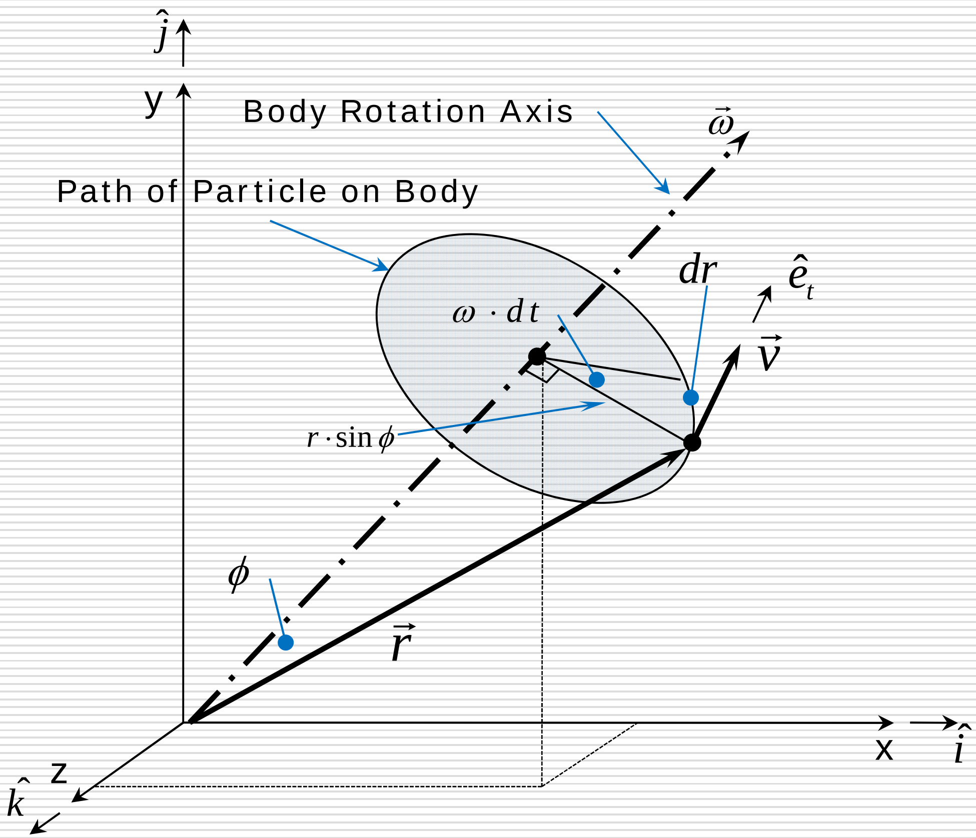
Omega Theorem

This theorem shows how to find the time derivative of any vector of constant magnitude whose direction changes as a result of rotation with a body in which it is fixed.

This leads to the development of a method for taking the derivative of a UNIT VECTOR

Configuration Description: 3D Body Rotating About a Point



A body is rotating about a fixed point O with an absolute angular velocity ω coinciding with the axis or rotation.

- The axis of rotation will move or precess
- Any point P located by $\vec{r}$ describes a circular path perpendicular to the axis of rotation.
- With only one point fixed in space as in a top
  - the axis of rotation will move or precess
  - the path of P will be circular only for dt
  - during this time the radius of the path is

