

MTH 344-001 Winter 2022 HW 1

Randi Bolt

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1. Do the following define operations on the given set? Why or why not? (Your answers only need to be a sentence or two; if these do define operations, you do not need to check if they're commutative, associative, ect. in this problem.)

(a) $a * b = 4a - b$ on \mathbb{Z}

No, because the operation $*$ is neither commutative, or associative.

Commutative : Let $a = 1$, and $b = 2$,

$$a * b = 4a - b = 4(1) - 2 = 4 - 2 = 2$$

$$b * a = 4b - a = 4(2) - a = 8 - 1 = 7$$

$$2 \neq 7$$

□

Associative :

$$(a * b) * c = (4a - b) * c = 4(4a - b) - c = 16a - 4b - c$$

$$a * (b * c) = 4a - (4b - c) = 4a - 4b + c$$

$$16a - 4b - c \neq 4a - 4b + c$$

□

(b) $a * b = 3^b$ on \mathbb{Q}

No, because $*$ is neither commutative, or associative.

Commutative : Let $a = 1$ and $b = 2$ then,

$$a * b = 3^2 = 9 \neq 3 = 3^1 = b * a$$

□

Associative :

$$a * (b * c) = (3^c)^{3^c} \neq 3^c = (a * b) * c$$

□

(c) $a * b = \frac{a}{ab+2}$ on \mathbb{R}

No, because the operation $*$ is neither commutative, or associative.

Commutative : Let $a = 1$, and $b = 2$,

$$a * b = \frac{a}{ab+2} = \frac{1}{4} \neq \frac{1}{2} = \frac{b}{ba+2} = b * a$$

□

Associative :

$$a * (b * c) = a * \frac{b}{bc+2} = \frac{a}{a\frac{b}{bc+2}+2} = \frac{\frac{a}{1}}{\frac{ab+2(bc+2)}{bc+2}} = \frac{a(bc+2)}{ab+2(bc+2)} = \frac{abc+2}{ab+2bc+4}$$

$$(a * b) * c = \frac{a}{ab+2} * c = \frac{\frac{a}{ab+2}}{\frac{a}{ab+2}c+2} = \frac{\frac{a}{ab+2}}{\frac{ac+2(ab+2)}{ab+2}} = \frac{a(ab+2)}{(ac+2(ab+2))(ab+2)} = \frac{a^2b+2a}{(ac+2ab+4)(ab+2)}$$

$$a * (b * c) \neq (a * b) * c$$

□

(d) $a * b = \frac{1}{ab+2}$ on \mathbb{R}^+

No, because the operation $*$ is not associative.

$$a * (b * c) = a * \left(\frac{1}{bc+2}\right) = \frac{1}{a\left(\frac{1}{bc+2}\right)+2} = \frac{\frac{1}{1}}{\frac{a+2(bc+2)}{bc+2}} = \frac{bc+2}{a+2bc+4}$$

Let $a = 1$, $b = 2$, $c = 3$,

$$\frac{bc+2}{a+2bc+4} = \frac{2(3)+2}{1+2(2)(3)+4} = \frac{8}{17} \approx 0.470588$$

$$(a * b) * c = \left(\frac{1}{ab+2}\right) * c = \frac{1}{\left(\frac{1}{ab+2}\right)c+2} = \frac{\frac{1}{1}}{\frac{c+2(ab+2)}{ab+2}} = \frac{ab+2}{c+2ab+4}$$

Let $a = 1$, $b = 2$, $c = 3$,

$$\frac{ab+2}{c+2ab+4} = \frac{4}{3+4+4} = \frac{4}{11} \approx 0.3636$$

$$\frac{8}{17} \neq \frac{4}{11}$$

□

2. Define an operation $*$ on \mathbb{R} by $a * b = 3a - 4b + 2$

(a) Is $*$ commutative?

No, let $a = 1$, and $b = 2$.

$$\begin{aligned} 3a - 4b + 2 &= 3b - 4a + 2 \\ 3(1) - 4(2) + 2 &= 3(2) - 4(1) + 2 \\ 3 - 8 + 2 &= 6 - 4 + 2 \\ -3 &\neq 4 \end{aligned} \tag{1}$$

Since $a * b \neq b * a$, then $*$ is NOT commutative.

□

(b) Is $*$ associative?

No,

$$\begin{aligned} (a * b) * c &= (3a - 4b + 2) * c \\ &= 3(3a - 4b + 2) - 4c + 2 \\ &= 9a - 12b + 6 - 4c + 2 \\ &= 9a - 12b - 4c + 8 \end{aligned} \tag{2}$$

$$\begin{aligned} a * (b * c) &= a * (3b - 4c + 2) \\ &= 3a - 4(3b - 4c + 2) + 2 \\ &= 3a - 12b + 16c - 8 + 2 \\ &= 3a - 12b + 16c - 6 \end{aligned} \tag{3}$$

Since $(a * b) * c \neq a * (b * c)$, then $*$ isn't associative.

□

(c) Is there an identity element $e \in \mathbb{R}$ w.r.t. $*$?

No, let $e \in \mathbb{R}$ such that $e * a = a \forall a \in \mathbb{R}$.

$$\begin{aligned} e * a = a &\Rightarrow 3a - 4e + 2 = a \\ &\Rightarrow 4e = 2a + 2 \\ &\Rightarrow e = \frac{1}{4}(2a + 2) = \frac{1}{2}a + \frac{1}{2} \end{aligned} \tag{4}$$

(d) Does every element $a \in \mathbb{R}$ have an inverse w.r.t. $*$?

3. Define an operation $*$ on \mathbb{R} by $a * b = a + b - ab$.

(a) Is $*$ commutative?

$$\begin{aligned} a * b &= a + b - ab \\ &= b + a - ab \quad \text{since } + \text{ is commutative} \\ &= b + a - ba \quad \text{since } \cdot \text{ is commutative} \\ &= b * a \end{aligned} \tag{5}$$

(b) Is $*$ associative?

(c) Is there an identity element $e \in \mathbb{R}$ w.r.t. $*$?

(d) Does every element $a \in \mathbb{R}$ have an inverse w.r.t. $*$?

4. Define an operation $*$ on the set $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \neq 0\}$ by

$$(a, b) * (c, d) = (ac, ad + bc).$$

Prove that $\langle G, * \rangle$ is an abelian group. (You do not need to check associativity. We will show this in class.)

5. Let a, b, c, x be elements of a group G . Solve the following system of equations for x :

(a) $bx^2 = ax^{-1}$ and $x^4 = c$

(b) $x^2c = bxa^{-1}$ and $xca = cax$

6. This problem asks you to consider the importance of using proper notation for inverses in groups. Your answer only needs to be a sentence or two.

Suppose a, b , and x are elements of a *nonabelian* group G , and that we want to solve the equation $ax = b$ for x . Why would it be incorrect and unclear to say that the solution is $x = \frac{b}{a}$