Again, apply the definition of variance to $E(Y_1|Y_2)$,

$$Var[E(Y_1|Y_2)] = E\left[[E(Y_1|Y_2)]^2\right] - \left[E[E(Y_1|Y_2)]\right]^2 \quad \text{④}$$

Now add equations ③ and ④,

$$E[Var(Y_1|Y_2)] + Var[E(Y_1|Y_2)] = E[E(Y_1^2|Y_2)] - \left[E[E(Y_1|Y_2)]\right]^2$$

$$= E(Y_1^2) - [E(Y_1)]^2 \quad \text{substitute from equation ① and ②}$$

$$= Var(Y_1) \blacksquare$$

If $Y_2 = 5$

Example 9: (Problem 5.153) Suppose the (Y_2) number of eggs laid by a certain insect has a Poisson distribution with mean λ. The probability that any one egg hatches is p. Assume the eggs hatch independently of one another.

What do we know?

Y_2 has a Poisson(λ). Therefore, $E(Y_2) = \lambda$, $Var(Y_2) = \lambda$

Let $Y_1 =$ the number of eggs that hatch. Then $Y_1 | Y_2$ has a Binomial (Y_2, p)

(a) Find the expected value of Y_1, the total number of eggs that hatch. $E(Y_1) = ?$

From Theorem 5.14, $E(Y_1) = E[\underbrace{E(Y_1|Y_2)}_{Y_2 p}]$ $Y_2 p$ because it's the mean of Binomial

$$= E(Y_2 p)$$

$$= pE(Y_2) = p\lambda$$

(b) Find the variance of Y_1.

$$Var(Y_1) = E[\underbrace{Var(Y_1|Y_2)}_{Y_2 p(1-p)}] + Var[\underbrace{E(Y_1|Y_2)}_{Y_2 p}] \quad \text{from Theorem 5.15}$$

because $Y_1|Y_2$ has Binomial (Y_2, p)

$$= E[Y_2 p(1-p)] + Var(Y_2 p)$$

$$= p(1-p)\underbrace{E(Y_2)}_{\lambda} + p^2 \underbrace{Var(Y_2)}_{\lambda}$$

because Y_2 has a Poisson (λ)

$$Var(Y_1) = p(1-p)\lambda + p^2\lambda = \underline{\underline{p\lambda}}$$ ← This is non-negative because $\lambda > 0$, $0 < p < 1$

Recall: variance cannot be negative.

Previously, we found if Y_2 has a Poisson (λ) and $Y_1 | Y_2$ has a Binomial with (Y_2, p), then Y_1 has a Poisson (λp).

Therefore, $E(Y_1) = \lambda p$ and $Var(Y_1) = \lambda p$.

Example 10: (Problem 5.135) A quality control plan randomly select 3 items from the daily production (assumed large) of a certain machine and observe the number of defectives.

The proportion of defectives produced by machines is denoted by Y_1 and it varies from day to day. Y_1 has a uniform distribution on the interval $(0,1)$.

Y_1 has continuous uniform $(0,1)$

Therefore, $E(Y_1) = \frac{0+1}{2} = \frac{1}{2}$ and $Var(Y_1) = \frac{(1-0)^2}{12} = \frac{1}{12}$.

Let Y_2 be the number of defectives among 3 items.

$Y_2 | Y_1$ has a Binomial $(n=3, p=Y_1)$

$\Rightarrow E(Y_2|Y_1) = 3Y_1$ and $Var(Y_2|Y_1) = 3Y_1(1-Y_1)$.

(a) Find the expected number of defectives among 3 samples items. $E(Y_2) = ?$

From Theorem 5.14, $E(Y_2) = E[\underbrace{E(Y_2|Y_1)}_{3Y_1}]$

$$= E(3Y_1)$$
$$= 3E(Y_1)$$
$$= 3\left(\tfrac{1}{2}\right) = \underline{\underline{1.5}}$$

(b) Find the variance and then standard deviation of the number of defectives among 3 samples items.

$Var(Y_2)$ and Standard Deviation (Y_2).

From Theorem 5.15,

$$Var(Y_2) = E[\underbrace{Var(Y_2|Y_1)}_{3Y_1(1-Y_1)}] + Var[\underbrace{E(Y_2|Y_1)}_{3Y_1}]$$

$$= E[3Y_1(1-Y_1)] + Var(3Y_1)$$

$$= 3E\left[Y_1 - Y_1^2\right] + 3^2\, Var(Y_1)$$

$$= 3\left[E(Y_1) - E(Y_1^2)\right] + 9\, Var(Y_1)$$

$$= 3\left[\frac{1}{2} - \frac{1}{3}\right] + 9\left(\frac{1}{12}\right)$$

$$= \underline{\underline{1.25}}$$

Recall:

$$Var(Y_1) = E(Y_1^2) - [E(Y_1)]^2$$

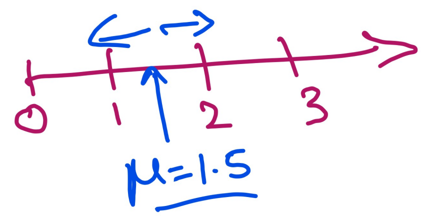
$$\frac{1}{12} = E(Y_1^2) - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow E(Y_1^2) = \frac{1}{12} + \left(\frac{1}{2}\right)^2 = \frac{4}{12} = \frac{1}{3}$$

$$\Rightarrow \text{Standard Deviation}\,(Y_2) = \sqrt{Var(Y_2)}$$

$$= \sqrt{1.25}$$

$$= \underline{\underline{1.12}}$$



<u>Covariance and Correlation (Sections 5.7 & 5.8)</u>

If two random variables are not independent, we are often interested in finding their conditional distributions. However, we are often also interested in **quantifying the strength of the relationship** between them.

