Q05 To start we are given that:

$$\lim_{x \to 4} f(x) = 7$$

From our $\epsilon - \delta$ definition we can rewrite this as:

$$\text{if } 0 < |x-4| < \delta \text{ then } |f(x) - 7| < \epsilon$$

We can further expand out our equation for epsilon:

$$\begin{aligned} |f(x) - 7| &< \epsilon \\ 7 - \epsilon < f(x) &< 7 + \epsilon \\ 15 - \epsilon < f(x) + 8 &< 15 + \epsilon \\ |f(x) + 8| &< \epsilon + 15 \end{aligned}$$

So we could rewrite the given arbitrary equation as:

$$|f(x) + 8||f(x) - 7| < (15 + \epsilon)\epsilon$$

We can thus define a new epsilon:

$$\epsilon' = (15 + \epsilon)\epsilon$$

Which results in the previous equation becoming:

$$\begin{aligned} |f(x) + 8||f(x) - 7| &< \epsilon' \\ |(f(x))^2 + f(x) - 56| &< \epsilon' \end{aligned}$$

This implies from the $\epsilon - \delta$ definition that for $0 < |x - 4| < \delta$:

$$\begin{aligned} \lim_{x \to 4}[(f(x))^2 + f(x) - 56] &= 0 \\ \lim_{x \to 4}[(f(x))^2 + f(x) - 56] + 57 &= 0 + 57 \\ \lim_{x \to 4}[(f(x))^2 + f(x) + 1] &= 57 \end{aligned}$$