Question 2 [15 marks]

a) We know from the hint that $0 = (0)(0)$, thus our equation becomes:

$$G(0) = G((0)(0))$$

We also know that G is a linear function, so by the second property the equation becomes:

$$\begin{aligned} G(0) &= G((0)(0)) \\ &= (0)G((0)) \\ &= 0 \end{aligned}$$

Thus proving that if G is a linear function the G(0) = 0.

b) To prove this is false we will use proof by contradiction. To start we will assume that sin(x) is linear, that is to say that the first property will hold:

$$\forall x_1, x_2 \in \mathbb{R},\ \ \sin(x_1 + x_2) = \sin(x_1) + \sin(x_2)$$

If we consider $x_1 = \frac{\pi}{2}$ and $x_2 = \frac{\pi}{2}$ and apply this to the first property we find that:

$$\begin{aligned} \sin(x_1 + x_2) &= \sin(x_1) + \sin(x_2) \\ \sin(\frac{\pi}{2} + \frac{\pi}{2}) &= \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) \\ \sin(\pi) &= \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) \\ 0 &= 1 + 1 \end{aligned}$$

However this is a clear contradiction as $0 \neq 2$, thus showing that since the first property does not hold sin(x) is not a linear function.

c) To prove this is false we will use proof by contradiction. To start we will assume that 2x + 3 is linear, that is to say that the first property will hold:

$$\forall x_1, x_2 \in \mathbb{R},\ 2(x_1 + x_2) + 3 = 2(x_1) + 3 + 2(x_2) + 3$$

If we consider $x_1 = 3$ and $x_2 = 1$, then by the first property the equation will become:

$$\begin{aligned} 2(x_1 + x_2) + 3 &= 2(x_1) + 3 + 2(x_2) + 3 \\ 2(3 + 1) + 3 &= 2(3) + 3 + 2(1) + 3 \\ 2(4) + 3 &= 6 + 2 + 6 \\ 8 + 3 &= 14 \\ 11 &= 14 \end{aligned}$$

However this is a clear contradiction as $11 \neq 14$, thus showing that since the first property does not hold 2x + 3 is not a linear function.

d) To prove that f(x) = 2x is linear, we will prove that the two properties hold for any x. Starting with the first property let $\forall x_1, x_2 \in \mathbb{R}$, such that:

$$\begin{aligned} f(x_1 + x_2) &= f(x_1) + f(x_2) \\ 2(x_1 + x_2) &= 2(x_1) + 2(x_2) \\ 2(x_1) + 2(x_2) &= 2(x_1) + 2(x_2) \end{aligned}$$

Thus showing that the first property holds for any x_1 and x_2. To check for the second property let $x_3, c \in \mathbb{R}$, such that:

$$\begin{aligned} f(c \times x_3) &= c \times f(x_3) \\ 2(c \times x_3) &= c \times 2(x_3) \\ c(2 \times x_3) &= c(2 \times x_3) \end{aligned}$$

Since both properties holds for any $x_1, x_2, x_3, c \in \mathbb{R}$, f(x) = 2x is proven to be linear.

e) To start we will use the hypothesis to prove that F(0) = 0, assuming the hypothesis we know that:

$$\forall y_1, y_2 \text{ and } d \in \mathbb{R}, F(dy_1 + y_2) = dF(y_1) + F(y_2)$$

To prove the first hypothesis we let d = 1, therefore we get that for every $y_1, y_2 \in \mathbb{R}$:

$$\begin{aligned} F(dy_1 + y_2) &= dF(y_1) + F(y_2) \\ F((1)y_1 + y_2) &= (1)F(y_1) + F(y_2) \\ F(y_1 + y_2) &= F(y_1) + F(y_2) \end{aligned}$$

Thus showing that the first property holds in F for any y_1 and y_2. To check for the second property let $y_2 = 0$ such that for every $y_1, d \in \mathbb{R}$, such that:

$$\begin{aligned} F(dy_1 + y_2) &= dF(y_1) + F(y_2) \\ F(dy_1 + 0) &= dF(y_1) + F(0) \\ F(dy_1) &= dF(y_1) + F(0) \end{aligned}$$

