

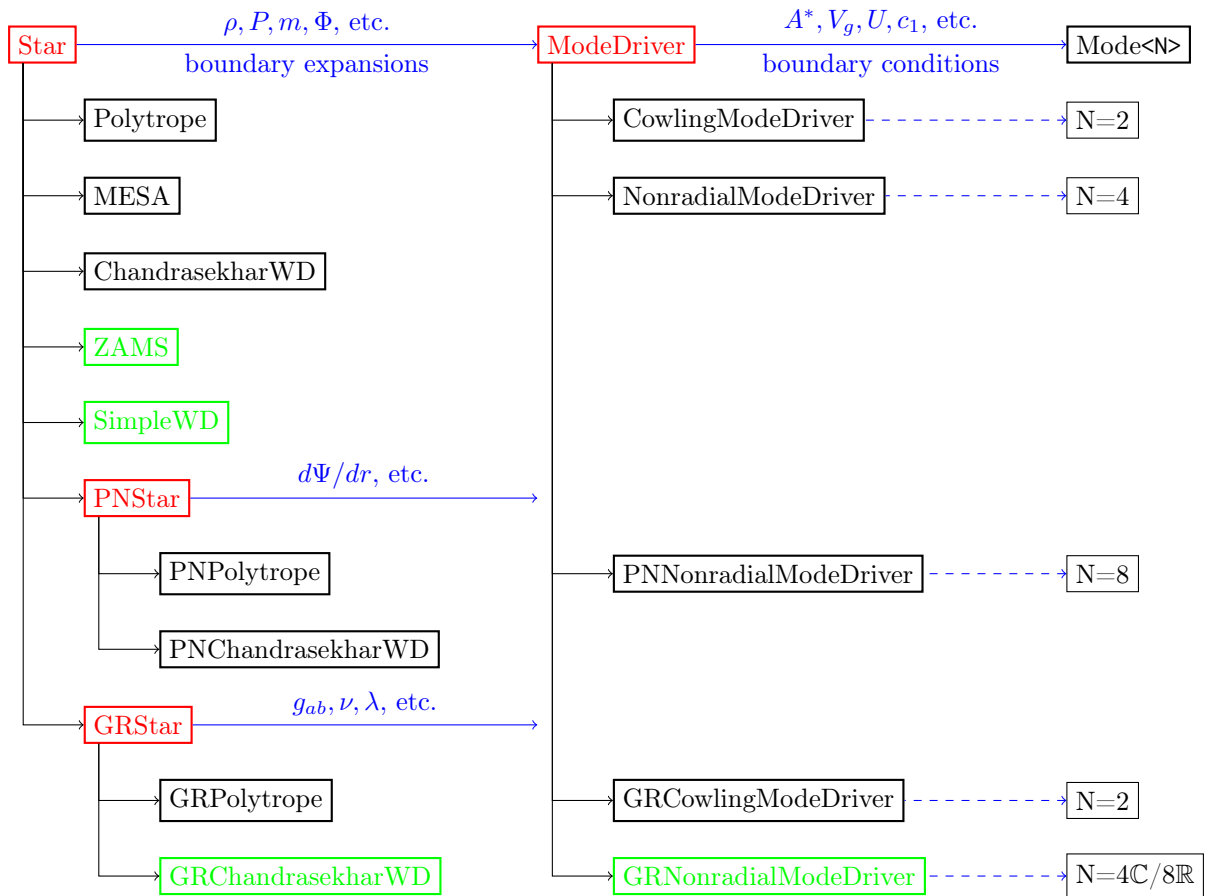
GRPulse Documentation v1.0

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Diagrammatic Table of Contents

Click on a box in the below code diagram to jump to the relevant chapter. Green boxes are not yet included.



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Chapter 1

Introduction

This document contains development notes and user notes for the `GRPulse` program developed as part of my thesis work. This document explains the equations and algorithms used, why certain choices are made, and how to use and further develop the program.

1.1 How to Cite This Program

If you use this program in scientific research, please cite it by the paper introducing it: Boston (2021)

In bibtex

```
@author{Boston2021a,  
author = "{\{Boston\}, S.~R.~ and {\{Clemens\}, J.~C.~}",  
title = "{\{The Relativistic Limit of White Dwarf Asteroseismology\}}",  
year = {2021}  
}
```

1.2 Version 1.0

This documentation was created for release with version 1.0, which was released with the above paper. Later updates will include functionality for realistic simple WD models, and for asteroseismology in full general relativity.

1.3 Structure of Program

The program follows OOP design. Most control of the program is contained within methods attached to classes. The classes act to encapsulate data and methods.

One of the benefits of OOP design is polymorphism; the ability to use different classes interchangeably. This is facilitated by the use of abstract classes. There are three main abstract classes in the code:

- **Star**: Represents an equilibrium model of a star used as background in the wave equations. Described in 3.
- **Mode<N>**: Represents a single *klm*-mode, described by an N-dimensional eigenfunction (e.g. for a Cowling mode N=2). Described in 5.
- **ModeDriver**: Represents a particular form of a stellar wave equation (e.g. Cowling modes, Dziembowski equations, or non-adiabatic waves). Described in 4.

Classes `Star` and `ModeDriver` are abstract, meaning they only provide a template that other classes have to follow. It is up to the children of `Star` and `ModeDriver` to actually implement those methods.

The structure of the code, the polymorphism pattern, and the relationship between the classes, is illustrated in Figure 1.2.

In this document, we separate two facets of the classes, as they relate to the OOP structure:

Interface External-facing elements (data or functions) that specify how outside code can interact with the object.

Implementation The internal logic that calculates all relevant values inside of the object, and cannot be seen by external code.

As an example, all *Stars* have an method `rho()` in their interface, which must return the density ρ ; however, the implementation can vary on how ρ is found, whether by storing ρ in an array, or calculating it from other quantities.

This is illustrated in Figure 1.1.

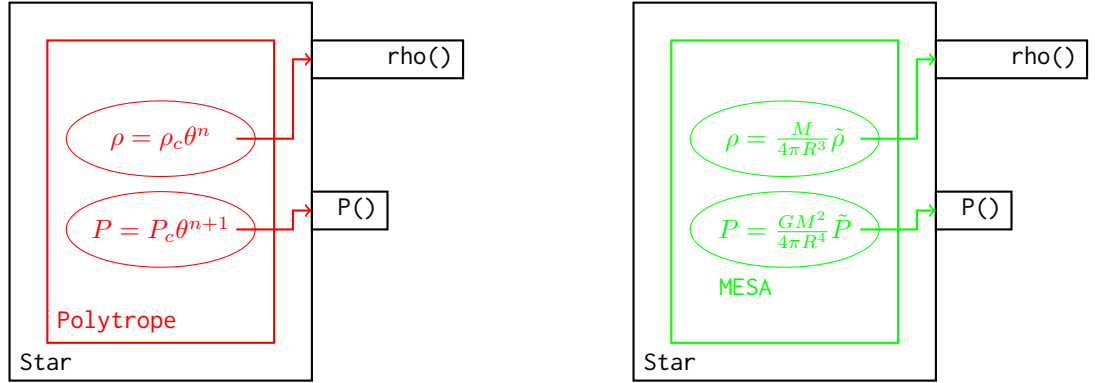


Figure 1.1: An illustration of a class (the boxes labeled *Star*). The interface (the “pipes” `rho()`, `P()`) can connect to external code needing a star’s density and pressure. The internal logic of the implementation (the circles) is illustrated for two *Star* daughters: the *Polytrope* class (red) and the *MESA* class (green). External code cannot access the internal implementation, and does not care what kind of star is returning ρ , P .

1.4 How to Use This Program

Two uses of this code are intended:

1. using the `GRPulse` program to calculate eigenfrequencies of the included stellar models. The I/O is handled by text files and explained in Chapter 2. To run the `GRPulse` program, you must first compile it by running the included make file.
2. as a library of C++ classes for asteroseismology. Several programs in the `testing` suite will show the individual class files being used in this way. This document hopes to explain ways the library can be adapted or extended to other purposes.

1.5 Diagram of Program Structure

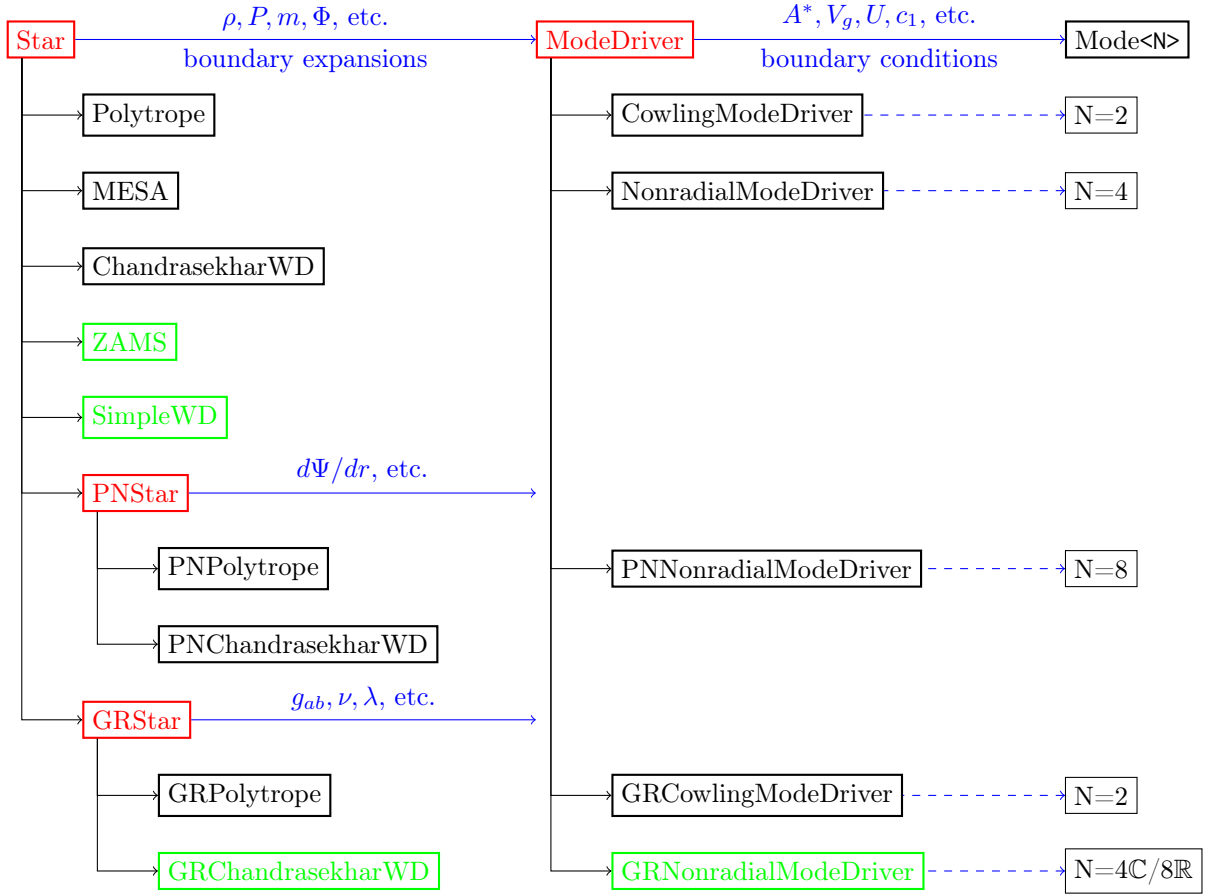


Figure 1.2: Structure of program, showing hierarchies. Red boxes are abstract classes, which provide rules for child classes to follow. Blue lines show friendship. Green planned but unimplemented. Any figure lower in the hierarchy (a child) can be used in place of a parent or grandparent class. Therefore, a ChandrasekharWD can be used anywhere the program calls for a Star.

Chapter 2

Using the command line program

2.1 Installation and Set Up

This program was developed in macOS and originally compiled with `gcc`. If your `gcc` is up-to-date, then the included makefile should prepare everything for you. Linux users should be able to make this work without too much intervention. Windows users should consider installing a real operating system.

To compile, first `cd` to the `lib` directory and type

```
make -f makelib
```

This will create the library. Next `cd` back to main directory and type `make`. That will create the entire program.

If you have persistent problems, check that you have the latest version of `gcc` installed, capable of C++11. If so but there are still problems, please email me.

2.2 Creating an input file

The program includes a UI meant to be run from the command line, called `GRPulse`. User specification for the stellar background and the modes should be placed in the input file. The input file should be structured as follows:

- Line 1: the word `Name:` followed by a keyword to be used when naming files related to this calculation.
- Line 2: the word `Model:` followed by the following specifications, separated by a space:
 - a keyword for the *regime* of physics:
 - * `newtonian` – Newtonian physics
 - * `1pn` – the first post-Newtonian approximation
 - * `gr` – Einstein’s general relativity
 - a keyword for the stellar background model:
 - * `polytrope` – a polytrope (works in all regimes)
 - * `MESA` – a wrapper for a MESA model (only in newtonian)
 - * `CHWD` – a Chandrasekhar WD (only in newtonian, 1pn)
 - model-specific parameters. See Sec 2.3 below.
- Line 3: in **some models**, the word `Params:` followed by two of the following:
 - the word `mass` and a real number indicating the total mass of the star
 - the word `radius` and a real number indicating the total radius of the star

- the word `logg` and a real number indicating the $\log_{10} g$ surface gravity of star
- the word `zsurf` and a real number indicating the surface redshift
- Line 4: in **all models**, the word `Units:` followed by a keyword specifying:
 - CGS, currently the main one working
 - geo, for using geometric units as in GR
 - SI, for using SI units
 - **astro**, for using “astronomical units”, where mass is in M_{\odot} , distance is in km, and time in seconds

Currently, only CGS and **astro** are confirmed to behave properly.

- Line 5: this must be a blank line.
- Line 6: in **all models** the word `Frequencies:` followed by the mode type to use and the adiabatic index to use. Mode type options are:
 - `cowling` (only in newtonian or gr)
 - `nonradial`

The adiabatic index is a number. The special cases $\Gamma_1 = 5/3$ and $\Gamma_1 = 4/3$ can be specified by writing the fractions 5/3 and 4/3. Other indices must be specified in decimal format. To use the adiabatic index $\Gamma_1 = (\partial \log P / \partial \log \rho)_{ad}$ calculated from the stellar background then use 0.

- Line 7: Beginning here and onward, write the mode numbers for each mode you wish to calculate on a separate line. These are specified as ℓ, k , where k counts the nodes and ℓ the angular momentum. Negative k specify g-modes, positive specify p-modes. Note that there is no 1,0 mode, so specifying one can cause the program to hang.

The file should be saved in the same file as `GRPulse`. Assuming `GRPulse` has been properly compiled and the file called `myinput.txt`, then you can run with

```
./GRPulse myinput.txt
```

2.2.1 Comments in Input Files

Comments may be placed in an input file, subject to the following restrictions:

- Comments must begin with #.
- Comments may only be placed **at the top of the file**.
- Blank lines may be used within the comment section, but not after.

You can see this examples of comment use within the sample files.

2.3 Parameters for stellar models

For the following model, on the same line as the model keyword, give the following parameters in order, separated by a space.

- **polytrope**
 1. real number for polytrope index n
 2. integer for number of grid points

The next line should be `Params:` as above.

- CHWD

1. real number for initial value y_o (see ChandrasekharWD)
2. integer for number of grid points

Do **not** include Params: line.

- MESA

1. string with name of dat file (omitting the “.dat”)
2. integer for number of grid points to use – this number is a *suggestion*

All MESA data files must be moved to the GRPulse directory. Do **not** include the Params: line.

2.4 The output

When the calculation begins, GRPulse will create a directory specific to this calculation based on the given name, located in the `output` directory. The first output it makes is an echo of the input file, for re-running the specific calculation. Assuming your calculation was given the name `myname`, this file will be called `myname_in.txt`. The main output file will be called `myname.txt`. This contains data about the star, and summary results for all modes calculated.

The program will also create subdirectories called `modes` and `star`. These contain graphs and data files relevant either to the stellar background or to the modes. The `modes` directory can take up a lot of memory, so if you do not think you will need graphs of every mode, you may want to delete this directory’s contents.

2.5 Sample Input

There are several sample inputs included in the distribution. They perform the following:

sampleinput1.txt Will create results for an $n = 0$ polytrope to compare against Pekeris formula.

sampleinput2.txt Will create the post-newtonian results for $n = 1$ in Boston (2021)

sampleinput3.txt Will create the Newtonian results for $n = 2$ in Boston (2021)

sampleinput4.txt Will create a table which can be compared to results in Christensen-Dalsgaard & Mullan (1994) paper for polytrope eigenmodes.

sampleinput5.txt Shows how to use MESA data files.

Part I

Base Classes

Chapter 3

The Star Abstract Class

This is an abstract class. In any function that requires a `Star` object, any child of `Star` – such as `Polytrope` or `MESA` – may be used without any additional changes to the program.

3.1 The Star Interface

The `Star` class has the following methods for external code to interact with:

3.1.1 Physics

Global stellar properties:

- `double Radius()`: total radius R_\star of the surface of the star.
- `double Mass()`: total mass M_\star of the star.
- `double Gee()`: value of Newton’s constant G in correct units for the star.
- `double light_speed2()`: the squared speed of light in correct units for the star.

Basic stellar properties:

- `double rad(int X)`: radial coordinate r at index X .
- `double rho(int X)`: density $\rho(r)$.
- `double drhodr(int X)`: derivative of density $\frac{d\rho}{dr}(r)$.
- `double P(int X)`: pressure $P(r)$.
- `double dPdr(int X)`: derivative of pressure $\frac{dP}{dr}$.
- `double Phi(int X)`: gravitational potential $\Phi(r)$.
- `double dPhidr(int X)`: gravitational field $g = \frac{d\Phi}{dr}(r)$.
- `double mr(int X)`: interior mass $m(r)$.
- `double sound_speed2(int X, double gam1)`: squared sound speed inside star, $v_s^2 = \frac{\text{gam1}P}{\rho}$.

Stellar properties used in wave equations:

- `double Schwarzschild_A(int X, double gam1)`: the Schwarzschild discriminant A , using $\Gamma_1 = \text{gam1}$.

- `double getAstar(int X, double gam1)`: a dimensionless Schwarzschild discriminant $A^* = -rA$, using $\Gamma_1 = \text{gam1}$.
- `double getU(int X)`: the dimensionless variable $U(r) = \frac{d \ln m(r)}{d \ln r}$.
- `double getVg(int X, double gam1)`: the dimensionless variable $V_g(r) = -\frac{1}{\Gamma_1} \frac{d \ln P}{d \ln r}$ using $\Gamma_1 = \text{gam1}$.
- `double getC(int X)`: the dimensionless variable $c_1 = \frac{M_*}{m(r)} \frac{r^3}{R_*^3}$.
- `double Gamma1(int X)`: the adiabatic exponent $\Gamma_1 = \left(\frac{\partial P}{\partial \rho} \right)_{\text{ad}}$.

3.1.2 Miscellaneous

- `int indexFit`: the index where shoot-to-center algorithms should try to match solutions.
- `int length()`: the maximum index of arrays in the star.
- `double SSR()`: returns a fractional root-mean-square residual of the stellar model when returned to original equilibrium equations.
- `char* name[]`: a C-string holding the star name, used in printing plots and files. Must comply with UNIX directory conventions.
- `void graph_title(char* c)`: replaces `c` with a name to be used in printed graphs of the stellar profile.
- `void writeStar(char *c)`: outputs values of `Star` and plots in gnuplot. The optional argument `c` specifies a directory path where to print the star.
- `void printStar(char *c)`: same as `writeStar()`, but also opens the plot image for easy viewing.

3.1.3 Boundary conditions for modes

- `void getXYZCenter(double*, int& maxPow)`: where XYZ could be $A^* = A^*$, $U = U$, $V_g = V_g$, $C1 = c_1$, with U, V_g, c_1 as above, and $A^* = -rA$. Returns coefficients from power-series expansion of XYZ in $x = r/R$ near the center. The highest power desired is `maxPow`; if the stellar model does not implement that power, then `maxPow` will be lowered to match. The coefficients are returned within the array. [In the future, this should include methods for \$\rho, P, m\$.](#)
- `void getXYZSurface(double*, int& maxPow)`: as above, but coefficients from power-series expansion in $t = 1 - r/R$ near the surface.

3.2 The Star Implementation

This is an abstract class, meaning that (almost) all of its methods are declared `virtual`, so that children classes must specify how all of these functions work. The only methods that `Star` actually defines are:

`writeStar(char* c)` writes out r, ρ, P, Φ as columns in a tab-separated `.txt` file, then plots ρ and P vs r into a `.png` file with gnuplot. The output files are named according to the model description in `name`.

`printStar()` as above, but additionally opens the plot to the screen. The optional argument `c` specifies a directory path where output should be written.

`SSR()` explained in Sec. 3.2.1.

