Conditional Expectations (Section 5.11)

Conditional distributions can be used in the same way unconditional distributions are used. For example, conditional distributions may be used to compute conditional expectations.

Definition 5.13: Conditional Expectations
If Y_1 and Y_2 are two random variables and $g(Y_1)$ is a function of Y_1, then the **conditional expectation** of $g(Y_1)$ given that $Y_2 = y_2$ is:

$$E[\,g(Y_1) \mid (Y_2 = y_2)] = \begin{cases} \displaystyle\sum_{all\ y_1} g(y_1) \times p(y_1 \mid y_2) & ;\ \text{if } Y_1 \text{ and } Y_2 \text{ are DISCRETE} \\ \displaystyle\int_{-\infty}^{\infty} g(y_1) \times f(y_1 \mid y_2)\, dy_1 & ;\ \text{if } Y_1 \text{ and } Y_2 \text{ are CONTINUOUS} \end{cases}$$

In particular, the **conditional mean** of Y_1 given that $Y_2 = y_2$ is denoted by and computed using

$$\mu_{Y_1|Y_2} = E(Y_1 \mid Y_2 = y_2)$$

and the **conditional variance** of Y_1 given that $Y_2 = y_2$ is denoted by and computed as

$$Var(Y_1 \mid Y_2 = y_2) = E\left[(Y_1 - \mu_{Y_1|Y_2})^2 \mid Y_2 = y_2\right]$$

Further, the shortcut formula for $Var(Y_1 \mid Y_2 = y_2)$:

$$Var[\,Y_1 \mid Y_2 = y_2\,] = E\left[Y_1^2 \mid Y_2 = y_2\right] - (\mu_{Y_1|Y_2})^2$$

Example 7: Consider the joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = e^{-y_2} \quad ; \quad 0 < y_1 < y_2 < \infty$$

Previously, we showed the conditional distribution of $Y_1 \mid Y_2$ is $f(y_1|y_2) = \begin{cases} \frac{1}{y_2} & ;\ 0<y_1<y_2<\infty \\ 0 & ;\ \text{otherwise} \end{cases}$

Find $E(Y_1 \mid (Y_2 = y_2))$. Here Y_1 is a function only Y_1.

$$E(Y_1 \mid (Y_2 = y_2)) = \int_{-\infty}^{\infty} y_1 f(y_1|y_2)\, dy_1 = \int_0^{y_2} y_1 \left(\frac{1}{y_2}\right) dy_1 = \underline{\underline{\frac{y_2}{2}}}$$

$$\Rightarrow E(Y_1 \mid Y_2 = 5) = \underline{\underline{\frac{5}{2}}}$$

Let $E(Y_1^2 \mid (Y_2 = y_2)) = \int_{-\infty}^{\infty} y_1^2 f(y_1|y_2)\, dy_1$