Conditional Expectations (Section 5.11)

Conditional distributions can be used in the same way unconditional distributions are used. For example, conditional distributions may be used to compute conditional expectations.

Definition 5.13: **Conditional Expectations**
If Y_1 and Y_2 are two random variables and $g(Y_1)$ is a function of Y_1, then the **conditional expectation** of $g(Y_1)$ given that $Y_2 = y_2$ is:

function of ONLY Y_1

$$E[\underbrace{\boldsymbol{g(Y_1)} \mid (\boldsymbol{Y_2 = y_2})}_{\text{This is a function of ONLY } Y_2}] = \begin{cases} \sum_{all\ y_1} \boldsymbol{g(y_1) \times p(y_1 \mid y_2)} & ; \textbf{if } Y_1 \textbf{ and } Y_2 \textbf{ are DISCRETE} \\ \int_{-\infty}^{\infty} \boldsymbol{g(y_1) \times f(y_1 \mid y_2)\, dy_1} & ; \textbf{if } Y_1 \textbf{ and } Y_2 \textbf{ are CONTINUOUS} \end{cases}$$

In particular, the **conditional mean** of $\boldsymbol{Y_1}$ given that $\boldsymbol{Y_2 = y_2}$ is denoted by and computed using

$$\boldsymbol{\mu_{Y_1 \mid Y_2} = E(Y_1 \mid (Y_2 = y_2))}$$

and the **conditional variance** of $\boldsymbol{Y_1}$ given that $\boldsymbol{Y_2 = y_2}$ is denoted by and computed as

$$\boldsymbol{Var(Y_1 \mid (Y_2 = y_2)) = E\left[(Y_1 - \mu_{Y_1 \mid Y_2})^2 \mid (Y_2 = y_2)\right]}$$

Further, the shortcut formula for $\boldsymbol{Var(Y_1 \mid Y_2 = y_2)}$:

$$\boldsymbol{Var[Y_1 \mid (Y_2 = y_2)] = E\left[Y_1^2 \mid (Y_2 = y_2)\right] - (\mu_{Y_1 \mid Y_2})^2}$$

Recall: If Y is a random variable with mean μ and variance σ^2

$\mu = E(Y)$ and $\sigma^2 = Var(Y) = E[(Y-\mu)^2] = E(Y^2) - \mu^2$

Example 7: Consider the joint pdf

$$f_{Y_1, Y_2}(y_1, y_2) = e^{-y_2} \quad ; \quad 0 < y_1 < y_2 < \infty$$

Previously, we showed the conditional distribution of $Y_1 \mid Y_2$ is $f(y_1 \mid y_2) = \begin{cases} \frac{1}{y_2} ; & 0 < y_1 < y_2 < \infty \\ 0 ; & \text{otherwise} \end{cases}$

Find $E(Y_1 \mid (Y_2 = y_2))$. Here Y_1 is a function only Y_1.

$$E[Y_1 \mid (Y_2 = y_2)] = \int_{-\infty}^{\infty} y_1 f(y_1 \mid y_2)\, dy_1$$

$$= \int_0^{y_2} y_1 \left(\frac{1}{y_2}\right) dy_1 = \frac{y_2}{2}$$

$$\Rightarrow E(Y_1 \mid Y_2 = 5) = \frac{5}{2}$$

Let $E(Y_1^2 \mid (Y_2 = y_2)) = \int_{-\infty}^{\infty} y_1^2 f(y_1 \mid y_2)\, dy_1$

Example 6 Contd.: Let Y_1 denote the number of luxury cars sold on a given day and let Y_2 denote the number of extended warranties sold. $Y_1 = 1$

Find the expected number of warranties sold if only 1 car was sold on a given day. That is, find

$E[Y_2 | (Y_1 = 1)]$

In example 6, we found that conditional pmf of $Y_2|Y_1 = 1$ is given by

$$p(y_2 | (y_1 = 1)) = \begin{cases} 1/3 & ; y_2 = 0 \\ 2/3 & ; y_2 = 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Y_1 \ Y_2	0	1	2
0			
1	1/12	1/6	0
2			

$$E(Y_2 | (Y_1 = 1)) = \sum_{\text{all } y_2} [y_2 \, p(y_2 | y_1 = 1)]$$

$$= \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{2}{3}\right)$$

$$= \frac{2}{3}$$

* Note that when Y_1 is 1, Y_2 can be 0 or 1. Therefore, $0 < E[Y_2 | Y_1 = 1] < 1$.

