STAT 461

INTRODUCTION TO MATHEMATICAL STATISTICS I

Fall 2021

Extra Credit 1

A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. The amount of drink is normally distributed with a standard deviation equal to 15 milliliters.

(a) What is the median amount filled by this machine?
(b) What is the probability that a cup contains more than 230 milliliters?
(c) What is the probability that a cup contains between 191 and 209 milliliters?
(d) Below what value do we get the smallest 25% of the drinks?
(e) What is the expected number of cups that will overflow if 230 milliliter cups are used for the next 1000 drinks?

Answers

Let Y is the amount of soft-drink discharged from the machine. Then Y has a $N(\mu = 200, \;\; \sigma = 15)$

a) For a symmetric distribution, **median = mean**. Therefore, median amount filled by this machine is 200 ml.

b) $\mathrm{P(Y > 30) = 1 - P(Y \le 30) = 1 - P\left(Z \le \frac{30-200}{15}\right) =}$

c) $\mathrm{P(191 < Y < 209) = P\left(\frac{191-200}{15} \le Z \le \frac{209-200}{15}\right) =}$

d) $\mathrm{P(Y < k) = 25\% = 0.25}$, what is k?

Answers

e)

There are 1000 cups of 230 ml.

Each cup can overflow or not (two outcomes of interest).

p = P(overflow) = 0.0228   found in part (a)

Let X be the number of overflow cups out of 1000 cups of 230 ml.

Then what is the probability distribution of X?

$$Binomial(n = 1000, \quad p = 0.0228)$$

Find $E(X) = np = 1000 \times 0.0228 = 23$ (round properly to an integer)

Extra credit 2

The operator of a pumping station has observed that demand for water during the early afternoon hours has an approximately exponential distribution with **mean 100cfs (cubic feet per second).**

(a) Find the parameter(s) of the exponential distribution.
(b) What is the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.
(c) What water-pumping capacity should the station maintain during the early afternoons so that the probability that demand will exceed capacity on a randomly selected day is 0.02?

Answers

Let Y be the demand for water.

a) Then $Y \sim Exp(100)$

b) $P(Y > 200) = \int_{200}^{\infty} f(y)\, dy$

$$\boldsymbol{f(y) = \frac{1}{\Gamma(1)100^1} y^{1-1}\, e^{-y/100} = 0.01\, e^{-0.01y} \quad ; \quad y > 0}$$

$$\boldsymbol{P(Y > 200) = \int_{200}^{\infty} 0.01\, e^{-0.01y}\, dy = \left(-e^{-0.01y}\right)\Big|_{200}^{\infty} = e^{-2} \approx 0.1353}$$

Answers

c) **Let capacity is k.**

Then find k so that $P(Y > k) = 0.02$

According to part (b) above,

$$P(Y > k) = e^{-k/100} = 0.02$$

$$-\frac{k}{100} = ln(0.02)$$

$$k = -100 \times ln(0.02) \approx 391.20$$

The station should maintain about 391.20 cfs water capacity during early afternoons to satisfy above requirement.

CHAPTER 5:

BIVARIATE PROBABILITY DISTRIBUTIONS

Introduction

- Often, we may be interested simultaneously in two outcomes of a random experiment rather than one.

- For example, we might be interested in…

a) SAT score and GPA for a student

b) Number of customers waiting in two lines at the grocery store

c) Number of hours spent studying and test score

d) Dosage of a drug and blood pressure

The concept of pmfs/pdfs, CDFs, and independent random variables (discrete/continuous) can be extended to **bivariate** random variables, as well as **n-dimensional** random variables.

Bivariate CDF

- Let Y_1 and Y_2 be two random variables. The joint cumulative distribution function (**bivariate CDF**) of Y_1 and Y_2 is

$$\boxed{\begin{array}{c} \boldsymbol{F_{Y_1,Y_2}(y_1, y_2) = P(Y_1 \le y_1 \;, Y_2 \le y_2)} \,; \\ \boldsymbol{-\infty < y_1 < \infty \,, \qquad -\infty < y_2 < \infty} \end{array}}$$

The function $\boldsymbol{F_{Y_1,Y_2}(y_1, y_2)}$ is a **bivariate CDF** if and only if the following conditions hold:

1. $F_{Y_1,Y_2}(-\infty, -\infty) = F_{Y_1,Y_2}(y_1, -\infty) = F_{Y_1,Y_2}(-\infty, y_2) = 0$ [That is, minimum is zero]