3.2.1 The generic stellar SSR()

One method of quantifying the fitness of a numerical solution is by re-substituting the numerical solution into the original equations and calculating the residual. For a general Newtonian star, the hydrostatic conditions are

$$0 = \frac{dP}{dr} + \rho \frac{d\Phi}{dr} \quad (3.1a)$$

$$\nabla^2 \Phi = 4\pi G \rho. \quad (3.1b)$$

At each location X , we calculate a scaled residual

$$\text{res}_1[X] = \frac{|\text{dPdr}(X) + \rho(X) \cdot \text{dPhidr}(X)|}{|\text{dPdr}(X)| + |\rho(X) \cdot \text{dPhidr}(X)|} \quad (3.2a)$$

$$\text{res}_2[X] = \frac{|4\pi G \cdot \rho(X) \cdot \text{rad}(X) - 2\text{dPhidr}(X) - \text{d2Phi}|}{|4\pi G \cdot \rho(X) \cdot \text{rad}(X)| + |2\text{dPhidr}(X)| + |\text{d2Phi}|} \quad (3.2b)$$

where $\text{d2Phi} = \frac{d^2\Phi}{dr^2}$ and is computed numerically. The SSR is then

$$SSR = \sqrt{\frac{1}{2 \cdot \text{len}} \sum_{X=0}^{\text{len}-1} [\text{r}_1[X]^2 + \text{r}_2[X]^2]}. \quad (3.3)$$

For practicality, the first few points near either border are omitted from the sum. Note that SSR is a misnomer, as this is actually an RMSR (root-mean-square residual).

3.3 How to make new Star children

If you wish to add a new object representing a stellar model to the code, you need to program the above functions. You need a way of determining ρ, P, \dots , and also A, V_g, c_1, U , the total mass and radius M, R , and *find power series expansions* for A^*, V_g, c_1, U near the surface and the center. The easiest method is to copy `Polytrope` and change as necessary. If the star model solves different equilibrium equations as those in (3.1) (such as in rotating stars, or 1PN stars), then the `SSR()` method must be overwritten.

3.4 References

- Christensen-Dalsgaard, J. and D. J. Mullan (1994), in the appendix, for an example of the process described in 3.1.3.
- Tassoul, Fontaine, Winget (1990), in their equations 40, 41 for examples of the SR method of measuring numerical error described in ??.

Chapter 4

The ModeDriver Abstract Class

This is an abstract class, representing a set of wave equations within a star. The equations are assumed to be in the dimensionless linear form

$$x \frac{d}{dx} \vec{Y} = \mathbf{A} \vec{Y}, \quad (4.1)$$

where $x = r/R$ is a radial variable, \vec{Y} is the normal mode eigenvector and $\mathbf{A} = (a_{ij})$ is a square matrix, so that

$$x \frac{dy_i}{dx} = \sum_j a_{ij} y_j. \quad (4.2)$$

The job of `ModeDriver` objects is to provide the coefficients a_{ij} and boundary conditions to a `Mode<N>` object. This division provides greater flexibility and intercompatibility for different physics within the code.

4.1 The ModeDriver Interface

The `ModeDriver` definition specifies the following methods for external code to be able to interact with it:

4.1.1 Constants

- `const size_t num_var`: a constant integer specifying the number of eigenfunctions in this form of wave equation.

4.1.2 Physics

- `double rad(int X)`: the radial coordinate $\text{rad}[X] = x = r/R_*$.
- `double Gamma1()`: the adiabatic exponent of the perturbations, Γ_1 , as in $\frac{\delta P}{P} = \Gamma_1 \frac{\delta \rho}{\rho}$. If set to zero, then $\Gamma_1 = (\partial \log P / \partial \log \rho)_{ad}$ from the stellar background is used. This can also be set to a specific number (e.g. $\Gamma_1 = 5/3$), useful with models like polytropes.
- `void getCoeff(double *CC, int X, int b, double w2, int l)`: replaces $\text{CC} = \mathbf{A}$, the coefficient matrix at the index X and interpolated by spacing $0.5*b$ for use in an RK4 routine. Uses frequency $\tilde{\omega}^2 = w2$ and angular momentum number ℓ given. The equation form `Mode<N>` expects is shown in (4.2) (do **not** divide the a_{ij} by the x on the LHS).

4.1.3 Calculation

- `int length()`: returns the correct array size for modes.

- `void getBoundaryMatrix(int, double*, double*, double**, int*)`: return a matrix that allows `Mode<N>` to calculate a Wronskian, and an array in integers that allows `Mode<N>` to rescale solutions to match in the center. See Sec. 4.3 below and Sec. 5.2.5 in `Mode<N>` for a fuller explanation.

4.1.4 Miscellaneous

- `double SSR(double, int, ModeBase*)`: returns a sum-square-residual from back-substitution of mode solution into the original physical equations, normalized by mode amplitude. Used as a check of accuracy of the mode solution.
- `void varnames(std::strings* vn)`: populates a vector of strings `vn` with names to use for each of the eigenfunction variables when forming plots.

4.1.5 Constructor

There is only one constructor:

- `ModeDriver(int nv, Star *s)`: initialize the number of variables needed, and set stellar background.

Because this class is **abstract**, the constructor does not actually create an object; it is instead needed in order for daughter classes to initialize the `const` variables `num_var` and `*star`.

4.1.6 Boundary conditions for modes

- `setupBoundaries()`: populate arrays containing power series coefficients of relevant stellar properties near center and surface, so that mode eigenfunctions can be calculated at these boundaries.
- `int CentralBC(double **y, double *yo, double w2, int l, int m=0)`: calculate eigenfunction `y[l][m]` at center, so that outward mode integration begins at the returned integer. Near the center of a spherical star, we can expand each eigenfunction y_i in the form

$$y_i(x) = y_{i0} + y_{i2}x^2 + y_{i4}x^4 + \dots \quad (4.3)$$

which allows us to find values near the center using only y_{i0}, y_{i2}, y_{i4} . The goal of `CentralBC()` is to find those coefficients given the equation form represented in **A** and the frequency `w2`; this normally takes the form of a recursion algorithm involving series expansions of **A**. We then populate

$$y[i][X] = y_{i0} + y_{i2} \cdot \text{rad}[X]^2 + y_{i4} \cdot \text{rad}[X]^4 + \dots$$

The maximum position `X` where this procedure is used is the returned value, and normal integration should take over from there.

- `int SurfaceBC(double **y, double *ys, double w2, int l, int m=0)`: calculate eigenfunction `y[l][m]` at surface, so that outward mode integration begins at the returned integer. Near the surface of a star, we can expand each eigenfunction y_i in the form

$$y_i(x) = y_{i0} + y_{i1}t + y_{i2}t^2 + \dots \quad (4.4)$$

for $t = 1 - x$, similarly to the center. `SurfaceBC()` works similarly to its central counterpart, finding these coefficients and populating the eigenfunction near the surface

$$y[i][\text{len}-X] = y_{i0} + y_{i1} \cdot t + y_{i2} \cdot t^2 + \dots$$

The minimum position `len-X` where this procedure is used is the returned value, and standard numerical integration takes over from there.

For concrete description, see the included Mathematica notebook `NewtonianBC.nb`.

4.2 The ModeDriver Implementation

The above section specifies all the ways external code can interact with a `ModeDriver` object. Internally, a `ModeDriver` must also specify:

- `Star* star`: a pointer to the `Star` object to serve as stellar background.
- `int num_var`: the number of eigenfunctions y_i for this mode type.
- `int central_bc_order, surface_bc_order`: the desired order of series expansions in y_i to use at the center and surface.

Because mode types can differ greatly one from the other, `ModeDriver` does not actually define any functions (unlike `Star`). Everything must be implemented by child classes.

4.3 Solution method

This class was made to interface with `Mode<N>` to solve for normal modes of stellar oscillation. Because `Mode<N>` is physics-agnostic, all physics is carried by `ModeDriver` objects.

At each grip point and half-grid point, the `ModeDriver` must provide a coefficient matrix \mathbf{A} , through the function `getCoeff()`. The exact values of the coefficients in \mathbf{A} will depend on the physics of the waves being implemented.

If there are $N = \text{numvar}$ eigenfunctions in the wave solution, then there are also N independent solutions. However, there is only one physical solution, which must satisfy all N of the boundary conditions. There are usually $N/2$ BCs at the center, and $N/2$ at the surface; there are also $N/2$ degrees of freedom at the center and $N/2$ at the surface. We can then relate each d.f. to one BC, and to one independent solution. A schematic of this is shown in Fig. 4.1.

It is up to `getBoundaryMatrix()` to explain how to make these independent solutions in each region, by specifying the d.f.s needed to create each independent solution, the order to create them, and then how to match them at the center. For example, if $N = 4$ and `getBoundaryMatrix()` returns the matrix below, it leads to the following specifications of BCs to create the independent solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{ll} \vec{Y}^{(c1)} &= (y_1, y_2, 0, 0) \\ \vec{Y}^{(c2)} &= (0, 0, y_3, y_4) \\ \vec{Y}^{(s1)} &= (0, y_2, y_3, y_4) \\ \vec{Y}^{(s2)} &= (y_1, y_2, 0, 0) \end{array} \quad (4.5)$$

where $\vec{Y}^{(c1)}, \vec{Y}^{(c2)}$ are solutions in the center and $\vec{Y}^{(s1)}, \vec{Y}^{(s2)}$ are solutions at the surface. The further output of `getBoundaryMatrix()` is a list of indices to combine the four solutions to make the physical solution. The process is explained further in Sec. 5.2.5 of `Mode<N>`.

4.4 How to make new ModeDriver children

If you wish to add a new object representing different stellar waves to the code, you need to add the above functions. If your mode type does not use the same functions as the Dziembowski equation (i.e. A^*, V_g, c_1, U), then you will need for `setupBoundaries()` to find a way to extract power series expansions of the functions you use to the desired order: this is usually possible with numerical derivatives of quantities such as ρ, P, m, g , which can be found through the equivalent methods in `Star`. You will further need to work out a recursion algorithm that calculates coefficients y_{i0}, y_{i1} , etc. near the center and surface based on the form of the equations you are using. The easiest method is to copy `NonradialModeDriver` and change whatever necessary.

4.5 References

- Unno et al., 1979
- Dziembowski (1971), for the standard equation set assumed throughout the program, which describes the wave equations using structure variables A^* , V_g , c_1 , U .
- Cox (1980), *Theory of Stellar Pulsation*, Section 17.6, for a brief of the recursion process of boundary conditions of the wave equations. The coefficients calculated in `setupBoundaries()` are similar to those in Cox's equation 17.63a-e, and the coefficients y_{in} calculate by `CentralBC()` are similar to those in Cox's equation 17.60a-c.
- Christiansen-Dalsgaard & Mullan (1994), for the same boundary recursion process at the surface.

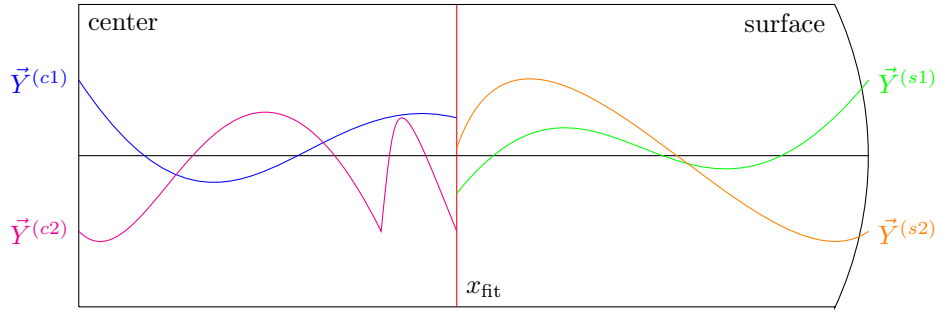


Figure 4.1: A sketch of the solution process for $N = 4$. The matching of the four independent solutions at x_{fit} allows for finding the eigenvalue $\bar{\omega}^2$ and the physical solution. To create the independent solutions, `getBoundaryMatrix()` gives `Mode<N>` a matrix for assigning boundary values.

Chapter 5

The Mode<N> Class

This class represents a physical mode of oscillation within a star. Each `Mode` has `num_var=N` variables y_1, \dots, y_N , a dimensionless frequency $\tilde{\omega}^2 = \text{omega2}$, and mode numbers $k, \ell, m = \text{k}, \text{l}, \text{m}$, where k is the radial mode order and ℓ, m are the angular harmonic numbers. The mode can optionally have Γ_1 defined separately from the stellar background, as is necessary for polytrope calculations.

`Mode<N>` expects to be solving mode equations of the form

$$x \frac{d\vec{Y}}{dx} = \mathbf{A} \vec{Y}, \quad (5.1)$$

where $\vec{Y} = \langle y_1, \dots, y_N \rangle$ is the eigenvector and $\mathbf{A} = (a_{ij})$ is a square $N \times N$ matrix, so that

$$x \frac{dy_i}{dx} = \sum_j a_{ij} y_j. \quad (5.2)$$

It is necessary for `ModeDriver` objects to provide the coefficients a_{ij} , which contain all of the physics of the wave behavior. Note the factor of x on the LHS of the differential equation.

5.1 The Mode<N> Interface

The `Mode<N>` definitions specifies the following methods for external code to be able to interact with it:

5.1.1 Mode Properties

Methods related to the mode numbers k, ℓ, m :

- `int modeOrder()`: radial mode order, k .
- `void modeNumbers(int& k, int& l, int& m)`: replaces arguments with all mode numbers k, ℓ, m .

Methods related to the frequency:

- `double getOmega2()`: the square dimensionless frequency $\tilde{\omega}^2$ ($\omega^2 = \frac{GM}{R^3} \tilde{\omega}^2$).
- `double getFreq()`: returns the angular frequency ω in rad/s.
- `double getPeriod()`: the period $\Pi = 2\pi/\omega$, in s.

5.1.2 Physics

- `double getRad(int X)`: the radial coordinate $\text{rad}[X] = x = r/R_\star$.
- `double getY(int i, int X)`: the eigenfunction $y[i][X] = y_i(\text{rad}[X])$.
- `double tidal_overlap()`: calculates a tidal overlap coefficient.

5.1.3 Miscellaneous

- `double SSR()`: returns a root-mean-square-residual from back-substitution of mode solution into the original physical wave equations (e.g. the LAWE), normalized by mode amplitude. Used as a check of accuracy of the mode solution.
- `void writeMode(char *c)`: prints x, y_1, \dots, y_N to external files and forms plots in gnuplot. The point `xfit` where the double-shooting method is matched is highlighted by a **red vertical line** – be sure . The optional argument `c` specifies a directory path where to print the star.
- `void printMode(char *c)`: writes the mode as above, and then opens the plotted file in an external window.

5.1.4 Constructors

There are three constructors for creating modes. They are very similar, and differ only in how they find an initial guess for $\bar{\omega}^2$:

- `Mode<num_var>(int k, int l, int m, ModeDriver*)`: creates a mode with anticipated mode numbers k, l, m . The starting value of ω^2 is picked based on the formula (5.3) for a uniform-density stellar model.
- `Mode<num_var>(double w2, int l, int m, ModeDriver*)`: creates a mode with initial guess $\bar{\omega}^2 = w2$.
- `Mode<num_var>(double omegMin, double omegMax, int l, int m, ModeDriver*)`: creates a mode where the eigenfrequency $\bar{\omega}^2$ is known to fall in the range $\bar{\omega}^2 \in [\text{omegMin}, \text{omegMax}]$.

In all cases, $\bar{\omega}^2$ is found with a bisection search aided by Newton's method.

5.2 The Mode<N> Implementation

The above section specifies all the ways external code can interact with a `Mode` object. In this section, we will explain how the details of the calculation are performed.

5.2.1 Constants

- `int num_var`: number of variables, `num_var = N`, which is specified by `ModeDriver`.
- `int k, int l, int m`: the spherical harmonic quantum numbers k, l, m of the oscillation.
- `double cee2, Gee`: values for c^2, G in appropriate units, for scaling laws.
- `double sig2omeg`: a conversion factor, $\bar{\omega}^2 = \text{sig2omeg} * \omega^2$.
- `int len`: the maximum index of the perturbation arrays.
- `int len_star`: the number of grid points in the stellar background model.
- `int xfit`: the index where shoot-to-center methods at surface and center are to match.
- `double Gamma1`: the adiabatic exponent of the perturbations, $\Gamma_1 = \delta \log P / \delta \log \rho$, as set from the `ModeDriver`.

5.2.2 Physics

- `double omega2`: the square of the dimensionless eigenfrequency, $\bar{\omega}^2 = \frac{R^3}{GM}\omega^2$.
- `double *rad`: an array holding the radial variable $x = r/R_*$.
- `double **y`: 2D array holding the perturbation variables of the star: $y[0][x] = y_1(x)$, $y[1][x] = y_2(x)$, etc.
- `Star *star`: a pointer to a `Star` object containing the background equilibrium model. Must be the same `Star` object as the related `ModeDriver`.
- `ModeDriver *driver`: a pointer to a `ModeDriver` object, which handles all quantities pertinent to the background stellar model, including implementing the boundary conditions.

5.2.3 Calculation

- `void basic_setup()`: a demiurge constructor used by all three constructors to setup `r[]`, `y[][]` arrays, calculate `xfit`, and prepare the boundary matrix.
- `double boundaryMatrix[num_var][num_var]`: a matrix used to specify how to form independent solutions for constructing a Wronskian.
- `int indexOrder[num_var]`: an array of indices for rescaling solutions to match at center.
- `double yCenter[num_var]`, `ySurface[num_var]`: starting values of \vec{Y} at center and surface.
- `void converge()`: the routine to calculate eigenfrequency $\bar{\omega}^2$. Does not need to be called outside of the constructor.
- `bool converged`: a flag, indicates if mode calculation reached the end of convergence process. [Deprecated](#).
- `double RK4out(int xmax, double, double y0[num_var])`: given central values `y0[]`, integrate using RK4 from the center up to index `xmax`.
- `double RK4in(int xmin, double, double ys[num_var])`: given surface values `ys[]`, integrate using RK4 from the surface down to index `xmin`.
- `double RK4center(double, double y0[num_var], double ys[num_var])`: given central and surface values `y0[]`, `ys[]`, calculate a Wronskian W by shooting outward and inward to `xfit`. The Wronskian is the return value.
- `convergeBisect(double tol)`: find $\bar{\omega}^2$ using a bisection search for the zero of Wronskian W , stopping when the brackets are within `tol` of one another. Newton's method is used to find the starting brackets.
- `convergeNewton(double tol, int term)`: find $\bar{\omega}^2$ using Newton's method to search for the zero of Wronskian W , stopping when $|W| < \text{tol}$, or until after `term` steps. [Deprecated; use convergeBisect\(\) instead](#).
- `linearMatch(double w2, double y0[num_var], double ys[num_var])`: if frequency `w2` leads to a vanishing Wronskian, then this will solve for coefficients of independent solutions needed to cause outward and inward solutions to match at the center.
- `int verifyMode()`: a function which classifies the mode k of the eigenfunction based on its nodes using the Osaki-Sculfaire method. Does not need to be called outside of the constructor.

5.2.4 Constructors

All constructors begin with

- `basic_setup()`

which initializes everything necessary except for the frequency. The array `rad[]` is populated with the dimensionless radius $x = r/R$ from the `ModeDriver`. Because `Mode<N>` uses RK4, requiring stellar properties at half-steps, the grid for `Mode<N>` is chosen to be double that of `Star`, so that the stellar properties are defined at the half-steps; this avoids inaccuracies from interpolation.

The index `xfit` for matching the shoot-to-center solutions is chosen from the stellar background. In the output graph, the location of `xfit` is marked by a red vertical line – this can help to diagnose failures to converge.

The three constructors differ only in how they find the eigenfrequency $\bar{\omega}$. They are:

- `Mode<num_var>(int k, int l, int m, ModeDriver*)`: creates a mode with anticipated mode numbers k, l, m . The starting value of $\bar{\omega}^2$ is picked based on the Pekeris formula for uniform stellar frequencies (see Cox, section 17.7),¹

$$D_k = -2 + \Gamma_1[k(l + k + 5/2) + l + 3/2] \quad (5.3a)$$

$$\bar{\omega}_{k,l}^2 = D_k \pm \sqrt{D_k^2 + l(l + 1)}. \quad (5.3b)$$

The actual k is found at the end from `verifyMode()`, and will not necessarily correspond to the desired k ; ensuring $k = k$ takes place outside the class.²

- `Mode<num_var>(double w2, int l, int m, ModeDriver*)`: creates a mode with initial guess $\bar{\omega}^2 = w2$. It will then converge to the eigenfrequency closest to the initial guess. The mode number k is found at the end from `verifyMode()`.
- `Mode<num_var>(double omegMin, double omegMax, int l, int m, ModeDriver*)`: creates a mode where the eigenfrequency $\bar{\omega}^2$ is known to fall in the range $\bar{\omega}^2 \in [\text{omegMin}, \text{omegMax}]$. A bisection search is then used to find the $\bar{\omega}^2$. Note that this range must bound *exactly* one eigenfrequency or the constructor will fail. If the brackets are *bad* (i.e., they enclose multiple roots, or no root), then this merely calls `converge()` to find $\bar{\omega}^2$ using the midpoint as a starting guess.

5.2.5 The solution process

The process of finding the solution is handled by `converge()`, and can be broken into two steps handled in two methods:

1. find the eigenfrequency $\bar{\omega}^2$, handled by `convergeBisect()`.
2. find the physical solution y_1, \dots, y_N for this $\bar{\omega}^2$, handled by `linearMatch()`.

`Mode<N>` uses a shoot-to-center process to calculate the eigenfunctions. Because the system is of order N , there will in general be N independent solutions; however, there is only one physical solution which satisfies the boundary conditions. At the center, we must have $N/2$ boundary conditions³, which we can write in the form

$$c_{ij}y_j(0) = 0, \quad i = 1, \dots, N/2 \quad (5.4)$$

¹ Note that this guess tends to overestimate $\bar{\omega}^2$ for non-uniform stars, so that the resulting mode will actually have a radial number k much larger than anticipated. To account for this

$$ks = k - 21 + 4$$

is used instead of k in (5.3) to better land at anticipated radial mode k (there is no magic to this, it just seemed to work). For g-modes, where $k < 0$, we use an empirical formula based on the data tables in Christensen-Dalsgaard and Mullan 1994.

