ECE 1259 - Electromagnetics
Fall 2018

Vector Calculus

Vector Algebra

$$\mathbf{a_A} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

$$|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$$

$$\begin{aligned}\mathbf{A}\pm\mathbf{B} &= (A_x \pm B_x)\mathbf{a_x} \pm (A_y \pm B_y)\mathbf{a_y} \\ &\quad \pm (A_z \pm B_z)\mathbf{a_z}\end{aligned}$$

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= A_xB_x + A_yB_y + A_zB_z \\ &= AB\cos\theta\end{aligned}$$

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \begin{vmatrix}\mathbf{a_x} & \mathbf{a_y} & \mathbf{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z\end{vmatrix} \\ &= (A_yB_z - A_zB_y)\mathbf{a_x} + (A_zB_x - A_xB_z)\mathbf{a_y} \\ &\quad + (A_xB_y - A_yB_x)\mathbf{a_x} \\ &= (AB\sin\theta)\mathbf{a_n}\end{aligned}$$

$$proj_{\mathbf{B}}\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|}$$

Coordinate Systems

Cartesian ⇔ Cylindrical

$$x = \rho\cos\phi \quad y = \rho\sin\theta \quad z = z$$

$$\mathbf{a_x} = \cos\phi\mathbf{a}_\rho - \sin\phi\mathbf{a}_\phi$$

$$\mathbf{a_y} = \sin\phi\mathbf{a}_\rho + \cos\phi\mathbf{a}_\phi$$

$$\mathbf{a_z} = \mathbf{a_z}$$

$$\rho = \sqrt{x^2+y^2} \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}} \quad z = z$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\tan\phi = \frac{y}{x}$$

$$\mathbf{a}_\rho = \cos\phi\mathbf{a_x} + \sin\phi\mathbf{a_y}$$

$$\mathbf{a}_\phi = -\sin\phi\mathbf{a_x} + \cos\phi\mathbf{a_y}$$

$$\mathbf{a_z} = \mathbf{a_z}$$

Cartesian ⇔ Spherical

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\theta \quad z = r\cos\theta$$

$$\mathbf{a_x} = \sin\theta\cos\phi\mathbf{a_r} + \cos\theta\cos\phi\mathbf{a}_\theta - \sin\theta\mathbf{a}_\phi$$

$$\mathbf{a_y} = \sin\theta\sin\phi\mathbf{a_r} + \cos\theta\sin\phi\mathbf{a}_\theta + \cos\phi\mathbf{a}_\phi$$

$$\mathbf{a_z} = \cos\theta\mathbf{a_r} - \sin\theta\mathbf{a}_\theta$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}} \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z} \qquad \tan\phi = \frac{y}{x}$$

$$\mathbf{a_r} = \sin\theta\cos\phi\mathbf{a_x} + \sin\theta\sin\phi\mathbf{a_y} + \cos\theta\mathbf{a_z}$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\mathbf{a_x} + \cos\theta\sin\phi\mathbf{a_y} - \sin\phi\mathbf{a_z}$$

$$\mathbf{a}_\phi = -\sin\phi\mathbf{a_x} + \cos\phi\mathbf{a_y}$$

Cylindrical ⇔ Spherical

$$\rho = r\sin\theta \quad \phi = \phi \quad z = r\sin\theta$$

$$\mathbf{a}_\rho = \sin\theta\mathbf{a_r} + \cos\theta\mathbf{a}_\theta$$

$$\mathbf{a}_\phi = \mathbf{a}_\phi$$

$$\mathbf{a_z} = \cos\theta\mathbf{a_r} - \sin\theta\mathbf{a}_\theta$$

$$r = \sqrt{\rho^2+z^2} \quad \sin\theta = \frac{\rho}{r} \quad \phi = \phi$$

$$\cos\theta = \frac{z}{r}$$

$$\tan\theta = \frac{\rho}{z}$$

$$\mathbf{a_r} = \sin\theta\mathbf{a}_\rho + \cos\theta\mathbf{a_z}$$

$$\mathbf{a}_\theta = \cos\theta\mathbf{a}_\rho - \sin\theta\mathbf{a_z}$$

$$\mathbf{a}_\phi = \mathbf{a}_\phi$$

Differential Length, Surface, Volume

$$
\begin{aligned}
d\mathbf{L} =& dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z} \\
=& d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a_z} \\
=& dr\mathbf{a_r} + rd\theta\mathbf{a}_\theta + r\sin\theta\mathbf{a}_\phi
\end{aligned}
$$

