Q02a Let b be a real number. We know from the "Critical Point" Theorem (Page 202) that a critical point will happen at some c when:

$$f'(c) = 0 \text{ or } f'(c) = DNE$$

Thus in order to solve we must first find the first derivative.

$$\begin{aligned}
f(x) &= x^{\frac{1}{3}} + bx^{\frac{4}{3}} \\
\frac{dy}{dx}(f(x)) &= \frac{dy}{dx}(x^{\frac{1}{3}} + bx^{\frac{4}{3}}) \\
f'(x) &= \frac{dy}{dx}(x^{\frac{1}{3}}) + \frac{dy}{dx}(bx^{\frac{4}{3}}) \\
f'(x) &= \frac{1}{3}x^{(\frac{1}{3}-1)} + b\frac{4}{3}x^{(\frac{4}{3}-1)} \\
f'(x) &= \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}}
\end{aligned}$$

Since we can see an x is in the denominator for the first term, this would imply that f'(x) is undefined at 0 or:

$$f'(0) = DNE$$

Since there is only one possible value such that $f'(c) = DNE$, we will now try to solve $f'(c) = 0$, starting by setting f'(x) to be zero:

$$\begin{aligned}
0 &= \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}} \\
-\frac{1}{3x^{\frac{2}{3}}} &= b\frac{4}{3}x^{\frac{1}{3}} \\
-\frac{1}{3x^{\frac{2}{3}}}x^{\frac{2}{3}} &= b\frac{4}{3}x^{\frac{1}{3}}x^{\frac{2}{3}} \\
-\frac{1}{3} &= b\frac{4}{3}x \\
-\frac{1}{4b} &= x
\end{aligned}$$

Therefore our possible critical points will be when:

$$x = 0 \text{ or } -\frac{1}{4b}$$

Q02b We know we have two possible values to test if a local minimum can exist, point 0 and point $-\frac{1}{4b}$ (for some real number b). At the end I also prove this using an interval table.

To see if 0 in a minimum we shall use "The First Derivative Test" (Page 223) we will use the interval $(0^-, 0^+)$ to check if:

$$f'(0^-) < f'(x) < 0 \text{ for all } x \in (0^-, 0)$$
$$f'(0^+) > f'(x) > 0 \text{ for all } x \in (0, 0^+)$$

Evaluating the first equation we find a contradiction at $f'(0^-)$ we get:

$$\begin{aligned}
f'(0^-) &< f'(x) < 0 \\
\frac{1}{3(0^-)^{\frac{2}{3}}} + b\frac{4}{3}(0^-)^{\frac{1}{3}} &< f'(x) < 0 \\
\frac{1}{3(0^+)} + b\frac{4}{3}(0^-) &< f'(x) < 0 \\
\frac{1}{(0^+)} + (0^-) &< f'(x) < 0 \\
\infty + (0^-) &< f'(x) < 0 \\
\infty &< f'(x) < 0
\end{aligned}$$

Because this contradicts the First Derivative Test this means that 0 can not be a local minimum (as well this follows logically as the value to the will always be smaller left is smaller).

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}}$$

Moving on we can test the critical point $-\frac{1}{4b}$ using the "Second Derivative Test", in order to solve this we will first find the second derivative.

$$\begin{aligned}
f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} + b\frac{4}{3}x^{\frac{1}{3}} \\
\frac{dy}{dx}f'(x) &= \frac{dy}{dx}(\frac{1}{3}x^{-\frac{2}{3}}) + \frac{dy}{dx}(b\frac{4}{3}x^{\frac{1}{3}}) \\
f''(x) &= \frac{1}{3}\frac{-2}{3}x^{-\frac{5}{3}} + b\frac{4}{3}\frac{1}{3}x^{-\frac{2}{3}} \\
f''(x) &= \frac{-2}{9}x^{-\frac{5}{3}} + b\frac{4}{9}x^{-\frac{2}{3}}
\end{aligned}$$

Now we will plug in the critical point (c) and simplify:

$$f''(c) = \frac{-2}{9}(-\frac{1}{4b})^{-\frac{5}{3}} + b\frac{4}{9}(-\frac{1}{4b})^{-\frac{2}{3}}$$
$$f''(c) = \frac{2}{9}(-\frac{1}{4b})^{-\frac{2}{3}}(-(-\frac{1}{4b})^{-1} + 2b)$$
$$f''(c) = \frac{2}{9}(-4b)^{\frac{2}{3}}(-(-4b)^{1} + 2b)$$
$$f''(c) = \frac{2}{9}\sqrt[3]{(16b^2)}(6b)$$

The "Second Derivative Test" tells us that a minimum will exist at the critical point if:

$$f''(c) > 0$$

As we can see from the above equation, the positivity is of the equation is solely reliant on the $6b$ term as (b^2 will always be positive). Therefore f'(c) ¿ 0 if b is also positive. In other words the minimum will exist as along as:

$$b > 0$$

Therefore we have shown that the only possible local exist when b is positive and the minimum will exists at:

$$-\frac{1}{4b}$$