PROBLEM 7.55 A 500mm LONG COLUMN WITH PINNED-PINNED ENDS HAS A RECTANGULAR CROSS-SECTION $b \times h = 30 \times 50$ mm.. THE PINS GO THROUGH THE 30-mm DIMENSION. CONSIDER THAT THE COLUMN IS FIXED-FIXED RELATIVE TO BUCKLING ABOUT THE WEAK AXIS. THE COLUMN IS MADE OF A MATERIAL FOR WHICH THE COMPRESSIVE STRESS-STRAIN CURVE IS GIVEN BY THE DATA GIVEN. THE DATA ARE LINEAR TO $\sigma = 294$ MPa. DETERMINE THE CRITICAL LOAD FOR BUCKLING.

$\epsilon\ (10^{-3})$	0	1.1	1.2	1.3	1.4	1.5	1.6	1.8	2.0	2.2	2.5	2.8	3.2	3.6	4.0
σ MPa	0	231	252	273	294	314.3	333.4	367.7	397.3	422.6	453	473.7	495.5	506	510

GIVEN:

1. 500mm LONG COLUMN
2. RECTANGULAR CROSS-SECTION $b \times h = 30 \times 50$ mm
3. PIN THROUGH 30mm DIMENSION
4. STRESS-STRAIN DATA GIVEN
5. DATA LINEAR UP TO 294 MPa

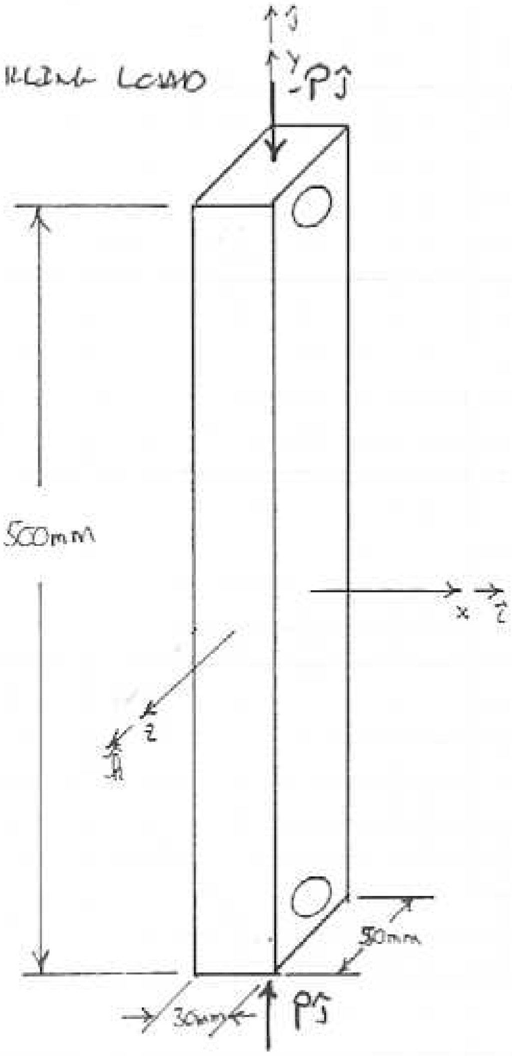
ASSUMPTIONS:

1. WEAK AXIS FIXED-FIXED CONDITION
2. STRONG AXIS PINNED-PINNED CONDITION

FIND:

1. CRITICAL BUCKLING LOAD

FIGURE:



SOLUTION:

THE DATA PROVIDED WAS ENTERED INTO AN EXCEL SPREADSHEET. THE SUBSET OF THIS DATA FROM 0 TO 294 MPa WAS PLOTTED ON A σ VERSUS ϵ DIAGRAM AND FIT WITH A STRAIGHT LINE TO DETERMINE THE MODULUS OF THE MATERIAL. THE RESULTING MODULUS IS

$$E = 200 \times 10^9 \frac{N}{m^2} = 200 \times 10^9 Pa = 200 GPa \text{ (STEEL)} \qquad (1)$$

