Probability

Probs and Stats

- Statistical inference provides estimates and predictions about population's characteristics, based on data from a *sample* of that *population*.
- Probability theory is the main instrument of inferential stats

Random experiments

- Random experiment is an experiment (physical or conceptual) that produces an uncertain outcome called an **event**

Examples of random experiments are: rolling a die, tossing a coin or taking a light bulb of a certain brand and measuring its lifetime, …

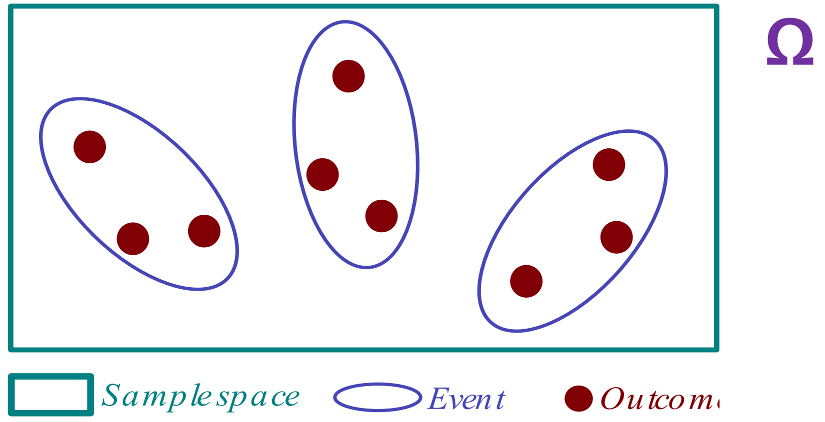
- We assume that an experiment can be replicated in similar conditions and all the possible outcomes are (in principle) known

Sample space and Event

Sample space: is the set of all the possible outcomes of a random experiment and it is denoted by **Ω**

A single outcome of Ω is called **elementary event**

In general an event is a subset of the sample space i.e. $A \subseteq \Omega$



Example 1: Sample Space and Events

Random Experiment: rolling a regular die

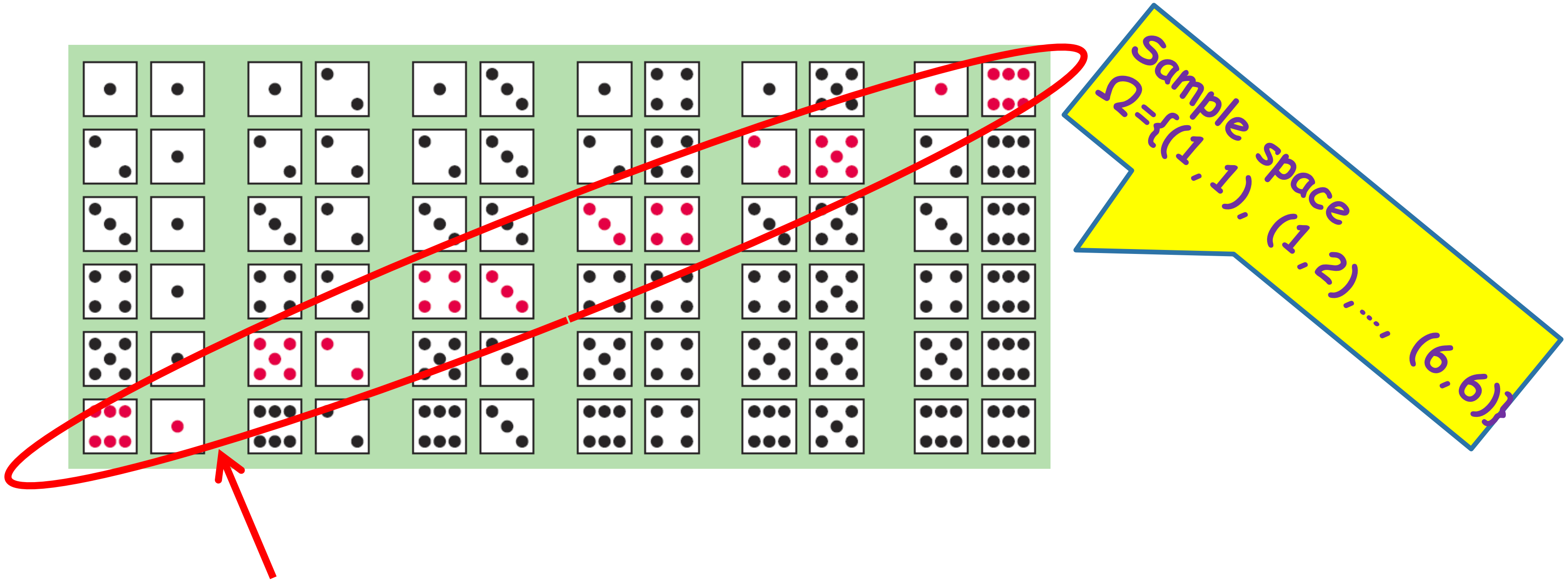
Outcome: a number (colour, letter,…)

