Note on Jan 11

CDF $F(y_1, y_2) = P(Y_1 \le y_1, Y_2 \le y_2)$

$$= \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} f(t_1, t_2)\, dt_1\, dt_2$$

Relationship between Bivariate CDFs and PDFs

- Bivariate CDFs and PDFs have the following relationship:

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = \frac{\partial^2 \left[F_{Y_1,Y_2}(y_1, y_2)\right]}{\partial y_1 \partial y_2}; \quad -\infty < y_1 < \infty, \quad -\infty < y_2 < \infty}$$

where f_{Y_1,Y_2} is a **joint (bivariate) probability density function (joint pdf).**

- This means that …

$$\boxed{f_1(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2}$$

- That is, if we **integrate** Y_2 continuous variable **out** of the bivariate/joint pdf, we are left with the pdf of the Y_1 variable. **This is the marginal** probability density function of Y_1 variable.

- The **marginal** probability density function of Y_2 variable is obtained by **integrating** Y_1 continuous variable **out** of the bivariate/joint pdf as shown below.

$$\boxed{f_2(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_1}$$

Example 1:

$$f(y_1, y_2) = \begin{cases} \frac{6}{5}\,(y_1 + y_2^2); & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0\ ;\ Otherwise \end{cases}$$

Compute and report the marginal pdf of Y_1.

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$$

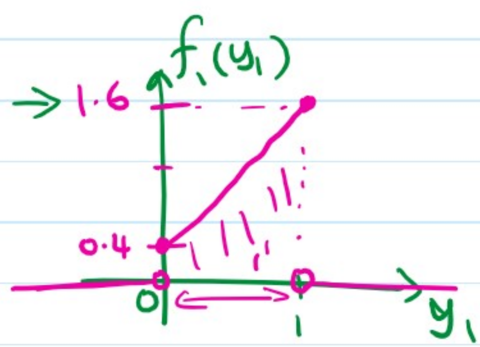
$$= \int_0^1 \frac{6}{5}(y_1 + y_2^2)\, dy_2$$

$$= \frac{6}{5}\left(y_1 y_2 + \frac{y_2^3}{3}\right)\Big|_{y_2=0}^{1}$$

$$= \frac{6}{5}\left(y_1 + \frac{1}{3}\right)$$

The complete pdf of Y_1 is

$$f_1(y_1) = \begin{cases} \frac{6}{5}\left(y_1 + \frac{1}{3}\right) & ;\ 0 \le Y_1 \le 1 \\ 0 & ;\ Otherwise \end{cases}$$



Check if $f_1(y_1)$ is a pdf:

$f_1(y_1) = \frac{6}{5}\left(y_1 + \frac{1}{3}\right) > 0$ when $0 \le y_1 \le 1$ and

$$\int_{-\infty}^{\infty} f_1(y_1)\, dy_1 = \int_0^1 \frac{6}{5}\left(y_1 + \frac{1}{3}\right) dy_1 = \frac{6}{5}\left(\frac{y_1^2}{2} + \frac{y_1}{3}\right)\Big|_0^1 = 1$$

So, $f_1(y_1)$ is a pdf.

Independence of Random Variables

Two random variables, Y_1 and Y_2, are **independent** <u>if and only if</u>

- **Using CDF:**

$$\boxed{F_{Y_1,Y_2}(y_1, y_2) = F_1(y_1) \times F_2(y_2)}$$

where $F_1(y_1)$ and $F_2(y_2)$ **marginal CDFs**.

- **Using PDF:**

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = f_1(y_1) \times f_2(y_2)}$$

where $f_1(y_1)$ and $f_2(y_2)$ **marginal PDFs**.

- If the support is **dependent**, then variables are **NOT independent.**
- Further, we **don't need the marginal CDFs and pmfs/pdfs to show independence**! We can show independence using non-negative functions of the variables themselves.

Example 1 Continued:

$$f(y_1, y_2) = \begin{cases} \frac{6}{5}\left(y_1 + y_2^2\right); & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0 \ ; & Otherwise \end{cases}$$

Are the random variables Y_1 and Y_2 independent according to probability? Explain.