2. $F_{Y_1,Y_2}(\infty, \infty) = 1$ [That is, maximum is 1]

3. If $y_1 \le a$ and $y_2 \le b$, then

$$F_{Y_1,Y_2}(a, b) - F_{Y_1,Y_2}(a, y_2) - F_{Y_1,Y_2}(y_1, b) + F_{Y_1,Y_2}(y_1, y_2) \ge 0$$

because $\boldsymbol{F_{Y_1,Y_2}(a, b) - F_{Y_1,Y_2}(a, y_2) - F_{Y_1,Y_2}(y_1, b) + F_{Y_1,Y_2}(y_1, y_2)}$

$$\boldsymbol{= P(y_1 < Y_1 \le \mathrm{a} \,, y_2 < Y_2 \le \mathrm{b}) \ge 0}$$

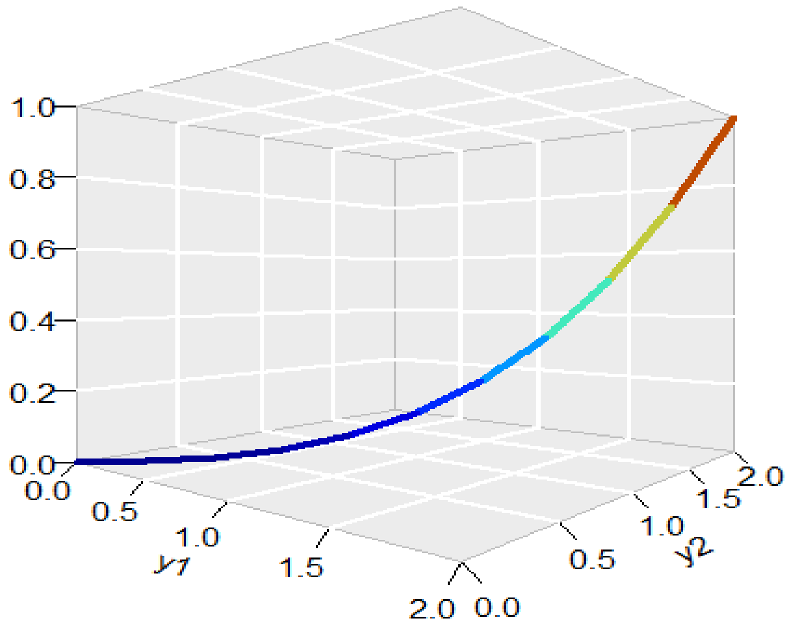
Example

Verify that the following function is a bivariate CDF.

$$F_{Y_1,\,Y_2}(y_1, y_2) = \frac{1}{16} y_1 y_2 \,(y_1 + y_2\,);$$

$$0 \le y_1 \le 2, \qquad 0 \le y_2 \le 2$$

$\boldsymbol{F(y_1, y_2)}$



Answers

- Property 1:

$$\boldsymbol{F_{Y_1,Y_2}(-\infty,-\infty) = F_{Y_1,Y_2}(y_1,-\infty) = F_{Y_1,Y_2}(-\infty,y_2) = 0}$$

$$\begin{aligned}
F_{Y_1,Y_2}(-\infty,-\infty) &= F_{Y_1,Y_2}(0,0) = 0 \\
F_{Y_1,Y_2}(y_1,-\infty) &= F_{Y_1,Y_2}(y_1,0) = 0 \\
F_{Y_1,Y_2}(-\infty,y_2) &= F_{Y_1,Y_2}(0,y_2) = 0
\end{aligned}$$

Answers

- Property 2:

$$\boldsymbol{F_{Y_1, Y_2}(\infty, \infty) = 1}$$

$$F_{Y_1, Y_2}(\infty, \infty) \;=\; F_{Y_1, Y_2}(2,2) = \frac{\mathbf{1}}{\mathbf{16}} \times \mathbf{2} \times \mathbf{2} \times (\mathbf{2} + \mathbf{2}) \;= \mathbf{1}$$

Answers

- Property 3:

$$\boldsymbol{F_{Y_1,Y_2}(a,b) - F_{Y_1,Y_2}(a,y_2) - F_{Y_1,Y_2}(y_1,b) + F_{Y_1,Y_2}(y_1,y_2) \geq 0}$$

if $y_1 \leq a$ and $y_2 \leq b$

Note that here $0 \leq y_1$, $0 \leq y_2$.

