then the joint pdf of U_1 and U_2 is

$$f_{U_1,U_2}(u_1,u_2) = \underbrace{f_{Y_1,Y_2}(\ h_1^{-1}(u_1,u_2),\quad h_2^{-1}(u_1,u_2)\ )}_{\text{The joint pdf of } y_1, y_2 \text{ evaluated at } y_1 = h_1^{-1}(u_1,u_2) \text{ and } y_2 = h_2^{-1}(u_1,u_2)} \times \underbrace{|J|}_{\text{Absolute value of Jacobian.}}$$

Example 1: Let Y_1 and Y_2 be independent random variables with $Y_1 \sim N(0,1)$ and $Y_2 \sim N(0,1)$.

Let $U_1 = \dfrac{Y_1+Y_2}{2}$ and $U_2 = \dfrac{Y_2-Y_1}{2}$

(a) Find joint pdf of U_1 and U_2.

(b) Find the marginal pdf of U_1

(a) $Y_1 \sim N(0,1) \Rightarrow$ pdf of Y_1 is $f_1(y_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}}$; $-\infty < y_1 < \infty$

$Y_2 \sim N(0,1) \Rightarrow$ pdf of Y_2 is $f_2(y_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}}$; $-\infty < y_2 < \infty$

Since Y_1 and Y_2 are independent, the joint pdf of Y_1 and Y_2 is,

$$f(y_1,y_2) = f_1(y_1)\, f_2(y_2) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y_2^2}{2}}\right)$$

$$= \frac{1}{2\pi} e^{-\left(\frac{y_1^2+y_2^2}{2}\right)} \ ; \ -\infty < y_1 < \infty,\ -\infty < y_2 < \infty$$

When $-\infty < y_1 < \infty$ and $-\infty < y_2 < \infty$,
$-\infty < u_1 < \infty$ and $-\infty < u_2 < \infty$.

Solve $u_1 = \frac{y_1+y_2}{2}$ and $u_2 = \frac{y_2-y_1}{2}$ for y_1 and y_2.

$$\begin{cases} u_1 - u_2 = \frac{y_1+y_2}{2} - \frac{y_2-y_1}{2} = \frac{y_1}{2} + \frac{y_2}{2} - \left(\frac{y_2}{2} - \frac{y_1}{2}\right) = \frac{y_1}{2} + \frac{y_1}{2} = y_1 \\ u_1 + u_2 = \frac{y_1+y_2}{2} + \frac{y_2-y_1}{2} = \frac{y_1}{2} + \frac{y_2}{2} + \left(\frac{y_2}{2} - \frac{y_1}{2}\right) = \frac{y_2}{2} + \frac{y_2}{2} = y_2 \end{cases}$$

$$\begin{cases} y_1 = u_1 - u_2 = h_1^{-1}(u_1,u_2) \\ y_2 = u_1 + u_2 = h_2^{-1}(u_1,u_2) \end{cases}$$

Then the Jacobian is

$$J = \det\begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial (u_1-u_2)}{\partial u_1} & \frac{\partial (u_1-u_2)}{\partial u_2} \\ \frac{\partial (u_1+u_2)}{\partial u_1} & \frac{\partial (u_1+u_2)}{\partial u_2} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = (1\times 1) - (-1\times 1) = 2$$

The joint pdf of U_1 and U_2 is

$$f(u_1,u_2) = f_{Y_1,Y_2}\left(y_1 = h_1^{-1}(u_1,u_2),\ y_2 = h_2^{-1}(u_1,u_2)\right) \times |J|$$

$$= \left[\frac{1}{\cancel{2}\pi} e^{-\frac{\left((u_1-u_2)^2+(u_1+u_2)^2\right)}{2}}\right] \times |\cancel{2}|$$

$$= \frac{1}{\pi} e^{-\left[\frac{2u_1^2+2u_2^2}{2}\right]}$$

$$= \frac{1}{\pi} e^{-(u_1^2+u_2^2)} \quad ; \quad -\infty < u_1 < \infty, \ -\infty < u_2 < \infty$$

(b) pdf of U_1 is

$$f(u_1) = \int_{-\infty}^{\infty} f(u_1,u_2)\, du_2$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(u_1^2+u_2^2)}\, du_2$$

$$= \frac{1}{\pi} e^{-u_1^2} \underbrace{\int_{-\infty}^{\infty} e^{-u_2^2}\, du_2}_{\sqrt{\pi}}$$

