Question 1 [15 marks]

a) Let u and v be vectors in $\mathbb{R}^n$, we will start by assuming that $(u \bullet v) = 0$, it thus follows that:

$$(u \bullet v) = 0$$
$$(u \bullet v) + (u \bullet v) = 0 + 0$$
$$(u \bullet v) + (v \bullet u) = 0$$

We will now add $(u \bullet u)$ and $(v \bullet v)$ to both sides:

$$(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) = (u \bullet u) + 0 + (v \bullet v)$$
$$(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) = (u \bullet u) + (v \bullet v)$$

We know from the properties of the dot product that $(u \bullet u) = ||u||^2$, so our equation becomes:

$$(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) = ||u||^2 + (v \bullet v)$$
$$(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) = ||u||^2 + ||v||^2$$

We know from the proprieties of the dot product that $(u \bullet v) + (u \bullet u) = u \bullet (u + v)$, so we will distribute out u and then v to get:

$$u \bullet (u \bullet v) + (v \bullet u) + (v \bullet v) = ||u||^2 + ||v||^2$$
$$u \bullet (u + v) + v \bullet (u + v) = ||u||^2 + ||v||^2$$

As well since $(u + v) \bullet (u + v) = u \bullet (u + v) + v \bullet (u + v)$, it implies that our equation will become:

$$(u + v) \bullet (u + v) = ||u||^2 + ||v||^2$$
$$||u + v||^2 = ||u||^2 + ||v||^2$$

b) The converse of this statement would be (for vectors u and v in $\mathbb{R}^n$)

$$\text{if } ||u + v||^2 = ||u||^2 + ||v||^2 \text{ then } (u \bullet v) = 0$$

c) To start we will assume the hypothesis (for vectors u and v in $\mathbb{R}^n$):

$$\text{let } ||u + v||^2 = ||u||^2 + ||v||^2$$

By the definition of length for a vector, it implies that the equation becomes:

$$||u + v||^2 = ||u||^2 + ||v||^2$$
$$(\sqrt{(u + v) \bullet (u + v)})^2 = (\sqrt{(u \bullet u)})^2 + (\sqrt{(v \bullet v)})^2$$
$$(u + v) \bullet (u + v) = (\sqrt{(u \bullet u)})^2 + (\sqrt{(v \bullet v)})^2$$
$$(u + v) \bullet (u + v) = (u \bullet u) + (v \bullet v)$$

We know by definition that $(u+v) \bullet (u+v) = u \bullet (u+v) + v \bullet (u+v)$, thus:

$$\begin{aligned}(u+v) \bullet (u+v) &= (u \bullet u) + (v \bullet v) \\ u \bullet (u+v) + v \bullet (u+v) &= (u \bullet u) + (v \bullet v)\end{aligned}$$

Furthermore we know that $(u \bullet v) + (u \bullet u) = u \bullet (u+v)$, which thus implies:

$$\begin{aligned}u \bullet (u+v) + v \bullet (u+v) &= (u \bullet u) + (v \bullet v) \\ (u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= (u \bullet u) + (v \bullet v) \\ (u \bullet v) + (v \bullet u) &= (u \bullet u) + (v \bullet v) - (v \bullet v) - (u \bullet u) \\ (u \bullet v) + (v \bullet u) &= 0\end{aligned}$$

From the properties of dot product we know this can simplify to:

$$\begin{aligned}(u \bullet v) + (v \bullet u) &= 0 \\ (u \bullet v) + (u \bullet v) &= 0 \\ 2(u \bullet v) &= 0 \\ (u \bullet v) &= 0\end{aligned}$$

Therefore proving that for all vectors u, v in $\mathbb{R}^n$, that if $||u+v||^2 = ||u||^2 + ||v||^2$ then $(u \bullet v) = 0$

d) For vectors $u, v, w \in \mathbb{R}^n$, the statement can be extended to:

$$\text{if } (u \bullet v) = 0, (u \bullet w) = 0 \text{ and } (v \bullet w) \text{ then } ||u+v+w||^2 = ||u||^2 + ||v||^2 + ||w||^2$$

If we assume the hypothesis is correct we get the equation we get (and it thus follows that):

$$\begin{aligned}(u \bullet v) + (w \bullet u) + (v \bullet w) &= 0 + 0 + 0 \\ (u \bullet v) + (w \bullet u) + (v \bullet w) &= 0 \\ 2(u \bullet v) + 2(w \bullet u) + 2(v \bullet w) &= 0\end{aligned}$$