Let $y_1 \leq a$ and $y_2 \leq b$.

$$F_{Y_1,Y_2}(a,b) = \frac{1}{16} \times ab(a+b) = \frac{1}{16} \times (a^2b + ab^2)$$

$$F_{Y_1,Y_2}(a,y_2) = \frac{1}{16} \times ay_2(a+y_2) = \frac{1}{16} \times (a^2y_2 + ay_2^2)$$

$$F_{Y_1,Y_2}(y_1,b) = \frac{1}{16} \times by_1(y_1+b) = \frac{1}{16} \times (by_1^2 + b^2y_1)$$

$$F_{Y_1,Y_2}(y_1,y_2) = \frac{1}{16} \times y_1y_2(y_1+y_2) = \frac{1}{16} \times (y_1^2y_2 + y_1y_2^2)$$

Answers

$$F_{Y_1,Y_2}(a,b) - F_{Y_1,Y_2}(a,y_2) - F_{Y_1,Y_2}(y_1,b) + F_{Y_1,Y_2}(y_1,y_2)$$

$$= \frac{1}{16}\left[\, a^2b + ab^2 \quad - \quad a^2y_2 - ay_2^2 \quad - \quad by_1^2 + b^2y_1 \quad + \quad y_1^2y_2 + y_1y_2^2 \,\right]$$

$$= \frac{1}{16}\left[\; a^2(b - y_2) \;+\; b^2(a - y_1) \;-\; y_2^2(a - y_1) \;-\; y_1^2(b - y_2) \;\right]$$

$$= \frac{1}{16}\left[\; (b - y_2)\,\left(a^2 - y_1^2\right) \;+\; (a - y_1)\,\left(b^2 - y_2^2\right) \;\right] \geq 0$$

because $0 \leq y_1 \leq a \quad \Longrightarrow \quad a - y_1 \geq 0$ and $a^2 - y_1^2 \geq 0$ and

$0 \leq y_2 \leq b \quad \Longrightarrow \quad b - y_2 \geq 0$ and $b^2 - y_2^2 \geq 0$

Since all the three conditions hold for the given $F_{Y_1,Y_2}(y_1,y_2)$, it is a valid joint CDF.

Joint PMF

If Y_1 and Y_2 are jointly distributed/bivariate (absolutely) **discrete** random variables, then

$$\boldsymbol{F_{Y_1,Y_2}(y_1,y_2) = P(Y_1 \le y_1 , Y_2 \le y_2) = \sum_{t_1 \le y_1} \sum_{t_2 \le y_2} p_{Y_1,Y_2}(t_1,t_2);}$$

$$\boldsymbol{-\infty < y_1 < \infty , \qquad -\infty < y_2 < \infty}$$

where $\boldsymbol{p_{Y_1,Y_2}(y_1,y_2) = P(Y_1 = y_1 , Y_2 = y_2)}$ is a **joint (bivariate) probability mass function (joint pmf)**.

Joint PMF

Further, the following conditions are hold for joint pmf:

$$0 \le p_{Y_1, Y_2}(y_1, y_2) \le 1$$

$$\textbf{for} \quad -\infty < y_1 < \infty \; , \qquad -\infty < y_2 < \infty$$

$$\sum_{all \;\, y_1} \sum_{all \;\, y_2} p_{Y_1, Y_2}(y_1, y_2) \; = 1$$

Example

A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car.

Let Y_1 denote the *number of luxury cars sold* on a given day and let Y_2 denote the *number of extended warranties sold*.

The joint probability function of Y_1 and Y_2 can be given by the following table or equation.

		Number of warranties ($\boldsymbol{Y_2}$)		
		0	1	2
Number of cars ($\boldsymbol{Y_1}$)	**0**	1/6		
	1	1/12	1/6	
	2	1/12	1/3	1/6

$$p(y_1, y_2) = \begin{cases} 1/6, & (y_1, y_2) = (0,0) \\ 1/12, & (y_1, y_2) = (1,0) \\ 1/6, & (y_1, y_2) = (1,1) \\ 1/12, & (y_1, y_2) = (2,0) \\ 1/3, & (y_1, y_2) = (2,1) \\ 1/6, & (y_1, y_2) = (2,2) \end{cases}$$

Questions

1. Fill in the blanks of the table with the joint probabilities given on the equation.
2. Verify that $\boldsymbol{p_{Y_1, Y_2}(y_1, y_2)}$ is a joint pmf.
3. Find the probability that 2 cars and at least 1 extended warranty are sold in a given day.
4. Find $\boldsymbol{F_{Y_1, Y_2}(1, 1)}$ and interpret this value.
5. What is the probability that the number of cars and warranties sold is same in one day?

