Question 1 [15 marks]

a) We will start by assuming that $(u \bullet v) = 0$, it thus follows that:

$$\begin{aligned}(u \bullet v) &= 0\\ (u \bullet v) + (u \bullet v) &= 0 + 0\\ (u \bullet v) + (v \bullet u) &= 0\end{aligned}$$

We will now add $(u \bullet u)$ and $(v \bullet v)$ to both sides:

$$\begin{aligned}(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= (u \bullet u) + 0 + (v \bullet v)\\ (u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= (u \bullet u) + (v \bullet v)\end{aligned}$$

We know from the properties of the dot product that $(u \bullet u) = ||u||^2$, so our equation becomes:

$$\begin{aligned}(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= ||u||^2 + (v \bullet v)\\ (u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= ||u||^2 + ||v||^2\end{aligned}$$

We know from the proprieties of the dot product that $(u \bullet v) + (u \bullet u) = u \bullet (u + v)$, so we will distrabute out u and then v to get:

$$\begin{aligned}u \bullet (u \bullet v) + (v \bullet u) + (v \bullet v) &= ||u||^2 + ||v||^2\\ u \bullet (u + v) + v \bullet (u + v) &= ||u||^2 + ||v||^2\end{aligned}$$

As well since $(u + v) \bullet (u + v) = u \bullet (u + v) + v \bullet (u + v)$, it implies that our equation will become:

$$\begin{aligned}(u + v) \bullet (u + v) &= ||u||^2 + ||v||^2\\ ||u + v||^2 &= ||u||^2 + ||v||^2\end{aligned}$$

b) The converse of this statement would be:

$$\text{if } ||u + v||^2 = ||u||^2 + ||v||^2 \text{ then } (u \bullet v) = 0$$

c) To start we know that with three vectors u, v, w the equation would become:

$$\text{if } u \bullet v \bullet w = 0 \text{ then } ||u + v + w|| = ||u||^2 + ||v||^2 + ||w||^2$$

It thus follows that:

$$\begin{aligned}(u \bullet v \bullet w) &= 0\\ (u \bullet v \bullet w) + (u \bullet v \bullet w) + (u \bullet v \bullet w) &= 0 + 0 + 0\\ (u \bullet (v \bullet w)) + (w \bullet (u \bullet v)) + (v \bullet (u \bullet w)) &= 0\\ (u \bullet v) + (v \bullet w) + (w \bullet u) + (w \bullet v) + (v \bullet u) + (v \bullet w) &= 0\end{aligned}$$

By the dot product's property of symmetry we know that this becomes: