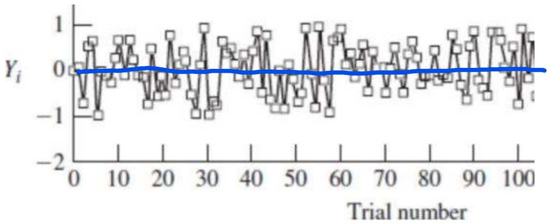
Bivariate Probability Distributions

Recall: A (univariate) random variable is defined to be a function from a sample space S into the real numbers, $\mathbb{R}$. Often we are interested simultaneously in two outcomes of a random experiment rather than one. For example, we might be interests in…

- SAT score and GPA for a student
- Number of customers waiting in two lines at the grocery store
- Number of hours spent studying and test score
- Dosage of a drug and blood pressure

In each of these examples, there are two random variables, and we are interested in the 2-dimnesinal random vector (X, Y). The concept of discrete/continuous random variable, independent, CDFs and pmfs/pdfs can be extended to **bivariate random variables,** as well as n-dimensinoal random variables. The following graph shows different values a random variable Y takes when the same experiment is repeated 100 times.



Zero

Expected value or Average of Y.

The Expected Value of a Function of Random Variables (Section 5.5)

Similar to the univariate case, we can also find the **expected value (average or mean)** of functions of multiple random variables.

Definition: (**Bivariate Expectations**)
Suppose $g(y_1,\ y_2)$ is a real-valued function. If Y_1 and Y_2 are random variables with joint pmf of $p_{1,2}(y_1\,,\ y_2)$ or pdf of $f_{1,2}(y_1\,,\ y_2)$, then

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{Y_1},\ \boldsymbol{Y_2})\ ] = \sum_{all\ y_1}\sum_{all\ y_2} \boldsymbol{g}(\boldsymbol{y_1},\ \boldsymbol{y_2}) \times \boldsymbol{p_{1,2}}(\boldsymbol{y_1}, \boldsymbol{y_2}) \quad ; \textbf{ if } Y_1 \textbf{ and } Y_2 \textbf{ are DISCRETE}$$

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{Y_1},\ \boldsymbol{Y_2})\ ] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{g}(\boldsymbol{y_1},\ \boldsymbol{y_2}) \times \boldsymbol{f_{1,2}}(\boldsymbol{y_1}, \boldsymbol{y_2})\, \boldsymbol{dy_1}\, \boldsymbol{dy_2} \quad ; \textbf{ if } Y_1 \textbf{ and } Y_2 \textbf{ are CONTINUOUS}$$

Recall:

$$E(g(Y)) = \begin{cases} \sum_{all\ y} [g(y) \times p(y)] & \text{if } Y \text{ is Discrete} \\ \int_{-\infty}^{\infty} g(y) \times f(y)\, dy & \text{if } Y \text{ is continuous.} \end{cases}$$

Independence can make our lives easier in lots of ways!

Theorem: (5.9)
If two random variables $\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ **are independent**, then

$$\boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{Y_1}) \times \boldsymbol{h}(\boldsymbol{Y_2})\ ] = \boldsymbol{E}[\ \boldsymbol{g}(\boldsymbol{Y_1})\ ] \times \boldsymbol{E}[\ \boldsymbol{h}(\boldsymbol{Y_2})\ ]$$

* This is used only if you are given Y_1 and Y_2 are independent or you can verify Y_1 and Y_2 are independent according to the probability.

Example 1: The joint probability function of Y_1 and Y_2 can be given by the following function or table.

$$p(y_1, y_2) = \begin{cases} 0.125, & (y_1, y_2) = (0,\ 0) \\ 0.250, & (y_1, y_2) = (1,\ 0) \\ 0.250, & (y_1, y_2) = (0,\ 2) \\ 0.375, & (y_1, y_2) = (1,\ 2) \\ 0, & otherwise \end{cases}$$

	Y_2		
Y_1	0	2	Total
0	0.125	0.25	0.375
1	0.25	0.375	0.625
Sum	0.375	0.625	1

(a) Are the variables Y_1 and Y_2 independent?

when $Y_1 = 0$ and $Y_2 = 0$;

$p_1(0) \times p_2(0) = 0.375 \times 0.375 = 0.140625$

and $p(0,0) = 0.125$.

$\Rightarrow p(0,0) \neq p_1(0) \times p_2(0)$

Therefore, Y_1 and Y_2 are not independent.

(b) Find $E(Y_1 Y_2)$

$Y_1 Y_2 = Y_1 \times Y_2 = g(Y_1) \times h(Y_2)$ where $g(Y_1) = Y_1$ and $h(Y_2) = Y_2$.

However, $E(Y_1 Y_2) \neq E(Y_1)E(Y_2)$ because Y_1, Y_2 are NOT independent.

$$E(Y_1 Y_2) = \sum_{\text{all } y_1} \sum_{\text{all } y_2} \left[(y_1 y_2)\, p(y_1, y_2)\right]$$

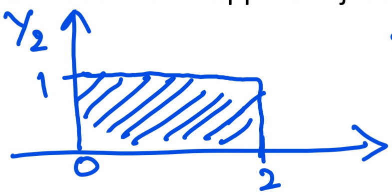
$$= [0 \times 0 \times 0.125] + [0 \times 2 \times 0.25] + [1 \times 0 \times 0.25] + [1 \times 2 \times 0.375]$$

$$= \underline{0.75}$$

Example 2: Let the joint pdf has the following form:

$$f(y_1, y_2) = \begin{cases} y_1 y_2 \; ; & 0 \le y_1 \le 2, \; 0 \le y_2 \le 1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

(a) Draw and shade the support of joint pdf.



