Analytical Linkage Synthesis

- Precision Points
- Two-Position Motion Generation
- Three Position Motion Generation by Analytical Synthesis

Precision Points

The points, or positions, prescribed for successive locations of the output (coupler or rocker) link in the plane are generally referred to as precision points or precision positions.

Two Position Motion Generation

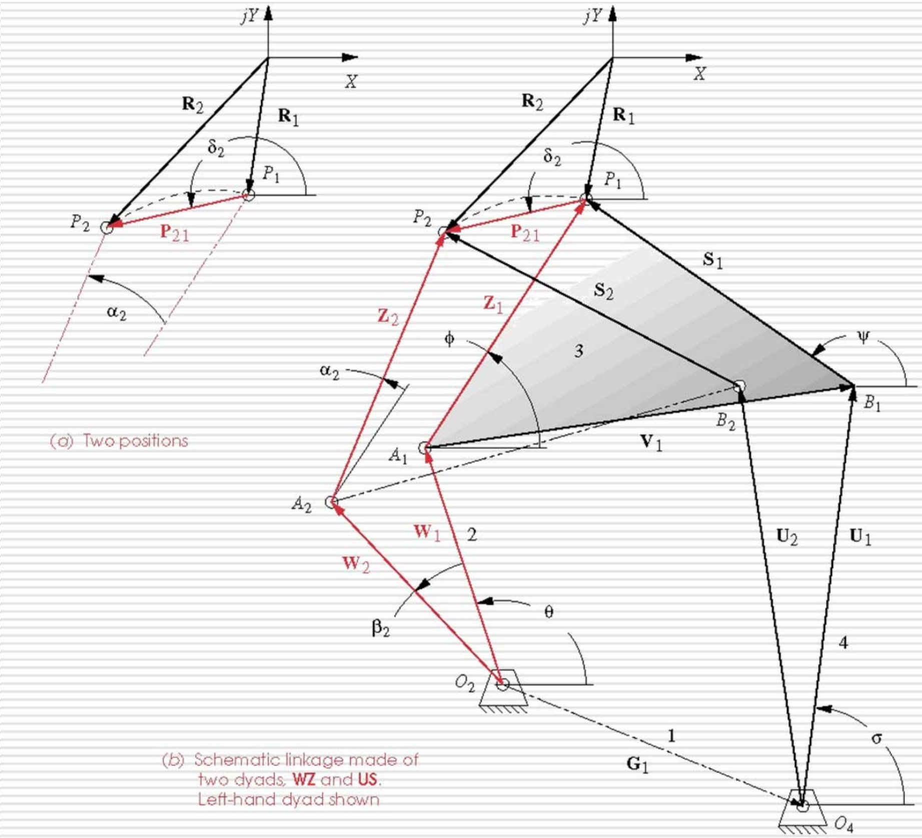
Problem Statement

Design a fourbar linkage which will move a line on its coupler link such that:

- a point P on that line will be first at P_1 and later at P_2
- the line will rotate through an angle α_2

Two-Position Motion Generation by Analytical Synthesis

- Point P on coupler moves from P_1 to P_2
- Coupler rotates through an angle α_2
- Given: p_{21}, δ_2, and α_2
- Solution
  - Write Loop Equations for Dyads W-Z and U-S
- Coupler
  - $\vec{V} = \vec{Z} - \vec{S}$
- Ground Link
  - $\vec{G} = \vec{W}_1 + \vec{V}_1 - \vec{U}_1$



