Q04 let n be an arbitrary positive integer and let k be a odd arbitrary integer $(1 \leq k \leq n)$, to start we will prove that the $\gcd(n, k)$ has no factors of 2.

Case 1 $(k = 1)$: By definition this means that:

$$\gcd(n, k) = \gcd(n, 1) = 1$$

Therefore if k = 1 then $\gcd(n, k)$ has no factors of 2.

Case 2 $(k > 1)$: We know from UFT that we can express both k and n in terms of their prime factors (g $\leq$ 1):

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_g^{\alpha_g}$$
$$k = p_1^{\beta_1} p_2^{\beta_2} \cdots p_g^{\beta_g}$$

where for p_i (for $0 \geq i \geq g$) represent the prime divisors of n where some of the exponents $(\alpha_i,\ \beta_i)$ could be zero. We also know that k will always be odd and that p_1 is 2, thus its equations become:

$$n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_g^{\alpha_g}$$
$$k = 2^0 p_2^{\beta_2} \cdots p_g^{\beta_g}$$

Applying GCD PF we find that:

$$k = 2^0 p_2^{\beta_2} \cdots p_g^{\beta_g}$$

Therefore if k = 1 then $\gcd(n, k)$ will be an odd integer

$$\gcd(n, k) = 2^{\min(0, \alpha_1)} p_2^{\min(\beta_2, \alpha_2)} \cdots p_g^{\min(\beta_g, \alpha_g)}$$
$$\gcd(n, k) = 2^0 p_2^{\min(\beta_2, \alpha_2)} \cdots p_g^{\min(\beta_g, \alpha_g)}$$

This means that 2 is not a factor of $\gcd(n, k)$ for all $k > 1$.

We know that n can be expressed as

$$n = \frac{n}{\gcd(n, k)} (\gcd(n, k))$$

Let s be the largest non negative integer such that

$$2^s | n \implies 2^s | \frac{n}{\gcd(n, k)} (\gcd(n, k))$$

We know that $\gcd(n, k)$ will never have a factor of 2, this implies that from GCD PF that

$$\gcd(2^s, \gcd(n, k)) = 1$$

Therefore from CAD we know that

$$2^s | \frac{n}{\gcd(n, k)}$$

We were given the definiton that

$$\frac{n}{\gcd(n,k)}|\binom{n}{k}$$

Since

$$2^s|\frac{n}{\gcd(n,k)} \text{ and } \frac{n}{\gcd(n,k)}|\binom{n}{k}$$

we thus know that TD applys and the equation becomes $2^s|\binom{n}{k}$ for all the every possible s, n and k.