$$E[Y_2^2 | (Y_1 = 1)] = \sum_{\text{all } y_2} [y_2^2 \, p(y_2 | y_1 = 1)] = \left(0^2 \times \frac{1}{3}\right) + \left(1^2 \times \frac{2}{3}\right) = \frac{2}{3}$$

Example 8: Let Y_1 and Y_2 have the following joint pmf.

$$p_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \dfrac{y_1 + y_2}{32} & ; \; y_1 = 1, 2 \; ; \; y_2 = 1, 2, 3, 4 \\ 0 & ; \text{otherwise} \end{cases}$$

These are two Discrete random variables and hence use Σ

Find $E(Y_2^2 \mid Y_1)$. $E[Y_2^2 | Y_1] = \sum_{\text{all } y_2} [y_2^2 \, p(y_2 | y_1)]$

First, need to find $p(y_2 | y_1) = \dfrac{p(y_1, y_2)}{p_1(y_1)}$

Therefore, need to find $p_1(y_1) = \sum_{\text{all } y_2} p(y_1, y_2) = \sum_{y_2 = 1}^{4} \left(\dfrac{y_1 + y_2}{32}\right) = \dfrac{4y_1 + 10}{32}$

Then $p(y_2 | y_1) = \dfrac{p(y_1, y_2)}{p_1(y_1)} = \dfrac{\frac{y_1 + y_2}{32}}{\frac{4y_1 + 10}{32}} = \dfrac{y_1 + y_2}{4y_1 + 10}$; $y_1 = 1, 2$, $y_2 = 1, 2, 3, 4$

Finally,

$$\underbrace{E(Y_2^2 | Y_1)} = \sum_{y_2 = 1}^{4} \left[y_2^2 \left(\frac{y_1 + y_2}{4y_1 + 10} \right) \right] = \frac{30y_1 + 100}{4y_1 + 10} = \frac{15y_1 + 50}{2y_1 + 5}$$

This must be a function of only Y_1.

In general, $E[g(Y_1)] = E[E(g(Y_1)|Y_2)]$ and $E[g(Y_2)] = E[E(g(Y_2)|Y_1)]$

Conditional expectations lend themselves to other useful results, such as finding the expected values and variances of univariate random variables.

Theorem 5.14: This is true for both Discrete and Continuous variables.
If Y_1 and Y_2 are two random variables, then

$$E(Y_1) = E[\ E(Y_1 \mid Y_2)\ ] \qquad \textbf{and} \qquad E(Y_2) = E[\ E(Y_2 \mid Y_1)\ ]$$

Proof:
Let Y_1 and Y_2 be jointly distributed **discrete** random variables.

$$E(Y_2) = \sum_{\text{all } y_1} \sum_{\text{all } y_2} y_2\, p(y_1, y_2) \quad \text{by definition of } E[g(Y_2)]$$

$$p(y_2|y_1) = \frac{p(y_1,y_2)}{p_1(y_1)}$$

$$= \sum_{\text{all } y_1} \sum_{\text{all } y_2} \left[y_2 \left[p(y_2|y_1) \times p_1(y_1) \right] \right]$$

$$= \sum_{\text{all } y_1} \left\{ p_1(y_1) \sum_{\text{all } y_2} \left[y_2\, p(y_2|y_1) \right] \right\}$$

$$= \sum_{\text{all } y_1} \left[p_1(y_1)\, E(Y_2|Y_1) \right]$$

$$= E\left[E(Y_2|Y_1) \right]$$

Note that $E(Y_2|Y_1)$ is a function of ONLY Y_1. Therefore, $g(Y_1) = E(Y_2|Y_1)$.

Example 7 Continued: Consider the joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = e^{-y_2} \quad ; \quad 0 < y_1 < y_2 < \infty \quad \text{and zero otherwise}$$

Previously, we mentioned that $f_2(y_2) = y_2 e^{-y_2}$; $0 < y_2 < \infty$ and zero otherwise

Also, we showed, $E(Y_1 \mid Y_2 = y_2) = \dfrac{Y_2}{2}$

Compute $E(Y_1)$ using $E(Y_1 \mid Y_2 = y_2)$.

