Q02A Let a, b be real numbers ($a, b \in \mathbb{R}$) such that the complex numbers w and z ($w, z \in \mathbb{R}$) can be represented as:

$$z = a + bi$$
$$w = 1 + z = (1 + a) + bi$$

We know that $|w|$ and $|z|$ can be expressed as:

$$|z| = \sqrt{a^2 + b^2}$$
$$|w| = \sqrt{(1 + a)^2 + b^2}$$

We also know that:

$$|w| = |z| = 1$$

We also know that we can split this into two equations:

$\lvert z\rvert = 1$	$\lvert w\rvert = 1$
$\sqrt{a^2 + b^2} = 1$	$\sqrt{(1 + a)^2 + b^2} = 1$
$a^2 + b^2 = 1^2$	$(1 + a)^2 + b^2 = 1^2$
$a^2 + b^2 = 1$	$1 + 2a + a^2 + b^2 = 1$
$a^2 = 1 - b^2$	$1 + 2a + a^2 = 1 - b^2$

This thus tells us that:

$$a^2 = 1 + 2a + a^2$$
$$0 = 1 + 2a$$
$$a = -\frac{1}{2}$$

Plunging this back into our two equations we get:

$\lvert z\rvert = 1$	$\lvert w\rvert = 1$
$\sqrt{(-\frac{1}{2})^2 + b^2} = 1$	$\sqrt{(1 - \frac{1}{2})^2 + b^2} = 1$
$(-\frac{1}{2})^2 + b^2 = 1^2$	$(\frac{1}{2})^2 + b^2 = 1^2$
$\frac{1}{4} + b^2 = 1$	$\frac{1}{4} + b^2 = 1$

Therefore we will get that:

$$b^2 = 1 - \frac{1}{4}$$
$$b = \pm\sqrt{\frac{3}{4}}$$

Therefore the only possible pairs of complex numbers w, z that satisfy:

$$|w| = |z| = 1$$

Are given by the following:

$$z = -\frac{1}{2} \pm \sqrt{\frac{3}{4}}i$$

$$w = \frac{1}{2} \pm \sqrt{\frac{3}{4}}i$$

Q02B Let z and w be arbitrary complex numbers, to start we will split the following equation:

$$|z + iw| = |z - iw| \iff z\overline{w} \in \mathbb{R}$$

Into the following implications:

$\lvert z + iw\rvert = \lvert z - iw\rvert \implies z\overline{w} \in \mathbb{R}$	$z\overline{w} \in \mathbb{R} \implies \lvert z + iw\rvert = \lvert z - iw\rvert$
$z = a + bi$ (for a,b $\in \mathbb{R}$) $w = c + di$ (for c,d $\in \mathbb{R}$) $\lvert z + iw\rvert = \lvert z - iw\rvert$ $\lvert(a + bi) + i(c + di)\rvert = \lvert(a + bi) - i(c + di)\rvert$ $\lvert a + bi + ci - d\rvert = \lvert a + bi - ic + d)\rvert$ $\lvert(a - d) + (c + b)i\rvert = \lvert(a + d) + (b - c)i\rvert$ $\sqrt{(a - d)^2 + (c + b)^2} = \sqrt{(a + d)^2 + (b - c)^2}$ $(a - d)^2 + (c + b)^2 = (a + d)^2 + (b - c)^2$ $a^2 - 2ad + d^2 + (c + b)^2 = a^2 + 2ad + d^2 + (b - c)^2$ $(c + b)^2 = 4ad + (b - c)^2$ $c^2 + 2bc + b^2 = 4ad + b^2 - 2bc + c^2$ $4bc = 4ad$ $bc = ad$ This implies that bc - ad = 0, if we expand: $z\overline{w} = (ac + bd) + (ad - bc)i$ $z\overline{w} = (ac + bd) - (bc - ad)i$ $z\overline{w} = (ac + bd) - 0i$ $z\overline{w} = (ac + bd)$ Which implies $z\overline{w} \in \mathbb{R}$	$z = a + bi$ (for a,b $\in \mathbb{R}$) $w = c + di$ (for c,d $\in \mathbb{R}$) $z\overline{w} = (ac + bd) + (ad - bc)i$ We know that $\text{im}(z\overline{w}) = 0$ $bc - ad = 0$ $4bc = 4ad$ $(a - d)^2 + (c + b)^2 = (a + d)^2 + (b - c)^2$ $\sqrt{(a - d)^2 + (c + b)^2} = \sqrt{(a + d)^2 + (b - c)^2}$ $\lvert(a - d) + (c + b)i\rvert = \lvert(a + d) + (b - c)i\rvert$ $\lvert a + bi + ci - d\rvert = \lvert a + bi - ic + d\rvert$ $\lvert(a + bi) + i(c + di)\rvert = \lvert(a + bi) - i(c + di)\rvert$ Thus proving the right hand side from the hypothesis (More detail on steps is given on the right)

Since both ways are proved we have also proved the if and only if statement.