(a) Two positions

(b) Schematic linkage made of two dyads, WZ and US. Left-hand dyad shown

Loop Equation for First Dyad

$$\vec{W}_2 + \vec{Z}_2 = \vec{W}_1 + \vec{Z}_1 + \vec{P}_{21}$$

$$\left|\vec{W}_1\right| = \left|\vec{W}_2\right| = w$$

$$\left|\vec{Z}_1\right| = \left|\vec{Z}_2\right| = z$$

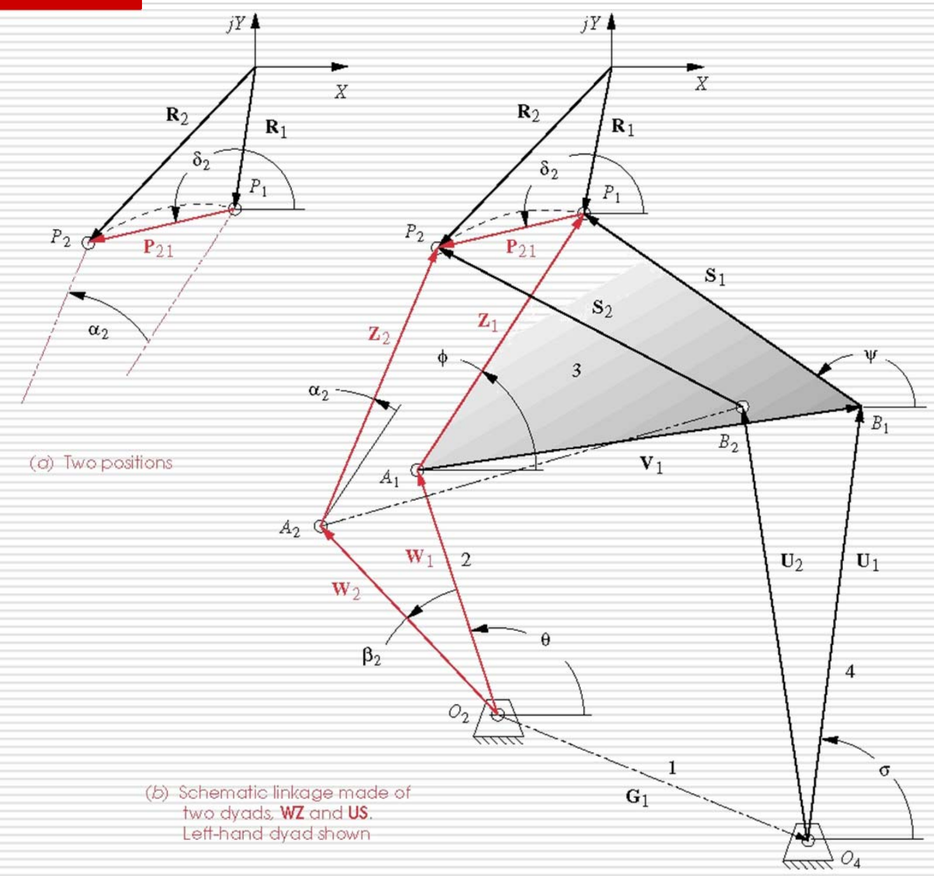
$$\vec{W}_1 = w \cdot \left[\cos(\theta)\hat{i} + \sin(\theta)\hat{j}\right]$$

$$\vec{W}_2 = w \cdot \left[\cos(\theta + \beta_2)\hat{i} + \sin(\theta + \beta_2)\hat{j}\right]$$

$$\vec{Z}_1 = z \cdot \left[\cos(\phi)\hat{i} + \sin(\phi)\hat{j}\right]$$

$$\vec{Z}_2 = z \cdot \left[\cos(\phi + \alpha_2)\hat{i} + \sin(\phi + \alpha_2)\hat{j}\right]$$

$$\vec{P}_{21} = p_{21} \cdot \left[\cos(\delta_2)\hat{i} + \sin(\delta_2)\hat{j}\right]$$



Resulting Equations

$$w \cdot \left[\cos\left(\theta+\beta_2\right)-\cos\left(\theta\right)\right]+z \cdot \left[\cos\left(\phi+\alpha_2\right)-\cos\left(\phi\right)\right]=p_{21} \cdot \cos\left(\delta_2\right)$$

$$w \cdot \left[\sin\left(\theta+\beta_2\right)-\sin\left(\theta\right)\right]+z \cdot \left[\sin\left(\phi+\alpha_2\right)-\sin\left(\phi\right)\right]=p_{21} \cdot \sin\left(\delta_2\right)$$

<u>Approach A</u>

Given: p_{21}, δ_2, α_2

Choose: θ, ϕ, β_2

Find: w, z

<u>Approach B</u>

Given: p_{21}, δ_2, α_2

Choose: z, ϕ, β_2

Find: w, θ

Solution Using Approach A First Dyad

<u>Approach A</u>

Given: p_{21}, $\delta_{2,}$ α_2

Choose: θ, ϕ, β_2

Find: w, z

$$\begin{bmatrix} \cos(\theta+\beta_2)-\cos(\theta) & \cos(\phi+\alpha_2)-\cos(\phi) \\ \sin(\theta+\beta_2)-\sin(\theta) & \sin(\phi+\alpha_2)-\sin(\phi) \end{bmatrix} \begin{Bmatrix} w \\ z \end{Bmatrix} = \begin{Bmatrix} p_{21}\cdot\cos(\delta_2) \\ p_{21}\cdot\sin(\delta_2) \end{Bmatrix}$$

Solution Using Approach B First Dyad

<u>Approach B</u>
Given: p_{21}, $\delta_{2,}$ α_2
Choose: z, ϕ, β_2
Find: w, θ or W_{1x}, W_{1y}

$$\begin{bmatrix} \cos(\beta_2)-1 & -\sin(\beta_2) \\ \sin(\beta_2) & \cos(\beta_2)-1 \end{bmatrix} \begin{Bmatrix} W_{1x} \\ W_{1y} \end{Bmatrix} = \begin{Bmatrix} p_{21}\cdot\cos(\delta_2)-z\cdot\left[\cos(\phi+\alpha_2)-\cos(\phi)\right] \\ p_{21}\cdot\sin(\delta_2)-z\cdot\left[\sin(\phi+\alpha_2)-\sin(\phi)\right] \end{Bmatrix}$$

