18a. To start we know that the modulus of z_0 is

$$\begin{aligned} r_{z_0} &= \sqrt{(2)^2 + (2sqrt(3))^2} \\ &= \sqrt{4 + (4)(3)} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

We also know that the θ of z_0 is:

$$\begin{aligned} \theta_{z_0} &= \tan^{-1}(\frac{2\sqrt{3}}{2}) \\ \theta_{z_0} &= \tan^{-1}(\sqrt{3}) \\ \theta_{z_0} &= \frac{\pi}{3} \end{aligned}$$

Thus we know z_0 will be:

$$\begin{aligned} z_0 &= r(\cos\theta + \sin\theta i) \\ z_0 &= 4(\cos(\frac{\pi}{3}) + \sin(\frac{\pi}{3})i) \end{aligned}$$

Moving on to z_1 we know its modulus is:

$$\begin{aligned} r_{z_1} &= \sqrt{(-\frac{1}{\sqrt{2}})^2 + (-\frac{\sqrt{3}}{\sqrt{2}})^2} \\ &= \sqrt{\frac{1}{2} + \frac{3}{2}} \\ &= \sqrt{\frac{4}{2}} \\ &= \sqrt{2} \end{aligned}$$

We also know that the θ of z_1 is:

$$\begin{aligned} \theta_{z_1} &= \tan^{-1}(\frac{\sqrt{3} \cdot \sqrt{2}}{\sqrt{2}}) \\ \theta_{z_1} &= \tan^{-1}(\sqrt{3}) \\ \theta_{z_1} &= \frac{\pi}{3} \end{aligned}$$

We know from the signs that it is in third quadrant, thus it becomes:

$$\theta_{z_1} = \frac{4\pi}{3}$$

Thus we know z_1 will be:

$$z_1 = r(\cos\theta + \sin\theta i)$$
$$z_1 = \sqrt{2}(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})i)$$

18b. To start we are given the equation:

$$z_n = (\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i)\frac{z_{n-1}}{|z_{n-2}|}$$

We can replace $\frac{1}{|z_{n-2}|}$ with $|z_{n-2}|^{-1}$ using Properties of Modulus (PM) and our equation becomes:

$$z_n = (\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i) \cdot z_{n-1} \cdot |z_{n-2}|^{-1}$$

However we are trying to solve for $|z_n|$ so our equation becomes:

$$|z_n| = |(\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i) \cdot z_{n-1} \cdot |z_{n-2}|^{-1}|$$

We can use PM to distribute out the modulus so we get:

$$|z_n| = |(\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i)| \cdot |z_{n-1}| \cdot ||z_{n-2}|^{-1}|$$

Notice that $|z| = ||z||$, as the imaginary component will not exist after the first modulus, so you will get (for some real number r:

$$|z| = r$$

Thus:

$$||z|| = \sqrt{r^2 + 0^2}$$
$$||z|| = r$$

So our equation can be rewritten as:

$$|z_n| = |(\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i)| \cdot |z_{n-1}| \cdot |z_{n-2}|^{-1}$$

We also know that $|(\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i)| = 1$ as its in polar form and it's $r = 1$, so:

$$|z_n| = 1 \cdot |z_{n-1}| \cdot |z_{n-2}|^{-1}$$
$$|z_n| = \frac{|z_{n-1}|}{|z_{n-2}|}$$

By finding the shortest non repeating sequence within z, we can find the positive integer p such that for any positive integer n:

$$|z_{n+p}| = |z_n|$$

To do this we will list the values of z_n until we find a repeating term (we wont list z_0 as 0 is not a positive integer):

$\lvert z_1 \rvert$	$\sqrt{2}$ (from Part A)
$\lvert z_2 \rvert$	$\frac{\lvert z_{n-1} \rvert}{\lvert z_{n-2} \rvert} = \frac{\sqrt{2}}{4}$
$\lvert z_3 \rvert$	$\frac{\lvert z_{n-1} \rvert}{\lvert z_{n-2} \rvert} = \frac{\sqrt{2}}{4 \cdot \sqrt{2}} = \frac{1}{4}$
$\lvert z_4 \rvert$	$\frac{\lvert z_{n-1} \rvert}{\lvert z_{n-2} \rvert} = \frac{4}{4 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}$
$\lvert z_5 \rvert$	$\frac{\lvert z_{n-1} \rvert}{\lvert z_{n-2} \rvert} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$
$\lvert z_6 \rvert$	$\frac{\lvert z_{n-1} \rvert}{\lvert z_{n-2} \rvert} = 2\sqrt{2} \cdot \sqrt{2} = 4$
$\lvert z_7 \rvert$	$\frac{\lvert z_{n-1} \rvert}{\lvert z_{n-2} \rvert} = \frac{4}{2\sqrt{2}} = \sqrt{2}$

