"What is the Most Effective Formation of Pikemen?"

Words: 3000 (approximately)

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Introduction:

I love videogames, especially a game called Napoleon Total War. In fact, I managed to log almost 2000 hours of playtime[1] on it over a two-year period. The game is classified as a real time strategy game, where you organize formations of your soldiers to fight the enemy's formations. There are two main types of formations; the first type is ranged combat where soldiers will stay in orderly lines firing at each other from a distance (see Figure 1 & 2)

Figure 1: A battle layout, with the two opposing forces facing off against each other in ranged formation.

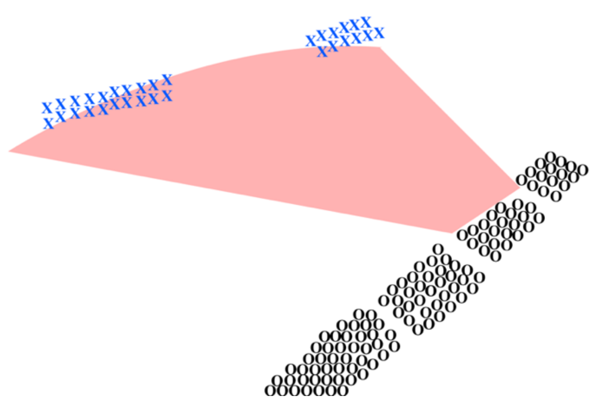


Figure 2: A representation of the same battle, where the red sector shows the range of fire of the highlighted unit above.

Ranged combat is relatively straightforward to model. More challenging is how to model melee combat, which is where soldiers fight in close proximity (See Figure 3 & 4 below).

[1] https://steamcommunity.com/profiles/76561198023367027/games/?tab=all

Figure 3: A blue coat squadron fighting in melee combat with bayonets against the approaching red coat army.

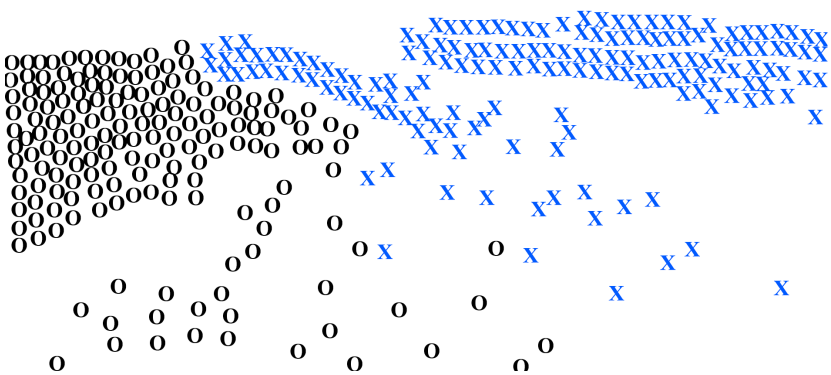


Figure 4: A representation of the same battle, illustrating how the opposing forces are now intermingled.

In melee combat every solider operates as a largely independent unit, attacking their immediate neighbors. Pikemen are my favorite unit for melee combat because the superior length of their pikes gives them an advantage over soldiers armed with conventional bayonets. This means that pikemen in the second and third rank are able to attack the first rank of enemy soldiers. In skirmishes, one unit of pikemen can take on three units of normal soldiers in melee!

This made me wonder about the most effective formation for pikemen. There are three main formations: a spread-out formation (where soldiers stand far apart), a staggered formation (where every even row of soldiers is one soldiers ahead of the odd rows), and the normal formation (where each soldier stands side by side). I decided that I would build a model to determine which formation would be most effective as measured by the number of pikemen left alive after the battle was over.