$d\mathbf{S}$ = multiply scalar components of $d\mathbf{L}$ tangential to $d\mathbf{S}$ with the unit vector normal to it

$$
\begin{aligned}
dV =& dxdydz \\
=& \rho d\rho d\phi dz \\
=& r^2\sin\theta drd\theta d\phi
\end{aligned}
$$

Unit Vector Derivatives

Cartesian Coordinates

Derivative of any unit vector with respect to any cartesian variable is 0.

Cylindrical Coordinates

$\dfrac{\partial\mathbf{a}_\rho}{\partial\phi} = \mathbf{a}_\phi, \dfrac{\partial\mathbf{a}_\phi}{\partial\phi} = -\mathbf{a}_\rho,$ 0 otherwise

Spherical Coordinates

$\dfrac{\partial\mathbf{a_r}}{\partial\theta} = \mathbf{a}_\theta, \dfrac{\partial\mathbf{a}_\theta}{\partial\theta} = -\mathbf{a_r}, \dfrac{\partial\mathbf{a_r}}{\partial\phi} = \sin\theta\mathbf{a}_\phi$

$\dfrac{\partial\mathbf{a}_\theta}{\partial\phi} = \cos\theta\mathbf{a}_\phi \dfrac{\partial\mathbf{a}_\phi}{\partial\phi} = -(\sin\theta\mathbf{a_r} + \cos\theta\mathbf{a}_\theta),$

0 otherwise

Del Operator (∇) Applications

Gradient:

$$
\begin{aligned}
\nabla V =& \frac{\partial V}{\partial x}\mathbf{a_x} + \frac{\partial V}{\partial y}\mathbf{a_y} + \frac{\partial V}{\partial z}\mathbf{a_z} \\
=& \frac{\partial V}{\partial\rho}\mathbf{a}_\rho + \frac{\partial V}{\rho\partial\phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a_z} \\
=& \frac{\partial V}{\partial r}\mathbf{a_r} + \frac{\partial V}{r\partial\theta}\mathbf{a}_\theta + \frac{\partial V}{r\sin\theta\partial\phi}\mathbf{a}_\phi
\end{aligned}
$$

Divergence:

$$
\begin{aligned}
\nabla\cdot\mathbf{A} =& \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
=& \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\
=& \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) \\
&+ \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}
\end{aligned}
$$

Laplacian:

$$
\begin{aligned}
\nabla^2 V =& \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\
=& \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} \\
=& \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) \\
&+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}
\end{aligned}
$$

Curl:

$$
\begin{aligned}
\nabla\times\mathbf{A} =& \begin{vmatrix} \mathbf{a_x} & \mathbf{a_y} & \mathbf{a_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
=& \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a_z} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \\
=& \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a_r} & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}
\end{aligned}
$$

Electrostatics

Constants

$\epsilon_0 = 8.854 * 10^{-12} F/m$

Electric Field Intensity

Coulomb's Law

$$\mathbf{F_{12}} = \frac{Q_1 Q_2}{4\pi\epsilon R_{12}^2}\mathbf{a_{12}}$$

$$\mathbf{E_1} = \frac{\mathbf{F_{12}}}{Q_2} = \frac{Q_1}{4\pi\epsilon R^2}\mathbf{a_R}$$

