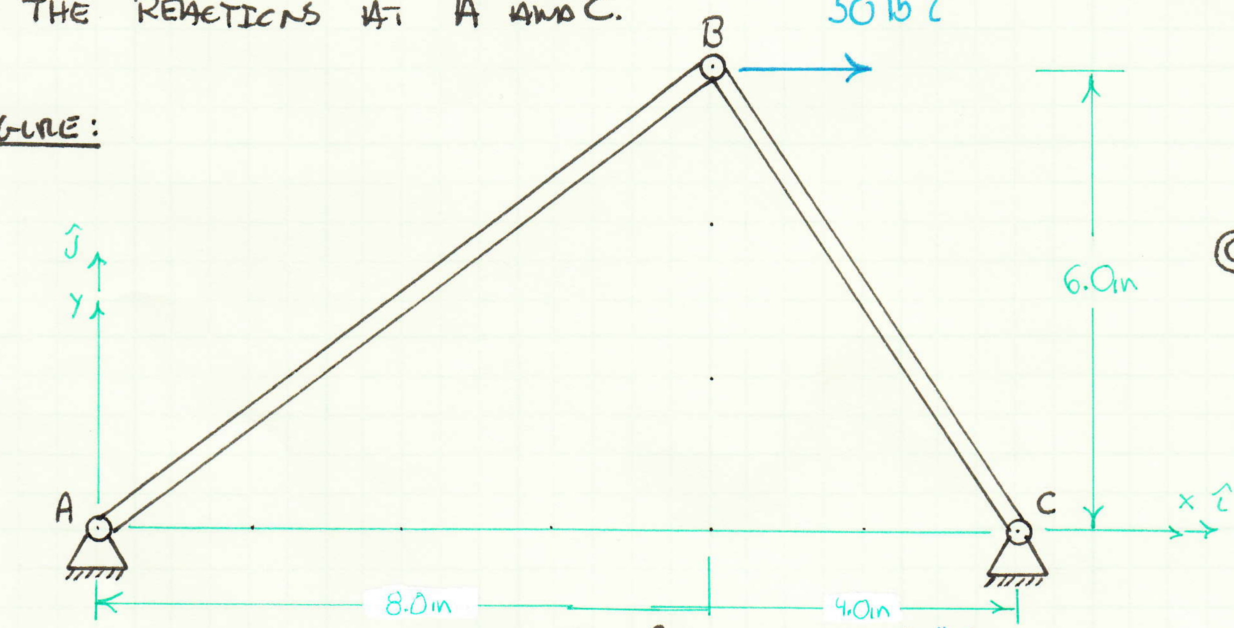
PROBLEM STATEMENT | THE TWO MEMBER TRUSS SHOWN IS BEING MODELED IN CLASS USING THE FINITE ELEMENT TECHNIQUE. USE THE TWO DIMENSIONAL TRUSS DEVELOPED IN THE LAB TO SOLVE FOR THE DEFLECTION AT B AND THE REACTIONS AT A AND B. THE TRUSS MEMBERS ~~ORE~~ ARE MADE OF CIRCULAR CROSS-SECTION ($d = 0.25 in$) ALLOY STEEL.

