Misc. Notes

July 31, 2024

1 System Model and Dynamics

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix}\mathbf{q} & \dot{\mathbf{q}}\end{bmatrix}^\top = \begin{bmatrix}\theta_s & \theta_e & \dot{\theta}_s & \dot{\theta}_e\end{bmatrix}^\top && (1)\\
\mathbf{u} &= \begin{bmatrix}a_{\text{brach}} & a_{\text{lattri}} & a_{\text{antdel}} & a_{\text{postdel}} & a_{\text{bic}} & a_{\text{lattri}}\end{bmatrix} && (2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}, \mathbf{w}) \quad \mathbf{w} \sim \mathcal{N}(0, \Sigma_w) && (3)\\
&= \begin{bmatrix} \dot{\mathbf{q}} \\ M(\mathbf{q})^{-1}\left(C(\mathbf{q}, \dot{\mathbf{q}}) + T_M(\tilde{\mathbf{u}}, \mathbf{q})\right)\end{bmatrix} && (4)\\
\tilde{\mathbf{u}} &= \mathbf{u} + \operatorname{diag}(\mathbf{u})\mathbf{w} && (5)
\end{aligned}
$$

2 Discretization & Linearization

$$
\begin{aligned}
\mathbf{x}_{k+1} &= f(\mathbf{x_k}, \mathbf{u_k}, \mathbf{w_k})\\
&= g(\mathbf{x_k}, \tilde{\mathbf{u}}_\mathbf{k}) = g(\mathbf{x_k}, \mathbf{u_k} + \operatorname{diag}(\mathbf{u})\mathbf{w_k})\\
&\approx f^* + A_k(\mathbf{x_k} - \mathbf{x}^*) + B_k(\mathbf{u_k} - \mathbf{u}^*) + C_k(\mathbf{w_k} - \mathbf{w}^*)\\
\mathbf{w}^* &= 0\\
\mathbf{x}_{k+1} &\approx f^* + A_k(\mathbf{x_k} - \mathbf{x}^*) + B_k(\mathbf{u_k} - \mathbf{u}^*) + C_k\mathbf{w_k}\\
f^* &= g(\mathbf{x}^*, \mathbf{u}^*)\\
A_k &= \left.\frac{\partial g}{\partial \mathbf{x}}\right|_{\mathbf{x}^*, \mathbf{u}^*}\\
B_k &= \left.\frac{\partial g}{\partial \tilde{\mathbf{u}}}\frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{u}}\right|_{\mathbf{x}^*, \mathbf{u}^*, \mathbf{w}^*} = \left.\frac{\partial g}{\partial \tilde{\mathbf{u}}}\left(I + \operatorname{diag}(\mathbf{w})\right)\right|_{\mathbf{x}^*, \mathbf{u}^*, \mathbf{w}^*} = \left.\frac{\partial g}{\partial \tilde{\mathbf{u}}}\right|_{\mathbf{x}^*, \mathbf{u}^*}
\end{aligned}
$$

$$C_k = \frac{\partial g}{\partial \tilde{\mathbf{u}}} \frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{w}}\bigg|_{\mathbf{x}^*,\mathbf{u}^*,\mathbf{w}^*} = \frac{\partial g}{\partial \tilde{\mathbf{u}}} \operatorname{diag}(\mathbf{u})\bigg|_{\mathbf{x}^*,\mathbf{u}^*} = B_k \operatorname{diag}(\mathbf{u}^*)$$

$$\begin{aligned}
\mathbf{x}_{k+1} &= f^* + A_k(\mathbf{x_k} - \mathbf{x}^*) + B_k(\mathbf{u_k} - \mathbf{u}^*) + B_k\Lambda(\mathbf{u}^*)\mathbf{w_k} \\
P_{k+1} &= A_k P_k A_k^\top + C_k \Sigma_w C_k^\top \\
&= A_k P_k A_k^\top + B_k\Lambda(\mathbf{u}^*)\Sigma_w\Lambda(\mathbf{u}^*)B_k^\top
\end{aligned}$$

$$\begin{aligned}
B_k\Lambda(\mathbf{u}^*)\Sigma_w\Lambda(\mathbf{u}^*)B_k^\top &= B \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix} B^\top \\
&= B \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix} \begin{bmatrix} q_{11}u_1 & q_{12}u_2 & q_{13}u_3 \\ q_{12}u_1 & q_{22}u_2 & q_{23}u_3 \\ q_{13}u_1 & q_{23}u_2 & q_{33}u_3 \end{bmatrix} B^\top \\
&= B \begin{bmatrix} q_{11}u_1^2 & q_{12}u_1u_2 & q_{13}u_1u_3 \\ q_{12}u_1u_2 & q_{22}u_2^2 & q_{23}u_2u_3 \\ q_{13}u_1u^3 & q_{23}u_2u_3 & q_{33}u_3^2 \end{bmatrix} B^\top \\
&= B\left(\Sigma_w \odot \mathbf{u}\mathbf{u}^\top\right) B^\top
\end{aligned}$$

Where $\odot$ refers to element-wise multiplication.

Aside - derivative of diagonal matrix times vector:

$$\begin{aligned}
\operatorname{diag}(\mathbf{u})\mathbf{w} &= \begin{bmatrix} u_1w_1 \\ u_2w_2 \\ \vdots \\ u_nw_n \end{bmatrix} \\
\left[\tfrac{\partial \operatorname{diag}(\mathbf{u})\mathbf{w}}{\partial \mathbf{u}}\right]_{ij} &= \frac{\partial [\operatorname{diag}(\mathbf{u})\mathbf{w}]_i}{\partial u_j} = \begin{cases} w_i & \text{if i = j} \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \operatorname{diag}(\mathbf{u})\mathbf{w}}{\partial \mathbf{u}} &= \operatorname{diag}(\mathbf{w})
\end{aligned}$$