Event: to get an even number $A=\{2,4,6\}$

Sample space $\Omega=\{1,2,3,4,5,6\}$

Example 2: Sample Space and Events

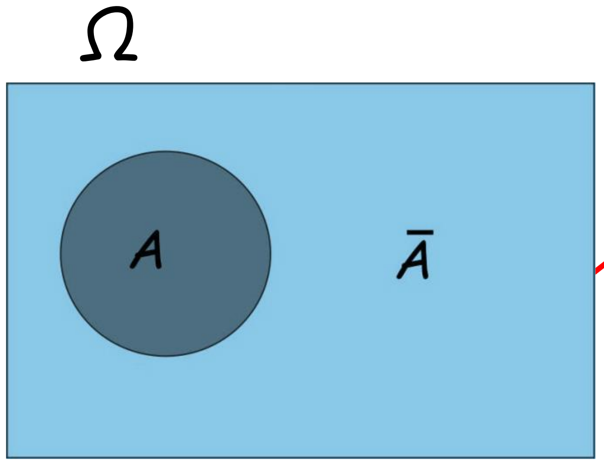
Random Experiment: rolling two regular dice



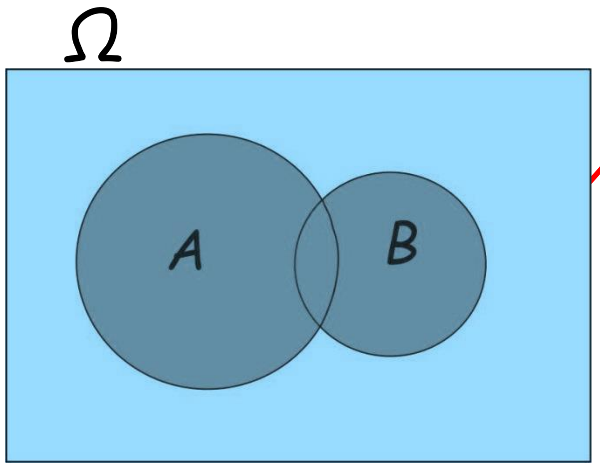
Event: «the sum of the two outcomes is 7"

A **= { (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}**

Set operations

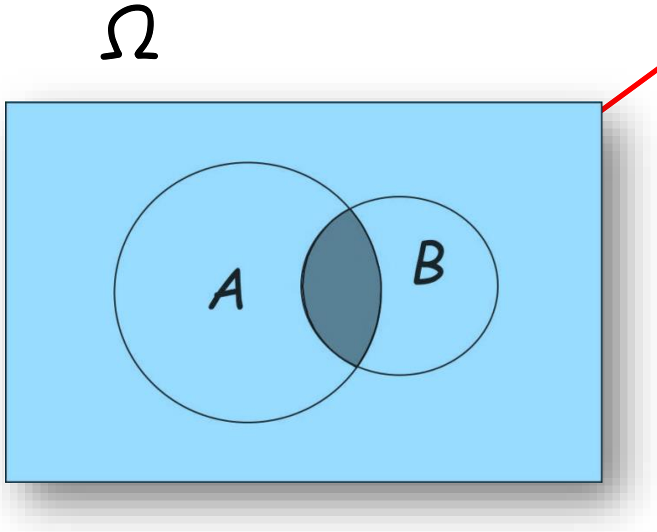


The **complement of *A*** is the set of elements of Ω not in *A*



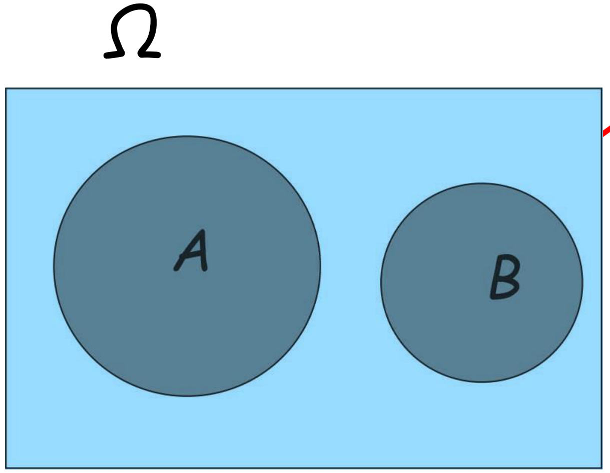
the **union** of *A* and *B*, denoted by ***A* ∪ *B***, is the set of all elements that are members of *A* or of *B* or of both. Hence *A* ∪ *B* occurs if A **or** B occurs

