Section 6.5: The Method of Moment-Generating Functions

Recall:

A **special expected value** that is quite useful is the moment-generating function (**mgf**).

Definitions 3.14 and 4.14: (Moment-Generating Function)

Let Y be a random variable. The moment-generating function (mgf) of Y is $M(t)$, where

$M_x(t) = E(e^{tx})$

$$M_Y(t) = E(e^{tY}) = \begin{cases} \sum_{\text{all } y} [\, e^{ty}\, p(y) \,] & \text{if } Y \text{ is a } \textbf{discrete} \\ \int_{-\infty}^{\infty} [\, e^{ty}\, f(y) \,]dy & \text{if } Y \text{ is a } \textbf{continuous} \end{cases}$$

for $-b \le t \le b$ where $b(> 0)$ is constant.

Recall:

Distribution of Y with Parameter(s)	mgf of Y, $M_Y(t)$	pmf or pdf of Y
Binomial (n, p)	$(pe^t + 1 - p)^n$	$\binom{n}{y} p^y (1-p)^{n-y}$
Geometric (p)	$\frac{pe^t}{1-(1-p)e^t}$	$p(1-p)^{y-1}$
Negative Binomial (r, p)	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\binom{y-1}{r-1} p^r (1-p)^{y-r}$
Hypergeometric (N, r, n)	Does not exist	$\frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$
Poisson (λ)	$e^{\lambda(e^t - 1)}$	$\frac{e^{-\lambda}\lambda^y}{y!}$
Continuous Uniform (θ_1, θ_2)	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$	$\frac{1}{\theta_2 - \theta_1}$
Normal (μ, σ^2)	$e^{\left[\mu t + \frac{1}{2}\sigma^2 t^2\right]}$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}$
Gamma (α, β)	$(1-\beta t)^{-\alpha}$	$\frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$
Exponential (β) Mean $=\beta$	$(1-\beta t)^{-1}$	$\frac{1}{\beta} e^{-y/\beta}$
Chi-square (v)	$(1-2t)^{-v/2}$	$\frac{1}{\Gamma(v/2)\, 2^{v/2}} y^{v/2 - 1} e^{-y/2}$

Discrete

$t \neq 0 \rightarrow$

We can extend this to the multivariate case. The mgf can be used to recognize the probability distribution of function of random variables. However, this method is **limited** to known probability distributions.

Theorem 6.1:

Let $M_X(t)$ and $M_Y(t)$ are moment-generating functions of X and Y, respectively, random variables. **If $M_X(t) = M_Y(t)$ for all the values of t, then X and Y have the same probability distribution.**

Example 3: Let Y be the number of successes in a binomial experiment with n independent trials and p probability of success.

$\Rightarrow$ mgf of Y is $M_Y(t) = (pe^t + 1-p)^n$

Let $U = n - Y$. Number of Failures in n independent trials.

Find the probability distribution of U using mgf.

By definition of mgf, mgf of U is

$$
\begin{aligned}
M_U(t) &= E(e^{tU}) \\
&= E(e^{t(n-Y)}) \\
&= E(e^{tn-tY}) \\
&= E(e^{tn} e^{-tY}) \\
&= e^{tn} E(e^{-tY}) \quad \text{because } e^{tn} \text{ is a constant.} \\
&= e^{tn} \underbrace{E(e^{(-t)Y})} \\
&= e^{tn} M_Y(-t) \quad \text{since } M_Y(t) = E(e^{tY}) \\
&= e^{tn} (pe^{-t} + 1-p)^n \\
&= [e^t(pe^{-t} + 1-p)]^n \\
&= [p + e^t(1-p)]^n = [(1-p)e^t + \underbrace{(1-(1-p))}_{p}]^n
\end{aligned}
$$

$1-(1-p) \rightarrow p$

Now compare this mgf of U with mgf of known distributions and recognize the probability distribution of U.

Therefore, mgf U has the form of a mgf of Binomial distribution with n trials and (1-p) probability.

$\Rightarrow$ U has a Binomial $(n, \underbrace{1-p})$ probability of a failure in one trial.

$\Rightarrow$ pmf of U is

$$p(u) = \begin{cases} \binom{n}{u} (1-p)^u (p)^{n-u} & ; u = 0, 1, 2, \cdots, n \\ 0 & ; \text{otherwise} \end{cases}$$

Example 4: The length of time necessary to tune up a car is exponentially distributed with a $\beta =$ mean of 0.5 hour. If two cars are waiting for a tune up and the service times are independent,
(a) find the probability distribution of total time for the two tune ups.
(b) find the probability that total time for the two tune ups will exceed 1.5 hours.

(a) Let Y_i be the time to tune up i^{th} car. For $i = 1, 2$ we know $E(Y_i) = 0.5$.
Since Y_i has an exponential distribution, its mgf

$$M_{Y_i}(t) = (1 - \beta t)^{-1} = (1 - 0.5t)^{-1} \text{ for } i = 1, 2$$

Then let the total time to tune up 2 cars be U.

$U = Y_1 + Y_2 =$ Sum of 2 independent random variables.

