CS370, Winter 2023
Assignment 5: Question 4

Q4a) Write the coefficient system matrix.

We can convert the system of equations we are given to get:

$$\begin{bmatrix} -24 & 12 & 36 & -12 \\ -12 & 30 & -30 & -18 \\ -12 & -2 & 40 & 22 \\ 6 & -15 & 3 & 33 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ -18 \\ 18 \\ -39 \end{bmatrix}$$

Where A is the left most matrix, and b is the right most vector.

Q4a) Compute the LU factorization of A:

To begin with we will combine A with the unit-diagonal matrix as they are equivalent:

$$\begin{bmatrix} -24 & 12 & 36 & -12 \\ -12 & 30 & -30 & -18 \\ -12 & -2 & 40 & 22 \\ 6 & -15 & 3 & 33 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ -12 & 30 & -30 & -18 \\ -12 & -2 & 40 & 22 \\ 6 & -15 & 3 & 33 \end{bmatrix}$$

We will then subtract the second row by half of the first row:

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ -12 & -2 & 40 & 22 \\ 6 & -15 & 3 & 33 \end{bmatrix}$$

We will then subtract the third row by half of the first row:

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & -8 & 22 & 28 \\ 6 & -15 & 3 & 33 \end{bmatrix}$$

We will then subtract the negative quarter of the first row to the fourth row:

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & -8 & 22 & 28 \\ 0 & -12 & 12 & 30 \end{bmatrix}$$

We will then subtract a negative third of the first second to the third row:

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 6 & 24 \\ 0 & -12 & 12 & 30 \end{bmatrix}$$

We will then subtract a negative half of the second row to the fourth row:

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 6 & 24 \\ 0 & 0 & -12 & 24 \end{bmatrix}$$

We will then subtract 2 times the third row from the fourth row:

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 72 \end{bmatrix}$$

This gives us our L (lower triangular and unit diagonal) matrix on the left and our U (upper triangular) matrix on the right. Since we didnt need to do any swaps, P is still unit diagonal
Q4a) Solve the system using the LU factorization:

Plugging LU into A we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 72 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ -18 \\ 18 \\ -39 \end{bmatrix}$$

Lets now assume that U*X = Y, such that:

$$\begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 72 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Plugging this back into our equation we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 36 \\ -18 \\ 18 \\ -39 \end{bmatrix}$$

Solving this, by expanding out the system of equations we then get

$$y_0 = 36, y_1 = -36, y_2 = -12, y_3 = -72$$

Plugging this back into our equation for Y we then get:

$$\begin{bmatrix} -24 & 12 & 36 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 72 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 36 \\ -36 \\ -12 \\ -72 \end{bmatrix}$$

Solving this directly again gives us the following values for x:

$$x_0 = 3, x_1 = 2, x_2 = 2, x_3 = -1$$

As required!