Y_1, Y_2

Functions of Random Variables and Their Distributions

Recall: A (univariate) random variable is defined to be a function from a sample space S into the real numbers, $\mathbb{R}$. We've talked **a lot** about distributions of random variables (including bivariate random variables), and we've used these distributions to find probabilities and other quantities like expected values that can be used to make decisions.

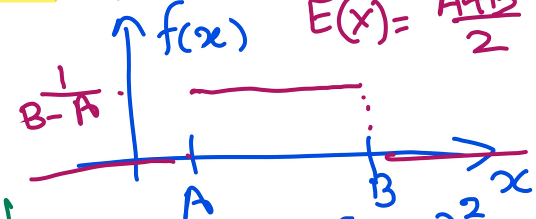
Often, we are not interested in the random variable(s) themselves, but instead in functions of the random variable(s), such as the sample mean, sample proportion and sample variance. In these cases, we need to know the probability distribution for the resulting function in order to make decisions. Let's start by looking at some examples.

Section 6.3: The Method of Distribution Function (that is, the CDF learned in Chapter 5)

Recall: The CDF of Y is $F_Y(y) = P(Y \le y)$

Example 1: Assume a continuous random variable $X \sim \boldsymbol{Uniform}(0,10)$. We are interested in various functions of X.

$E(X) = \frac{A+B}{2}$



$Var(X) = \frac{(B-A)^2}{12}$

For each of the random variables defined below, find the

- Expected Value
- Variance
- pdf
- CDF

CDF of X,

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{0}(0)\,dt}_{0} + \underbrace{\int_{0}^{x} 0.1\,dt}_{0.1x}$$

If $X \sim \boldsymbol{Uniform}(0, 10)$, we know

pdf of X is $f_X(x) = \begin{cases} \frac{1}{10-0} = 0.1 & ; 0 < x < 10 \\ 0 & ; \text{otherwise} \end{cases}$

Expected value of X is $E(X) = \frac{0+10}{2} = 5$

Variance of X is $Var(X) = \frac{(10-0)^2}{12} = \frac{100}{12} = \frac{25}{3}$

CDF of X is $F_X(x) = \begin{cases} 0 & ; x < 0 \\ 0.1x & ; 0 \le x < 10 \\ 1 & ; x \ge 10 \end{cases}$

(a) Let $W = 2X + 4$. When $x = 0$, $w = 4$ and when $x = 10$, $w = 24$.

Using properties of expected value (learned in ch 4),

$$E(W) = E(2X+4) = 2E(X) + 4 = (2 \times 5) + 4 = \underline{14}$$

Using properties of variance (learned in ch 4)

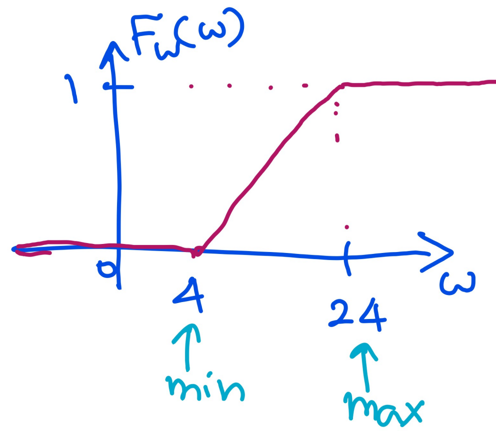
$$Var(W) = Var(2X+4) = 2^2 Var(X) = 2^2 \times \frac{25}{3} = \underline{\underline{\frac{100}{3}}}$$

By the definition of CDF, the CDF of W is,

$$(*)\quad F_W(\omega) = P(W \le \omega)$$
$$= P(2X+4 \le \omega)$$
$$= P\left(X \le \frac{\omega-4}{2}\right) \qquad \frac{\omega-4}{2} \rightarrow 0.5\omega-2$$
$$= \int_{-\infty}^{0.5\omega-2} f(x)\,dx \quad \text{where } f(x) = \begin{cases} 0.1 & ; 0<x<10 \\ 0 & ; \text{else} \end{cases}$$
$$= \int_{-\infty}^{0} (0)\,dx + \boxed{\int_{0}^{0.5\omega-2} (0.1)\,dx}$$
$$= 0.1\,(0.5\omega-2) \quad \text{where } 4 \le \omega < 24$$

The complete CDF of W is

$$F_W(\omega) = \begin{cases} 0 & ; \omega < 4 \\ 0.05\omega-0.2 & ; 4 \le \omega < 24 \\ 1 & ; \omega \ge 24 \end{cases}$$