    $$r \cdot \sin\phi$$

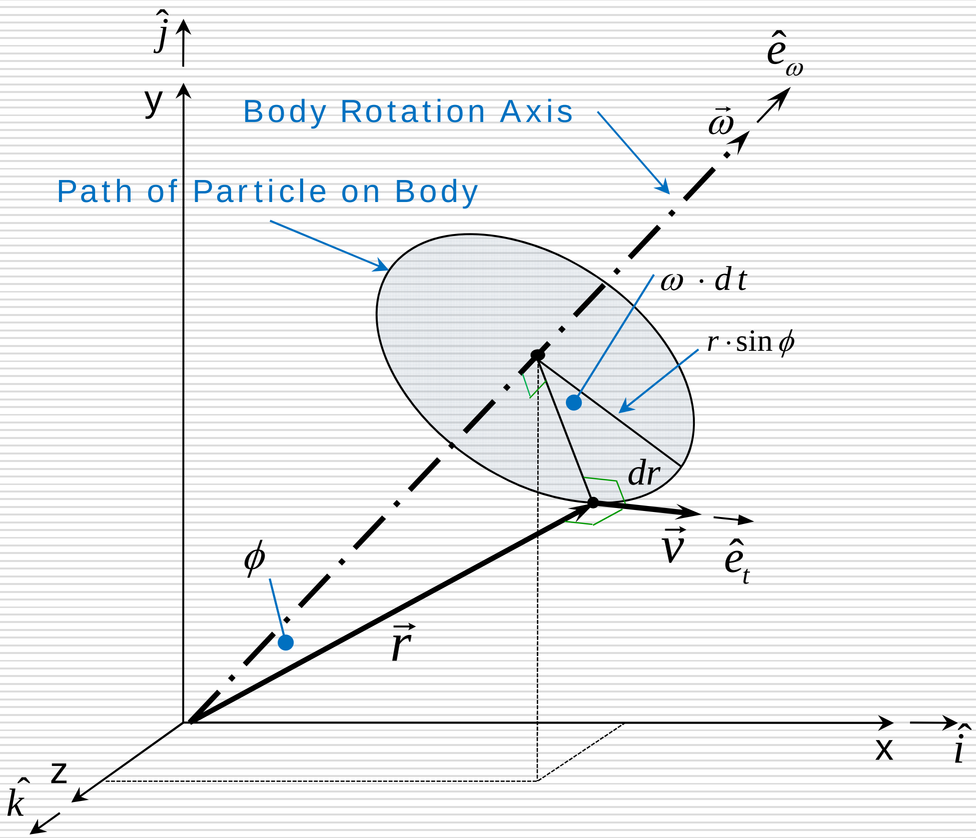
  - the angle of sweep is

    $$\omega \cdot dt$$

  - the displacement of P is (arc length)

    $$d\vec{r} = (\omega \cdot dt) \cdot (r \cdot \sin\phi) \cdot \hat{e}_t$$

Velocity:
3D Body Rotating About a Point



The velocity is found by dividing the displacement $d\vec{r}$ by the time interval dt.

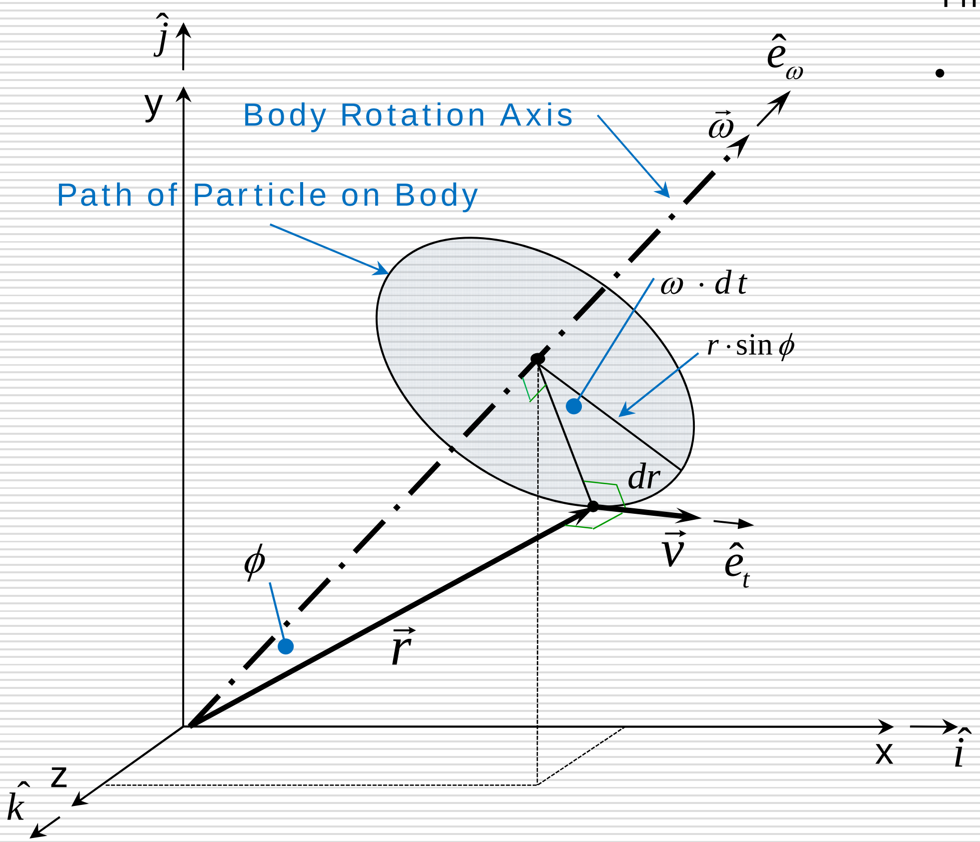
$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{(\omega \cdot dt)\cdot(r \cdot \sin\phi)\cdot \hat{e}_t}{dt} = \omega \cdot r \cdot \sin\phi \cdot \hat{e}_t \quad (1)$$

