Robert (Robbie) Knowles MATH 135 Fall 2020: WA06

Q01. To start we will use EEA to find x,y and $d = gcd(2996, 1520)$

x	y	r	q
1	0	2996	0
0	1	1520	0
1	-1	1476	1
-1	2	44	1
34	-67	-24	33
-35	69	20	1
69	-136	-4	1
-380	749	0	5

According to EEA, the last second row will provide the x,y and d. in other words this means that $x = 69$, $y = -136$ and $d = gcd(2996, 1520) = 4$. Plugging this the original equation we find that:

$$\begin{aligned}
2996x + 1520y &= \gcd(2996, 1520) \\
2996x + 1520y &= d \\
2996 * (69) + 1520 * (-136) &= 4 \\
4 &= 4
\end{aligned}$$

Therefore $x = 69$, $y = -136$ and $d = gcd(2996, 1520) = 4$.

Q02A We will disprove by counter example, consider when $a = -2$ and $b = 4$, Note that:

1. $\min(a, b) = \min(-2, 4) = -2$.
2. $\gcd(a, b) = \gcd(-2, 4) = 2$.

However since we know that $-2 < 2$, this would imply that for $a = -2$ and $b = 4$, $\min(a, b) < \gcd(a, b)$. Therefore since this violates the universality of the original statement, the origonal statement is disproved.

Q02B We will disprove by counter example, consider when $a = 0$ and $b = 2$, Note that:

1. $\min(|a|, |b|) = \min(|0|, |2|) = \min(0, 2) = 0$.
2. $\gcd(a, b) = \gcd(0, 2) = 2$.

However since we know that $0 < 2$, this would imply that for $a = 0$ and $b = 2$, $\min(|a|, |b|) < \gcd(a, b)$. Therefore since this violates the universality of the original statement, the origonal statement is disproved.

Q02C We will prove by using case analysis, note for all cases $a, b \neq 0$ and a,b are integers. **The first case will be when a = b:**

1. $\min(|a|, |b|) = a$
2. $\gcd(a, b) = a$

We thus have that $\min(|a|, |b|) = \gcd(a, b)$ when $a = b$, proving the statement for the first case. **The second case will be when $\boldsymbol{a > b}$:**

1. $\min(|a|, |b|) = b$
2. Let d be an integer such that $\gcd(a, b) = d$

Since d must be a common divisor of a and b, d is bounded by:

$$d \leq b < a$$

As d is the $\gcd(a, b)$ and b is the $\min(|a|, |b|)$ we have that:

$$d \leq b < a$$
$$gcd(a, b) \leq \min(|a|, |b|) < a$$

Therefore when $a > b$, we will have $gcd(a, b) \leq \min(|a|, |b|)$ proving the statement for the second case. **The third case will be when $\boldsymbol{a < b}$:**

1. $\min(|a|, |b|) = a$
2. Let d be an integer such that $\gcd(a, b) = d$

Since d must be a common divisor of a and b, d is bounded by:

$$d \leq a < b$$

As d is the $\gcd(a, b)$ and a is the $\min(|a|, |b|)$ we have that:

$$d \leq a < b$$
$$gcd(a, b) \leq \min(|a|, |b|) < b$$

Therefore when $a < b$, we will have $gcd(a, b) \leq \min(|a|, |b|)$ proving the statement for the third case.

Since the three cases cover all the possible values of a,b and they are are true for each case, it must be that for all non-zero integers a and b, $\gcd(a, b) \leq \min(|a|, |b|)$.

Q03. To prove the implication we will split it into two parts:

$$a \iff b$$

Q04. We will prove the the binomial equation is divisible by $n+1$, by showing that for any $n \geq 1$ that $n+1$ is a factor of the binomial equation

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\ &= \frac{2n!}{n! * n!}\end{aligned}$$

We will divide the numerator by $n!$, notice that this will produce a series of multiplies from $(n+1)$ to $(2n)$ on the numerator and we are left with a series of multiples from 1 to n on the denominator:

$$\begin{aligned}\frac{2n!}{n! * n!} &= \frac{(2n) * 2n - 1 * (2n-2) * ... * (n+2) * (n+1)}{n!} \\ &= \prod_{k=1}^{n} (\frac{n+k}{k}) \\ &= \prod_{k=1}^{n} (\frac{n}{k} + \frac{k}{k}) \\ &= \prod_{k=1}^{n} (\frac{n}{k} + 1)\end{aligned}$$

We will now factor out the first term (when $k = 1$) to show that if $n \geq 2$, the binomial expression will equal some integer times $(n+1)$ (we know this because all binomial coefficients are integers):

$$\begin{aligned}\prod_{k=1}^{n} (\frac{n}{k} + 1) &= (\frac{n}{1} + 1) \prod_{k=2}^{n} (\frac{n}{k} + 1) \\ &= (n+1) \prod_{k=2}^{n} (\frac{n}{k} + 1)\end{aligned}$$

However if we have $n = 1$, we will get $(n+1)$ multiplied by 1

$$(n+1) \prod_{k=2}^{1} (\frac{n}{k} + 1) = (n+1) * 1$$

Therefore as long as $n \geq 1$, the binomial expression will be some integer times $n+1$. Since $n+1$ is thus a factor of the binomial equation, it will always be divisible by $n+1$.

Q05. (5 marks) Prove that, for all positive integers d, m and n, if $d = \gcd(m, n)$, then $\gcd(m, nk) = \gcd(m, dk)$ for any positive integer k.

d = ma + nb m = nb + r

if m ¿= n: d = gcd(m,n) m = qn + r (gcd n, r)

m = nk + ma+nkb = ma + dkb