University of Waterloo
CS 480
r2knowle: 2023-10-22

Exercise # 3

Q1) To begin we are given the kernel function:

$$k(x,y) = e^{-\alpha(x-y)^2}$$

To find the feature map we will start by converting it into its Taylor series approximation:

$$\begin{aligned}
e^{-\alpha(x-y)^2} &= \sum_{k=0}^{\infty} \frac{-\alpha^k (x-y)^2 k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{-\alpha^k}{k!} (x-y)^2 k \\
&= \sum_{k=0}^{\infty} \frac{-\alpha^k}{k!} (x^2 - 2xy + y^2)^k \\
&= \sum_{k=0}^{\infty} (\frac{(-\alpha x^2)^k}{k!} + \frac{(-2\alpha xy)^k}{k!} + \frac{(-\alpha y^2)^k}{k!}) \\
&= \sum_{k=0}^{\infty} (\frac{(-\alpha x^2)^k}{k!} + \frac{(\sqrt{-2\alpha})^k x^k}{\sqrt{k!}} \frac{(\sqrt{-2\alpha})^k y^k}{\sqrt{k!}} + \frac{(-\alpha y^2)^k}{k!}) \\
< \phi(x), \phi(y) > &= e^{(-\alpha x^2)} \dot{e}^{(-\alpha x^2)} \sum_{k=0}^{\infty} \frac{(\sqrt{-2\alpha})^k x^k}{\sqrt{k!}} \frac{(\sqrt{-2\alpha})^k y^k}{\sqrt{k!}}
\end{aligned}$$

Thus we get the feature mappings of:

$$\phi(x) = e^{(-\alpha x^2)} \left[1, \frac{\sqrt{-2\alpha} x}{\sqrt{1}}, \frac{\sqrt{-2\alpha} x^2}{\sqrt{2}}, ...\right]^t$$

$$\phi(y) = e^{(-\alpha y^2)} \left[1, \frac{\sqrt{-2\alpha} y}{\sqrt{1}}, \frac{\sqrt{-2\alpha} y^2}{\sqrt{2}}, ...\right]$$

We would want to use the primal in SVM, as the feature space for the kernel is infinite whereas the primal it is not.

Q2) Since $x, y \in (-1, 1)$, it follows that $|xy| < 1$ Because of this we can use the taylor series expansions to get that:

$$\frac{1}{1-xy} = \sum_{k=0}^{\infty} (xy)^k$$

This will give us the feature maps of:

$$\phi(x) = \left[1, x, x^2, x^3, ...\right]^T$$

$$\phi(y) = \left[1, y, y^2, y^3, ...\right]$$

Q3) This is not a valid kernel, consider x = 12 and y = 1. Thus we get that:

$$M = \begin{bmatrix} \log(145) & \log(13) \\ \log(13) & \log(2) \end{bmatrix}$$

If we then consider the vector $v = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$ when we multiply we will get:

$$\begin{aligned} \begin{bmatrix} 1 & -6 \end{bmatrix} \begin{bmatrix} \log(145) & \log(13) \\ \log(13) & \log(2) \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix} &= \begin{bmatrix} \log(145) - 6\log(13) & \log(13) - 6\log(2) \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix} \\ &= \log(145) + 36\log(2) - 12\log(13) \\ &\approx -0.37 < 0 \end{aligned}$$

Thus since its not positive semi-definite it cant be a valid kernel.

Q4) This is not a valid kernel, consider x = 1 and y = 2. Thus we get that:

$$M = \begin{bmatrix} \cos(2) & \cos(3) \\ \cos(3) & \cos(4) \end{bmatrix}$$

If we then consider the vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ when we multiply we will get:

$$\begin{aligned} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(2) & \cos(3) \\ \cos(3) & \cos(4) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} \cos(2) + \cos(3) & \cos(3) + \cos(4) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \cos(2) + 2\cos(3) + \cos(4) \\ &\approx -3.05 < 0 \end{aligned}$$

Thus since its not positive semi definite it cant be a valid kernel.