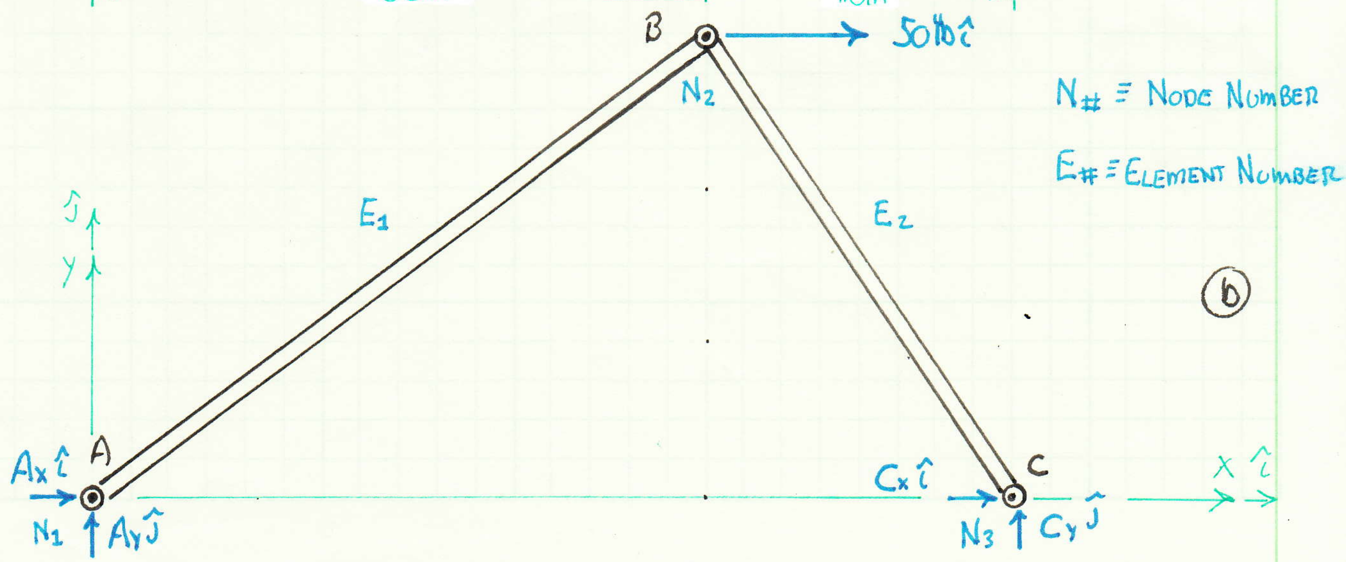
GIVEN:
1. TWO MEMBER TRUSS.
2. ~~A & B/ON~~ JOINTS A & C ON THE TRUSS ARE RESTRICTED FROM HORIZONTAL OR VERTICAL TRANSLATIONS.
3. BOTH MEMBERS OF THE ARE ATTACHED WITH A PIN JOINT AT B.
4. A 50 lb LOAD IS APPLIED AT C

ASSUMPTIONS:
1. SMALL DEFLECTIONS
2. MEMBERS ARE LINEAR ELASTIC AN CAN ONLY CARRY AXIAL LOADS.
3. THE JOINTS ARE FRICTIONLESS

FIND:
1. THE DEFLECTION OF POINT B.
2. THE REACTIONS AT A AND C.

FIGURE:





SOLUTION:

STARTING WITH THE COMPUTATION OF THE STIFFNESS OF THE TWO ELEMENTS IN THIS TRUSS.

$$k_1 = \frac{A_1 \cdot E_1}{L_1} = \frac{\pi \cdot (0.25\,in)^2}{4} \cdot \frac{30(10^6)\,lb/in^2}{\sqrt{(6.0in)^2 + (8.0in)^2}} = \underline{\underline{147.3(10^3)\frac{lb}{in}}} \quad (1)$$

$$k_2 = \frac{A_2 \cdot E_2}{L_2} = \frac{\pi \cdot (0.25\,in)^2}{4} \cdot \frac{30(10^6)\,lb/in^2}{\sqrt{(4.0in)^2 + (6.0in)^2}} = \underline{\underline{204.2(10^3)\,lb/in}} \quad (2)$$

