

Analytic Gradient of Snake Robot Inverse Kinematic Objective

For sake of clarity, we describe our derivation of the analytic gradient of a cost function for doing inverse kinematics for a simple snake robot in the form of an illustrative example of a snake with 3 links and 3 sets of joint angles (parameterized as Euler angles). Let the orientation of link l_i w.r.t. the orientation of the previous link be $\theta_i = [\alpha_i, \beta_i, \gamma_i]$. We assume that the orientation of first link is with respect to the x-axis.

Now denote the equivalent rotation matrix for link rotation θ_i as \mathbf{R}_i . Our objective is to identify the set of angles $\{\theta_1, \theta_2, \theta_3\}$ that place the tip position of the third link \mathbf{p}_3 as close as possible to a target \mathbf{t} with a final orientation of the third link as close to a target orientation \mathbf{q}_t while avoiding a set of spherical obstacles and obeying a set of hard constraints places on the joint angles, $\{\theta_1^{\min}, \theta_2^{\min}, \theta_3^{\min}\}, \{\theta_1^{\max}, \theta_2^{\max}, \theta_3^{\max}\}$.

The overall cost function is composed of distance, orientation, and obstacle components whose proportion to the overall cost can be adjusted by a set of scaling factors λ_x , $\sum_x \lambda_x = 1$

$$C = \lambda_d C_d + \lambda_o C_o$$

We wish to compute the gradient of the overall cost with respect to each of the angle sets θ_i . Because the gradient is a linear operator, we have

$$\nabla_{\theta_i} C = \nabla_{\theta_i} \lambda_d C_d + \nabla_{\theta_i} \lambda_o C_o$$

The distance cost is the sum of squared differences of the final effector position \mathbf{p}_3 and target \mathbf{t} , where \mathbf{p}_j denotes the final effector position after each successive rotation j .

$$C_d = (\mathbf{p}_3 - \mathbf{t})^T (\mathbf{p}_3 - \mathbf{t})$$

And its gradient with respect to a set of joint angles,

$$\nabla_{\theta_i} C_d = \nabla_{\mathbf{p}_3} C_d J_{\mathbf{p}_3}(\theta_i)$$

First note that $\mathbf{p}_0 = [0 \ 0 \ 0]^T$ and $\mathbf{l}_i = [l_i \ 0 \ 0]^T$. We can make use of the recursive relationship of end effector position between rotations.

$$\begin{aligned} \mathbf{p}_0 &= [0 \ 0 \ 0]^T \\ \mathbf{p}_1 &= \mathbf{R}_3(\mathbf{p}_0 + \mathbf{l}_3) \\ \mathbf{p}_2 &= \mathbf{R}_2(\mathbf{p}_1 + \mathbf{l}_2) \\ \mathbf{p}_3 &= \mathbf{R}_1(\mathbf{p}_2 + \mathbf{l}_1) \end{aligned}$$

Therefore we can simply compute the Jacobian $J_{\mathbf{p}_3}(\theta_i)$ using the chain rule

$$\begin{aligned} J_{\mathbf{p}_3}(\theta_1) &= J_{\mathbf{p}_3}(\theta_1) \\ J_{\mathbf{p}_3}(\theta_2) &= J_{\mathbf{p}_3}(\mathbf{p}_2) J_{\mathbf{p}_2}(\theta_2) \\ J_{\mathbf{p}_3}(\theta_3) &= J_{\mathbf{p}_3}(\mathbf{p}_2) J_{\mathbf{p}_2}(\mathbf{p}_1) J_{\mathbf{p}_1}(\theta_3) \end{aligned}$$

More generally this is,

$$J_{\mathbf{p}_n}(\theta_i) = \left[\prod_{k=n-i}^n J_{\mathbf{p}_k}(\mathbf{p}_{k-1}) \right] J_{\mathbf{p}_{n-i+1}}(\theta_i)$$

Where

$$J_{\mathbf{p}_k}(\mathbf{p}_{k-1}) = \mathbf{R}_{n-k+1}$$

We also find (Using symbolic MATLAB) that

$$\begin{aligned}
J_{\mathbf{p}}(\theta)[1, 1] &= \sin(\alpha) \sin(\beta) + \cos(\alpha) \cos(\gamma) \sin(\beta)(l_2 + p_2) + (\cos(\alpha) \sin(\gamma) - \cos(\gamma) \sin(\alpha) \sin(\beta))(l_3 + p_3) \\
J_{\mathbf{p}}(\theta)[1, 2] &= \cos(\alpha) \cos(\beta) \cos(\gamma)(l_3 + p_3) - \cos(\gamma) \sin(\beta)(l_1 + p_1) + \cos(\beta) \cos(\gamma) \sin(\alpha)(l_2 + p_2) \\
J_{\mathbf{p}}(\theta)[1, 3] &= (\cos(\gamma) \sin(\alpha) - \cos(\alpha) \sin(\beta) \sin(\gamma))(l_3 + p_3) - (\cos(\alpha) \cos(\gamma) + \sin(\alpha) \sin(\beta) \sin(\gamma))(l_2 + p_2) - \cos(\beta) \sin(\gamma)(l_1 + p_1) \\
J_{\mathbf{p}}(\theta)[2, 1] &= -(\cos(\gamma) \sin(\alpha) - \cos(\alpha) \sin(\beta) \sin(\gamma))(l_2 + p_2) - (\cos(\alpha) \cos(\gamma) + \sin(\alpha) \sin(\beta) \sin(\gamma))(l_3 + p_3) \\
J_{\mathbf{p}}(\theta)[2, 2] &= \cos(\beta) \sin(\alpha) \sin(\gamma)(l_2 + p_2) - \sin(\beta) \sin(\gamma)(l_1 + p_1) + \cos(\alpha) \cos(\beta) \sin(\gamma)(l_3 + p_3) \\
J_{\mathbf{p}}(\theta)[2, 3] &= (\sin(\alpha) \sin(\gamma) + \cos(\alpha) \cos(\gamma) \sin(\beta))(l_3 + p_3) - (\cos(\alpha) \sin(\gamma) - \cos(\gamma) \sin(\alpha) \sin(\beta))(l_2 + p_2) + \cos(\beta) \cos(\gamma)(l_1 + p_1) \\
J_{\mathbf{p}}(\theta)[3, 1] &= \cos(\alpha) \cos(\beta)(l_2 + p_2) - \cos(\beta) \sin(\alpha)(l_3 + p_3) \\
J_{\mathbf{p}}(\theta)[3, 2] &= -\cos(\beta)(l_1 + p_1) - \cos(\alpha) \sin(\beta)(l_3 + p_3) - \sin(\alpha) \sin(\beta)(l_2 + p_2) \\
J_{\mathbf{p}}(\theta)[3, 3] &= 0
\end{aligned}$$