Set operations



The **intersection** of *A* and *B*, denoted by ***A* ∩ *B***, is the set of all events that are members of both *A* and *B.*

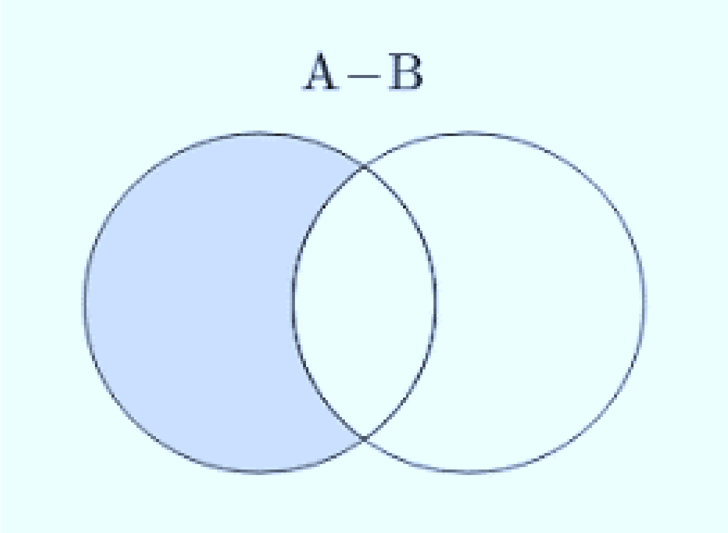
Hence the event A ∩ B occurs if **both** A **and** B occur



If $A \cap B = \emptyset$, then *A* and *B* are said to be **disjoint** or **incompatible events**

Set operations

Ω



The **difference** between *A* and *B*, denoted by ***A* – *B*** or *A* \ *B* *(aka relative complement of A in B)* is the set of elements in A but not in B

Note: $A - B = A \cap \bar{B}$

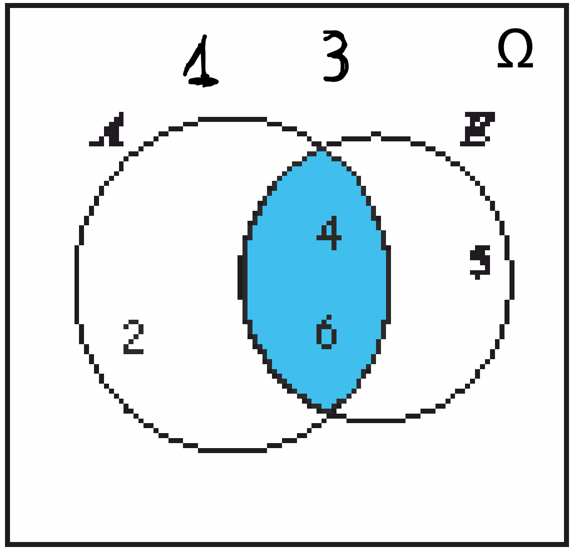
Examples

$\Omega = \{1, 2, 3, 4, 5, 6\}$

A: "the result of rolling a regular die is an even number" ➔ $A = \{2, 4, 6\}$

E: "the result of rolling a regular die is ≥ 4" ➔ $E = \{4, 5, 6\}$

- **COMPLEMENT SET:** $\bar{A} = \{1, 3, 5\}$; $\bar{E} = \{1, 2, 3\}$
- **UNION:** $A \cup E = \{2, 4, 5, 6\}$; $\bar{A} \cup E = \{1, 3, 4, 5, 6\}$
- **INTERSECTION:** $A \cap E = \{4, 6\}$; $\bar{A} \cap E = \{5\}$
- **INCOMPATIBLE (DISJOINT) EVENTS:** $A \cap \bar{A} = \varnothing$; $E \cap \bar{E} = \varnothing$



Probability

Probability is a function $P(\cdot)$ defined on the subsets of Ω such that (axioms)

- $P(\Omega) = 1$;

  the probability that at least one of the elementary events in the entire sample space will occur is 1

- $P(A) \geq 0$, for every A;

  The probability of an event is a non-negative real number

- $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

  for any sequence $A_1, A_2, \ldots$ of disjoint subsets (i.e. $A_i \cap A_j = \emptyset$, *mutually exclusive events*) of Ω (σ- additivity axiom)

Probability: some properties

1. $P(\bar{A}) = 1 - P(A)$, for any A *in* Ω (complement rule)
2. $P(\emptyset) = 0$, $\emptyset = \bar{\Omega}$: *empty set*;
3. $P(A) \leq 1$, for any A *in* Ω
4. $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$, for any A_1, A_2 (sum rule).

