Let's consider the following example

Equilibrium of the entire structure

$\Sigma F_x = \emptyset = A_x + B_x + P = \emptyset$

$\Rightarrow A_x + B_x = -P$ (1)

$\Sigma M_{z|at A} = \emptyset = -2a B_x - 3a P$

$\Rightarrow \boxed{B_x = -\frac{3}{2} P \quad A_x = \frac{1}{2} P}$

$\Sigma F_y = \emptyset = A_y$

Equilibrium for internal loads, $\emptyset < y < 2a$

$\Sigma F_x = \emptyset = \frac{P}{2} + V \Rightarrow V = -\frac{P}{2}$

$\Sigma M_{z|at p} = \emptyset = M + \frac{P}{2} y \Rightarrow M = -\frac{P}{2} y$

Equilibrium for internal loads, $2a < y < 3a$

$\Sigma F_x = \emptyset = -V + P \Rightarrow V = P$

$\Sigma M_{z|at p} = \emptyset = -M_p - (3a-y) P$

$\Rightarrow M_p = -(3a-y) P$

Now let's consider the slope and displacement, $\emptyset < y < 2a$

$$\frac{d^2u}{dy^2} = \frac{Py}{2EI}$$

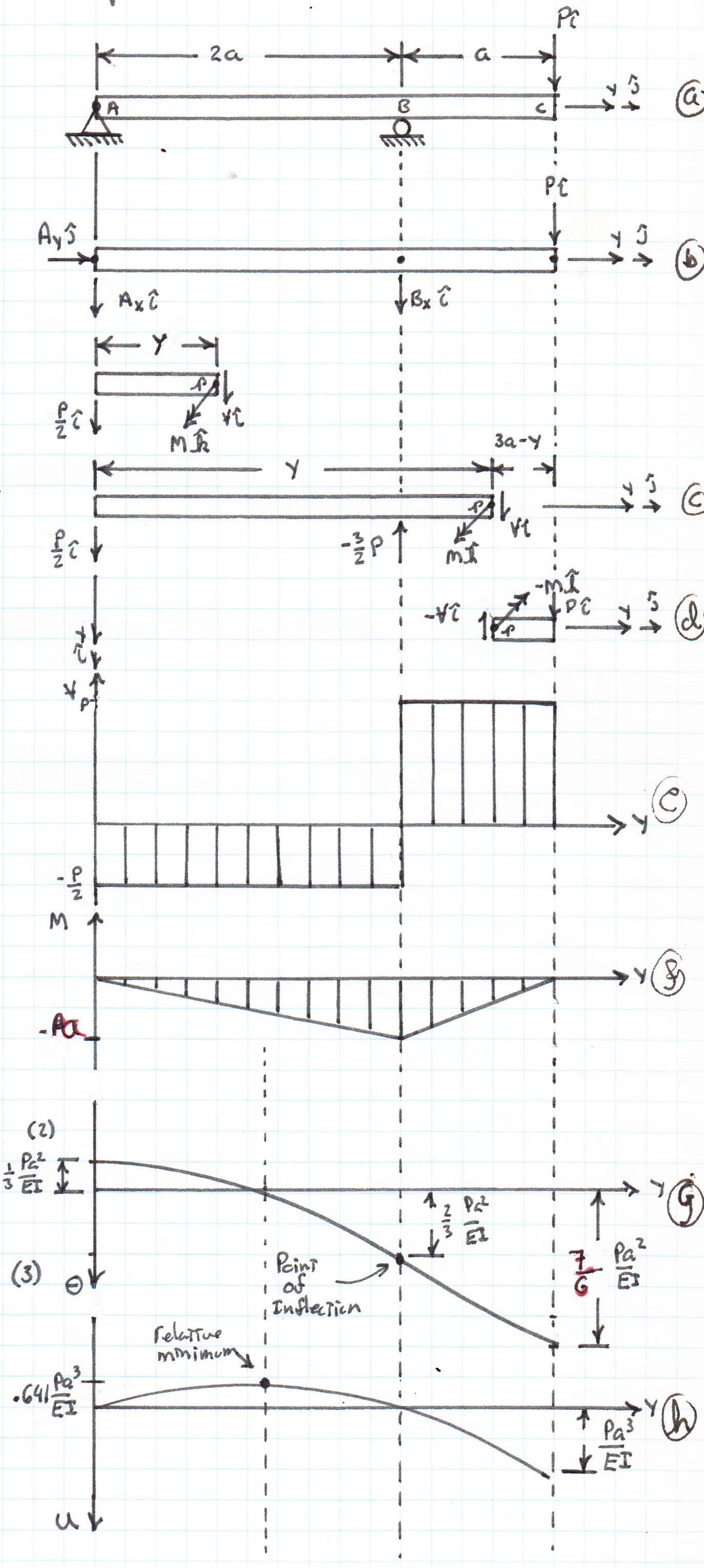
$$\frac{du}{dy} = \int \frac{Py}{2EI} dy = \frac{Py^2}{4EI} + C_1 = \Theta \quad (2)$$