For example, we could quantify the strength of the relationship between a randomly chosen person's height and weight or the strength of the relationship between a randomly chosen person's height and GPA.
We would expect the relationship between a randomly chosen person's **height and weight** would be much Stronger than that between a randomly chosen person's **height and GPA**.
Covariance and correlation are two commonly used **numerical measures of a <u>linear</u> relationship between two random variables.**

Positive or Negative

- **Covariance** described **only the <u>nature (or direction)</u> of a <u>linear</u> relationship between two random variables.**
- Pearson's **Correlation** describes **<u>both</u> <u>nature</u> and <u>strength</u> of a <u>linear</u> relationship between two random variables.**

Definition 5.10 and Theorem 5.10: (**Covariance**)
Let Y_1 and Y_2 be jointly distributed random variables such that $E(Y_1) = \mu_{Y_1}$, $Var(Y_1) = \sigma_{Y_1}^2$, $E(Y_2) = \mu_{Y_2}$, and $Var(Y_2) = \sigma_{Y_2}^2$.
The **covariance** between Y_1 and Y_2 is

$$\boldsymbol{Cov\,(Y_1,\ Y_2) = E\left[\left(Y_1 - \mu_{Y_1}\right)\left(Y_2 - \mu_{Y_2}\right)\right] = E(Y_1Y_2) - [\underbrace{E(Y_1)}_{\mu_1}\underbrace{E(Y_2)}_{\mu_2}]}$$

$$= E(Y_1Y_2) - \mu_1\mu_2$$

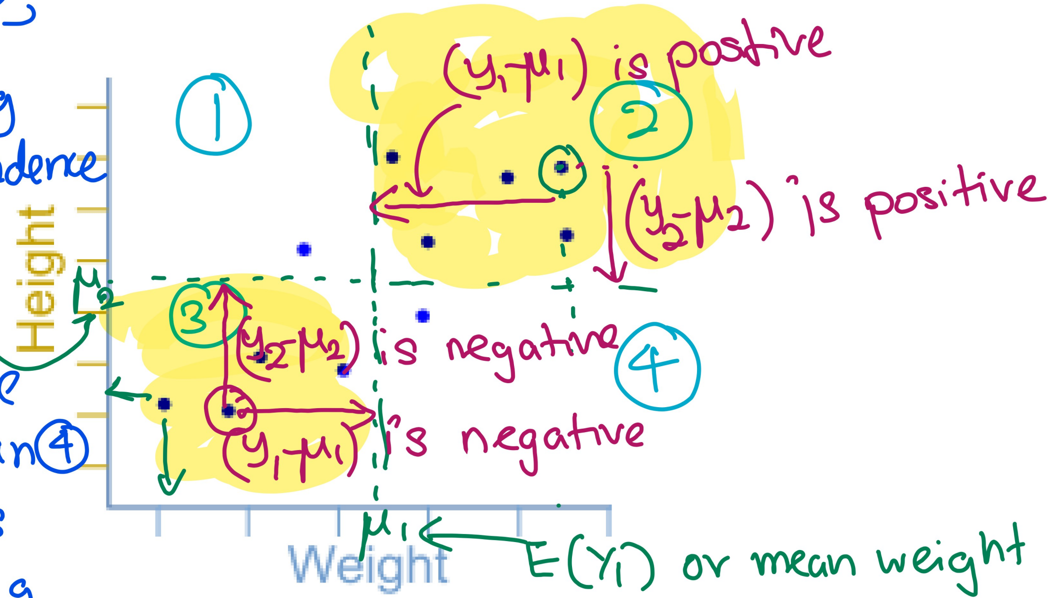
$$E[g(Y_1, Y_2)] = ?$$

 $(y_1-\mu_1)(y_2-\mu_2)$

(*) If more points are in quadrants ② and ③, then Covariance is positive indicating an increasing Linear dependence

$E(Y_2)$ = mean height

(*) If more points are in quadrants ① and ④ then Covariance is negative indicating a decreasing Linear dependence.



Pearson's **Correlation:**

"Rho"

$$Corr\,(Y_1,\ Y_2) = \rho_{Y_1Y_2} = \frac{Cov\,(Y_1,\ Y_2)}{\sigma_{Y_1}\,\sigma_{Y_2}} = \frac{E(Y_1Y_2)-E(Y_1)E(Y_2)}{\sqrt{Var(Y_1)\,Var(Y_2)}}$$

Note: $-1 \le \rho_{Y_1Y_2} \le 1$

Note: $Cov\,(Y_1,\ Y_2) = Cov\,(Y_2,\ Y_1)$**.**

What is $Cov\,(Y_1,\ Y_1)$**?**

Theorem 5.11:
If Y_1 **and** Y_2 are **independent** random variables, then

$$Cov\,(Y_1,\ Y_2) = 0$$

because $E(Y_1Y_2) - [E(Y_1)\,E(Y_2)] = E(Y_1)\,E(Y_2) - [E(Y_1)\,E(Y_2)]$

Thus, **independent random variables must be uncorrelated.**

However, the **converse is not true**. That is, **if the covariance** between two random variables **is zero**, **this does necessarily imply they are independent**.