²The reason it takes place outside the class is that the mode found will be an actual eigenmode solution, just not the one we asked for now. Since we may want it later, outside programs may want to store it before trying again.

³ This exact split is not mathematically necessary, and the code can be modified to accept different divisions of BCs; this will require `ModeDriver` to specify the division by an integer, and to edit the `numvar/2` lines in `linearMatch()` and `RK4center()`.

and at the surface the other $N/2$ boundary conditions, which we can write in the form

$$s_{ij}y_j(1) = 0, \quad i = N/2 + 1, \dots, N. \quad (5.5)$$

The center BC removes $N/2$ degrees of freedom, leaving $N/2$ remaining; these make $N/2$ independent solutions, $\vec{Y}^{(ci)}$; likewise for the surface BCs, leading to $N/2$ independent solutions $\vec{Y}^{(si)}$. The physical solution will be a combination of all N independent solutions that matches all boundary conditions.

This can be most concretely explained by referring to nonradial pulsation modes, as described by `NonradialModeDriver` 13. This is a 4th-order system, so there must be four boundary conditions – two at the center and two at the surface (See Fig. 5.1). Because the system is fourth order, we need four solutions to form a Wronskian. At the center, two of these are taken to be (see section 3.1 of Christensen-Dalsgaard's notes distributed with the `adiphs` program, or chapter 6 of his lecture notes)

$$y_1(0) = 0, \quad y_3(0) = y\theta[2] \rightarrow \vec{Y}^{(s1)} = (y_1^{(c1)}, y_2^{(c1)}, y_3^{(c1)}, y_4^{(c1)}), \quad (5.6a)$$

$$y_1(0) = y\theta[0], y_3(0) = 0 \rightarrow \vec{Y}^{(s2)} = (y_1^{(c2)}, y_2^{(c2)}, y_3^{(c2)}, y_4^{(c2)}) \quad (5.6b)$$

and at the surface

$$y_1(1) = 0, \quad y_3(1) = ys[2] \rightarrow \vec{Y}^{(s1)} = (y_1^{(s1)}, y_2^{(s1)}, y_3^{(s1)}, y_4^{(s1)}), \quad (5.6c)$$

$$y_1(1) = ys[0], y_3(1) = 0 \rightarrow \vec{Y}^{(s2)} = (y_1^{(s2)}, y_2^{(s2)}, y_3^{(s2)}, y_4^{(s2)}) \quad (5.6d)$$

The condition for these to match is that

$$a\vec{Y}^{(c1)}[\text{xfit}] + b\vec{Y}^{(c2)}[\text{xfit}] = c\vec{Y}^{(s1)}[\text{xfit}] + \vec{Y}^{(s2)}[\text{xfit}] \quad (5.7)$$

for some variables a, b, c at the fitting point (the scale of \vec{Y}^{s2} is arbitrarily set to 1). It can be shown that this occurs when the Wronskian W vanishes, where W is calculated as

$$W = \begin{vmatrix} y_1^{(c1)}[\text{xfit}] & y_2^{(c1)}[\text{xfit}] & y_3^{(c1)}[\text{xfit}] & y_4^{(c1)}[\text{xfit}] \\ y_1^{(c2)}[\text{xfit}] & y_2^{(c2)}[\text{xfit}] & y_3^{(c2)}[\text{xfit}] & y_4^{(c2)}[\text{xfit}] \\ y_1^{(s1)}[\text{xfit}] & y_2^{(s1)}[\text{xfit}] & y_3^{(s1)}[\text{xfit}] & y_4^{(s1)}[\text{xfit}] \\ y_1^{(s2)}[\text{xfit}] & y_2^{(s2)}[\text{xfit}] & y_3^{(s2)}[\text{xfit}] & y_4^{(s2)}[\text{xfit}] \end{vmatrix}. \quad (5.8)$$

The solution is considered converged when $W = 0$, which means inward and outward solutions are linearly dependent on one another. When $W = 0$, it is *not* the case that inward and outward solutions have the same value at `xfit`, just that the four sets of solutions can be written in terms of each other. For the purposes of finding $\bar{\omega}^2$, it is sufficient to use 0 or 1 for the initial values. It is only necessary to give these specific values when matching the inward and outward solutions for mode counting.

Each constructor specifies an initial guess for $\bar{\omega}^2$. The method `convergeBisect()` then uses bisection to find a value $\bar{\omega}^2$ so that the Wronskian of (5.8) vanishes. The brackets for the bisection search are determined using Newton's method. If $W = 0$, then $\bar{\omega}^2$ is an eigenfrequency.

Once an eigenfrequency $\bar{\omega}^2$ has been discovered, we must find the physical solution with `linearMatch()`. Because W vanishes, we know that (5.7) is satisfied for some choice of a, b, c . To find these values, write the homogeneous linear system of (5.7) as a matrix equation,

$$\begin{bmatrix} y_1^{(c1)} & y_1^{(c2)} & -y_1^{(s1)} & -y_1^{(s2)} \\ y_2^{(c1)} & y_2^{(c2)} & -y_2^{(s1)} & -y_2^{(s2)} \\ y_3^{(c1)} & y_3^{(c2)} & -y_3^{(s1)} & -y_3^{(s2)} \\ y_4^{(c1)} & y_4^{(c2)} & -y_4^{(s1)} & -y_4^{(s2)} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (5.9)$$

which can be solved with Gauss-Jordan elimination. Then the values

$$y_1(0) = a \cdot y\theta[0], \quad y_3(0) = b \cdot y\theta[2], \quad y_1(1) = ys[0], \quad y_3(1) = c \cdot ys[2] \quad (5.10)$$

are the correct boundary values to use to find the physical solution. Other values are determined from the boundary conditions, (5.4),(5.5). In order for this to work, we *must* have a vanishing Wronskian. If mode solutions are behaving poorly, check the Wronskian values to ensure they are nearly zero.

We have illustrated this for a nonradial Newtonian mode with $N=4$. For different modes, the process is analogous, but with different boundary conditions and different choices of independent solutions.

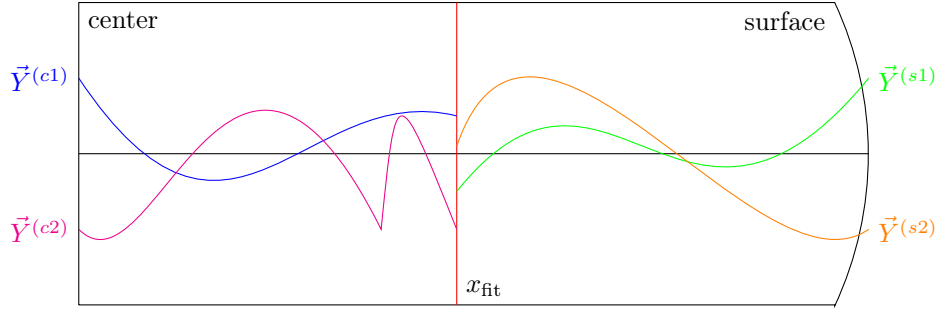


Figure 5.1: A sketch of the solution process for nonradial modes with $N = 4$. The two BCs at each boundary give the two solutions in each region, and the values of these four solutions at x_{fit} are used in the Wronskian for find $\bar{\omega}^2$, then fit together to create the physical solution.

5.3 How to extend Mode<N>

The class `Mode<N>` is actually a child class of the class `ModeBase` (found in `ModeDriver.h`). To make use of polymorphism with a template class, C++ syntax requires the non-template class `ModeBase` to be the parent. The solution method for `Mode<N>` is deliberately old-fashioned, relying on RK4 integration and other basic methods. To use a more sophisticated method to solve for modes, such as the Henyey method, it is recommended to make a new mode type inheriting directly from `ModeBase` instead of from `Mode<N>`.

5.4 A note about editing Mode<N>

Because `Mode<N>` is a template class, C++ does not allow a true separation of header/source files, so that the `Mode.h` file includes the `Mode.cpp` file. This means object files which include `Mode.h` will also include the definitions in `Mode.cpp`. The net of this is that whenever you make edits to `Mode.cpp`, you must make `clean` before recompiling.

5.5 References

- Scuflaire, R. (1974) and Osaki, Y. (1975); these two papers jointly describe the method to classify mode order and distinguish p and g-modes.
- Christensen-Dalsgaard, J. and D. J. Mullan (1994), for the table of polytrope g-modes used to predict g-mode frequencies.
- Christensen-Dalsgaard, J. (2008), in *Astrophysics and Space Science* for the official publication describing `adipls`. It contains a (much less complete) description of the Wronskian process.
- Christensen-Dalsgaard lecture notes

Part II

Stellar Models

Chapter 6

The Polytrope Class

This class represents a pure polytropic star with given index n . Polytropes are described by the Lane-Emden equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \quad (6.1)$$

where the function $\theta(x)$ ¹ is related to density by

$$\rho(r) = \rho_c \theta(r)^n \quad (6.2)$$

and where

$$\xi = r \sqrt{\frac{4\pi G \rho_c^2}{(n+1)P_c}}. \quad (6.3)$$

The boundary conditions are that $\theta = 1$ at the center $\xi = 0$, and that $\theta = 0$ at the surface $\xi = \xi_1$.

6.1 The Polytrope Interface

`Polytrope` is a child of `Star`, which means it must implement the `Star` interface – i.e., the functions returning ρ, P, Φ , the `printStar()` method, etc. as in 3.

In addition, `Polytrope` offers the following methods accessible to external code:

- `double getX(int k)`: return position variable $\xi = x[k]$.
- `double getY(int k)`: return solution $\theta(\xi[k]) = y[k]$.
- `double getYderiv(int k)`: return derivative $\frac{d\theta}{d\xi}|_{\xi[k]} = z[k]$.

These are mostly used for testing purposes, such as comparing to known analytic equations or to relativistic polytropes.

6.1.1 Constructors

There are three constructors available for a `Polytrope`:

- `Polytrope(double M, double R, double n, int len)`: creates a polytrope of specific grid size (length) `len`, of polytropic index $n = n$, with total mass $M_\star = M$ and $R_\star = R$. The step size dx needed to reach the surface ($y = 0$) in exactly `len` steps is found by bisection.
- `Polytrope(double n, int len)`: as above, but maintaining dimensionless variables where $G = \rho_c = P_c = 1$.

¹Within the code we use ASCII-friendly characters $\xi = x$ and $\theta = y$

- `Polytrope(double n, int len, double dx)`: as above, but specifies the step size to use as `dx`. There is no guarantee of reaching the “true” surface. [This is used for testing scaling relations. Do not use for production runs.](#)

6.2 The Polytrope Implementation

The Polytrope definition specifies the following class data and methods used in the implementation:

6.2.1 Constants

- `int len`: the length of arrays for this calculation, which is returned by `length()` function.
- `double n`: the polytropic index n .
- `double Gamma`: the adiabatic exponent Γ of the polytrope, $\Gamma = 1 + \frac{1}{n}$, where $P \sim \rho^\Gamma$.
- `double rho0`: the central density ρ_c .
- `double P0`: the central pressure P_c .
- `double Rn`: the scale factor $R_n = \sqrt{\frac{(n+1)P_c}{4\pi G \rho_c^2}}$ with $r = R_n \xi$.

6.2.2 Solution Variables

- `double **Y`: an array holding the solution to (6.1) in the form $(\xi, \theta, \frac{d\theta}{d\xi})$. The first index is the grid point X .
- `enum VarName {x=0, y, z, numvar}`: indices for solution variables within `**Y`, where $x \rightarrow \xi, y \rightarrow \theta, z \rightarrow \frac{d\theta}{d\xi}$, and where `numvar` is used to indicate the number of variables.
- `double *mass`: an array containing the interior mass $m(r)$ as a function of radius, $m(r) = \int_0^r 4\pi r^2 \rho(r) dr$, or removing dimensional factors, $\text{mass}[X] = -4\pi \xi^2 \frac{d\theta}{d\xi} = -4\pi Y[X][x]^2 Y[X][z]$.
- `double *base`: an array used in calculation of stellar variables. Defined as $\text{base}[X] = Y[X][y]^{n-1}$. Allows for fewer calls to `pow()` in calculation.

6.2.3 Constructors

There are three constructors:

- `Polytrope(double M, double R, double n, int len)`
- `Polytrope(double n, int len)`
- `Polytrope(double n, int len, double dx)`

In all, `n` is the polytropic index and `len` is the number of grid points to use. In the third, `dx` is the stepsize to use; this latter method is intended for testing errors when the grid size is changed.

`Polytrope(double M, double R, double n, int len)` begins by doing a check on n to be sure it corresponds to a finite star, so that $0 \leq n < 5$. If not, the constructor sends an error message and quits – future work should provide better error handling here. This constructor is intended for real stars, and will not construct $n = 5$ models.

The `name[]` for polytropes will be prefixed by `polytrope`, and appended with the index n rounded to the first decimal place. So for $n = 3$, the `name[]` will be `polytrope.3.0`.

The first step of the constructor is to find the appropriate step size `dx` so the stellar surface occurs within exactly `len` steps. This is done using bisection with the variable $\theta_1 = Y[\text{len}-1][y]$, beginning with an initial

guess of $dx = \frac{\sqrt{6}}{10^{n-1}}$ (which is the correct value for an $n = 0$ polytrope). The bisection search will stop either when a dx is found so that $Y[10^n-1][y]=0$, or when the brackets have stopped moving.

Once the surface is found, we pick `indexFit` to be the place where $\theta(\xi) = 0.5$; because the stellar grid is twice the mode grid, this index must then be divided in half. Shooting methods in the `Mode` class will match solutions at this index value. At the same time, we also calculate `base[X] = $\theta^{n-1} = Y[X][y]^{n-1}$` and `mass[X]`. For `mass[]`, we can use (6.1) to simplify the integration, noting

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr = \int_0^x 4\pi R_n^3 \xi^2 \rho_c \theta^n d\xi = 4\pi \rho_c R_n^3 \int_0^\xi \frac{d}{d\xi} \left(-\xi^2 \frac{d\theta}{d\xi} \right) = -4\pi R_n^3 \rho_c \xi^2 \frac{d\theta}{d\xi}; \quad (6.4)$$

thus we can calculate `mass` directly without needing to use numerical integration techniques.

The scale of the star (including central values of $P(0)$, $\rho(0)$ and radius) are fixed so that the total mass and radius correspond to `M,R`.

`Polytrope(double n, int len)` works the same, except that instead of using homology to fix mass and radius, everything is left dimensionless, where $G = P_c = \rho_c = 1$.

This is capable of handling the $n = 5$ case. In that case there is no surface, so the bisection method would never finish. We use the dreaded `goto` to skip over the bisection process, and use `dx` from our first guess. This index cannot model a realistic star, so is only useful when testing against the $n = 5$ analytic solution, $\theta_5(\xi) = (1 + \frac{1}{3}\xi^2)^{-1/2}$.

`Polytrope(double n, int len, double dx)` does not have a bisection search for `dx`, which is specified as a parameter. The fitting index is also chosen to be an exact power of 2, for easier comparisons. This constructor should only be used when testing errors or how quantities scale with changes in the grid size.

6.2.4 Calculation

- `void centerInit(double y[numvar]):` set initial values $\xi = Y[0][x] = 0, \theta = Y[0][y] = 1, \frac{d\theta}{d\xi} = Y[0][z] = 0$.
- `void RK4step(double dx, double yin[numvar], double yout[numvar]):` performs a single RK4 step of the Lane-Emden equations, with step size $\Delta\xi = dx$ from `yin[]` and ending at `yout[]`. Because the equations call for θ^n , the program can produce NaNs at the surface if $\theta < 0$ and n is not an integer. To prevent NaN errors, we use a complex variable $YCN = \theta^n$, and take only the real part of YCN in equations.
- `double RK4integrate(int Len, double dx):` integrate from the center out to position `Len` using `RK4step()` above. Returns the value $\theta[Len-1]$ at the end of the integration, to be used in bisection search of `dx`. Since only $\theta \geq 0$ is physical, the integration stops when $\theta[X+1] < 0$ and returns $\theta[X+1]$.
- `int RK4integrate(int Len, double dx, int g):` as above, but all the way to the end of `Len`; it does not stop if $\theta < 0$. The return value is an integer, `Len`. [The integer `g` now mainly serves the role of distinguishing this method from the other method. Better naming would also work.](#)

6.2.5 Boundary conditions for modes

The boundary conditions of the modes require series expressions for A^*, U, c_1, V_g . Because of the very simple mathematical structure of a polytrope it is possible to derive analytic formulas for these coefficients.

- `double ac[4]:` coefficients of power series for θ near the center, in terms of dimensionless $x = r/R_\star = \xi/\xi_1$:

$$\theta(x) \approx \theta_0 + \theta_2 x^2 + \theta_4 x^4 + \theta_6 (r/R)^6 = ac[0] + ac[1]x^2 + ac[2]x^4 + ac[3]x^6.$$

- `double as[6]:` coefficients of power series for θ near the surface (with $t = 1 - r/R_\star = 1 - \xi/\xi_1$)

$$\theta(t) \approx \theta_0 + \theta_1 t + \theta_2 t^2 + \dots = as[0] + as[1]t + as[2]t^2 + \dots.$$

- `setupCenter():` initialize the coefficients above. See `NewtonianBC.nb` for further information.

- `setupSurface()`: initialize the coefficients above. See `NewtonianBC.nb` for further information.
- `getXYZCenter(double* xyz, int& maxPow)`: where XYZ is Astar, Vg, U, C1 Returns terms of the power series expansion in x near the center. For a polytrope with index n , these are pretty easy to calculate, and are given up to fourth order. See the Mathematica notebook `NewtonianBC.nb` for detailed explanation. The coefficients are:

$$A^*(r) = 0 + A_V \left(\frac{n+1}{3\Gamma_1} \right) \xi_1^2 x^2 + A_V \left(\frac{2(n+1)}{36\Gamma_1} - \frac{2n(n+1)}{60\Gamma_1} \right) \xi_1^4 x^4 + \dots \quad (6.5)$$

$$U(r) = 3 + \left(-\frac{n}{5} \right) \xi_1^2 x^2 + 54n \left(\frac{7n}{8100} - \frac{1}{1296} + \frac{5-8n}{15120} \right) \xi_1^4 x^4 + \dots \quad (6.6)$$

$$c_1(r) = \left(-\frac{3z_1}{x_1} \right) + \left(\frac{3nz_1}{10x_1} \right) \xi_1^2 x^2 + \left(\frac{n(2n+25)z_1}{1400x_1} \right) \xi_1^4 x^4 + \dots \quad (6.7)$$

$$V_g(r) = 0 + \left(\frac{n+1}{3\Gamma_1} \right) \xi_1^2 x^2 + \left(\frac{2(n+1)}{36\Gamma_1} - \frac{2n(n+1)}{60\Gamma_1} \right) \xi_1^4 x^4 + \dots \quad (6.8)$$

Here $\xi_1 = Y[\text{len}-1][x]$ and $z_1 = \frac{d\theta}{d\xi}|_{\xi_1} = Y[\text{len}-1][z]$. The coefficients are stored in the input `xyz[]` array to be returned. We have found terms up to order x^4 ; if more are requested with `maxPow`, then we set `maxPow = 4` so that the `ModeDriver` only implements the BC to order x^4 .

- `getXYZSurface(double* xyz, int& maxPow)`: as above, but near the surface expanded in powers of $t = 1 - x$. Unlike the central boundary, there is no requirement that only even powers appear. Two of the stellar variables, A^*, V_g have a pole at $t = 0$, so require a t^{-1} term. If n is not an integer, there are also terms t^n appearing in the expansions (I have included the t^n term in the past, but it is not currently set up - need a method to include at least the first non-integer part of an expansion).

Due to the nature of the recursion relation at the surface, to implement the BCs to 4th order, we need A^*, V_g to order t^3 (which includes the t^{-1} term), we need c_1 to t^3 , and we need U to order t^4 . The values of the U terms will depend nontrivially on the polytropic index n ; for certain n , many terms will simply vanish. The details of the calculations are too tedious to write out here, but can be found in the Mathematica notebook `NewtonianBC.nb`.

In both the center and the surface, the BC is given to **fourth order**.

6.3 Internal Logic Behind the Interface

With the above variables, we can now define how the following `Star` quantities are computed:

- `double rad(int X)`: $r = R_n \xi$.
- `double rho(int X)`: $\rho = \rho_c \theta^n = \text{rho0} * \text{base}[X] * Y[X][y]$.
- `double drhodr(int X)`: $\frac{d\rho}{dr} = \frac{n\rho_c}{R_n} * \theta^{n-1} \frac{d\theta}{d\xi} = n * \text{rho0} * \text{base}[X] * Y[X][z] / R_n$.
- `double P(int X)`: $P = P_c \theta^{n+1} = P0 * \text{base}[X] * Y[X][y] * Y[X][y]$.
- `double dPdr(int X)`: $\frac{dP}{dr} = \frac{(n+1)}{R_n} \theta^n \frac{d\theta}{d\xi} = (n+1) * \text{rho0} * \text{base}[X] * Y[X][y] * Y[X][z] / R_n$.
- `double Phi(int X)`: Φ which has been matched so that $\Phi(R_*) = -\frac{GM_*}{R_*}$. Returns $\Phi = -\frac{(n+1)P_c}{\rho_c} \theta + \Phi_c = (n+1) * P0 / \text{rho0} * (Y[\text{len}-1][x] * Y[\text{len}-1][z] - Y[X][y])$.
- `double dPhidr(int X)`: $g = \frac{d\Phi}{dr} = -\frac{(n+1)P_c}{\rho_c} \frac{d\theta}{d\xi} = -(n+1) * P0 / \text{rho0} * Y[X][z] / R_n$.
- `double mr(int X)`: $m(r) = -4\pi R_n^3 \rho_c \xi^2 \frac{d\theta}{d\xi} = R_n^3 * \text{rho0} * \text{mass}[X]$.

