CS240, Spring 2022
Assignment 1: Question 1

Q1a) Prove that $7n^4 - 5n^2 + 6 \in O(n^4)$

We will start with the following observations:

$$1)\ n \leq n^4 \text{ which implies } n \leq 6n^4 \qquad \forall n \geq 1$$
$$2)\ -n \leq 0 \text{ which implies } -5n^2 \leq 0 \qquad \forall n \geq 1$$

Applying both identities to the equation we started with gives us:

$$7n^4 - 5n^2 + 6 \leq 7n^4 + 6n^4 \qquad \forall n \geq 1$$

Simplifying we get:

$$7n^4 - 5n^2 + 6 \leq 13n^4 \qquad \forall n \geq 1$$

Thus $\boldsymbol{c = 13}$ and $\boldsymbol{n_0 = 1}$, which by first principles proves that $7n^4 - 5n^2 + 6 \in O(n^4)$.

Q1b) Prove that $7n^4 - 5n^2 + 6 \in \Omega(n^4)$

Again we will start with some more observations:

$$1)\ 7n^4 = 2n^4 + 5n^4$$
$$2)\ 6 \geq 0$$

Applying both identities to the equation we started with gives us:

$$7n^4 - 5n^2 + 6 \geq 2n^4 + 5n^4 - 5n^2$$

We will now isolate the terms $5n^4 - 5n^2$ and discover when the equation is positive:

$$5n^4 - 5n^2 \geq 0$$
$$5n^4 \geq 5n^2$$
$$n^2 \geq 1$$

Therefore when $n \geq 1$ the overall equation would be smaller if the terms are removed, so it follows that:

$$7n^4 - 5n^2 + 6 \geq 2n^4 + 5n^4 - 5n^2 \geq 2n^4 \qquad \forall n \geq 1$$

Thus $\boldsymbol{c = 2}$ and $\boldsymbol{n_0 = 1}$, which by first principles proves that $7n^4 - 5n^2 + 6 \in \Omega(n^4)$.

Q1c) Prove that $5n^2 + 15 \in o(n^3)$

We can start by making the following observation:

$$1)\ 1 \leq 1 * n^2 \text{ which implies } 15 \leq 15n^2 \qquad \forall n \geq 1$$

Therefore we can get the following equality:

$$\begin{aligned} 5n^2 + 15 &\leq 5n^2 + 15n^2 \qquad && \forall n \geq 1 \\ 5n^2 + 15 &\leq 20n^2 && \forall n \geq 1 \end{aligned}$$

In order to get a n^3 on the right side we can do the following:

$$5n^2 + 15 \leq \frac{20}{n} n^3 \qquad \forall n \geq 1$$

For $5n^2 + 15 \in o(n^3)$, it must be the case that for any value of $c > 0$:

$$5n^2 + 15 \leq \frac{20}{n} n^3 \leq cn^3 \qquad \forall n \geq 1$$

At the moment $5n^2 + 15$ is not relevant so we can remove it and solve for n_0:

$$\frac{20}{n} n^3 \leq cn^3 \qquad \forall n \geq 1$$

$$n \geq \frac{20}{c} \quad \forall n \geq 1$$

Therefore $n \geq \frac{20}{c}$ and $n \geq 1$ which tells us that:

$$\boldsymbol{n_0 = \lceil \frac{20}{c} \rceil}$$

Thus for any value $c > 0$ we can get a value for n_0 such that:

$$5n^2 + 15 \leq \frac{20}{n} n^3 \leq cn^3 \qquad \forall n \geq n_0$$

Which by first principles proves that $5n^2 + 15 \in o(n^3)$.