Note on Jan 6

Joint PDF

If Y_1 and Y_2 are jointly distributed/bivariate (absolutely) **continuous** random variables, then

$$F_{Y_1,Y_2}(y_1,y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f_{Y_1,Y_2}(t_1,t_2)\ dt_1\, dt_2\,;$$

$$-\infty < y_1 < \infty\,, \qquad -\infty < y_2 < \infty$$

where f_{Y_1,Y_2} is a **joint (bivariate) probability density function (joint pdf).**

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Properties of a joint pdf.

Further, the following conditions are hold for joint pdf:

$$f_{Y_1,Y_2}(y_1,y_2) \ge 0$$

$$\text{for} \quad -\infty < y_1 < \infty\,, \quad -\infty < y_2 < \infty$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_1\, dy_2 = 1$$ [Total volume under the joint pdf is 1]

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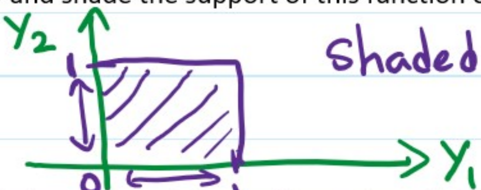
PMF : $0 \le p(y_1,y_2) \le 1$ for Discrete variables.

Note that maximum of a pdf is NOT necessary to be 1.

Example 1:

$$f(y_1,y_2) = \begin{cases} \frac{6}{5}(y_1 + y_2^2) & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0\ ; & Otherwise \end{cases}$$

(a) Draw and shade the support of this function on Cartesian plane.



Shaded area is the support.

(b) Verify that $f(y_1,y_2)$ is a legitimate (or valid) joint pdf.

$f(y_1,y_2) = \frac{6}{5}(y_1 + y_2^2) \ge 0$ for $0 \le y_1 \le 1$ and $0 \le y_2 \le 1$ ✓

Next, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)\, dy_1\, dy_2 = \int_0^1\left(\int_0^1 \frac{6}{5}(y_1+y_2^2)\, dy_1\right) dy_2$

$$= \frac{6}{5}\int_0^1 \left(\frac{y_1^2}{2} + y_2^2 y_1\right)\Big|_{y_1=0}^{1} dy_2$$

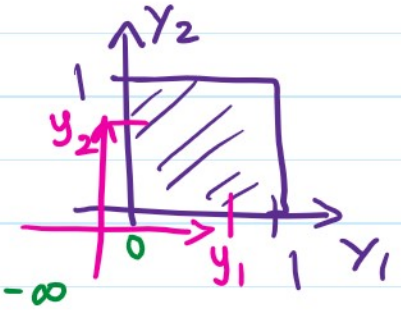
$$= \frac{6}{5}\int_0^1 \left(\frac{1}{2} + y_2^2\right) dy_2$$

$$= \frac{6}{5}\left(\frac{y_2}{2} + \frac{y_2^3}{3}\right)\Big|_{y_2=0}^{1}$$

$$= 1 \checkmark$$

Therefore, this $f(y_1,y_2)$ is a valid joint pdf.

(c) Find the joint CDF for this joint pdf.



$$F(y_1, y_2) = P(Y_1 \le y_1 , Y_2 \le y_2)$$

$$= \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} f(y_1,y_2)\, dt_1\, dt_2$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} \underbrace{f(y_1,y_2)}_{0}\, dt_1 dt_2 + \int_{0}^{y_2}\int_{0}^{y_1} f(y_1,y_2)\, dt_1\, dt_2$$

$$= \int_{0}^{y_2}\int_{0}^{y_1} \frac{6}{5}(t_1+t_2^2)\, dt_1\, dt_2$$

$$= \frac{6}{5}\int_{0}^{y_2} \left(\frac{t_1^2}{2} + t_2^2 t_1\right)\Big|_{t_1=0}^{y_1} dt_2$$