$\vec{v}$ is perpendicular to the plane formed by the vectors $\vec{\omega}$ and $\vec{r}$.

- The right hand side of Equation 1 represents the definition of the cross-product.

$$\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r} \quad (2)$$

Velocity Gerneralization: 3D Body Rotating About a Point



2

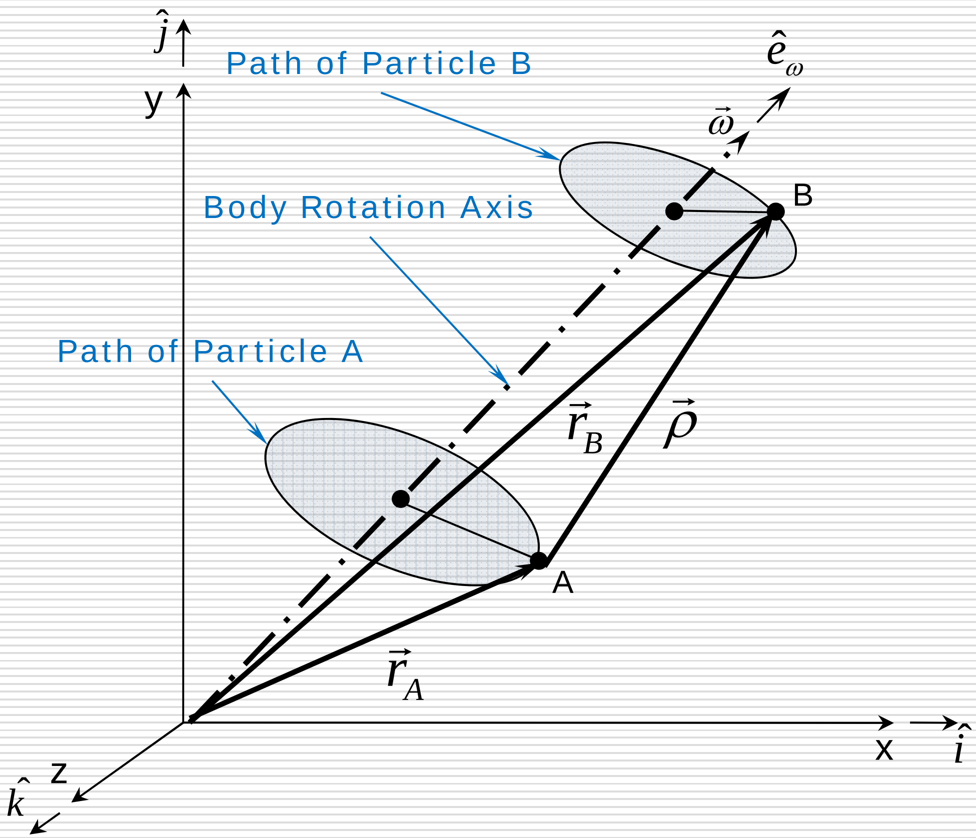
The velocity

$$\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

- $\vec{r}$ may be the position vector to P from any origin on the axis of rotation regardless of whether the axis of rotation is fixed or moving.
  - If the $\hat{e}_\omega$ axis has constant spatial orientation (fixed), $\vec{\omega}$ can change only in magnitude.