From Theorem 5.14, $E(Y_1) = E[E(Y_1|Y_2)]$

$$= E\left[\frac{Y_2}{2}\right]$$

$$= \frac{1}{2} E(Y_2)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[y_2 f_2(y_2)\, dy_2 \right]$$

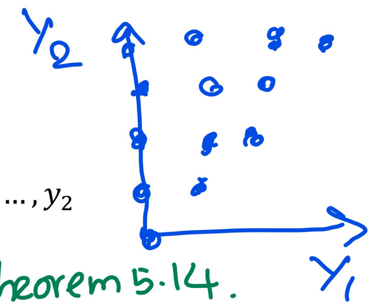
$$= \frac{1}{2} \int_{0}^{\infty} \left[y_2 (y_2 e^{-y_2})\, dy_2 \right]$$

$$= 1$$

Recall:
For any constant k,
$E(kY) = k E(Y)$.

To confirm this answer, do $E(Y_1) = \int_{-\infty}^{\infty} y_1 f_1(y_1)\, dy_1$

where $f_1(y_1) = e^{-y_1}$; $0 < y_1 < \infty$.



Example 5 Continued: Consider the joint pmf

$$p_{Y_1,Y_2}(y_1,y_2) = \binom{y_2}{y_1} p^{y_1}(1-p)^{(y_2-y_1)}\left(\frac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right) \quad ; y_2 = 0,1,\ldots ;\ y_1 = 0,1,\ldots,y_2$$

Find $E(Y_1)$ using $E[\ E(Y_1 \mid Y_2)\ ]$? $E(Y_1) = E[E(Y_1|Y_2)]$ from Theorem 5.14.

Since we previously found $Y_1|Y_2$ has a Binomial (Y_2, p) we know $E(Y_1|Y_2) = Y_2 p$.

Therefore, $E(Y_1) = E(Y_2 p)$

$= pE(Y_2)$ because p is a constant

Since we previously found Y_2 has a Poisson (λ) we know $E(Y_2) = \lambda$.

Therefore, $E(Y_1) = \underline{\underline{p\lambda}}$.

If you did not know specific distribution of $Y_1|Y_2$ and Y_2, compute $E(Y_1|Y_2) = \sum_{\text{all } y_1} [y_1 p(y_1|y_2)] = Y_2 p$ and then

$$E(Y_1) = E[E(Y_1|Y_2)]$$
$$= E(Y_2 p) = pE(Y_2) = p\sum_{\text{all } y_2}[y_2 p_2(y_2)] = \underline{\underline{p\lambda}}$$

Theorem 5.15: This is true for both Discrete and Continuous variables

If Y_1 and Y_2 are two random variables, then

$$\boldsymbol{E[Var(Y_1 \mid Y_2)] + Var[E(Y_1 \mid Y_2)] = Var(Y_1)}$$

Similarly, $E[Var(Y_2|Y_1)] + Var[E(Y_2|Y_1)] = Var(Y_2)$

Proof:

From Theorem 5.14, $E(Y_1) = E[E(Y_1|Y_2)]$ —— (1)

Similarly, $E(Y_1^2) = E[E(Y_1^2|Y_2)]$ —— (2)

By definition, $Var(Y) = E(Y^2) - [E(Y)]^2$

similarly, $Var(Y_1|Y_2) = E(Y_1^2|Y_2) - [E(Y_1|Y_2)]^2$

Take expected value on both sides,

$$E[Var(Y_1|Y_2)] = E\{E(Y_1^2|Y_2) - [E(Y_1|Y_2)]^2\}$$
$$= E[E(Y_1^2|Y_2)] - E[[E(Y_1|Y_2)]^2] \text{ —— (3)}$$

Again, apply the definition of variance to $E(Y_1|Y_2)$,

$$Var\left[E(Y_1|Y_2)\right] = E\left[\left[E(Y_1|Y_2)\right]^2\right] - \left[E\left[E(Y_1|Y_2)\right]\right]^2 \quad \text{—} ④$$

Now add equations ③ and ④,

$$E\left[Var(Y_1|Y_2)\right] + Var\left[E(Y_1|Y_2)\right] = E\left[E(Y_1^2|Y_2)\right] - \left[E\left[E(Y_1|Y_2)\right]\right]^2$$

$$= E(Y_1^2) - \left[E(Y_1)\right]^2 \quad \text{substitute from equation ① and ②}$$

$$= Var(Y_1) \blacksquare$$

Example 9: (Problem 5.153) Suppose the (Y_1) number of eggs laid by a certain insect has a Poisson distribution with mean λ. The probability that any one egg hatches is p. Assume the eggs hatch independently of one another.
What do we know?

Let $Y_2 =$ the number of eggs that hatch. Then $Y_2 \mid Y_1$

(a) Find the expected value of Y_2, the total number of eggs that hatch.

(b) Find the variance of Y_2.