- `double Schwarzschild_A(int X, double GamPert)`: the Schwarzschild discriminant

$$A = \frac{d\theta/d\xi}{\theta R_n} \left(n - \frac{n+1}{\Gamma_1} \right) = \frac{Y[X][z]}{Y[X][y]R_n} \left(n - \frac{n+1}{\text{GamPert}} \right).$$

If $\text{GamPert} = 0$ is given, uses $\Gamma_1 = \Gamma$ (which will result in $A = 0$).

- `double getAstar(int X, double GamPert)`: dimensionless Schwarzschild discriminant

$$A^* = \frac{\xi}{\theta} \frac{d\theta}{d\xi} \left(\frac{n+1}{\Gamma_1} - n \right) = \frac{Y[X][z]*Y[X][x]}{Y[X][y]} \left(\frac{n+1}{\text{GamPert}} - n \right).$$

If $\text{GamPert} = 0$ is given, uses $\Gamma_1 = \Gamma$ (which will result in $A^* = 0$).

- `double getU(int X)`: $U = \frac{4\pi\rho r}{d\Phi/dr} = -\frac{\xi\cdot\theta^n}{d\theta/d\xi} = -\frac{Y[X][x]*Y[X][y]*\text{base}[X]}{Y[X][z]}$. Special care must be take at $X = 0$, where $U = 3$.
- `double getVg(int X, double GamPert)`: returns $V_g(r) = \frac{rg}{v_s^2} = -\frac{n+1}{\Gamma_1} * \frac{\xi}{\theta} \frac{d\theta}{d\xi} = -\frac{n+1}{\text{GamPert}} \frac{Y[X][x]}{Y[X][y]}$. If $\text{GamPert} = 0$, uses $\Gamma_1 = \Gamma$
- `double getC(int X)`: returns

$$c_1(r) = \frac{M_\star}{m(r)} \frac{r^3}{R_\star^3} = \frac{z_1}{\xi_1} * \frac{\xi}{d\theta/d\xi} = \frac{Y[\text{len}-1][z]*Y[X][x]}{Y[\text{len}-1][x]*Y[X][z]}.$$

Special care must be take at $X = 0$, where $c_1 = -\frac{3z_1}{\xi_1}$.

- `double sound_speed2(int X, double GamPert)`: returns $v_s^2 = \frac{\Gamma_1 P}{\rho} = \frac{\Gamma_1 P_c}{\rho_c} * \theta(x) = \frac{\Gamma_1 P_c}{\rho_c} Y[X][y]$, where $\Gamma_1 = \text{GamPert}$. If no Γ_1 is specified, uses $\Gamma_1 = \Gamma$
- `double Gamma1(int X)`: returns $\Gamma = 1 + \frac{1}{n}$.

These values are all calculated at $r = \text{rad}(X)$. Since $\text{base}[X] = \theta^{n-1}$, this is used in the code to avoid calls to `pow()`. The other methods, `length()`, `Radius()`, etc., are trivially defined.

6.4 References

- Lane, J. H. (1870) for the first publication of Eqn. (6.1).
- Hansen, Kewaler, and Trimble (2004), in particular chapter 7 for a discussion of using RK4 method to calculate polytropic stellar models.
- Christensen-Dalsgaard and Mullan (1994) in their appendix, for a description of a similar recursive boundary process as that described in ?? (though there for a different form of the equations).
- Cox (1980), *Theory of Stellar Pulsation*, Section 17.6, for a brief of the recursion process of boundary conditions of the wave equations.

Chapter 7

The ChandrasekharWD Class

This class represents a zero-temperature white dwarf star, held up entirely by electron degeneracy pressure, as first described by Chandrasekhar (1935).

Treated as a homogeneous gas, the exclusion principle indicates that each cell of phase space (quantal volume $d^3x d^3p = h^3$) can accommodate at most two electrons (two to account for the two spin states). In the limit of cold matter where $T \rightarrow 0$, all electrons will settle to the lowest possible unfilled state and the number density of electrons becomes

$$n_e = 2 \frac{\int d^3p}{h^3} = \int_0^{p_F} \frac{8\pi p^2}{h^3} dp = \frac{8\pi p_F^3}{3h^3}, \quad (7.1)$$

where here p_F is defined from the energy

$$E_F = \sqrt{p_F^2 c^2 + m_e^2 c^4}. \quad (7.2)$$

Define the parameters x, θ by

$$x = \frac{p_F}{m_e c}, \quad p = m_e c \sinh \theta. \quad (7.3)$$

In terms of this the electron number density becomes

$$n_e = \frac{8\pi}{3h^3} m_e^3 c^3 x^3 = \frac{8\pi}{3} \left(\frac{m_e c}{h} \right)^3 x^3. \quad (7.4)$$

We can use kinetic theory to calculate the electron pressure

$$P_e = \frac{1}{3} \int_0^{p_F} \frac{p^2 c^2}{E} \frac{8\pi p^2}{h^3} dp = \frac{8\pi m_e^4 c^5}{3h^3} \int_0^{\theta_F} \sinh^4 \theta d\theta. \quad (7.5)$$

Most of the density is going to come from the baryons. Define a parameter μ_e , the mean atomic mass per electron. This number is determined by the chemical composition: for pure hydrogen, $\mu_e = 1$, as there is a single nucleon for each electron; for helium-4, $\mu_e = 2$; likewise for carbon-12, $\mu_e = 2$. In general,

$$\frac{1}{\mu_e} = \sum_i X_i \frac{Z_i}{A_i}, \quad (7.6)$$

where the sum is over all elements inside the star and X_i is the relative mass abundance. In terms of μ_e , we can express the baryon density as

$$\rho_B = \mu_e m_H n_e = \mu_e m_H \frac{8\pi}{3} \left(\frac{m_e c}{h} \right)^3 x^3 = B x^3, \quad (7.7)$$

where m_H is the mass of a hydrogen atom.

Define

$$f(x) = 8 \int_0^{\theta_F} \sinh^4 \theta d\theta = \left[(2x^3 - 3x) \sqrt{1+x^2} + \sinh^{-1} x \right]. \quad (7.8)$$

The electron pressure becomes

$$P_e = \frac{\pi m_e^4 c^5}{3h^3} \left[(2x^3 - 3x) \sqrt{1+x^2} + \sinh^{-1} x \right] = Af(x). \quad (7.9)$$

Note that

$$\frac{df(x)}{dx} = 8 \frac{d\theta}{dx} \frac{d}{d\theta} \int_0^{\theta_F} \sinh^4 \theta d\theta = \frac{8 \sinh^4 \theta_F}{\cosh \theta_F} = \frac{8x^4}{\sqrt{1+x^2}}. \quad (7.10)$$

Consider the equations of hydrostatic equilibrium:

$$\frac{d\Phi}{dr} = -\frac{1}{\rho} \frac{dP}{dr}, \quad \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho. \quad (7.11)$$

Here, the only contribution to pressure we are considering is the partial pressure from electrons P_e , and the only density we are considering is the baryon mass density ρ_B . Therefore, combining these,

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{Ar^2}{Bx^3} \frac{df(x)}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left(\frac{Ar^2}{Bx^3} \frac{8x^4}{\sqrt{1+x^2}} \frac{dx}{dr} \right) = -4\pi GBx^3. \quad (7.12)$$

Define a new radius ξ by $r = R_n \xi$, analogous to in the Lane-Emden equation, with

$$R_n = \sqrt{\frac{2A}{\pi GB^2}}. \quad (7.13)$$

Further define the new variable y by

$$y^2 = 1 + x^2. \quad (7.14)$$

Note

$$\frac{dy}{d\xi} = \frac{d}{d\xi} \sqrt{1+x^2} = \frac{x}{\sqrt{1+x^2}} \frac{dx}{d\xi}. \quad (7.15)$$

Then at last we have the equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{dy}{d\xi} \right) = -(y^2 - 1)^{3/2}. \quad (7.16)$$

This is the equation for Chandrasekhar's white dwarf. It is very similar to the Lane-Emden equation (6.1), and can be easily integrated numerically using the same techniques. At the center of the star the central value $y_o = y(\xi = 0)$ provides a free parameter; the surface will occur when $P_e = 0$, which happens when $f(x) = 0$, or when $x = 0$, which is to say when $y_1 = y(\xi_1) = 1$. Since y will decrease, this means y_o must be larger than 1.

7.1 The ChandrasekharWD Interface

ChandrasekharWD is a child of Star and must offer the same interface. In addition, ChandrasekharWD offers the following methods accessible to external code:

- `double getXi(int k):` return position variable $x_i = \text{xi}[k]$.
- `double getX(int k):` return Fermi-momentum variable $x = \text{x}[k]$.
- `double getY(int k):` return solution $y(\xi[k]) = \text{y}[k]$.
- `double getYderiv(int k):` return derivative $\frac{dy}{d\xi}|_{\xi[k]} = \text{z}[k]$.

These are used for testing against other solutions, such as tabulated values or relativistic solutions.

7.1.1 Constructors

There are two constructors available for a ChandrasekharWD:

- `ChandrasekharWD(double y0, double mue, int len)`: creates a WD of specific grid size (length) `len` with an initial value $y(0) = y0$, and with mean baryons-per-electron (μ_e) given by `mue`. To reproduce Chandrasekhar’s famous result of the limiting mass, use `mue = 2`. The step size `dx` needed to reach the surface ($y = 0$) in exactly `len` steps is found by bisection.
- `ChandrasekharWD(double y0, int len, double dx)`: as above, but specifies the step size to use as `dx`. There is no guarantee of reaching the “true” surface. [This is used for testing scaling relations. Do not use for production runs.](#)

7.2 The ChandrasekharWD Implementation

The ChandrasekharWD definition specifies the following class data and methods used in the implementation. Because the mathematical structure is very similar to a polytrope, the implementation is also very similar to the Polytrope.

7.2.1 Constants

- `int len`: the maximum index of the polytrope, which is returned by `length()` function.
- `double Y0, Y02`: the central value $y(0)$ and its square.
- `double X0, X02`: the central value $x(0) = \sqrt{y(0)^2 - 1}$ and its square.
- `double A0`: the central pressure. [This is set from physical variables described in Chandrasekhar 1939; to get better SSR, set to 1.](#)
- `double B0`: the central density. [This is set from physical variables described in Chandrasekhar 1939; to get better SSR, set to 1.](#)
- `double Rn`: the scale factor $R_n = \sqrt{\frac{2A}{\pi G B^2}}$ with $r = R_n \xi$, where ξ is the independent variable of equation (7.16).

7.2.2 Solution Variables

- `double *xi`: an array with the position variable ξ from (7.16).
- `double *x`: an array with the relativistic parameter $x = p_F/mc = \sqrt{y^2 - 1}$.
- `double *y`: an array with the solution y from (7.16).
- `double *z`: an array containing $z = \frac{dy}{d\xi}$, which is used in the integration method, and also allows the calculation of e.g. $\frac{d\rho}{dr}$ without the need for finite differencing.
- `double *f`: an array containing $f(x) = (2x^3 - 3x)\sqrt{1 + x^2} + 3 \sinh^{-1}(x)$.
- `double *mass`: an array containing the interior mass $m(r)$ as a function of radius, $m(r) = \int_0^r 4\pi r^2 \rho(r) dr$, or removing dimensional factors, $\text{mass}[X] = -4\pi \text{xi}[X]^2 * \text{z}[X]$.

7.2.3 Constructors

There are two constructors:

- `ChandrasekharWD(double Y0, double mu_electron, int len)`
- `ChandrasekharWD(double Y0, int len, double dx)`

In both, `Y0` is the initial value $y_o = Y0$ and `len` is the number of grid points to use. The parameter `mu_electron` = μ_e , and `dx` is the stepsize to use. The second method is intended for testing errors when the grid size is changed.

Excepting the equations appearing in the integration, the internal logic of the constructors is identical to `Polytrope`. See Sec. ?? for more info.

7.2.4 Calculation

- `double RK4integrate(int Len, double dx)`: integrate equation (7.16) from the center out to position `Len` using RK4. Returns the value `y[Len-1]` at the end of the integration, to be used in dissection search of `dx`. Because the equations call for $(y^2 - 1)^{3/2}$, the program can produce NaNs if $y < 1$. To prevent NaN errors, we use a complex variable $YCN = (y^2 - 1)^{3/2}$, and consider the real part of YCN . Since only $y \geq 1$ is physical, the integration stops when $y[X+1] < 1$ and returns `y[X+1]` instead.
- `int RK4integrate(int Len, double dx, int g)`: This method integrates as described above, all the way to the end of `Len`; it does not stop if $y < 1$. The return value is an integer, `Len`. [The integer `g` now mainly serves the role of distinguishing this method from the other method. Better naming would also work.](#)

7.2.5 Boundary conditions for modes

In order to implement the boundary conditions of modes, it is necessary to have power series expressions for A^*, U, c_1, V_g . For the case of a Chandrasekhar WD, because of the very simple mathematical structure it is possible to derive analytic formulas for these coefficients.

- `double yc[4]`: coefficients of power series for y near the center, in terms of dimensionless $s = r/R_\star = \xi/\xi_1$:

$$y(\xi) \approx y_0 + y_2 s^2 + y_4 s^4 + y_6 s^6 = yc[0] + yc[1]s^2 + yc[2]s^4 + yc[3]s^6.$$

- `double xc[2]`: coefficients of

$$x(\xi) \approx x_0 + x_2 s^2 = xc[0] + xc[1]s^2.$$

- `double fc[2]`: coefficients of

$$f(x(\xi)) \approx f_0 + f_2 s^2 = fc[0] + fc[1]s^2.$$

- `setupCenter()`: initialize the coefficients above. See `NewtonianBC.nb` for further information.
- `setupSurface()`: an empty method. Only for conformity.
- `getXYZCenter(double* xyz, int& maxPow)`: where `XYZ` is `Astar`, `Vg`, `U`, `C1`. Returns terms of the power series expansion in $s = r/R_\star$ near the center. The coefficients are stored in the `xyz[]` array to be returned. We have found terms up to order s^4 ; if more are requested with `maxPow`, then we set `maxPow = 4` so that the `ModeDriver` only implements the BC to order s^4 . The details of the calculations can be found in the Mathematica notebook `NewtonianBC.nb`.

- `getXYZSurface(double* xyz, int& maxPow)`: as above, but near the surface expanded in powers of $t = 1 - r/R_* = 1 - s$. Unlike the central boundary, there is no requirement that only even powers appear. The stellar variable A^* has a pole at $t = 0$, so requires a t^{-1} term.

Due to the nature of the recursion relation at the surface, to implement the BCs to 4th order, we need A^*, V_g, c_1 to t^3 (which includes the t^{-1} term of A^*). All U terms will simply vanish. The details of the calculations can be found in the Mathematica notebook `NewtonianBC.nb`.

In both the center and the surface, the BC is given to **fourth order**.

7.3 Internal Logic Behind the Interface

With the above variables, we can now define how the following `Star` quantities are computed:

- `double rad(int X)`: $r = R_n \xi$.
- `double rho(int X)`: $\rho = B x^3$.
- `double drhodr(int X)`: $\frac{d\rho}{dr} = \frac{3B}{R_n} x^2 \frac{dx}{d\xi} = \frac{3B}{R_n} x^2 \frac{y}{x} \frac{dy}{d\xi} = \frac{3B}{R_n} x y \frac{dy}{d\xi} = 3B\theta * x[X] * y[X] * z[X] / R_n$.
- `double P(int X)`: $P = A * f(x) = A\theta * f[X]$.
- `double dPdr(int X)`: $\frac{dP}{dr} = \frac{8A}{R_n} * x^3 \frac{dy}{d\xi}$, see eqn. 9 in Chandrasekhar 1935.
- `double Phi(int X)`: Φ which has been matched so that $\Phi(R) = -\frac{GM}{R}$.
- `double dPhidr(int X)`: $g = \frac{d\Phi}{dr} = -\frac{1}{\rho} \frac{dP}{dr} = -\frac{8A}{BR_n} * \frac{dy}{d\xi}$.
- `double mr(int X)`: $m(r) = -4\pi R_n^3 B * x^2 \frac{dy}{d\xi} = B\theta * R_n^3 * \text{mass}[X]$.
- `double Schwarzschild_A(int X, double GamPert)`:

$$A = -\frac{dy/d\xi}{R_n} \left[\frac{8x^3}{\Gamma_1 f(x)} - \frac{3y}{x^2} \right]$$

If `GamPert=0`, uses Γ_1 from the equation of state (which will result in $A^* = 0$).

- `double getAstar(int X, double GamPert)`:

$$A^* = \xi \frac{dy}{d\xi} \left(\frac{8x^3}{f(x)\Gamma_1} - \frac{3y}{x^2} \right).$$

- `double getU(int X)`: $U = \frac{4\pi\rho r}{d\Phi/dr} = -\frac{\xi x^3}{dy/d\xi}$. Special care must be taken at $X = 0$.
- `double getVg(int X, double GamPert)`: returns $V_g(r) = \frac{rg}{v_s^2} = -\frac{8\xi x^3}{\Gamma_1 f(x)} \frac{dy}{d\xi}$, where $\Gamma_1 = \text{GamPert}$. If no Γ_1 is specified, uses $\Gamma_1 = \text{Gamma1}(X)$, leading to $V_g(r) = -\frac{3\xi y}{x^2} \frac{dy}{d\xi}$.
- `double getC(int X)`: returns $c_1(r) = \frac{M}{m(r)} \frac{r^3}{R^3} = \frac{z_1}{\xi_1} * \frac{\xi}{dy/d\xi}$. Special care must be take at $X = 0$.
- `double sound_speed2(int X, double GamPert)`: $v_s^2 = \frac{\Gamma_1 P}{\rho} = \frac{\Gamma_1 A}{B} * \frac{f(x)}{x^3}$, where $\Gamma_1 = \text{GamPert}$. If no Γ_1 is specified, then $v_s^2 = \frac{8A}{3B} * \frac{x^2}{y}$.
- `double Gamma1(int X)`: $\Gamma = \frac{8}{3} * \frac{x^5}{y f(x)}$.

These values are all calculated at $r = \text{rad}(X)$. The other methods, `length()`, `Radius()`, etc., are trivially defined.

7.4 References

- Chandrasekhar (1935), MNRAS **95** for the first publication of (7.16).
- Chandrasekhar (1939), *An Introduction to the Study of Stellar Structure*, a classic text available from Dover, which includes any tabulated values as a means of testing.

Chapter 8

The MESA Class

This class is used to read in data from MESA output to be used with our mode-calculation routines. We will assume we have calculated a stellar profile with MESA which has been saved with name `mymesa.dat`. The code is assuming output given by MESA v 1.01 in the `.dat` file – output columns explained here.

GRPulse is distributed with two sample profiles calculated with MESA, one for a cold and another for a hot WD model. Both of these were calculated by simply modifying the `wd_cool_0.6M` example in the standard MESA test suite that comes with the MESA distribution. The sample input file `sampleinput5.txt` will calculate several modes for a MESA model.

8.1 The MESA Interface

MESA is a child of `Star`, which means it must implement the `Star` interface. This is a minimal stellar model, as all work has been done for us by MESA.

8.1.1 Constructors

There is only one constructor:

- `MESA(const char* file, int L)`: opens and reads in the MESA output stored in the file, whose address is given by `file`; the final model is to be interpolated to have a specific grid size (length) *close to* `L`.

8.2 The MESA Implementation

The MESA definition specifies the following class data and methods used in the implementation.

Many of the variables below are objects called `Splinors`, which are spline interpolators. These all use the radial coordinate as the independent variable (x), and different columns of the MESA file (or calculated quantities) as the dependent variable (y).

These `Splinors` can be called like a function, taking the argument $x = r/R_*$ as in `radi[]`. `Splinors` also have a derivative method (`deriv()`), which returns the derivative of the interpolating spline at the given x .

8.2.1 Constants

- `int len`: The desired gridsize of the model to be used in pulsations.
- `int subgrid`: The number of subgrid points needed to achieve the closest proximity to `len`. Due to the nature of integer division, and the need for half-steps, the actual gridsize will not be exactly as desired.
- `int Ntot`: The original number of grid points in the first line of MESA file.

- `int Mtot`: the mass of the star M_\star in CGS units, read from first line of MESA file.
- `int Rtot`: the radius of the star R_\star in CGS units, read from first line of MESA file.
- `double Dscale`: density scale $\frac{M_\star}{4\pi R_\star^3}$.
- `double Pscale`: pressure scale $\frac{GM_\star^2}{4\pi R_\star^4}$.
- `double Gscale`: gravitational scale GM_\star/R_\star^2 .

