Q02A If our we set [a] to be [7] our system of equations for $\mathbb{Z}_{12}$ and multiply the first equation by 5 and the second by 3 we get:

$$\begin{cases} [35][x] + [15][y] & \equiv [5](1) \\ [6][x] + [15][y] & \equiv [-3](2) \end{cases}$$

Subtracting equation (1) by equation (2) we will get the new equation: [29][y] = [8]. Note that [29] = [5] our equation will become:

$$[5][x] \equiv [8]$$

We know that [5] and 12 are co-prime, and thus by INV (with integers) we know that [5] will have a mathimatical inverse, thus if we multiply both sides we will get:

$$[5]^{-1}[5][y] \equiv [8][5]^{-1}$$

By definition $[5]^{-1}[5] = 1$ and we know that $[5][5] = [1] \pmod{12}$ and thus the mathimatical inverse of [5] is [5]. Our equation thus becomes:

$$[x] \equiv [40] \equiv [4]$$

Plugging this into equation (2) we get that:

$$\begin{aligned} [6][x] + [15][y] &\equiv [-3] \\ [15][y] &\equiv [-3] - [6][4] \\ [3][y] &\equiv [-3] - [0] \\ [3][y] &\equiv [9] \end{aligned}$$

We know this condition will be satisfied if [y] = [3] and thus our solution set is given by both:

$$\{x \in \mathbb{Z} : x \equiv 4 \pmod{12}\}$$

$$\{y \in \mathbb{Z} : y \equiv 3 \pmod{12}\}$$

Q02B To start we will mutliply the first equation by 5 and the second by 3 so we will get:

$$\begin{cases} [5][a][x] + [15][y] & \equiv [5](1) \\ [6][x] + [15][y] & \equiv [-3](2) \end{cases}$$

substracting equation (1) by equation (2) we will get the new equation:

$$\begin{aligned} [5][a][x] - [6][x] &\equiv [8] \\ [x]([5][a] - [6]) &\equiv [8] \end{aligned}$$

Note that we cancelled out the y's, this implies that number of possible solution set pairs is limited by the possible [x] solutions. Let [b] = [5][a] - [6]:

$$[b][x] \equiv [8]$$

MAT tells us a solution [x] will only exist if $d = \gcd(b, 12)$ such that $d|8$. MAT also tells us that for a solution (x_0) the set of solutions is given by:

$$[x_0], [x_0 + \frac{m}{d}], [x_0 + 2\frac{m}{d}]...[x_0 + (d-1)\frac{m}{d}]$$

In other words this shows us that the number of possible solutions for [x] will be equal to d. Therefore all congruence classes that have 2 solutions will have $d = 2$, this can only happen in these cases:

$$\begin{cases} 2 \equiv \gcd(b, 12) \equiv \gcd(2, 12) \\ 2 \equiv \gcd(b, 12) \equiv \gcd(10, 12) \end{cases}$$

Thus we see that b has to equal either [2] or [10], solving for [2] we find that:

$$\begin{aligned} [5][a] - [6] &\equiv [b] \\ [5][a] &\equiv [2] - [6] \\ [5][a] &\equiv [8] \end{aligned}$$

We know from the previous question that the mathimatical inverse of [5] is [5] so if we multiply both sides by the inverse:

$$\begin{aligned} [5]^{-1}[5][a] &\equiv [8][5]^{-1} \\ [a] &\equiv [8][5] \\ [a] &\equiv [4] \end{aligned}$$

If we know solve for the second case, when b is [10] we get:

$$\begin{aligned} [5][a] - [6] &\equiv [b] \\ [5][a] &\equiv [10] - [6] \\ [5][a] &\equiv [4] \\ [5]^{-1}[5][a] &\equiv [4][5]^{-1} \\ [a] &\equiv [8][5] \\ [a] &\equiv [8] \end{aligned}$$

Thus we know there are only 2 [x] solutions when [a] is [8] or [2], we also know there are only going to be 2 solution pairs ([x],[y]) when [a] is [8] or [2]

Q02C We know from the previous equations that the equation for a possible [x] value will be:

$$[x]([5][a] - [6]) \equiv [8]$$

We also know that we want to find an [a] such that [x] has 6 solutions. MAT says this can only happen if

$$d = \gcd(([5][a] - [6]), 12) = 6$$

This is impossible as MAT tells us that a solution can only happen when $d \mid 8$ and since $6 \not| 8$, we know that no [a] exists such that there are 6 solution pairs ([x],[y]).