Q02 Let a be a positive integer, note that any a can be expressed as:

$$a = a_{rest}(10) + a_0$$

where a_0 is the last digit and a_{rest} is all the other digits. If we take the apply (mod 10) we thus get:

$$\begin{aligned} a \pmod{10} &\equiv a_{rest}(10) + a_0 \pmod{10} \\ &\equiv a_{rest}(0) + a_0 \pmod{10} \\ &\equiv a_0 \pmod{10} \end{aligned}$$

Thus we have shown that for any real number a, the last digit of a will be equal to a (mod 10). Since it has the same functionality as D(a) (as a is a positive integer) we can write D(a) as:

$$D(a) = a_0 \pmod{10}$$

From this, we can see that for an arbitrary natural m our original equation will become:

$$D(19^m + \sum_{k=1}^{m} k!) = 19^m + \sum_{k=1}^{m} k! \pmod{10}$$

CAM tells us this equation can be split into two parts:

1. $19^m \pmod{10}$
2. $\sum_{k=1}^{m} k! \pmod{10}$

Starting with [1] we can see that for every natural m, the equation becomes:

$$\begin{aligned} 19^m \pmod{10} &= 9^m \pmod{10} \\ &= 9^{m-1}(9) \pmod{10} \end{aligned}$$

If we apply CAM, this asserts that the last digit of 9^m is equal to the last digit of $9^{m-1}9$, Plugging in 1 (odd) and 2 (even) for m we find that:

$$(m = 1) : (9^{1-1})(9) \pmod{10} \implies (1)(9) \pmod{10} \implies 9$$

$$(m = 2) : (9^{2-1})(9) \pmod{10} \implies (9)(9) \pmod{10} \implies 1$$

Lets split m into two cases, if m is even then by definition an integer a exists such that $2a = m$. Our equation thus becomes:

$$\begin{aligned} 9^m \pmod{10} &= (9^2a) \pmod{10} \\ &= (9^2)^a \pmod{10} \\ &= (1)^a \pmod{10} \\ &= 1 \pmod{10} \end{aligned}$$

if m is odd then by definition an integer a exists such that $2a = m - 1$. Our equation thus becomes:

$$\begin{aligned} 9^m \pmod{10} &= 9^{m-1}(9) \pmod{10} \\ &= 9^{2a}(9) \pmod{10} \\ &= (9^2)^a(9) \pmod{10} \\ &= 1^a(9) \pmod{10} \\ &= 9 \pmod{10} \end{aligned}$$

Thus we can see, if m is odd then [1] is 9. On the other hand if m is even then [1] is 1.

Moving on to [2], we will again split the equation into cases, if m is greater then 5 we see the equation will become:

$$\sum_{k=1}^{m} k! \pmod{10} \equiv 1! + 2! + 3! + 4! + 5! + ... + m! \pmod{10}$$

Notice however that for any $(k \geq 5)$, that k! (because of CAM) will be equal to:

$$\begin{aligned} k! &= 1 \times 2 \times 3 \times 4 \times 5 \times ... \times (k-1) \times (k) \pmod{10} \\ &= (2 \times 5)(1 \times 3 \times 4 \times ... \times (k-1) \times (k)) \pmod{10} \\ &= (10)(1 \times 3 \times 4 \times ... \times (k-1) \times (k)) \pmod{10} \\ &= (0)(1 \times 3 \times 4 \times ... \times (k-1) \times (k)) \pmod{10} \\ &= 0 \end{aligned}$$

Thus for any $(m \geq 5)$ our equation will become:

$$\begin{aligned} \sum_{k=1}^{m} k! \pmod{10} &\equiv 1! + 2! + 3! + 4! + 5! + ... + m! \pmod{10} \\ &\equiv 1! + 2! + 3! + 4! + 0 + ... + 0 \pmod{10} \\ &\equiv 1! + 2! + 3! + 4! \pmod{10} \end{aligned}$$

Thus we have shown that if $(m \geq 5)$, the last digit will be equal to the last digit when m = 4. Thus the only possible unique solutions can exists when $(m = 1)$, $(m = 2)$, $(m = 3)$, $(m = 4)$:

$$(m = 1) : \sum_{k=1}^{1} k! \pmod{10} \equiv 1! \pmod{10} \equiv 1 \pmod{10}$$

$$(m = 2) : \sum_{k=1}^{2} k! \pmod{10} \equiv 1! + 2! \pmod{10} \equiv 3 \pmod{10}$$

$$(m = 3) : \sum_{k=1}^{3} k! \pmod{10} \equiv 1! + 2! + 3! \pmod{10} \equiv 9 \pmod{10}$$

$$(m=4): \sum_{k=1}^{4} k! \pmod{10} \equiv 1! + 2! + 4! \pmod{10} \equiv 3 \pmod{10}$$

Thus we have shown that [2] will have unique solutions when m is 1,2,3 or 4.
Therefore we have 6 possible unique combinations which correspond to possible last digits:

$$(\text{m} = 1, \text{ where [1] is odd and [2]} < 5): 19^1 + \sum_{k=1}^{1} k! \pmod{10} \equiv 9+1 \pmod{10} \equiv 0 \pmod{10}$$

$$(\text{m} = 2, \text{ where [1] is even and [2]} < 5): 19^2 + \sum_{k=1}^{2} k! \pmod{10} \equiv 1+3 \pmod{10} \equiv 4 \pmod{10}$$

$$(\text{m} = 3, \text{ where [1] is odd and [2]} < 5): 19^3 + \sum_{k=1}^{3} k! \pmod{10} \equiv 9+9 \pmod{10} \equiv 8 \pmod{10}$$

$$(\text{m} = 4, \text{ where [1] is even and [2]} < 5): 19^4 + \sum_{k=1}^{4} k! \pmod{10} \equiv 1+3 \pmod{10} \equiv 4 \pmod{10}$$

$$(\text{m} = 5, \text{ where [1] is odd and [2]} \geq 5): 19^5 + \sum_{k=1}^{5} k! \pmod{10} \equiv 9+3 \pmod{10} \equiv 2 \pmod{10}$$

$$(\text{m} = 6, \text{ where [1] is even and [2]} \geq 5): 19^6 + \sum_{k=1}^{6} k! \pmod{10} \equiv 1+3 \pmod{10} \equiv 4 \pmod{10}$$

Since this exausts every possible combination for the natural m, only the elements [0,2,4,8] willl be contained in the set S:

$$S = \left\{ D\left(19^m + \sum_{k=1}^{m} k!\right) : m \in \mathbb{N} \right\}$$