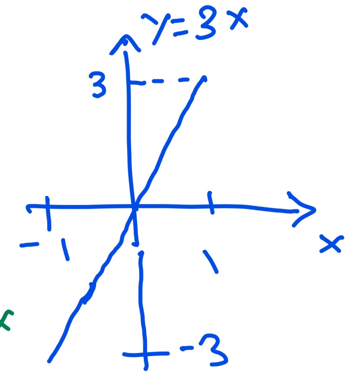
The complete pdf of z is

$$f_Z(z) = \begin{cases} \frac{4}{z^5} ; & 1 < z < \infty \\ 0 & \text{, otherwise} \end{cases}$$



Example 3: Let X be a random variable with pdf

X is continuous.

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 ; & -1 < x < 1 \\ 0 ; & elsewhere \end{cases}$$

(a) Find the pdf of $Y = 3X$. This is a linear function of x

When $x=-1$, $y=-3$ and when $x=1$, $y=3$.

$Y=3x$ is $g(x)$ where g is monotone and increasing for $-1<x<1$.

Further, $g^{-1}(y) = \frac{y}{3}$ This is found solving $Y=3X$ for X.

$$\Rightarrow \frac{d}{dy}\left[g^{-1}(y)\right] = \frac{d}{dy}\left(\frac{y}{3}\right)$$ This is the Jacobian.

$$= \frac{1}{3}$$

Then the pdf of y is

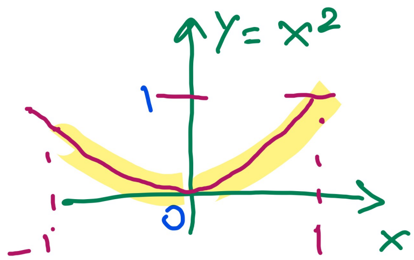
$$f_Y(y) = f_X\left(g^{-1}(y)\right) * \left|\frac{d}{dy}\left(g^{-1}(y)\right)\right|$$

$$= \frac{3}{2}\left(\frac{y}{3}\right)^2 * \left|\frac{1}{3}\right|$$

$$= \frac{y^2}{18}$$

The complete pdf of Y is

$$f_Y(y) = \begin{cases} \frac{y^2}{18} & ; -3 < y < 3 \\ 0 & ; \text{else} \end{cases}$$



This is NOT linear function of X

(b) Find the pdf of $Y = X^2$. $Y = X^2$

when $x = -1$, $Y = 1$ and when $x = 1$, $Y = 1$.

$Y = X^2$ is $g(X)$ and NOT monotone for $-1 < x < 1$.

However, g is monotone and decreasing for $-1 < X \leq 0$ and g is monotone and increasing for $0 \leq X < 1$.

Therefore, Find pdf of Y when $-1 < X < 0$ and then find pdf of Y when $0 < X < 1$. Then add those to get the complete pdf.

When $-1 < X \leq 0$, $0 < Y < 1$ and

$$g^{-1}(y) = -\sqrt{y}$$

Find this by solving $Y = X^2$ for X. and $-1 < X < 0$.

$$\Rightarrow \frac{d}{dy}\left(g^{-1}(y)\right) = \frac{d}{dy}\left(-\sqrt{y}\right)$$

$\leftarrow$ This is the Jacobian.

$$= -\frac{1}{2\sqrt{y}}$$

When $0 \leq X < 1$, $0 < Y < 1$ and

$$g^{-1}(y) = \sqrt{y}$$

Positive because $x > 0$

$$\frac{d}{dy}\left(g^{-1}(y)\right) = \frac{d}{dy}\left(\sqrt{y}\right)$$

$$= \frac{1}{2\sqrt{y}}$$

Therefore, the pdf of Y is

$$f_Y(y) = f_X\left(g^{-1}(y)\right) * \left|\frac{d}{dy}\left(g^{-1}(y)\right)\right|$$

$$= \underbrace{\frac{3}{2}\left(-\sqrt{y}\right)^2 * \left|-\frac{1}{2\sqrt{y}}\right|}_{\text{when } -1 < x < 0} + \underbrace{\frac{3}{2}\left(\sqrt{y}\right)^2 * \left|\frac{1}{2\sqrt{y}}\right|}_{\text{when } 0 < x < 1}$$

$$= \frac{3}{2}\sqrt{y}$$

The complete pdf of Y is

$$f_Y(y) = \begin{cases} \dfrac{3\sqrt{y}}{2} & ; 0 \leq y < 1 \\ 0 & ; \text{otherwise} \end{cases}$$