Q02b Let n be a positive integer such that $n > 1$, we know from UFT that n can be expressed as:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

Where p_1,p_2,...,p_k, $k \geq 1$ are a list of prime divisors of n and α_1,α_2,...,α_k are all non-negative integers. We also know from Part A that the number of positive divisors of n (or $d(n)$) will equal:

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_k + 1)$$

For all coprime integers a,b such that $ab =$ n. This implies that:

$$ab = n$$

$$ab = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

Applying UFT to a and b we find that (where β_1,β_2,...,β_k and ω_1,ω_2,...,ω_k are non negative integers) :

$$(p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k})(p_1^{\omega_1} p_2^{\omega_2} \cdots p_k^{\omega_k}) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

However before we go futher we know that a, b are coprime this means that $\gcd(a, b) = 1$, using GCD PF we find that:

$$gcd(a, b) = p_1^{\theta_1} p_2^{\theta_2} \cdots p_k^{\theta_k} \text{ where } \theta_i = \min(\beta_i, \omega_i) \text{ for i = 1,2,...,k}$$

$$1 = p_1^{\theta_1} p_2^{\theta_2} \cdots p_k^{\theta_k}$$

This means that for any prime factor p_i, $\theta_i = 0$ or that $\min(\beta_i, \omega_i) = 0$. Going back to our UFT expansion of a and b this means that:

$$(p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k})(p_1^{\omega_1} p_2^{\omega_2} \cdots p_k^{\omega_k}) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$$(p_1^{\beta_1+\omega_1} p_2^{\beta_2+\omega_2} \cdots p_k^{\beta_k+\omega_k}) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

For any prime factor p_i, (either $\beta = 0$ or $\omega = 0$) the equation becomes:

$$p_i^{\beta_i+\omega_i} = p_i^{\alpha_i}$$

$$p_i^{\beta_i+0} = p_i^{\alpha_i} \text{ or } p_i^{0+\omega_i} = p_i^{\alpha_i}$$

$$p_i^{\beta_i} = p_i^{\alpha_i} \text{ or } p_i^{\omega_i} = p_i^{\alpha_i}$$

This means that the combinations at each p_i is:

$$(\alpha_i + 1) = (\beta_i + 1) \text{ or } (\alpha_i + 1) = (\omega_i + 1)$$

The total combinations of n will thus become (where $b_{any} \neq c_{any}$):

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_k + 1)$$

$$d(n) = ((\beta_{b1} + 1)(\beta_{b2} + 1)...(\beta_{bk} + 1))((\omega_{c1} + 1)(\omega_{c2} + 1)...(\omega_{ck} + 1))$$

From Part A we know that:

$$d(a) = ((\beta_{b1} + 1)(\beta_{b2} + 1)...(\beta_{bk} + 1)) \text{ and } d(b) = ((\omega_{c1} + 1)(\omega_{c2} + 1)...(\omega_{ck} + 1))$$

So this means for all n and all coprime integers a,b such that $ab =$ n,

$$d(n) = ((\beta_{b1} + 1)(\beta_{b2} + 1)...(\beta_{bk} + 1))((\omega_{c1} + 1)(\omega_{c2} + 1)...(\omega_{ck} + 1))$$

$$d(ab) = ((\beta_{b1} + 1)(\beta_{b2} + 1)...(\beta_{bk} + 1))((\omega_{c1} + 1)(\omega_{c2} + 1)...(\omega_{ck} + 1))$$
$$d(ab) = (d(a))(d(b))$$

Proving that d(n) is multiplicative.