Q03a We know that $f'(x)$ is defined for all $x \in \mathbb{R}$, and we also know that differentiability applies continuity for $f(x)$. This means that f(x) is both continuous and differentiable for all $x \in \mathbb{R}$.
Therefore we can use the bounded derivative theorem (BDT) to find the smallest interval of f(6), and since we know that:

$$1 \leq f'(x) \leq 5 \text{ for every } x \in (3, 6)$$

Therefore by BDT this implies:

$$\begin{aligned} f(3) + 1(x-3) \leq f(x) &\leq f(3) + 5(x-3) \\ 1 + 1(x-3) \leq f(x) &\leq 1 + 5(x-3) \\ 1 + x - 3 \leq f(x) &\leq 1 + 5x - 15 \\ x - 2 \leq f(x) &\leq 5x - 14 \end{aligned}$$

Therefore if we let x = 6, we find that:

$$\begin{aligned} 6 - 2 \leq f(6) &\leq 30 - 14 \\ 4 \leq f(6) &\leq 16 \end{aligned}$$

We can thus say that the smallest interval that $f(6)$ is bounded by is [4,16]

Q03b The textbook defines the second order Taylor Polynomial as (for some real x = a):

$$T_{2,a} = \sum_{k=0}^{2} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If we expand this out we get that:

$$T_{2,a} = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$$

Since we are finding this polynomial centered at x = 3, the polynomial becomes:

$$T_{2,3} = \frac{f(3)}{0!}(x-3)^0 + \frac{f'(3)}{1!}(x-3)^1 + \frac{f''(3)}{2!}(x-3)^2$$

From definition we know that $f(3) = 1$, $f'(3) = 5$ and $f''(3) = 7$, so we can simplify:

$$\begin{aligned} T_{2,3} &= \frac{1}{0!} + \frac{5}{1!}(x-3)^1 + \frac{7}{2!}(x-3)^2 \\ T_{2,3} &= 1 + 5(x-3) + \frac{7}{2}(x^2 - 6x + 9) \\ T_{2,3} &= 1 + 5x - 15 + \frac{7x^2}{2} + \frac{42x}{2} + \frac{63}{2} \end{aligned}$$