Note that since we have proven the first hypothesis, we can use it to expand the $F(dy_1)$, we thus get y_1 repeated d times:

$$\begin{aligned} F(y_1 + y_1 + \cdots + y_1) &= dF(y_1) + F(0) \\ F(y_1) + F(y_1) + \cdots + F(y_1) &= dF(y_1) + F(0) \\ dF(y_1) &= dF(y_1) + F(0) \\ F(0) &= 0 \end{aligned}$$

If we plug this onto our previous equation for property 2 we get that:

$$\begin{aligned} F(dy_1) &= dF(y_1) + F(0) \\ F(dy_1) &= dF(y_1) + 0 \\ F(dy_1) &= dF(y_1) \end{aligned}$$

Thus proving the second property for any y_1 and $d \in \mathbb{R}$. Since both properties hold, we have shown that F is linear, that is to say that:

$$\begin{aligned} F(0) &= F((0)(0)) \\ &= (0)F(0)) \\ &= (0) \end{aligned}$$

f) In order to prove the if and only if we will start by proving:

$$\text{if G is linear } \implies \forall y_1, y_2 \text{ and } d \in \mathbb{R}, G(dy_1 + y_2) = dG(y_1) + G(y_2)$$

If we assume the hypothesis then we know from property 1 that ($\forall y_1, y_2 \in \mathbb{R}$):

$$G(y_1 + y_2) = G(y_1) + G(y_2)$$

If we let $y_1 = d \times y_3$, where $\forall d, y_3 \in \mathbb{R}$, we get:

$$\begin{aligned} G(y_1 + y_2) &= G(y_1) + G(y_2) \\ G(dy_3 + y_2) &= G(dy_3) + G(y_2) \end{aligned}$$

Since G is linear we can thus apply the second property:

$$\begin{aligned} G(dy_3 + y_2) &= G(dy_3) + G(y_2) \\ G(dy_3 + y_2) &= dG(y_3) + G(y_2) \end{aligned}$$

Thus proving that if G is linear that for all y_2, y_3 and $d \in \mathbb{R}$ that $G(dy_3 + y_2) = dG(y_3) + G(y_2)$.
Next we need to prove the reverse direction or:

$$\forall y_1, y_2 \text{ and } d \in \mathbb{R}, G(dy_1 + y_2) = dG(y_1) + G(y_2) \implies \text{if G is linear}$$

To prove this we will show that we can derive both properties from the hypothesis, starting with property 1. To prove the first hypothesis we let d = 1, therefore we get that for every $y_1, y_2 \in \mathbb{R}$:

$$\begin{aligned} G(dy_1 + y_2) &= dG(y_1) + G(y_2) \\ G((1)y_1 + y_2) &= (1)G(y_1) + G(y_2) \\ G(y_1 + y_2) &= G(y_1) + G(y_2) \end{aligned}$$

Thus showing that the first property holds in G for any y_1 and y_2. To check for the second property let $y_2 = 0$ such that for every $y_1, d \in \mathbb{R}$, such that:

$$\begin{aligned} G(dy_1 + y_2) &= dG(y_1) + G(y_2) \\ G(dy_1 + 0) &= dG(y_1) + G(0) \\ G(dy_1) &= dG(y_1) + G(0) \end{aligned}$$

Note that since we have proven the first hypothesis, we can use it to expand the $F(dy_1)$, we thus get y_1 repeated d times:

$$\begin{aligned}
G(y_1 + y_1 + \cdots + y_1) &= dG(y_1) + G(0) \\
G(y_1) + G(y_1) + \cdots + F(y_1) &= dG(y_1) + G(0) \\
dG(y_1) &= dG(y_1) + G(0) \\
G(0) &= 0
\end{aligned}$$

If we plug this onto our previous equation for property 2 we get that:

$$\begin{aligned}
G(dy_1) &= dG(y_1) + F(0) \\
G(dy_1) &= dG(y_1) + 0 \\
G(dy_1) &= dG(y_1)
\end{aligned}$$

Thus proving the second property for any y_1 and $d \in \mathbb{R}$.
$\therefore$ Since both property's are proved G is linear. Therefore we have shown both directions are true and proved the if and only if statement.