Bivariate Probability Distributions

Recall: A (univariate) random variable is defined to be a function from a sample space S into the real numbers, $\mathbb{R}$. Often we are interested simultaneously in two outcomes of a random experiment rather than one. For example, we might be interests in…

- SAT score and GPA for a student
- Number of customers waiting in two lines at the grocery store
- Number of hours spent studying and test score
- Dosage of a drug and blood pressure

In each of these examples, there are two random variables, and we are interested in the 2-dimnesinal random vector (X, Y). The concept of discrete/continuous random variable, independent, CDFs and pmfs/pdfs can be extended to **bivariate random variables,** as well as n-dimensinoal random variables.

The Expected Value of a Function of Random Variables (Section 5.5)

Similar to the univariate case, we can also find the **expected value (average or mean)** of functions of multiple random variables.

Definition: (**Bivariate Expectations**)
Suppose $g(y_1,\ y_2)$ is a real-valued function. If Y_1 and Y_2 are random variables with joint Pmf of $p_{1,2}(y_1,\ y_2)$ or pdf of $f_{1,2}(y_1,\ y_2)$, then

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{y_1},\ \boldsymbol{y_2})\ ] = \sum_{\boldsymbol{all}\ \boldsymbol{y_1}} \sum_{\boldsymbol{all}\ \boldsymbol{y_2}} \boldsymbol{g}(\boldsymbol{y_1},\ \boldsymbol{y_2}) \times \boldsymbol{p_{1,2}}(\boldsymbol{y_1}, \boldsymbol{y_2}) \quad ;\ \textbf{if } Y_1 \text{ and } Y_2 \textbf{ are DISCRETE}$$

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{y_1},\ \boldsymbol{y_2})\ ] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \boldsymbol{g}(\boldsymbol{y_1},\ \boldsymbol{y_2}) \times \boldsymbol{f_{1,2}}(\boldsymbol{y_1}, \boldsymbol{y_2})\, \boldsymbol{dy_1}\, \boldsymbol{dy_2} \quad ;\ \textbf{if } Y_1 \text{ and } Y_2 \textbf{ are CONTINUOUS}$$

Independence can make our lives easier in lots of ways!

Theorem: (5.9)
If two random variables $\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ **are independent**, then

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{Y_1}) \times\ \boldsymbol{h}(\boldsymbol{Y_2})\ ]\ =\ \boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{Y_1})\ ] \times \boldsymbol{E}[\ \boldsymbol{h}(\boldsymbol{Y_2})\ ]$$

Example 1: The joint probability function of Y_1 and Y_2 can be given by the following function or table.

$$p(y_1, y_2) = \begin{cases} 0.125, & (y_1, y_2) = (0,\ 0) \\ 0.250, & (y_1, y_2) = (1,\ 0) \\ 0.250, & (y_1, y_2) = (0,\ 2) \\ 0.375, & (y_1, y_2) = (1,\ 2) \\ 0, & otherwise \end{cases}$$

	Y_2	
Y_1		

(a) Are the variables Y_1 and Y_2 independent?

(b) Find $E(Y_1Y_2)$

Example 2: Let the joint pdf has the following form:

$$f(y_1, y_2) = \quad y_1 y_2 \ ; \quad 0 \leq y_1 \leq 2, \ \ 0 \leq y_2 \leq 1$$

(a) Draw and shade the support of joint pdf.

(b) Are the variables Y_1 and Y_2 independent?

(c) Find $E(\,Y_1^2\ Y_2)$

Example 3: Let the joint pdf has the following form:

$$f(y_1, y_2) = \quad y_1 + y_2 \ ; \quad 0 \le y_1 \le 1, \ \ 0 \le y_2 \le 1$$

(a) Draw and shade the support of joint pdf.

(b) Are the variables Y_1 and Y_2 independent?

(c) Find $E[\, Y_1^2 \ (Y_2 + 1)\,]$

Conditional Distributions (Section 5.3)

Often, two random variables are not independent.

For example, a randomly selected person's height is typically related to his/her weight. If we looked at the conditional distribution of a randomly selected person's weight given that we knew a randomly selected person was 63 inches tall, it would be different from the distribution of any randomly selected person's weight. The additional information about height changes the distribution of weight, so these two random variables are not independent.

In such cases, we are often interested in the conditional probability of Y_1 given knowledge that $Y_2 = y_2$.

Recall the definition of conditional probability:

$$\boldsymbol{P(A \mid B)} =$$

This definition can be extended to conditional distributions.

Definitions 5.5 & 5.7: (**Conditional PMF/PDF**)

If Y_1 and Y_2 are bivariate **continuous** variables, the **conditional pdf** of Y_1 given $Y_2 = y_2$ is

$$\boldsymbol{f(y_1 \mid y_2) = \frac{f(y_1\,,\ y_2)}{f_2(y_2)}} \quad , \quad \boldsymbol{f_2(y_2)} > 0$$

If Y_1 and Y_2 are bivariate **discrete** variables, the **conditional pmf** of Y_1 given $Y_2 = y_2$ is

$$\boldsymbol{p(y_1 \mid y_2) = \frac{p(y_1\,,\ y_2)}{p_2(y_2)}} \quad , \quad \boldsymbol{p_2(y_2)} > 0$$

$\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ **are independent** random variables **if and only if**

$$\boldsymbol{f(y_1 \mid y_2) = f_1(y_1)} \qquad \textbf{or} \qquad \boldsymbol{p(y_1 \mid y_2) = p_1(y_1)} \quad \textbf{or}$$
$$\boldsymbol{f(y_2 \mid y_1) = f_2(y_2)} \qquad \textbf{or} \qquad \boldsymbol{p(y_2 \mid y_1) = p_2(y_2)}$$

Example 4: Consider the joint pdf

$$f(y_1, y_2) = \begin{cases} e^{-y_2} \;; & 0 < y_1 < y_2 < \infty \\ 0; & Otherwise \end{cases}$$

We can show the marginal pdfs of Y_1 and Y_2 (left as practice for you!) are:

(a) Are Y_1 and Y_2 independent?

(b) For $y_2 > 0$, find the conditional pdf of $Y_1|Y_2 = y_2$.

(c) Find the conditional pdf of $Y_1|Y_2 = 5$.

Example 5: Consider the joint pmf

$$p(y_1, y_2) = \begin{cases} \binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2 - y_1)} \left(\dfrac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right) & ; \ y_2 = 0, 1, \dots \ ; \ y_1 = 0, 1, \dots, y_2 \\ 0 \ ; \ otherwise \end{cases}$$

(a) Are Y_1 and Y_2 independent?

We can show that (left as practice for you!)

(b) For $y_2 = 0, 1, \ldots$, find the conditional pmf of $Y_1|Y_2 = y_2$.

❖ **If $\boldsymbol{Y_2} \sim$ and $\boldsymbol{Y_1|Y_2 = y_2} \sim$**

then $\boldsymbol{Y_1} \sim$

(c) Find $P(Y_1 = 2 \mid Y_2 = 3)$.

Example 6: Let Y_1 denote the number of luxury cars sold in a given day and let Y_2 denote the number of extended warranties sold. The joint probability function is given below.

		Number of warranties (Y_2)			
		0	**1**	**2**	**Total**
Number of cars (Y_1)	**0**	**1/6**			
	1	**1/12**	**1/6**		
	2	**1/12**	**1/3**	**1/6**	
Total					

Find the conditional pmf of $Y_2 | Y_1 = 1$.
This is the pmf of number of warranties sold if only 1 car was sold on a given day.