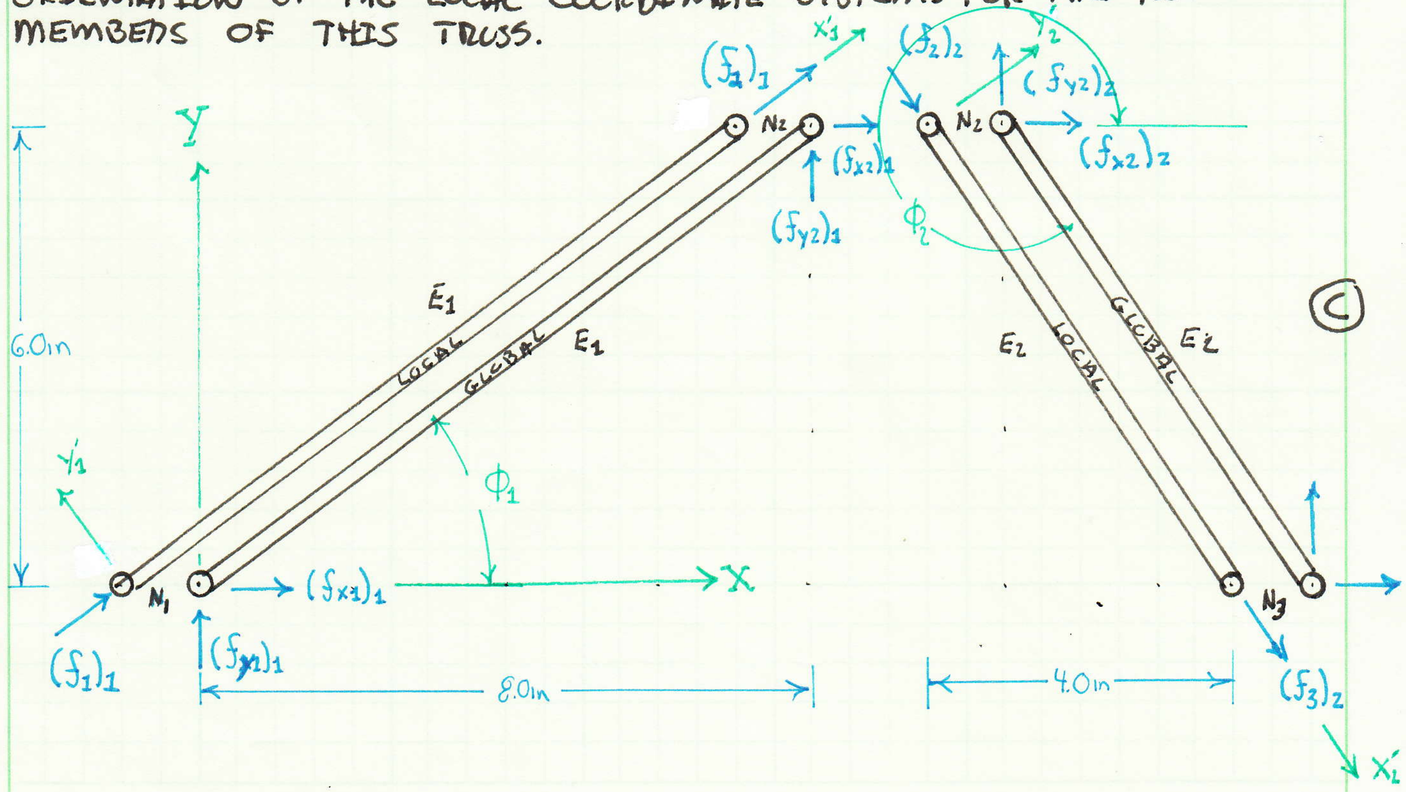
THE ANGLE EACH ELEMENT MAKES WITH THE POSITIVE HORIZONTAL AXIS (X). IT IS IMPORTANT THAT IN THE GENERAL FORM OF OF THE CALCULATION OF THIS ANGLE

$$\phi_I = \tan^{-1}\left(\frac{y_j - y_i}{x_j - x_i}\right)$$

WHERE "I" IS THE ELEMENT NUMBER, AND i & j ARE NODE NUMBERS, THAT THE CONVENTION THAT

- THE NODE "i" ~~IS~~ THE LOWER NODE NUMBER, AND
- THE NODE "j" IS THE LARGER NODE NUMBER

GIVEN THIS STANDARD CONVENTION (C) ILLUSTRATES ϕ_I AND THE ORIENTATION OF THE LOCAL COORDINATE SYSTEMS FOR THE TWO MEMBERS OF THIS TRUSS.



FOR THE TRUSS MEMBERS SHOWN IN (C)

$$\phi_1 = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \tan^{-1}\left(\frac{6in - 0}{8in - 0}\right) = \tan^{-1}\left(\frac{6in}{8in}\right) = 36.87^\circ \quad (3)$$

$$\phi_2 = \tan^{-1}\left(\frac{y_3 - y_2}{x_3 - y_3}\right) = \tan^{-1}\left(\frac{0in - 6in}{12in - 8in}\right) = \tan^{-1}\left(\frac{-6.0in}{4.0in}\right) = 303.69^\circ \quad (4)$$

THE TRANSFORMATION MATRICES FOR EACH ELEMENT CAN NOW BE CALCULATED.

