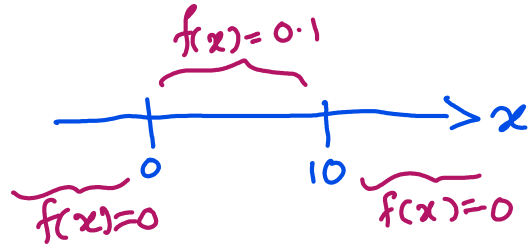
Recall: $f_X(x) = \begin{cases} 0.1 & ; 0<x<10 \\ 0 & ; \text{else} \end{cases}$ and $F_X(x) = \begin{cases} 0 & ; x<0 \\ 0.1x & ; 0 \le x < 10 \\ 1 & ; x \ge 10 \end{cases}$

(c) Let $R = -\ln(X)$. When $X \to 0$, $R \to \infty$ and when $X=10$, $R = -\ln(10)$

$R = -\ln(X)$ is a Non-linear function of X and therefore, we cannot use properties of expected value and variance to compute $E(R)$ and $Var(R)$.

By definition of CDF, the CDF of R is,

$$F_R(r) = P(R \le r)$$
$$= P(-\ln(X) \le r)$$
$$= P(\ln(X) \ge -r)$$
$$= P(X \ge e^{-r})$$
$$= \int_{e^{-r}}^{10} 0.1 \; x$$
$$= 0.1(10 - e^{-r}) \quad \text{where } -\ln(10) < r < \infty$$



The complete CDF of R is

$$F_R(r) = \begin{cases} 0 & ; r < -\ln(10) \\ 0.1(10 - e^{-r}) & ; -\ln(10) \le r < \infty \end{cases}$$

Then pdf of R is

$$f_R(r) = \frac{d}{dr}\left[F_R(r)\right]$$
$$= \frac{d}{dr}\left[0.1(10 - e^{-r})\right]$$
$$= 0.1 e^{-r}$$

The complete pdf of R is,

$$f_R(r) = \begin{cases} 0.1 e^{-r} & ; -\ln(10) < r < \infty \\ 0 & ; \text{else} \end{cases}$$

$$E(R) = \int_{-\infty}^{\infty} r f(r) dr$$
$$= \int_{-\ln(10)}^{\infty} r(0.1 e^{-r}) dr$$
$$= \underline{-0.1303}$$

$$Var(R) = E(R^2) - [E(R)]^2$$
$$= \int_{-\infty}^{\infty} r^2 f(r) \, dr - (-0.1303)^2$$
$$= \int_{-\ln(10)}^{\infty} r^2 (0.1 e^{-r}) \, dr - (-0.1303)^2$$
$$= \underline{2.6797}$$

$$E(R^2) = 2.6967$$

All the previous examples featured monotone, one-to-one functions. **If the function of interest is monotone, we can use the CDF of the original random variable to find the distribution of the transformed random variable.**

⊛ When finding $g^{-1}(y)$, simply solve for x.

Univariate Transformations Using CDFs

Let X have CDF $F_X(x)$, let $Y = g(X)$.

1. If g is an increasing function on X,

$$\textbf{CDF of } Y,\ F_Y(y) = P(Y \le y) = \underbrace{F_X(g^{-1}(y))}_{\text{plug } x = g^{-1}(y) \text{ in the CDF of } X} \quad \textbf{for } y \textbf{ in the support of } Y$$

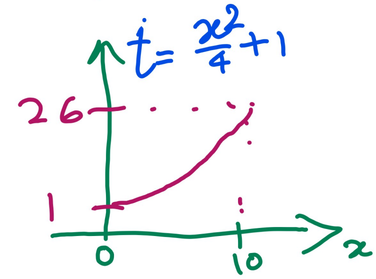
2. If g is a decreasing function on X,

$$\textbf{CDF of } Y,\ F_Y(y) = P(Y \le y) = 1 - F_X(g^{-1}(y)) \quad \textbf{for } y \textbf{ in the support of } Y$$

