Conditional Distributions (Section 5.3)

Often, two random variables are not independent.

For example, a randomly selected person's height is typically related to his/her weight. If we looked at the conditional distribution of a randomly selected person's weight given that we knew a randomly selected person was 63 inches tall, it would be different from the distribution of any randomly selected person's weight. The additional information about height changes the distribution of weight, so these two random variables are not independent.

In such cases, we are often interested in the conditional probability of Y_1 given knowledge that $Y_2 = y_2$.

Recall the definition of conditional probability:

$$\boldsymbol{P(A \mid B)} = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) > 0 \quad \Big| \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

joint probability

This definition can be extended to conditional distributions.

Definitions 5.5 & 5.7: (**Conditional PMF/PDF**)

If Y_1 and Y_2 are bivariate **continuous** variables, the **conditional pdf** of Y_1 given $Y_2 = y_2$ is

$$\boldsymbol{f(y_1 | y_2)} = \frac{\boldsymbol{f(y_1, y_2)}}{\boldsymbol{f_2(y_2)}} \quad , \quad \boldsymbol{f_2(y_2)} > 0$$

If Y_1 and Y_2 are bivariate **discrete** variables, the **conditional pmf** of Y_1 given $Y_2 = y_2$ is

$$\boldsymbol{p(y_1 | y_2)} = \frac{\boldsymbol{p(y_1, y_2)}}{\boldsymbol{p_2(y_2)}} \quad , \quad \boldsymbol{p_2(y_2)} > 0$$

$\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ **are independent** random variables **if and only if**

$$\boldsymbol{f(y_1 | y_2) = f_1(y_1)} \quad \textbf{or} \quad \boldsymbol{p(y_1 | y_2) = p_1(y_1)} \quad \textbf{or}$$
$$\boldsymbol{f(y_2 | y_1) = f_2(y_2)} \quad \textbf{or} \quad \boldsymbol{p(y_2 | y_1) = p_2(y_2)}$$

Example 4: Consider the joint pdf

$$f(y_1, y_2) = \begin{cases} e^{-y_2} & ; \quad 0 < y_1 < y_2 < \infty \\ 0; & Otherwise \end{cases}$$

Y_1, Y_2 are Continuous variables.

We can show the marginal pdfs of Y_1 and Y_2 (left as ~~practice~~ extra credit for you!) are:

$$f_1(y_1) = \begin{cases} e^{-y_1} & ; \; 0 < y_1 < \infty \\ 0 & ; \; otherwise \end{cases} \qquad Y_1 \, ?$$

$$f_2(y_2) = \begin{cases} y_2 e^{-y_2} & ; \; 0 < y_2 < \infty \\ 0 & ; \; otherwise \end{cases} \qquad Y_2 \, ?$$

(a) Are Y_1 and Y_2 independent? NO, because support of Y_1 and support of Y_2 are NOT independent. Further, the product of marginals is NOT equal to the joint pdf.

(b) For $y_2 > 0$, find the conditional pdf of $Y_1|Y_2 = y_2$.

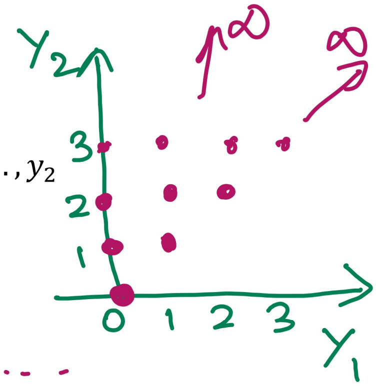
⊛ $f(y_1|y_2)$ must be a function of both y_1, y_2

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{e^{-y_2}}{y_2 e^{-y_2}} = \begin{cases} \frac{1}{y_2} & ; \ 0 < y_1 < y_2 < \infty \\ 0 & ; \ \text{otherwise} \end{cases}$$

(c) Find the conditional pdf of $Y_1|Y_2 = 5$. Take $f(y_1|y_2)$ and plug $y_2 = 5$.

$$f(y_1|(y_2=5)) = \begin{cases} \frac{1}{5} & ; \ 0 < y_1 < 5 \ \cancel{< \infty} \\ 0 & ; \ \text{otherwise} \end{cases}$$

(d) $P(1 < Y_1 < 3 \mid Y_2 = 5) = \int_1^3 f(y_1|y_2=5)\, dy_1 = \int_1^3 \frac{1}{5}\, dy_1 = \underline{\underline{\frac{2}{5}}}$



Example 5: Consider the joint pmf

$$p(y_1, y_2) = \begin{cases} \binom{y_2}{y_1} p^{y_1} (1-p)^{(y_2 - y_1)} \left(\frac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right) & ; \ y_2 = 0, 1, \ldots \ ; \ y_1 = 0, 1, \ldots, y_2 \\ 0 \ ; \ otherwise \end{cases}$$

Y_1, Y_2 are Discrete variables.