Loop Equation for Second Dyad

$$\vec{U}_2 + \vec{S}_2 = \vec{U}_1 + \vec{S}_1 + \vec{P}_{31}$$

$$\left|\vec{U}_1\right| = \left|\vec{U}_2\right| = u$$

$$\left|\vec{S}_1\right| = \left|\vec{S}_2\right| = s$$

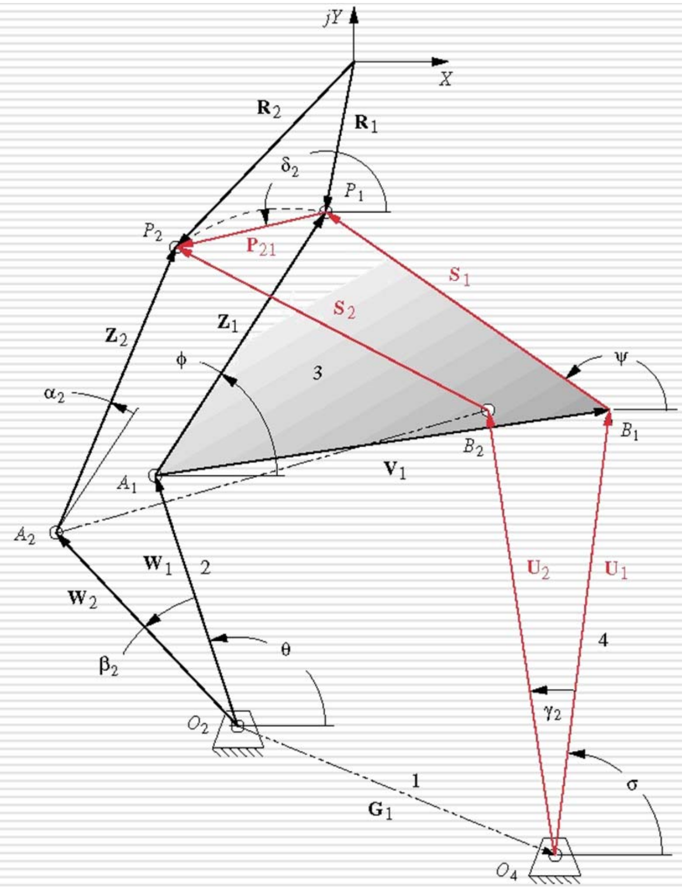
$$\vec{U}_1 = u \cdot \left[\cos(\sigma)\hat{i} + \sin(\sigma)\hat{j}\right]$$

$$\vec{U}_2 = u \cdot \left[\cos(\sigma + \gamma_2)\hat{i} + \sin(\sigma + \gamma_2)\hat{j}\right]$$

$$\vec{S}_1 = s \cdot \left[\cos(\psi)\hat{i} + \sin(\psi)\hat{j}\right]$$

$$\vec{S}_2 = s \cdot \left[\cos(\psi + \alpha_2)\hat{i} + \sin(\psi + \alpha_2)\hat{j}\right]$$

$$\vec{P}_{21} = p_{21} \cdot \left[\cos(\delta_2)\hat{i} + \sin(\delta_2)\hat{j}\right]$$



Resulting Equations

$$u\cdot\left[\cos\left(\sigma+\gamma_2\right)-\cos\left(\sigma\right)\right]+s\cdot\left[\cos\left(\psi+\alpha_2\right)-\cos\left(\psi\right)\right]=p_{21}\cdot\cos\left(\delta_2\right)$$

$$u\cdot\left[\sin\left(\sigma+\gamma_2\right)-\sin\left(\sigma\right)\right]+s\cdot\left[\sin\left(\psi+\alpha_2\right)-\sin\left(\psi\right)\right]=p_{21}\cdot\sin\left(\delta_2\right)$$

