r2knowle_a1q4

January 21, 2023

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# Standard imports
import numpy as np
np.seterr(all='ignore');  # allows floating-point exceptions
import matplotlib.pyplot as plt
```

1.1 Q4: Upper Bound on Error

We begin with the equation:

$$\frac{\left|(a \otimes b) \ominus c - (ab - c)\right|}{|ab - c|} \tag{1}$$

To determine the error we will first determine the error for $(a \otimes b)$. We will set the realtive error to be S_1, and so by rearraning we get:

$$\begin{align} S_1 &= \text{ Relative error} \tag{2}\\ S_1 &= \frac{(a \otimes b) - (a \times b)}{(a \times b)} \tag{3}\\ S_1 &= \frac{(a \otimes b) - (ab)}{(ab)} \tag{4}\\ S_1(ab) &= (a \otimes b) - (ab) \tag{5}\\ (ab)(1 + S_1) &= (a \otimes b) \tag{6} \end{align}$$

Lets do the same for $ a c $ (where a represents some value). We will set the realtive error to be S_2, and so by rearraning we get:

$$\begin{align} S_2 &\geq \text{ Relative error} \tag{7}\\ S_2 &\geq \frac{(a \ominus b) - (a - b)}{(a - b)} \tag{8}\\ S_2(a - b) &\geq (a \otimes b) - (a - b) \tag{9}\\ (a - b)(S_2 + 1) &\geq (a \otimes b) \tag{10} \end{align}$$

And so with these two identities proven we can now move forward. Starting again with our equation we have:

$$
\begin{aligned}
\frac{\left|(a \otimes b) \ominus c - (ab - c)\right|}{|ab - c|} &\leq \frac{\left|(ab)(1 + S_1) \ominus c - (ab - c)\right|}{|ab - c|} && (11)\\
&\leq \frac{\left|[(ab)(1 + S_1) - c](S_2 + 1) - (ab - c)\right|}{|ab - c|} && (12)\\
&\leq \frac{\left|[ab + abS_1 - c](S_2 + 1) - (ab - c)\right|}{|ab - c|} && (13)\\
&\leq \frac{\left|ab + abS_1 - c + abS_2 + abS_1S_2 - cS_2 - (ab - c)\right|}{|ab - c|} && (14)\\
&\leq \frac{\left|abS_1 + abS_2 + abS_1S_2 - cS_2\right|}{|ab - c|} && (15)\\
&\leq \frac{\left|abS_1 + abS_1S_2 + abS_2 - cS_2\right|}{|ab - c|} && (16)\\
&\leq \frac{\left|abS_1 + abS_1S_2 + (ab - c)S_2\right|}{|ab - c|} && (17)\\
&\leq \frac{\left|(ab)S_1(1 + S_2) + (ab - c)S_2\right|}{|ab - c|} && (18)\\
& && (19)
\end{aligned}
$$

We will first apply the triangle inequality and thus we will use that $S_1 \leq E$ and $S_2 \leq E$ where E is machine epsilon to get:

$$
\begin{aligned}
\frac{\left|(ab)S_1(1 + S_2) + (ab - c)S_2\right|}{|ab - c|} &\leq \frac{|ab|}{|ab - c|}\left|S_1(1 + S_2)\right| + \left|S_2\right| && (20)\\
&\leq \frac{|ab|}{|ab - c|}\left|E(1 + E)\right| + \left|E\right| && (21)
\end{aligned}
$$

Which gives us the required result.

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