FROM THE GEOMETRY GIVEN, THE MOMENTS OF INERTIA ABOUT THE TWO PRINCIPAL AXES OF THE CROSS-SECTION ARE CALCULATED.

$$I_{zz} = \frac{1}{12}(0.050m)(0.030m)^3 = 112.5 \times 10^{-9} m^4 \qquad (2)$$

$$I_{xx} = \frac{1}{12}(0.030m)(0.050m)^3 = 312.5 \times 10^{-9} m^4 \qquad (3)$$

USING THE RESULTS IN (1) THROUGH (3), CRITICAL BUCKLING LOADS ABOUT THE X-X AND Z-Z AXES CAN BE COMPUTED. BECAUSE THE END CONDITIONS ARE NOT THE SAME IN THE TWO DIRECTIONS, THEY BOTH NEED TO BE EVALUATED. STARTING WITH THE Z-Z AXIS, THE END CONSTRAINTS IN THIS DIRECTION ARE APPROXIMATED BY FIXED-FIXED CONSTRAINTS. THE LINEAR-ELASTIC CRITICAL BUCKLING LOAD IS

$$P_{CR,z} = \frac{\pi^2 \cdot E \cdot I}{K^2 \cdot L^2} = \frac{\pi^2 \cdot (200 \times 10^9 \frac{N}{m^2}) \cdot (112.5 \times 10^{-9} m^4)}{(0.7)^2 (0.5m)^2} = 1.8128 \times 10^6 N$$

$$\sigma_{CR,z} = \frac{P_{CR,z}}{A} = \frac{1.8128 \times 10^6 N}{(0.03m)(0.05m)} = 1.2085 \times 10^9 \frac{N}{m^2} = 1.2085 GPa \qquad (4)$$

FROM THIS CALCULATION IT IS CLEAR THAT THE COLUMN WILL NOT BUCKLE IN A LINEAR ELASTIC MODE

THE END CONSTRAINTS OF THE COLUMN IN THE X-X DIRECTION ARE BEST APPROXIMATED BY PINNED-PINNED CONSTRAINTS. THE LINEAR-ELASTIC BUCKLING LOAD IS

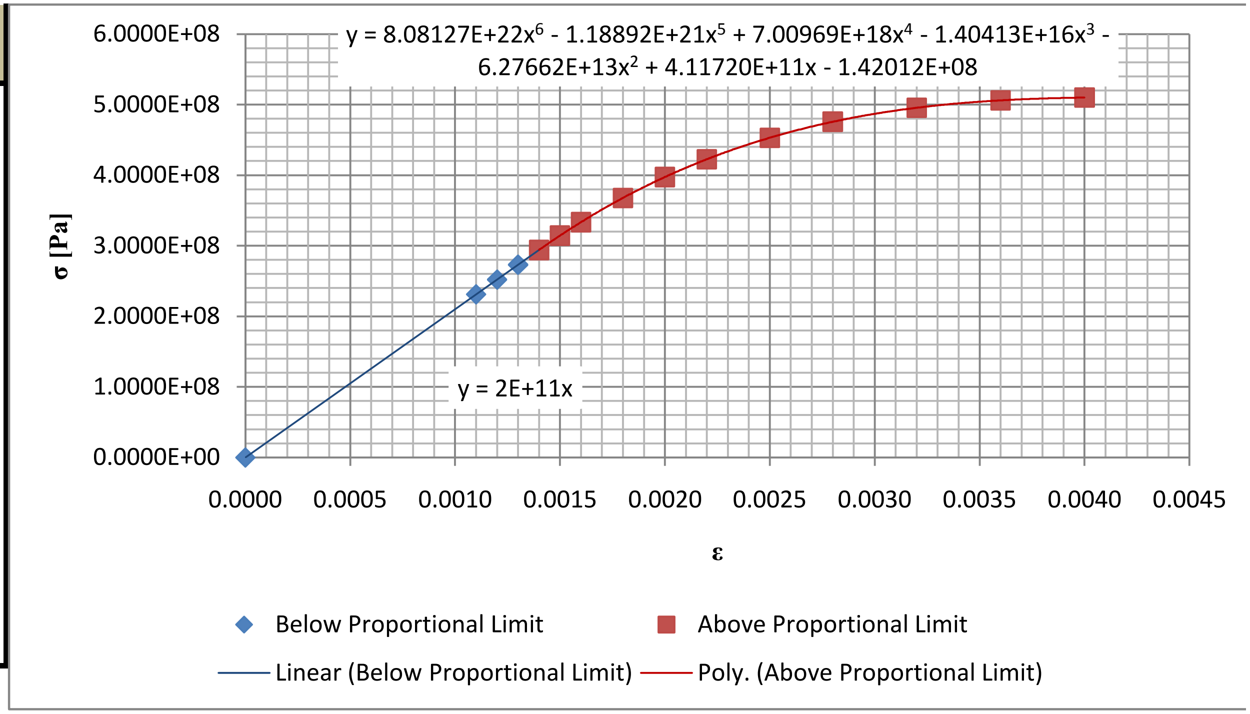
$$P_{CR,x} = \frac{\pi^2 \cdot E \cdot I}{k^2 \cdot L^2} = \frac{\pi^2 \cdot (200 \times 10^9 \frac{N}{m^2})(312.5 \times 10^{-9} m^4)}{(0.5m)^2} = 2.467 \times 10^6 N$$