Support of Y_1 is $(0,1)$ while support of Y_2 is $(0,1)$ and they are independent. ✓

However, the joint pdf $f(y_1,y_2) = \frac{6}{5}(y_1+y_2^2)$ cannot be written as a product of non-negative function of only Y_1 and non-negative function of only Y_2.

Therefore, Y_1 and Y_2 are NOT independent.

Another way is to find marginal pdf of Y_1 and marginal pdf of Y_2, and then check if $f(y_1,y_2) = f_1(y_1) \times f_2(y_2)$.

Example 2 : Consider the following function.

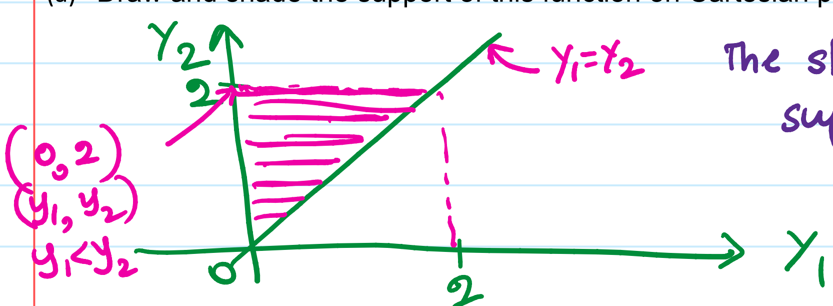
$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 0.3\left(1 - y_1^2 + y_2\right); & 0 \le y_1 \le y_2 \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

$Y_1 \le Y_2$

plot $Y_1 = Y_2$ line

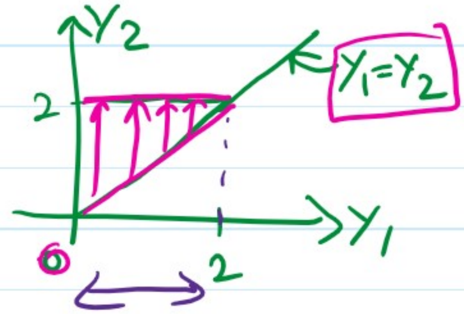
Then find and shade where $Y_1 < Y_2$

(a) Draw and shade the support of this function on Cartesian plane.



The shaded area is the support.

(b) Find and report the marginal pdf of Y_1.



$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1,y_2)\,dy_2$$

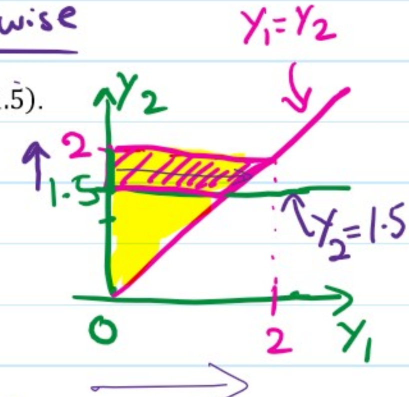
$$= \int_{y_1}^{2} 0.3(1-y_1^2+y_2)\,dy_2$$

$$= 0.3\left(y_2 - y_1^2 y_2 + \frac{y_2^2}{2}\right)\Big|_{y_2=y_1}^{2}$$

$$= 0.3\left((2-2y_1^2+2) - \left(y_1 - y_1^3 + \frac{y_1^2}{2}\right)\right)$$

$$= \begin{cases} 0.3(4-y_1-2.5y_1^2+y_1^3) & ;\ 0<y_1<2 \\ 0 & ;\ \text{otherwise} \end{cases}$$

(c) Find the probability that Y_2 is higher than 1.5. That is, find $P(Y_2 > 1.5)$.



$$P(Y_2>1.5) = P(0<Y_1<Y_2,\ 1.5<Y_2<2)$$

$$= \int_{1.5}^{2}\int_{0}^{y_2} 0.3(1-y_1^2+y_2)\,dy_1\,dy_2$$

$$= \int_{1.5}^{2}\left(0.3\left(y_1 - \frac{y_1^3}{3} + y_1 y_2\right)\Big|_{y_1=0}^{y_2}\right)dy_2$$

$$= \int_{1.5}^{2} 0.3\left(y_2 - \frac{y_2^3}{3} + y_2^2\right)dy_2$$

$$= 0.3\left(\frac{y_2^2}{2} - \frac{y_2^4}{12} + \frac{y_2^3}{3}\right)\Big|_{1.5}^{2}$$

$$= 0.451562$$ Less likely

Another way is, first, find marginal pdf of Y_2 as

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1,y_2)\,dy_1$$

and then do $P(Y_2>1.5) = \int_{1.5}^{2} f_2(y_2)\,dy_2$.