These properties follow from probability axioms

Probability: meaning

There are several possible **interpretation** of probability

With a randomized experiment, the probability of a particular outcome is the proportion of times that an outcome would occur **in a very long** sequence of like observations (frequentist definition).

We often think that the probability of an event is the ratio between the number of cases favourable to it, to the number of all cases possible when all these events are equally possible (classical definition). (note that the overall number of cases must be finite in this case).

Probability calculation for a finite sample space

Let N be the number of elementary event in Ω and $p(\omega_{\mathrm{i}}) = 1/N$, for any elementary event ω_i in Ω

$$P(A) = \frac{n(A)}{N}$$

where $n(A)$ is the number of elementary events in A.

In (even moderately) more complex situations combinatorics is necessary to determine N and $n(A)$

Example: probability calculation

$\Omega = \{1, 2, 3, 4, 5, 6\}$ $N = 6$

A: "the result of rolling a regular die is an even number" ➔ $A = \{2, 4, 6\}$ $n(A)=3$

E: "the result of rolling a regular die is ≥ 4" ➔ $E = \{4, 5, 6\}$

We get

- $P(A)= \frac{n(A)}{N} = 3/6 = 0{,}5$
- $P(E) = 3/6 = 0{,}5$
- $P(A \cup B) = P(A) + P(E) - P(A \cap E) = 3/6 + 3/6 - 2/6 = 4/6$ (via the sum rule)

  Otherwise $A \cup E = \{2, 4, 5, 6\}$; ➔ $P(A \cup B) = 4/6$ (using the definition of prob)

Conditional probability

Definition Let A *and* B *be two events of* Ω and $P(A) > 0$, *then the conditional probability* of B *given* A is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad P(A) \neq 0$$

When you condition on event A, you «change» the sample space and also the event B you want to probabilize.

Example: $A = \{2, 4, 6\}$, $E = \{4, 5, 6\}$ ➔ $A|E = \{4, 6\}$, **i.e. if we know that** ***E will have occurred,*** **2 is no more an admissible elementary event. Hence** $P(A|E) = 2/3$.

From the formula above we also have (chain rule) $P(A \cap B) = P(A)P(B|A)$

Example: conditional probability

It is known that in a certain population 70% of the mothers get their first child after age 19 (Older Moms); 60% of the women do not experience hospitalization for problems after birth, either of the mother, of the child or both; and 20% of the older mothers are hospitalized.

Calculate the probability that a Teen Mom will be hospitalized after birth.

	Hospitalization (H)	No Hospitalization (NH)	Overall
Older Moms (O)	**0,2**		**0,7**
Teen Moms (T)			
Overall		**0,6**	

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$$P(H|T) = \frac{P(H \cap T)}{P(T)} = \frac{0{,}2}{0{,}3} = 0{,}67.$$

Among teen mothers 2/3 experience hospitalisation

Independent events

Let $P(A)$ and $P(B)$ be larger than 0, A and B are **independent** if

$$P(B|A) = P(B) \text{ and } P(A|B) = P(A)$$

i.e. knowledge on A does not affect the probability that B occurs and viceversa.

Equivalently

$$P(B \cap A) = P(B) \times P(A)$$

Example: independent events

Moms/Hospitalization	Hospitalization (H)	No Hospitalization (NH)	Overall
Older Moms (O)	**0,2**	0,5	**0,7**
Teen Moms (T)	**0,2**	0,1	**0,3**
Overall	0,4	**0,6**	1

Are "*Teen Moms*" and "Hospitalization" independent from each other?

From the table above we have

$P(T \cap H) = \mathbf{0{,}2}$ $\quad P(T) = \mathbf{0{,}3}$; $\quad P(H) = 0{,}40$ **hence**

$$P(T \cap H) = 0{,}20 \neq P(T)P(H) = 0{,}12.$$

Therefore, being a teen mother (T) and being hospitalized (H) are not independent events