If you wanted total volume shipped per week, then it is $27Y_1 + 125Y_2 + 512Y_3$ and this is a linear function of Y_1, Y_2, Y_3 and NOT sum of Y_1, Y_2, Y_3.
Cannot use Theorem 6.2 here.

❖ If $Y_i \sim$ **Poisson**(λ_i) for $i = 1, 2, \dots, n$ and all Y_i **are independent,** then

$\sum_{i=1}^{n} Y_i$ has a Poisson$\left(\sum_{i=1}^{n} \lambda_i\right)$

$\Rightarrow$ Sum of independent Poisson random variables has a Poisson distribution.

Theorem 6.3: pdf of linear function of independent Normal random variables

Let $Y_1, Y_2, \dots, Y_n$ be **independent** random variables each with a **normal** distribution, $E(Y_i) = \mu_i$ and $Var(Y_i) = \sigma_i^2$, for $i = 1,2, \dots n$.

Let $a_1, a_2, \dots, a_n$ be constants.

$$\text{If } U = \sum_{i=1}^{n} a_i Y_i = a_1Y_1 + a_2Y_2 + \dots + a_nY_n, \text{ then}$$

U has a **Normal** distribution with **mean** = $E(U) = \sum_{i=1}^{n} a_i\mu_i$ and **variance** = $Var(U) = \sum_{i=1}^{n} a_i^2\sigma_i^2$

Proof: By definition of mgf, the mgf of U is

$$M_U(t) = E\left[e^{tU}\right]$$
$$= E\left[e^{t\left(\sum_{i=1}^{n} a_iY_i\right)}\right]$$
$$= E\left[e^{t(a_1Y_1)} e^{t(a_2Y_2)} \dots e^{t(a_nY_n)}\right]$$
$$= E\left[e^{t(a_1Y_1)}\right] E\left[e^{t(a_2Y_2)}\right] \dots E\left[e^{t(a_nY_n)}\right]$$ because Y_i are independent
$$= \underbrace{E\left[e^{(ta_1)Y_1}\right]}_{M_{Y_1}(ta_1)} \underbrace{E\left[e^{(ta_2)Y_2}\right]}_{M_{Y_2}(ta_2)} \dots \underbrace{E\left[e^{(ta_n)Y_n}\right]}_{M_{Y_n}(ta_n)}$$

Recall: Mgf of normal (μ, σ) is $M_Y(t) = e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}$

$$\Rightarrow M_U(t) = e^{\left[\mu_1(ta_1)+\frac{1}{2}\sigma_1^2(ta_1)^2\right]} * e^{\left[\mu_2(ta_2)+\frac{1}{2}\sigma_2^2(ta_2)^2\right]} * \cdots * e^{\left[\mu_n(ta_n)+\frac{1}{2}\sigma_n^2(ta_n)^2\right]}$$

$$= e^{\left[t(a_1\mu_1+a_2\mu_2+\cdots+a_n\mu_n)+\frac{1}{2}t^2(a_1^2\sigma_1^2+a_2^2\sigma_2^2+\cdots+a_n^2\sigma_n^2)\right]}$$

$$= e^{\left[t\underbrace{\left(\sum_{i=1}^{n} a_i\mu_i\right)}_{\mu}+\frac{1}{2}t^2\underbrace{\left(\sum_{i=1}^{n} a_i^2\sigma_i^2\right)}_{\sigma^2}\right]}$$

$$= e^{\left(\mu t+\frac{1}{2}\sigma^2 t^2\right)}$$ which is a mgf of normal (μ, σ^2)

$\Rightarrow$ U has a normal distribution with

mean $= \sum_{i=1}^{n} a_i\mu_2$ and variance $= \sum_{i=1} a_i^2\sigma_i^2$

Example 6:

Five automobiles of the same type are to be driven on a 300-mile trip. The first two will use an economy brand of gasoline, and the other three will use a name brand.

Let Y_1, Y_2, Y_3, Y_4, and Y_5 be the observed fuel efficiencies (mpg) for the five cars.

Suppose these variables are independent and normally distributed with

$$\mu_1 = \mu_2 = 20, \quad \mu_3 = \mu_4 = \mu_5 = 21, \quad \sigma_1 = \sigma_2 = 2 \text{ and } \sigma_3 = \sigma_4 = \sigma_5 = 1.2$$

U is a measure of the difference in efficiency between economy gas and name-brand gas and defined as

$$U = \frac{Y_1 + Y_2}{2} - \frac{Y_3 + Y_4 + Y_5}{3} = \underbrace{\frac{1}{2}}_{a_1}Y_1 + \underbrace{\frac{1}{2}}_{a_2}Y_2 \underbrace{-\frac{1}{3}}_{a_3}Y_3 \underbrace{-\frac{1}{3}}_{a_4}Y_4 \underbrace{-\frac{1}{3}}_{a_5}Y_5$$

Find the pdf of U and then compute $P(U \le 0)$ and $P(-1 \le U \le 1)$.