Answers

1.

		Number of warranties ($\boldsymbol{Y_2}$)		
		0	1	2
Number of cars ($\boldsymbol{Y_1}$)	**0**	1/6	**0**	**0**
	1	1/12	1/6	**0**
	2	1/12	1/3	1/6

2.

- $\mathbf{0 < p_{Y_1, Y_2}(y_1, y_2) < 1}$ **for all the values of y_1 and y_2, and zero otherwise**
- $\sum_{all\ y_1} \sum_{all\ y_2} p_{Y_1, Y_2}(y_1, y_2) = \frac{1}{6} + \frac{1}{12} + \frac{1}{6} + \frac{1}{12} + \frac{1}{3} + \frac{1}{6} + 0 = 1$

Since both conditions hold for the given $p_{Y_1, Y_2}(y_1, y_2)$, it is a joint pmf.

Answers

3. That is, find $P(Y_1 = 2 \ , Y_2 \geq 1)$

$$P(Y_1 = 2 \ , Y_2 \geq 1) = P(Y_1 = 2 \ , Y_2 = 1) + P(Y_1 = 2 \ , Y_2 = 2)$$

$$= \frac{1}{3} + \frac{1}{6} = 0.5$$

4.

$$F_{Y_1, Y_2}(1, 1) = P(Y_1 \leq 1 \, , Y_2 \leq 1) = \frac{1}{6} + 0 + \frac{1}{12} + \frac{1}{6} = 0.4167$$

There is about 42% chance <u>at most 1 car</u> and <u>at most 1 extended warranty</u> is sold in a given day.

5.

$$P(Y_1 = Y_2) = P(Y_1 = 0 \ , Y_2 = 0) + P(Y_1 = 1 \ , Y_2 = 1) + P(Y_1 = 2 \ , Y_2 = 2)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 0.5$$

Joint PDF

If Y_1 and Y_2 are jointly distributed/bivariate (absolutely) **continuous** random variables, then

$$\boldsymbol{F_{Y_1,Y_2}(y_1,y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f_{Y_1,Y_2}(t_1,t_2)\ dt_1\, dt_2}\,;$$

$$-\infty < \boldsymbol{y_1} < \infty\,, \qquad -\infty < \boldsymbol{y_2} < \infty$$

where $\boldsymbol{f_{Y_1,Y_2}}$ is a **joint (bivariate) probability density function (joint pdf).**

Joint PDF

Further, the following conditions are hold for joint pdf:

$$f_{Y_1,Y_2}(y_1, y_2) \geq 0$$

$$\textbf{for} \quad -\infty < y_1 < \infty \ , \qquad -\infty < y_2 < \infty$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_1\, dy_2 = 1$$ [Total volume under the joint pdf is 1]

Example

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

a) Verify that $f(x, y)$ is a legitimate joint pdf
b) Find the joint CDF for this joint pdf.
c) Find $P\left(0 \le X \le \frac{1}{4}, 0 \le Y \le \frac{1}{4}\right)$

Answer

a) $\int_0^1 \left(\int_0^1 \frac{6}{5}(x+y^2)\,dy \right) dx$

$\therefore f(x,y)$ is a valid pdf

$= \int_0^1 \frac{6}{5}\left[xy + \frac{y^3}{3} \Big|_0^1 \right] dx$

$= \frac{6}{5}\int_0^1 \left[x + \frac{1}{3} \right] dx = \frac{6}{5}\left[\frac{x^2}{2} + \frac{1}{3}x \Big|_0^1 \right]$

$= \frac{6}{5}\left[\frac{1}{2} + \frac{1}{3} \right] = \frac{6}{5}\left[\frac{5}{6} \right] = 1$

