CS240, Spring 2022
Assignment 1: Question 2

Q2a) Determine the relationship between $f(n) = n^2 + 22n(\log(n)) + 13$ and $g(n) = n^2 \log(n) + 14$:

In order to find the relationship between $f(n)$ and $g(n)$ we will use limit L such that:

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)}$$

Replacing $f(n)$ and $g(n)$ with the polynomials they represent gives us:

$$L = \lim_{n \to \infty} \left(\frac{n^2 + 22n(\log(n)) + 13}{n^2 \log(n) + 14} \right)$$

The Comparison Test tells us that since $n^2 \log(n) < n^2 \log(n) + 14$ for all $n > 0$ that:

$$\text{if } \lim_{n \to \infty} \left(\frac{n^2 + 22n(\log(n)) + 13}{n^2 \log(n)} \right) = 0 \implies \lim_{n \to \infty} \left(\frac{n^2 + 22n(\log(n)) + 13}{n^2 \log(n) + 14} \right) = 0$$

Therefore we can simplify the left most equation to get:

$$\begin{aligned}
\lim_{n \to \infty} \left(\frac{n^2 + 22n(\log(n)) + 13}{n^2 \log(n)} \right) &= \lim_{n \to \infty} \left(\frac{n^2}{n^2 \log(n)} + \frac{22n(\log(n))}{n^2 \log(n)} + \frac{13}{n^2 \log(n)} \right) \\
&= \lim_{n \to \infty} \left(\frac{1}{\log(n)} + \frac{22}{n} + \frac{13}{n^2 \log(n)} \right) \\
&= \frac{1}{\log(\infty)} + \frac{22}{\infty} + \frac{13}{\infty^2 \log(\infty)} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

Thus by the Comparison test we get:

$$\lim_{n \to \infty} \left(\frac{n^2 + 22n(\log(n)) + 13}{n^2 \log(n) + 14} \right) = L = 0$$

Since $L = 0$ this means that:

$$n^2 + 22n(log(n)) + 13 \in \mathbf{o}(n^2 log(n) + 14)$$

Q2b) Determine the relationship between $f(n) = \sqrt{n}$ and $g(n) = \log(n^4)$:

We will find the relationship between $f(n)$ and $g(n)$ by finding the limit L where:

$$L = \lim_{n \to \infty} \frac{\sqrt{n}}{\log(n^4)}$$

Log rules show us that $\log(n^4) = 4\log(n)$ therefore our limit formula becomes:

$$L = \lim_{n \to \infty} \frac{\sqrt{n}}{4\log(n)}$$

If we apply L'Hopital's rule we get:

$$\begin{aligned}
\lim_{n \to \infty} \frac{\sqrt{n}}{4\log(n)} &= \lim_{n \to \infty} \frac{\frac{1}{2}n^{-\frac{1}{2}}}{\frac{4}{n\ln(2)}} \\
&= \lim_{n \to \infty} \frac{\frac{1}{2} \times \frac{1}{n^{\frac{1}{2}}}}{\frac{4}{n\ln(2)}} \\
&= \lim_{n \to \infty} \frac{\frac{1}{2} \times \frac{1}{n^{\frac{1}{2}}} \times n}{\frac{4}{\ln(2)}} \\
&= \lim_{n \to \infty} \frac{\frac{1}{2}\sqrt{n}}{\frac{4}{\ln(2)}} \\
&= \frac{\frac{1}{2}\sqrt{\infty}}{\frac{4}{\ln(2)}} \\
&= \infty
\end{aligned}$$

Therefore by the limit test since $L = \infty$ it implies that the relationship is:

$$\sqrt{n} \in \boldsymbol{\omega}(log(n^4))$$

Q2c) Determine the relationship between $f(n) = 10^n + 99n^{10}$ and $g(n) = 75^n + 25n^{27}$:

We will find the relationship between $f(n)$ and $g(n)$ by finding the limit L where:

$$L = \lim_{n\to\infty} \frac{10^n + 99n^{10}}{75^n + 25n^{27}}$$

We will apply L'Hopital's rule to see if we notice any trends in the derivative:

$$\lim_{n\to\infty} \frac{10^n + 99n^{10}}{75^n + 12n^{27}} = \lim_{n\to\infty} \frac{10^n \times \ln(10) + 990n^9}{75^n \times \ln(75) + 650n^{26}}$$

As we can see each time we take the derivative of 75^n or 10^n we will get a constant factor ($\ln(75)$ or $\ln(10)$) which wont impact future derivatives. Since the derivative will continue much the same we will skip to the 10th derivative:

$$\lim_{n\to\infty} \frac{10^n + 99n^{10}}{75^n + 12n^{27}} = \lim_{n\to\infty} \frac{10^n \times (\ln(10))^{10} + 359251200}{75^n \times (\ln(75))^{10} + 25 \times \frac{27!}{17!}n^{17}}$$

If we take the next derivative we will get rid of the constant term (359251200) as seen below:

$$\lim_{n\to\infty} \frac{10^n + 99n^{10}}{75^n + 12n^{27}} = \lim_{n\to\infty} \frac{10^n \times (\ln(10))^{11}}{75^n \times (\ln(75))^{11} + 25 \times \frac{27!}{16!}n^{16}}$$

To get rid of the $\frac{27!}{16!}n^{16}$ term on the denominator we will skip to the 28th derivative:

$$\begin{aligned}
\lim_{n\to\infty} \frac{10^n + 99n^{10}}{75^n + 12n^{27}} &= \lim_{n\to\infty} \frac{10^n \times (\ln(10))^{28}}{75^n \times (\ln(75))^{28}} \\
&= \lim_{n\to\infty} (\frac{10}{75})^n \times (\frac{\ln(10)}{\ln(75)})^{25} \\
&= (\frac{10}{75})^\infty \times (\frac{\ln(10)}{\ln(75)})^{25} \\
&= 0 \times (\frac{\ln(10)}{\ln(75)})^{25} \\
&= 0
\end{aligned}$$

Therefore by the limit test since $L = 0$ it implies that the relationship is:

$$10^n + 99n^{10} \in \mathbf{o}(75^n + 12n^{27})$$