8.2.2 Solution Variables

- `double *radi`: The increased radial grid, $x = r/R_\star$, of length `len`. Takes initial data from MESA file, column 2; augmented with subgrids between points.
- `Spliner *dens`: density from MESA column 7, quantity $\hat{\rho} = \rho/\text{Dscale}$.
- `Spliner *pres`: pressure from MESA column 5, quantity $\hat{P} = P/\text{Pscale}$.
- `Spliner *mass`: mass from MESA column 3, quantity $\hat{m} = m/\text{Mtot}$.
- `Spliner *grav`: gravitational field strength calculated as $\hat{g} = \hat{m}/x^2$.
- `Spliner *BVfq`: Brunt-Väisälä frequency from MESA column 9, quantity $\hat{N}^2 = N^2 * R_\star^2/(GM_\star)$.
- `Spliner *Gam1`: adiabatic index Γ_1 from MESA column 10.

8.2.3 Stellar Properties for Waves

- `Spliner *aSpline`: stellar variable $A^* = -\hat{N}^2 x/\hat{g}$.
- `Spliner *vSpline`: stellar variable $V = \hat{g}x\hat{\rho}/\hat{p}$ (note: this is *not* V_g).
- `Spliner *cSpline`: stellar variable $c_1 = x^3/\hat{m}$.
- `Spliner *uSpline`: stellar variable $U = \hat{\rho}x^3/\hat{m}$.

8.2.4 Constructor

There is only one constructor:

- `MESA(char *file, int len)`

It begins by opening the file specified by `file`, and reads in `Ntot`, which MESA prints in the 1st line. The data contains $r, m(r), \rho, P, N^2, \Gamma_1$ in CGS units, and we calculate $g = Gm/r^2$, along with A, U, c_1, V .

While CGS units are great for scales of a tablespoon of water at STP, they are not so great when talking about literally astronomical pressures, temperatures and volumes. For this reason, all quantities are converted into units where $M_\star = R_\star = G = 1$, which perform better numerically. We define the Schwarzschild dimensionless units, represented by a hat, as (see Schwarzschild 1958)

$$r = R_\star x, \quad m = M_\star \hat{m}, \quad P = \frac{GM_\star^2}{4\pi R_\star^4} \hat{P}, \quad \rho = \frac{M_\star}{4\pi R_\star^3} \hat{\rho}, \quad g = \frac{GM_\star}{R_\star^2} \hat{g}. \quad (8.1)$$

We now augment the grid so that there are *at least* `len` points. The simplest way to do this is to add evenly spaced subgrid points between each pair of data points. We choose to divide each subgrid into a binary fraction, to ensure there are half-steps for use in the mode solution. This means you can not exactly specify how many grid points you can use, and you must use twice as many as are available in the MESA output.

Once the grid is formed, we initialize all `Spliner` quantities.

We next set `indexFit`, the fitting index for mode calculations, as the location where $\rho = 0.5\text{dens}[0]$.

8.2.5 Calculation

The only calculation to be performed is the spline-fitting, which is handled automatically by `Splnor`. Everything else has been handled by MESA to generate the `.dat` file.

This class exists only to allow mode calculations with data computed by MESA. The mode routines we have defined rely on simple RK4, which requires evaluation of the derivatives at half-grid points (i.e. for $x[1]$, requires an evaluated at $x[1] + \frac{1}{2}[x[2] - x[1]]$). It is highly unlikely that the grid chosen by MESA will give these values. Therefore, in this program we form our own grid, which must at minimum define the half-grid points. See Figure 8.1.



Figure 8.1: The circles labeled represent the grid as computed by MESA. These are drawn to be unevenly spaced, as they are in most MESA data files. The squares labeled are the half-grid points needed by the modes' RK4 algorithms. These must be determined by spline-fitting.

8.2.6 Boundary conditions for modes

In order to implement the boundary conditions of modes, it is necessary to have power series expressions for A^*, U, c_1, V_g . This is more difficult to do for the complicated physical processes taking place inside a white dwarf star.

- `setupCenter()`: In the center of the star, suppose we have a general equation of state $\hat{P} = F(\hat{\rho}, \hat{T})$ in the center, with \hat{T} a dimensionless temperature. We can then find expansions for $\hat{\rho}, \hat{P}$ in terms of $x = r/R_*$,

$$\hat{T} = \hat{T}_0 + \hat{T}_2 x^2 + \hat{T}_4 x^4 + \dots \quad (8.2a)$$

$$\hat{\rho} = \hat{\rho}_0 + \hat{\rho}_2 x^2 + \hat{\rho}_4 x^4 + \dots \quad (8.2b)$$

$$= \hat{\rho}(0) + \left(\frac{\hat{P}_2 - \hat{T}_2 F_T}{F_\rho} \right) x^2 + \left(\frac{2\hat{P}_4 - 2\hat{T}_4 F_T - \hat{T}_2^2 F_{TT} - 2\hat{T}_2 \hat{\rho}_2 F_{T\rho} - \hat{\rho}_2^2 F_{\rho\rho}}{2F_\rho} \right) x^4 \quad (8.2c)$$

$$\hat{P} = \hat{P}_0 + \hat{P}_2 x^2 + \hat{P}_4 x^4 + \dots \quad (8.2d)$$

$$= \hat{P}(0) + \left(-\frac{1}{6} \hat{\rho}_0^2 \right) x^2 + \left(-\frac{2}{15} \hat{\rho}_0 \hat{\rho}_2 \right) x^4 + \dots, \quad (8.2e)$$

in terms of which

$$c_1 = \frac{3}{\hat{\rho}_0} - \frac{9}{5} \frac{\hat{\rho}_1}{\hat{\rho}_0^2} x^2 + \left(\frac{27}{25} \frac{\hat{\rho}_2^2}{\hat{\rho}_0^3} - \frac{9}{7} \frac{\hat{\rho}_4}{\hat{\rho}_0^2} \right) x^4 + \dots \quad (8.3a)$$

$$U = 3 + \frac{6}{5} \frac{\hat{\rho}_2}{\hat{\rho}_0} x^2 + \left(\frac{12}{7} \frac{\hat{\rho}_4}{\hat{\rho}_0} - \frac{18}{25} \frac{\hat{\rho}_2^2}{\hat{\rho}_0^2} \right) x^4 + \dots \quad (8.3b)$$

$$V_g = -\frac{2\hat{P}_2}{\hat{P}_0 \Gamma_1} x^2 + \left(\frac{2\hat{P}_2^2}{\hat{P}_0^2 \Gamma_1} - \frac{4\hat{P}_4}{\hat{P}_0 \Gamma_1} \right) x^4 + \dots \quad (8.3c)$$

$$A^* = \left(\frac{2\hat{P}_2}{\hat{P}_0 \Gamma_1} - \frac{2\hat{\rho}_2}{\hat{\rho}_0 \Gamma_1} \right) x^2 + \left(\frac{4\hat{P}_4}{\hat{P}_0^2 \Gamma_1} - \frac{2\hat{P}_2^2}{\hat{P}_0^2 \Gamma_1} - \frac{4\hat{\rho}_4}{\hat{\rho}_0} + \frac{2\hat{\rho}_2^2}{\hat{\rho}_0^2} \right) x^4 + \dots \quad (8.3d)$$

These are general. For MESA, we approximate the equation of state in the center as a polytrope with $n = \frac{1}{\Gamma_1(0)-1}$. We can then find series coefficients for θ around x (see Sec. 6.2.5)

$$\theta = \theta_0 + \theta_2 x^2 + \theta_4 x^4 + \dots \quad (8.4)$$

in terms of which $\hat{\rho}_2 = n\hat{\rho}_0\theta_2$ and $\hat{\rho}_4 = \frac{1}{2}\hat{\rho}_0(2n\theta_4 + n(n-1)\theta_2^2)$. The rest then follows.

- `getXYZCenter(double* xyz, int& maxPow)`: where XYZ is Astar, Vg, U, C1 Returns terms of the power series expansion as above. The coefficients are replaced into the xyz[] array to be returned. If more are requested with maxPow, then we set maxPow = 4 so that the ModeDriver only implements the BC to order x^4 .
- `setupSurface()`: Follows a similar method as for the center above, except at the surface expanded in powers of $t = 1 - r/R_*$. The EOS near the surface is assumed to be an ideal gas plus radiation pressure. See the `NewtonianBCs.nb` notebook for more details.
- `getXYZSurface(double* xyz, int& maxPow)`: as above, but near the surface expanded in powers of $t = 1 - r/R$.

Within the directory `output/calc_name/star/wave_coefficient`, there are several files for testing the goodness of these power series expansions. The coefficients of the expansion are printed to files `center.txt`, `surface.txt`. The first few points of $\rho, P, A^*, U, V_g, c_1$ are printed, from the stellar data and using the expansion function, for comparison, and the errors printed in the corresponding .png files for easier viewing.

8.3 Internal Logic Behind the Implementation

With the above variables, we can now define how the following Star quantities are computed:

- `double rad(int X)`: $r = R_{\text{tot}} * \text{radi}[X]$.
- `double rho(int X)`: $\rho = D_{\text{scale}} * \text{dens}(X)$.
- `double drhodr(int X)`: $\frac{d\rho}{dr} = D_{\text{scale}}/R_{\text{tot}} * \text{dens}'(X)$.
- `double P(int X)`: $P = P_{\text{scale}} * \text{pres}(X)$.
- `double dPdr(int X)`: $\frac{dP}{dr} = P_{\text{scale}}/R_{\text{tot}} * \text{pres}'(X)$.
- `double Phi(int X)`: **Not yet implemented (07/15/2021)** .
- `double dPhidr(int X)`: $g = \frac{d\Phi}{dr} = G_{\text{scale}} * \text{grav}(X)$.
- `double mr(int X)`: $m(r) = M_{\text{tot}} * \text{mass}(X)$.
- `double Schwarzschild_A(int X, double GamPert)`: If GamPert=0, $A = \text{aSpline}(X)/\text{radi}[X]/R_{\text{tot}}$. Otherwise,

$$A = \frac{\text{dens}'(X)}{\text{dens}(X) * R_{\text{tot}}} - \frac{\text{pres}'(X)}{\text{Gam1}(X) * \text{pres}(X) * R_{\text{tot}}}.$$

(Note units! A^* is dimensionless, but Schwarzschild discriminant is not.)

- `double getAstar(int X)`: If GamPert=0, $A^* = \text{aSpline}(X)$. Otherwise

$$A = \frac{\text{radi}[X] \text{pres}'(X)}{\text{Gam1}(X) * \text{pres}(X)} - \frac{\text{radi}[X] \text{dens}'(X)}{\text{dens}(X)}.$$

- `double getU(int X)`: $U = \text{uSpline}(X)$.
- `double getVg(int X, double GamPert)`: $V_g = \text{vSpline}(X)/\text{GamPert}$. If GamPert=0, $V_g = \text{vSpline}(X)/\text{Gam1}(X)$.

- `double getU(int X): U = uSpline(X).`
- `double getC(int X): c1(r) = cSpline(X).`
- `double sound_speed2(int X, double GamPert):` $v_s^2 = \frac{\Gamma_1 P}{\rho}$. If `GamPert=0`, $v_s^2 = \frac{\text{Gam1}(X)*\text{pres}(X)}{\text{dens}(X)}$. Otherwise, $v_s^2 = \frac{\text{GamPert}*\text{pres}(X)}{\text{dens}(X)}$.

For brevity's sake, all calls to interpolation are written as `func(X)`; in C++ syntax, they must actually be written `func->interp(radi[X])`. All calls to interpolated derivatives are written as `func'(X)`, instead of `func->deriv(radi[X])`.

8.4 Limitations

The `SSR()` method used for modes assumes a uniform grid spacing when calculating derivatives; for this reason, the printed mode `SSR` in output from this model will be higher than expected, due to numerical errors introduced in the derivatives. For modes with $l \geq 2$, I suggest using the c_0 column (the last one) from the `tidal_overlap.txt` file as a better measurement of error.

For hot WD stars in the instability strip, for some reason many modes will fail to calculate, and errors are high. Cold stars below the instability strip seem to calculate well. When referring to eigenmodes calculated from MESA, check the graph of the mode. The central **red vertical line** shows the fitting point of the double-shooting method; ensure regular behavior around this line.

8.5 References

- Paxton et al. (2010), the original paper introducing MESA to the world, describing the physics and numerical methods used within the code.
- Paxton et al. (2013), describes extensions to MESA, including new capabilities for small planets.
- Paxton et al. (2015), further extends MESA, including modeling supernovae.
- Schwarzschild (1958), *Structure and Evolution of the Stars*, available from Dover, as a reference for the dimensionless variables chosen here.
- Fuller & Lai (2011), for an explanation of the c_0 column (and Q_α column) of `tidal_overlap.txt`. See especially their Table 1.

Chapter 9

The PNStar Extension of Star

The class `Star` represents a Newtonian star in equilibrium. In the limit of Newtonian physics, the gravitational field can be represented by a single scalar field, Φ , the Newtonian potential, sourced by

$$\nabla^2 \Phi = 4\pi G \rho, \quad (9.1a)$$

where ρ is the total mass-energy density (i.e. $\rho = \varepsilon c^2$). For situations where the 1PN limit of gravity holds, it is necessary to include additional fields: the gravitomagnetic potential \vec{W} and the 1PN scalar potential Ψ , sourced by

$$\nabla^2 \vec{W} = 4\pi G \rho \vec{v} \quad (9.1b)$$

$$\nabla^2 \Psi = 4\pi G [2\rho v^2 - \rho \Phi + 3P] + \frac{\partial^2 \Phi}{\partial t^2}. \quad (9.1c)$$

The three fields are necessary to describe the gravitational structure of a star at 1PN order.

Any pulsation mode utilizing 1PN physics will need to overwrite its `*star` pointer to be of type `PNStar` in order to get the correct gravitational fields.

9.1 The PNStar Interface

In addition to the `Star` methods, `PNStar` defines the following methods for external code to interact with it:

9.1.1 Physics

- `double Psi(int X)`: the 1PN scalar potential $\Psi(r)$.
- `double dPside(int X)`: the 1PN gravitational field $\frac{d\Psi}{dr}(r)$.
- `double Wx(int X), Wy(int X), Wz(int X)`: the indicated component of the gravitomagnetic potential \vec{W} .
- `double dWxdr(int X), dWydr(int X), dWzdr(int X)`: the derivative of the indicated component of \vec{W} .

9.1.2 Boundary conditions for modes

- `void getXYZCenter(double* xyz, int& maxPow)`: where XYZ could be $\text{Beta} = \beta_s^2 = v_s^2/\Phi_s$, $\text{Phi} = \Phi/\Phi_s$, with $\Phi_s = \Phi(x=1)$. Returns coefficients from power-series expansion of the stated quantity in $x = r/R_\star$ near the center. The highest power desired is `maxPow`; if the stellar model does not implement that power, then `maxPow` must be lowered to match. The coefficients are returned within the array `xyz`.
- `void getXYZSurface(double* xyz, int& maxPow)`: as above, but coefficients from power-series expansion in $t = 1 - r/R_\star$ near the surface.

9.2 How to make new PNStar children

To make new models of stars intended to implement 1PN physics, in addition to all of the methods of `Star`, you must implement the above methods for the 1PN potentials, as well as the boundaries.

The definition of the field must match our own convention. Our choice of symbol \vec{W} for the gravito-magnetic field is somewhat non-standard, but our definition matches that across the literature. For the 1PN scalar field Ψ , there is no standard definition, even though almost everyone uses the symbol Ψ . Note that the definition of Ψ in Weinberg or in Poisson & Will do not match ours. When we speak of density ρ for a 1PN star, we mean the total mass-energy density; it is common in 1PN literature to see baryon density also called ρ , and Poisson & Will prefer to use a coordinate invariant ρ^* instead. The `PNStar` class is indifferent to these definitions, provided they are consistent with the 1PN mode equations being used.

9.3 References

- Poisson and Will
- Weinberg
- Chandrasekhar

Chapter 10

The PNPolytrope Class

This class represents a First Post-Newtonian (1PN) polytropic star with given index n and relativistic parameter σ (same as for the GR case). Because it uses 1PN physics, it inherits from `PNStar`. Relativistic polytropes are described by the 1PN Lane-Emden equations, Boston (2021).

$$\frac{1}{\xi^2} \frac{d}{dx} \left(\xi^2 \frac{d\theta}{dx} \right) = -\theta^n + \sigma \left[2(n+1)\theta^n \phi - \frac{d\phi}{d\xi} \frac{d\theta}{d\xi} - 4\theta^{n+1} \right] \quad (10.1a)$$

$$\frac{1}{\xi^2} \frac{d}{dx} \left(\xi^2 \frac{d\phi}{dx} \right) = \theta^n \quad (10.1b)$$

where the function $\theta(\xi)$ is related to density by

$$\rho(r) = \rho_c \theta(\xi)^n, \quad (10.2)$$

the function $\phi(x)$ is related to the Newtonian potential by

$$\Phi(r) = \frac{(n+1)P_c}{\rho_c} \phi(\xi), \quad (10.3)$$

the function $z(x) = \frac{d\theta}{d\xi}$ as in the case of `Polytrope`, $u(x) = \frac{d\phi}{dx}$, and the function $v(x)$ is defined by

$$\frac{d\Psi(r)}{dx} = \frac{(n+1)P_c^2}{\rho_c^2} v(x). \quad (10.4)$$

This set of equations is integrated numerically by methods defined within this class.

The parameter σ is a tunable parameter for specifying the strength of relativity, and is related to the surface redshift, $z = \frac{GM}{c^2 R}$. For purposes of using astrophysically meaningful quantities, we define the polytrope in terms of z . The relationship between σ and z depends on u_1 , the value of $u(x)$ at the surface as in

$$z = \frac{GM}{c^2 R} = \frac{4\pi G \rho_c R_n^3 x_1^2 u_1}{c^2 R_n x_1} = (n+1)\sigma x_1 u_1. \quad (10.5)$$

Given a desired value of z , the correct σ to use can only be known after having integrated the polytrope all the way to the surface, for which σ is needed in advance. Therefore, to ensure consistency, an iterative process is used, updating from an initial guess for σ until it corresponds to the desired redshift z .

10.1 The PNPolytrope Contract

`PNPolytrope` is a child of `PNStar`, which means it must uphold the `PNStar` contract, which includes the `Star` contract – i.e., the functions returning ρ, P, Φ , and functions returning η, V_p etc.

In addition, `PNPolytrope` offers the following methods accessible to external code:

- `double getX(int k)`: return position variable $x = x[k]$.
- `double getY(int k)`: return solution $y(x[k]) = y[k]$.
- `double getYderiv(int k)`: return derivative $\frac{dy}{dx}(x[k]) = z[k]$.
- `double getXproper(int k)`: returns a proper radial distance

$$\bar{x} = \int_0^x e^{\lambda/2} dx = \int_0^x [1 - 2\sigma(n+1)\nu(x)/x]^{-1/2} dx,$$

which is used in comparison to literature values. Performed using trapezoidal integration.

These are used for testing against analytic solutions, or for testing against Newtonian and GR solutions. The radius defined by $r = R_n \xi$ is an isotropic radial coordinate with no global meaning; therefore the proper radius \bar{r} must be used in comparisons to other solutions.

10.1.1 Constructors

There are two constructors available for a `PNPolytrope`:

- `PNPolytrope(double n, double zsurf, int len)`: creates a 1PN polytrope of specific grid size (length) `len`, of polytropic index $n = n$, with surface redshift $z = GM/c^2 R = zsurf$. The step size `dx` needed to reach the surface ($y = 0$) in exactly `len` steps is found by bisection, and $\sigma = \sigma$ is found from an iterative process.
- `PNPolytrope(double n, double sigma, int len, double dx)`: as above, but specifies the step size to use as `dx`. There is no guarantee of reaching the “true” surface. [This is used for testing scaling relations. Do not use for production runs.](#)

10.2 The PNPolytrope Implementation

10.2.1 Physics

Because the polytropic equation of state $P = K\rho^\Gamma$ is the same in the relativistic case, the definitions of ρ, P , and their derivatives will remain the same. The mass is also calculated similarly to `GRPolytrope`, as

$$m(r) = 4\pi\rho_c R_n^3 x^2 u(x).$$

The `PNPolytrope` class defines the following:

- `zsurf`: the surface gravitational redshift $z = \frac{GM}{c^2 R}$.
- `sigma`: the relativistic parameter as defined by Tooper (1964), $\sigma = \frac{P_c}{\rho_c c^2}$.
- `*phi`: an array with ϕ as in (10.1).¹
- `*u`: an array with $u = d\phi/dx$
- `*v`: an array with v as in (10.4).

Other variables (e.g. `x`, `y`, `n`, etc.) are the same as for `Polytrope`. With the above, we now define how the following `PNStar` quantities are computed:

- `double deriv1PN(double x, double y, double z, double u, double phi)`: the derivative $\frac{d^2 y}{dx^2}$ as determined by (10.1).
- `double phi0`: the central value of ϕ needed to make Φ match the surface redshift. Chosen by iteration.

¹ Note that the program listing for this class contains non-ASCII characters. This should be supported by the gcc compiler.