Answer

b) $F_{X,Y}(x,y) = \int_0^x \int_0^y \frac{6}{5}\left[t_1 + t_2^2\right] dt_2\, dt_1$

$$= \frac{6}{5}\int_0^x \left[t_1 t_2 \Big|_0^y + \frac{t_2^3}{3}\Big|_0^y\right] dt_1$$

$$= \frac{6}{5}\left[\int_0^x y t_1 + \frac{y^3}{3}\right] dt_1 = \frac{6}{5}\left[y\int_0^x t_1\, dt_1 + \frac{y^3}{3}\int_0^x dt_1\right]$$

$$= \frac{6}{5}\left[\frac{x^2 y}{2} + \frac{x y^3}{3}\right]$$

Answer

$$F_{X,Y}(x,y) = \begin{cases} \frac{6}{5}\left[\frac{x^2 y}{2} + \frac{x y^3}{3}\right] & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Answer

c) $P(0 \le X \le \frac{1}{4}, 0 \le Y \le \frac{1}{4})$

$$= \int_0^{1/4} \int_0^{1/4} \frac{6}{5}(x+y^2)\,dy\,dx$$

$$= \int_0^{1/4} \frac{6}{5}\left[xy + \frac{y^3}{3} \Big|_0^{1/4} \right] dx = \frac{6}{5}\int_0^{1/4} \left[\frac{x}{4} + \frac{1}{192} \right] dx$$

$$= \frac{6}{5}\left[\frac{x^2}{8} + \frac{x}{192} \Big|_0^{1/4} \right] = \frac{7}{640} = 0.01094$$

(5 d.p)

Discrete Marginal distributions

Let Y_1 and Y_2 are jointly distributed/bivariate (absolutely) **discrete** random variables with **joint pmf** $\boldsymbol{p_{Y_1,Y_2}(y_1, y_2)}$**.**

Then the **marginal** probability mass function (pmf) **of** $\boldsymbol{Y_1}$ variable is given by

$$\boxed{\boldsymbol{p_1(y_1) = \sum_{all\ y_2} p_{Y_1,Y_2}(y_1, y_2)}}$$

The **marginal** probability mass function (pmf) of $\boldsymbol{Y_2}$ variable is given by

$$\boxed{\boldsymbol{p_2(y_2) = \sum_{all\ y_1} p_{Y_1,Y_2}(y_1, y_2)}}$$

Example

$p(x,y)$		$y=1$	$y=2$	$y=3$
x	0	.1	0	.15
	1	.2	.2	.05
	2	.05	.15	.1

$$P_X(0) = \sum_y p(x,y) = 0.10 + 0 + 0.15$$
$$= 0.25$$

$$P_X(1) = 0.2 + 0.20 + 0.05$$
$$= 0.45$$

$$P_X(2) = 0.05 + 0.15 + 0.10$$
$$= 0.30$$

x	0	1	2
$p(x)$	0.25	0.45	0.30

What about $p_Y(y)$?

Relationship between Bivariate CDFs and PDFs

- Bivariate CDFs and PDFs have the following relationship:

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \frac{\partial^2\left[F_{Y_1,Y_2}(y_1,y_2)\right]}{\partial y_1 \partial y_2}; \qquad -\infty < y_1 < \infty, \quad -\infty < y_2 < \infty}$$

where f_{Y_1,Y_2} is a **joint (bivariate) probability <u>density</u> function (joint pdf).**

- This means that …

$$\boxed{f_1(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_2}$$

Relationship between Bivariate CDFs and PDFs

- That is, if we **integrate** $\boldsymbol{Y_2}$ continuous variable **out** of the bivariate/joint pdf, we are left with the pdf of the Y_1 variable. This is the **marginal** probability density function of $\boldsymbol{Y_1}$variable**.**

- The **marginal** probability density function of $\boldsymbol{Y_2}$variable is obtained by **integrating** $\boldsymbol{Y_1}$ continuous variable **out** of the bivariate/joint pdf as shown below.

$$\boxed{\boldsymbol{f_2(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_1}}$$

Example

Consider the function

$$\boxed{F_{Y_1,Y_2}(y_1, y_2) = \frac{1}{16} y_1 y_2 (y_1 + y_2) \quad ; \quad 0 \le y_1 \le 2, \qquad 0 \le y_2 \le 2}$$

(a) Find the corresponding joint pdf and verify it holds the conditions for a joint pdf.