    $\vec{\alpha} = \dot{\vec{\omega}}$ (will be collinear with $\vec{\omega}$)
  - For a moving $\hat{e}_\omega$ axis
    - $\vec{\omega}$ will always change direction
    - $\vec{\omega}$ will possibly be changing magnitude.
    - $\vec{\alpha}$ will have different direction from $\vec{\omega}$

Significance of Equation 2



Equation 2 ($\vec{v} = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$) represents the rate at which a constant length vector fixed in a rotating body changes its direction

- $\vec{\rho}$ is fixed in the rotating body joining particles A and B

$$\vec{\rho} = \vec{r}_B - \vec{r}_A \quad (3)$$

- Taking the time derivative of Equation 3

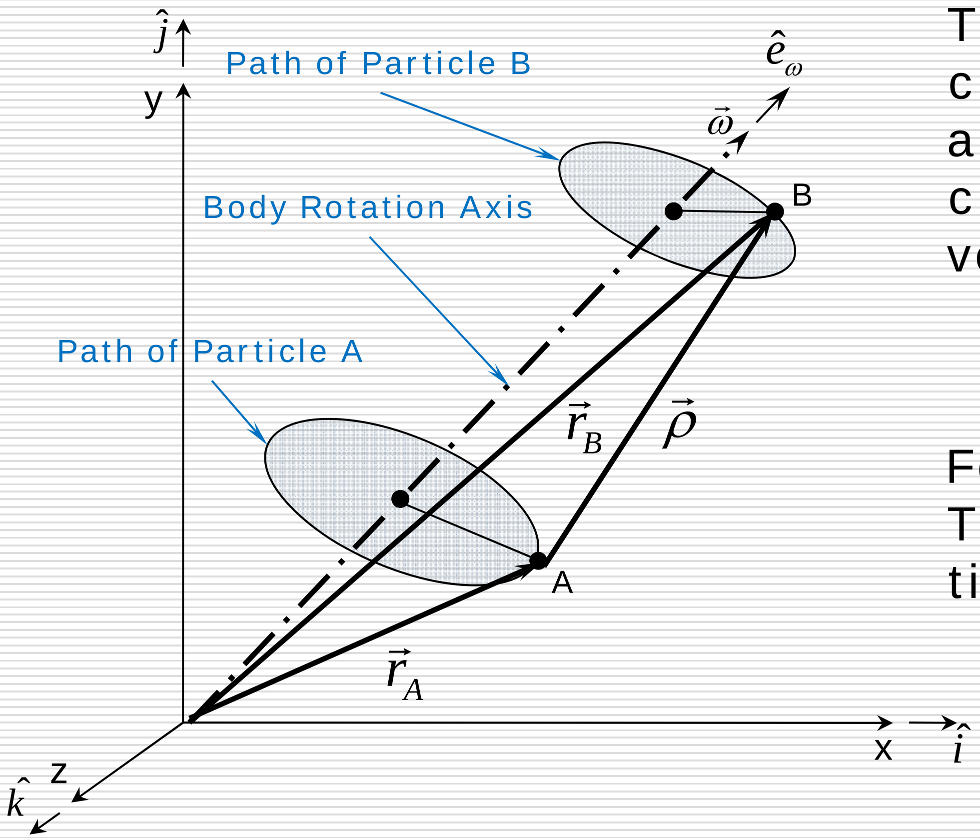
$$\frac{d}{dt}(\vec{\rho}) = \frac{d}{dt}(\vec{r}_B - \vec{r}_A)$$

$$\dot{\vec{\rho}} = \dot{\vec{r}}_B - \dot{\vec{r}}_A$$

- Substituting in Equation 2

$$\begin{aligned}\dot{\vec{\rho}} &= (\vec{\omega} \times \vec{r}_B) - (\vec{\omega} \times \vec{r}_A) \\ &= \vec{\omega} \times (\vec{r}_B - \vec{r}_A) \\ &= \vec{\omega} \times \vec{\rho} \quad (4)\end{aligned}$$

