Interpolation

Due at 11:59pm on Monday 6 February 2023

What you need to get

- `YOU_a2q1.ipynb`
- **https://www.overleaf.com/read/vztpkwrpmtcn**: (optional) Overleaf template for Q2
- `YOU_a2q3.ipynb`
- `YOU_a2q4.ipynb`

What to do

1. [25 marks] **Complete the Python function** `MySpline` that reads in a set of `x` and `y` values (each as arrays or lists), and outputs a piecewise-cubic function with natural boundary conditions. The function prototype is

   ```
   cs = MySpline(x, y)
   ```

   The returned function can be called using `cs(2.4)` or `cs([2.4, 3.7])`, for example. You will have to find the parameter values for the as, bs, and cs so that the cubic spline can be evaluated using

   $$p_i(x) = a_{i-1}\frac{(x_{i+1}-x)^3}{6h_i} + a_i\frac{(x-x_i)^3}{6h_i} + b_k\,(x_{i+1}-x) + c_i\,(x-x_i)\ ,$$

   where $h_i = x_{i+1} - x_i$ and $i = 1, 2, \ldots, n-1$. Note that all Python indexing starts at 0, so all the indices can be decremented by 1. However, some care is needed to handle $a_0, \ldots, a_{n-1}$. See the documentation for `MySpline` for some guidance on this issue.

   The function `MySpline` is already set up to return a cubic spline function. But it's not a very interesting one, and it does not pass through the points (x_i, y_i).

   The notebook has a small sample set of points to interpolate. Feel free to choose your own set of points. **Create a figure that plots the interpolation points overtop of the smooth cubic spline.**

2. [25 marks] Consider the following alternative representation for a cubic spline, $S(x)$:

   $$S(x) = \begin{cases} a + b(x-1) + c(x-1)^2 + \frac{1}{4}(x-1)^2(x-2) & 1 \le x \le 2 \\ \\ e + f(x-2) + g(x-2)^2 - \frac{1}{4}(x-2)^2(x-3) & 2 \le x \le 3. \end{cases}$$

   We also wish our cubic spline to satisfy the boundary conditions:

   $$\frac{d^2S}{dx^2}(1) = 0\ , \quad \frac{d^2S}{dx^2}(3) = 0\ .$$

   (a) What are the conditions on the coefficients a, b, c, e, f and g such that $S(x)$ interpolates the points (1,2), (2,1), and (3,1)? Deduce the values of a and e.

   (b) What is the condition on the coefficients such that $S'(x)$ is continuous at $x = 2$?

   (c) Show that enforcing the boundary conditions at $x = 1$ and $x = 3$ leads to $c = \frac{1}{4}$ and $g = \frac{1}{2}$.

   (d) Compute the values of b and f from part (a).

   (e) To ensure that $S(x)$ is a cubic spline, what other condition needs to be checked? (It is not necessary to actually verify this condition for the purpose of this exercise.)

3. [30 marks] Create a parametric curve representation of your nickname written in your handwriting. This representation should be based on parametric spline interpolation. Your nickname must have at least two curved segments. Figure 1 below shows an example using 5 coarsly-sampled segments.

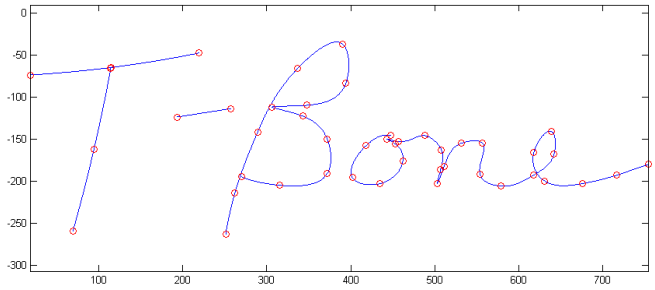

Figure 1: The nickname "T-Bone" written with 5 parametric cubic spline segments

Steps:

(a) Draw your nickname and select interpolation points. You can do this on graph paper by hand, or you can use `ginput`. Make sure the interpolation points are not too close together. Notice that the loops of the "B" in Figure 1 are made from only 6 points each. Your curved segments should be sampled sparingly.

(b) Add code to the notebook to initialize and store the interpolation points for each segment. For example, you could store the coordinates of the interpolation points for the first segment in the arrays `x1` and `y1`, and you would have one `x-y` pair of arrays for each segment. Your data arrays should be hardcoded into your script.

