**Q04a** Let $a$ be an arbitrary integer, we start with an if and only if statement which can be expressed as:

$$37|a \iff = 37|S(a)$$

To prove this we will split it up by cases starting with:

$$37|a \implies 37|S(a)$$

Let assume that $37|a$, this would mean that from CTR:

$$a \pmod{37} = 0 \pmod{37}$$

We also know that $a$ can be expressed in terms of its digits (where $k$ is an integer and $3|k$, if any digit $d_k$ is greater then what exists in $a$ then $d_k = 0$) :

$$\begin{aligned} 0 \pmod{37} &= a \pmod{37} \\ &= d_k(10^{k-1}) + d_{k-1}(10^{k-2}) + ... + d^2(10^2) + d^1(10) + d_0 \pmod{37} \end{aligned}$$

We notice that the digit expression (mod 37) must also be equal to 0, from CAM. We also notice the following pattern in terms of the coefficents of 10 ($n$) and (mod 37):

$$(n = 0) : (10^0) \pmod{37} \implies 1 \pmod{37} \implies 1 \pmod{37}$$

$$(n = 1) : (10^1) \pmod{37} \implies 10 \pmod{37} \implies 10 \pmod{37}$$

$$(n = 2) : (10^2) \pmod{37} \implies 100 \pmod{37} \implies 26 \pmod{37}$$

$$(n = 3) : (10^3) \pmod{37} \implies 1000 \pmod{37} \implies 1 \pmod{37}$$

Notice that this pattern repeats. Returning back to our digit expansion, we know we can group terms into threes like:

$$\begin{aligned} 0 \pmod{37} &= d_k(10^{k-1}) + d_{k-1}(10^{k-2}) + d_{k-2}(10^{k-3}) + ... + d_2(10^2) + d_1(10) + d_0 \pmod{37} \\ &= (d_k(10^2) + d_{k-1}(10^1) + d_{k-2}(10^0))(10^{k-3}) + ... + (d^2(10^2) + d_1(10) + d_0)(10^0) \pmod{37} \end{aligned}$$

Since $3|k$ (so for some integer $a$, $3a = k$) this can be better expressed in sigma notation where:

$$\begin{aligned} 0 \pmod{37} &= \sum_{n=0}^{a} 10^{3n}(d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} \\ &= \sum_{n=0}^{a} (10^3)^n(d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} \\ &= \sum_{n=0}^{a} (1)^n(d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} \\ &= \sum_{n=0}^{a} (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} \end{aligned}$$

Thus we can see that our equation is equivilent to S(a), and thus:

1. $S(a) = \sum_{n=0}^{a} (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37}$

2. $\sum_{n=0}^{a} (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} = 0 \pmod{37}$

This infers that S(a) = 0 (mod 37), which because of CTR proves that $37|S(a)$. We have thus proved one the first implication.

We will now try to prove the second case:

$$37|S(a) \implies 37|a$$

Let $37|S(a)$ this means that:

$$S(a) \pmod{37} = 0 \pmod{37}$$

We know that we can express S(a) as:

$$0 \pmod{37} = \sum_{n=0}^{a} (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37}$$

We know from CAM that we can multiply both sides by 1 (mod 37), which could then become:

$$\begin{aligned}
0 \pmod{37} &= \sum_{n=0}^{a} (1)^n (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} \\
&= \sum_{n=0}^{a} (10^3)^n (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37} \\
&= \sum_{n=0}^{a} 10^{3n} (d_{2+3n}(10^2) + d_{1+3n}(10) + d_{3n}) \pmod{37}
\end{aligned}$$

Expanding out the sigma notation we get that:

$$0 \pmod{37} = d_k(10^{k-1}) + d_{k-1}(10^{k-2}) + ... + d^2(10^2) + d^1(10) + d_0 \pmod{37}$$

$$0 \pmod{37} = a$$

Therefore CTR shows us that $37|a$ we have thus proved the second implications. Since we have proven both implication for all possible values of $a$, we have proved the if and only if statements.

**Q04b** Notice that in order for the previous if and only if statement to be true, you need a repeating pattern where $10^3 \pmod{27} = 1$ plus $10^2, 10^1$ and $10^0$ need to have unquie endings. Thus if we plug in the numbers 0-4 we should notice a similar pattern to the (mod 37') pattern:

$$(n = 0) : (10^0) \pmod{27} \implies 1 \pmod{27} \implies 1 \pmod{27}$$

$$(n=1):(10^1) \pmod{27} \implies 10 \pmod{27} \implies 10 \pmod{27}$$
$$(n=2):(10^2) \pmod{27} \implies 100 \pmod{27} \implies 19 \pmod{27}$$
$$(n=3):(10^3) \pmod{27} \implies 1000 \pmod{27} \implies 1 \pmod{27}$$

Since $10^3$ is 1, this means that you could get rid of $10^{3n}$ term, which along with the previous conditions and CAM means that if question 1's mod was replaced with 27 the if and only if statement would be true.