} \tag{11}$$

Therefore  $\forall a \in \mathbb{R}$  except for 1 has an inverse given by:  $a^{-1} = \frac{-a}{1-a}$ .

□

**4. Define an operation  $*$  on the set  $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$  by**

$$(a, b) * (c, d) = (ac, ad + bc).$$

**Prove that  $\langle G, * \rangle$  is an abelian group. (You do not need to check associativity. We will show this in class.)**

Let  $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$ . Define  $*$  on  $G$  by

$$(a, b) * (c, d) = (ac, ad + bc)$$

For  $\langle G, * \rangle$  is an abelian group it must be commutative, associative, have an identity element, and each element has an inverse.

(i) Commutative

(ii) Associative

done in class.

(iii) Identity

Solve  $(a, b) * (e_1, e_2) = (a, b)$  for  $e_1, e_2 \in \mathbb{R}$ .

- $e_1$  and  $e_2$  are the identities

(iv) Inverse

By parts i - iv,  $G$  is commutative, associative, has an identity element, and every element has an inverse.

**5. Let  $a, b, c, x$  be elements of a group  $G$ . Solve the following system of equations for  $x$  :**

(a)  $bx^2 = ax^{-1}$     and  $x^4 = c$

(b)  $x^2c = bxa^{-1}$     and  $xca = cax$

**6. This problem asks you to consider the importance of using proper notation for inverses in groups. Your answer only needs to be a sentence or two.**

Suppose  $a, b$ , and  $x$  are elements of a *nonabelian* group  $G$ , and that we want to solve the equation  $ax = b$  for  $x$ . Why would it be incorrect and unclear to say that the solution is  $x = \frac{b}{a}$