<u>Approach A</u>
Given: p_{21}, $\delta_{2,}$ α_2
Choose: σ, ψ, γ_2
Find: u, s

<u>Approach B</u>
Given: p_{21}, $\delta_{2,}$ α_2
Choose: u, ψ, γ_2
Find: s, σ

Solution Using Approach A Second Dyad

<u>Approach A</u>

Given: p_{21}, δ_2, α_2

Choose: σ, ψ, γ_2

Find: u, s

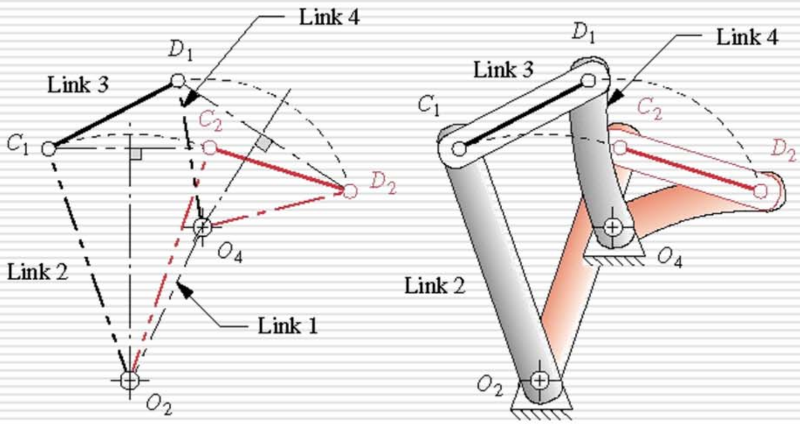
$$\begin{bmatrix} \cos(\sigma+\gamma_2)-\cos(\sigma) & \cos(\psi+\alpha_2)-\cos(\psi) \\ \sin(\sigma+\gamma_2)-\sin(\sigma) & \sin(\psi+\alpha_2)-\sin(\psi) \end{bmatrix} \begin{Bmatrix} u \\ s \end{Bmatrix} = \begin{Bmatrix} p_{21}\cdot\cos(\delta_2) \\ p_{21}\cdot\sin(\delta_2) \end{Bmatrix}$$

Solution Using Approach B Second Dyad

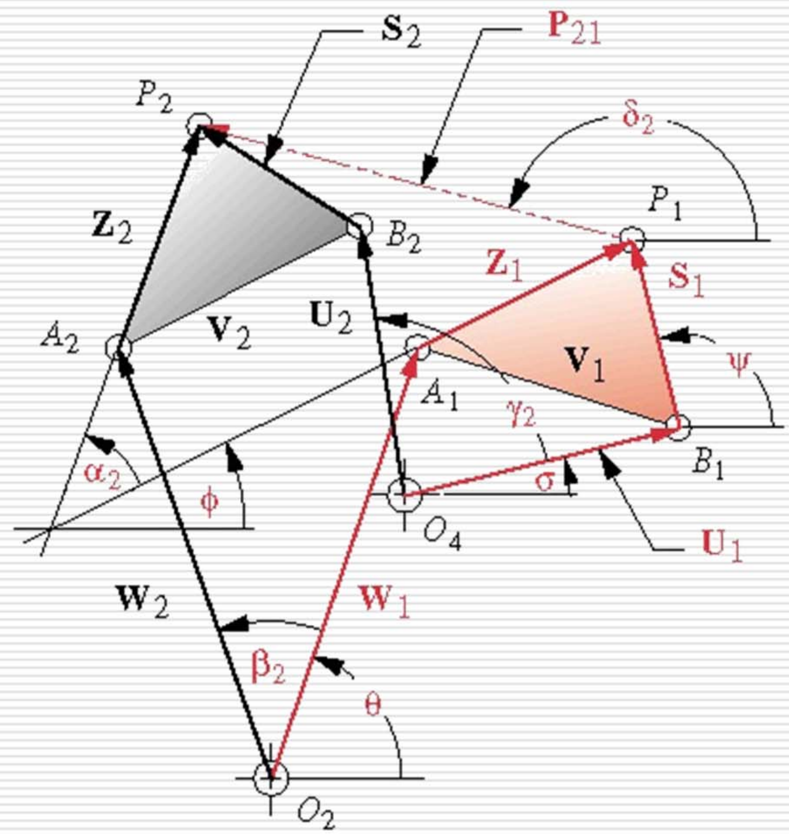
<u>Approach B</u>
Given: p_{21}, δ_2, α_2
Choose: s, ψ, γ_2
Find: u, σ or U_{1x}, U_{1y}

$$\begin{bmatrix} \cos(\gamma_2)-1 & -\sin(\gamma_2) \\ \sin(\gamma_2) & \cos(\gamma_2)-1 \end{bmatrix} \begin{Bmatrix} U_{1x} \\ U_{1y} \end{Bmatrix} = \begin{Bmatrix} p_{21}\cdot\cos(\delta_2)-s\cdot\left[\cos(\psi+\alpha_2)-\cos(\psi)\right] \\ p_{21}\cdot\sin(\delta_2)-s\cdot\left[\sin(\psi+\alpha_2)-\sin(\psi)\right] \end{Bmatrix}$$