Therefore we notice a pattern that for any z_n where n is an integer:

$$n \equiv n_0 \pmod{6} \implies |z_n| = |z_{n_0}|$$

Therefore be definition this tells us that for some k:

$$n \equiv n + 6k \implies |z_n| = |z_{n+6k}|$$

Let $p = 6k$, the smallest positive integer of p will happen when $k = 1$. Thus:

$$|z_n| = |z_{n+p}| \text{ (where p = 6)}$$

18c We need to find a positive integer q such that for every positive integer $n, z_n = z_{n+q}$. This will happen when both:

$$|z_n| = |z_{n+q}| \text{ and } \arg(z_n) = \arg(z_{n+q})$$

From Part B we know that q must be be a multiple of 6. In order to get when the arguments are equal we must come up with a general expression for $\arg(z_n)$:

$$z_n = (\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i) \cdot z_{n-1} \cdot |z_{n-2}|^{-1}$$

We know that real components will have no impact on our argument, so we can remove $|z_{n-2}|^{-1}$, so we get:

$$z_n = (\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i) \cdot z_{n-1}$$

We can express z_{n-1} in polar form such that for some integer r_{n-1}, θ_{n-1}:

$$z_{n-1} = r_{n-1}(\cos(\theta_{n-1}) + \sin(\theta_{n-1}))i)$$

Plugging this back into our equation we get that:

$$z_n = (\cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4})i)(r_{n-1}(\cos(\theta_{n-1}) + \sin(\theta_{n-1}))i))$$

Thus by Polar Multiplication in C (PMC), we know that this will simplify to:

$$z_n = r_{n-1}(\cos(\frac{3\pi}{4} + \theta_{n-1}) + \sin(\frac{3\pi}{4} + \theta_{n-1})i)$$

And the argument of this term will thus be:

$$\begin{aligned}
\arg(z_n) &= \arg[r_{n-1}(\cos(\frac{3\pi}{4} + \theta_{n-1}) + \sin(\frac{3\pi}{4} + \theta_{n-1})i)] \\
&= (\cos(\frac{9\pi}{12} + \theta_{n-1}) + \sin(\frac{9\pi}{12} + \theta_{n-1})i) \\
&= \frac{9\pi}{12} + \theta_{n-1}
\end{aligned}$$

Thus we will now find the smallest y such that:

$$\arg(z_n) = \arg(z_{n+y})$$

In order to do this we will find the shortest non repeating sequence within z by creating a list of all $\arg(z_n)$ up to the first repeat:

$\arg(z_1)$	$\frac{4\pi}{3} = \frac{16\pi}{12}$ (from Part A)
$\arg(z_2)$	$\frac{9\pi}{12} + \theta_{n_1} = \frac{9\pi}{12} + \frac{16\pi}{12} = \frac{1\pi}{12}$
$\arg(z_3)$	$\frac{9\pi}{12} + \theta_{n_2} = \frac{9\pi}{12} + \frac{1\pi}{12} = \frac{10\pi}{12}$
$\arg(z_4)$	$\frac{9\pi}{12} + \theta_{n_3} = \frac{9\pi}{12} + \frac{10\pi}{12} = \frac{19\pi}{12}$
$\arg(z_5)$	$\frac{9\pi}{12} + \theta_{n_4} = \frac{9\pi}{12} + \frac{19\pi}{12} = \frac{4\pi}{12}$
$\arg(z_6)$	$\frac{9\pi}{12} + \theta_{n_5} = \frac{9\pi}{12} + \frac{4\pi}{12} = \frac{13\pi}{12}$
$\arg(z_7)$	$\frac{9\pi}{12} + \theta_{n_6} = \frac{9\pi}{12} + \frac{13\pi}{12} = \frac{22\pi}{12}$
$\arg(z_8)$	$\frac{9\pi}{12} + \theta_{n_7} = \frac{9\pi}{12} + \frac{22\pi}{12} = \frac{7\pi}{12}$
$\arg(z_9)$	$\frac{9\pi}{12} + \theta_{n_8} = \frac{9\pi}{12} + \frac{7\pi}{12} = \frac{16\pi}{12}$

Therefore we notice a pattern that for any z_n where n is an integer:

$$n \equiv n_0 \pmod 8 \implies \arg(z_n) = \arg(z_{n_0})$$

Therefore be definition this tells us that for some k:

$$n \equiv n + 8k \implies \arg(z_n) = \arg(z_{n+9k})$$

We thus know that for some positive integer q:

$$\begin{aligned} q = 0 \pmod 6 &\implies |z_n| = |z_{n+q}| \\ q = 0 \pmod 8 &\implies \arg(z_n) = \arg(z_{n+q}) \end{aligned}$$

Thus by CTR, both of these conditions will be met (both conditions being met implies $z_n = z_{n+q}$) when:

$$q = 0 \pmod{24} \implies z_n = z_{n+q}$$

Q is thus equal to (for some integer t):

$$q = 24t$$

Thus the smallest positive integer of q happens when $t = 1$ and so the smallest positive q is 24. So we get:

$$z_n = z_{n+q} \text{ where q} = 24$$