Rearrange this function as a pdf of a normal distribution to get $\sqrt{\pi}$.

$$= \frac{1}{\sqrt{\pi}} e^{-u_1^2} \quad ; \quad -\infty < u_1 < \infty$$

$$f(u_1) = \frac{1}{\sqrt{\pi}} e^{-u_1^2}$$

$$= \frac{1}{\sqrt{2\pi}\left(\frac{1}{\sqrt{2}}\right)} e^{-\frac{(u_1 - 0)^2}{2\left(\frac{1}{\sqrt{2}}\right)^2}}$$

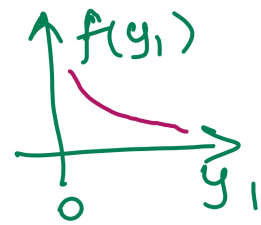
$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(u_1-\mu)^2}{2\sigma^2}}$$

$\Rightarrow U_1$ has a Normal with mean = 0 and standard deviation of $\frac{1}{\sqrt{2}}$.

(*) Are U_1 and U_2 independent? Yes because $f(u_1) \times f(u_2) = f(u_1, u_2)$.

Example 2: Let Y_1 and Y_2 be independent **exponential** random variables, both with **mean** $\beta = 1$.

Let $U_1 = \dfrac{Y_1}{Y_2}$. Find the pdf of U_1.



$Y_1 \sim Exp(\beta = 1) \Rightarrow f_1(y_1) = e^{-y_1}$; $0 < y_1 < \infty$

$Y_2 \sim Exp(\beta = 1) \Rightarrow f_2(y_2) = e^{-y_2}$; $0 < y_2 < \infty$

Since Y_1 and Y_2 are independent, the joint pdf of Y_1 and Y_2 is

$$f(y_1, y_2) = f_1(y_1) f_2(y_2) = \begin{cases} e^{-(y_1 + y_2)} & ; 0 < y_1 < \infty ; 0 < y_2 < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

Note that U_1 is a function of Y_1 and Y_2, and we are NOT given a new 2nd variable. Then let's define a new 2nd variable so that it is easy to compute Jacobian.

Let's define $U_2 = Y_2$. Then $y_2 = u_2$ and

Since $u_1 = \frac{y_1}{y_2} = \frac{y_1}{u_2} \Rightarrow y_1 = u_1 u_2$

When $0 < y_1 < \infty$ and $0 < y_2 < \infty$, $0 < u_2 < \infty$ and $0 < u_1 < \infty$

Then the Jacobian is

$$J = \det\begin{bmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial (u_1 u_2)}{\partial u_1} & \frac{\partial (u_1 u_2)}{\partial u_2} \\ \frac{\partial (u_2)}{\partial u_1} & \frac{\partial (u_2)}{\partial u_2} \end{bmatrix}$$

$$= \det\begin{bmatrix} u_2 & u_1 \\ 0 & 1 \end{bmatrix} = (u_2 \times 1) - (0 \times u_1) = u_2$$

The joint pdf of U_1 and U_2 is

$$f(u_1,u_2) = f_{Y_1,Y_2}\Big(y_1 = \underbrace{h_1^{-1}(u_1,u_2)}_{u_1u_2},\ y_2 = \underbrace{h_2^{-1}(u_1,u_2)}_{u_2}\Big) \times |J|$$

$$= e^{-(u_1u_2 + u_2)} * |u_2|$$

$$= \begin{cases} u_2 e^{-u_2(u_1+1)} & ; 0<u_1<\infty,\ 0<u_2<\infty \\ 0 & ; \text{otherwise} \end{cases}$$

Marginal pdf of U_1 is

$$f(u_1) = \int_{-\infty}^{\infty} f(u_1,u_2)\, du_2$$

$$= \int_0^{\infty} u_2 e^{-u_2(u_1+1)}\, du_2$$

Use integration by parts.

$$= \begin{cases} \frac{1}{(u_1+1)^2} & ; 0<u_1<\infty \\ 0 & ; \text{otherwise} \end{cases}$$

Are U_1 and U_2 independent? No because

$f(u_2) = f(y_2)$ and $f(u_1)f(u_2) = \left[\frac{1}{(u_1+1)^2}\right]\left[e^{-u_2}\right]$

$\neq f(u_1,u_2)$