Modeling each Formation

In order to determine the efficiency of each military formation and compare them, we will be using Lancaster's square law[2] which states that:

$$\frac{dA}{dt} = \beta B \tag{1}$$

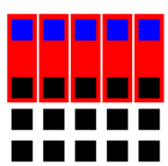
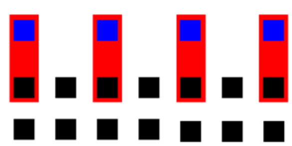
$$\frac{dB}{dt} = \alpha A \tag{2}$$

Where A represents the number of soldiers in the pikemen formation, α represents the fire power of each pikeman. To contrast this, B represents the number of soldiers that the enemy has, and β represents the enemy's fire power. For our model, α and β are described as:

$\alpha =$ The number of pikemen able to attack each enemy $\{\alpha \in \mathbb{R} \mid 0 \leq \alpha \leq 12\}$

$\beta =$ The number of enemies able to attack each pikeman $\{\beta \in \mathbb{R} \mid 0 \leq \beta \leq 2\}$

For the sake of simplicity in our model, we will assume that the enemy soldiers have only short-range weapons (such as a sword), and they can only attack the soldier immediately in front of them. Figure 3 and Figure 4 show the difference in the enemies' offensive firepower (β) as the formation of the pikemen gets expanded.

		■ = Enemy Soldier (Sword) ■ = Combat Zone of 2 Soldiers ■ = Pike Man (Pike)
Figure 3: $\beta = 1$, as each enemy is able to attack one pikeman	**Figure 4:** $\beta = 0.5$; each enemy is able to attack every other pikeman	**Legend:** This legend is universal across all the figures in this paper.

The calculation of α (how many of the enemy each pikeman can reach) is much more complicated[3]. α is clearly greater than the number of pikemen who have enemy soldiers directly

[2] Jaiswal N.K. (1997) Homogeneous Combat Models

[3] https://ocw.mit.edu/courses/mathematics/18-01sc-single-variable-calculus-fall-2010/unit-4-techniques-of-integration/part-a-trigonometric-powers-trigonometric-substitution-and-completing-the-square/session-70-preview-of-trig-substitution-and-polar-coordinates/MIT18_01SCF10_Ses70a.pdf

in front of them because a pikeman in the second row has enough range in his weapon to also attack enemies. In fact, a pikeman can attack an enemy if the following equation is satisfied:

$$\text{Enemy Position} - \text{Pikeman Position} \leq \text{Length of the Pike/Spear}$$

Since the pikeman can operate across 2 dimensions, the equation becomes:

$$(PP_{row} - EP_{row})^2 + (PP_{col} - EP_{col})^2 \leq ROS^2 \tag{3}$$

where PP_{row} is the position in the row of the pikeman, EP_{row} is the position in the row of the enemy. PP_{col} is the position in the column of the pikeman, EP_{col} is the position in the column of the enemy. In Figure 5, the black squares (pikemen) which intersect the red circular radius satisfies equation (1). Whereas, the black square (pikemen) outside the red circle do not satisfy (3). Lastly, ROS represents the radius of the pikeman's spear.

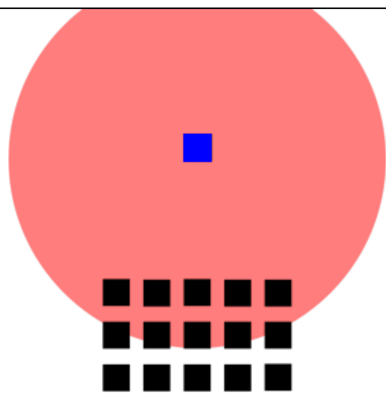


Figure 5: Positions in the formation where pikemen are able to reach the enemy

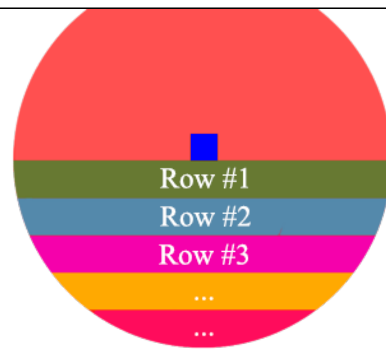


Figure 6: The areas in which the enemy can be attacked, separated by row.

