Q05 Let n be a non negative integer. We know the original equation can be split into parts:

$$\mathrm{Re}(z_n)\,\mathrm{Im}(z_n) = 0 \iff 3 \mid n$$

The first case we shall consider is when:

$$3 \mid n \implies \mathrm{Re}(z_n)\,\mathrm{Im}(z_n) = 0$$

If we assume that $3|n$, we can also express n (for some integer k) as:

$$n = 3k$$

We can thus prove this using induction on z_n. Consider the base case $n = 0$: By definition we know that $z_0 = i$, which means that the real component is 0.
Assume that for any p which is an integer and n = 3p, such that $\mathrm{Re}(z_n)\,\mathrm{Im}(z_n) = 0$
Prove p + 1 satisfies the condition:

$$\mathrm{Re}(z_{3(p+1)})\,\mathrm{Im}(z_{3(p+1)}) = 0$$

We know that we can express:

$$\begin{aligned} z_{3p+3} &= z_{3p+2}z_{3p+1} \\ &= z_{3p+1}z_{3p}z_{3p+1} \\ &= (z_{3p+1})^2 z_{3p} \end{aligned}$$

As well we know that $(z_{3p+1})^2$ will contain either only real values or imaginary values. By definition we know that z_{3p} has only real values or imaginary values. this means that:

$$\mathrm{Re}((z_{3p+1})^2 z_{3p}) = 0 \text{ or } \mathrm{Im}((z_{3p+1})^2 z_{3p}) = 0$$

In the opposite case we shall consider:

$$\mathrm{Re}(z_n)\,\mathrm{Im}(z_n) = 0 \implies 3 \mid n$$

Lets start by splitting this again into two cases:

$$\mathrm{Re}(z_n) = 0 \text{ or } \mathrm{Im}(z_n) = 0$$

If we assume the real component is zero, this would mean that (for some real value or imaginary value m):

$$\begin{aligned} m &= z_n \\ &= z_{n-1}z_{n-2} \\ &= z_{n-2}z_{n-2}z_{n-3} \end{aligned}$$

Notice that $z_{n-2}z_{n-2}$ could be m. This pattern will repeat until:

$$z_n = (z_{n-2}z_{n-2}) \cdot (z_{n-5}z_{n-5}) \ldots z_b$$

where b is either 0, 1, 2. Notice that the real or imaginary component being zero will depend on z_b. Thus we know:

$$z_0 = i$$

$$z_1 = 1 + i$$

$$z_2 = -1 + i$$

Thus we can tell that the only when z_0 that:

$$\text{Re}(z_0) = 0 \text{ or } \text{Im}(z_0) = 0$$

This means that for any n where $3|n$ the base case will be zero or in other words:

$$\text{Re}(z_n) = 0 \text{ or } \text{Im}(z_n) = 0$$

Therefore since we proved both sides we have proved the if and only if.