Example



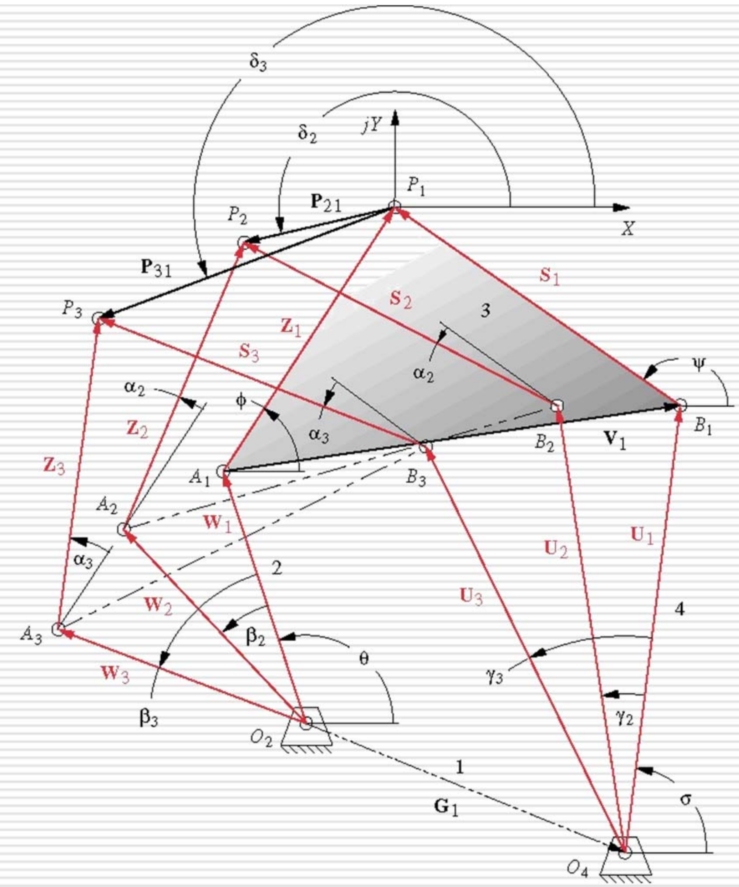
(a) Two-position synthesis (b) Finished non-Grashof fourbar



Three-Position Motion Generation by Analytical Synthesis

- Point P on coupler moves from P_1 to P_2 to P_3
- Coupler rotates through an angle α_2 to α_3
- Given: p_{21}, p_{31}, δ_2, δ_3, α_3 and α_2

- Solution
  - Write Loop Equations for Dyads W-Z and U-S
- Coupler
  - $\vec{V} = \vec{Z} - \vec{S}$
- Ground Link
  - $\vec{G} = \vec{W}_1 + \vec{V}_1 - \vec{U}_1$



Loop Equation for First Dyad

$$\vec{W}_2 + \vec{Z}_2 = \vec{W}_1 + \vec{Z}_1 + \vec{P}_{21}; \quad \vec{W}_3 + \vec{Z}_3 = \vec{W}_1 + \vec{Z}_1 + \vec{P}_{31}$$

$$\left|\vec{W}_1\right| = \left|\vec{W}_2\right| = \left|\vec{W}_3\right| = w; \quad \left|\vec{Z}_1\right| = \left|\vec{Z}_2\right| = \left|\vec{Z}_3\right| = z$$

$$\vec{W}_1 = w \cdot \left[\cos(\theta)\hat{i} + \sin(\theta)\hat{j}\right]$$

$$\vec{W}_2 = w \cdot \left[\cos(\theta + \beta_2)\hat{i} + \sin(\theta + \beta_2)\hat{j}\right]$$

$$\vec{W}_3 = w \cdot \left[\cos(\theta + \beta_3)\hat{i} + \sin(\theta + \beta_3)\hat{j}\right]$$

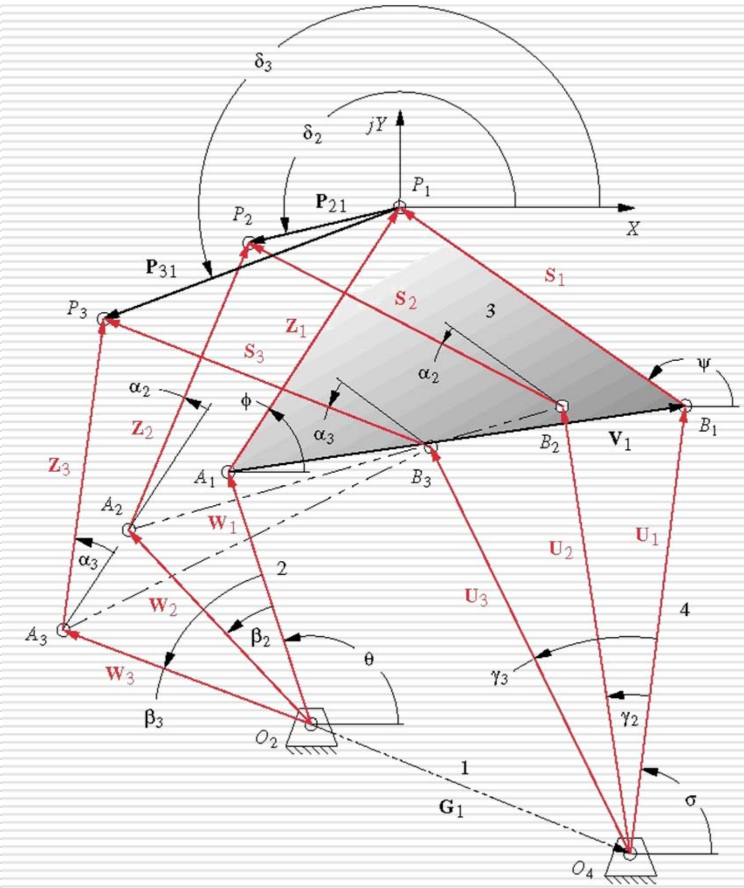
$$\vec{Z}_1 = z \cdot \left[\cos(\phi)\hat{i} + \sin(\phi)\hat{j}\right]$$