Charge Distributions

Continuous Distribution: $E = \int \frac{dQ}{4\pi\epsilon R^2}\mathbf{a_R}$

Point Charge at Origin: $E = \frac{Q}{4\pi\epsilon r^2}\mathbf{a_r}$

Line of Charge on Z-Axis: $E = \frac{\rho_L}{2\pi\epsilon\rho}\mathbf{a}_\rho$

Infinite Sheet of Charge: $E = \frac{\rho_s}{2\epsilon}\mathbf{a_N}$

Electric Flux Density

$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_r\epsilon_0\mathbf{E}$

$\psi = \int \mathbf{D} \cdot d\mathbf{S}$

Gauss's Law

$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc} = \int \rho_v dV$

$\rho_v = \nabla \cdot \mathbf{D}$

Voltage

$V_{ba} = -\int_a^b \mathbf{E} \cdot d\mathbf{L} = V_a - V_b$

$V = \int \frac{dQ}{4\pi\epsilon r}$

$\mathbf{E} = -\nabla V$

Current

$I = \int \mathbf{J} \cdot d\mathbf{S}$

Resistance

Ohm's Law

$\mathbf{J} = \sigma\mathbf{E}$

$R = \frac{-\int \mathbf{E} \cdot d\mathbf{L}}{\int \sigma\mathbf{E} \cdot d\mathbf{S}}$

$P = \int \mathbf{E} \cdot \mathbf{J} dV$

Boundary Conditions

Dielectric-Dielectric

$\mathbf{E_{T1}} = \mathbf{E_{T2}}$

$\mathbf{a_{21}} \cdot (\mathbf{D_1} - \mathbf{D_2}) = \rho_s$

For $\rho_s = 0$: $\mathbf{D_{N1}} = \mathbf{D_{N2}}$

Dielectric-Conductor

$E_T = 0$

$D_N = 0$

Capacitance

$\nabla^2 V = \frac{-\rho_v}{\epsilon}$

For $\rho_v = 0$: $\nabla^2 V = 0$

$C = \frac{Q}{V}$

For parallel plate capacitors: $C = \frac{\epsilon S}{d}$

$\tau = RC = \frac{\epsilon}{\sigma}$

Work and Power

$W = -Q\int_a^b \mathbf{E} \cdot d\mathbf{L} = QV_{ba}$

Electrostatic potential energy: $W_E = \frac{1}{2}CV^2$

$P = \iiint \mathbf{E} \cdot \mathbf{J}\, dV = \iiint \sigma E^2 dV = \text{IV}$

Magnetostatics

Constants

$\mu_0 = 4\pi \times 10^{-7}$ H/m

Magnetic Field Intensity

Law of Biot-Savart

$$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$\mathbf{H} = \int \frac{\mathbf{K}dS \times \mathbf{a}_R}{4\pi R^2}$$

$$\mathbf{H} = \int \frac{\mathbf{J}dV \times \mathbf{a}_R}{4\pi R^2}$$

Current Distributions

Infinite line of current on Z-Axis: $\mathbf{H} = \dfrac{I\mathbf{a}_\phi}{2\pi\rho}$

Center of a solenoid: $\mathbf{H} = \dfrac{NI\mathbf{a}_z}{h}$

Infinite sheet of current: $\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N$

Magnetic Flux Density

$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_0\mathbf{E}$

$\phi = \int \mathbf{B} \cdot d\mathbf{S}$

Ampere's Law

$\oint \mathbf{H} \cdot d\mathbf{L} = I_{enc} = \int\int \mathbf{J} \cdot d\mathbf{S}$

$\mathbf{J} = \nabla \times \mathbf{H}$

Dynamic Fields

Continuity Equation

$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$

$\rho_v = \rho_0 e^{-t/\tau}$

Relaxation time: $\tau = \frac{\epsilon}{\sigma}$

Faraday's Law

Integral form: $\oint \mathbf{E} \cdot d\mathbf{L} = -\frac{\partial}{\partial t} \int\int \mathbf{B} \cdot d\mathbf{S}$

Point form: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$