Q05 Let a and b be a arbitrary positive integers and c be an integer such that $\gcd(a,b)|c$, we thus know from definition that an integer g exisits so:

$$\gcd(a,b)g = c$$

We know from LDET 1 that some integers s,t exist such that:

$$as + bt = \gcd(a,b)$$

Multiplying this all by g we find that:

$$a(gs) + b(gt) = \gcd(a,b)g$$

$$a(gs) + b(gt) = c$$

A pair of integers x_0, y_0 will exist such that $x_0 = gs$ and $y_0 = gt$ and:

$$a(x_0) + b(y_0) = c$$

From LDET 2 it follows that the set of all possible solutions of $ax + by = c$ is given by (for every natural h):

$$x = x_0 + \frac{b}{\gcd(a,b)}h \text{ and } y = y_0 - \frac{a}{\gcd(a,b)}h$$

We know that a solution of x (represented by x') will be of the form:

$$x' = x_0 + \frac{b}{\gcd(a,b)}n \text{ (where n is some integer)}$$

If we give the restriction that:

$$0 \leq x' < \frac{b}{\gcd(a,b)}$$

It implies that:

$$0 \leq x_0 + \frac{b}{\gcd(a,b)}n < \frac{b}{\gcd(a,b)}$$

If we divide all three sides by $\frac{b}{\gcd(a,b)}$ we get:

$$\frac{(0)(gcd(a,b))}{b} \leq \frac{(x_0)(gcd(a,b))}{b} + \frac{(b)(\gcd(a,b))}{(\gcd(a,b))(b)}n < \frac{(b)(\gcd(a,b))}{(b)(\gcd(a,b))}$$

$$\frac{(0)}{b} \leq \frac{(x_0)(gcd(a,b))}{b} + (1)n < (1)$$

$$0 \leq \frac{(x_0)(gcd(a,b))}{b} + n < 1$$

$$-\frac{(x_0)(gcd(a,b))}{b} \leq n < 1 - \frac{(x_0)(gcd(a,b))}{b}$$

We know that b is a positive number (non zero), which means that a real number r exists such that:

$$r = -\frac{(x_0)(gcd(a,b))}{b}$$

Plugging this in we find that for some real number r, a integer n exists such that:

$$-\frac{(x_0)(gcd(a,b))}{b} \leq n < 1 - \frac{(x_0)(gcd(a,b))}{b}$$

$$r \leq n < 1 + r$$

I will use proof by contradiction to show that only one integer n exists such that $n \in [r, r+1)$. We know that at least one number exists within the interval as if we set n = $\lceil r \rceil$, we find that:

$$r \leq \lceil r \rceil < 1 + r$$

Which will always satisfy the equation, so we will let the natural k represent the number of integers within the interval $[r, r+1)$:

Assume that $k \geq 2$. Since the smallest possible integer (n_0) that exists within the interval will always be $\lceil r \rceil$, we also know that the largest integer (n_{k-1}) will be equal to:

$$\text{largest integer} = \text{smallest integer} + (\text{amount of integers - 1})$$

$$n_{k-1} = n_0 + (k-1)$$

We also know that since $k \geq 2$ that:

$$n_0 + (k-1) \geq n_0 + ((2) - 1)$$

$$n_0 + (k-1) \geq n_0 + 1$$

Which thus implies:

$$n_{k-1} \geq n_0 + 1$$

$$n_{k-1} \geq \lceil r \rceil + 1$$

Since $n_{k-1} \in [r, r+1)$ this means that it is bounded by:

$$r \leq n_{k-1} < 1 + r$$

$$r \leq \lceil r \rceil + 1 \leq n_{k-1} < 1 + r$$

$$r \leq \lceil r \rceil + 1 < 1 + r$$

Splitting this into two equations we get that:

1. $r \leq \lceil r \rceil + 1$
2. $\lceil r \rceil + 1 < 1 + r$

We know that (1) is correct, but notice that equation (2) simplifies to:

$$\lceil r \rceil + 1 < 1 + r$$

$$\lceil r \rceil < r$$

Therefore whenever $k \geq 2$ a contradiction will occour as the $\lceil r \rceil$ is can never be less then r.
We know that since k is a natural number and $k \nleq 2$, so k = 1. This means that there is only one possible n such that by LDET 2:

$$x' = x_0 + \frac{b}{d} * n$$

$$y' = y_0 - \frac{a}{d} * n$$

Since there is only one possible integer value of n and x_0, y_0 are integers and $\frac{b}{\gcd(a,b)}$, $\frac{a}{\gcd(a,b)}$ are also both integers it means that x' and y' are unique integer solutions that both:

$$ax' + by' = c \text{ and } x' < \frac{b}{\gcd(a, b)}$$

For any positive integer a,b and integer c such that $\gcd(a, b)|c$.