Robert (Robbie) Knowles MATH 137 Fall 2020: WA06

Q01. In order for $f(x)$ to be differentiable at a

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This must also hold for the left and right limits, such that

$$f'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

We will start by finding the value right side limit

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Notice that as $\lim_{h \to 0^+}$, the value of $a + h > a$, which means that according to the piecewise function $f(a+h) = (a+h)^2$, also note that $f(a) = a^2$ by definition as well. This means the rightside limit becomes:

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0^+} \frac{(a+h)^2 - a^2}{h} \\
&= \lim_{h \to 0^+} \frac{(a^2 + 2ah + h^2) - a^2}{h} \\
&= \lim_{h \to 0^+} \frac{2ah + h^2}{h} \\
&= \lim_{h \to 0^+} 2a + h \\
&= 2a
\end{aligned}$$

Now that we have a value for the righthand limit, we will find the values of a and b such that the righthand limit equals the left hand limit, or that:

$$2a = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Notice that as $\lim_{h \to 0^-}$, the value of $a + h < a$, which means that according to the piecewise function $f(a+h) = 2(a+h) + b$, and $f(a) = a^2$ so we get:

$$2a = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0^-} \frac{2(a+h) + b - a^2}{h}$$

$$= \lim_{h \to 0^-} \frac{2a + 2h + b - a^2}{h}$$

$$= \lim_{h \to 0^-} \frac{2h}{h} + \frac{2a + b - a^2}{h}$$

In order to find the limit, the numerator of $\frac{2a+b-a^2}{h}$ must equal zero. This means that we will make $2a + b = a^2$, and that results in:

$$2a = \lim_{h \to 0^-} \frac{2h}{h} + \frac{2a + b - a^2}{h}$$

$$= \lim_{h \to 0^-} 2 + \frac{0}{h}$$

$$= 2$$

Therefore if $2a = 2$, then $\boldsymbol{a = 1}$, if we plug this into $2a + b = a^2$ we get that $2 + b = 1$ or that $\boldsymbol{b = -1}$. In conclusion, since when $\boldsymbol{a = 1}$ and $\boldsymbol{b = -1}$ $f'(a)$ exists as the left and right derivative of f are equal, it must mean that for those values of a and b that $f(x)$ is differentiable at a.