$$\vec{Z}_2 = z \cdot \left[\cos(\phi + \alpha_2)\hat{i} + \sin(\phi + \alpha_2)\hat{j}\right]$$

$$\vec{Z}_3 = z \cdot \left[\cos(\phi + \alpha_3)\hat{i} + \sin(\phi + \alpha_3)\hat{j}\right]$$

$$\vec{P}_{21} = p_{21} \cdot \left[\cos(\delta_2)\hat{i} + \sin(\delta_2)\hat{j}\right]$$

$$\vec{P}_{31} = p_{31} \cdot \left[\cos(\delta_3)\hat{i} + \sin(\delta_3)\hat{j}\right]$$



Resulting Equations for First Dyad

Approach A

Given: p_{21}, p_{31}, δ_2, δ_2, α_2, α_2

Choose: β_2, β_3

Find: w, θ, z, ϕ

First Loop

$$w \cdot \left[\cos(\theta+\beta_2)-\cos(\theta)\right]+z \cdot \left[\cos(\phi+\alpha_2)-\cos(\phi)\right]=p_{21} \cdot \cos(\delta_2)$$

$$w \cdot \left[\sin(\theta+\beta_2)-\sin(\theta)\right]+z \cdot \left[\sin(\phi+\alpha_2)-\sin(\phi)\right]=p_{21} \cdot \sin(\delta_2)$$

Second Loop

$$w \cdot \left[\cos(\theta+\beta_3)-\cos(\theta)\right]+z \cdot \left[\cos(\phi+\alpha_3)-\cos(\phi)\right]=p_{31} \cdot \cos(\delta_3)$$

$$w \cdot \left[\sin(\theta+\beta_3)-\sin(\theta)\right]+z \cdot \left[\sin(\phi+\alpha_3)-\sin(\phi)\right]=p_{31} \cdot \sin(\delta_3)$$

Solution Using Approach A First Dyad

Approach A

Given: p_{21}, p_{31}, δ_2, δ_2, α_2, α_2

Choose: β_2, β_3

Find: w, θ, z, ϕ or W_{1x}, W_{1y}, Z_{1x}, Z_{1y}

$$\begin{bmatrix} \cos\beta_2 - 1 & -\sin\beta_2 & \cos\alpha_2 - 1 & -\sin\alpha_2 \\ \sin\beta_2 & \cos\beta_2 - 1 & \sin\alpha_2 & \cos\alpha_2 - 1 \\ \cos\beta_3 - 1 & -\sin\beta_3 & \cos\alpha_3 - 1 & -\sin\alpha_3 \\ \sin\beta_3 & \cos\beta_3 - 1 & \sin\alpha_3 & \cos\alpha_3 - 1 \end{bmatrix} \cdot \begin{Bmatrix} W_{1x} \\ W_{1y} \\ Z_{1x} \\ Z_{1y} \end{Bmatrix} = \begin{Bmatrix} p_{21} \cdot \cos\delta_2 \\ p_{21} \cdot \sin\delta_2 \\ p_{31} \cdot \cos\delta_3 \\ p_{31} \cdot \sin\delta_3 \end{Bmatrix}$$

Loop Equation for Second Dyad

$$\vec{U}_2+\vec{S}_2=\vec{U}_1+\vec{S}_1+\vec{P}_{21}; \quad \vec{U}_3+\vec{S}_3=\vec{U}_1+\vec{S}_1+\vec{P}_{31}$$

$$\left|\vec{U}_1\right|=\left|\vec{U}_2\right|=\left|\vec{U}_3\right|=u; \quad \left|\vec{S}_1\right|=\left|\vec{S}_2\right|=\left|\vec{S}_3\right|=s$$

$$\vec{U}_1=u\cdot\left[\cos\left(\sigma\right)\hat{i}+\sin\left(\sigma\right)\hat{j}\right]$$

$$\vec{U}_2=u\cdot\left[\cos\left(\sigma+\gamma_2\right)\hat{i}+\sin\left(\sigma+\gamma_2\right)\hat{j}\right]$$

$$\vec{U}_3=u\cdot\left[\cos\left(\sigma+\gamma_3\right)\hat{i}+\sin\left(\sigma+\gamma_3\right)\hat{j}\right]$$

