18.03 Fourier Series Using Complex Exponentials

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For ease of notation we assume periodic functions of period 2π and work on the interval $[-\pi, \pi]$. The extension to other periods is easy.

First we'll give the Fourier theorem and then we'll motivate and prove it.

Theorem (Fourier): Suppose $f(t)$ is continuous and periodic with period 2π. Then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{int}, \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-int}\,dt$$

The proof of this uses the

Orthogonality relations: $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{int}\cdot \mathrm{e}^{-imt} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

The proof of the orthogonality relations is a trivial integration. (And the statement and proof are much easier than the analogous statements with sines and cosines.)

Proof of the formula for the Fourier coefficients c_n:

If $f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{int}$. then $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot \mathrm{e}^{-in_0t}\,dt = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} c_n \mathrm{e}^{int}\cdot \mathrm{e}^{-in_0t} = c_{n_0}$

The first equality is just a substitiution of the series for $f(t)$ and the second follows from the orthogonality relations.

Proof every continuous (period 2π) function equals its Fourier series:

See the note on Fourier completeness for this. Since $\cos t$ is a sum of complex exponentials the proof there suffices.

Comments

1. The *bilinear form* $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\bar{g}(t)\,dt$ is an inner product on the vector space of periodic functions. The Fourier theorem and orthogonality relations show the functions $\{\mathrm{e}^{int}\}$ form an *orthonormal basis.* They are analogous to the standard vectors **i**, **j**, **k** in three-dimensional space.

2. Also see the note on Fourier completeness for the definition of convolution and the periodic delta function.

3. The following notation is often used for the Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot \mathrm{e}^{-int}\,dt.$$

With this notation the Fourier theorem says

$$f(t) = \sum_{-\infty}^{\infty} \hat{f}(n)\,\mathrm{e}^{int}.$$

Theorem: Just like the Laplace transform, Fourier connects convolution and multiplication:

$$\widehat{f * g}(n) = 2\pi\, \hat{f}(n) \cdot \hat{g}(n).$$

Proof: The proof is simple algebra, but it gives some insight into the algebraic meaning of convolution.

First note that

$$\mathrm{e}^{int} * \mathrm{e}^{imt} = \int_{-\pi}^{\pi} \mathrm{e}^{in(t-u)} \mathrm{e}^{imu}\, du = \mathrm{e}^{int} \int_{-\pi}^{\pi} \mathrm{e}^{i(m-n)u}\, du = \begin{cases} 2\pi \mathrm{e}^{int} & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Now

$$\begin{aligned} (f * g)(t) &= \sum_n \hat{f}(n)\mathrm{e}^{int} * \sum_m \hat{g}(m)\mathrm{e}^{imt} \\ &= \sum_{n,\, m} \hat{f}(n)\hat{g}(m)\, \mathrm{e}^{int} * \mathrm{e}^{imt} \\ &\quad \text{by the formula just above, most of these terms are 0} \\ &= \sum_n 2\pi \hat{f}(n)\hat{g}(n)\mathrm{e}^{int} \end{aligned}$$

This last equality shows the Fourier coefficients of $f * g$ are $2\pi\, \hat{f}(n)\hat{g}(n)$ as claimed.

Note: In the Fourier game the factors of 2π that occur throughout are sometimes put in other places. For example, some authors define

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\mathrm{e}^{-int}\, dt$$

and then

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum \hat{f}(n)\mathrm{e}^{int}$$