$$\{u\}_{LOCAL} = [T]\{u\}_{GLOBAL} \quad \Rightarrow \quad (5)$$

$$\{f\}_{LOCAL} = [T]\{f\}_{GLOBAL} \quad (6)$$

LETTING $l = \cos\phi$ AND $m = \sin\phi$, THE TRANSFORMATION MATRICES FOR THE TWO ELEMENTS CAN BE WRITTEN

$$[T_1] = \begin{bmatrix} l_1 & m_1 & 0 & 0 \\ 0 & 0 & l_1 & m_1 \end{bmatrix} = \begin{bmatrix} 0.800 & 0.600 & 0 & 0 \\ 0 & 0 & 0.800 & 0.600 \end{bmatrix} \quad (7)$$

$$[T_2] = \begin{bmatrix} l_2 & m_2 & 0 & 0 \\ 0 & 0 & l_1 & m_2 \end{bmatrix} = \begin{bmatrix} 0.5547 & -0.8320 & 0 & 0 \\ 0 & 0 & 0.5547 & -0.8320 \end{bmatrix} \quad (8)$$

$\{u\}_{LOCAL}$ AND $\{f\}_{LOCAL}$ ARE ONE DIMENSIONAL ELEMENTS AND $\{u\}_{GLOBAL}$ AND $\{f\}_{GLOBAL}$ ARE TWO DIMENSIONAL. THE SECOND DIMENSION IS NOW ADDED TO THE LOCAL SYSTEM EVEN THOUGH IT WILL ALWAYS BE 0.

$$\{u\}_{LOCAL} = \begin{Bmatrix} u_{x'i} \\ u_{x'j} \end{Bmatrix}_1 \Rightarrow \begin{Bmatrix} u_{x'i} \\ u_{y'i} \\ u_{x'j} \\ u_{y'j} \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \begin{Bmatrix} u_{xi} \\ u_{yi} \\ u_{yj} \\ u_{yj} \end{Bmatrix} \quad (9)$$

$$\{f\}_{LOCAL} = \begin{Bmatrix} f_{x'i} \\ f_{x'j} \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{x'i} \\ f_{y'i} \\ f_{x'j} \\ f_{y'j} \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \begin{Bmatrix} f_{xi} \\ f_{yi} \\ u_{xj} \\ u_{yj} \end{Bmatrix} \quad (10)$$

THE TRANSFORMATION MATRICES IN (7) & (8) CAN NOW BE ~~RE~~ EXPANDED

$$[T_1] = \begin{bmatrix} 0.800 & 0.600 & 0 & 0 \\ -0.600 & 0.800 & 0 & 0 \\ 0 & 0 & 0.800 & 0.600 \\ 0 & 0 & -0.600 & 0.800 \end{bmatrix} \quad (11)$$

$$[T_2] = \begin{bmatrix} 0.5547 & -0.8320 & 0 & 0 \\ 0.8320 & 0.5547 & 0 & 0 \\ 0 & 0 & 0.5547 & -0.8320 \\ 0 & 0 & 0.8320 & 0.5547 \end{bmatrix} \quad (12)$$

THE LOCAL STIFFNESS MATRICES $[k]_{LOCAL}$ FOR EACH ELEMENT IN THE TRUSS NEEDS TO BE CALCULATED

$$\{f\}_{LOCAL} = [k]_{LOCAL}\{u\}_{LOCAL} \tag{13}$$

GIVEN THE STIFFNESS CALCULATIONS IN (1) & (2) AND THE DEFINITION OF THE LOCAL MATRIX IN (13)

$$[k]_{LOCAL} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

THE LOCAL STIFFNESS MATRICES CAN NOW BE CALCULATED

$$[k_1]_{LOCAL} = 147.3(10^3)\,\frac{lb}{in}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{14}$$

$$[k_2]_{LOCAL} = 204.2(10^3)\,\frac{lb}{in}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15}$$

(14) AND (15) NEED TO BE EXPANDED SO THEY ARE THE SAME DIMENSION AS THE GLOBAL ~~FA~~ PARAMETERS, ~~BUT~~ AND INSURE THAT THE LOCAL LOADS ARE AXIAL.

$$[k_1]_{LOCAL} = 147.3(10^3)\,\frac{lb}{in}\cdot\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

$$[k_2]_{LOCAL} = 204.2(10^3)\,\frac{lb}{in}\cdot\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

