PROBLEM SET 2
GMM

Exercise 1 A representative consumer, with CRRA utility function, $u(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, who discounts future utility at rate β, solves the following portfolio two-period allocation problem:

$$\max_{\xi_1,\xi_2,\ldots,\xi_N} u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})]$$

$$c_t = Y_t - \sum_{i=1}^{N} p_{i,t}\xi_i \qquad c_{t+1} = Y_{t+1} + \sum_{i=1}^{N}(p_{i,t+1} + d_{i,t+1})\xi_i$$

Where the choice is which quantity to purchase from assets in t, indexed by i, to transfer wealth from t to $t+1$, where Y_t is an exogenous income of the consumer. Answer the following items:

a) Let $R_{i,t+1} = (p_{i,t+1} + d_{i,t+1})/p_{i,t}$ the gross return from asset i in $t+1$, function of prices and profits. Express the first order conditions as a system of equations in the format:

$$\mathbb{E}_t[f(\theta_0, \omega_{t+1})] = 0 \tag{1}$$

To propose a GMM estimator for θ, of dimension $a \times 1$, show that, following from (1), you can show:

$$\mathbb{E}[h(\theta_0, \omega_{t+1})] = 0 \tag{2}$$

assuming that θ_0 is the only θ that satisfies (1). Describe step by step the transformation from (1) to (2), explaining the format of $h(\theta_0, \omega_{t+1})$ from the format of $f(\theta_0, \omega_{t+1})$, the dimensions of each, the format and dimensions of θ_0 and ω_{t+1} and which instrumental variables are used to identify θ_0.

b) What restriction must be met for the parameters in θ_0 to be econometrically identifiable in (2)? To have an over-identified model, what condition must be met?

c) From the moment condition of item (1), write the GMM $\hat{\theta}_T$ of θ_0 estimator optimization problem, using the corresponding optimal weight matrix.

d) Suppose that the regularity conditions for $\hat{\theta}_T \xrightarrow{p} \theta_0$ are valid. Let $g_T(\theta, \omega) = \frac{1}{T}\sum_{t=1}^{T} h(\theta, \omega_{t+1})$. Prove that $\sqrt{T} g_T(\theta_0, \omega) \xrightarrow{d} N(0, S)$ and use it to show that $\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{d} N(0, V)$.

Compute V.

Exercise 2 Take the linear model

$$y_i = x_i'\beta + e_i$$

$$\mathbb{E}(z_i\, e_i) = 0$$

And consider the GMM estimator $\hat{\beta}$ of β. Let:

$$J = n\, \overline{g}_n(\hat{\beta})'\, \hat{\Omega}^{-1}\, \overline{g}_n(\hat{\beta})$$

denote the test of overidentifying restrictions. Show that $J \xrightarrow{d} \chi^2_{l-k}$ by demonstrating each of the following:

a) Since $\Omega > 0$, we can write $\Omega^{-1} = CC'$ and $\Omega = C^{-1'}C^{-1}$

b) $J = n(C'\overline{g}_n(\hat{\beta}))'(C'\hat{\Omega}C)^{-1}C'\overline{g}_n(\hat{\beta})$

c) $C'\overline{g}_n(\hat{\beta}) = D_n C'\overline{g}_n(\beta)$ where

$$D_n = I_l - C'(\frac{1}{n}Z'X)((\frac{1}{n}X'Z)\hat{\Omega}^{-1}(\frac{1}{n}Z'X))^{-1}(\frac{1}{n}X'Z)\hat{\Omega}^{-1}C^{-1'}$$

$$\overline{g}_n(\beta) = \frac{1}{n}Z'e$$

d) $D_n \xrightarrow{p} I_l - R(R'R)^{-1}R'$ where $R = C'\mathbb{E}(z_i x_i')$

e) $\sqrt{n}C'\overline{g}_n(\beta) \xrightarrow{d} u \sim N(0, I_l)$

f) $J \xrightarrow{d} u'(I_l - R(R'R)^{-1}R')u$

g) $u'(I_l - R(R'R)^{-1}R')u \sim \chi^2_{l-k}$

Exercise 3 The observations are i.i.d. $(y_i, x_i, q_i : i = 1, \ldots, n)$, where x_i is $k \times 1$ and q_i is $m \times 1$. The model is:

$$y_i = x_i'\beta + e_i$$

$$\mathbb{E}(x_i e_i) = 0$$

$$\mathbb{E}(q_i e_i) = 0$$

a) Find the efficient GMM estimator.

b) Compare with OLS estimator. Are they always different?

c) Let $z_i = (q_i x_i)$, and consider that

$$\mathbb{E}(z_i\, e_i) \neq 0$$

Using your knowledge of exercise 2, can we say that the J statistic diverges to infinity?