**Section 6.6: Bivariate Transformations using Jacobians**

Often, we are interested in find the probability distribution of a function of two or more random variables and/or the joint distribution of functions of multivariate random variables.

For example, suppose two random variables $Y_1$ and $Y_2$ have joint pdf/pmf $f_{Y_1,Y_2}(y_1, y_2)$.

We are sometimes interested in a new bivariate random vector $(U, V)$ defined by $U = g_1(Y_1, Y_2)$ and $V = g_2(Y_1, Y_2)$, and we want to find either the CDF/pdf/pmf of $U$ or $V$, or the joint pdf/pmf of $(U, V)$.

In such situations, we can extend the methods used in the univariate case to the bivariate (or multivariate) case.

**Example:**

A shipping company handles containers in three different sizes:
(1) 27 ft$^3$, (2) 125 ft$^3$, and (3) 512 ft$^3$.
Let $Y_i$ $(i = 1, 2, 3)$ denote the number of type $i$ containers shipped during a given week.
Total volume shipped = $27Y_1 + 125Y_2 + 512Y_3 = g(y_1, y_2, y_3)$

**Example:**

A gas station sells three grades of gasoline: regular, extra, and super. These are priced at \$3.00, \$3.20, and \$3.40 per gallon, respectively. Let $Y_1$, $Y_2$, and $Y_3$ denote the amounts of these grades purchased (gallons) on a particular day.
Suppose the $Y_i$ are independent and each $Y_i$ has pdf of $f(y_i)$.
The revenue from sales is $U = 3Y_1 + 3.2Y_2 + 3.4Y_3 = g(y_1, y_2, y_3)$

Find the probability that revenue exceeds 4500. That is, find $P(U > 4500) = ?$
Need to know pdf of $U$.

**Example:**

Five automobiles of the same type are to be driven on a 300-mile trip. The first two will use an economy brand of gasoline, and the other three will use a name brand.
Let $Y_1$, $Y_2$, $Y_3$, $Y_4$, and $Y_5$ be the observed fuel efficiencies (mpg) for the five cars.

Suppose these variables are independent and normally distributed with
$\mu_1 = \mu_2 = 20$, $\mu_3 = \mu_4 = \mu_5 = 21$, $\sigma_1 = \sigma_2 = 2$ and $\sigma_3 = \sigma_4 = \sigma_5 = 1.1$.
$U$ is a measure of the difference in efficiency between economy gas and name-brand gas and defined as

$$U = \frac{Y_1 + Y_2}{2} - \frac{Y_3 + Y_4 + Y_5}{3}$$

Compute $P(U \leq 0)$ and $P(-1 \leq U \leq 1)$. Need to know pdf of $U$.

**Example:**

Suppose your waiting time for a bus in the morning is denoted by $Y_1$ and uniformly distributed on [0, 8] minutes, whereas waiting time in the evening is denoted by $Y_2$ and uniformly distributed on [0, 10] minutes independent of morning waiting time.

a. If you take the bus each morning and evening for a week, what is your total expected waiting time? $U = 7(Y_1 + Y_2) = g(y_1, y_2)$

b. What is the variance of your total waiting time per week? $Var(U) = ?$

c. What is the probability that the difference between morning and evening waiting times on a given day is less than 5 minutes? $P(|Y_1 - Y_2| < 5) = ?$

Need to know pdf of $Y_1 - Y_2$.

Now, suppose $Y_1$ and $Y_2$ are (absolutely) **continuous** random variables. If there is a one-to-one transformation from the support of $(Y_1, Y_2)$ to the support of $(U_1, U_2)$, the **Jacobian** method discussed for the univariate case can be extended to find the joint distribution of $(U_1, U_2)$ in the bivariate case.

**Bivariate Transformations for <u>Continuous</u> Random Variables using Jacobian:**

Suppose $Y_1$ and $Y_2$ are (absolutely) continuous random variables with joint pdf $f_{Y_1,Y_2}(y_1, y_2)$.

Let $U_1 = h_1(Y_1, Y_2)$

Let $U_2 = h_2(Y_1, Y_2)$

These are two different functions of $Y_1$ and $Y_2$.

Suppose the transformation pair $u_1 = h_1(y_1, y_2)$ and $u_2 = h_2(y_1, y_2)$ is **one-to-one.**

Then, for each $(u_1, u_2)$ in the support of $(U_1, U_2)$,

$y_1 = h_1^{-1}(u_1, u_2)$

and $y_2 = h_2^{-1}(u_1, u_2)$

These are functions of $u_1$ and $u_2$. These are found by solving $u_1 = h_1(y_1, y_2)$ and $u_2 = h_2(y_1, y_2)$ for $y_1$ and $y_2$.

