Q03. Given that $d = gcd(m, n)$, we can also find that $d = \gcd(m, d)$ as $m|d$. Therefore we can say that

To start we will prove $(\{px + qy \colon x, y \in \mathbb{Z}\} = \mathbb{Z}) \implies (p \neq q)$, we will use proof by contrapositive, so the function becomes:

$$(p = q) \implies (\{px + qy \colon x, y \in \mathbb{Z}\} \neq \mathbb{Z})$$

If we assume the hypthoesis, the series:

$$(\{px + qy \colon x, y \in \mathbb{Z}\} \neq \mathbb{Z})$$

becomes:

$$(\{qx + qy \colon x, y \in \mathbb{Z}\} \neq \mathbb{Z})$$
$$(\{q(x + y) \colon x, y \in \mathbb{Z}\} \neq \mathbb{Z})$$

Notice that:

1. for every value of $q, 2 \leq q$ (as 1, 0 are not prime)
2. for every value of $x, y, (x + y)$ will be every integer

This means that $q(x+y)$ or $qx+qy$ will be every integer expect for 1 and -1. As such this means that it is a subset of $\mathbb{Z}$ but not equal to $\mathbb{Z}$, or that:

$$(\{qx + py \colon x, y \in \mathbb{Z}\} \subsetneq \mathbb{Z})$$

And so we can conclude that:

$$(\{qx + py \colon x, y \in \mathbb{Z}\}) \neq \mathbb{Z})$$

We have thus proved the first implication of the if and only if statement.

The second impication states that: $(p \neq q) \implies (\{px + qy \colon x, y \in \mathbb{Z}\} = \mathbb{Z})$. Since p and q are unique primes such that $d = \gcd(p, q) = 1$, we can find that:

$$\exists d, a, b \in \mathbb{Z}\,, pa + bq = d = 1$$

We know since $x = ad$ and $y = bd$, that $px + qy$ becomes:

$$\begin{aligned} px + qy &= p(ad) + q(bd) \\ &= pad + qbd \\ &= d(ap + qb) \end{aligned}$$

As $d = 1$ this will become

$$\exists a, b \in \mathbb{Z}\,, (ap + qb)$$

Since a and b can be any integer $(ap + qb)$ will equal every integer, this means that $(ap + qb) = \mathbb{Z}$. Note that $(ap + qb)$ will have the same domain as $px + qy$ which means that $\{px + qy \colon x, y \in \mathbb{Z}\} = \mathbb{Z}$, thus proving the second implication.

Therefore since the first and second implication is true, we have proved the if and only if statement.