Q03a We know that $f'(x)$ is defined for all $x \in \mathbb{R}$, and we also know that differentiability applies continuity for $f(x)$. This means that f(x) is both continuous and differentiable for all $x \in \mathbb{R}$.
Therefore we can use the bounded derivative theorem (BDT) to find the smallest interval of f(6), and since we know that:

$$1 \leq f'(x) \leq 5 \text{ for every } x \in (3, 6)$$

Therefore by BDT this implies:

$$\begin{gathered} f(3) + 1(x-3) \leq f(x) \leq f(3) + 5(x-3) \\ 1 + 1(x-3) \leq f(x) \leq 1 + 5(x-3) \\ 1 + x - 3 \leq f(x) \leq 1 + 5x - 15 \\ x - 2 \leq f(x) \leq 5x - 14 \end{gathered}$$

Therefore if we let x = 6, we find that:

$$\begin{gathered} 6 - 2 \leq f(6) \leq 30 - 14 \\ 4 \leq f(6) \leq 16 \end{gathered}$$

We can thus say that the smallest interval that $f(6)$ is bounded by is [4,16]

Q03b The textbook defines the second order Taylor Polynomial as (for some real x = a):

$$T_{2,a} = \sum_{k=0}^{2} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If we expand this out we get that:

$$T_{2,a} = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$$

Since we are finding this polynomial centered at x = 3, the polynomial becomes:

$$T_{2,3} = \frac{f(3)}{0!}(x-3)^0 + \frac{f'(3)}{1!}(x-3)^1 + \frac{f''(3)}{2!}(x-3)^2$$

From definition we know that $f(3) = 1$, $f'(3) = 5$ and $f''(3) = 7$, so we can simplify:

$$\begin{aligned}
T_{2,3} &= \frac{1}{0!} + \frac{5}{1!}(x-3)^1 + \frac{7}{2!}(x-3)^2 \\
T_{2,3} &= 1 + 5(x-3) + \frac{7}{2}(x^2 - 6x + 9) \\
T_{2,3} &= 1 + 5x - 15 + \frac{7x^2}{2} - \frac{42x}{2} + \frac{63}{2} \\
T_{2,3} &= \frac{2}{2} + 5x - \frac{30}{2} + \frac{7x^2}{2} - 21x + \frac{63}{2} \\
T_{2,3} &= \frac{7x^2}{2} - 16x + \frac{35}{2}
\end{aligned}$$

Thus giving us the answer for the second order Taylor Polynomial centered around 3.

Q03c We know from the previous question that the second order Taylor Polynomial centered around 3 is:

$$T_{2,3} = \frac{7x^2}{2} - 16x + \frac{35}{2}$$

Therefore we can approximate $f(5)$ by letting $x = 5$, so we get:

$$T_{2,3} = \frac{7(5)^2}{2} - 16(5) + \frac{35}{2}$$
$$T_{2,3} = \frac{175}{2} - 80 + \frac{35}{2}$$
$$T_{2,3} = \frac{210}{2} - 80$$
$$T_{2,3} = 105 - 80$$
$$T_{2,3} = 25$$

In order to find the magnitude of error for this polynomial we will use Taylor Remainder (page 271) such that:

$$|R_{2,3}| = |f(5) - T_{2,3}(5)|$$

We can thus apply Taylor's Theorem (page 272), such that this becomes:

$$|R_{2,3}| = f(5) - T_{2,3}(5)|$$
$$|R_{2,3}| = |\frac{f'''(c)}{(2+1)!}(5-3)^(2+1)|$$
$$|R_{2,3}| = |\frac{f'''(c)}{(3!}(2)^(3)|$$
$$|R_{2,3}| = |\frac{f'''(c)}{6}8|$$

In order to find the maximum error we thus need to find the largest possible magnitude of f"'(c), and since:

$$-4 \leq f'''(c) \leq -1$$

The largest possible magnitude is —-4—, so the largest possible error will be when $f'''(c) = -4$. Plugging this in we get:

$$|R_{2,3}| = |\frac{-4}{6}8|$$
$$|R_{2,3}| = |\frac{-16}{3}|$$
$$R_{2,3} = \frac{16}{3}$$

Therefore the upper bound on the error is $\frac{16}{3}$.

Q03d We know from c that when we subtract the Taylor Polynomial from the original function we get:

$$f(5) - T_{2,3}(5) = \frac{-16}{3}$$

(we can get rid of the absolute values as we are no longer looking at just the magnitude), as well this implies that:

$$\begin{aligned} f(5) - T_{2,3}(5) &< 0 \\ f(5) &< T_{2,3}(5) \end{aligned}$$

Since our Taylor Polynomial is larger then the original function we can state with certainty that it is an overestimate.