Q02a Let b be a real number. We know from the "Critical Point" Theorem (Page 202) that a critical point will happen at some c when:

$$f'(c) = 0 \text{ or } f'(c) = DNE$$

Thus in order to solve we must first find the first derivative.

$$\begin{aligned}
f(x) &= x^{\frac{1}{3}} + bx^{\frac{4}{3}} \\
\frac{dy}{dx}(f(x)) &= \frac{dy}{dx}(x^{\frac{1}{3}} + bx^{\frac{4}{3}}) \\
f'(x) &= \frac{dy}{dx}(x^{\frac{1}{3}}) + \frac{dy}{dx}(bx^{\frac{4}{3}}) \\
f'(x) &= \frac{1}{3}x^{(\frac{1}{3}-1)} + b\frac{4}{3}x^{(\frac{4}{3}-1)} \\
f'(x) &= \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}}
\end{aligned}$$

Since we can see an x is in the denominator for the first term, this would imply that f'(x) is undefined at 0 or:

$$f'(0) = DNE$$

Since there is only one possible value such that $f'(c) = DNE$, we will now try to solve $f'(c) = 0$, starting by setting f'(x) to be zero:

$$\begin{aligned}
0 &= \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}} \\
-\frac{1}{3x^{\frac{2}{3}}} &= b\frac{4}{3}x^{\frac{1}{3}} \\
-\frac{1}{3x^{\frac{2}{3}}}x^{\frac{2}{3}} &= b\frac{4}{3}x^{\frac{1}{3}}x^{\frac{2}{3}} \\
-\frac{1}{3} &= b\frac{4}{3}x \\
-\frac{1}{4b} &= x
\end{aligned}$$

Therefore our possible critical points will be when:

$$x = 0 \text{ or } -\frac{1}{4b}$$

Q02b We know we have two possible values to test if a local minimum can exist, 0 and $-\frac{1}{4b}$ (for some real number b). This is because minimum values points can only exist at critical points

We know that f(x) is continuous as if you replace x with the $z = x^1/3$ (which is continuous), you get:

$$z + bz^4$$

Which is also continuous as its a polynomia,l thus $f(x)$ is a continuous. Therefore in order to see if 0 in a minimum we shall use "The First Derivative Test" (Page 223), we will use the interval $(0^-, 0^+)$ to check if:

$$f'(0^-) < f'(x) < 0 \text{ for all } x \in (0^-, 0)$$
$$f'(0^+) > f'(x) > 0 \text{ for all } x \in (0, 0^+)$$

Evaluating the first equation we find a contradiction at $f'(0^-)$:

$$\begin{aligned}
f'(0^-) &< f'(x) < 0 \\
\frac{1}{3(0^-)^{\frac{2}{3}}} + b\frac{4}{3}(0^-)^{\frac{1}{3}} &< f'(x) < 0 \\
\frac{1}{3(0^+)} + b\frac{4}{3}(0^-) &< f'(x) < 0 \\
\frac{1}{(0^+)} + (0^-) &< f'(x) < 0 \\
\infty + (0^-) &< f'(x) < 0 \\
\infty &< f'(x) < 0
\end{aligned}$$

Because this contradicts the First Derivative Test this means that 0 can not be a local minimum.
Moving on we can test the critical point $-\frac{1}{4b}$ using the "Second Derivative Test" (we can use this as f'($-\frac{1}{4b}$) = 0). But before we do this we must first find the second derivative:

$$\begin{aligned}
f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} + b\frac{4}{3}x^{\frac{1}{3}} \\
\frac{dy}{dx}f'(x) &= \frac{dy}{dx}(\frac{1}{3}x^{-\frac{2}{3}}) + \frac{dy}{dx}(b\frac{4}{3}x^{\frac{1}{3}}) \\
f''(x) &= \frac{1}{3}\frac{-2}{3}x^{-\frac{5}{3}} + b\frac{4}{3}\frac{1}{3}x^{-\frac{2}{3}} \\
f''(x) &= \frac{-2}{9}x^{-\frac{5}{3}} + b\frac{4}{9}x^{-\frac{2}{3}}
\end{aligned}$$

Now we will plug in the critical point (c) and simplify:

$$f''(c) = \frac{-2}{9}(-\frac{1}{4b})^{-\frac{5}{3}} + b\frac{4}{9}(-\frac{1}{4b})^{-\frac{2}{3}}$$
$$f''(c) = \frac{2}{9}(-\frac{1}{4b})^{-\frac{2}{3}}(-(-\frac{1}{4b})^{-1} + 2b)$$
$$f''(c) = \frac{2}{9}(-4b)^{\frac{2}{3}}(-(-4b)^{1} + 2b)$$
$$f''(c) = \frac{2}{9}\sqrt[3]{(16b^2)}(6b)$$

The "Second Derivative Test" tells us that a minimum will exist at the critical point if:

$$f''(c) > 0$$

As we can see from the above equation, the positivity is of the equation is solely reliant on the $6b$ term as (b^2 will always be positive). Therefore $f'(c) > 0$ if b is also positive. In other words the minimum will exist as along as:

$$b > 0$$

Therefore we have shown that the only possible local minimum exists when b is positive and the minimum will exist at:

$$x = -\frac{1}{4b}$$

Q02c We know that intervals (of increase and decrease) will only change sign values critical points, therefore our possible intervals are :

$(-\infty, -\frac{1}{4b})$	$-\frac{1}{4b}$	$(-\frac{1}{4b}, 0)$	0	$(0, \infty)$
	0	+	Undefined	