By definition of mgf of U,

$$M_U(t) = E\left[e^{tU}\right]$$
$$= E\left[e^{t(Y_1+Y_2)}\right]$$
$$= E\left[e^{tY_1} e^{tY_2}\right] = E\left[g(Y_1) * h(Y_2)\right]$$
$$= E(e^{tY_1})\, E(e^{tY_2}) \quad \text{because } Y_1 \text{ and } Y_2 \text{ are independent.}$$
$$= \underbrace{M_{Y_1}(t)}\ \underbrace{M_{Y_2}(t)}$$

$$M_U(t) = (1-0.5t)^{-1}(1-0.5t)^{-1}$$
$$= (1-0.5t)^{-2} = (1-\beta t)^{-\alpha}$$

This has the form of mgf of a Gamma (α, β) distribution where $\alpha = 2$, $\beta = 0.5$ here. Therefore, the total time (U) has a Gamma $(\alpha = 2, \beta = 0.5)$.

$\Rightarrow$ The sum of independent exponential random variables with the same mean has a gamma distribution.

$0 < Y_1 < \infty$
$0 < Y_2 < \infty$

(b) $P(U > 1.5) = P(1.5 < U < \infty)$ where

$$f(u) = \frac{1}{\Gamma 2\,(0.5)^2} u^{2-1} e^{-\frac{u}{0.5}}$$
$$= 4ue^{-2u} \;;\; u > 0$$

$$= \int_{1.5}^{\infty} 4ue^{-2u}\,du$$
$$= 4e^{-3}$$
$$\approx 0.1991 \quad \text{Unlikely}$$

OR $P(U > 1.5) = 1 - P(U \le 1.5)$
$$= 1 - P(0 < U < 1.5)$$
$$= 1 - \int_0^{1.5} 4ue^{-2u}\,du$$

Theorem 6.2: Mgf of sum of independent random variables

Let $Y_1, Y_2, \dots, Y_n$ be **independent** random variables with mgf $M_{Y_1}(t), M_{Y_2}(t), \dots, M_{Y_n}(t)$, respectively.

If $U = Y_1 + Y_2 + \dots + Y_n$, This is a sum of independent random variables (NOT a Linear function)

then mgf of U is

$$M_U(t) = \underbrace{M_{Y_1}(t) \times M_{Y_2}(t) \times \dots \times M_{Y_n}(t)}_{\text{product of individual mgf.}}$$

because,

$$M_U(t) = E(e^{tU})$$
$$= E(e^{t(Y_1+Y_2+\dots+Y_n)})$$
$$= E(e^{tY_1} e^{tY_2} \dots e^{tY_n})$$
$$= E(e^{tY_1}) E(e^{tY_2}) \dots E(e^{tY_n})$$ because $Y_1, Y_2, \dots, Y_n$ are independent
$$= M_{Y_1}(t) M_{Y_2}(t) \dots M_{Y_n}(t)$$

Example 5:

A shipping company handles containers in three different sizes:

(1) 27 ft^3, (2) 125 ft^3, and (3) 512 ft^3.

Let Y_i $(i = 1, 2, 3)$ denote the number of type i containers shipped during a given week and each has the following distributions:

$Y_1 \sim Poisson\ (\lambda_1 = 200)$ $\Longrightarrow$ **mgf of Y_1 is, $M_{Y_1}(t) =$** $e^{200(e^t-1)}$

$Y_2 \sim Poisson\ (\lambda_2 = 250)$ $\Longrightarrow$ **mgf of Y_2 is, $M_{Y_2}(t) =$** $e^{250(e^t-1)}$

$Y_3 \sim Poisson\ (\lambda_3 = 100)$ $\Longrightarrow$ **mgf of Y_3 is, $M_{Y_3}(t) =$** $e^{100(e^t-1)}$

The number of each type of containers shipped is independent from others.

$\Rightarrow Y_1, Y_2, Y_3$ are independent.

Find the probability distribution of total number of containers shipped per week.

Let U be the total number of containers shipped per week

$U = Y_1 + Y_2 + Y_3$ This is a sum of independent random variables.

From Theorem 6.2, the mgf of U is

$$M_U(t) = M_{Y_1}(t) M_{Y_2}(t) M_{Y_3}(t)$$
$$= e^{200(e^t-1)} e^{250(e^t-1)} e^{100(e^t-1)}$$
$$= e^{550(e^t-1)} = e^{\lambda(e^t-1)}$$

This has the form of mgf of a Poisson $(\underbrace{\lambda = 550}_{\sum_{i=1}^{3} \lambda_i})$.

If you wanted total volume shipped per week, then it is $27Y_1 + 125Y_2 + 512Y_3$ and this is a linear function of Y_1, Y_2, Y_3 and NOT sum of Y_1, Y_2, Y_3.

Cannot use Theorem 6.2 here.

❖ If $Y_i \sim$ **Poisson**(λ_i) for $i = 1, 2, \ldots, n$ and all Y_i **are independent,** then

$\sum_{i=1}^{n} Y_i$ has a Poisson$\left(\sum_{i=1}^{n} \lambda_i\right)$

$\Rightarrow$ Sum of independent Poisson random variables has a Poisson distribution.

Theorem 6.3: pdf of <u>linear function of independent Normal</u> random variables

Let $Y_1, Y_2, \ldots, Y_n$ be **independent** random variables each with a **normal** distribution, $E(Y_i) = \mu_i$ and $Var(Y_i) = \sigma_i^2$, for $i = 1,2, \ldots n$.

Let $a_1, a_2, \ldots, a_n$ be constants.

$$\text{If } U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + a_2 Y_2 + \ldots + a_n Y_n, \text{ then}$$

U has a **Normal** distribution with **mean** $= \mathrm{E(U)} = \sum_{i=1}^{n} a_i \mu_i$ and **variance** $= \mathrm{Var(U)} = \sum_{i=1}^{n} a_i^2 \sigma_i^2$