If $y_1$ and $y_2$ have continuous partial derivatives with respect to $u_1$ and $u_2$, and **absolute value of the Jacobian**:

$|J|$

$$|\,J\,| = \left| \; determinant \begin{bmatrix} \dfrac{\partial h_1^{-1}(u_1,u_2)}{\partial u_1} & \dfrac{\partial h_1^{-1}(u_1,u_2)}{\partial u_2} \\ \dfrac{\partial h_2^{-1}(u_1,u_2)}{\partial u_1} & \dfrac{\partial h_2^{-1}(u_1,u_2)}{\partial u_2} \end{bmatrix} \right|$$

$$= \left| \; det \begin{bmatrix} \dfrac{\partial y_1}{\partial u_1} & \dfrac{\partial y_1}{\partial u_2} \\ \dfrac{\partial y_2}{\partial u_1} & \dfrac{\partial y_2}{\partial u_2} \end{bmatrix} \right|$$

because $y_1 = h_1^{-1}(u_1, u_2)$ and $y_2 = h_2^{-1}(u_1, u_2)$.

$$= \left| \left( \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_1} \times \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_2} \right) - \left( \frac{\partial h_1^{-1}(u_1,u_2)}{\partial u_2} \times \frac{\partial h_2^{-1}(u_1,u_2)}{\partial u_1} \right) \right|$$

$$= \left| \left( \frac{\partial y_1}{\partial u_1} \times \frac{\partial y_2}{\partial u_2} \right) - \left( \frac{\partial y_1}{\partial u_2} \times \frac{\partial y_2}{\partial u_1} \right) \right| \neq 0$$

Recall: $det \begin{vmatrix} a & b \\ c & b \end{vmatrix} = (ab) - (bc)$

then the joint pdf of $U_1$ and $U_2$ is

$$f_{U_1,U_2}(u_1,u_2) = \underbrace{f_{Y_1,Y_2}(\ h_1^{-1}(u_1,u_2),\quad h_2^{-1}(u_1,u_2)\ )}_{\text{The joint pdf of } y_1, y_2 \text{ evaluated at } y_1 = h_1^{-1}(u_1,u_2) \text{ and } y_2 = h_2^{-1}(u_1,u_2)} \times \underbrace{|J|}_{\text{Absolute Value of Jacobian.}}$$

**Example 1:** Let $Y_1$ and $Y_2$ be independent random variables with $Y_1 \sim N(0,1)$ and $Y_2 \sim N(0,1)$.

Let $U_1 = \dfrac{Y_1+Y_2}{2}$ and $U_2 = \dfrac{Y_2-Y_1}{2}$

(a) Find joint pdf of $U_1$ and $U_2$.

(b) Find the marginal pdf of $U_1$

(a) $Y_1 \sim N(0,1) \Rightarrow$ pdf of $Y_1$ is $f_1(y_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}}$ ; $-\infty < y_1 < \infty$

$Y_2 \sim N(0,1) \Rightarrow$ pdf of $Y_2$ is $f_2(y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}}$ ; $-\infty < y_2 < \infty$

Since $Y_1$ and $Y_2$ are independent, the joint pdf of $Y_1$ and $Y_2$ is,

$$f(y_1,y_2) = f_1(y_1)\, f_2(y_2) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}}\right)$$

$$= \frac{1}{2\pi} e^{-\left(\frac{y_1^2+y_2^2}{2}\right)} \quad ; -\infty < y_1 < \infty, \ -\infty < y_2 < \infty$$

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When $-\infty < y_1 < \infty$ and $-\infty < y_2 < \infty$,

$-\infty < u_1 < \infty$ and $-\infty < u_2 < \infty$.

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Solve $u_1 = \frac{y_1+y_2}{2}$ and $u_2 = \frac{y_2-y_1}{2}$ for $y_1$ and $y_2$.

$$\begin{cases} u_1 - u_2 = \frac{y_1+y_2}{2} - \frac{y_2-y_1}{2} = \frac{y_1}{2} + \frac{y_2}{2} - \left(\frac{y_2}{2} - \frac{y_1}{2}\right) = \frac{y_1}{2} + \frac{y_1}{2} = y_1 \\ u_1 + u_2 = \frac{y_1+y_2}{2} + \frac{y_2-y_1}{2} = \frac{y_1}{2} + \frac{y_2}{2} + \left(\frac{y_2}{2} - \frac{y_1}{2}\right) = \frac{y_2}{2} + \frac{y_2}{2} = y_2 \end{cases}$$

$$\begin{cases} y_1 = u_1 - u_2 = h_1^{-1}(u_1,u_2) \\ y_2 = u_1 + u_2 = h_2^{-1}(u_1,u_2) \end{cases}$$

Then the Jacobian is

$$J = \det\begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial (u_1 - u_2)}{\partial u_1} & \frac{\partial (u_1 - u_2)}{\partial u_2} \\ \frac{\partial (u_1 + u_2)}{\partial u_1} & \frac{\partial (u_1 + u_2)}{\partial u_2} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = (1 \times 1) - (-1 \times 1) = 2$$