$$\sigma_{CR,x} = \frac{P_{CR,x}}{A} = \frac{(2.467 \times 10^6 N)}{(0.03m)(0.05m)} = 1.6449 \times 10^9 \frac{N}{m^2} = 1.6449 GPa \qquad (5)$$

FROM THIS CALCULATION IT IS CLEAR THAT THE COLUMN WILL NOT BUCKLE IN A LINEAR-ELASTIC MODE.

BEFORE THE COLUMN CAN BE CONSIDERED FREE FROM THE POSSIBILITY OF BUCKLING, BUCKLING BEYOND THE ELASTIC LIMIT MUST BE CONSIDERED.

ε		σ	
mε	ε	MPa	Pa
0.0	0.0000	0	0.0000E+00
1.1	0.0011	231	2.3100E+08
1.2	0.0012	252	2.5200E+08
1.3	0.0013	273	2.7300E+08
1.4	0.0014	294	2.9400E+08
1.5	0.0015	314.3	3.1430E+08
1.6	0.0016	333.4	3.3340E+08
1.8	0.0018	367.7	3.6770E+08
2.0	0.0020	397.3	3.9730E+08
2.2	0.0022	422.6	4.2260E+08
2.5	0.0025	453	4.5300E+08
2.8	0.0028	475.7	4.7570E+08
3.2	0.0032	495.5	4.9550E+08
3.6	0.0036	506	5.0600E+08
4.0	0.0040	510	5.1000E+08



CALCULATING BUCKLING BEYOND THE PROPORTIONAL LIMIT REQUIRES THE USE OF THE TANGENT MODULUS IN PLACE OF YOUNG'S MODULUS IN THE BUCKLING EQUATION. THE TANGENT MODULUS IS EVALUATED ABOVE THE PROPORTIONAL LIMIT BY FIRST FITTING A 6TH DEGREE POLYNOMIAL THROUGH THE DATA GIVEN FROM 394 MPa AND BEYOND. THE RESULTS OF THE NUMERICAL CALCULATION IS SHOWN ON THE ATTACHED EXCEL SPREADSHEET.

$$\sigma = 8.081\times10^{22}\,Pa\cdot\epsilon^6 - 1.1889\times10^{21}\,Pa\cdot\epsilon^5 + 7.010\times10^{18}\,Pa\cdot\epsilon^4 - 1.4041\times10^{16}\,Pa\cdot\epsilon^3 - 6.277\times10^{13}\,Pa\cdot\epsilon^2 + 4.117\times10^{11}\,Pa\cdot\epsilon - 1.4201\times10^{8}\,Pa \qquad (6)$$