If we then add $(u \bullet u)$, $(v \bullet v)$ and $(w \bullet w)$ to both sides and apply the definition of length of a vector we thus get:

$$\begin{aligned}2(u \bullet v) + 2(w \bullet u) + 2(v \bullet w) + (u \bullet u) + (v \bullet v) + (w \bullet w) &= 0 + (u \bullet u) + (v \bullet v) + (w \bullet w) \\ 2(u \bullet v) + 2(w \bullet u) + 2(v \bullet w) + (u \bullet u) + (v \bullet v) + (w \bullet w) &= ||u||^2 + (v \bullet v) + (w \bullet w) \\ 2(u \bullet v) + 2(w \bullet u) + 2(v \bullet w) + (u \bullet u) + (v \bullet v) + (w \bullet w) &= ||u||^2 + ||v||^2 + (w \bullet w) \\ 2(u \bullet v) + 2(w \bullet u) + 2(v \bullet w) + (u \bullet u) + (v \bullet v) + (w \bullet w) &= ||u||^2 + ||v||^2 + ||w||^2\end{aligned}$$

From the definition we know that $(u \bullet v) + (u \bullet u) + (u \bullet w) = u \bullet (u + v + w)$, so by arragning we get that:

$$\begin{aligned}
u \bullet (u+v+w) + (u \bullet v) + (w \bullet u) + 2(v \bullet w) + (v \bullet v) + (w \bullet w) &= ||u||^2 + ||v||^2 + ||w||^2 \\
u \bullet (u+v+w) + v \bullet (v+u+w) + (w \bullet u) + (v \bullet w) + (w \bullet w) &= ||u||^2 + ||v||^2 + ||w||^2 \\
u \bullet (u+v+w) + v \bullet (u+v+w) + w \bullet (u+v+w) &= ||u||^2 + ||v||^2 + ||w||^2
\end{aligned}$$

We know that $(u+v+w) \bullet (u+v+w) = u \bullet (u+v+w) + v \bullet (u+v+w) + w \bullet (u+v+w)$ and by the definition of length this becomes:

$$\begin{aligned}
u \bullet (u+v+w) + v \bullet (u+v+w) + w \bullet (u+v+w) &= ||u||^2 + ||v||^2 + ||w||^2 \\
(u+v+w) \bullet (u+v+w) &= ||u||^2 + ||v||^2 + ||w||^2 \\
||u+v+w||^2 &= ||u||^2 + ||v||^2 + ||w||^2
\end{aligned}$$

This thus proves for all vectors $u, v, w \in \mathbb{R}^n$ if $(u \bullet v) = 0, (u \bullet w) = 0$ and $(v \bullet w)$, then:

$$||u+v+w||^2 = ||u||^2 + ||v||^2 + ||w||^2$$

e) The extension to b) for vectors u, v, w in $\mathbb{R}^2$ can be written as:

if $||u+v+w||^2 = ||u||^2 + ||v||^2 + ||w||^2$ then $(u \bullet v) = 0, (y \bullet w) = 0$ and $(y \bullet w) = 0$

f) For the counter example let the following vectors (in $\mathbb{R}^3$) equal:

$$u = \begin{pmatrix} -1/2 \\ -2 \\ -2 \end{pmatrix} \quad v = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad w = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

To start we will check if the hypothesis holds, thus we will check if:

$$\begin{aligned}
||u+v+w||^2 &= ||u||^2 + ||v||^2 + ||w||^2 \\
||u+v+w||^2 &= (u \bullet u) + (v \bullet v) + (w \bullet w) \\
(u+v+w) \bullet (u+v+w) &= (u \bullet u) + (v \bullet v) + (w \bullet w)
\end{aligned}$$

Plugging our values our equation becomes:

$$\begin{pmatrix} -1/2 + -1 - 1 \\ -2 - 1 + 1 \\ -2 - 1 + 1 \end{pmatrix} \bullet \begin{pmatrix} -1/2 - 1 - 1 \\ -2 - 1 + 1 \\ -2 - 1 + 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1/2 \\ -2 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} + \begin{pmatrix}-1\\-1\\-1\end{pmatrix} \bullet \begin{pmatrix}-1\\-1\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\\1\end{pmatrix} \bullet \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} + \begin{pmatrix}-1\\-1\\-1\end{pmatrix} \bullet \begin{pmatrix}-1\\-1\\-1\end{pmatrix} + (-1)*(-1)+1*1+1*1$$

$$\begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} + (-1)(-1)+(-1)(-1)+(-1)(-1)+3$$

$$\begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-5/2\\-2\\-2\end{pmatrix} = (-1/2)(-1/2)+(-2)(-2)+(-2)(-2)+6$$

$$\begin{aligned}(-\frac{5}{2})(-\frac{5}{2})+(-2)(-2)+(-2)(-2) &= \frac{57}{4}\\ \frac{25}{4}+4+4 &= \frac{57}{4}\\ \frac{57}{4} &= \frac{57}{4}\end{aligned}$$

Thus we have proved the hypothesis, however note that the conclusion does not hold as:

$$u \bullet v = \begin{pmatrix}-1/2\\-2\\-2\end{pmatrix} \bullet \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

$$\begin{aligned}u \bullet v &= (-\frac{1}{2})(-1)+(-2)(-1)+(-2)(-1)\\ u \bullet v &= \frac{1}{2}+2+2\\ u \bullet v &= \frac{9}{2}\end{aligned}$$

Because $(u \bullet v) \neq 0$, this shows that the counter examples disproves the origional statement.