CS370, Winter 2023
Assignment 4: Question 2

Prove that the vector of Fourier coefficients F is real, even vector:

We are given that f is a real even vector and that for all $k = 0, 1, ..., N-1$:

$$F_k = \sum_{n=0}^{N-1} f_n e^{\frac{-2ink\pi}{N}}$$

F is thus equal to $[F_0, F_1, ..., F_{N-1}]$. F is even vector if and only if for any integer $k < N$:

$$F_k = F_{N-k}$$

Therefore we will start by replacing F_k with its Discrete Fourier Transform:

$$\begin{aligned}
F_k &= \sum_{n=0}^{N-1} f_n e^{\frac{-2ink\pi}{N}} \\
&= f_0 e^0 + f_1 e^{\frac{-2ik\pi}{N}} + \cdots + f_{N-2} e^{\frac{-2i(N-2)k\pi}{N}} + f_{N-1} e^{\frac{-2i(N-1)k\pi}{N}}
\end{aligned}$$

However since f is even and $f_k = f_{N-k}$ we can rearrange to get:

$$\begin{aligned}
F_k &= f_0 e^0 + f_1 e^{\frac{-2ik\pi}{N}} + \cdots + f_{N-2} e^{\frac{-2i(N-2)k\pi}{N}} + f_{N-1} e^{\frac{-2i(N-1)k\pi}{N}} \\
&= f_0 e^{\frac{-2i(N-1)k\pi}{N}} + f_1 e^{\frac{-2i(N-2)k\pi}{N}} + \cdots + f_{N-2} e^{\frac{-2ik\pi}{N}} + f_{N-1} e^0 \\
&= \sum_{n=0}^{N-1} f_n e^{\frac{-2i(N-n)k\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n e^{\frac{-2iNk\pi}{N}} e^{\frac{2ink\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n e^{2ik\pi} e^{\frac{2ink\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n (1^k) e^{\frac{2ink\pi}{N}}
\end{aligned}$$

Using the fact that 1^k will also be equivalent to 1^n as both are just 1, we will now get:

$$
\begin{aligned}
F_k &= \sum_{n=0}^{N-1} f_n(1^k)e^{\frac{2ink\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n(1^n)e^{\frac{2ink\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n e^{2in\pi} e^{\frac{2ink\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n e^{\frac{-2inN\pi}{N}} e^{\frac{-2in(-k)\pi}{N}} \\
&= \sum_{n=0}^{N-1} f_n e^{\frac{-2in(N-k)\pi}{N}} \\
&= F_{N-k}
\end{aligned}
$$

Which proves that F is an even function.