THE LOCAL STIFFNESS MATRICES IN (16) AND (17) NEED TO BE TRANSFORMED INTO THE GLOBAL SYSTEM. STARTING WITH (13)

$$\{f\}_{LOCAL} = [k]_{LOCAL}\{u\}_{LOCAL}$$

SUBSTITUTING IN (5) & (6)

$$[T]\cdot\{f\}_{GLOBAL} = [k]_{LOCAL}\cdot[T]\cdot\{u\}_{GLOBAL}$$

$$[T]^{-1}\cdot[T]\cdot\{f\}_{GLOBAL} = [T]^{-1}[k]_{LOCAL}[T]\cdot\{u\}_{GLOBAL}$$

$$\{f\}_{GLOBAL} = \underbrace{[T]^{-1}[k]_{LOCAL}[T]}_{\text{GLOBAL STIFFNESS } [K]}\{u_{GLOBAL}\}$$

$$\{f\}_{GLOBAL} = [K]\cdot\{u\}_{GLOBAL} \tag{18}$$

THE GLOBAL STIFFNESS MATRIX FOR EACH ELEMENT CAN NOW BE CALCULATED

$$[K]_{GLOBAL} = [T]^{-1}[k]_{LOCAL}[T] = [T]^{T}[k]_{LOCAL}[T] \tag{19}$$

$[T]^{-1} = [T]^{T}$ BECAUSE $[T]$ IS AN ORTHOGONAL MATRIX. (16) AND (17) CAN NOW BE TRANSFORMED INTO THE GLOBAL SYSTEM USING (11) & (12)

$$[K_1] = \begin{bmatrix} 0.800 & -0.600 & 0 & 0 \\ 0.600 & 0.800 & 0 & 0 \\ 0 & 0 & 0.800 & -0.600 \\ 0 & 0 & 0.600 & 0.800 \end{bmatrix} \cdot 147.3(10^3)\,\frac{lb}{in} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 0.800 & 0.600 & 0 & 0 \\ -0.600 & 0.800 & 0 & 0 \\ 0 & 0 & 0.800 & 0.600 \\ 0 & 0 & -0.600 & 0.800 \end{bmatrix}$$

$$= 147.3(10^3)\,\frac{lb}{in} \begin{bmatrix} 0.6400 & 0.4800 & -0.6400 & -0.4800 \\ 0.4800 & 0.3600 & -0.4800 & -0.3600 \\ -0.6400 & -0.4800 & 0.6400 & 0.4800 \\ -0.4800 & -0.3600 & 0.4800 & 0.3600 \end{bmatrix} \tag{20}$$

$$[K_2] = \begin{bmatrix} 0.5547 & 0.8320 & 0 & 0 \\ -0.8320 & 0.5547 & 0 & 0 \\ 0 & 0 & 0.5547 & 0.8320 \\ 0 & 0 & -0.8326 & 0.5547 \end{bmatrix} \cdot 204.2(10^3)\,\frac{lb}{in} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 0.5547 & -0.8326 & 0 & 0 \\ 0.8326 & 0.5547 & 0 & 0 \\ 0 & 0 & 0.5547 & -0.8326 \\ 0 & 0 & 0.8320 & 0.5547 \end{bmatrix}$$

$$= 204.2(10^3)\,\frac{lb}{in} \begin{bmatrix} 0.3077 & -0.4615 & -0.3077 & 0.4615 \\ -0.4615 & 0.6923 & 0.4615 & -0.6923 \\ -0.3077 & 0.4615 & 0.3077 & -0.4615 \\ 0.4615 & -0.6923 & -0.4615 & 0.6923 \end{bmatrix} \tag{21}$$

Global Stiffness w/o Constant

Does not include stiffness constant

1

$[T]^T=$

0.8	-0.6	0	0
0.6	0.8	0	0
0	0	0.8	-0.6
0	0	0.6	0.8

[k*]=

1	0	-1	0
0	0	0	0
-1	0	1	0
0	0	0	0

[T]=

0.8	0.6	0	0
-0.6	0.8	0	0
0	0	0.8	0.6
0	0	-0.6	0.8

[K]=

0.6400	0.4800	-0.6400	-0.4800
0.4800	0.3600	-0.4800	-0.3600
-0.6400	-0.4800	0.6400	0.4800
-0.4800	-0.3600	0.4800	0.3600

2

$[T]^T=$

0.554747	0.83207	0	0
-0.83207	0.554747	0	0
0	0	0.554747	0.83207
0	0	-0.83207	0.554747

[k*]=

1	0	-1	0
0	0	0	0
-1	0	1	0
0	0	0	0

[T]=

0.5547	-0.832	0	0
0.832	0.5547	0	0
0	0	0.5547	-0.832
0	0	0.832	0.5547

[K]=

0.3077	-0.4615	-0.3077	0.4615
-0.4615	0.6923	0.4615	-0.6923
-0.3077	0.4615	0.3077	-0.4615
0.4615	-0.6923	-0.4615	0.6923