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{\partial^2 \left[F_{Y_1,Y_2}(y_1, y_2)\right]}{\partial y_1 \partial y_2} = \frac{\partial^2 \left[\frac{1}{16}\left(y_1^2 y_2 + y_1 y_2^2\right)\right]}{\partial y_1 \partial y_2}$$

$$= \frac{\partial \left[\frac{1}{16}\left(2y_1 y_2 + y_2^2\right)\right]}{\partial y_2} = \frac{1}{16}\left(2y_1 + 2y_2\right)$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{8}\left(y_1 + y_2\right) \quad ; \quad 0 \le y_1 \le 2, \qquad 0 \le y_2 \le 2$$

Example

Verify it holds the condition that $\boldsymbol{f_{Y_1,Y_2}(y_1,y_2) \geq 0}$

Since both y_1 and y_2 are non-negative, $\boldsymbol{f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{8}(y_1 + y_2) \geq 0}$

Verify it holds the condition that $\boldsymbol{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_1\, dy_2 = 1}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_1\, dy_2 = \int_{0}^{2}\int_{0}^{2} \frac{1}{8}(y_1 + y_2)\ dy_1\ dy_2 = 1$$

Both conditions for a joint pdf hold for $\boldsymbol{f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{8}(y_1 + y_2)}$

(b) Find the marginal density function (pdf) of Y_1.

What about Y_2?

$$f_1(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_2 = \int_{0}^{2} \frac{1}{8}(y_1 + y_2)\ dy_2$$

$$f_1(y_1) = \frac{1}{4}(1 + y_1) \quad ; \quad 0 \leq y_1 \leq 2$$

Independence of Random Variables

"How to tell whether Y_1 and Y_2 are independent ? "

Is the support of Y_1 **and** Y_2 are dependent ? $\xrightarrow{\textbf{YES}}$ Y_1 **and** Y_2 **are dependent**

NO $\downarrow$

Is the **joint** CDF, PDF or PMF **a product of Marginals?** $\xrightarrow{\textbf{YES}}$ Y_1 **and** Y_2 **are independent**

NO $\downarrow$

Y_1 **and** Y_2 **are dependent**

Independence of Random Variables

Two random variables, Y_1 and Y_2, are **independent** if and only if

- **Using CDF**:

$$\boxed{\boldsymbol{F_{Y_1,Y_2}(y_1, y_2) = F_1(y_1) \times F_2(y_2)}}$$

where $\boldsymbol{F_1(y_1)}$ and $\boldsymbol{F_2(y_2)}$ **marginal CDFs**.

- **Using PDF**:

$$\boxed{\boldsymbol{f_{Y_1,Y_2}(y_1, y_2) = f_1(y_1) \times f_2(y_2)}}$$

where $\boldsymbol{f_1(y_1)}$ and $\boldsymbol{f_2(y_2)}$ **marginal PDFs**.

Independence of Random Variables

Using PMF:

$$\boxed{\boldsymbol{p_{Y_1,Y_2}(y_1,y_2) = p_1(y_1) \times p_2(y_2)}}$$

where $\boldsymbol{p_1(y_1)}$ and $\boldsymbol{p(y_2)}$ marginal pmfs.

for every pair of real numbers (y_1, y_2). **Be careful with the support!!!**

- If the support is **dependent**, then variables are **NOT independent.**
- Further, we **don't need the marginal CDFs and pmfs/pdfs to show independence**! We can show independence using non-negative functions of the variables themselves.

Example

State if the random variables, Y_1 and Y_2, are independent for the following joint functions:

$$\text{(a)} \quad f_{Y_1,Y_2}(y_1, y_2) = \frac{3}{16} y_1 y_2^2 \quad ; \quad 0 \le y_1 \le 2\,, \qquad 0 \le y_2 \le 2$$

The support of each variable is **independent** of the other and

$$\boldsymbol{f_{Y_1,Y_2}(y_1, y_2)} = \frac{3}{16} y_1 y_2^2 = (\boldsymbol{a y_1}) \times (\boldsymbol{b y_2^2}) \quad \text{so that } \frac{3}{16} = ab$$

The joint pdf can be written as a product of non-negative separate functions of Y_1 and Y_2

Therefore, the two variables are independent.

Example

$$\text{(b) } F_{Y_1,Y_2}(y_1, y_2) = \frac{1}{16} y_1 y_2 (y_1 + y_2) \quad ; \quad 0 \le y_1 \le 2, \qquad 0 \le y_2 \le 2$$

The **support of each variable is independent** of the other **but**

$$\boldsymbol{F_{Y_1,Y_2}(y_1, y_2) = \frac{1}{16} y_1 y_2 (y_1 + y_2) \neq F_1(y_1) \times F_2(y_2)}$$

The joint CDF **cannot** be written as a product of separate functions of Y_1 and Y_2

Therefore, the two variables are NOT independent.