If the enemy is located at (0,0), the function for determining if a pikeman is within attack range is:

$$(PP_{row} - 0)^2 + (PP_{col} - 0)^2 \leq ROS^2$$

We notice that this formula is the same as the formula for all points in a circle centered around (0,0).

$$(PP_{row})^2 + (PP_{col})^2 \leq ROS^2$$

Now that we have a general equation, we need to solve for area of this "radius of attack". We need this to find the number of enemy soldiers that each pikeman can attack:

$$(PP_{col})^2 \leq ROS^2 - (PP_{row})^2$$
$$PP_{col} \leq \sqrt{ROS^2 - (PP_{row})^2} \tag{4}$$

By integrating this in terms of PP_{row} we can find the vertical slices of the "radius of attack", also giving us the area and the number of enemies each pikeman can attack:

$$Area = \int_{-ROS}^{ROS} \sqrt{ROS^2 - (PP_{row})^2}\, d(PP_{row})$$

As stated above, this equation will integrate from the left most point (-ROS), to the right most point (+ROS). This will give us the total area (above the *x* axis) from which the enemy can be attacked. However, we can't just find the total area and divide it by the area of each solider to find the number of soldiers. This is because the area is circular, and would try to place soldiers in gaps by splitting the area of each soldier up.

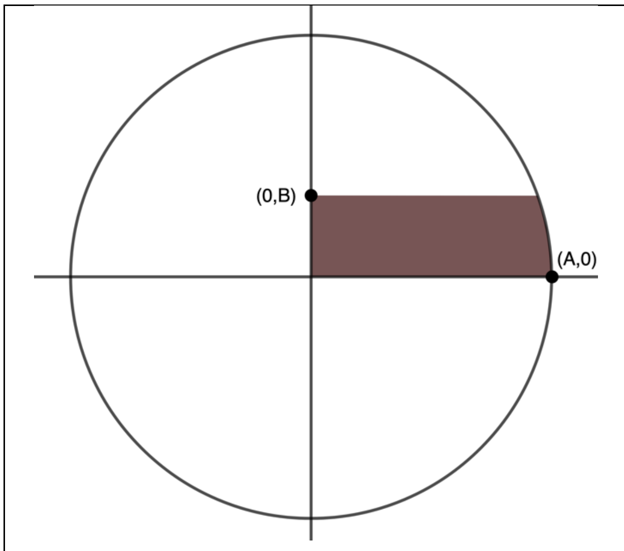


Figure 7: A cross section of the area of a row

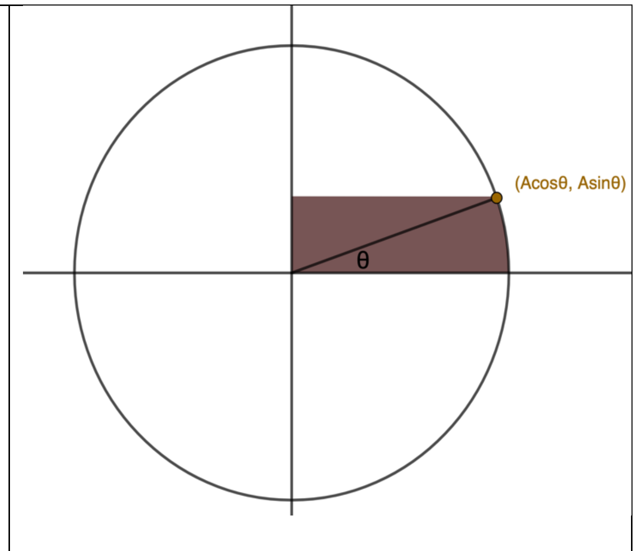


Figure 8: Polar coordinates of a point

To address this, I will instead split the area of attack into rows, where each one is equal to the shoulder length of one solider. If I find this area and divide by the area of each solider and round down, I will get a much more practical answer. Half of this area can be seen in Figure 7.

To find the area of a circle above the x axis and below a certain row height, I will integrate using horizontal strips, starting by substituting the radius of the spear (ROS) with A in (4):

$$PP_{row} = \sqrt{(A)^2 - (PP_{col})^2}$$

As seen by Figure 8s, the height of the row or PP_{col} is equal to $A\sin(\theta)$:

$$PP_{row} = \sqrt{A^2 - Asin(\theta)^2}$$
$$PP_{row} = A\sqrt{1 - sin(\theta)^2}$$
$$PP_{row} = A\sqrt{(cos(\theta)^2}$$
$$PP_{row} = Acos(\theta)$$