$$u = \int \left[\frac{Py^2}{4EI} + C_1 \right] dy = \frac{Py^3}{12EI} + C_1 y + C_2 \quad (3)$$

Now we have to determine the constants of integration

at $x = \emptyset$, $u = \emptyset$

$u(\emptyset) = \emptyset = \emptyset + \emptyset + C_2 \Rightarrow C_2 = \emptyset$



Therefore we can rewrite (3)

$$u(y) = \frac{Py^3}{12EI} - c_1 y$$

We also know that at $y = 2a$, $u = \emptyset$

$$u(2a) = \emptyset = \frac{P \cdot 8 \cdot a^3}{12 \cdot EI} - c_1 \cdot 2 \cdot a \quad \Rightarrow \quad c_1 = \frac{Pa^2}{3EI}$$

Therefore the equations for the displacement and slope from $\emptyset < y < 2a$ can be written

$$\underline{\underline{u(y) = \frac{Py^3}{12EI} - \frac{Pa^2 y}{3EI}}} \qquad (4)$$

$$\underline{\underline{\emptyset < y < 2a}}$$

$$\underline{\underline{\theta(y) = \frac{du}{dy} = \frac{Py^2}{4EI} - \frac{Pa^2}{3EI} = \frac{P}{EI}\left(\frac{y^2}{4} - \frac{a^2}{3}\right)}} \qquad (5)$$

Let's look at the beginning and end of this region

$$u(\emptyset) = \emptyset \quad ; \quad \theta(\emptyset) = -\frac{Pa^2}{3EI} \qquad (6)$$

$$u(2a) = \emptyset \quad ; \quad \theta(2a) = \frac{2}{3}\frac{Pa^2}{EI} \qquad (7)$$

Now let's look at the region of the beam from $\underline{\underline{2a < y < 3a}}$

$$\frac{d^2u}{dy^2} = \frac{1}{EI}(-y + 3a)P = \frac{yP}{EI} - \frac{3aP}{EI}$$

$$\frac{du}{dy} = \int\left[\frac{-yP}{EI} + \frac{3aP}{EI}\right]dy = \frac{-Py^2}{2EI} + \frac{P \cdot 3 \cdot a \cdot y}{EI} + c_1 = \theta \qquad (8)$$

$$u = \int\left[-\frac{Py^2}{2EI} + \frac{3Pay}{EI} + c_1\right]dy = \frac{-Py^3}{6EI} + \frac{3}{2}\frac{Pay^2}{EI} + c_1 y + c_2 \qquad (9)$$