THE TANGENT MODULUS E_T IS FOUND BY TAKING THE DERIVATIVE OF (6) WITH RESPECT TO ϵ

$$E_T = \frac{d\sigma}{d\epsilon} = 6\cdot 8.081\times10^{22}\,Pa\cdot\epsilon^5 - 5\cdot 1.889\times10^{21}\,Pa\cdot\epsilon^4 + 4\cdot 7.010\times10^{18}\,Pa\cdot\epsilon^3 - 3\cdot 1.4041\times10^{16}\,Pa\cdot\epsilon^2 - 2\cdot 6.277\times10^{13}\,Pa\cdot\epsilon + 4.117\cdot10^{11}\,Pa$$

$$= 484.9\times10^{21}\,Pa\cdot\epsilon^5 - 5.944\times10^{21}\,Pa\cdot\epsilon^4 + 28.04\times10^{18}\,Pa\cdot\epsilon^3 - 42.12\times10^{15}\,Pa\cdot\epsilon^2 - 125.54\times10^{12}\,Pa\cdot\epsilon + 411.7\times10^{9}\,Pa \qquad (7)$$

THE BUCKLING EQUATION IN THE GENERAL FORM IS WRITTEN IN TERMS OF THE TANGENT MODULUS AS

$$P_{CR} = \frac{\pi^2\cdot E_T I}{K^2\cdot L^2}$$

THIS EQUATION CAN BE FURTHER MODIFIED BY DIVIDING BOTH SIDES BY THE CROSS-SECTIONAL AREA YIELDING A BUCKLING EQUATION THAT PREDICTS THE CRITICAL BUCKLING STRESS

$$\frac{P_{CR}}{A} = \sigma_{CR} = \frac{\pi^2\cdot E_T\cdot I}{A\cdot K^2\cdot L^2} = \frac{\pi^2\cdot I}{A\cdot K^2\cdot L^2}\,E_T \qquad (8)$$

(6) AND (7) CAN NOW BE SUBSTITUTED INTO (8)

$$80.81\times10^{21}\,Pa\cdot\epsilon^6 - 1.1889\times10^{21}\,Pa\cdot\epsilon^5 + 7.010\times10^{18}\,Pa\cdot\epsilon^4 - 14.041\times10^{15}\,Pa\cdot\epsilon^3 - 62.77\times10^{12}\,Pa\cdot\epsilon^2 + 411.7\times10^{9}\,Pa\cdot\epsilon - 142.01\times10^{6}\,Pa$$
$$= \left(\frac{I}{K^2}\right)\frac{\pi^2}{(.03m)(.05m)(5m)^2}\left[484.9\times10^{21}\,Pa\cdot\epsilon^5 - 5.944\times10^{21}\,Pa\cdot\epsilon^4 + 28.04\times10^{18}\,Pa\cdot\epsilon^3 - 42.12\times10^{15}\,Pa\cdot\epsilon^2 - 125.54\times10^{12}\,Pa\cdot\epsilon + 411.7\times10^{9}\,Pa\right]$$

$$= \left(\frac{I}{K^2}\right)\left[12.762\times10^{27}\,\frac{Pa}{m^4}\cdot\epsilon^5 - 156.44\times10^{24}\,\frac{Pa}{m^4}\cdot\epsilon^4 + 738.0\times10^{21}\,\frac{Pa}{m^4}\cdot\epsilon^3 - 1.1086\times10^{21}\,\frac{Pa}{m^4}\cdot\epsilon^2 - 3.304\times10^{18}\,\frac{Pa}{m^4}\cdot\epsilon + 10.836\times10^{15}\,\frac{Pa}{m^4}\right] \qquad (9)$$

THE VALUE OF I AND K ARE DEPENDENT ON THE AXIS ABOUT WHICH THE BUCKLING IS BEING CONSIDERED. ONCE THE AXES IS IDENTIFIED (9)'S ROOTS CAN BE DETERMINED AND THE CRITICAL BUCKLING LOAD CALCULATED.