Recall: Ex 1: We know CDF of X is $F_X(x) = \begin{cases} 0 & ; x < 0 \\ 0.1x & ; 0 \le x < 10 \\ 1 & ; x \ge 10 \end{cases}$

Example 1 Part (b) (revisited): $X \sim Uniform(0,10)$, and $T = \frac{X^2}{4} + 1$

$T = \frac{X^2}{4} + 1$ is $g(X)$ where g is monotone and increasing function of X for $0 < x < 10$.



$\Rightarrow g^{-1}(t) = \sqrt{4(t-1)}$ for $0 < x < 10$.

Therefore, CDF of T is $F_T(t) = F_X(g^{-1}(t))$

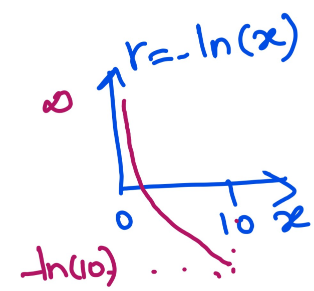
$= 0.1\left(\sqrt{4(t-1)}\right) = 0.2\sqrt{t-1}$

The complete CDF of T;

$$F_T(t) = \begin{cases} 0 & ; t < 1 \\ 0.2\sqrt{t-1} & ; 1 \le t < 26 \\ 1 & ; t \ge 26 \end{cases}$$

Example 1 Part (c) (revisited): $X \sim Uniform(0,10)$, and $R = -\ln(X)$

$R = -\ln(X)$ is $g(X)$ where g is monotone and decreasing function of x for $0 < x < 10$.



$\Rightarrow g^{-1}(r) = e^{-r}$ for $0 < x < 10$ and $-\ln(10) < r < \infty$.

Therefore, CDF of R is

$F_R(r) = 1 - F_X(g^{-1}(r))$

$= 1 - 0.1e^{-r}$

The complete CDF of R is,

$$F_R(r) = \begin{cases} 0 & ; r < -\ln(10) \\ 1 - 0.1e^{-r} & ; -\ln(10) \le r < \infty \end{cases}$$

In each of the previous examples, we differentiated the CDF with respect to the random variable to get the pdf.

- If $Y = g(X)$ is an **increasing** function of X over the support of X, then
  $F_Y(y) = F_X(g^{-1}(y))$ where $g^{-1}(y)$ is found simply by solving for x.
  Then pdf of Y is $f_Y(y) = \frac{d}{dy}\left[F_X(g^{-1}(y))\right]$
  $= f_X(g^{-1}(y)) * \frac{d}{dy}(g^{-1}(y))$
  Since $f_X(x)$ is a pdf, $f_X(g^{-1}(y))$ is non-negative and therfore, $\frac{d}{dy}[g^{-1}(y)]$ must be positive to make $f_Y(y)$ non-negative.
- If $Y = g(X)$ is a **decreasing** function of X over the support of X, then
  $F_Y(y) = 1 - F_X(g^{-1}(y))$
  Then pdf of Y is $f_Y(y) = \frac{d}{y}\left[1 - F_X(g^{-1}(y))\right]$
  $= \frac{d}{dy}\left[-F_X(g^{-1}(y))\right]$
  $= -f_X(^{-1}(y)) * \frac{d}{dy}[g^{-1}(y)]$
  Note that $f_Y(y)$ must be non-negative and therefore, $\frac{d}{dy}[g^{-1}(y)]$ must be negative here.
  Therefore, if we remove the negative sign at the beginning and take the absolute value of $\frac{d}{dy}[g^{-1}(y)]$, then $f_Y(y)$ is non-negative here.