Since U is a Linear function of independent normal random variables, From Theorem 6.3, U has a normal distribution with

mean $= \sum_{i=1}^{n} a_i\mu_i = \frac{1}{2}(20)+\frac{1}{2}(20)+\left(-\frac{1}{3}\right)(21)+\left(-\frac{1}{3}\right)(21)+\left(-\frac{1}{3}\right)(21)$

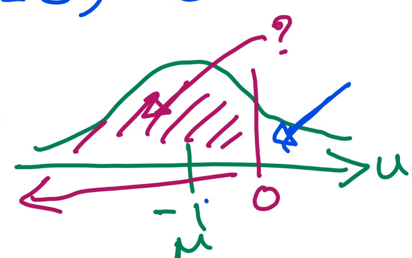
$= -1$

Variance $= \sum_{i=1}^{n} a_i^2\sigma_i^2 = \left(\frac{1}{2}\right)^2(2^2)+\left(\frac{1}{2}\right)^2(2^2)+\left(-\frac{1}{3}\right)^2(1.2)^2+\left(-\frac{1}{3}\right)^2(1.2)^2+\left(-\frac{1}{3}\right)^2(1.2)^2$

$= 2.48$.

Therefore, pdf of U is $f(u) = \frac{1}{\sqrt{2\pi}\sqrt{2.48}} e^{-\frac{1}{2}\left[\frac{(u-(-1))^2}{2.48}\right]}$; $-\infty < u < \infty$

$P(U \le 0) = P(U < 0)$ because $P(U=0) = 0$.

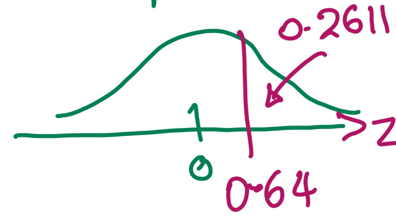


If using the standard normal table (Table 4 in your textbook), first compute z for 0.

work

$Z = \frac{X-\mu}{\sigma} = \frac{0-(-1)}{\sqrt{2.48}} = 0.64$ (round to 2)

From table 4, $P(Z > 0.64) = 0.2611$



However, $P(U < 0) = P(Z < 0.64)$

$= 1 - P(Z > 0.64)$

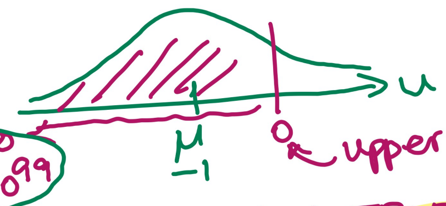
$= 1 - 0.2611$

$= \underline{\underline{0.7389}}$ or $\underline{73.89}$% Likely

On Ti calculator, use

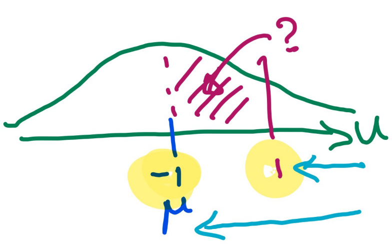
work → $\boxed{\text{normalcdf}(-10^{99}, 0, -1, \sqrt{2.48})}$

and get $\underline{\underline{0.737286}}$.



In R or RStudio, use pnorm(0,-1,sqrt(2.48),lower.tail=TRUE)

$P(-1 < U < 1) = ?$ On Ti calculator use normalcdf$(-1, 1, -1, \sqrt{2.48})$ and get $\underline{\underline{0.397958}}$.



If using Table 4, first compute z for each value here. Since $\mu = -1$,

z for -1 is $z = \frac{-1-(-1)}{\sqrt{2.48}} = 0$ and

z for 1 is $z = \frac{1-(-1)}{\sqrt{2.48}} = 1.27$ (round to 2 decimal)

$P(-1 < U < 1) = P(0 < Z < 1.27)$

$= \underbrace{P(Z > 0)}_{0.5} - \underbrace{P(Z > 1.27)}_{0.1020}$ from Table 4

$= \underline{\underline{0.398 \text{ or } 39.8}}$ unlikely.