THEREFORE

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = 147.3(10^3)\frac{\text{lb}}{\text{in}} \begin{bmatrix} 0.6400 & 0.4800 & -0.6400 & -0.4800 \\ 0.4800 & 0.3600 & -0.4800 & -0.3600 \\ -0.6400 & -0.4800 & 0.6400 & 0.4800 \\ -0.4800 & -0.3600 & 0.4800 & 0.3600 \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{Bmatrix} \quad (22)$$

$$\begin{Bmatrix} f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = 204.2(10^3)\frac{\text{lb}}{\text{in}} \begin{bmatrix} 0.3077 & -0.4615 & -0.3077 & 0.4615 \\ -0.4615 & 0.6923 & 0.4615 & -0.6923 \\ -0.3077 & 0.4615 & 0.3077 & -0.4615 \\ 0.4615 & -0.6923 & -0.4615 & 0.6923 \end{bmatrix} \begin{Bmatrix} u_{2x} \\ u_{2y} \\ u_{3y} \\ u_{3y} \end{Bmatrix} \quad (23)$$

(22) AND (23) ARE NOW COMBINED INTO THE SYSTEM STIFFNESS MATRIX.

$$\{F\} = [K_s]\{U\} \quad (24)$$

WHERE

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3y} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} (f_{1x})_1 \\ (f_{1y})_1 \\ (f_{2x})_1 + (f_{2x})_2 \\ (f_{2y})_1 + (f_{2y})_2 \\ (f_{3x})_2 \\ (f_{3y})_2 \end{Bmatrix} = \begin{Bmatrix} A_x \\ A_y \\ 50\text{ lb} \\ 0\text{ lb} \\ C_x \\ C_y \end{Bmatrix} \quad (25)$$

$$\begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3y} \\ u_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_{2x} \\ u_{2y} \\ 0 \\ 0 \end{Bmatrix} \quad (26)$$

$$[K_s] = \begin{bmatrix} 942.7 & 707.0 & -942.7 & -707.0 & 0 & 0 \\ 707.0 & 530.3 & -707.0 & -530.3 & 0 & 0 \\ -942.7 & -707.0 & 1571.1 & -235.4 & -628.4 & 942.5 \\ -707.0 & -530.3 & -235.4 & 1944.0 & 942.5 & -1413.6 \\ 0 & 0 & -628.4 & 942.5 & 628.4 & -942.5 \\ 0 & 0 & 942.5 & -1413.6 & -942.5 & 1413.6 \end{bmatrix} \times 10^3 \frac{\text{lb}}{\text{in}} \quad (27)$$

constant

Global Stiffness Matrix w/ Constant

1.47E+05	94272	70704	-94272	-70704
	70704	53028	-70704	-53028
	-94272	-70704	94272	70704
	-70704	-53028	70704	53028

94272	70704	-94272	-70704	0	0
70704	53028	-70704	-53028	0	0
-94272	-70704	94272	70704	0	0
-70704	-53028	70704	53028	0	0
0	0	0	0	0	0
0	0	0	0	0	0

2.04E+05	62836	-94248	-62836	94248
	-94248	141364	94248	-141364
	-62836	94248	62836	-94248
	94248	-141364	-94248	141364

0	0	0	0	0	0
0	0	0	0	0	0
0	0	62836	-94248	-62836	94248
0	0	-94248	141364	94248	-141364
0	0	-62836	94248	62836	-94248
0	0	94248	-141364	-94248	141364

SYSTEM STIFFNESS MATRIX

94272	70704	-94272	-70704	0	0
70704	53028	-70704	-53028	0	0
-94272	-70704	157108	-23544	-62836	94248
-70704	-53028	-23544	194392	94248	-141364
0	0	-62836	94248	62836	-94248
0	0	94248	-141364	-94248	141364

