## Conditional Expectations (Section 5.11)

Conditional distributions can be used in the same way unconditional distributions are used. For example, conditional distributions may be used to compute conditional expectations.

**Definition 5.13**: **Conditional Expectations**
If $Y_1$ and $Y_2$ are two random variables and $g(Y_1)$ is a function of $Y_1$, then the **conditional expectation** of $g(Y_1)$ given that $Y_2 = y_2$ is:

$$E[\,\boldsymbol{g(Y_1)}|\ \ \boldsymbol{Y_2 = y_2}\ ] = \begin{cases} \displaystyle\sum_{\boldsymbol{all\ y_1}} \boldsymbol{g(y_1)} \times \boldsymbol{p(y_1|\,y_2)} & ;\ \textbf{if } Y_1 \text{ and } Y_2 \textbf{ are DISCRETE} \\ \displaystyle\int_{-\infty}^{\infty} \boldsymbol{g(y_1)} \times \boldsymbol{f(y_1|\,y_2)\,dy_1} & ;\ \textbf{if } Y_1 \text{ and } Y_2 \textbf{ are CONTINUOUS} \end{cases}$$

In particular, the **conditional mean** of $\boldsymbol{Y_1}$ given that $\boldsymbol{Y_2} = \boldsymbol{y_2}$ is denoted by and computed using

$$\boldsymbol{\mu_{Y_1|\,Y_2}} \quad = \quad \boldsymbol{E(\,Y_1|\ \ Y_2 = y_2\ \ )}$$

and the **conditional variance** of $\boldsymbol{Y_1}$ given that $\boldsymbol{Y_2} = \boldsymbol{y_2}$ is denoted by and computed as

$$\boldsymbol{Var(\,Y_1|\ \ Y_2 = y_2\ \ )} \ = \ \boldsymbol{E\left[\left(Y_1 - \mu_{Y_1|\,Y_2}\right)^2 |\ Y_2 = y_2\right]}$$

Further, the shortcut formula for $\boldsymbol{Var(\,Y_1|\ \ Y_2 = y_2\ )}$**:**

$$\boldsymbol{Var[\ \ Y_1\ \ |\ \ Y_2 = y_2\ \ ] = E\left[\,Y_1^2\ \ |\ \ Y_2 = y_2\ \ \right] - \left(\mu_{Y_1|\,Y_2}\right)^2}$$

**Example 7**: Consider the joint pdf

$$f_{Y_1,Y_2}(y_1,y_2) \ = \ e^{-y_2} \quad ; \quad 0 < y_1 < y_2 < \infty$$

Previously, we showed the conditional distribution of $Y_1 \mid Y_2$ is

Find $E(\,Y_1 \mid\ Y_2 = y_2\ )$.

**Example 6 Contd.:** Let $Y_1$ denote the number of luxury cars sold on a given day and let $Y_2$ denote the number of extended warranties sold.
Find the expected number of warranties sold if only 1 car was sold on a given day. That is, find

In example 6, we found that conditional pmf of $Y_2|Y_1 = 1$ is given by

**Example 8:** Let $Y_1$ and $Y_2$ have the following joint pmf.

$$p_{Y_1,Y_2}(y_1, y_2) = \frac{y_1 + y_2}{32} \quad ; \; y_1 = 1, 2 \; ; \; y_2 = 1, 2, 3, 4$$

Find $E(Y_2^2 \mid Y_1)$.

Conditional expectations lend themselves to other useful results, such as finding the expected values and variances of univariate random variables.

**Theorem 5.14:**
If $Y_1$ and $Y_2$ are two random variables, then

$$\boldsymbol{E(Y_1) = E[\ E(\ Y_1 \mid Y_2\ )\ ]} \qquad \textbf{and} \qquad \boldsymbol{E(Y_2) = E[\ E(\ Y_2 \mid Y_1\ )\ ]}$$

**Proof:**
Let $Y_1$ and $Y_2$ be jointly distributed **discrete** random variables.

**Example 7 Continued:** Consider the joint pdf

$$f_{Y_1,Y_2}(y_1, y_2) = e^{-y_2} \quad ; \quad 0 < y_1 < y_2 < \infty$$

Previously, we mentioned that $f_2(y_2) =$

Also, we showed, $E(\ Y_1 |\ Y_2 = y_2\ ) =$

Compute $E(\ Y_1)$ using $E(\ Y_1 |\ Y_2 = y_2\ )$.

**Example 5 Continued**: Consider the joint pmf

$$p_{Y_1,Y_2}(y_1, y_2) = \binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2 - y_1)} \left(\frac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right) \; ; y_2 = 0, 1, \dots ; \; y_1 = 0, 1, \dots, y_2$$

Find $E(Y_1)$ using $E[\ E(Y_1 \mid Y_2)\ ]$?