STARTING WITH THE $\underline{Z\text{-}Z}$ AXIS THAT IS SUBJECTED TO FIXED-FIXED CONDITIONS

$$80.81\times10^{21}Pa\cdot\epsilon^6 - 1.1889\times10^{21}Pa\cdot\epsilon^5 + 7.010\times10^{18}Pa\cdot\epsilon^4 - 14.041\times10^{15}Pa\cdot\epsilon^3 - 62.77\times10^{12}Pa\cdot\epsilon^2 + 411.7\times10^9 Pa\cdot\epsilon - 142.01\times10^6 Pa$$

$$= \left(\frac{112.5\times10^{-9}m^4}{(0.5)^2}\right)\left[12.762\times10^{27}\tfrac{Pa}{m^4}\cdot\epsilon^5 - 156.44\times10^{24}\tfrac{Pa}{m^4}\cdot\epsilon^4 + 738.0\times10^{21}\tfrac{Pa}{m^4}\cdot\epsilon^3 - 1.1086\times10^{21}\tfrac{Pa}{m^4}\cdot\epsilon^2 - 3.304\times10^{18}\tfrac{Pa}{m^4}\cdot\epsilon + 10.836\times10^{15}\tfrac{Pa}{m^4}\right]$$

$$= 5.743\times10^{21}Pa\cdot\epsilon^5 - 70.40\times10^{18}Pa\cdot\epsilon^4 + 332.1\times10^{15}Pa\cdot\epsilon^3 - 498.9\times10^{12}Pa\cdot\epsilon^2 - 1.4868\times10^{12}Pa\cdot\epsilon + 4.876\times10^9 Pa$$

$$0 = 80.81\times10^{21}Pa\cdot\epsilon^6 - 6.932\times10^{21}Pa\cdot\epsilon^5 + 77.41\times10^{18}Pa\cdot\epsilon^4 - 346.1\times10^{15}Pa\cdot\epsilon^3 + 436.1\times10^{12}Pa\cdot\epsilon^2 + 1898.5\times10^9 Pa\cdot\epsilon - 5.018\times10^9 Pa$$

SOLVING FOR THE ROOTS OF THIS EQUATION ON A CALCULATOR (REAL ROOTS ONLY)

$$3121\mu\epsilon,\ \underbrace{5.078\mu\epsilon,\ -2323\mu\epsilon,\ 73530\mu\epsilon}_{\text{OUTSIDE THE DOMAIN OF THE PROBLEM}} \quad (10)$$

FOR THE $\underline{X\text{-}X}$ AXIS THAT IS SUBJECT TO PINNED-PINNED CONSTRAINTS

$$80.81\times10^{21}Pa\cdot\epsilon^6 - 1.1889\times10^{21}Pa\cdot\epsilon^5 + 7.010\times10^{18}Pa\cdot\epsilon^4 - 14.041\times10^{15}Pa\cdot\epsilon^3 - 62.77\times10^{12}Pa\cdot\epsilon^2 + 411.7\times10^9 Pa\cdot\epsilon - 142.01\times10^6 Pa$$

$$= \left(\frac{312.5\times10^{-9}m^4}{(1.0)^2}\right)\left[12.762\times10^{27}\tfrac{Pa}{m^4}\cdot\epsilon^5 - 156.44\times10^{24}\tfrac{Pa}{m^4}\cdot\epsilon^4 + 738.0\times10^{21}Pa/m^4\cdot\epsilon^3 - 1.1086\times10^{21}Pa/m^4\cdot\epsilon^2 - 3.304\times10^{18}Pa/m^4\cdot\epsilon + 10.836\times10^{15}Pa/m^4\right]$$