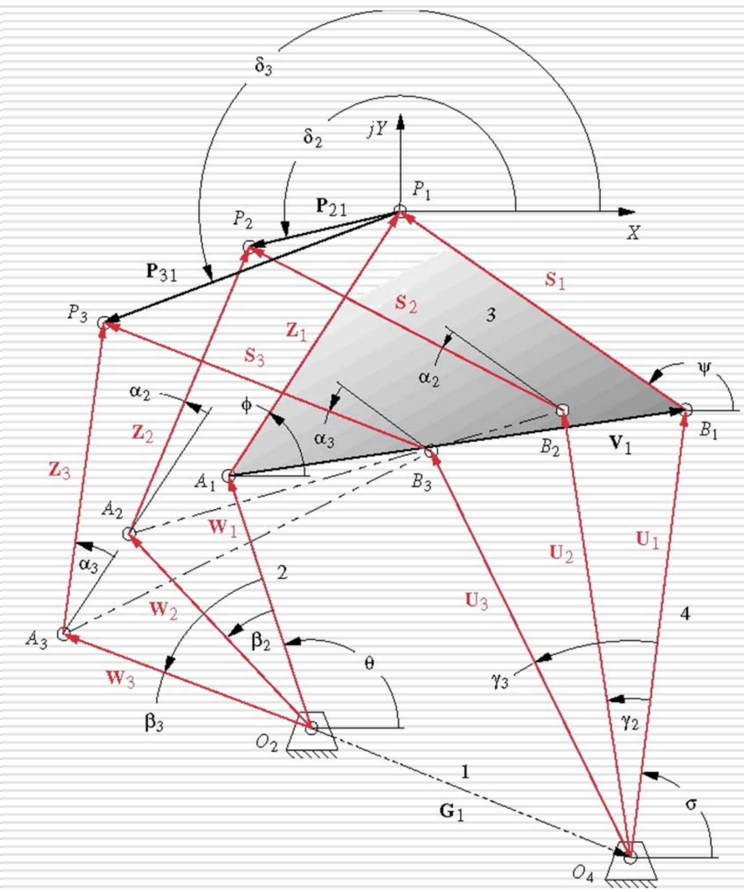
$$\vec{S}_1=s\cdot\left[\cos\left(\psi\right)\hat{i}+\sin\left(\psi\right)\hat{j}\right]$$

$$\vec{S}_2=s\cdot\left[\cos\left(\psi+\alpha_2\right)\hat{i}+\sin\left(\psi+\alpha_2\right)\hat{j}\right]$$

$$\vec{S}_3=s\cdot\left[\cos\left(\psi+\alpha_3\right)\hat{i}+\sin\left(\psi+\alpha_3\right)\hat{j}\right]$$

$$\vec{P}_{21}=p_{21}\cdot\left[\cos\left(\delta_2\right)\hat{i}+\sin\left(\delta_2\right)\hat{j}\right]$$

$$\vec{P}_{31}=p_{31}\cdot\left[\cos\left(\delta_3\right)\hat{i}+\sin\left(\delta_3\right)\hat{j}\right]$$



Resulting Equations for Second Dyad

Approach

Given: p_{21}, p_{31}, δ_2, δ_2, α_2, α_2

Choose: β_2, β_3

Find: u, σ, s, ψ or U_{1x}, U_{1y}, S_{1x}, S_{1y}

First Loop

$$u \cdot \left[\cos(\sigma+\gamma_2)-\cos(\sigma)\right]+s\cdot\left[\cos(\psi+\alpha_2)-\cos(\psi)\right]=p_{21}\cdot\cos(\delta_2)$$

$$u \cdot \left[\sin(\sigma+\gamma_2)-\sin(\sigma)\right]+s\cdot\left[\sin(\psi+\alpha_2)-\sin(\psi)\right]=p_{21}\cdot\sin(\delta_2)$$

Second Loop

$$u \cdot \left[\cos(\sigma+\gamma_3)-\cos(\sigma)\right]+s\cdot\left[\cos(\psi+\alpha_3)-\cos(\psi)\right]=p_{31}\cdot\cos(\delta_3)$$

$$u \cdot \left[\sin(\sigma+\gamma_3)-\sin(\sigma)\right]+s\cdot\left[\sin(\psi+\alpha_3)-\sin(\psi)\right]=p_{31}\cdot\sin(\delta_3)$$