- `double Phi(int X)`: returns $\Phi(r) = \frac{(n+1)P_c}{\rho_c} \phi(x)$.
- `double dPhidr(int X)`: returns $\frac{d\Phi}{dr}(r) = \frac{(n+1)P_c}{\rho_c} u(x)/R_n$.
- `double mr(int X)`: returns the interior mass, $m(r) = 4\pi\rho_c R_n^3 x^2 u(x)$.
- `double dPsidr(int X)`: returns $\frac{d\Psi(r)}{dr} = \frac{(n+1)P_c^2}{\rho_c^2} v(x)/R_n$.
- `double Schwarzschild_A(int X, double GamPert)`: returns $A(r) = \frac{dy/dx}{y(x)R_n} \left(\frac{n}{1+\sigma y(x)} - \frac{n+1}{\Gamma_1} \right)$, where $\Gamma_1 = \text{GamPert}$. If no Γ_1 is specified, uses $\Gamma_1 = \Gamma$.
- `double getU(int X)`: returns $U(r) = \frac{d \log g}{d \log r} + 2 = x[X]y[X]^n (1 + \sigma(3y[X] - 2(n+1)\phi[X])) / (u[X] + \sigma v[X])$.
- `double getVg(int X, double GamPert)`: returns $V_g = -\frac{1}{\Gamma_1} \frac{d \log P}{d \log r} = -\frac{(n+1)x}{\Gamma_1} \frac{y'}{y} = -(n+1)x[X]dydx[X]/y[X]/\text{GamPert}$. When no `GamPert` is specified, uses $\Gamma_1 = \Gamma$.
- `double getC(int X)`: returns $c_1 = \frac{GM}{R^3} \frac{r}{g} = \frac{x[X]}{x[\text{len}-1]} u[\text{len}-1] / (u[X] + \sigma v[X])$.
- `double sound_speed(int X, double GamPert)`: returns $v_s^2 = \frac{\Gamma_1 P}{\rho + P/c^2} = \frac{\text{GamPert} P_0}{r h o 0} \frac{y[X]}{1 + \sigma y[X]}$. When no `GamPert` is specified, uses $\Gamma_1 = \Gamma$.
- `double Gamma1(int X)`: returns $\Gamma = 1 + \frac{1}{n}$.

The other variables are calculated similarly as for `Polytrope` and aren't repeated.

10.2.2 Constructors

There are two constructor methods:

- `PNPolytrope(double n, double zsurf, int len)`
- `PNPolytrope(double n, double sigma, int len, double dx)`

These each work very much like their counterpart in the `Polytrope` class, with some minor differences.

Because there is no analytic solution when $n = 5$, that case is excluded from the range of allowed values. When either z or σ is too large, the stars likewise become unphysical, so certain values are also excluded.

The proper value for dx is found using bisection, as in ???. The principle difference is an additional step where the integration is reiterated several times to ensure that $z = GM/c^2 R = \text{zsurf}$. We start with a guess $\text{sigma} = \frac{\text{zsurf}}{1+n}$ corresponding to a guess $u_1 = x_1 = 1$ in (10.5), and integrate. The end result gives the true $u_1 = u[\text{len}-1]$ (which is likely not 1), and thence to a guess for z . If this does not match `zsurf`, then a new σ is chosen as in (10.5), and the process repeats.

Simultaneously with looking for a proper σ so that $z = (n+1)\sigma u_1 x_1$, we also need to determined a central value $\phi_c = \phi c0$ such that $\Phi_1 = \frac{(n+1)P_c}{\rho_c} \phi_1 = -\frac{GM}{R}$, to ensure consistency of the internal gravitational field with the external gravitational field vanishing at infinity. This occurs when $\phi_1 = -u_1 x_1$.

The alternative constructor `Polytrope(double n, double sigma, int len, double dx)` works nearly identically to its counterpart, but with a specification of `sigma` and `dx`. This constructor should only be used when testing how quantities scale with changes in the number of grid points.

10.2.3 Calculation

There are three methods to integrate the entire length of the star:

- `complex<double> RK4integrate(double dx)`: integrates in step sized of dx until the end is found, defined by $y \leq 0$. It then adjusts the guesses for $\phi c0$ and σ using the terminal values of x , $\frac{dy}{dx}$ and ϕ . The return is a complex number whose two values are guesses of $z = (n+1)\sigma u_1 \xi_1 = -(n+1)\sigma \phi_1$. Running this near the beginning of the constructor will help make sure σ, ϕ have reasonable values.

- `double RK4integrate(double n, double zsurf, int len)`
- `int RK4integrate(double n, double sigma, int len, double dx)`

These last two methods work very much like their counterparts in the `Polytrope` class, described in ??, with some minor differences. The set of first-order ODEs that are solved by the RK4 method are:

$$K_a = dx \cdot y'(x_{c,a}) = z_{c,a} dx \quad (10.6)$$

$$L_a = dx \cdot z'(x_{c,a}) = \left(-y_{c,a}^n - \frac{2z_{c,a}}{x_{c,a}} + \sigma [2(n+1)y_{c,a}^n \phi_{c,a} - 4y_{c,a}^{n+1} - u_{c,a} z_{c,a}] \right) dx \quad (10.7)$$

$$P_a = dx \cdot \phi'(x_{c,a}) = u_{c,a} dx \quad (10.8)$$

$$M_a = dx \cdot u'(x_{c,a}) = \left(y_{c,a}^n - \frac{2u_{c,a}}{x_{c,a}} \right) dx \quad (10.9)$$

$$N_a = dx \cdot v'(x_{c,a}) = \left(3y_{c,a}^{n+1} - 2(n+1)y_{c,a}^n \phi_{c,a} - \frac{2v_{c,a}}{x_{c,a}} \right) dx \quad (10.10)$$

These are solved at each step a in the algorithm. At the end of the integration, the value of σ and the starting value of $\phi(x=0)$ is changed to ensure consistency with `zsurf`.

10.2.4 Boundary conditions for modes

`PNPolytrope` is responsible for specifying the boundary conditions for `Modes` that use it as a background. These methods are

- `double theta[2]: coefficients`

$$y(x) \approx y_0 + y_2(r/R)^2 = \theta c[0] + \theta c[1](r/R)^2.$$

- `double phi[2]: coefficients`

$$\phi(x) \approx \phi_0 + \phi_2(r/R)^2 = \phi c[0] + \phi c[1](r/R)^2.$$

- `double psi[2]: coefficients`

$$\psi(x) \approx y_0 + y_2(r/R)^2 = \psi c[0] + \psi c[1](r/R)^2.$$

- `double theta_s[4]: coefficients (with $t = 1 - r/R$)`

$$y(t) \approx y_0 + y_1 t + y_2 t^2 + \dots = \theta s[0] + \theta s[1]t + \theta s[2]t^2 + \dots$$

- `double phi_s[4]: coefficients`

$$\phi(r) \approx \phi_0 + \phi_2 t^2 + \dots = \phi c[0] + \phi c[1]t^2 + \dots$$

- `double psi_s[4]: coefficients`

$$\psi(x) \approx \psi_0 + \psi_2 t^2 + \dots = \psi c[0] + \psi c[1]t^2 + \dots$$

- `double igs[4]: coefficient of $\frac{1}{y(x)}$ expanded in a power series in $t = 1 - x$ near the surface.`
- `setupCenter(): initialize the center coefficients above.`
- `setupSurface(): initialize the surface coefficients above.`
- `getXYZCenter(double* xyz, int& maxPow):` where XYZ is Astar, Vg, U, C1, Beta, Phi. Same as for `Polytrope`. Implemented to second order.
- `getXYZSurface(double* xyz, int& maxPow):` as above, but near the surface expanded in powers of $t = 1 - r/R$.

As opposed to `Polytrope`, the coefficients in the 1PN case are much more difficult; therefore, they are given up to **second order**. There is too much to write out; see the Mathematica notebook `1pn3_wave_BC.nb` for detailed explanation.

10.3 Limitations

The 1PN equations of stellar dynamics are only a good choice in situations where relativity is mild. Errors in these models scales like z^2 . For this, 1PN polytropes should only be used as toy models for stars such as white dwarfs or asymptotic giants, where $z^2 \sim \mathcal{O}(10^{-8})$ or smaller. They do not make good toy models for neutron stars or other highly compact objects, there $z^2 \sim \mathcal{O}(10^{-2})$ or larger.

10.4 References

- Poisson & Will (2004), for a rigorous treatment of the Post-Newtonian theory.
- Weinberg (1972), for an alternate look at Post-Newtonian theory.
- Tooper (1964, 1965), for the fully relativistic polytrope equations. Note that the post-newtonian form he gives in the (1965) paper is calculated with a different definition of the radial coordinate as we use, making his 1PN equations non-comparable to ours. They can be compared *numerically*, using the proper radius \bar{x} defined in `getXproper()` above.
- Cutler
- Cutler & Lindblom

Chapter 11

The GRStar Extension of Star

In GR, gravity is a tensor-field theory, where the metric tensor $g_{\mu\nu}$, describing the curvature of spacetime, can be taken as a basic quantity. To model a stellar interior within the limit of relativity, it will be necessary to model the components of the metric. For the static, non-rotating stars we consider, the metric can be written in the form

$$ds^2 = g_{00}c^2dt^2 + 2g_{tr}cdtdr + g_{rr}dr^2 + g_{\theta\theta}d\Omega^2. \quad (11.1)$$

Following Tolman (1934) and Oppenheimer & Volkoff (1939), it is customary to write this in the form

$$ds^2 = -e^{\nu(r)}c^2dt^2 + e^{\lambda(r)}dr^2 + r^2d\Omega^2, \quad (11.2)$$

where $\nu = \log(-g_{00})$ and $\lambda = \log(g_{rr})$. Therefore, to describe a star within GR, we need expressions for all of the metric components. The class `GRStar` is meant to account for those fields, and serve as a stand-in for all stars implementing general relativity.

Any pulsation modes utilizing GR will need for its `*star` pointer to be of type `GRStar` instead of `Star` in order to get the correct gravitational fields.

11.1 The GRStar Interface

In addition to the `Star` methods, `GRStar` defines the following methods for external code to be able to interact with it:

11.1.1 Physics

- `double grr(int X), gtt(int X), gtr(int X), g\theta\theta`: the metric components.
- `double dgrdr(int X)`, etc.: the derivatives of the metric components.
- `double Nu(int X), double dNudr(int X), double d2Nudr2(int X)`: the function ν from (11.2) and its first and second derivative.
- `double Lambda(int X), double dLambdadr(int X), double d2Lambdadr2(int X)`: the function λ from (11.2) and its first and second derivative.
- `double getRedshift()`: the surface gravitational redshift $z = [1 - 2GM_\star/c^2R_\star]^{-1/2} - 1$.

11.1.2 Boundary conditions for modes

- `void getXYZCenter(double*, int& maxPow)`: where XYZ could be `Nu` or `Lambda`. Returns coefficients from power-series expansion of the stated quantity in $x = r/R_\star$ near the center. The highest power desired is `maxPow`; if the stellar model does not implement that power, then `maxPow` must be lowered to match. The coefficients are returned within the array.

- `void getXYZSurface(double*, int& maxPow)`: as above, but coefficients from power-series expansion in $t = 1 - r/R$ near the surface.

11.2 How to make new GRStar children

To make new models of stars using GR, in addition to all of the methods of `Star`, you must implement the above methods, as well as the boundaries. The definition of the fields must match our own convention. Be sure the definitions from other sources matches that above. It is common to see, e.g., $g_{00} = -e^{2\nu}$ instead of $-e^\nu$ as we have it. When we speak of density ρ for a GR star, we mean the total mass-energy density; it is common in GR literature so see baryon density also called ρ . Make sure the definition of ρ from other sources matches.

11.3 References

- Tolman
- Oppenheimer and Volkoff
- Tooper 1964, Tooper 1965
- Yoshida and Lee 2001

Chapter 12

The GRPolytrope Class

This class represents a relativistic polytropic star with given index n and relativistic parameter σ , following Tooper (1964). Defining functions θ, v ¹ such that

$$\rho(r) = \rho_c \theta^n, \quad P(r) = P_c \theta^{n+1}, \quad m(r) = 4\pi \rho_c R_n^3 v, \quad (12.1)$$

where $r = R_n x = x \sqrt{\frac{(n+1)P_c}{4\pi G \rho_c^2}}$ as in the Newtonian case, and defining the relativistic parameter

$$\sigma = \frac{P_c}{c^2 \rho_c}, \quad (12.2)$$

then relativistic polytropes will be described by the relativistic Lane-Emden equations (see Tooper 1964),

$$0 = \xi^2 \frac{d\theta}{d\xi} \frac{1 - 2\sigma(n+1)v/x\xi}{1 + \sigma\theta} + v + \sigma \xi^2 \theta^{n+1} \quad (12.3a)$$

$$\frac{dv}{d\xi} = \xi^2 \theta^n. \quad (12.3b)$$

From the TOV equations, we can also find

$$\frac{d\lambda}{d\xi} = e^\lambda \left(2\sigma(n+1)\theta^n \xi - \frac{1}{\xi} \right) + \frac{1}{\xi} \quad (12.4a)$$

$$\frac{d\nu}{d\xi} = e^\lambda \left(2\sigma^2(n+1)\theta^{n+1}\xi + \frac{1}{\xi} \right) - \frac{1}{\xi}, \quad (12.4b)$$

from which a slightly simpler equation for θ follows,

$$\frac{d\theta}{d\xi} = -\frac{1 + \sigma\theta}{2(n+1)\sigma} \frac{d\nu}{d\xi}. \quad (12.5)$$

The boundary conditions are that, at the center, $\xi = 0, \theta = 1, v = 0, \lambda = 0$, and (see Tooper's eq. 2.18)

$$\nu(\xi = 0) = -2(n+1) \log(1 + \sigma) - 2 \log(1 + z),$$

where z is the surface gravitational redshift. At the surface, we must have $\theta = 1, e^\lambda = e^{-\nu} = 1 - \frac{2GM}{c^2 R}$.

There are two relativistic parameters, σ, z , which are related at the surface by

$$z = [1 - 2(n+1)\sigma v_1/\xi_1]^{-1/2} - 1. \quad (12.6)$$

The parameter z is usually the value we want to set based on the star's mass, radius. However, σ is what actually appears in the equations that are integrated. We therefore have a further surface condition that needs to be set by adjusting σ .

The primary difference between **GRPolytrope** and **Polytrope** is the set of equations being solved. Nearly everything is identical to its counterpart in the **Polytrope** class, so attention will only be paid to differences.

¹ Note that within the code we use ASCII-friendly characters $\xi = x, \theta = y$.

12.1 The GRPolytrope Interface

This is a child of GRStar, which means it must implement the GRStar interface and in turn the Star interface.

In addition, GRPolytrope offers the following methods accessible to external code:

- `double getX(int k)`: return position variable $\xi = x[k]$.
- `double getY(int k)`: return solution $\theta(\xi[k]) = y[k]$.
- `double getV(int k)`: return mass-aspect $v(\xi[k]) = v[k]$.
- `double getXproper(int k)`: return the proper distance from center to $\xi[k]$,

$$\bar{\xi} = \int_0^\xi g_{rr}(R_n \xi) d\xi = \int_0^\xi e^{\lambda/2} d\xi = \int_0^\xi [1 - 2\sigma(n+1)v/\xi]^{-1/2} d\xi.$$

See Tooper (1964) eqn. 3.6, 3.8. Since this quantity is coordinate-invariant, it is useful for comparisons.

12.1.1 Constructors

There are two constructors available for a GRPolytrope:

- `GRPolytrope(double n, double zsurf, int len)`: create a GR polytrope of grid size (length) `len`, with polytropic index $n = n$, and matching the surface to have gravitational redshift $z = zsurf$.
- `GRPolytrope(double n, double sigma, int len, double dx)`: as above, but specifies the step size to use as `dx`, and uses $\sigma = \text{sigma}$. The value of z is left free to vary. There is no guarantee of reaching the “true” surface. [This is used for testing scaling relations. Do not use for production runs.](#)

12.2 The GRPolytrope Implementation

The GRPolytrope definition specifies the following class data and methods used in the implementation:

12.2.1 Constants

The following constants have values analogous to their Newtonian counterparts:

- `int len`: the length of arrays for this calculation, which is returned by `length()` function.
- `double n`: the polytropic index n .
- `double Gamma`: the adiabatic exponent Γ of the polytrope, $\Gamma = 1 + \frac{1}{n}$, where $P \sim \rho^\Gamma$.
- `double rho0`: the central density ρ_c .
- `double P0`: the central pressure P_c .
- `double Rn`: the scale factor $R_n = \sqrt{\frac{(n+1)P_c}{4\pi G\rho_c^2}}$ with $r = R_n \xi$.

The following constants are new for GR models:

- `double sigma`: the relativistic parameter $\sigma = \frac{P_c}{\rho_c c^2}$.
- `double zsurf`: the surface gravitational redshift $z = e^{-\lambda/2} - 1$.

12.2.2 Solution Variables

- `double **Y`: an array holding the solution to (6.1) in the form $(\xi, \theta, v, \nu, \lambda)$. The first index is the grid point X , the second index is the variable (either ξ, θ , etc.).
- `enum VarName {x=0, y, v, nu, la, numvar}`: indices for solution variables within `**Y`, where $x \rightarrow \xi$, $y \rightarrow \theta$, $v \rightarrow v$, $\nu \rightarrow \nu$, $\lambda \rightarrow \lambda$, and where `numvar` is used to indicate the number of variables.
- `double *dydx`: the derivative $\text{dydx}[X] = \frac{d\theta}{d\xi}|_{\xi[X]}$.
- `double *base`: defined as $\text{base}[X] = \theta^{n-1} = Y[X][y]^{n-1}$. Allows for fewer calls to `pow()` in calculation.

12.2.3 Constructors

There are two constructors:

- `GRPolytrope(double n, double zsurf, int len)`
- `GRPolytrope(double n, double sigma, int len, double dx)`

In all, n is the polytropic index and `len` is the number of grid points to use. In the third, `dx` is the stepsize to use; this latter method is intended for testing errors when the grid size is changed.

The `name[]` for objects of `Star` child will be prefixed by `GRpolytrope`, and appended with the indices n and σ rounded to first decimal. So for $n = 3$, $\sigma = 0.5$, the `name[]` will be `GRPolytrope.3.0_0.5`.

`GRPolytrope(double n, double zsurf, int len)` requires a value of `zsurf`, which is used to match to physical conditions that can be observed outside the star (such as M_*, R_*). In order to achieve this, a Newton's method is used to find a value of σ so that the terminal values match with the desired `zsurf`, according to (12.6). The remaining logic to find the proper `dx` from bisection is exactly as described in Sec. 6.2.3.

`GRPolytrope(double n, double sigma, int len, double dx)`: requires a value `sigma` instead. This method is used for testing scaling relations, and since σ appears directly in the equations this helps reduce errors in scale. A star is integrated several times to the surface to create consistency between `zsurf, sigma`. The logic is otherwise identical to the analogous method in Sec. 6.2.3.

12.2.4 Calculation

This method works exactly like its counterpart in the `Polytrope` class, described in Sec. 6.2.4. The only difference is that a different set of equations is being integrated.

- `double deriv(double s, double yy[numvar])`: the derivative $d\theta/d\xi$, according to (12.3).
- `double setSigma(double ysurface[numvar])`: returns the value of σ based on surface values, according to (12.6).
- `double setZsurf(double ysurface[numvar])`: returns the value of the redshift z based on the surface values, according to (12.6).
- `void centerInit(double y[numvar])`: set initial values in `y[numvar]`.
- `void RK4step(double dx, double s, double yin[numvar], double yout[numvar])`: performs a single RK4 step integrating the GR Lane-Emden equations (12.3),(12.4), with step size $\Delta\xi = dx$ from `yin[]` and ending at `yout[]`. Uses a complex-valued $YCN = \theta^n$ to avoid possible NaN errors when $\theta < 0$ near surface.
- `double RK4integrate(double s, double& dx)`: used to generate initial consistency between σ, z , which must be related at the surface, by integrating from the center to the surface. Values are not saved, just carried forward, until $\theta < 0$ is found, marking the surface. The value z from (12.6) is returned, and used to control a Newton's method search for the value of σ .

- `double RK4integrate(int Len, double dx)`: integrate from the center out to position `Len` using `RK4step()` above. Returns the value $\theta[\text{Len}-1]$ at the end of the integration, to be used in bisection search of dx . Since only $\theta \geq 0$ is physical, the integration stops when $\theta[X+1] < 0$ and returns $\theta[X+1]$.
- `int RK4integrate(int Len, double dx, int grid)`: as above, but all the way to the end of `Len`; it does not stop if $\theta < 0$. The return value is an integer, `Len`. [The integer `g` now mainly serves the role of distinguishing this method from the other methods. Better naming would also work.](#)

12.2.5 Boundary conditions for modes

The boundary conditions of the modes require series expressions for $A^*, c_1, V_g, \nu, \lambda$. Because of the very simple mathematical structure of a polytrope it is possible to derive analytic formulas for these coefficients.

- `double thc[5], nc[7], lc[5]`: coefficients of power series for θ, ν, λ near the center, in terms of dimensionless $x = r/R_\star = \xi/\xi_1$:

$$\theta(x) \approx \theta_0 + \theta_2 x^2 + \theta_4 x^4 + \theta_6 (r/R)^6 = \text{thc}[0] + \text{thc}[1]x^2 + \text{thc}[2]x^4 + \text{thc}[3]x^6.$$

- `double ths[5], ns[7], ls[5]`: coefficients of power series for θ, ν, λ near the surface (with $t = 1 - r/R_\star = 1 - \xi/\xi_1$)

$$\theta(t) \approx \theta_0 + \theta_1 t + \theta_2 t^2 + \dots = \text{ths}[0] + \text{ths}[1]t + \text{ths}[2]t^2 + \dots.$$

- `setupCenter()`: initialize the coefficients above. See the Mathematica notebook `GRBCs.nb` for further information.
- `setupSurface()`: initialize the coefficients above. See the Mathematica notebook `GRBCs.nb` for further information.
- `getXYZCenter(double* xyz, int& maxPow)`: where XYZ is `Astar`, `Vg`, `C1`, `Nu`, or `Lambda`. Returns terms of the power series expansion in x near the center. See the Mathematica notebook `GRBCs.nb` for detailed explanation. The coefficients are stored in the input `xyz[]` array to be returned. We have found terms up to order x^4 ; if more are requested with `maxPow`, then we set `maxPow = 4` so that the `ModeDriver` only implements the BC to order x^4 .
- `getXYZSurface(double* xyz, int& maxPow)`: as above, but near the surface expanded in powers of $t = 1 - x$. Unlike the central boundary, there is no requirement that only even powers appear. Two of the stellar variables, A^*, V_g have a pole at $t = 0$, so require a t^{-1} term. If n is not an integer, there are also terms t^n appearing in the expansions ([I have included the \$t^n\$ term in the past, but it is not currently set up - need a method to include at least the first non-integer part of an expansion](#)).