**Theorem 5.15:**

If $Y_1$ and $Y_2$ are two random variables, then

$$\boldsymbol{E\,[\,Var(\,Y_1\,|\,Y_2\,)\,] \quad + \quad Var[\,E(\,Y_1\,|\,Y_2\,)] \quad = \quad Var(\,Y_1\,)}$$

**Proof:**

**Example 9**: (Problem 5.153) Suppose the ($Y_1$) number of eggs laid by a certain insect has a Poisson distribution with mean $\lambda$. The probability that any one egg hatches is $p$. Assume the eggs hatch independently of one another.
*What do we know?*

Let $Y_2 =$ the number of eggs that hatch. Then $Y_2 \mid Y_1$

(a) Find the expected value of $Y_2$, the total number of eggs that hatch.

(b) Find the variance of $Y_2$.

**Example 10**: (Problem 5.135) A quality control plan randomly select 3 items from the daily production (assumed large) of a certain machine and observe the number of defectives.
The proportion of defectives produced by machines is denoted by $Y_2$ and it varies from day to day. $Y_2$ has a uniform distribution on the interval $(0,1)$.

(a) Find the expected number of defectives among 3 samples items.

(b) Find the variance and then standard deviation of the number of defectives among 3 samples items.

**Covariance and Correlation (Sections 5.7 & 5.8)**

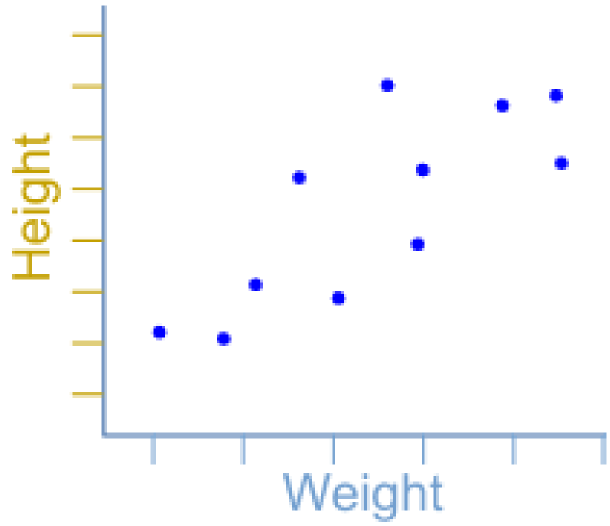
**If two random variables are not independent**, we are often interested in finding their conditional distributions. However, we are often also interested in **quantifying the strength of the relationship** between them.

For example, we could quantify the strength of the relationship between a randomly chosen person's height and weight or the strength of the relationship between a randomly chosen person's height and GPA.
We would expect the relationship between a randomly chosen person's **height and weight** would be much ___________ than that between a randomly chosen person's **height and GPA**.
**Covariance and correlation** are two commonly used **numerical measures of a linear relationship between two random variables**.

- **Covariance** described **only the nature (or direction) of a linear relationship between two random variables**.
- **Correlation** describes **both nature and strength of a linear relationship between two random variables**.

**Definition 5.10 and Theorem 5.10**: (**Covariance**)
Let $Y_1$ and $Y_2$ be jointly distributed random variables such that $E(Y_1) = \mu_{Y_1}$ , $Var(Y_1) = \sigma_{Y_1}^2$ , $E(Y_2) = \mu_{Y_2}$ , and $Var(Y_2) = \sigma_{Y_2}^2$ .
The **covariance** between $Y_1$ and $Y_2$ is

$$\boldsymbol{Cov\,(Y_1\,,\ Y_2) \;=\; E\big[\,(Y_1 - \mu_{Y_1})(Y_2 - \mu_{Y_2})\,\big] \;=\; E(Y_1 Y_2) - [E(Y_1)E(Y_2)]}$$




**Correlation:**

$$\boldsymbol{Corr\,(Y_1\,,\;\; Y_2)\;=\;\rho_{Y_1Y_2}\;=\;\frac{Cov\,(Y_1\,,\;\;Y_2)}{\sigma_{Y_1}\,\sigma_{Y_2}}}$$

**Note:** $\boldsymbol{-1 \leq \;\rho_{Y_1Y_2}\;\leq 1}$

**Note:** $\boldsymbol{Cov\,(Y_1\,,\;\;Y_2) = \;Cov\,(Y_2\,,\;\;Y_1)}$**.**

**What is** $\boldsymbol{Cov\,(Y_1\,,\;\;Y_1)}$**?**

**Theorem 5.11**:

If $\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ are **independent** random variables, then