And thus:

$$Area = \int_0^b \sqrt{(A)^2 - (PP_{col})^2}\, d(PP_{col})$$

But before we integrate, since we substituted PP_{col} with $A\sin(\theta)$ we must do the same with $d(PP_{col})$:

$$PP_{col} = Asin(\theta)$$
$$d(PP_{col}) = Acos(\theta)\, \mathrm{d}\theta$$

By plugging that back into our original equation for area, we get:

$$Area = \int Acos(\theta) \cdot Acos(\theta)\, d\theta$$
$$Area = A^2 \int cos^2(\theta)\ d\theta$$

Solving for the integral, we use the double angle identity to find that:

$$\int cos^2(\theta)\ d\theta = \int \frac{1}{2}\, d\theta + \int cos(2\theta)\ d\theta$$
$$= \frac{\theta}{2} + \frac{1}{2}\int cos(2\theta)\ d\theta$$

If we assume u = 2θ, we thus get:

$$\int cos^2(\theta) = \frac{\theta}{2} + \frac{1}{2}\int cos(u)\, d\theta$$
$$= \frac{\theta}{2} + \frac{\sin(u)}{2}$$
$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$

Plugging it back into our area equation we get that:

$$Area = A^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) + C$$
$$= A^2\left(\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}\right) + C$$
$$= \frac{A^2\theta}{2} + \frac{A^2\cos(\theta)\sin(\theta)}{2} + C$$

Since $PP_{col} = A\sin(\theta)$ we find that:

$$\theta = \arcsin\left(\frac{PP_{col}}{A}\right)$$

Since Asin(θ) = PP_{col} and Acos(θ) = PP_{row} = $\sqrt{(A)^2 - (SPos_{col})^2}$:

$$Area = \left(\frac{A^2 \arcsin\left(\frac{SPoscol}{A}\right)}{2} + \frac{SPoscol\sqrt{(A)^2 - (SPos_{col})^2}}{2}\right) + C$$

However, this represents only the area to the right of the x axis, so we have to multiply by 2 to find the total area in the top quadrant:

$$Area = 2\left(\frac{A^2 \arcsin\left(\frac{PP_{col}}{A}\right)}{2} + \frac{PP_{col}\sqrt{(A)^2 - (PP_{col})^2}}{2}\right) + C$$

$$= \left(A^2 \arcsin\left(\frac{PP_{col}}{A}\right) + PP_{col}\sqrt{(A)^2 - (PP_{col})^2}\right) + C$$

To find a row starting at (0,b), we take the definite integral:

$$Area = \left(A^2 \arcsin\left(\frac{PP_{col}}{A}\right) + PP_{col}\sqrt{(A)^2 - (PP_{col})^2}\right)\Big[_{\mathbf{0}}^{\boldsymbol{b}}$$

$$= \left(A^2 \arcsin\left(\frac{b}{A}\right) + b\sqrt{(A)^2 - (b)^2}\right)$$

So, to find the nth row, we can just take the area of n and subtract it by the area of n-1, because b is the difference between rows this can thus be expressed as:

$$Row\ Area = \left(A^2 \arcsin\left(\frac{PP_{col}}{A}\right) + SPos_{col}\sqrt{(A)^2 - (PP_{col})^2}\right)\Big[_{\boldsymbol{b} * \boldsymbol{n}}^{\boldsymbol{b} * (\boldsymbol{n} + \mathbf{1})}$$

Thus, to find the total row area, we can just take the summation from n=0, to n = 6 (as that's the maximum depth of the model we will be looking at).

$$Total\ Area = \sum_{n=0}^{n=6}\left(A^2 \arcsin\left(\frac{PP_{col}}{A}\right) + SPos_{col}\sqrt{(A)^2 - (PP_{col})^2}\right)\Big[_{\boldsymbol{b} * \boldsymbol{n}}^{\boldsymbol{b} * (\boldsymbol{n} + \mathbf{1})}$$

Now, to find the amount of solders in that row, we can divide the total area by the width of each solider and round down, where $\lfloor x \rfloor$ is the floor of the function:

$$Soilders = \left\lfloor \frac{Row\ Area}{Soldier\ Width} \right\rfloor$$

If we substitute this into our equation for total area, we can find the equation for total solders:

$$\text{Total Soldiers} = \sum_{n=0}^{n=6}\left\lfloor \frac{nth\ Row\ Area}{b} \right\rfloor$$

This represents our general equation for α, β. We will now see how each formation impacts this model.

