1b) According to the standard definition, if $\lim_{n\to\infty} a_n = L$ then

$$|a_n - L| < \epsilon \qquad (\epsilon > 0)$$

$\Rightarrow |(a_n + 2) - L - 2| < \epsilon$ ① add -2 and $+2$

$\Rightarrow |(a_n + 2) - (L + 2)| < \epsilon$ ② Sub $x_n = a_n + 2$ and $M = L + 2$

$\Rightarrow |x_n - M| < \epsilon$

- We will now check if $\left|\frac{1}{x_n} - \frac{1}{M}\right| < \epsilon$ is a valid statement

$\Rightarrow \left|\frac{1}{x_n} - \frac{1}{M}\right| < \epsilon$

$\Rightarrow \left|\frac{M - x_n}{x_n M}\right| = \frac{1}{|x_n||M|}|M - x_n|$ ③ distribute the absolute value

$\Rightarrow \frac{1}{|x_n||M|}|M - x_n| < \frac{2}{|M||M|}|M - x_n|$

* This is from subbing out defined equation Ⓑ

$\frac{2}{|M||M|}|M - x_n| < \frac{2}{|M||M|} \cdot \frac{M^2}{2}\epsilon$

* This is from subbing out defined equation Ⓐ

④ Defined equations:

Ⓐ $|M - x_n| < \epsilon_3 \quad \left(\epsilon_3 = \frac{M^2}{2}\epsilon\right)$

Triangle Inequality

$|x_n - M| < \epsilon_2 \quad \left(\epsilon_2 = \frac{|M|}{2}\right)$

$||x_n| - |M|| < |x_n - M| < \epsilon_2$

$||x_n| - |M|| < \epsilon_2$

$-\frac{|M|}{2} < |x_n| - |M| < \frac{|M|}{2}$

$\frac{|M|}{2} < |x_n| < \frac{3|M|}{2}$

Ⓑ $\left|\frac{1}{x_n}\right| < \frac{2}{|M|}$

- All together

$$\left|\frac{1}{x_n} - \frac{1}{M}\right| = \frac{1}{|x_n||M|}|M - x_n| < \frac{2}{|M||M|}|M - x_n| < \frac{2}{|M||M|} \cdot \frac{M^2}{2}\epsilon = \epsilon$$

$\Rightarrow \left|\frac{1}{x_n} - \frac{1}{M}\right| < \epsilon$ ⑤ Sub ϵ_4 in $\left(\epsilon_4 = \frac{\epsilon}{5}\right)$

$\Rightarrow \left|\frac{1}{x_n} - \frac{1}{M}\right| < \epsilon_4 = \frac{\epsilon}{5}$

$\Rightarrow \left|\frac{5}{x_n} - \frac{5}{M}\right| < \epsilon$

- From ② we sub back x_n and M:

$\Rightarrow \left|\frac{5}{a_n + 2} - \frac{5}{L + 2}\right| < \epsilon$

$\therefore$ By our the primary definition of the limit this means:

$$\lim_{n\to\infty} \frac{5}{a_n + 2} = \frac{5}{L + 2}$$