Due to the nature of the recursion relation at the surface, to implement the BCs to 4th order, we need A^*, V_g to order t^3 (which includes the t^{-1} term), we need c_1 to t^3 , and we need ν to order t^6 . The details of the calculations are too tedious to write out here, but can be found in the Mathematica notebook `GRBCs.nb`. [The expansions of \$\nu, \lambda\$ are currently unimplemented at the surface.](#)

In both the center and the surface, the BC is given to **fourth order**.

12.3 Internal Logic Behind the Interface

With the above variables, we can now define how the following `Star` quantities are computed:

- `double Phi(int X)`: returns a Newtonian-like potential $\Phi = -Gm/r = 4\pi\rho_c R_n^2 \frac{v}{\xi}$. This only really works at the surface.
- `double dPhidr(int X)`: $g = Gm/r^2 = 4\pi\rho_c R_n \frac{v}{\xi^2}$

- `double mr(int X)`: $m(r) = -4\pi R_n^3 \rho_c \xi^2 \frac{d\theta}{d\xi} = Rn^3 * rho0 * mass[X]$.

- `double Schwarzschild_A(int X, double GamPert)`: the Schwarzschild discriminant, which in GR is

$$A = \frac{1}{\rho + P/c^2} \frac{d\rho}{dr} - \frac{1}{P\Gamma_1} \frac{dP}{dr} = \frac{d\theta/d\xi}{\theta R_n} \left(\frac{n}{1 + \sigma\theta} - \frac{n+1}{\Gamma_1} \right) = \frac{Y[X][z]}{Y[X][y]R_n} \left(\frac{n}{1 + \sigma Y[X][y]} - \frac{n+1}{\text{GamPert}} \right).$$

If `GamPert = 0` is given, uses $\Gamma_1 = \Gamma$ (which will result in $A = 0$).

- `double getAstar(int X, double GamPert)`: dimensionless Schwarzschild discriminant, which in GR is

$$A^* = \frac{1}{\Gamma_1} \frac{d \log P}{d \log r} - \frac{1}{1 + P/\rho c^2} \frac{d \log \rho}{d \log r} = \frac{\xi}{\theta} \frac{d\theta}{d\xi} \left(\frac{n+1}{\Gamma_1} - \frac{n}{1 + \sigma\theta} \right) = \frac{Y[X][z] * Y[X][x]}{Y[X][y]} \left(\frac{n+1}{\text{GamPert}} - \frac{n}{1 + \sigma Y[X][y]} \right).$$

If `GamPert = 0` is given, uses $\Gamma_1 = \Gamma$ (which will result in $A^* = 0$).

- `double getU(int X)`: $U = \frac{d \log m}{d \log r} = \frac{\xi^3 \cdot \theta^n}{v}$. Special care must be take at $X = 0$, where $U = 3$. This is **not** the same U_1, U_2 used in the GR Cowling equations.
- `double getC(int X)`: returns

$$c_1(r) = \frac{GM_*}{c^2 R_*^3} \frac{2r}{\nu'} e^{-\nu} = \frac{v_1}{\xi_1^3} * \frac{\xi(1 + \sigma\theta)}{d\theta/d\xi} e^{-\nu}$$

as defined in Yoshida & Lee (2002) eqn. 39. Special care must be take at $X = 0$, where $c_1 = \frac{3v_1}{\xi_1^3} \frac{e^{-\nu_0}}{1 + 3\sigma}$.

- `double sound_speed2(int X, double GamPert)`: the sound speed, which in GR is $v_s^2 = \frac{\Gamma_1 P}{\rho + P/c^2} = \frac{\Gamma_1 P_c}{\rho_c(1 + \sigma\theta)} * \theta(x) = \frac{\Gamma_1 P_c}{\rho_c} \frac{Y[X][y]}{1 + \sigma Y[X][y]}$, where $\Gamma_1 = \text{GamPert}$. If no Γ_1 is specified, uses $\Gamma_1 = \Gamma$

All other quantities from `Star` are implemented exactly as in `Polytrope`.

We can also specify the following `GRStar` quantities:

- `double getRedshift()`: return z_{surf} , the surface redshift.
- `double gtt(int X)`: return $g_{tt} = -e^\nu = -\exp(Y[X][\text{nu}])$.
- `double grr(int X)`: return $g_{rr} = e^\lambda = \exp(Y[X][\text{la}])$.
- `double gtr(int X)`: return $g_{tr} = 0$ in the usual Schwarzschild coordinates.
- `double gtheta(int X)`: return $g_{\theta\theta} = r^2$.

The derivatives of the above terms are obvious. We also define

- `double Nu(int X)`: return $\nu = Y[X][\text{nu}]$.
- `double dNudr(int X)`: return $\frac{d\nu}{dr} = -\frac{2\sigma(n+1)}{1 + \sigma\theta} \frac{d\theta}{d\xi}$ as in Tooper (1964) eqn. 2.19.5.
- `double d2Nudr2(int X)` return $\frac{d^2\nu}{dr^2}$. The result is rather complicated, but by combining the three equations in Tooper (1965) eqn. 2.3-2.5, $\frac{d^2\nu}{dr^2}$ can be solved for.
- `double Lambda(int X)`: return $\lambda = Y[X][\text{la}]$.
- `double dLambdadr(int X)`: return $\frac{d\lambda}{dr} = \frac{1 - e^\lambda + 2e^\lambda(n+1)\xi^2\sigma^2\theta^{n+1}}{R_n\xi}$, as in Tooper (1964) eqn. 2.3.
- `double d2Lambdadr2(int X)`: not implemented – returns 0.

These values are all calculated at $r = \text{rad}(X)$. Since $\text{base}[X] = \theta^{n-1}$, this is used in the code to avoid calls to `pow()`. The other methods, `length()`, `Radius()`, etc., are trivially defined.

12.4 References

- Tolman
- Oppenheimer and Volkoff
- Tooper 1964
- Tooper 1965 for a slightly different interpretation of a polytrope
- Bludman
- Yoshida & Lee 2002

Part III

Mode Drivers

Chapter 13

The NonradialModeDriver Class

This class controls the nonradial, k, ℓ, m oscillation modes calculated for the full fourth-order Newtonian system, which consists of the LAWE and the perturbed Poisson equation:

$$\omega^2 \xi = -\nabla \chi + \frac{\vec{A}}{v_s^2} (\nabla \cdot \xi) \quad (13.1a)$$

$$\nabla^2 \Delta \Phi = 4\pi G \Delta \rho. \quad (13.1b)$$

With some algebra this can be put in the form of four first-order equations

$$x \frac{dy_1}{dx} = [V_g - 3 + (2 - \ell)] y_1 + \left[\frac{\ell(\ell + 1)}{c_1 \bar{\omega}^2} - V_g \right] y_2 + V_g y_3 \quad (13.2a)$$

$$x \frac{dy_2}{dx} = [c_1 \bar{\omega}^2 - A^*] y_1 + [1 + A^* - U + (2 - \ell)] y_2 - A^* y_3 \quad (13.2b)$$

$$x \frac{dy_3}{dx} = [1 - U + (2 - \ell)] y_3 + y_4 \quad (13.2c)$$

$$x \frac{dy_4}{dx} = A^* U y_1 + U V_g y_2 + [\ell(\ell + 1) - U V_g] y_3 + [(2 - \ell) - U] y_4 \quad (13.2d)$$

where

$$y_1 = \frac{1}{x^{\ell-2}} \frac{\xi^r}{r}, \quad y_2 = \frac{1}{x^{\ell-2}} \frac{\chi}{gr}, \quad y_3 = \frac{1}{x^{\ell-2}} \frac{\Delta \Phi}{gr}, \quad y_4 = \frac{1}{x^{\ell-2}} \frac{1}{g} \frac{d\Delta \Phi}{dr}, \quad (13.3)$$

with $x = r/R$ and where U, V_g, A^*, c_1 are unitless stellar variables defined as

$$U = \frac{d \log g}{d \log r} + 2, \quad V_g = -\frac{1}{\Gamma_1} \frac{d \log P}{d \log r}, \quad A^* = \frac{1}{\Gamma_1} \frac{d \log P}{d \log r} - \frac{d \log \rho}{d \log r}, \quad c_1 = \frac{M}{m(r)} \left(\frac{r}{R} \right)^3. \quad (13.4)$$

We could also write this set of equations in the form

$$x \frac{d}{dx} \vec{Y} = x \frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{13} & 0 \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{A} \vec{Y}. \quad (13.5)$$

using matrix notation. Notice that the LHS of (13.2) is in terms of $x \frac{d}{dx}$, and not merely $\frac{d}{dx}$. This is to preserve unitlessness of the equations, and preserves stability near the center.

13.1 The NonradialModeDriver Interface

`NonradialModeDriver` is a child of `ModeDriver`, which means it must uphold the `ModeDriver` contract. It does not define anything beyond that.

13.1.1 Constructor

There is only one constructor:

- `NonradialModeDriver(Star *s, double Gamma1)`

This single constructor initializes all arrays and sets up the boundaries.

13.2 The NonradialModeDriver Implementation

The `NonradialModeDriver` definition specifies the following class data and methods used in the implementation:

13.2.1 Constants

- `int len`: the number of gridpoints of the `Mode<N>` calculation.
- `int len_star`: the number of gridpoints of the underlying `Star` model.
- `double adiabatic_index`: the adiabatic exponent Γ_1 : if set to zero uses the background star's value of Γ_1 .
- `int BC_C, BC_S = 4`: the order of the power series expansions we would like for the central, surface boundary conditions. The actual order will depend on the background stellar model.

13.2.2 Physics

We need the variables A^*, U, c_1, V_g defined in (13.4):

- `double r[len_star]`: the dimensionless radius $r[X] = x = r/R$.
- `double A[len_star]`: the dimensionless Schwarzschild discriminant $A[X] = A^* = \frac{1}{\Gamma_1} \frac{d \log P}{d \log r} - \frac{d \log \rho}{d \log r}$.
- `double U[len_star]`: the variable $U[X] = U = \frac{d \log m}{d \log r}$.
- `double C[len_star]`: the variable $C[X] = c_1 = \frac{M}{R^3} \frac{r^3}{m}$.
- `double V[len_star]`: the variable $V[X] = V_g = -\frac{1}{\Gamma_1} \frac{d \log P}{d \log r}$.

These variables are called from the stellar model through `*star`.

13.2.3 Calculation

- `void initializeArrays()`: fill out values of $A[], U[], C[], V[]$ from `*star`.
- `void getCoeff(double *CCI, int X, int b, double omeg2, int L)`: return the matrix $\mathbf{A} = \text{CCI}$ shown in (14.3).
- `void getBoundaryMatrix(int nv, double* y0, double* ys, double** y, int* indexOrder)`: the boundary matrix y specifies how the `Mode<N>`'s shoot-to-center algorithm described in Sec. 5.2.5 specifies values at each boundary as in (5.6). For the fourth-order nonradial mode, we pick

$$\vec{Y}^{(c1)} = (1, y_2, 0, 0), \vec{Y}^{(c2)} = (0, 0, 1, y_4), \vec{Y}^{(s1)} = (0, y_2, 1, y_4), \vec{Y}^{(s2)} = (1, y_2, 0, 0), \quad (13.6)$$

which leads to the matrix y returned. The array `indexOrder = {y1, y3, y3, y1}` indicates the first solution comes by scaling y_1 , the second by scaling y_3 , the third by scaling y_3 , and the final by y_1 , so that these BCs can be rescaled to make the physical solution.

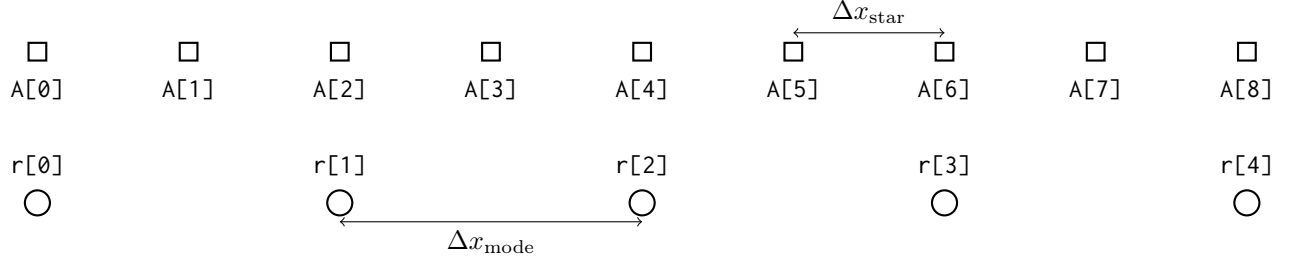


Figure 13.1: The squares are stellar grid points with spacing Δx_{star} containing values such as $A^* = A[N]$. The circles are mode grid points with spacing Δx_{mode} , and contain values such as $r = r[N]$ or mode values $y_1(r[N]) = yy[y1][N]$.

- `double SSR(double omeg2, int 1, ModeBase* mode)`: calculate the RMSR in equations (13.1), where the variables in (13.1) double are calculated from the background star and the dimensionless variables in (13.2). This requires numerical differentiation to evaluate quantities such as $\frac{d^2}{dr^2} \Delta\Phi$. To make the derivatives as accurate as possible, this uses a 7-point stencil for the differences; however, **this requires an assumption of a uniform grid**. For modes calculated for background stars without a uniform grid, the RMSR will record additional numerical error, making these values larger. This is important for modes calculated with MESA data.
- `double innerproduct(ModeBase* mode1, ModeBase* mode2)`: calculates an inner product between two nonradial modes, defined by $\langle \vec{\xi}_1 | \vec{\xi}_2 \rangle = \int d^3r \rho \vec{\xi}_1^* \cdot \vec{\xi}_2$. Must first transform to dimensionless y_i into physical perturbations $\xi^r, \chi, \Delta\Phi$, then uses trapezoidal integration. See Press & Teukolsky (1977).
- `double tidal_overlap(ModeBase* mode)`: as above, but calculates an overlap coefficient with an external tidal force. See Press & Teukolsky (1977).

13.2.4 Constructor

The only constructor does not do much more than initialize the arrays $A[], U[], C[], V[]$ described above and which appear in (13.2), and then set up the boundary conditions described below. The only point of it worth mentioning is the selection of the grid. As mentioned above, there are two grids, one of length `len` and the other of length `len_star`.

In the RK4 integration, it is necessary to have values corresponding to midpoints, e.g. $x[n] + 0.5dx$. To achieve this without the error inherent in spline fitting, the mode calculation uses half the number of grid points as the stellar background code (see Fig 13.1).

It is necessary to avoid fence-post issues at the boundaries. If the background star has an odd number of points, then we can evenly pick the mode grid so that there is always a stellar grid point on the half-grid,

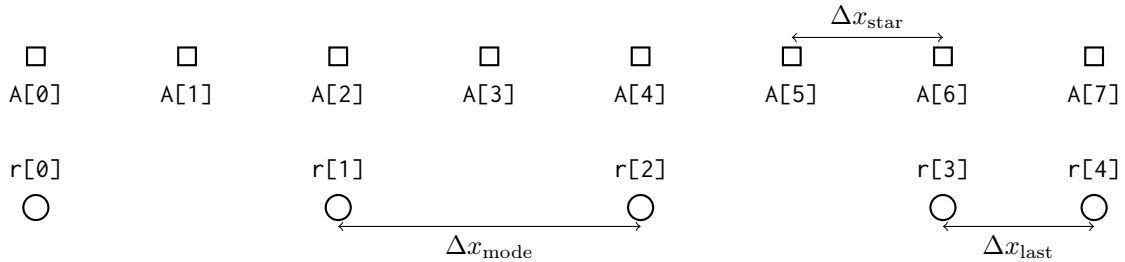


Figure 13.2: As in Fig. 13.1 the stellar grid is odd and can't be evenly divided. To fix this, the final grid spacing is $\Delta x_{\text{last}} = \frac{1}{2} \Delta x_{\text{mode}}$. Because of the power series solutions used there, this does not impact the accuracy of the calculation.

by picking the grid with total length $\text{len} = (\text{len_star} + 1)/2$. If the background star has an even number of points, the situation is a bit more complicated (see Fig. 13.2). The mode grid cannot be chosen so that there is always a stellar grid point on the half-grid; there has to be one point left over. We make the leftover point the last grid point, and the spacing there will be half the spacing elsewhere. To achieve this, choose $\text{len} = \text{len_star}/2 + 1$.

13.2.5 Miscellaneous

- `enum VarNames {y1=0, y2, y3, y4}`: the conventional names for the variables are y_1, \dots, y_4 ; however, C++ is zero-indexed. To avoid calling, e.g. `yy[1]` (being y_2) when y_1 is intended, one can instead call `yy[y1]`.
- `void varnames(std::string *names)`: allows for `Mode<N>` to know what labels to assign the mode variables when it is printing graphs. The input array of strings will be set with the names described here.

13.2.6 Boundary conditions

- `Ac[3], Vc[3], cc[3], Uc[3]`: coefficients from power series expansions of A^*, V_g, c_1, U to order $x^4 = (r/R)^4$ at the center of the star. Will come from the particular `Star` model.
- `As[5], Vs[5], cs[5], Us[5]`: coefficients from power series expansions of A^*, V_g, c_1, U to order $t^4 = (1 - r/R)^4$ at the surface of the star. Will come from the particular `Star` model.
- `cProdc[3], cProds[5]`: coefficients in expansion of $c\text{Prod} = \frac{1}{c_1}$ at the center and surface of the star.
- `setupBoundaries()`: initialize the above arrays at both boundaries.
- `CenterBC(double** ymode, double* y0, double omeg2, int l, int m)`: Initialize the first few steps of the eigenmode using a power series expansion. The array `y0[]` specifies the values of y_i at the very center. Once the coefficients are found, the first `start` values are calculated in `ymode[][]` using the series expansion. Then `start` is returned, indicating where the usual integration method of `Mode<N>` begins.
- `SurfaceBC(double** ymode, double* y0, double omeg2, int l, int m)`: As above, but at the surface.

13.3 References

- Dziembowski, 1971, for the first publication of the equations in (13.2).
- Unno et al., *Nonradial Oscillations of Stars*, 1979, for detailed description of the equations (13.2) and their boundary conditions.
- Cox, *Theory of stellar pulsation*, 1980, for a detailed discussion of nonradial oscillations.
- Press & Teukolsky, 1977, for a better description of the inner product and a definition of the overlap coefficients.

Chapter 14

The CowlingModeDriver Class

This class represents n, l, m oscillation modes calculated in the Cowling approximation, where $\Delta\Phi \rightarrow 0$. In this approximation the LAWE (13.1) simplifies to the equations of motion for an oscillation mode

$$\frac{d\xi^r}{dr} = \left[\frac{g}{v_s^2} - \frac{2}{r} \right] \xi^r + \left[\frac{\ell(\ell+1)}{r^2\sigma^2} - \frac{1}{v_s^2} \right] \chi \quad (14.1a)$$

$$\frac{d\chi}{dr} = [\sigma^2 - N^2] \xi^r - A\chi \quad (14.1b)$$

which we rewrite in the form

$$x \frac{dy_1}{dx} = [V_g - 3 + (2 - \ell)]y_1 + \left[\frac{\ell(\ell+1)}{c_1\bar{\omega}^2} - V_g \right] y_2 \quad (14.2a)$$

$$x \frac{dy_2}{dx} = [c_1\bar{\omega}^2]y_1 + [1 + A^* - U + (2 - \ell)]y_3 \quad (14.2b)$$

or in matrix form

$$x \frac{d}{dx} \vec{Y} = x \frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A} \vec{Y}. \quad (14.3)$$

These variables have the same meaning as in (13.2).

This class works exactly as the `NonradialModeDriver` class, except that it uses `num_var=2`. The boundary conditions are the same, but cut off to two variables.

Because there are only two variables, there are only two BCs, one at each boundary. This means that the outward and inward solutions are the only ones in their regions, which we can label

$$\vec{Y}^{(c)} = (1, y_2), \vec{Y}^{(s)} = (1, y_3). \quad (14.4)$$

These two solutions can be used in a Wronskian, which will vanish when the correct $\bar{\omega}^2$ is found, and can be rescaled to match at the fitting point, as described in Sec. 5.2.5.