Section 6.4: The Method of Transformations

Univariate Transformations Using PDFs

Let X have CDF $F_X(x)$ and let $Y = g(X)$, where g **is monotone and increasing or decreasing**. Suppose $f_X(x)$ is continuous on the support of X and $g^{-1}(y)$ has continuous derivative on Y. Then the pdf of Y is given by:

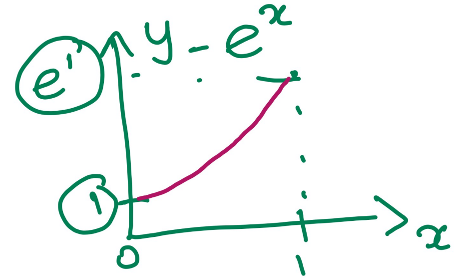
absolute value

$$f_Y(y) = f_X(g^{-1}(y)) \times \left| \frac{d\,(g^{-1}(y))}{dy} \right|$$

where $\frac{d\,[g^{-1}(y)]}{dy}$ is called the **Jacobian** of the transformation.

$f_X(g^{-1}(y))$ is the pdf of X evaluated at $g^{-1}(y)$.

To find $g^{-1}(y)$, simply solve $g(x)$ for x.



Example 2: Let X be a random variable with pdf

$$f_X(x) = \begin{cases} 4x^3 \ ; \ 0 < x < 1 \\ 0 \ ; \ elsewhere \end{cases}$$

X is continuous

Find the pdf for each of the functions of X below.

(a) $Y = e^X$ when $x=0$, $Y = e^0 = 1$ and when $x = 1$, $Y = e^1 = e$

$Y = e^x$ is $g(x)$ where g is monotone and increasing function of x when $0 < x < 1$.

$\Rightarrow g^{-1}(y) = \ln(y)$ To find this, simply solve $y = e^x$ for x.

$\Rightarrow \frac{d}{dy}\left(g^{-1}(y)\right) = \frac{d}{dy}\left(\ln(y)\right)$ This is the Jacobian.

$= \frac{1}{y}$

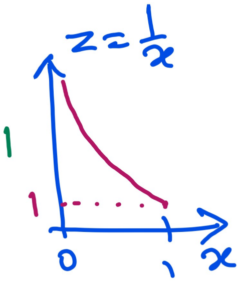
Therefore, pdf of Y is $f_Y(y) = f_X\left(g^{-1}(y)\right) * \left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$

$= 4\left[\ln(y)\right]^3 * \left|\frac{1}{y}\right|$

$= \frac{4}{y}\left[\ln(y)\right]^3$

The complete pdf of Y is

$$f_Y(y) = \begin{cases} \frac{4}{y}\left[\ln(y)\right]^3 \ ; \ 1 < y < e \\ 0 \ ; \ else \end{cases}$$



(b) $Z = \frac{1}{X}$ when $x \to 0$, $z \to \infty$ and when $x = 1$, $z = 1$

$Z = \frac{1}{X}$ is $g(x)$ where g is monotone and decreasing function of x for $0 < x < 1$.

$\Rightarrow g^{-1}(z) = \frac{1}{z}$

$\Rightarrow \frac{d}{dz}\left(g^{-1}(z)\right) = \frac{d}{dz}\left(\frac{1}{z}\right)$

$= -\frac{1}{z^2}$

Then pdf of Z is $f_Z(z) = f_X\left(g^{-1}(z)\right) * \left|\frac{d}{dz}\left(g^{-1}(z)\right)\right|$

$= 4\left(\frac{1}{z}\right)^3 * \left|-\frac{1}{z^2}\right| = \frac{4}{z^5}$

The complete pdf of z is

$$f_Z(z) = \begin{cases} \frac{4}{z^5} & ; \ 1 < z < \infty \\ 0 & ; \ \text{otherwise} \end{cases}$$

Example 3: Let X be a random variable with pdf

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 & ; \ -1 < x < 1 \\ 0 & ; \ elsewhere \end{cases}$$

(a) Find the pdf of $Y = 3X$.