(d) Are the random variables Y_1 and Y_2 independent according to probability? Explain.

$f(y_1, y_2) = 0.3\,(1 - y_1^2 + y_2)$; $0 \le y_1 \le y_2 \le 2$

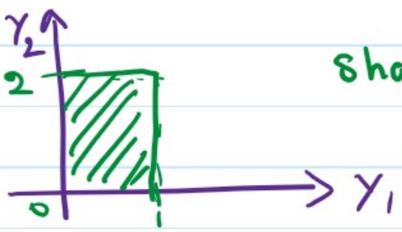
Support of Y_1 is $(0, y_2)$ while support of Y_2 is $(y_1, 2)$ and they are NOT independent.

Therefore, Y_1 and Y_2 are NOT independent.

Example : Consider the following function.

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} c y_1 y_2^2 \; ; \; 0 < y_1 < 1, 0 < y_2 < 2 \\ 0 \; ; \; \text{otherwise} \end{cases}$$

(a) Draw and shade the support of this function on Cartesian plane.



Shaded area is the support

(b) Find the value of c so that $f_{Y_1,Y_2}(y_1, y_2)$ is valid joint pdf.

First, $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1\, dy_2 = \int_0^2 \int_0^1 (c y_1 y_2^2)\, dy_1\, dy_2$$

$$= c\int_0^2 \left(\frac{y_1^2}{2} y_2^2\right)\Big|_{y_1=0}^{1} dy_2$$

$$= \frac{c}{2}\int_0^2 y_2^2\, dy_2$$

$$= \frac{c}{2}\left(\frac{y_2^3}{3}\right)\Big|_0^2$$

$$= \frac{4c}{3}$$

set $\frac{4c}{3} = 1$ and then $c = \frac{3}{4}$.

Then $f(y_1, y_2) = \frac{3}{4} y_1 y_2^2 > 0$ for $0 < y_1 < 1$ and $0 < y_2 < 2$.

Therefore, $c = \frac{3}{4}$ exists and makes $f(y_1, y_2)$ a valid joint pdf.

(c) Find the marginal pdf of Y_2.

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1$$

$$= \int_0^1 \left(\frac{3}{4} y_1 y_2^2\right) dy_1$$

$$= \begin{cases} \frac{3y_2^2}{8} & ; 0 < y_2 < 2 \\ 0 & ; \text{otherwise} \end{cases}$$

If you were given Y_1 and Y_2 are independent, then you can find the joint pdf as

$$f(y_1, y_2) = f_1(y_1) \times f_2(y_2).$$

For example, if Y_1 has an exponential distribution with mean μ_1, Y_2 has an exponential distribution with mean μ_2 and we are given Y_1, Y_2 are independent, then

pdf of Y_1 is $f_1(y_1) = \frac{1}{\mu_1} e^{-\frac{y_1}{\mu_1}}$; $y_1 > 0$

pdf of Y_2 is $f_2(y_2) = \frac{1}{\mu_2} e^{-\frac{y_2}{\mu_2}}$; $y_2 > 0$

The joint pdf is

$$f(y_1, y_2) = \frac{1}{\mu_1} e^{-\left(\frac{y_1}{\mu_1}\right)} \times \frac{1}{\mu_2} e^{-\left(\frac{y_2}{\mu_2}\right)}$$

$$= \begin{cases} \frac{1}{\mu_1 \mu_2} e^{-\left(\frac{y_1}{\mu_1} + \frac{y_2}{\mu_2}\right)} & ; y_1 > 0, y_2 > 0 \\ 0 & ; \text{otherwise} \end{cases}$$