Functions of Random Variables and Their Distributions

Recall: A (univariate) random variable is defined to be a function from a sample space S into the real numbers, $\mathbb{R}$. We've talked **a lot** about distributions of random variables (including bivariate random variables), and we've used these distributions to find probabilities and other quantities like expected values that can be used to make decisions.

Often, we are not interested in the random variable(s) themselves, but instead in functions of the random variable(s), such as the sample mean, sample proportion and sample variance. In these cases, we need to know the probability distribution for the resulting function in order to make decisions. Let's start by looking at some examples.

Section 6.3: The Method of Distribution Function (that is, the CDF learned in Chapter 5)

Example 1: Assume a continuous random variable $X \sim \boldsymbol{Uniform}(0,10)$. We are interested in various functions of X.
For each of the random variables defined below, find the

- Expected Value
- Variance
- pdf
- CDF

If $X \sim \boldsymbol{Uniform}(0, 10)$**,** we know

pdf of $\boldsymbol{X}$ is $f_X(x) =$

Expected value of $\boldsymbol{X}$ is $E(X) =$

Variance of $\boldsymbol{X}$ is $Var(X) =$

CDF of $\boldsymbol{X}$ is $F_X(x) =$

(a) Let $W \ = 2X + 4.$

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(b) Let $T = \frac{X^2}{4} + 1$.

(c) Let $R = -\ln(X)$.

All the previous examples featured monotone, one-to-one functions. **If the function of interest is monotone, we can use the CDF of the original random variable to find the distribution of the transformed random variable.**

Univariate Transformations Using CDFs

Let X have CDF $F_X(x)$, let $Y = g(X)$.

1. If $\boldsymbol{g}$ **is an increasing function on** $\boldsymbol{X}$,

$$\textbf{CDF of } \boldsymbol{Y},\ \boldsymbol{F_Y(y) = P(Y \leq y) = F_X(\, g^{-1}(y)\,)} \quad \textbf{for } \boldsymbol{y} \textbf{ in the support of } \boldsymbol{Y}$$

2. If $\boldsymbol{g}$ **is a decreasing function on** $\boldsymbol{X}$,

$$\textbf{CDF of } \boldsymbol{Y},\ \boldsymbol{F_Y(y) = P(Y \leq y) = 1 - \ F_X(\, g^{-1}(y)\,)} \quad \textbf{for } \boldsymbol{y} \textbf{ in the support of } \boldsymbol{Y}$$

Example 1 Part (b) (revisited): $X \sim Uniform(0{,}10)$, and $T = \frac{X^2}{4} + 1$

Example 1 Part (c) (revisited): $X \sim Uniform(0{,}10)$, and $R = -\ln{(X)}$

In each of the previous examples, we differentiated the CDF with respect to the random variable to get the pdf.

- If $Y = g(X)$ is an **increasing** function of X over the support of X, then

  $F_Y(y) =$

- If $Y = g(X)$ is a **decreasing** function of X over the support of X, then

  $F_Y(y) =$

Section 6.4: The Method of Transformations

Univariate Transformations Using PDFs

Let X have CDF $F_X(x)$ and let $Y = g(X)$, where ***g* is monotone and increasing or decreasing**. Suppose $f_X(x)$ is continuous on the support of X and $g^{-1}(y)$ has continuous derivative on Y. Then the pdf of Y is given by:

$$f_Y(y) = f_X(g^{-1}(y)) \times \left| \frac{d\,(g^{-1}(y))}{dy} \right|$$

where $\frac{d\,[g^{-1}(y)]}{dy}$ is called the **Jacobian** of the transformation.

Example 2: Let X be a random variable with pdf

$$f_X(x) = \begin{cases} 4x^3 & ; \ 0 < x < 1 \\ 0 & ; \ elsewhere \end{cases}$$

Find the pdf for each of the functions of X below.

$(\mathbf{a})$ $Y = e^X$

$(\mathbf{b})$ $Z = \dfrac{1}{X}$

Example 3: Let X be a random variable with pdf

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 \ ; & -1 < x < 1 \\ 0 \ ; & elsewhere \end{cases}$$

(**a) Find the pdf of** $Y = 3X$.

(b) Find the pdf of $Y = X^2$.