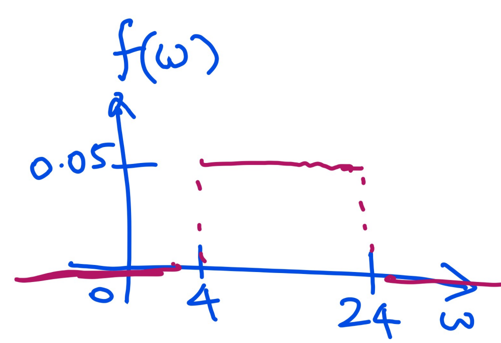
when $\omega = 4$, $F_W(4) = 0.05(4) - 0.2 = 0$
$\omega = 24$, $F_W(24) = 0.05(24) - 0.2 = 1$

Take the 1st derivative of CDF of W to get pdf of w,

$$f_W(\omega) = \frac{d\,F_W(\omega)}{d\omega}$$
$$= \frac{d\left[0.05\omega - 0.2\right]}{d\omega}$$
$$= 0.05$$

Then the complete pdf of w is,

$$f_W(\omega) = \begin{cases} 0.05 & ; 4 < \omega < 24 \\ 0 & ; \text{otherwise} \end{cases}$$



W has a continuous uniform (4, 24)

Parameters.

$$\boxed{Var(Y) = E(Y^2) - [E(Y)]^2}$$

when $x=0$, $t=1$ and when $x=10$, $t=26$.

(b) Let $T = \frac{X^2}{4} + 1$. Using properties of expected v we (in ch4)

$$* \; E(T) = E\left[\frac{X^2}{4}+1\right] = \frac{1}{4}\underbrace{E(X^2)}_{[Var(X)+[E(X)]^2]}+1 = \frac{1}{4}\Big[\underbrace{\frac{25}{3}+(5)^2}_{E(X^2)}\Big]+1 = \frac{28}{3}$$

Using properties of variance (in ch 4),

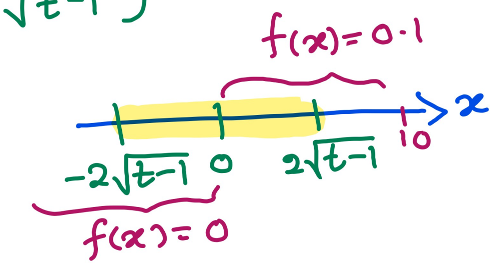
$$* \; Var(T) = Var\left(\frac{X^2}{4}+1\right) = \frac{1}{4^2}Var(X^2) = \frac{1}{16}\left[E[(X^2)^2] - [E(X^2)]^2\right]$$

$$= \frac{1}{16}\left[E(X^4) - \left(\frac{100}{3}\right)^2\right]$$

$$= \frac{1}{16}\Big[\underbrace{\int_0^{10} x^4 (0.1)\, dx}_{2000} - \left(\frac{100}{3}\right)^2\Big] = \frac{8000}{9}$$

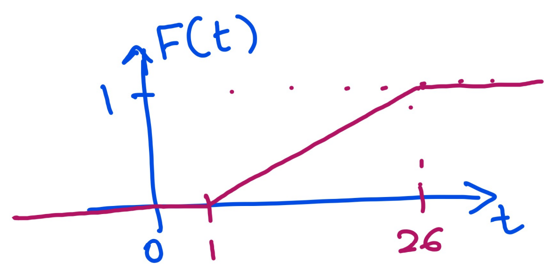
By definition of CDF, the CDF of T is,

$$F_T(t) = P(T \le t)$$
$$= P\left(\frac{X^2}{4}+1 \le t\right)$$
$$= P\left(X \le \pm\sqrt{4(t-1)}\right) = P\left(X \le \pm 2\sqrt{t-1}\right)$$
$$= P\left(-2\sqrt{t-1} \le X \le 2\sqrt{t-1}\right)$$
$$= \int_{-2\sqrt{t-1}}^{2\sqrt{t-1}} f(x)\, dx$$
$$= \int_{-2\sqrt{t-1}}^{0} (0)\, dx + \int_0^{2\sqrt{t-1}} (0.1)\, dx = 0.2\sqrt{t-1}$$



The complete CDF of T is

$$F_T(t) = \begin{cases} 0 & ; t < 1 \\ 0.2\sqrt{t-1} & ; 1 \le t < 26 \\ 1 & ; t \ge 26 \end{cases}$$



Then take the 1^{st} derivative of $F_T(t)$ and get pdf of T as,

$$f_T(t) = \frac{d}{dt}[F_T(t)] = \frac{d}{dt}\left(0.2\sqrt{t-1}\right) = \frac{0.1}{\sqrt{t-1}}$$ where $t \ne 1$

The complete pdf of T is

$$f_T(t) = \begin{cases} \frac{0.1}{\sqrt{t-1}} & ; 1 < t < 26 \\ 0 & ; \text{else} \end{cases}$$