For this region the boundary conditions are

$$u(2a) = \emptyset \quad , \quad \theta(2a) = \frac{2}{3}\frac{P}{EI}$$

From (8)

$$+\frac{2}{3}\frac{Pa^2}{EI} = \frac{-P \cdot \overset{2}{\cancel{4}} \cdot a^2}{\cancel{2}EI} + \frac{P \cdot 3 \cdot a \cdot 2a}{EI} + c_1 = +\frac{4Pa^2}{EI} + c_1$$

$$c_1 = \frac{-4Pa^2}{EI} + \frac{2}{3}\frac{Pa^2}{EI} \quad \Rightarrow \quad c_1 = -\frac{10}{3}\frac{Pa^2}{EI}$$

Therefore (8) and (9) become

$$\frac{du}{dy} = \theta = \frac{Py^2}{2EI} - \frac{3Pay}{EI} + \frac{10}{3}\frac{Pa^2}{EI} = \frac{P}{EI}\left[-\frac{y^2}{2} + 3ay - \frac{10}{3}a^2\right] \qquad (10)$$

$$u = \frac{-Py^3}{6EI} + \frac{3}{2}\frac{Pay^2}{EI} - \frac{10}{3}\frac{Pa^2}{EI}y + c_2 = \frac{P}{EI}\left[\frac{-y^3}{6} + \frac{3}{2}ay^2 - \frac{10}{3}a^2 y\right] + c_2 \qquad (11)$$

APPLYING THE SECOND BOUNDARY CONDITION TO (11).

$$u(2a) = 0 = \frac{P}{EI}\left[-\frac{8a^3}{6} + \frac{12a^3}{2} - \frac{20 \cdot a^3}{3}\right] + C_2 = -\frac{2Pa^3}{EI} + C_2$$

$$\Rightarrow C_2 = \frac{2 \cdot Pa^3}{EI}$$

$$\boxed{\begin{aligned} u(y) &= \frac{P}{EI}\left[-\frac{y^3}{6} + \frac{3}{2} \cdot a \cdot y^2 - \frac{10}{3} \cdot a^2 \cdot y + 2a^3\right] \quad (12) \\ \Theta(y) &= -\frac{P}{EI}\left[\frac{y^2}{2} - 3 \cdot a \cdot y + \frac{10}{3} a^2\right] \quad (13) \end{aligned}} \qquad 2a < y < 3a$$

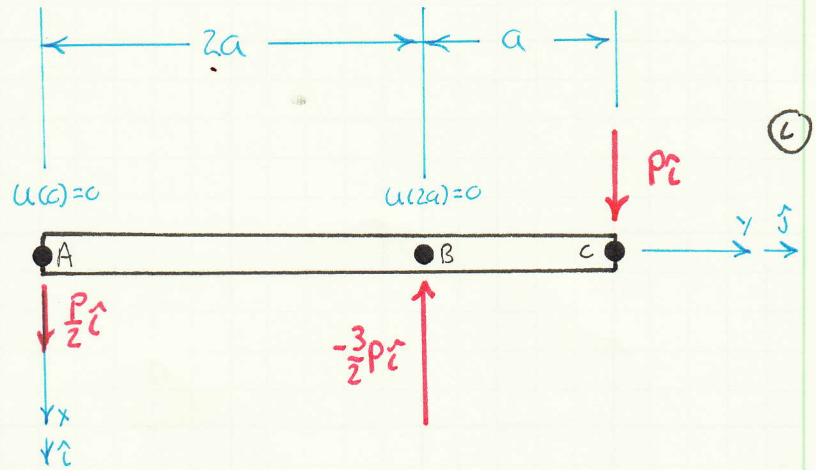
NOW THE VALUES OF u & Θ AT THE BOUNDARIES OF THIS REGION CAN BE DETERMINED.

$$u(2a) = 0 \qquad \Theta(2a) = \frac{2}{3}\frac{Pa^2}{EI}$$

$$u(3a) = \frac{Pa^3}{EI} \qquad \Theta(3a) = \frac{7}{6}\frac{Pa^2}{EI} \qquad (14)$$

THIS SAME EXAMPLE CAN NOW BE SOLVED USING SINGULARITY (MACAULAY) FUNCTIONS. THESE RESULTS WILL BE COMPARED WITH THE RESULTS DERIVED FROM THE DIRECT INTEGRATION APPROACH.

