Extra Credit 1

Problem 1: Consider the joint pdf

$$f(y_1, y_2) = \begin{cases} e^{-y_2}\,;\ 0 < y_1 < y_2 < \infty \\ 0;\ Otherwise \end{cases}$$

(a) Prove that the marginal pdf of Y_1 is

$$f_1(y_1) = \begin{cases} e^{-y_1}\,;\ 0 < y_1 < \infty \\ 0;\ Otherwise \end{cases}$$

And then recognize the **name** of this probability distribution along with the **value(s) of its parameter(s)**.

$$f_1(y_1) = \int_{y_1}^{\infty} e^{-y_2}\,dy_2 = \left(-e^{-y_2}\right)\Big|_{y_1}^{\infty} = e^{-y_1}\ ;\ 0<y_1<\infty \text{ and Zero otherwise.}$$

Y_1 has an Exponential distribution with parameter $=1$.

(b) Prove that the marginal pdf of Y_2 is

$$f_2(y_2) = \begin{cases} y_2 e^{-y_2}\,;\ 0 < y_2 < \infty \\ 0;\ Otherwise \end{cases}$$

And then recognize the **name** of this probability distribution along with the **value(s) of its parameter(s)**.

$$f_2(y_2) = \int_0^{y_2} e^{-y_2}\,dy_1 = e^{-y_2}\,(y_1)\Big|_0^{y_2} = y_2 e^{-y_2}\ ;\ 0<y_2<\infty \text{ and Zero otherwise}$$

Y_2 has a Gamma distribution with parameters $\alpha = 2$ and $\beta = 1$.

Problem 2: Consider the joint pmf

$$p(y_1, y_2) = \begin{cases} \binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2 - y_1)} \left(\dfrac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right);\ y_2 = 0, 1, \dots\ ; y_1 = 0, 1, \dots, y_2 \\ 0\ ;\ otherwise \end{cases}$$

(a) Prove that the marginal pmf of Y_1 is

$$p_1(y_1) = \begin{cases} \dfrac{e^{-\lambda p}\,(\lambda p)^{y_1}}{y_1!}\ ;\ y_1 = 0, 1, 2, \dots \\ 0;\ Otherwise \end{cases}$$

$$p_1(y_1) = \sum_{y_2=y_1}^{\infty} \left[\binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2-y_1)} \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \right]$$

$$= p^{y_1} e^{-\lambda} \sum_{y_2=y_1}^{\infty} \left[\frac{y_2!}{y_1!\,(y_2-y_1)!} (1-p)^{(y_2-y_1)} \frac{\lambda^{y_2}}{y_2!} \right]$$

$$= \frac{p^{y_1} e^{-\lambda}}{y_1!} \sum_{y_2=y_1}^{\infty} \frac{(1-p)^{(y_2-y_1)} \lambda^{y_2}}{(y_2-y_1)!}$$

Let $x = y_2 - y_1$. Then when $y_2 = y_1$, $x = 0$ and when $y_2 \to \infty$, $x \to \infty$.
Further, $y_2 = x + y_1$. Substitute these in the summation.

$$p_1(y_1) = \frac{p^{y_1} e^{-\lambda}}{y_1!} \sum_{x=0}^{\infty} \frac{(1-p)^{x} \lambda^{x+y_1}}{x!}$$

$$= \frac{p^{y_1} e^{-\lambda} \lambda^{y_1}}{y_1!} \underbrace{\sum_{x=0}^{\infty} \frac{[(1-p)\lambda]^{x}}{x!}}_{e^{[(1-p)\lambda]}}$$

because $\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^{z}$

$$= \frac{p^{y_1} e^{-\lambda} \lambda^{y_1} e^{(1-p)\lambda}}{y_1!}$$

$$= \frac{(\lambda p)^{y_1} e^{-\lambda p}}{y_1!} \quad ; y_1 = 0, 1, \dots \text{ and zero otherwise}$$

(b) Prove that the marginal pmf of Y_2 is

$$p_2(y_2) = \begin{cases} \dfrac{e^{-\lambda} (\lambda)^{y_2}}{y_2!} \; ; \; y_2 = 0, 1, 2, \dots \\ 0; \; Otherwise \end{cases}$$

$$p_2(y_2) = \sum_{y_1=0}^{y_2} \left[\binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2-y_1)} \frac{e^{-\lambda} \lambda^{y_2}}{y_2!} \right]$$

$$= \frac{e^{-\lambda} \lambda^{y_2}}{y_2!} \underbrace{\sum_{y_1=0}^{y_2} \left[\binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2-y_1)} \right]}_{1}$$

because $\binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2-y_1)}$ is pmf of Binomial distribution with y_2 and p.

$$= \frac{e^{-\lambda} \lambda^{y_2}}{y_2!} \quad ; y_2 = 0, 1, \dots \text{ and zero otherwise}$$