(c) Complete the function `ParametricSpline`; it takes the interpolation points of one segment as input, and outputs a pair of cubic spline functions (one for `x(t)` and one for `y(t)`), along with an array, `t`, that holds the parameter value at the interpolation points. See the function's documentation for more details. The parameter `t` must reflect the pseudo-arclength of the curve. You can use `scipy.make_interp_spline` (and return the spline functions that it creates). Alternatively, if you feel confident in your `MySpline` function, you can use that instead.

(d) Add lines to the notebook to call `ParametricSpline` for each segment in your nickname.

(e) Add more lines to to the notebook to plot your nickname. The plot should graph the parametric spline segments using a refined selection of parameter values (eg. 1000 points per segment). The plot should also display the original interpolation points, and the `x` and `y` axes should be rendered using the same scale ("`plt.axis('equal')`"). Figure 1 is an example of what your output should look like.

4. [20 marks] James Carver, a dentist, is now the main suspect for the murder of Robert Durst, but police are still watching Samatha Brundi carefully. More solid evidence is needed to convict someone of the murder of Robert Durst.

   The police would like to know the unlock pattern so they can access Mr. Carver's phone while he is in the shower at the gym. The phone is guarded by a lock screen that unlocks via a pattern that the user draws on a 4x4 grid of circles. The user begins at a circle and drags their finger across four other circles in one continuous motion, as illustrated in the figure.

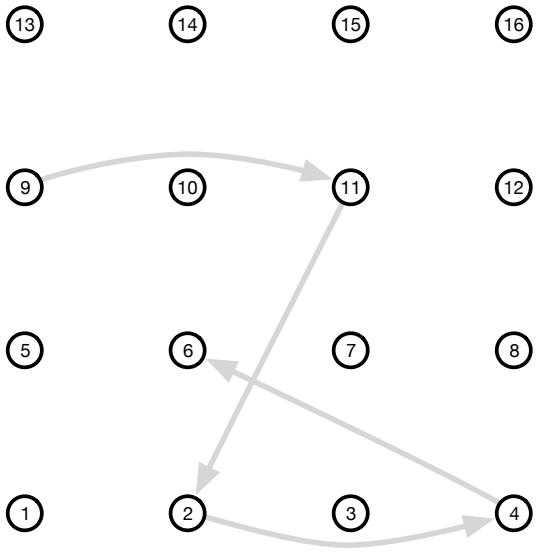

Figure 2: Example of an unlock pattern, 9–11–2–4–6

The police have managed to intercept some data packets sent from Mr. Carver's cell phone. The packets, collected by Gooble (the company that manufactures the phone), convey usage data in a logfile, including some screen digitizer data. Luckily, some of the logfile data pertains to his unlock pattern. The logfile includes the position of the finger at 5 different times, listed in the table below:

Time (milliseconds)	x position (mm)	y position (mm)
0	1	59
878	60	24
1424	32	22
1969	29	42
2737	44	62

The coordinates in the table are with respect to a lock-screen coordinate system in which the bottom left circle (labelled "1") is the origin, $(0, 0)$. Each subsequent circle in the x-axis direction increments the x-position by 20 mm, and each subsequent circle in the y-axis direction likewise increments the y-position by 20 mm. For example, in figure 2, the pattern starts at $(0, 40)$ and goes to $(40, 40)$, $(20, 0)$, $(60, 0)$ before ending at $(20, 20)$. The notebook includes a function called `DrawGrid()` that draws the unlock grid.

As a specialist working for the police department, your job is to interpolate the data given in the logfile to determine the unlock pattern of the dentist's phone. You may assume that the dentist's finger is stationary for the first and last points.

(a) Edit the notebook so that it creates a parametric cubic spline that interpolates the points from the logfile. You may assume that the suspect's finger was at rest at the beginning and end of the swipe pattern (i.e. velocity was zero).

(b) Create a plot of the path of the parametric spline on top of the unlock-grid. Include this plot in your assignment.

(c) What do you think is the unlock pattern? List the five circles in order.

What to submit

Rename each of your jupyter notebooks, replacing "`YOU`" with your WatIAM ID. For example, I would rename `YOU_a2q1.ipynb` to `kfountou_a2q1.ipynb`. Export each jupyter notebook as a PDF, and submit each PDF to Crowdmark. If you want, you can typeset your solutions in a LaTeX or Word document, or write electronically (as on a tablet), and hand in your document as a PDF. **Photographs or scans of handwritten solutions should be legible, otherwise, the TAs might deduct marks.** Submit a separate PDF document for each of Q1, Q2, Q3, and Q4.

Finally, upload your Python notebooks on Learn dropbox <—— Do not forget this, otherwise I will apply 10% penalty to the assignment.