THE ROWS ASSOCIATED WITH KNOWN DISPLACEMENTS ARE PARTITIONED OUT OF THE GLOBAL EQUATION, (27)

$$\begin{Bmatrix} A_x \\ A_y \\ C_x \\ C_y \end{Bmatrix} = \begin{bmatrix} 942.7 & 707.0 & -942.7 & -707.0 & 0 & 0 \\ 707.0 & 530.3 & -707.0 & -530.3 & 0 & 0 \\ 0 & 0 & -628.4 & 942.5 & 628.4 & -942.5 \\ 0 & 0 & 942.5 & -1413.6 & -942.5 & 1413.6 \end{bmatrix} \times 10^3 \frac{lb}{in} \begin{Bmatrix} 0 \\ 0 \\ U_{2x} \\ U_{2y} \\ 0 \\ 0 \end{Bmatrix} \quad (28)$$

$$\begin{Bmatrix} A_x \\ A_y \\ C_x \\ C_y \end{Bmatrix} = \begin{bmatrix} -942.7 & -707.0 \\ -707.0 & -530.3 \\ -628.4 & 942.5 \\ 942.5 & -1413.6 \end{bmatrix} \times 10^3 \frac{lb}{in} \begin{Bmatrix} U_{2x} \\ U_{2y} \end{Bmatrix}$$

IN THE REMAINING MATRIX,

$$\begin{Bmatrix} 50 lb \\ 0 lb \end{Bmatrix} = \begin{bmatrix} -942.7 & -707.0 & 1571.1 & -235.4 & -628.4 & 942.5 \\ -707.0 & -530.3 & -235.4 & 1944.0 & 942.5 & -1413.6 \end{bmatrix} \times 10^3 \frac{lb}{in} \begin{Bmatrix} 0 \\ 0 \\ U_{2x} \\ U_{2y} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 50 lb \\ 0 lb \end{Bmatrix} = \begin{bmatrix} 1571.1 & -235.4 \\ -235.4 & 1944.0 \end{bmatrix} \times 10^3 \frac{lb}{in} \begin{Bmatrix} U_{2x} \\ U_{2y} \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} U_{2x} \\ U_{2y} \end{Bmatrix} = \begin{Bmatrix} 0.3241 \\ 0.03926 \end{Bmatrix} \times 10^{-3} in \quad (29)$$

(29) CAN NOW BE SUBSTITUTED INTO (28) TO DETERMINE THE REACTIONS.

$$\begin{Bmatrix} A_x \\ A_y \\ C_x \\ C_y \end{Bmatrix} = \begin{bmatrix} 942.7 & 707.0 & -942.7 & -707.0 & 0 & 0 \\ 707.0 & 530.3 & -707.0 & -530.3 & 0 & 0 \\ 0 & 0 & -628.4 & 942.5 & 628.4 & -942.5 \\ 0 & 0 & 942.5 & -1413.6 & -942.5 & 1413.6 \end{bmatrix} \times 10^3 \frac{lb}{in} \begin{Bmatrix} 0 \\ 0 \\ 0.3241 \\ 0.03926 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-3} in$$

$$= \begin{Bmatrix} -33.3 \\ -25.0 \\ -16.7 \\ 25.0 \end{Bmatrix} lb$$

<u>SUMMARY:</u> ALL CALCULATIONS IN THIS EXAMPLE WERE PERFORMED USING EXCEL.

$$\begin{bmatrix} 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 157108 & -23544 \\ -23544 & 194392 \end{bmatrix} \bullet \begin{bmatrix} U_{2x} \\ U_{2y} \end{bmatrix}$$

$$\begin{bmatrix} U_{2x}= \\ U_{2y}= \end{bmatrix} = \begin{bmatrix} 6.48271E-06 & 7.8517E-07 \\ 7.85172E-07 & 5.2393E-06 \end{bmatrix} \bullet \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

$$\begin{matrix} U_{2x}= \\ U_{2y}= \end{matrix} \boxed{\begin{matrix} \mathbf{3.2414E-04} \\ \mathbf{3.9259E-05} \end{matrix}}$$

$$\begin{bmatrix} A_x \\ A_y \\ C_x \\ C_y \end{bmatrix} = \begin{bmatrix} 94272 & 70704 & -94272 & -70704 & 0 & 0 \\ 70704 & 53028 & -70704 & -53028 & 0 & 0 \\ 0 & 0 & -62836 & 94248 & 62836 & -94248 \\ 0 & 0 & 94248 & -141364 & -94248 & 141364 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 3.2414E-04 \\ 3.9259E-05 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ C_x \\ C_y \end{bmatrix} = \boxed{\begin{matrix} \mathbf{-33.33} \\ \mathbf{-25.00} \\ \mathbf{-16.67} \\ \mathbf{25.00} \end{matrix}}$$