$(y_1, y_2) = (0,0), (0,1), (1,1), (0,2), (1,2), (2,2), \ldots$

(a) Are Y_1 and Y_2 independent? NO, because Support of Y_1 and support of Y_2 are NOT independent.

$$\binom{y_2}{y_1} = \frac{y_2!}{y_1!\,(y_2 - y_1)!}$$

We can show that (left as practice for you!)

$$p_1(y_1) = \frac{e^{-(\lambda p)}(\lambda p)^{y_1}}{y_1!} \; ; \; y_1 = 0, 1, \ldots$$ Y_1 has a Poisson with λp

$$p_2(y_2) = \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \; ; \; y_2 = 0, 1, 2, \ldots$$ Y_2 has a Poisson with λ

(b) For $y_2 = 0, 1, \ldots$, find the conditional pmf of $Y_1 | Y_2 = y_2$.

$$p(y_1 | y_2) = \frac{p(y_1, y_2)}{p_2(y_2)}$$

$$= \frac{\binom{y_2}{y_1} p^{y_1}(1-p)^{(y_2-y_1)}\left(\frac{e^{-\lambda}\lambda^{y_2}}{y_2!}\right)}{\frac{e^{-\lambda}\lambda^{y_2}}{y_2!}}$$

$$= \begin{cases} \binom{y_2}{y_1} p^{y_1}(1-p)^{(y_2-y_1)} & ; y_2 = 0, 1, \ldots \; ; y_1 = 0, 1, \ldots, y_2 \\ 0 & ; \text{otherwise} \end{cases}$$

$Y_1 | Y_2$ has a Binomial distribution with $n = y_2$, p.

❖ **If** $Y_2 \sim$ Poisson(λ) **and** $Y_1 | Y_2 = y_2 \sim$ Binomial(y_2, p)
then $Y_1 \sim$ Poisson(λp).

(c) Find $P(Y_1 = 2 \mid Y_2 = 3)$.

$$P(Y_1 = 2 | Y_2 = 3) = p(y_1 = 2 | y_2 = 3)$$

$$= \binom{3}{2} p^2 (1-p)^{(3-2)}$$

$$= \underline{\underline{3p^2(1-p)}}$$

Example 6: Let Y_1 denote the number of luxury cars sold in a given day and let Y_2 denote the number of extended warranties sold. The joint probability function is given below.

		Number of warranties (Y_2)			marginal
		0	**1**	**2**	**Total**
Number of cars (Y_1)	**0**	**1/6**	0	0	$\frac{1}{6}$
	1	**1/12**	**1/6** $\frac{2}{12}$	0	$\frac{3}{12}=\frac{1}{4}$
	2	**1/12**	**1/3**	**1/6**	$\frac{7}{12}$
Total marginal		$\frac{4}{12}=\frac{1}{3}$	$\frac{3}{6}=\frac{1}{2}$	$\frac{1}{6}$	1

Find the conditional pmf of $Y_2|Y_1 = 1$.
This is the pmf of number of warranties sold if only 1 car was sold on a given day.

$$p(y_2|(y_1=1)) = \frac{p(y_1=1, y_2)}{p_1(y_1=1)}$$

$$\text{For } y_2=0,\ p(y_2|y_1=1) = \frac{p(y_1=1, y_2=0)}{p_1(y_1=1)} = \frac{\frac{1}{12}}{\frac{3}{12}} = \frac{1}{3}$$

$$\text{For } y_2=1,\ p(y_2|y_1=1) = \frac{p(y_1=1, y_2=1)}{p_1(y_1=1)} = \frac{\frac{1}{6}}{\frac{3}{12}} = \frac{2}{3}$$

The complete conditional pmf of $Y_2|(Y_1=1)$

$$p(y_2|(y_1=1)) = \begin{cases} \frac{1}{3}; & y_2=0 \\ \frac{2}{3}; & y_2=1 \\ 0; & \text{otherwise} \end{cases}$$