

Q04A Let n be a non negative integer. We know from BT we know that we can express $(1 + i)^n$ and $(1 - i)^n$ as:

$$(1 + i)^n = \sum_{k=0}^n \binom{n}{k} (i)^k$$

$$(1 - i)^n = \sum_{k=0}^n \binom{n}{k} (-i)^k$$

Adding the two together we find that:

$$(1 + i)^n + (1 - i)^n = \sum_{k=0}^n \binom{n}{k} (i)^k + \sum_{k=0}^n \binom{n}{k} (-i)^k$$

$$= \sum_{k=0}^n \binom{n}{k} [-i^k + i^k]$$

Notice that we can thus split this sigma into two cases, when k is even and when k is odd. Lets start when k is odd, we know that we can express k as $2d + 1$ (where $d \in \mathbb{N}$), so our equation becomes:

$$\sum_{k=0}^n \binom{n}{k} [-i^k + i^k] = \binom{n}{2d+1} [-i^{2d+1} + i^{2d+1}]$$

$$= \binom{n}{2d+1} [(-i^2)^d (-i) + (i^2)^d (i)]$$

$$= \binom{n}{2d+1} [(-1)^d (-i) + (-1)^d (i)]$$

$$= \binom{n}{2d+1} [-(-1)^d (i) + (-1)^d (i)]$$

$$= \binom{n}{2d+1} [0]$$

$$= 0$$

This means that for each odd k , the value of the summation will be unchanged.

In the second case we will let k be even, it can thus be expressed as k as $2d$ (where $d \in \mathbb{N}$), so our equation becomes:

$$\begin{aligned}\sum_{k=0}^n \binom{n}{k} [-i^k + i^k] &= \binom{n}{2d} [-i^{2d} + i^{2d}] \\ &= \binom{n}{2d} [(-i^2)^d + (i^2)^d] \\ &= \binom{n}{2d} [(-1)^d + (-1)^d] \\ &= \binom{n}{2d} [2(-1)^d]\end{aligned}$$

We thus know that only even values will impact the summation, the equation for all the even values is:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} [2(-1)^k] \text{ or } 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} [(-1)^k]$$

And because odd values are redundant we find that:

$$(1+i)^n + (1-i)^n = 2 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k.$$

Q04B Let n be a non negative integer, to start we can split the following into cases:

$$(1+i)^n + (1-i)^n = 0 \iff n \equiv 2 \pmod{4}$$

The first case we will consider is:

$$n \equiv 2 \pmod{4} \implies n, (1+i)^n + (1-i)^n = 0$$

If we assume $n \equiv 2 \pmod{4}$, this would mean for some integer j we could also express n as:

$$n = 4j + 2$$

We can then plug this into our original equation:

$$\begin{aligned}(1+i)^n + (1-i)^n &= (1+i)^{4j+2} + (1-i)^{4j+2} \\ &= (1+i)^{4j} \cdot (1+i)^2 + (1-i)^{4j} \cdot (1-i)^2 \\ &= (-4)^j \cdot (1+i)^2 + (-4)^j \cdot (1-i)^2 \\ &= (-4)^j [1+2i+i^2 + 1-2i+i^2] &= (-4)^j [0] = 0\end{aligned}$$

This thus proves the first case.

To find the second case we will consider:

$$(1+i)^n + (1-i)^n = 0 \implies n \equiv 2 \pmod{4}$$

We know from the hint that $(1+i)^4 = (1-i)^4 = -4$, this means that we can consider n have 4 possibilities (corresponding with mod 4), let $k \in \mathbb{R}$:

$$n = 4k + 0, 4k + 1, 4k + 2, 4k + 3$$

If we plug it in we get the following table:

4k	4k + 1
$(1+i)^{4j} + (1-i)^{4j}$ $(-4)^j + (-4)^j$ Not Zero	$(1+i)^{4j} \cdot (1+i)^1 + (1-i)^{4j} \cdot (1-i)^1$ $(-4)^j \cdot (1+i)^1 + (-4)^j \cdot (1-i)^1$ $(-4)^j[(1+i) + (1-i)]$ $(-4)^j[2]^1$ Not Zero
4k+2	4k + 3
$(1+i)^{4j} \cdot (1+i)^2 + (1-i)^{4j} \cdot (1-i)^2$ $(-4)^j \cdot (1+i)^2 + (-4)^j \cdot (1-i)^2$ $(-4)^j[1 + 2i + i^2 + 1 - 2i + i^2]$ $(-4)^j[0]$ Zero	$(1+i)^{4j} \cdot (1+i)^3 + (1-i)^{4j} \cdot (1-i)^3$ $(-4)^j \cdot (1+i)^3 + (-4)^j \cdot (1-i)^3$ $(-4)^j[(1+i)^3 + (1-i)^3]$ $(-4)^j[-4]$ Not Zero

Thus showing that the only possible solution is when $n \equiv 2 \pmod{4}$. Since both implications are proved the iff statement is also proved.