$f_1(x)$ is a piecewise version of $f(x)$ with the following definition:

$$f_1(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \{3 \le x \le 4\} \\ -\frac{1}{2}x + & \{4 < x < 6\} \\ \frac{1}{2}x + \frac{1}{2} & \{6 \le x \le 7\} \end{cases}$$

lets test if the conditions hold for $f_1(x)$:

1) $f_1(3) = 2$ and $f_2(7) = 4$

Proof: $f_1(3) = \frac{3}{2} + \frac{1}{2} = 2$

$f_1(7) = \frac{7}{2} + \frac{1}{2} = 4$

2) $f_1(x)$ achieves each value on $[3, 7]$ once

We know that if $f_1(x)$ is on the domain $[3, 4]$ each value will be unique and the maximum value is 2.5. We also know that if $f_1(x)$ is on the domain $\{6 \le x \le 7\}$, then each value is unique and the minimum value is 3.5. Note that this is from our proof of $f_1(x)$.

If $f_1(x)$ is on the domain $(4, 6)$, we will have a linear decreasing function. This means for $a, b \in \mathbb{R}$ if

if $4 < a < b < 6$, $\Rightarrow 2.5 < f_1(b) < f_1(a) < 3.5$

if $4 < b < a < 6$, $\Rightarrow 2.5 < f_1(a) < f_1(b) < 3.5$

$\therefore$ As long as $a \neq b$, both $f_1(a)$ and $f_2(b)$ will be values that only exist once as if they are in the domain $(4, 6)$ then they will always be greater than values $f_1(x)$ for $[3, 4]$ and less than values $f_1(x)$ for $[6, 7]$

$f_1(x)$ is discontinuous at 4 and 6 as:

$$\lim_{x \to 4} f_1(x) = \lim_{x \to 4^+} f_1(x) = \lim_{x \to 4^-} f_1(x)$$

$$-\frac{4}{2} + \frac{11}{2} \qquad = \frac{4}{2} + \frac{1}{2}$$

$$3.5 \neq 2.5 \leftarrow \lim_{x \to 4} f_1(x) = DNE$$

$$\lim_{x \to 6} f_1(x) = \lim_{x \to 6^+} f_1(x) = \lim_{x \to 6^-} f_1(x)$$

$$-\frac{6}{2} + \frac{11}{2} \qquad = \frac{6}{2} + \frac{1}{2}$$

$$2.5 \neq 3.5 \leftarrow \lim_{x \to 6} f_1(x) = DNE$$

$\therefore$ $f_1(x)$ is a discontinuous function where $f_1(3) = 2$ and $f_1(7) = 4$ and every value $[2, 4]$ is achieved once in the domain $[3, 7]$. $f(x)$ does not need to be continuous.