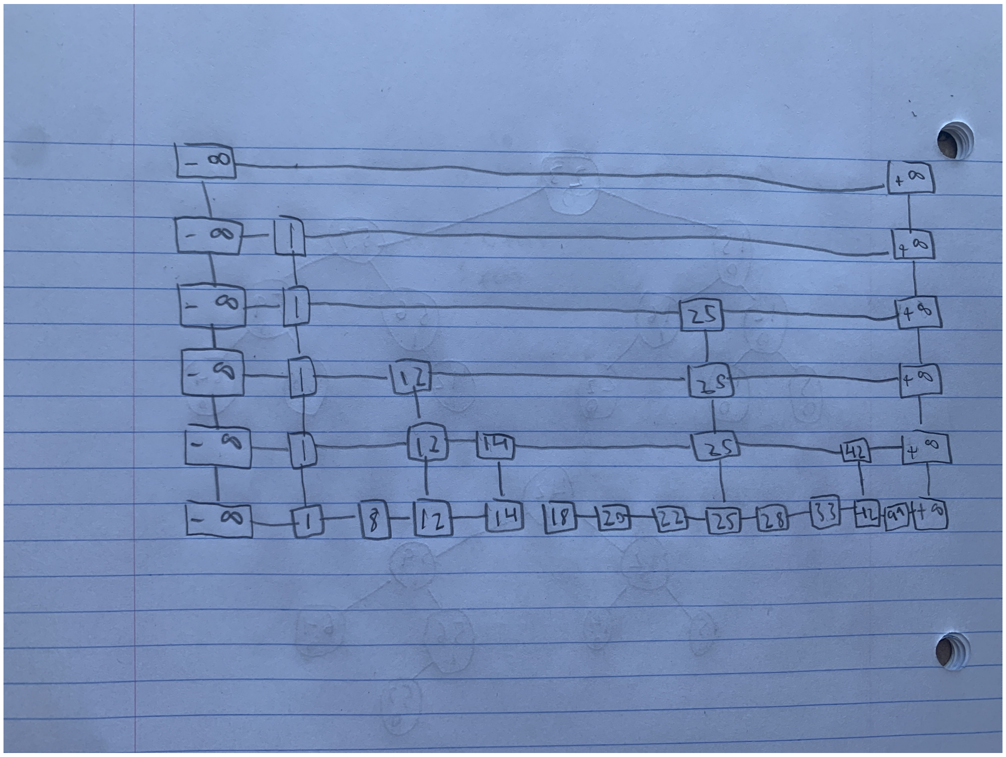
CS240, Spring 2022
Assignment 3: Question 5

Q5b)



Key	28	42	8	1	22	99
Comparisons	8	9	7	7	11	9

Q5c) We are told from the lectures that the height of the tower is given by the number of coin flips until you get a tails. In other words

$$\text{height of tower} = \text{number of heads before a tails}$$

We can reorganize this to get:

$$\text{probability tower of height i} = \text{probability of i heads}$$

We are told in the question that p represents the probability of adding a new layer to the tower, in other words this means $p =$ probability of rolling heads. Since the probability of each coin flip is independent we get:

$$\text{probability tower of height i} = p^i$$

Q5d) For each element in the skip list, the expected number of nodes is given by:

$$= i * p^i$$

Since there are n elements in the skip list we know our space complexity will be proportional to:

$$= n * i * p^i$$

From e) we are given that the max height of the the list is log(n), therefore if we know that i can be the values from [0, ..., log(n)]. Thus our expected space complexity is given by:

$$= \frac{1}{\log(n)} \sum_{i=0}^{\log(n)} n * i * p^i$$

$$= \frac{n}{\log(n)} \sum_{i=0}^{\log(n)} i * p^i$$

Note that since p is always less then 1, it follows that $i * p^i <= \frac{i}{log(n)}$, thus it follows that:

$$\leq \frac{n}{\log(n)} \sum_{i=0}^{\log(n)} \frac{i}{log(n)}$$

$$= \frac{n}{\log(n)} \frac{log(n) * (log(n) - 1)}{2\log(n)}$$

Simplifying we get:

$$\frac{n}{2} - \frac{n}{2\log n} \in O(n)$$

which proves out space complexity will be linear.

Q5e) For each element in the skip list, the number of elements with a height of i or greater is given by:

$$E(height) = n * p^i$$

We will create an indicator variable such that:

$$I(i) = \begin{matrix} 0, & \text{if no elements have height i } 0 \leq n \leq 1 \\ 1, & \text{if some elements have height i } 0 \leq n \leq 1 \end{matrix}$$

It thus follows that the height can be solved by finding:

$$= 1 + \sum_{i=1}^{\infty} I(i)$$

Since n is a power of $\frac{1}{p}$ we can split this summation into 2 parts:

$$= 1 + \sum_{i=1}^{log_{\frac{1}{p}}(n)} I(i) + \sum_{i=\log^{n}_{\frac{1}{p}}}^{\infty} I(i)$$

Using the equation for probability of a height we had at the top, our equation thus becomes:

$$= 1 + \sum_{i=1}^{log_{\frac{1}{p}}(n)} 1 + \sum_{i=\log_{\frac{1}{p}}^{n}}^{\infty} n * p^i$$

Which then becomes:

$$= 1 + log_{\frac{1}{p}}(n) + p * \sum_{i=\log_{\frac{1}{p}}^{n}}^{\infty} p^i$$

We can update the bounds to get:

$$\text{height} <= 1 + log_{\frac{1}{p}}(n) + p * \sum_{i=0}^{\infty} p^i$$

By the taylor series we know that our infinite summation is equivalent to $\frac{1}{1-p}$ so w get:

$$\text{height} <= 1 + log_{\frac{1}{p}}(n) + p * \frac{1}{1-p}$$

$$\text{height} <= log_{\frac{1}{p}}(n) + \frac{1-p}{1-p} + p * \frac{1}{1-p}$$

$$\text{height} <= log_{\frac{1}{p}}(n) + \frac{1}{1-p}$$

Which is as required.