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Q01a. To start we will assume that for any x that $L_a(x) = L_b(x)$. Therefore we know that for any real number a, b where $(a \neq b)$:

$$L_b(b) = L_a(b) \text{ and } L_a(a) = L_b(a)$$

We then re-write as:

$$L_b(b) - L_a(b) = 0 \text{ and } L_a(a) - L_b(a) = 0$$

Setting the $0 = 0$ we find that:

$$0 = 0$$

$$L_b(b) - L_a(b) = L_a(a) - L_b(a)$$

Re-arranging we find that:

$$L_b(b) + L_b(a) = L_a(a) + L_a(b)$$

By the definition of Linear Approximation that we can expand this as:

$$\begin{aligned}
f(b) + f'(b)(b-b) + f(b) + f'(b)(a-b) &= f(a) + f'(a)(a-a) + f(a) + f'(a)(b-a) \\
f(b) + f(b) + f'(b)(-(b-a)) &= f(a) + f(a) + f'(a)(b-a) \\
2f(b) - f'(b)(b-a) &= 2f(a) + f'(a)(b-a) \\
2f(b) - 2f(a) &= f'(b)(b-a) + f'(a)(b-a) \\
2(f(b) - f(a)) &= (f'(b) + f'(a))(b-a) \\
\frac{f(b) - f(a)}{b-a} &= \frac{f'(b) + f'(a)}{2}
\end{aligned}$$

Therefore we have proved that if $L_a(x) = L_b(x)$ then for all x and for any real a, b where $(a \neq b)$ that $\frac{f(b)-f(a)}{b-a} = \frac{f'(b)+f'(a)}{2}$

Q01b. Note that the question (for $a \neq b$ and all x) can be re-written as:

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(b) + f'(a)}{2} \implies L_a(x) = L_b(x)$$

In order to disprove this we need to find a counter example where the the hypothesis is true but the conclusion is false. For this case let $f(x) = x^2$ (note that $f'(x) = 2x$) and let $a = 3$ and $b = -2$, the hypothesis thus becomes:

$$\begin{aligned}
\frac{f(b) - f(a)}{b - a} &= \frac{f'(b) + f'(a)}{2} \\
\frac{b^2 - a^2}{b - a} &= \frac{2b + 2a}{2} \\
\frac{(-2)^2 - (3^2)}{-2 - 3} &= \frac{2(-2) + 2(3)}{2} \\
\frac{4 - 9}{-5} &= \frac{2}{2} \\
1 &= 1
\end{aligned}$$

Now that we know the hypothesis is correct, we will show that the Linear Approximation of $L_a(x)$ and $L_b(x)$ is equal to:

$$L_a(x) = f(a) + f'(a)(x - a) = 3^2 + (6)(x - 3) = 9 + 6(x - 3)$$

$$L_b(x) = f(b) + f'(b)(x - b) = -2^2 + (-4)(x + 2) = 4 - 4(x + 2)$$

When $x = 1$ notice that:

$$L_a(1) = 9 + 6(1 - 3) = 9 + 6(-2) = -3$$

$$L_b(1) = 4 - 4(1 + 2) = 4 - 4(3) = -8$$

Therefore at least one x exists such that for a given a and b that satisfies the hypothesis and where $l_a(x) \neq l_b(x)$ for that given x. This thus disproves that $l_a(x) = l_b(x)$ for all real values of x.