Robert (Robbie) Knowles MATH 137 Fall 2020: WA09

Q01A let f be a differentiable function such that $L \neq 0$ and that $\lim_{x\to\infty} f'(x)$ exists. We know that we can multiple any given number by 1:

$$f(x) \equiv f(x) \cdot 1 \equiv f(x) \cdot \frac{e^x}{e^x} \equiv \frac{f(x)e^x}{e^x}$$

Thus we can say:

$$f(x) \equiv \frac{f(x)e^x}{e^x}$$

If we try to solve for the limit as x approaches infinity:

$$\begin{aligned}\lim_{x\to\infty} f(x) &\equiv \lim_{x\to\infty} \frac{f(x)e^x}{e^x}\\ &\equiv \frac{\lim_{x\to\infty}(f(x)e^x)}{\lim_{x\to\infty} e^x}\end{aligned}$$

In order to simplify we will let:

$$\frac{\lim_{x\to\infty}(f(x)e^x)}{\lim_{x\to\infty} e^x} \equiv \frac{\lim_{x\to\infty} g(x)}{\lim_{x\to\infty} z(x)}$$

We know that $\lim_{x\to\infty} g(x)$ will become:

$$\begin{aligned}\lim_{x\to\infty} g(x) &\equiv \lim_{x\to\infty}(f(x)e^x)\\ &\equiv (\lim_{x\to\infty} f(x))(\lim_{x\to\infty} e^x)\\ &\equiv L \cdot e^{\infty}\\ &\equiv L \cdot \infty\end{aligned}$$

We know that L is non-zero, thus we know that $L \cdot \infty$ is not indefinite, this would imply that $L \cdot \infty \equiv \infty$. Which means that $\lim_{x\to\infty} g(x)$ is equivalent to:

$$\lim_{x\to\infty} g(x) \equiv L \cdot \infty \equiv \infty$$

If we move to $\lim_{x\to\infty} z(x)$ we know it will become:

$$\begin{aligned}\lim_{x\to\infty} z(x) &\equiv \lim_{x\to\infty} e^x\\ &\equiv L \cdot e^{\infty}\\ &\equiv L \cdot \infty\end{aligned}$$

Thus we know that $\lim_{x\to\infty} \frac{g(x)}{z(x)}$ will be equivalent to $\frac{\infty}{\infty}$. We also know that the derivative of g(x) and z(x) is:

$$\begin{cases} g'(x) \equiv \frac{dy}{dx}(f(x)e^x) \equiv e^x f(x) + f'(x)e^x \\ z'(x) \equiv \frac{dy}{dx}(e^x) \equiv e^x \end{cases}$$

We thus know that since g'(x) and z'(x) exist and $\lim_{x\to\infty} \frac{g(x)}{z(x)} = \frac{\infty}{\infty}$, that we can apply L'Hopital's Rule, such that:

$$\begin{aligned} \lim_{x\to\infty} f(x) \equiv \lim_{x\to\infty} \frac{g(x)}{z(x)} &\equiv \lim_{x\to\infty} \frac{g'(x)}{z'(x)} \\ &\equiv \lim_{x\to\infty} \frac{e^x f(x) + f'(x)e^x}{e^x} \\ &\equiv \lim_{x\to\infty} (f(x) + f'(x)) \end{aligned}$$

If we rearrange we get that:

$$\begin{aligned} \lim_{x\to\infty} f'(x) \equiv &\lim_{x\to\infty} (f(x) - f(x)) \\ \equiv &(L - L) \\ \equiv &0 \end{aligned}$$

This thus proves the hypothesis using L'Hopital's Rule.

Q01B We know that f is differentiable everywhere and as a result must also be continous everwhere. For any given $x \in \mathbb{R}$ we can define the interval:

$$(x, x+1)$$

MVT thus tells us that a c ($c \in \mathbb{R}$) which is bounded by $x < c < x + 1$ will also exist such that:

$$f'(c) \equiv \frac{f(x+1) - f(x)}{x + 1 - x}$$

Simplfying we find that:

$$\begin{aligned} f'(c) \equiv &\frac{f(x+1) - f(x)}{x + 1 - x} \\ \equiv &\frac{f(x+1) - f(x)}{x - x + 1} \\ \equiv &f(x+1) - f(x) \end{aligned}$$

We know that the conditions are satisfied to use the linear approximation on f(x + 1) as we know its differentible at (x+1):

$$f'(c) \equiv f(x) + f'(x)(x + 1 - x) - f(x)$$

$$f'(c) \equiv f'(x)$$

Thus we subsitute f'(x) for f'(c) and so our original equation becomes:

$$f'(x) \equiv f(x+1) - f(x)$$

If we let x approach inifinty we get that:

$$\lim_{x\to\infty} f'(x) \equiv \lim_{x\to\infty} f(x+1) - \lim_{x\to\infty} f(x)$$

$$\lim_{x\to\infty} f'(x) \equiv f(\infty+1) - f(\infty)$$

$$\lim_{x\to\infty} f'(x) \equiv f(\infty) - L$$

$$\lim_{x\to\infty} f'(x) \equiv L - L$$

$$\lim_{x\to\infty} f'(x) \equiv 0$$

This thus proves the hypothesis using MVT.