Q04A Let n be a non negative integer. We know from BT we know that we can express $(1+i)^n$ and $(1-i)^n$ as:

$$(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} (i)^k$$
$$(1-i)^n = \sum_{k=0}^{n} \binom{n}{k} (-i)^k$$

Adding the two together we find that:

$$\begin{aligned}(1+i)^n + (1-i)^n &= \sum_{k=0}^{n} \binom{n}{k} (i)^k + \sum_{k=0}^{n} \binom{n}{k} (-i)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} [-i^k + i^k]\end{aligned}$$

Notice that we can thus split this sigma into two cases, when k is even and when k is odd. Lets start when k is odd, we know that we can express k as $2d+1$ (where $d \in \mathbb{N}$), so our equation becomes:

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} [-i^k + i^k] &= \binom{n}{2d+1} [-i^{2d+1} + i^{2d+1}] \\ &= \binom{n}{2d+1} [(-i^2)^d(-i) + (i^2)^d(i)] \\ &= \binom{n}{2d+1} [(-1)^d(-i) + (-1)^d(i)] \\ &= \binom{n}{2d+1} [-(-1)^d(i) + (-1)^d(i)] \\ &= \binom{n}{2d+1} [0] \\ &= 0\end{aligned}$$

This means that for each odd k, the value of the summation will be unchanged.

In the second case we will let k be even, it can thus be expressed as k as $2d$ (where $d \in \mathbb{N}$), so our equation becomes:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}[-i^k+i^k] &= \binom{n}{2d}[-i^{2d}+i^{2d}]\\ &= \binom{n}{2d}[(-i^2)^d+(i^2)^d]\\ &= \binom{n}{2d}[(-1)^d+(-1)^d]\\ &= \binom{n}{2d}[2(-1)^d]\end{aligned}$$

We thus know that only even values will impact the summation, the equation for all the even values is:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}[2(-1)^k] \text{ or } 2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}[(-1)^k]$$

And because odd values are redundant we find that:

$$(1+i)^n+(1-i)^n = 2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(-1)^k.$$

Q04B Let n be a non negative integer, to start the following into cases:

$$(1+i)^n+(1-i)^n=0 \iff n\equiv 2 \pmod 4$$

The first case we will consider is:

$$n\equiv 2 \pmod 4 \implies n, (1+i)^n+(1-i)^n=0$$

If we assume $n\equiv 2 \pmod 4$, this would means for some integer j we could also express n as:

$$n=4j+2$$

We can then plug this into our original equation:

$$\begin{aligned}(1+i)^n+(1-i)^n &= (1+i)^{4j+2}+(1-i)^{4j+2}\\ &= (1+i)^{4j}\cdot(1+i)^2+(1-i)^{4j}\cdot(1-i)^2\\ &= (-4)^j\cdot(1+i)^2+(-4)^j\cdot(1-i)^2\\ &= (-4)^j[1+2i+i^2+1-2i+i^2] \qquad = (-4)^j[0]=0\end{aligned}$$

This thus proves the first case.

To find the second case we will consider:

$$(1+i)^n+(1-i)^n=0 \implies n\equiv 2 \pmod 4$$

We know from the hint that $(1+i)^4 = (1-i)^4 = -4$, this means that we can consider n have 4 possibilities (corresponding with mod 4), let $k \in \mathbb{R}$:

$$n = 4k + 0, 4k + 1, 4k + 2, 4k + 3$$

If we plug it in we get the following table:

4k	4k + 1
$(1+i)^{4j} + (1-i)^{4j}$	$(1+i)^{4j} \cdot (1+i)^1 + (1-i)^{4j} \cdot (1-i)^1$
$(-4)^j + (-4)^j$	$(-4)^j \cdot (1+i)^1 + (-4)^j \cdot (1-i)^1$
Not Zero	$(-4)^j[(1+i) + (1-i)]$
	$(-4)^j[2]^1$
	Not Zero

4k+2	4k + 3
$(1+i)^{4j} \cdot (1+i)^2 + (1-i)^{4j} \cdot (1-i)^2$	$(1+i)^{4j} \cdot (1+i)^3 + (1-i)^{4j} \cdot (1-i)^3$
$(-4)^j \cdot (1+i)^2 + (-4)^j \cdot (1-i)^2$	$(-4)^j \cdot (1+i)^3 + (-4)^j \cdot (1-i)^3$
$(-4)^j[1 + 2i + i^2 + 1 - 2i + i^2]$	$(-4)^j[(1+i)^3 + (1-i)^3]$
$(-4)^j[0]$	$(-4)^j[-4]$
Zero	Not Zero

Thus showing that the only possible solution is when$n \equiv 2 \pmod 4$. Since both implications are proved the iff statement is also proved