FIGURE (i) IS THE SOLUTION DIAGRAM THAT RESULTS FROM IMPOSING EQUILIBRIUM ON FIGURE (b). THIS IS THE STARTING POINT FOR THIS DEVELOPMENT.



USING THE MACAULAY FUNCTIONS, THE LOADING ON THE BEAM IS

$$q = \frac{P}{2}\langle y-0\rangle_{-1} - \frac{3}{2}P\langle y-2a\rangle_{-1} + P\langle y-3a\rangle_{-1} \quad (15)$$

FROM BEAM THEORY AND THE THEORY OF SINGULARITY FUNCTIONS

$$V(y) = \int -q(y)\,dy = \int\left[-\frac{P}{2}\langle y-0\rangle_{-1} + \frac{3}{2}P\langle y-2a\rangle_{-1} - P\langle y-3a\rangle_{-1}\right]dy$$

$$\underline{\underline{V(y) = -\frac{P}{2}\langle y-0\rangle^{0} + \frac{3}{2}P\langle y-2a\rangle^{0} - P\langle y-3a\rangle^{0}}} \quad (16)$$

$$M(y) = \int V(y)\,dy = \int\left[-\frac{P}{2}\langle y-0\rangle^{0} + \frac{3}{2}P\langle y-2a\rangle^{0} - P\langle y-3a\rangle^{0}\right]dy$$

$$\underline{\underline{M(y) = -\frac{P}{2}\langle y-0\rangle^{1} + \frac{3}{2}P\langle y-2a\rangle^{1} - P\langle y-3a\rangle^{1}}} \quad (17)$$

$$\Theta(y) = \frac{-1}{EI}\int M(y)\,dy = \int\left[\frac{P}{2\cdot E\cdot I}\langle y-0\rangle^{1} - \frac{3P}{2EI}\langle y-2a\rangle^{1} + \frac{P}{EI}\langle y-3a\rangle^{1}\right]dy$$

$$\Theta(y) = \frac{P}{4\cdot E\cdot I}\langle y-0\rangle^{2} - \frac{3P}{4EI}\langle y-2a\rangle^{2} + \frac{P}{2EI}\langle y-3a\rangle^{2} + C_1 \quad (17)$$

SINCE THERE ARE NO BOUNDARY CONDITIONS RELATED TO THE CURVATURE, C_1 WILL HAVE TO BE DETERMINED USING DISPLACEMENT BOUNDARY CONDITIONS.

$$u(y) = \int\Theta(y)\,dy = \int\left[\frac{P}{4EI}\langle y-0\rangle^{2} - \frac{3P}{4EI}\langle y-2a\rangle^{2} + \frac{P}{2EI}\langle y-3a\rangle^{2} + C_1\right]dy$$

$$u(y) = \frac{P}{12\cdot EI}\langle y-0\rangle^{3} - \frac{3P}{12EI}\langle y-2a\rangle^{3} + \frac{P}{6EI}\langle y-3a\rangle^{3} + C_1\cdot y + C_2 \quad (18)$$

