CS240, Spring 2022
Assignment 1: Question 4

Q4a) Prove that $f(n) \in O'(g(n))$ implies that $f(n) \in O(g(n))$:

To prove the implication we will assume the hypothesis that $f(n) \in O'(g(n))$. Be definition this means that there exists a constant $c > 0$, that for all $n > 0$ we have:

$$f(n) \leq cg(n)$$

Since $f(n) > 0$ this means that $f(n) = |f(n)|$ and $g(n) = |g(n)|$ this implies:

$$|f(n)| \leq c|g(n)|$$

By first principles this means that:

$$f(n) \in O(g(n))$$

Q4b) Prove that $f(n) \in O(g(n))$ implies that $f(n) \in O'(g(n))$:

To prove the implication we will assume the hypothesis that $f(n) \in O(g(n))$. Be definition this means that there exists a constant $c > 0$, that for some $n_0 > 0$ we have:

$$|f(n)| \leq c|g(n)| \qquad \forall n > n_0$$

Since $f(n), g(n) > 0$ we can remove the absolute value signs to get:

$$f(n) \leq cg(n) \qquad \forall n > n_0$$

This tells us that if n_0 is bounded by $0 < n_0 < n$ we will get:

$$f(n) > cg(n) \qquad \forall n > n_0$$

We will now analyse the fraction:

$$\frac{g(n)}{cf(n)}$$

Since both $c > 0$ and $g(n) > 0$ this fraction will be continuous, moreover the range of this fraction is all the real numbers between 0 and n_0. If we take the maximum value (called x) of the range it makes sense that:

$$\frac{g(n)}{cf(n)} \leq x$$

If we rearrange we get:

$$g(n) \leq x \times c \times f(n)$$

Since x and c are constants this means that the equation is equvilient to the definition of O', which means that we have proved $g(n) \in O'(f(n))$ as wanted.