14.1 References

Chapter 15

The PNNonradialModeDriver Class

This class controls the nonradial, k, l, m oscillation modes calculated for the full eighth-order 1PN system, which consists of the LAWE and the perturbed Poisson equation:

$$0 = \Delta\rho \left(1 - \frac{4\Phi}{c^2}\right) - \frac{3\rho}{c^2}\Delta\Phi + \nabla \cdot \left[\left(\rho + \frac{P - 4\rho\Phi}{c^2}\right)\vec{\xi}\right] + \frac{2\rho}{c^2}\vec{\xi} \cdot \nabla\Phi \quad (15.1a)$$

$$0 = \frac{\partial^2 \vec{\xi}}{\partial t^2} \left(\rho + \frac{P - 4\rho\Phi}{c^2}\right) + \nabla\Delta P + \rho\nabla\Delta\Phi + \Delta\rho\nabla\Phi \\ + \frac{1}{c^2} \left\{ \rho\nabla\Delta\Psi + \Delta\rho\nabla\Pi + \Delta P\nabla\Phi + P\nabla\Delta\Phi + 4\rho\frac{\partial\Delta\vec{W}}{\partial t} \right\} \quad (15.1b)$$

$$\nabla^2\Delta\Phi = 4\pi G\Delta\rho \quad (15.1c)$$

$$\nabla^2\Delta\vec{W} = 4\pi G\rho\frac{\partial\vec{\xi}}{\partial t} \quad (15.1d)$$

$$\nabla^2\Delta\Psi = \frac{\partial^2\Delta\Phi}{\partial t^2} + 12\pi G\Delta P - 8\pi G\rho\Delta\Phi - 8\pi G\Phi\Delta\rho. \quad (15.1e)$$

With some algebra, and using the harmonic coordinate condition as a constraint, this can be put in the form of eight first-order equations, with the surface gravitational redshift $z \sim GM/c^2 R$,

$$x \frac{dy_1}{dx} = [V_g - 3 - 3zV_g\beta_*^2 + (2-l)]y_1 + \left[\frac{l(l+1)}{c_1\bar{\omega}^2} - V_g - 4zV_g\Phi^* \right] y_2 + [V_g - 3zV_g\beta_*^2]y_3 - z \left[\frac{4\bar{\omega}^2 x^2}{l(l+1)} \right] y_4 - zV_g(z_1 + z_5) \quad (15.2a)$$

$$x \frac{dy_2}{dx} = \left[c_1\bar{\omega}^2 - A^* + z \left(4A^*\Phi^* - \frac{4U\bar{\omega}^2 x^2}{l(l+1)} \right) \right] y_1 + [1 + (2-l) - U + A^* - 4zV_g\beta_*^2] y_2 + [-A^* + 4zA^*\Phi^*]y_3 + zA^*(z_1 + z_5) \quad (15.2b)$$

$$x \frac{dy_3}{dx} = [1 + (2-l) - U]y_3 + y_4 \quad (15.2c)$$

$$x \frac{dy_4}{dx} = A^*U \left[1 + 2z \left(\frac{\beta_*^2}{\Gamma_1} - \Phi^* \right) \right] y_1 + UV_g \left[1 + 2z \left(\frac{\beta_*^2}{\Gamma_1} + \Phi^* \right) \right] y_2 + \left[l(l+1) - UV_g \left(1 + 2z \left[\frac{\beta_*^2}{\Gamma_1} - \Phi^* \right] \right) \right] y_3 + [(2-l) - U]y_4 + zUV_g(z_1 + z_5) \quad (15.2d)$$

$$x \frac{dz_1}{dx} = [1 + (2-l) - U]z_1 + z_2 \quad (15.2e)$$

$$x \frac{dz_2}{dx} = -2A^*U\Phi^*y_1 + [3UV_g\beta_*^2 - 2UV_g\Phi^*]y_2 + [2UV_g\Phi^* - 5UV_g\beta_*^2 + \bar{\omega}^2 x^2] y_3 + l(l+1)z_1 + [(2-l) - U]z_2 \quad (15.2f)$$

$$x \frac{dz_3}{dx} = -4\bar{\omega}^2 x^2 y_3 + [(2-l) - U]z_3 + l(l+1)z_5 \quad (15.2h)$$

$$x \frac{dz_5}{dx} = \frac{4U\bar{\omega}^2 x^2}{l(l+1)}y_1 + \frac{4\bar{\omega}^2 x^2}{l(l+1)}y_4 + z_3 + (1 + (2-l) - U)z_5 \quad (15.2i)$$

where

$$y_1 = \frac{1}{x^{l-2}} \frac{\xi^r}{r}, \quad y_2 = \frac{1}{x^{l-2}} \frac{\chi}{gr}, \quad y_3 = \frac{1}{x^{l-2}} \frac{\Delta\Phi}{gr}, \quad y_4 = \frac{1}{x^{l-2}} \frac{1}{g} \frac{d\Delta\Phi}{dr}, \quad (15.3)$$

and

$$z_1 = \frac{1}{x^{l-2}} \frac{\Delta\Psi}{g\Phi_s}, \quad z_2 = \frac{1}{x^{l-2}} \frac{1}{g\Phi_s} \frac{d\Delta\Psi}{dr}, \quad z_3 = \frac{1}{x^{l-2}} \frac{4i\omega}{g\Phi_s} \Delta W^r, \quad z_5 = \frac{1}{x^{l-2}} \frac{4i\omega}{g\Phi_s} \Delta W^H, \quad (15.4)$$

with $x = r/R$, but where now

$$g = \frac{d\Phi}{dr} + \frac{1}{c^2} \frac{d\Psi}{dr}, \quad \chi = \frac{\Delta P}{\rho} + \Delta\Phi + \frac{1}{c^2} \left\{ \Delta\Psi + \frac{4\Phi\Delta P}{\rho} + 4i\omega r \Delta W^H + 4\Phi\Delta\Phi - \frac{P}{\rho^2} \Delta P \right\} \quad (15.5)$$

and where U, V_g, A^*, c_1 are unitless stellar variables defined as

$$U = \frac{d \ln g}{d \ln r} + 2, \quad V_g = -\frac{1}{\Gamma_1} \frac{d \ln P}{d \ln r}, \quad A^* = \frac{1}{\Gamma_1} \frac{d \ln P}{d \ln r} - \frac{1}{1 + P/\rho c^2} \frac{d \ln \rho}{d \ln r}, \quad c_1 = \frac{GM}{R^3} \frac{r}{g}. \quad (15.6)$$

These definition are analogous to those in `NonradialMode`. We have also introduced variables

$$\beta_*^2 = \frac{v_s^2}{\Phi_s} = \frac{\Gamma_1}{\Phi_s} \frac{P}{\rho + P/c^2}, \quad \Phi^* = \frac{\Phi}{\Phi_s}. \quad (15.7)$$

We could also write this set of equations in the matrix form

$$x \frac{d}{dx} \vec{Y} = \mathbf{A} \vec{Y} \quad (15.8)$$

analogous to (14.3). Notice the similarities between (13.2) and (15.2); in particular, notice that in the Newtonian limit $c \rightarrow \infty$ where the gravitational redshift goes to zero, $z \rightarrow 0$, that the 1PN equation will reduce to the Newtonian case.

15.1 The PNNonradialModeDriver Interface

PNNonradialModeDriver is a child of ModeDriver, which means it must uphold the ModeDriver contract. It has only the following change:

- PNStar *const star: this overwrites the normal *star pointer, requiring the background star to implement 1PN physics as defined by the PNStar class.

15.1.1 Constructor

There is only one constructor:

- PNNonradialModeDriver(PNStar *s, double Gamma1)

This single constructor initializes all arrays and sets up the boundaries. It behaves exactly like the constructor for NonradialModeDriver, including its selection of a grid size (see Section 13.2.4).

15.2 The PNNonradialModeDriver Implementation

The PNNonradialModeDriver definition specifies the following class data and methods used in the implementation:

15.2.1 Physics

We need the variables $A^*, U, c_1, V_g, \beta_*^2, \Phi^*$ defined in (15.6):

- double zsurf: the surface gravitational redshift $z = GM/c^2 R$, which parameterized the strength of gravity in the equations (15.2).
- double r[len_star]: the dimensionless radius $r[X] = r/R$.
- double A[len_star]: the dimensionless Schwarzschild discriminant $A[X] = A^* = \frac{1}{\Gamma_1} \frac{d \ln P}{d \ln r} - \frac{d \ln \rho}{d \ln r}$.
- double U[len_star]: the variable $U[X] = U = \frac{d \ln m}{d \ln r}$.
- double C[len_star]: the variable $C[X] = c_1 = \frac{M}{R^3} \frac{r^3}{m}$.
- double Vg[len_star]: the variable $Vg[X] = V_g = -\frac{1}{\Gamma_1} \frac{d \ln P}{d \ln r}$.
- double beta[len_star]: the variable $\beta[X] = \beta_*^2 = \frac{v_s^2}{\Phi_s}$ where Φ_s is the surface value of Φ .
- double Phi[len_star]: the variable $\Phi[X] = \Phi^* = \frac{\Phi}{\Phi_s}$ where Φ_s is the surface value of Φ .
- double xstar[len_star]: the dimensionless radius r/R as in $r[]$, but on the entire star grid.

15.2.2 Calculation

- int len: the number of grid points of the Mode<N> calculation.
- int len_star: the number of grid points of the underlying Star model.
- void initializeArrays(): fill out values of $A[], U[],$ etc. from Star background.
- void getCoeff(double *CCI, int X, int b, double omeg2, int L): return the matrix $\mathbf{A} = \text{CCI}$ shown in (14.3).

- `void getBoundaryMatrix(int nv, double* y0, double* ys, double** y, int* indexOrder)`: the boundary matrix `y` specifies how the `Mode<N>`'s shoot-to-center algorithm described in Sec. ?? specifies values at each boundary as in (5.6). For the eighth-order 1PN nonradial mode, we pick

$$\vec{Y}^{(c1)} = (1, y_2, 0, 0, 0, 0, 0, 0), \quad \vec{Y}^{(c2)} = (0, 0, 1, y_4, 0, 0, 0, 0), \quad (15.9a)$$

$$\vec{Y}^{(c3)} = (0, 0, 0, 0, 1, z_2, 0, 0), \quad \vec{Y}^{(c4)} = (0, 0, 0, 0, 0, 0, 1, z_5), \quad (15.9b)$$

$$\vec{Y}^{(s1)} = (0, y_2, 1, y_4, 0, 0, 0, 0), \quad \vec{Y}^{(s2)} = (0, 0, 0, 0, 1, z_2, 0, 0), \quad (15.9c)$$

$$\vec{Y}^{(s3)} = (0, 0, 0, 0, 0, 0, 1, z_5), \quad \vec{Y}^{(s4)} = (1, y_2, 0, 0, 0, 0, 0, 0) \quad (15.9d)$$

which leads to the matrix form of `y` returned. The array `indexOrder` specifies the order in which the solutions should be combined to make the physical solution satisfying all BCs.

- `double SSR(double omeg2, int l, ModeBase* mode)`: calculate the RMSR in equations (15.1), where the variables in (15.1) double are calculated from the background star and the dimensionless variables in (15.2). This requires numerical differentiation to evaluate quantities such as $\frac{d}{dr}\Delta P$. To make the derivatives as accurate as possible, this uses a 7-point stencil for the differences; however, **this requires an assumption of a uniform grid**. For modes calculated for background stars without a uniform grid, the RMSR will record additional numerical error, making these values larger.
- `double innerproduct(ModeBase* mode1, ModeBase* mode2)`: calculates an inner product between two nonradial modes, defined by $\langle \vec{\xi}_1 | \vec{\xi}_2 \rangle = \int d^3r \rho \vec{\xi}_1^* \cdot \vec{\xi}_2$. Must first transform the dimensionless y_i into physical perturbations $\xi^r, \chi, \Delta\Phi$, then uses trapezoidal integration. See Press & Teukosky (1977).
- `double tidal_overlap(ModeBase* mode)`: as above, but calculates an overlap coefficient with an external tidal force. See Press & Teukosky (1977).

15.2.3 Miscellaneous

- `enum VarNames {y1=0, y2, y3, y4, z1, z2, z3, z5}`: the conventional names for the variables are $y_1, \dots, y_4, z_1, \dots, z_5$. Because C++ is zero-indexed, and because the z_i variables start numbering at 1, to avoid calling, e.g. `yy[1]` (being y_2) when y_1 is intended, and to facilitate calling, e.g. $z_3 = yy[z3] = yy[6]$.
- `void varnames(std::string *names)`: allows for `Mode<N>` to know what labels to assign the mode variables when it is printing graphs. The input array of strings will be set with the names described here.

15.2.4 Boundary conditions

- `cc[2], Vgc2, Ac2, betac0, phic0`: coefficients from power series expansions to order $x^2 = (r/R)^2$ at the center of the star. Here `cc[]` contains both coefficients for c_1 , the `Vgc2, Ac2` contain the quadratic coefficient of V_g, A^* , and `betac0, phic0` contain the central values of β_*^2, Φ^* . These will come from the particular PNStar model.
- `cs[3], betas0, phis0`: coefficients from power series expansions to order $x^2 = (r/R)^2$ at the surface of the star. Here `cs[]` contains both coefficients for c_1 , and `betas0, phis0` contain the surface values of β_*^2, Φ^* . These will come from the particular PNStar model.
- `surfCoeff[3][8][8]`: a set of 3 matrix coefficients, each 8x8, from the expansion $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}t + \mathbf{A}t^2$, with \mathbf{A} as in (15.8).
- `surfCoeffN[8][8]`: an 8x8 matrix coefficient which corresponds to a possible non-integer power in the expansion, i.e. $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_{3/2}t^{3/2} + \mathbf{A}_1t + \dots$.
- `setupBoundaries()`: initialize the above arrays at both boundaries.

- `updateSurface(double (&A)[3][8][8], double omeg2, int l)`: because $l, \bar{\omega}^2$ appear in the \mathbf{A}_n coefficients, they need to be updated each pass-through depending on the k, l mode, and as $\bar{\omega}^2$ is refined. These are returned as the input parameter, `A[3][8][8]`.
- `CenterBC(double** ymode, double* y0, double omeg2, int l, int m)`: Initialize the first few steps of the eigenmode using a power series expansion. The array `y0[]` specifies the values of y_i at the very center. Once the coefficients are found, the first `start` values are calculated in `ymode[][]` using the series expansion. Then `start` is returned, indicating where the usual integration method of `Mode<N>` begins.
- `SurfaceBC(double** ymode, double* y0, double omeg2, int l, int m)`: As above, but at the surface.

15.3 Limitations

The 1PN equations of stellar dynamics are only a good choice in situations where relativity is mild. Errors in these modes scales like z^2 . For this, 1PN modes should only be used in situations where $z \sim \mathcal{O}(10^{-4})$ or smaller.

15.4 References

- Poisson & Will, 2004, for detailed information on the Post-Newtonian method and a derivation of 1PN hydrodynamics, which informs (15.1).
- Cutler, 1991, for one of the earlier attempts to write simple equations for nonradial modes to 1PN order.
- Cutler & Lindblom, 1992, for a small table of frequencies in the case of an $n = 2$ polytrope with very large gravitational redshift ($z = 0.225$).
- Lindblom, Mendell & Ipser 1997.

Chapter 16

The GRCowlingModeDriver Class

This class controls the nonradial, k, ℓ, m oscillation modes calculated in the GR Cowling approximation of Yoshida & Lee (2002, eqn. 42,43),

$$x \frac{dy_1}{dx} = [V_g - 3 + U_2 + (2 - \ell)] y_1 + \left[\frac{\ell(\ell + 1)}{c_1 \bar{\omega}^2} - V_g \right] y_2 \quad (16.1a)$$

$$x \frac{dy_2}{dx} = [c_1 \bar{\omega}^2 e^\lambda - A^*] y_1 + [1 + A^* - U_1 + (2 - \ell)] y_2 \quad (16.1b)$$

where

$$y_1 = \frac{1}{x^{\ell-2}} \frac{\xi^r}{r}, \quad y_2 = \frac{1}{x^{\ell-2}} \frac{\chi}{gr}, \quad (16.2)$$

with $x = r/R$ and where U_1, U_2, V_g, A^*, c_1 are unitless stellar variables defined as (see eqn. 38,39

$$U_1 = 2 + \frac{d \log \nu'}{d \log r}, \quad U_2 = \frac{r}{2} \frac{d\lambda}{dr}, \quad V_g = -\frac{1}{\Gamma_1} \frac{d \log P}{d \log r}, \quad A^* = \frac{1}{\Gamma_1} \frac{d \log P}{d \log r} - \frac{d \log \rho}{d \log r}, \quad c_1 = \frac{GM_\star}{c^2 R_\star^3} \frac{2r}{\nu'} e^{-\nu}. \quad (16.3)$$

We could also write this set of equations in the form

$$x \frac{d}{dx} \vec{Y} = x \frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A} \vec{Y}. \quad (16.4)$$

using matrix notation.

16.1 The GRCowlingModeDriver Interface

GRCowlingModeDriver is a child of ModeDriver, which means it must implement the same interface. It does not define anything beyond that.

16.1.1 Constructor

There is only one constructor:

- GRCowlingModeDriver(GRStar *s, double Gamma1)

This single constructor initializes all arrays and sets up the boundaries. It works equivalently to the constructor of NonradialModeDriver described in Sec. 13.2.4.

16.2 The GRCowlingModeDriver Implementation

The GRCowlingModeDriver definition specifies the following class data and methods used in the implementation:

16.2.1 Physics

We need the variables A^*, U, c_1, V_g defined in (16.3):

- double r[len_star]: the dimensionless radius $r[X] = r/R$.
- double A[len_star]: the dimensionless Schwarzschild discriminant $A[X] = A^* = \frac{1}{\Gamma_1} \frac{d \log P}{d \log r} - \frac{d \log \rho}{d \log r}$.
- double V[len_star]: the variable $V[X] = V_g = -\frac{1}{\Gamma_1} \frac{d \log P}{d \log r}$.
- double C[len_star]: the variable $C[X] = c_1 = \frac{M}{R^3} \frac{r^3}{m}$.
- double U1[len_star]: the variable $U1[X] = U1 = 2 + \frac{d \log \nu'}{d \log r}$.
- double U2[len_star]: the variable $U1[X] = U2 = \frac{r}{2} \frac{d \lambda}{dr}$.

The arrays U_1, U_2 are constructed from ν, λ using the methods of GRStar, while the others are set directly with the methods in GRStar.

16.2.2 Calculation

- int len: the number of gridpoints of the Mode<N> calculation.
- int len_star: the number of gridpoints of the underlying Star model.
- void initializeArrays(): fill out values of A[], U[], C[], V[] from Star background.
- void getCoeff(double *CCI, int X, int b, double omeg2, int L): return the matrix $\mathbf{A} = \text{CCI}$ shown in (16.4).
- void getBoundaryMatrix(int nv, double* y0, double* ys, double** y, int* indexOrder): Because the mode is only second-order, there is only one solution in the center and one in the surface which makes the boundary matrix trivial.
- double SSR(double omeg2, int l, ModeBase* mode): **not yet correctly implemented**
- double innerproduct(ModeBase* mode1, ModeBase* mode2): **not yet correctly implemented.**
- double tidal_overlap(ModeBase* mode): **not yet correctly implemented.**

16.2.3 Miscellaneous

- enum VarNames {y1=0, y2}: the conventional names for the variables are y_1, y_2 .
- void varnames(std::string *names): allows for Mode<N> to know what labels to assign the mode variables when it is printing graphs. The input array of strings will be set with the names described here.

16.2.4 Boundary conditions

- Ac[3], Vgc[3], cc[3], U1c[3], U2c[3]: coefficients from power series expansions of A^*, V_g, c_1, U_1, U_2 to order $x^4 = (r/R_*)^4$ at the center of the star. Will come from the particular GRStar model. The coefficients of U_1, U_2 are constructed from the power series for ν, λ .
- As[5], Vs[5], cs[5], U1s[5], U2s[3]: as above, but at surface up to order $t^4 = (1 - r/R_*)^4$.
- setupBoundaries(): initialize the above arrays at both boundaries.

- `CenterBC(double** ymode, double* y0, double omeg2, int l, int m)`: Initialize the first few steps of the eigenmode using a power series expansion,

$$\vec{Y}(x) = \vec{Y}_0 + \vec{Y}_2 x^2 + \vec{Y}_4 x^4 + \dots \quad (16.5)$$

The coefficients are found from the recursion relation

$$\vec{Y}_{2k} = (2k - \mathbf{A}_0)^{-1} \sum_{m=0}^{k-1} \mathbf{A}_{2k-2m} \vec{Y}_{2m} \quad (16.6)$$

with \mathbf{A} as in (16.4) also expanded in a power series. The array `y0[]` specifies the values of y_i in \vec{Y}_0 . Once the coefficients are found, the first `start` values are calculated in `ymode[][]` using the series expansion. Then `start` is returned, indicating where the usual integration method of `Mode<N>` begins. See the Mathematica notebook `GRBCs.nb`.

- `SurfaceBC(double** ymode, double* y0, double omeg2, int l, int m)`: As above, but at the surface and to only 2nd order in t . See the Mathematica notebook `GRBCs.nb`.

16.3 References

- Yoshida & Lee, 2002. Present the GR Cowling approximation in a very suitable dimensionless form that closely mirrors the Newtonian Dziembowski equations. Note that we are using a slightly different metric convention, in terms of ν, λ .
- McDermott, Van Horn, Scholl. The original paper on the GR Cowling approximation.
- Finn
- Lindblom & Splinter, 1992. Compare to their column of frequencies ω_{MVS} using $\Gamma_1 = 1 + 1/n$ (by setting adiabatic index to \emptyset in input file) – agreement is very close.
- Lindblom, Mendell, Ipser, 1997. The version of the GR Cowling approximation presented here actually corresponds closer to that of Finn, formed by setting $H_0 = 0$ – however, this does not mean that $H_1 = 0$ nor that $K = 0$. However, we can compare our GR Cowling results to their exact results, expecting errors around 10%.