Question 1 [15 marks]

a) Let u and v be vectors in $\mathbb{R}^n$, we will start by assuming that $(u \bullet v) = 0$, it thus follows that:

$$\begin{aligned}(u \bullet v) &= 0\\ (u \bullet v) + (u \bullet v) &= 0 + 0\\ (u \bullet v) + (v \bullet u) &= 0\end{aligned}$$

We will now add $(u \bullet u)$ and $(v \bullet v)$ to both sides:

$$\begin{aligned}(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= (u \bullet u) + 0 + (v \bullet v)\\ (u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= (u \bullet u) + (v \bullet v)\end{aligned}$$

We know from the properties of the dot product that $(u \bullet u) = ||u||^2$, so our equation becomes:

$$\begin{aligned}(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= ||u||^2 + (v \bullet v)\\ (u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= ||u||^2 + ||v||^2\end{aligned}$$

We know from the proprieties of the dot product that $(u \bullet v) + (u \bullet u) = u \bullet (u + v)$, so we will distrabute out u and then v to get:

$$\begin{aligned}u \bullet (u \bullet v) + (v \bullet u) + (v \bullet v) &= ||u||^2 + ||v||^2\\ u \bullet (u + v) + v \bullet (u + v) &= ||u||^2 + ||v||^2\end{aligned}$$

As well since $(u + v) \bullet (u + v) = u \bullet (u + v) + v \bullet (u + v)$, it implies that our equation will become:

$$\begin{aligned}(u + v) \bullet (u + v) &= ||u||^2 + ||v||^2\\ ||u + v||^2 &= ||u||^2 + ||v||^2\end{aligned}$$

b) The converse of this statement would be (for vectors u and v in $\mathbb{R}^n$)

$$\text{if } ||u + v||^2 = ||u||^2 + ||v||^2 \text{ then } (u \bullet v) = 0$$

c) To start we will assume the hypothesis (for vectors u and v in $\mathbb{R}^n$):

$$\text{let } ||u + v||^2 = ||u||^2 + ||v||^2$$

By the definition of length for a vector, it imples that the equation becomes:

$$\begin{aligned}||u + v||^2 &= ||u||^2 + ||v||^2\\ (\sqrt{(u + v) \bullet (u + v)})^2 &= (\sqrt{(u \bullet u)})^2 + (\sqrt{(v \bullet v)})^2\\ (u + v) \bullet (u + v) &= (\sqrt{(u \bullet u)})^2 + (\sqrt{(v \bullet v)})^2\\ (u + v) \bullet (u + v) &= (u \bullet u) + (v \bullet v)\end{aligned}$$

We know by definition that $(u+v) \bullet (u+v) = u \bullet (u+v) + v \bullet (u+v)$, thus:

$$\begin{aligned}
(u+v) \bullet (u+v) &= (u \bullet u) + (v \bullet v) \\
u \bullet (u+v) + v \bullet (u+v) &= (u \bullet u) + (v \bullet v)
\end{aligned}$$

Futhermore we know that $(u \bullet v) + (u \bullet u) = u \bullet (u+v)$, which thus implies:

$$\begin{aligned}
u \bullet (u+v) + v \bullet (u+v) &= (u \bullet u) + (v \bullet v) \\
(u \bullet u) + (u \bullet v) + (v \bullet u) + (v \bullet v) &= (u \bullet u) + (v \bullet v) \\
(u \bullet v) + (v \bullet u) &= (u \bullet u) + (v \bullet v) - (v \bullet v) - (u \bullet u) \\
(u \bullet v) + (v \bullet u) &= 0
\end{aligned}$$

From the properties of dot product we know this can simplify to:

$$\begin{aligned}
(u \bullet v) + (v \bullet u) &= 0 \\
(u \bullet v) + (u \bullet v) &= 0 \\
2(u \bullet v) &= 0 \\
(u \bullet v) &= 0
\end{aligned}$$

Therefore proving that for all vectors u, v in $\mathbb{R}^n$, that if $||u+v||^2 = ||u||^2 + ||v||^2$ then $(u \bullet v) = 0$

d)