CS240, Spring 2022
Assignment 1: Question 3

Q3a) Prove or disprove "If $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$":

In order to prove this implication we will start by assuming the hypothesis, formally this tells us that there exists constants $c > 0$ and $n_0 > 0$ such that:

$$|f(n)| \leq c|g(n)| \quad \forall n \geq n_0$$

Dividing both sides by our constant c gives us:

$$\frac{1}{c}|f(n)| \leq |g(n)| \quad \forall n \geq n_0$$

Since c can be any real number (and contains all the values between 1 and 0), this implies that c and $\frac{1}{c}$ will have the same range of values. In other words we can make a new variable $d = \frac{1}{c}$ and it will be the case that $d > 0$. Therefore our equation becomes:

$$d|f(n)| \leq |g(n)| \quad \forall n \geq n_0$$

Therefore by first principles it must be the case that (which proves the statement):

$$g(n) \in \boldsymbol{\Omega}(f(n))$$

Q3b) Prove or disprove "There exists $f(n), g(n)$ such that $f(n) \in o(g(n))$ and $f(n) \in \omega(g(n))$":

Assuming the statement is correct formally tells us that $\forall c > 0$, there exists an $n_0 > 0$ such that the two equations below are satisfied:

$$1)\ |f(n)| \leq c|g(n)| \quad \forall n \geq n_0$$
$$2)\ |f(n)| \geq c|g(n)| \quad \forall n \geq n_0$$

Combining the two equation gives us:

$$|f(n)| \leq c|g(n)| \leq |f(n)| \quad \forall n \geq n_0$$

For each value of $|f(n)|$, c will be every possible value. Therefore in order for $c|g(n)|$ to be bounded by $|f(n)|$, it must be the case that $|g(n)| = 0$. Therefore we get:

$$|f(n)| \leq 0 \leq |f(n)| \quad \forall n \geq n_0$$

This tells us that the only possible value for $|f(n)| = 0$. However since we are told $f(x), g(x) > 0$, it implies that our only solution for $f(x)$ and $g(x)$ is illegal and so:

There exists **no** $f(n), g(n)$ such that $f(n) \in o(g(n))$ and $f(n) \in \omega(g(n))$

Q3c) Prove or disprove "If $f(n) \in O(g(n))$ then $2^{f(n)} \in O(2^{g(n)})$":

We will start by considering the case where $f(n) = 2n^2$ and $g(n) = n^2$, in class we found these functions had the following relationship:

$$2n^2 \in O(n^2)$$

Assuming the implication is correct, this implies that if we take both functions to the power of 2 we should get:

$$2^{2n^2} \in O(2^{n^2})$$

By first principles this means that there exists a $c > 0$ and $n_0 > 0$ such that:

$$2^{2n^2} \leq c2^{n^2} \quad \forall n \geq n_0$$

If we simplify we get:

$$\begin{aligned}
2^{n^2} \times 2^{n^2} \leq c2^{n^2} \quad \forall n \geq n_0 \\
2^{n^2} \leq c \quad \forall n \geq n_0 \\
n^2 \leq \log(c) \quad \forall n \geq n_0 \\
n \leq \sqrt{\log c} \quad \forall n \geq n_0
\end{aligned}$$

This tells us that for any $n > \sqrt{\log c}$, that 2^{2n^2} grows faster then 2^{n^2}. This proves the **statement is false**, as for all $n_0 = \sqrt{\log c}$ where $c > 0$:

$$2^{2n^2} \geq c2^{n^2} \quad \forall n \geq n_0$$

This implies that:

$$2^{2n^2} \in \omega(2^{n^2}) \text{ and } 2^{2n^2} \notin O(2^{n^2})$$

And so we have **disproved** the statement by giving a counter example.

Q3d) Prove or disprove "$(\log(n))^{\log(n)} \in O(n^2)$":

We will start by evaluating the following function:

$$\log(n)^{\log(n)}$$

In order to simplify we will set the following variable such that:

$$a = \log(n) \text{ and } n = 2^a$$

Plugging this into the original equation gives us:

$$\log(n)^{\log(n)} = a^a$$

We can also make the observation that:

$$a^a \geq 4^a \quad \forall a \geq 4$$

Thus we can get the following inequality:

$$\begin{aligned} a^a &\geq 4^a \\ \log(n)^{\log(n)} &\geq 4^a \quad \forall a \geq 4 \\ \log(n)^{\log(n)} &\geq 4^{log(n)} \quad \forall a \geq 4 \\ \log(n)^{\log(n)} &\geq 2^{2log(n)} \quad \forall a \geq 4 \\ \log(n)^{\log(n)} &\geq 2^{log(n^2)} \quad \forall a \geq 4 \\ \log(n)^{\log(n)} &\geq n^2 \quad \forall n \geq 16 \end{aligned}$$

Therefore we have found $n_0 = 16$ and $c = 1$ which by first principles tell us that:

$$\log(n)^{\log(n)} \notin O(n^2)$$

And so we have **disproved** the statement.

Q3e) Prove or disprove "$\log(n) \times 2^{\sin(n^3)} \in O(n)$":

To start we can make the following observation:

$$\sin(n^3) \leq 1$$

Therefore we can make the following inequality:

$$\begin{aligned} \log(n) \times 2^{\sin(n^3)} &\leq \log(n) \times 2^1 \\ &\leq 2\log(n) \\ &\leq 2n \text{ (since } \log(n) \in O(n))\forall n \geq 1 \end{aligned}$$

Thus we have $c = 2$ and $n_0 = 1$ and so by first principles we get:

$$\log(n) \times 2^{\sin(n^3)} \in \boldsymbol{O}(n)$$