$$= 3.988\times10^{21}Pa\cdot\epsilon^5 - 48.89\times10^{18}Pa\cdot\epsilon^4 + 230.6\times10^{15}Pa\cdot\epsilon^3 - 346.4\times10^{12}Pa\cdot\epsilon^2 - 1.0325\times10^{12}\cdot\epsilon + 3.386\times10^9 Pa$$

$$0 = 80.81\times10^{21}Pa\cdot\epsilon^6 - 5.177\times10^{21}Pa\cdot\epsilon^5 + 55.50\times10^{18}Pa\cdot\epsilon^4 - 244.6\times10^{15}Pa\cdot\epsilon^3 + 283.6\times10^{12}Pa\cdot\epsilon^2 + 1.4442\times10^{12}Pa\cdot\epsilon - 3.528\times10^9 Pa$$

SOLVING FOR THE ROOTS OF THIS EQUATION ON A CALCULATOR (REAL ROOTS ONLY)

$$\boxed{2914\mu\epsilon},\ \underbrace{4.979\mu\epsilon,\ -2363\mu\epsilon,\ 51930\mu\epsilon}_{\text{OUTSIDE THE DOMAIN OF THE PROBLEM}} \quad (11)$$

THIS IS THE LOWEST VALUE OF STRAIN AT WHICH BUCKLING OCCURS THAT IS IN THE DOMAIN OF INTEREST

THE STRAIN IN (10) IS IDENTIFIED AS THE LOWEST VALUE OF STRAIN ABOVE THE PROPORTIONAL LIMIT WHERE BUCKLING WILL OCCUR. THE STRESS AT THIS STRAIN VALUE IS CALCULATED FROM (6)

$$\begin{aligned}\sigma_{CR} = {} & 8.081\times10^{22}\,Pa\cdot(2914\times10^{-6})^6 - 1.1889\times10^{21}\,Pa\cdot(2914\times10^{-6})^5 \\ & + 7.010\times10^{18}\,Pa\cdot(2914\times10^{-6})^4 - 1.4041\times10^{16}\,Pa\cdot(2914\times10^{-6})^3 \\ & - 6.277\times10^{13}\,Pa\cdot(2914\times10^{-6})^2 + 4.117\times10^{11}\,Pa\cdot(2914\times10^{-6}) \\ & - 1.4201\times10^{8}\,Pa\end{aligned}$$

$$= 482.4\times10^6\,Pa = 482.4\,MPa$$

$$P_{CR} = \sigma_{CR}\cdot A = (482.4\times10^6\,\tfrac{N}{m^2})(0.03\,m)(0.05\,m)$$

$$= 723.6\times10^3\,N = \boxed{723.6\,kN}$$

SUMMARY:

THE SOLUTION TO BUCKLING PROBLEMS REQUIRES THE CONSIDERATION OF BUCKLING ABOUT ALL AXES: X-X AND Z-Z. THESE TWO AXES ARE SUBJECTED WITH PINNED-PINNED AND FIXED-FIXED CONSTRAINTS RESPECTIVELY. BUCKLING IS FIRST CALCULATED AS IF IT OCCURS IN THE LINEAR-ELASTIC DOMAIN OF THE MATERIAL, BELOW THE PROPORTIONAL LIMIT. THE CRITICAL BUCKLING STRESS IS FOUND TO BE ABOVE THE PROPORTIONAL LIMIT SO THE BUCKLING EQUATION IS MODIFIED TO USE AN EXPRESSION FOR THE TANGENT MODULUS AND CRITICAL STRESS FOR LOADING. THE RESULTING ROOTS OF THIS EQUATION ARE FIRST CHECKED TO MAKE SURE THEY ARE REAL AND THEN TO SEE IF THEY ARE IN THE DOMAIN OF THE PROBLEM. THE LOWEST STRAIN VALUE IN THE DOMAIN OF THE PROBLEM IS USED TO CALCULATE THE CRITICAL LOAD.