THE OMEGA THEOREM



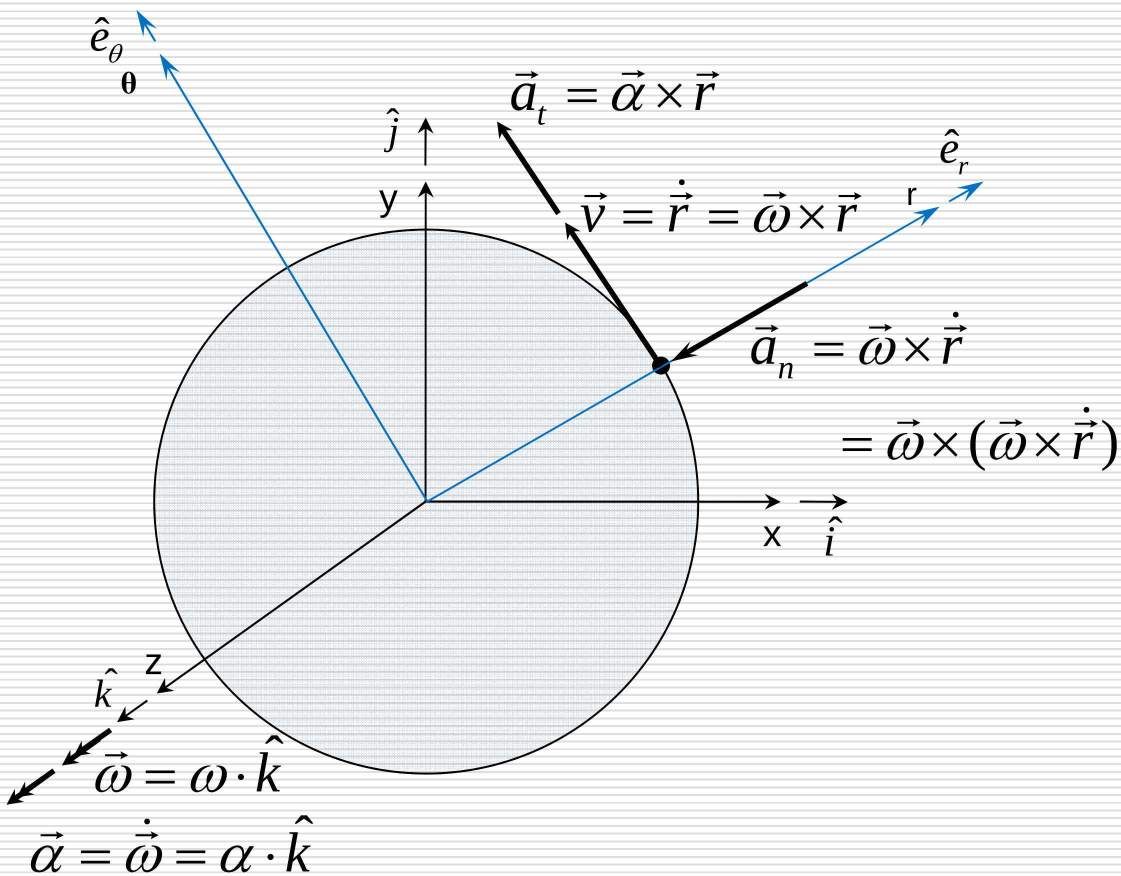
The time derivative of a constant length vector fixed in a rotating body is simply the cross product of the angular velocity with the vector

$$\dot{\vec{\rho}} = \vec{\omega} \times \vec{\rho} \tag{5}$$

For a generic UNIT VECTOR ,
The omega theorem gives it's time derivative as

$$\dot{\hat{e}} = \vec{\omega} \times \hat{e} \tag{6}$$

Derivatives of a Particle Traveling in Planar Circular Motion



$$\vec{v} = \frac{d}{dt}(\vec{r}) = \dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \dot{\vec{v}}$$

$$= \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}$$

$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \vec{a}_t + \vec{a}_n$$