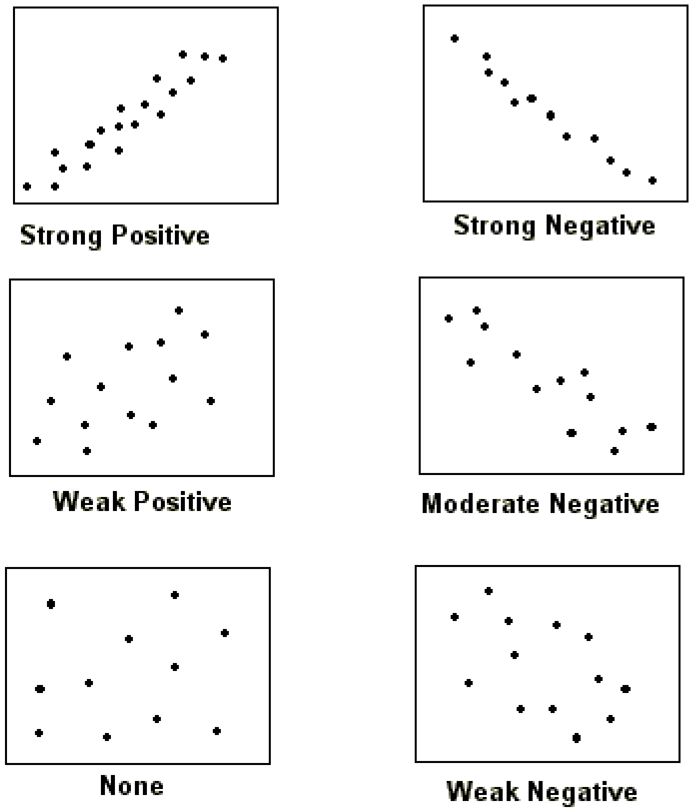
$$\boldsymbol{Cov\,(Y_1\,,\;\;Y_2)\;\;=\;0}$$

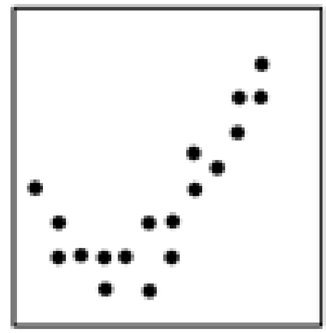
because $\quad E(Y_1Y_2) - [E(Y_1)\,E(Y_2)] \;\;=\;\; E(Y_1)\,E(Y_2) \;\;-\;\; [E(Y_1)\,E(Y_2)]$

Thus, **independent random variables must be uncorrelated**.

However, the **converse is not true**. That is, **if the covariance** between two random variables **is zero**, **this does necessarily imply they are independent**.

Degree of Correlation



Covariance is also useful when calculating the variance of linear functions of random variables.

**Theorem 5.12**:
If $Y_1$ and $Y_2$ are jointly distributed random variables, then

$$\boldsymbol{E(aY_1 + bY_2) = a\,E(Y_1) + b\,E(Y_2)}$$

$$\boldsymbol{Var(aY_1 + bY_2) = a^2\,Var(Y_1) + b^2\,Var(Y_2) + 2ab\,Cov\,(Y_1,\ Y_2)}$$

If $\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ are **independent** random variables, then

$$\boldsymbol{Var(aY_1 + bY_2) = a^2\,Var(Y_1) + b^2\,Var(Y_2)}$$

**Proof:**
Suppose $Y_1$ and $Y_2$ are jointly distributed random variables such that $E(Y_1) = \mu_{Y_1}$, $Var(Y_1) = \sigma^2_{Y_1}$, $E(Y_2) = \mu_{Y_2}$, and $Var(Y_2) = \sigma^2_{Y_2}$.

**Example 5 Continued:** Consider the joint pmf

$$p_{Y_1,Y_2}(y_1, y_2) \;=\; \binom{y_2}{y_1} p^{y_1}\,(1-p)^{(y_2-y_1)} \left(\frac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right) \quad ; \quad y_2 = 0, 1, \ldots ; \quad y_1 = 0, 1, \ldots, y_2$$

(a) Find $Corr(Y_1\,, Y_2)$.
(b) Find $E(3Y_1 - 4Y_2)$
(c) Find $Var(3Y_1 - 4Y_2)$

*What do we know?*

Sections 5.5-5.8 and 5.11 of the textbook

**Example 11**: (Problem 5.102) A firm purchases two types of chemicals. Type I chemical costs \$3 per gallon, whereas type II costs \$5 per gallon.
Let $Y_1$ be the number of gallons purchased from type I chemical.
The mean and variance for this are 40 and 4, respectively.
Let $Y_2$ be the number of gallons purchased from type II chemical.
The mean and variance for this are 65 and 8, respectively.
Assume that $Y_1$ and $Y_2$ are independent.
Find the mean and standard deviation of total amount of money spent on the two chemicals.

**Example 12**: (Problem 5.138) Assume that $Y$ denotes the number of bacteria per cubic centimeter in a particular liquid and that $Y$ has a Poisson distribution with parameter $\lambda$ for a given location.
Further, assume that $\lambda$ varies from location to location and has a gamma distribution with parameters $\alpha$ and $\beta$, where $\alpha$ is a positive integer.
If we randomly select a location, what is the
(a) expected number of bacteria per cubic centimeter?
(b) standard deviation of the number of bacteria per cubic centimeter?