Support is the shaded area.

(b) Are the variables Y_1 and Y_2 independent?

Support of Y_1 is $(0,2)$ while support Y_2 is $(0,1)$ and they are independent. ✓
Further, the joint pdf $f(y_1,y_2) = y_1 y_2$ can be written as a product of non-negative function of only Y_1 and a non-negative function of only Y_2. ✓
Therefore, Y_1 and Y_2 are independent.

$Y_1^2 Y_2 = g(Y_1) \times h(Y_2)$ where $g(Y_1) = Y_1^2$, $h(Y_2) = Y_2$.

(c) Find $E(Y_1^2 Y_2)$

Since we found Y_1, Y_2 are indenpendent,

$E(Y_1^2 Y_2) = E(Y_1^2) \times E(Y_2)$

Then find pdf of Y_1 as

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^1 (y_1 y_2)\, dy_2 = \frac{y_1 y_2^2}{2}\Big|_{y_2=0}^{1} = \frac{y_1}{2} \; ; 0 \le y_1 \le 2$$

$$E(Y_1^2) = \int_{-\infty}^{\infty} y_1^2 f_1(y_1)\, dy_1 = \int_0^2 y_1^2 \left(\frac{y_1}{2}\right) dy_1 = \frac{y_1^4}{8}\Big|_0^2 = 2$$

Next find the pdf of Y_2 as

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 = \int_0^2 (y_1 y_2)\, dy_1 = \frac{y_1^2}{2} y_2 \Big|_{y_1=0}^{2} = 2y_2 \; ; 0 \le y_2 \le 1$$

Then $E(Y_2) = \int_{-\infty}^{\infty} y_2 f_2(y_2)\, dy_2 = \int_0^1 y_2 (2y_2)\, dy_2 = 2\frac{y_2^3}{3}\Big|_0^1 = \frac{2}{3}$

$$E(Y_1^2 Y_2) = E(Y_1^2) E(Y_2) = (2)\left(\frac{2}{3}\right) = \boxed{\frac{4}{3}}$$

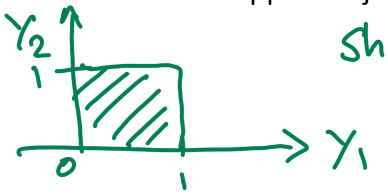
Another way to do this is using definition of $E[g(Y_1,Y_2)]$.

(*) $$E(Y_1^2 Y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [(y_1^2 y_2) f(y_1, y_2)]\, dy_1\, dy_2 = \int_0^1 \int_0^2 [(y_1^2 y_2)(y_1 y_2)]\, dy_1\, dy_2 = \frac{4}{3}$$

Example 3: Let the joint pdf has the following form:

$$f(y_1, y_2) = \begin{cases} y_1 + y_2 \ ; & 0 \le y_1 \le 1, \ 0 \le y_2 \le 1 \\ 0 \ ; & \text{otherwise} \end{cases}$$

(a) Draw and shade the support of joint pdf.



Shaded area is the support.

(b) Are the variables Y_1 and Y_2 independent?

Support of Y_1 is $(0,1)$ while support of Y_2 is $(0,1)$ and they are independent. ✓

However, the pdf $f(y_1, y_2) = y_1 + y_2$ cannot be written as a product of non-negative function of only y_1 and a non-negative function of only y_2.

Therefore, Y_1 and Y_2 are NOT independent.

(c) Find $E[\,Y_1^2\,(Y_2+1)\,]$ $\quad Y_1^2(Y_2+1) = g(Y_1) \times h(Y_2)$ where $g(Y_1) = Y_1^2$ and $h(Y_2) = (Y_2+1)$

However, since Y_1 and Y_2 are NOT independent,

$E[Y_1^2(Y_2+1)] \ne E(Y_1^2)\,E(Y_2+1)$.

Therefore,

$$E[Y_1^2(Y_2+1)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [y_1^2(y_2+1)] \times f(y_1, y_2)\, dy_1\, dy_2$$

$$\boxed{= \int_0^1\int_0^1 [y_1^2(y_2+1)(y_1+y_2)]\, dy_1\, dy_2}$$

Provide this before final answer if using a calculator to solve this.

$$= \int_0^1\int_0^1 (y_1^3 + y_1^2 y_2 + y_1^3 y_2 + y_1^2 y_2^2)\, dy_1\, dy_2$$

$$= \int_0^1 \left[\frac{y_1^4}{4} + \frac{y_1^3}{3} y_2 + \frac{y_1^4}{4} y_2 + \frac{y_1^3}{3} y_2^2\right]\Bigg|_{y_1=0}^{1} dy_2$$

$$= \int_0^1 \left(\frac{1}{4} + \frac{1}{3} y_2 + \frac{1}{4} y_2 + \frac{1}{3} y_2^2\right) dy_2$$

$$= \left(\frac{1}{4} y_2 + \frac{1}{3}\frac{y_2^2}{2} + \frac{1}{4}\frac{y_2^2}{2} + \frac{1}{3}\frac{y_2^3}{3}\right)\Bigg|_{y_2=0}^{1} = \frac{47}{72}$$

Find $E(Y_2+1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y_2+1) f(y_1, y_2)\, dy_1\, dy_2 = \int_0^1\int_0^1 [(y_2+1)(y_1+y_2)]\, dy_1\, dy_2 = \frac{19}{12}$

Find $E(Y_1+Y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(y_1+y_2) f(y_1, y_2)]\, dy_1\, dy_2 = \int_0^1\int_0^1 [(y_1+y_2)(y_1+y_2)]\, dy_1\, dy_2 = \frac{7}{6}$