Modeling the Classical Phalanx Formation:

In the phalanx formation, soldiers' line up side-by-side and up to 6-deep. If each of these soldiers uses a pike that is 2.7m long and stands a shoulder length apart (0.4m) from his companions, then each pikeman in the front row could reach up to 6 of the enemy (assuming that the enemy has the same formation). Figure 8 demonstrates this, based on a tightly-packed 3-deep formation of pikemen.

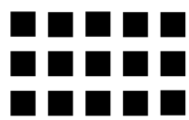


Figure 8: A 15 Man Phalanx ranged 3-deep

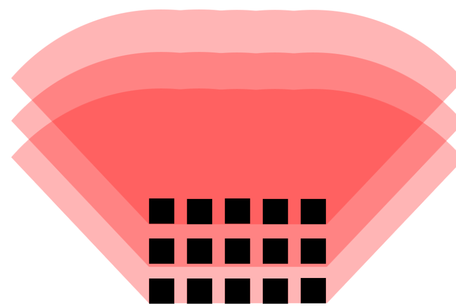


Figure 9: The Pike Range of the Phalanx

The more tightly packed the phalanx, the more overlap there is between the range of each pike and thus the more pikemen that are able to reach a given enemy soldier. Figure 9 shows the number of enemy soldiers that can be attacked by a pikeman depending on which row in the phalanx he is standing.

	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6
Row Area	2.152 m^2	2.104 m^2	2.004 m^2	1.844 m^2	1.605 m^2	1.242 m^2
Soldiers	5	5	5	4	4	3

Total Soldiers: 26

If the enemy falls back, this reduces the number of pikemen able to reach each enemy soldier. To calculate this, we simply integrate one row less, thus the table below gives the total soldiers for each row the solider moves back:

Position	0 Away	1 Away	2 Away	3 Away	4 Away	5 Away
Total Area	10.95 m^2	8.798 m^2	6.695m^2	4.691 m^2	2.847m^2	1.242 m^2
Soldiers	26	21	16	11	7	3

To find α we will simply take the average of the soldiers that are able to attack.

$$\alpha = \frac{26 + 21 + 16 + 11 + 7 + 3}{6}$$

$$\alpha = 14$$

As shown in Figure 2, each pikeman is facing off directly against the enemy and therefore vulnerable to attack by only one enemy ($\beta = 1$).

Modeling a Spread Formation:

This formation is similar to the situation above, except that the pikemen now stand two shoulder widths apart from each other (0.8M) and the formation is only 3 men deep. This differences in this formation can be seen in Figure 10.

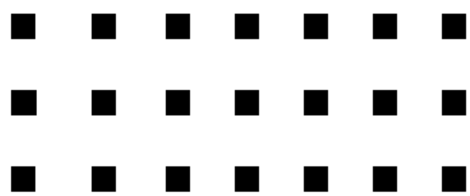


Figure 10: A 14 Man Spread Formation

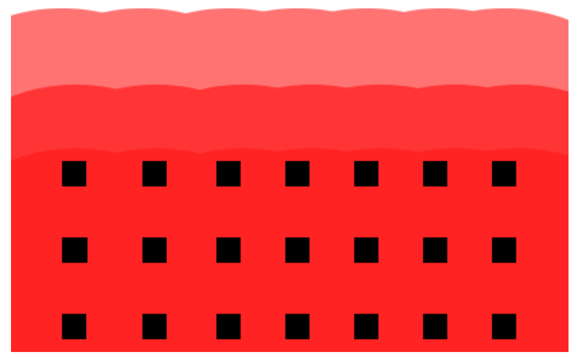


Figure 11: The Pike Range of the Spread Formation

Figure 11 illustrates how the reduced density of the formation affects the intersection between the range of each pike. By plugging the new values into our formula, I found that this dramatically reduces the number of enemy soldiers that can be attacked (assuming that the enemy retains the original formation of being 0.4m apart):

	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6
Row Area	4.256 m^2	3.848 m^2	2.847 m^2	0 m^2	0 m^2	0 m^2
Soldiers:	5	4	3	0	0	0

Total Soldiers: 12

If we follow the same method as we did in the previous table, we find that as the enemy moves back:

	0 Away	1 Away	2 Away	3 Away	4 Away	5 Away
Row Area	10.951 m^2	6.695 m^2	6.695 m^2	2.847 m^2	2.847 m^2	0 m^2
Soldiers:	12	7	7	3	3	0

To find α we will take the average of the row above:

$$\alpha = \frac{12 + 7 + 7 + 3 + 3 + 0}{6}$$

$$\alpha = 5.33$$

Also notice that this formation is twice as spread out then the classic phalanx formation, thus this would result in the enemy's attack power being weakened by a half. That is to say that $\beta = 0.5$.