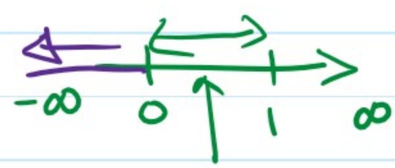
$$= \frac{6}{5}\int_{0}^{y_2} \left(\frac{y_1^2}{2} + t_2^2 y_1\right) dt_2$$

$$= \frac{6}{5}\left(\frac{y_1^2}{2}t_2 + \frac{t_2^3}{3}y_1\right)\Big|_{t_2=0}^{y_2}$$

$$= \frac{6}{5}\left(\frac{y_1^2 y_2}{2} + \frac{y_1 y_2^3}{3}\right)$$

$0 \le Y_1 \le 1$
$0 \le Y_2 \le 1$

Therefore, the complete joint CDF,



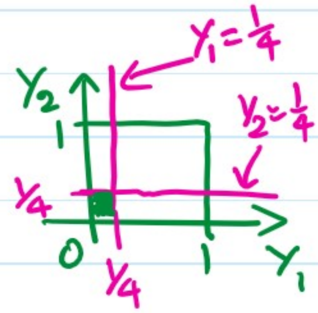
$$F(y_1, y_2) = \begin{cases} 0 \,; & \text{Otherwise} \\ \frac{6}{5}\left(\frac{y_1^2 y_2}{2} + \frac{y_1 y_2^3}{3}\right) ; & 0 \le y_1 < 1 ,\ 0 \le y_2 < 1 \\ 1 \,; & y_1 \ge 1 ,\ y_2 \ge 1 \end{cases}$$

(d) Find the probability that both variables are less than 1/4.

$$P\left(Y_1 < \tfrac{1}{4}, Y_2 < \tfrac{1}{4}\right)$$

$$= P\left(0 < Y_1 < \tfrac{1}{4}, 0 < Y_2 < \tfrac{1}{4}\right)$$

$$= \int_0^{1/4}\int_0^{1/4} \frac{6}{5}\left(y_1 + y_2^2\right) dy_1\, dy_2$$

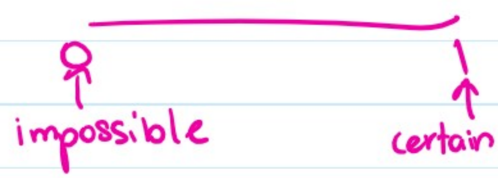


Integrate the joint pdf over the small shaded area.

$$= \frac{6}{5}\int_0^{1/4} \left(\frac{y_1^2}{2} + y_2^2 y_1\right)\Big|_{y_1=0}^{1/4} dy_2$$

$$= \frac{6}{5}\int_0^{1/4} \left(\frac{1}{32} + \frac{y_2^2}{4}\right) dy_2$$

$$= \frac{6}{5}\left(\frac{y_2}{32} + \frac{y_2^3}{12}\right)\Big|_{y_2=0}^{1/4}$$



$$= \boxed{0.0109375}$$ Rare

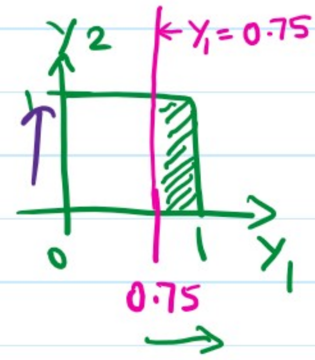
If you were to use joint CDF,

$$P\left(Y_1 \le \tfrac{1}{4}, Y_2 \le \tfrac{1}{4}\right) = F\left(\tfrac{1}{4}, \tfrac{1}{4}\right) = \frac{6}{5}\left[\frac{\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)}{2} + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^3}{3}\right]$$

(d) Find the probability that Y_1 is more than 0.75.

$$P(Y_1 > 0.75) = P(0.75 < Y_1 < 1, 0 < Y_2 < 1)$$

$$= \int_0^1\int_{0.75}^1 \frac{6}{5}\left(y_1 + y_2^2\right) dy_1\, dy_2$$



$$= 0.3675$$

If you were to use the CDF,

$$P(Y_1 > 0.75) = P(0.75 < Y_1 < 1, 0 < Y_2 < 1)$$

$$= F(1, \underbrace{1}_{y_2}) - F(0.75, \underbrace{1}_{y_2})$$