THE FIRST DISPLACEMENT BOUNDARY CONDITION IS

$$u(0)=0=\frac{P}{12EI}\langle \overset{0}{0-0}\rangle^3-\frac{3P}{12EI}\langle \overset{-2a}{0-2a}\rangle^3+\frac{P}{6EI}\langle \overset{-3a}{0-3a}\rangle^3\cdot C_1\cdot(0)+C_2$$

$$=\frac{P}{12EI}(0)-\frac{3P}{12EI}(0)+\frac{P}{6EI}(0)+C_1\,0+C_2$$

$$\Rightarrow \underline{\underline{C_2=0}}$$

THE SECOND BOUNDARY CONDITION ON THE DISPLACEMENT IS AT $2a$, $u(2a)=0$

$$u(2a)=0=\frac{P}{12EI}\langle \overset{2a}{2a-0}\rangle^3-\frac{3P}{12EI}\langle \overset{0}{2a-2a}\rangle^3+\frac{P}{6EI}\langle \overset{-a}{2a-3a}\rangle^3+C_1\cdot 2a$$

$$0=\frac{P}{12EI}(2a)^3-\frac{3P}{12EI}(0)^3+\frac{P}{6EI}(0)^3+C_1\cdot 2a$$

$$0=\frac{8\cdot Pa^3}{12\cdot EI}+C_1\cdot 2a \quad\Rightarrow\quad C_1=-\frac{1}{3}\frac{Pa^2}{EI}$$

NOW (17) AND (18) CAN BE REWRITTEN

$$(17)\rightarrow \theta=\frac{P}{4EI}\langle y-0\rangle^2-\frac{3}{4}\frac{P}{EI}\langle y-2a\rangle^2+\frac{P}{2EI}\langle y-3a\rangle^2-\frac{Pa^2}{3EI} \qquad (19)$$

$$(18)\rightarrow u(y)=\frac{P}{12EI}\langle y-0\rangle^3-\frac{3P}{12EI}\langle y-2a\rangle^3+\frac{P}{6EI}\langle y-3a\rangle^3-\frac{Pa^2}{3EI}\cdot y \qquad (20)$$

LET'S CHECK SOME CRITICAL POINTS ALONG BE BEAM TO VERIFY (19) & (20) WITH THE DIRECT INTEGRATION APPROACH.

$$\theta(0)=\frac{P}{4EI}\langle 0-0\rangle^2-\frac{3}{4}\frac{P}{EI}\langle 0-2a\rangle^2+\frac{P}{2EI}\langle 0-3a\rangle^2-\frac{Pa^2}{3EI}$$

$$=\frac{P}{4EI}(0)^2-\frac{3}{4}\frac{P}{EI}(0)+\frac{P}{2EI}(0)-\frac{Pa^2}{3EI}=\underline{\underline{-\frac{Pa^2}{3EI}\ \checkmark\ w/(6)}}$$

$$\theta(2a)=\frac{P}{4EI}\langle 2a-0\rangle^2-\frac{3}{4}\frac{P}{EI}\langle 2a-2a\rangle^2+\frac{P}{2EI}\langle 2a-3a\rangle^2-\frac{Pa^2}{3EI}$$

$$=\frac{4Pa^2}{4EI}-\frac{3}{4}\frac{P}{EI}(0)^2+\frac{P}{2EI}(0)-\frac{Pa^2}{3EI}=\frac{Pa^2}{EI}\left[1-\frac{1}{3}\right]$$

$$=\underline{\underline{\frac{2Pa^2}{3EI}\ \checkmark\ w/\ (7)\ \&\ (14)}}$$

$$u(3a) = \frac{P}{12EI}\langle 3a-0\rangle^3 - \frac{3P}{12EI}\langle 3a-2a\rangle^3 + \frac{P}{6EI}\langle 3a-3a\rangle^3 - \frac{Pa^2}{3EI}(3a)$$

$$= \frac{P}{12EI}(3a)^3 - \frac{3P}{12EI}(a)^3 + \frac{P}{6EI}(0)^3 - \frac{Pa^2}{3EI}(3a)$$

$$= \frac{27 \cdot Pa^3}{12EI} - \frac{3 \cdot Pa^3}{12EI} - \frac{12Pa^3}{12EI} = \frac{12Pa^3}{12EI} = \qquad P$$

$$= \frac{Pa^3}{EI} \quad \checkmark \text{ w/ } 14$$