Modeling a Staggered Formation:

In this formation every other row is moved forward and the pikemen stand one shoulder length apart. This is shown in Figure 11. This means that you will have a total of 7 rows in total, but the soldiers in the final row are unable to reach any enemies since each pike only has a maximum range of 6 rows of 0.4m.

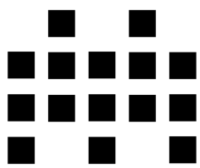
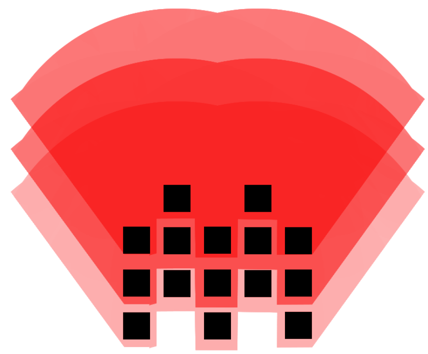
	

Figure 11: A 15 Man Staggered Formation	Figure 12: The Pike Range of the Staggered Formation

As seen in Figure 12, the spread is more intense than that of the spread formation, but the fact that the front row now consists of only two pikemen means that this is a less effective formation than the original phalanx formation. This can be seen in the reduction of enemy soldiers that can now be reached by the pikemen:

	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6
Row Area	1.076 m^2	2.104 m^2	2.004 m^2	1.844 m^2	1.605 m^2	1.242 m^2
Soldiers:	2	5	5	4	4	3

Total Soldiers: 23

We follow the integration pattern that we used before to find the total number of enemy soldiers that are vulnerable to attack as the enemy moves back:

	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6
Row Area	9.874 m^2	7.771 m^2	5.766 m^2	3.922 m^2	2.317 m^2	1.075 m^2
Soldiers:	23	21	16	11	7	3

To find α we will take the average of the row above:

$$\alpha = \frac{23 + 21 + 16 + 11 + 7 + 3}{6}$$

$$\alpha = 11.66$$

In this formation, the enemy is able to surround the formation easer; each enemy solider can only attack pikemen if they are bordering them, meaning they can attack if they are directly in front or beside the pikeman. This would mean that it would take 3 enemies are able to hit 5 phalanx members that means we would have a $\beta = 1.\overline{66}7$.

Finding the Optimal Formation:

After finding each of coefficients, equation (1) and (2) are now solvable. Remember we determine efficiency as the amount of phalanx soldiers left after the battle, as such, we used Euler's step method to determine the final army size. Both armies started at 400 soldiers each, and a step length of 0.001 was used.

