Q03a We will prove the results by using strong induction on n, where $P(n)$ is the statement:

$$a_{n+2} = 6a_n$$

Base Case: When n is 0, the statement $P(0)$ is given by:

$$a_2 \equiv 6a_0 \pmod{19}$$

This is from definition and satifies the general equation, thus proving $P(0)$. When n is 1, the statement $P(1)$ is given by:

$$a_3 \equiv 6a_1 \pmod{19}$$

This is also from definition and satifies the general equation thus proving $P(1)$. When n is 2, the statement $P(2)$ is given by:

$$\begin{aligned} a_4 &= a_3 + a_2 \pmod{19} \\ &= (6a_1) + (6a_0) \pmod{19} \\ &= 6(a_1 + a_0) \pmod{19} \\ &= 6(a_2) \pmod{19} \end{aligned}$$

We have thus shown that when n is 2, it satifies the general equation thus proving $P(1)$. Inductive Hypothesis: Let k be a positive integer such that ($k \geq 2$). Assume for all integers i = 1,2,3,...,k, $a_{i+2} = 6a_i$. We wish to prove that when P(k+1), we will get:

$$a_{k+3} = a_{k+1} \pmod{19}$$

We know that from our starting definition P(k+1):

$$\begin{aligned} a_{(k+1)+2} &= a_{(k+1)+2-1}ax_{(k+1)+2-2} \pmod{19} \\ a_{k+3} &= a_{k+2} + a_{k+1} \pmod{19} \end{aligned}$$

From our Inductive Hypthoesis we assume P(k) and P(k-1) are true, this means also that $a_{k+2} = 6a_k$ (P(k)) and $a_{k+1} = 6a_k$ (P(k-1)) hold and are true. Thus our equation becomes (thanks to CAM):

$$\begin{aligned} a_{k+3} &= a_{k+2} + a_{k+1} \pmod{19} \\ &= 6a_k + 6a_{k-1} \pmod{19} \\ &= 6(a_k + a_{k-1)} \pmod{19} \\ &= 6a_{k+1} \pmod{19} \end{aligned}$$

Thus since both the base case and inductive step hold, it follows by the Principle of Strong Mathematical Induction (POSI) that the statement it true.

Q03b We know from the fundimental definition given in the question that:

$$\begin{aligned} a_3 &= a_2 + a_1 \pmod{19} \\ &= 6a_0 + a_1 \pmod{19} \end{aligned}$$

However we also know that:

$$a_3 = 6a_1 \pmod{19}$$

So combining the two we find that:

$$6a_1 \pmod{19} = 6a_0 - a_1 \pmod{19}$$

$$0 = 6a_0 - 5a_1 \pmod{19}$$

All the possible interger combinations are given by the table:

a_0	a_1	r	r (mod 19)
0	0	0	0
1	5	-19	0
2	10	-38	0
3	-15	-57	0
5	6	0	0
7	16	-38	0
11	17	-19	0
13	8	38	0
17	9	57	0

Notice that any composite number can be created by multiplying a_0 and a_1 by some integer but the $r \pmod{19}$ will still be zero. Therefore all the prime and composite numbers are in the solution set so therefore their are 19 integer pairs which act as the solution