Solution Using Approach A Second Dyad

<u>Approach</u>

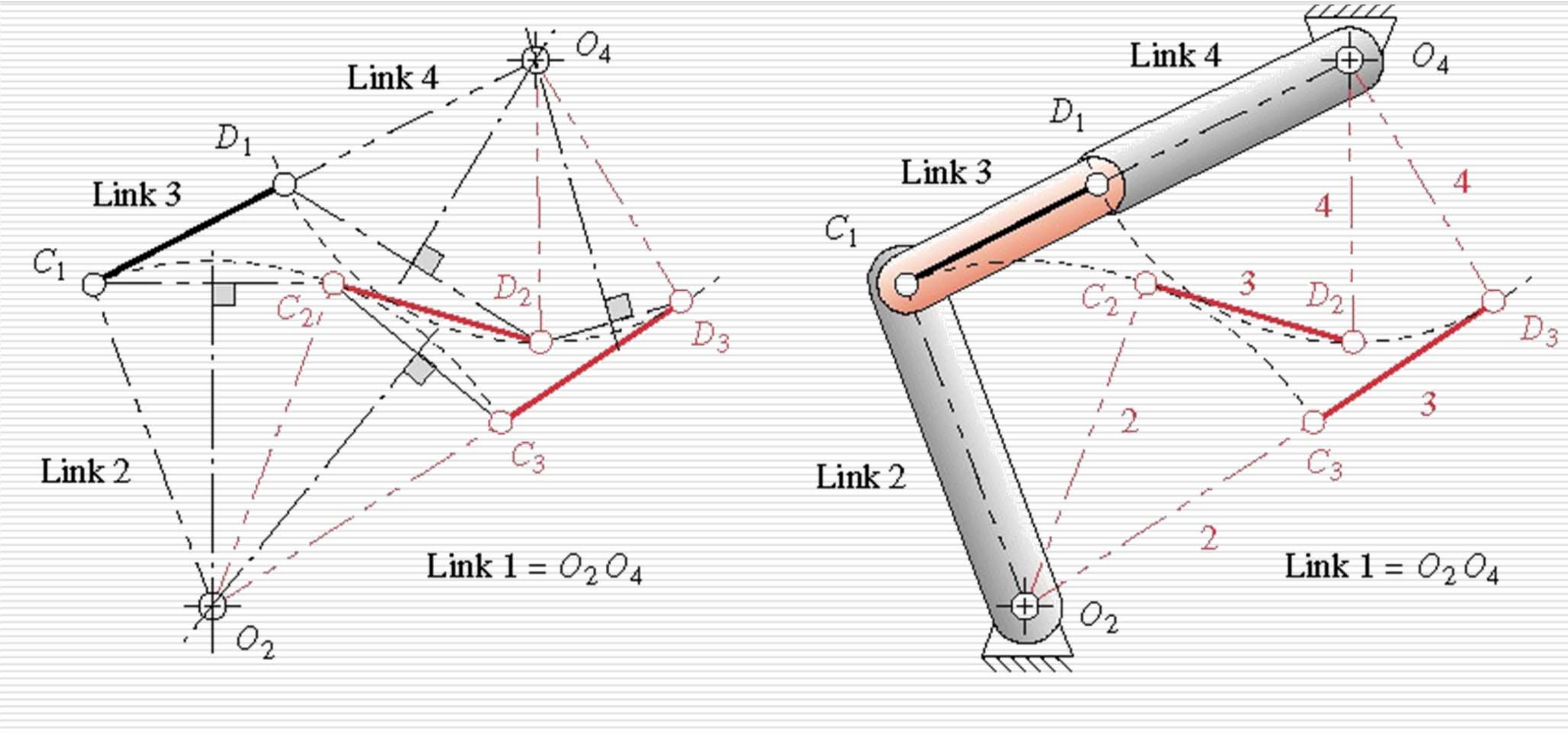
Given: p_{21}, p_{31}, δ_2, δ_2, α_2, α_2

Choose: γ_2, γ_3

Find: u, σ, s, ψ or U_{1x}, U_{1y}, S_{1x}, S_{1y}

$$\begin{bmatrix} \cos\gamma_2 - 1 & -\sin\gamma_2 & \cos\alpha_2 - 1 & -\sin\alpha_2 \\ \sin\gamma_2 & \cos\gamma_2 - 1 & \sin\alpha_2 & \cos\alpha_2 - 1 \\ \cos\gamma_3 - 1 & -\sin\gamma_3 & \cos\alpha_3 - 1 & -\sin\alpha_3 \\ \sin\gamma_3 & \cos\gamma_3 - 1 & \sin\alpha_3 & \cos\alpha_3 - 1 \end{bmatrix} \cdot \begin{Bmatrix} U_{1x} \\ U_{1y} \\ S_{1x} \\ S_{1y} \end{Bmatrix} = \begin{Bmatrix} p_{21}\cdot\cos\delta_2 \\ p_{21}\cdot\sin\delta_2 \\ p_{31}\cdot\cos\delta_3 \\ p_{31}\cdot\sin\delta_3 \end{Bmatrix}$$

Example
Previous Graphical Solution



(a) Construction method (b) Finished non-Grashof fourbar

Example
Analytical Solution