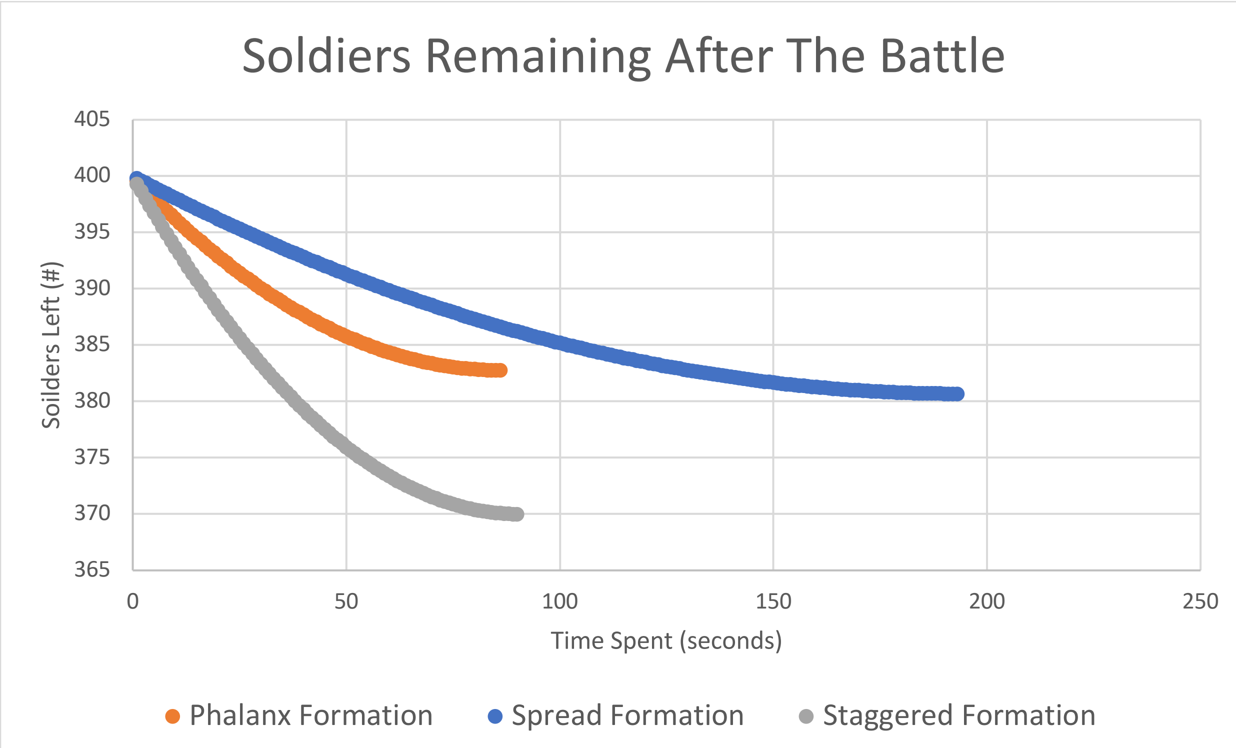


Figure 13: The number of soldiers left after the enemy army was nullified (*B = 0*)

As seen in Figure 13, both the phalanx and staggered formation involve much higher casualties because they both have large offensive coefficients ($\alpha > 10$) with casualties in the staggered formation being 2% higher as the enemy is able to attack more pikemen than in the phalanx formation. In our equation this is demonstrated by the enemies' coefficient of attack is larger in the staggered formation ($\beta = 1.\overline{667}$) then in the phalanx Formation ($\beta = 1$). The spread formation results in a much longer battle due to its diminished enemy offensive coefficient ($\beta = 0.5$) with a survivor count that is just under 0.5% less than the phalanx formation.

Before I conclude on which formation is most efficient, it's important to note that this model is heavily generalized. I made assumptions which make this model unrepresentative of the real world. For example, by using Lancaster's Laws, I model damage as a continuous stream gradually weakening both sides. In reality, damage is more like salvos, where each offensive result in soldiers dying and being unable to attack. A second major weakness of the model is that although I have established how many enemy soldiers each pikeman could attack in theory, the reality of battle is that, with a single weapon, you can only attack one enemy at a time. A third assumption is that each pikeman can attack any enemy up to the range of 2.7M away (the length of his pike). In practice, it would be very tough to fight someone directly in front of you with a weapon that is 2.7M long (especially if your companions are standing only 0.4M to the side of you and behind you).

Although this model makes some simplifying assumptions, I do believe that all of them are necessary for me to attempt to answer the problem that I set out to investigate. If I did not make the assumptions above, the math would become much more complex and outside the scope of this course, and while another solution could be to use probability along with the salvo equations, that would push this paper over the world limit!

In conclusion, I do believe that I have proved that the ancient Greek Phalanx was the most efficient formation for pikemen. As seen in Figure 11, the phalanx formation is the one in which the battle has the shortest duration and the greatest number of pikemen survive. The supremacy of this formation is due to the fact that it provides the greatest area in which each solider can attack an enemy. In the future I believe my model can be improved in several ways. The first would be comparing a larger sample of Greek formations to each other, the second would be substitute the Lancaster's square law equations with the Salvo equations (used in naval combat).

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