Q03 Let k be a perfect nth power and mth power (where both m and n are coprime), and let a and b both be arbitrary integers such that:

$$k = a^n = b^m$$

We know from UFT that k can be represented in terms of its prime divisors (p_i) and their non-negitive exponents (α_i), in other words the equation becomes (given g ≥ 1):

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_g^{\alpha_g} = a^n = b^m$$

If we split this into two equations we find that :

1. $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_g^{\alpha_g} = a^n$ and $p_1^{\frac{\alpha_1}{n}} p_2^{\frac{\alpha_2}{n}} \cdots p_g^{\frac{\alpha_g}{n}} = a$
2. $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_g^{\alpha_g} = b^m$ and $p_1^{\frac{\alpha_1}{m}} p_2^{\frac{\alpha_2}{m}} \cdots p_g^{\frac{\alpha_g}{m}} = b$

Since a and b are integers, equations 1 and 2 shows us that for each α_i (where $0 \leq i \leq g$):

$$n|\alpha_i \text{ and } m|\alpha_i$$

We know that we can rewrite α_i as:

$$\alpha_i = (\frac{\alpha_i}{n})(n)$$

Plugging in $m|\alpha_i$ we get that:

$$m|\alpha_i = m|(\frac{\alpha_i}{n})(n)$$

Notice that the gcd$(m, (n)) = 1$, thus from CAD we know that $m|(\frac{\alpha_i}{n})$. The definition of divisibilty thus imples:

$$m|\frac{\alpha_i}{n} \implies \text{(for some integer h) } mh = \frac{\alpha_i}{n} \implies mnh = \alpha_i \implies mn|\alpha_i$$

Since $mn|\alpha_i$, we also thus know that an integer c exists such that for every prime divisor:

$$p_i^{\frac{\alpha_i}{mn}} = c$$

As this holds for every prime divisor (p_i) we know that an integer d must exist such that:

$$p_1^{\frac{\alpha_1}{mn}} p_2^{\frac{\alpha_2}{mn}} \cdots p_k^{\frac{\alpha_g}{mn}} = d$$

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_g} = d^{mn}$$

Since this is the prime factor expansion of k, we can replace it with k to show that:

$$k = d^{mn}$$

Therefore we have shown that a positive integer d exsits such that k $= d^{mn}$ proving that if k is a perfect nth and mth power then it is also a perfect mnth power