These results were gotten from the previous question, to start solving for the rest we will find the first derivative at negative infinity:

$$f'(-\infty) = \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}} = \frac{1}{3(-\infty)^{\frac{2}{3}}} + b\frac{4}{3}(-\infty)^{\frac{1}{3}} \approx 0 - \infty$$

Thus the interval $(-\infty, -\frac{1}{4b})$ will be negative, moving onto positive infinity we know that:

$$f'(\infty) = \frac{1}{3x^{\frac{2}{3}}} + b\frac{4}{3}x^{\frac{1}{3}} = \frac{1}{3(\infty)^{\frac{2}{3}}} + b\frac{4}{3}(\infty)^{\frac{1}{3}} \approx 0 + \infty$$

Thus the interval $(0, \infty)$ is positive, so we will have the intervals of increase (+) and decrease (-) given by:

$(-\infty, -\frac{1}{4b})$	$-\frac{1}{4b}$	$(-\frac{1}{4b}, 0)$	0	$(0, \infty)$
-	0	+	Undefined	+

In order to find the points of concavity we must first find the inflection points, which are determined by:

$$f''(x) = 0 \text{ or } f''(x) = DNE$$

We know from our Part B that our equation for $f''(x)$ is:

$$f''(x) = \frac{-2}{9x^{\frac{5}{3}}} + b\frac{4}{9x^{\frac{2}{3}}}$$

Thus we can tell that this will be indeterminate when x = 0, so this means that x=0 is one of the inflection points. Setting f"(x) = 0 we find that:

$$\begin{aligned} f''(x) &= \frac{-2}{9x^{\frac{5}{3}}} + b\frac{4}{9x^{\frac{2}{3}}} \\ 0 &= \frac{-2}{9x^{\frac{5}{3}}} + b\frac{4}{9x^{\frac{2}{3}}} \\ \frac{-2}{9x^{\frac{5}{3}}} &= -b\frac{4}{9x^{\frac{2}{3}}} \\ -2 &= -b \cdot 4 \cdot x \\ x &= \frac{1}{2b} \end{aligned}$$

And thus our last inflection point will be when $x = \frac{1}{2b}$. When the second derivative is positive we know the function will be concave up (and the opposite for concave down), so we can create a table in order to show the concavity:

$(-\infty, 0)$	0	$(0, \frac{1}{2b})$	$\frac{1}{2b}$	$(\frac{1}{2b}, \infty)$
	undefined		0	

We know that 0 is undefined from what we saw previously and that $\frac{1}{2b}$ is zero, to start solving the rest we will consider negative infinity:

$$f''(-\infty) = \frac{-2}{9x^{\frac{5}{3}}} + b\frac{4}{9x^{\frac{2}{3}}} = \frac{-2}{9(-\infty)^{\frac{5}{3}}} + b\frac{4}{9(-\infty)^{\frac{2}{3}}} \approx 0^{+} + 0^{+}$$

Thus the interval $(-\infty, 0)$ will be positive. We will use $\frac{1}{2b}^{-}$ to check the next interval:

$$f''(\frac{1}{2b}^{-}) = \frac{-2}{9x^{\frac{5}{3}}} + b\frac{4x}{9x^{\frac{5}{3}}} = \frac{-2+4bx}{9x^{\frac{5}{3}}} = \frac{-2+4b\frac{1}{2b}^{-}}{9(\frac{1}{2b}^{-})^{\frac{5}{3}}} \approx -2 + 2^{-}$$

Since the denominator is positive it is redundant, we can see that the numerator is negative 2 plus a number a little less then 2 so the interval $(0, \frac{1}{2b})$ is negative. We will use $\frac{1}{2b}^-$ to check the next interval:

$$f''(\frac{1}{2b}^+) = \frac{-2}{9x^{\frac{5}{3}}} + b\frac{4x}{9x^{\frac{5}{3}}} = \frac{-2+4bx}{9x^{\frac{5}{3}}} = \frac{-2+4b\frac{1}{2b}^+}{9(\frac{1}{2b}^+)^{\frac{5}{3}}} \approx -2+2^+$$

Thus the interval $(\frac{1}{2b}, \infty)$ will be positive, so the finalized concave up and concave down table will look like:

$(-\infty, 0)$	0	$(0, \frac{1}{2b})$	$\frac{1}{2b}$	$(\frac{1}{2b}, \infty)$
+	undefined	-	0	+

Combining our interval of increase/decrease table and our concavity table we get:

	$(-\infty, -\frac{1}{4b})$	$-\frac{1}{4b}$	$(-\frac{1}{4b}, 0)$	0	$(0, \frac{1}{2b})$	$\frac{1}{2b}$	$(\frac{1}{2b}, \infty)$
f'(x)	-	0	+	undefined	+	+	+
f"(x)	+	+	+	undefined	-	0	+

So in summery the intervals and inflection points are:

$$\begin{cases} \text{Interval of Increase)} \quad (-\frac{1}{4b}, 0) \cup (0, \infty) \\ \text{Interval of Decrease)} \quad (-\infty, -\frac{1}{4b}) \\ \text{Interval of Concave Up)} \quad (-\infty, 0) \cup (\frac{1}{2b}, \infty) \\ \text{Interval of Concave Down)} \quad (0, \frac{1}{2b}) \\ \text{Inflection Points)} \quad x = 0, \frac{1}{2b} \end{cases}$$