Problem set 3
Time Series

Exercise 1 For a scalar time series y_t define the samples autocovariance and autocorrelation:

$$\hat{\gamma}(k) = \frac{1}{n} \sum_{t=k+1}^{n} (y_t - \overline{y})(y_{t-k} - \overline{y})$$

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} = \frac{\sum_{i=1}^{n} (y_t - \overline{y})\,(y_{t-k} - \overline{y})}{\sum_{i=1}^{n} (y_t - \overline{y})^2}$$

Assume the series is strictly stationary, ergodic and $\mathbb{E}(y_t^2) < \infty$.

Show that $\hat{\gamma}(k) \xrightarrow{p} \gamma(k)$ and $\hat{\rho}(k) \xrightarrow{p} \rho(k)$ as $n \to \infty$. (Use the Ergodic Theorem).

Exercise 2 Let $\sigma_t^2 = \mathbb{E}(e_t^2 | I_{t-1})$ and (e_t, I_t) be a MDS (Martingale Difference Sequence). Define $u_t = e_t^2 - \sigma_t^2$.

a) Show that (u_t, I_t) is a MDS.

b) If $\mathbb{E}(e_t^4) \leq \infty$, then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (e_t^2 - \sigma_t^2) \xrightarrow{d} N(0, v^2)$$

Express v^2 in terms of the moments of e_t.

Exercise 3 Let e_t be an iid normal distributed error, with mean zero and variance 1.

Consider an $AR(p)$ process given by:

$$y_t = \alpha_0 + \alpha_1\, y_{t-1} + \cdots + \alpha_p\, y_{t-p} + e_t$$

And an $MA(q)$ process given by:

$$y_t = \beta_0 + e_t + \beta_1\, e_{t-1} + \cdots + \beta_q\, e_{t-q}$$

a) Under what conditions the $MA(q)$ process is strictly stationary and ergodic?

b) Under what conditions the $AR(p)$ process is strictly stationary and ergodic?