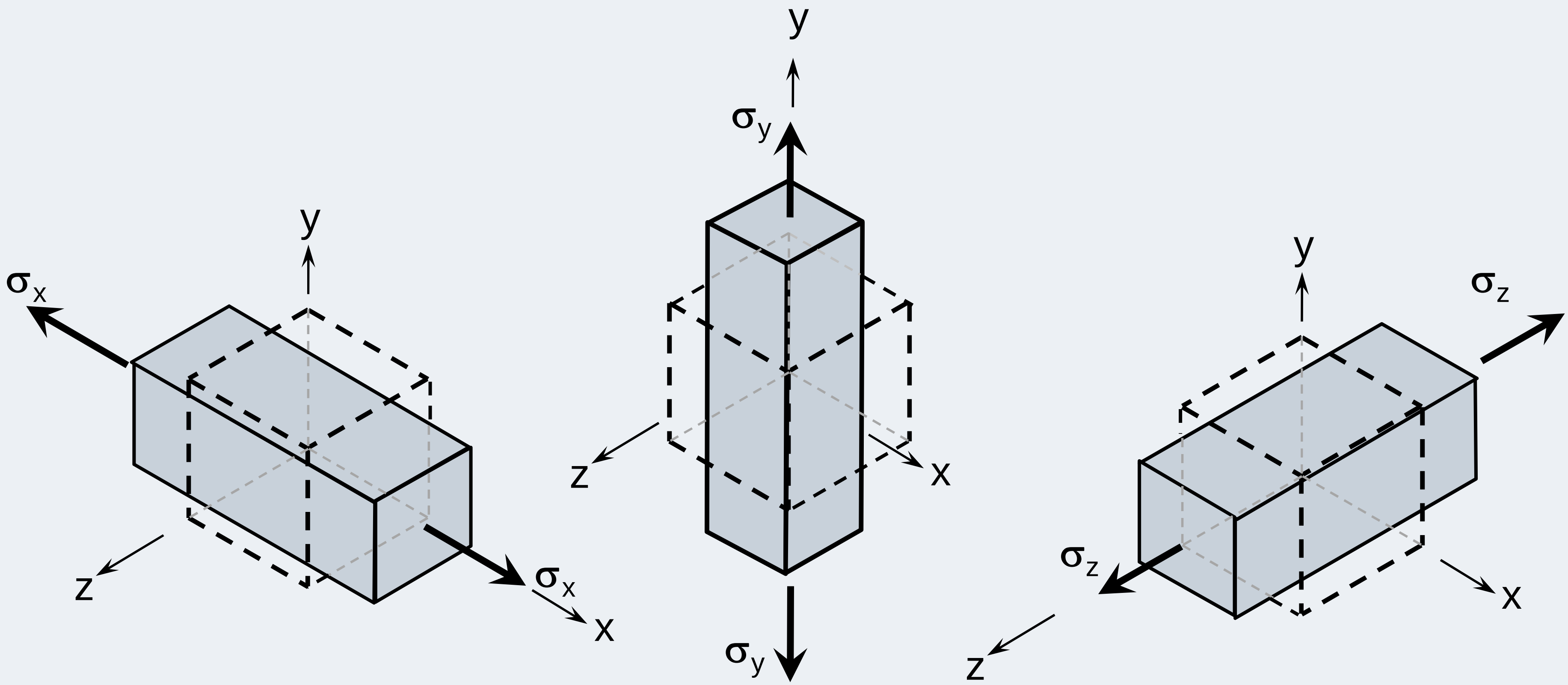
Strain – Displacement Relations

Relationship Between Stress and Strain



Stress-Strain Relations

$$\varepsilon_x = \frac{1}{E}\cdot\left[\sigma_x - \nu\cdot\left(\sigma_y + \sigma_z\right)\right] \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_y = \frac{1}{E}\cdot\left[\sigma_y - \nu\cdot\left(\sigma_x + \sigma_z\right)\right] \qquad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\varepsilon_z = \frac{1}{E}\cdot\left[\sigma_z - \nu\cdot\left(\sigma_y + \sigma_x\right)\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

Matrix Form of Stress-Strain Relations

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Matrix Form of Stress-Strain Relations

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\cdot(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\cdot(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\cdot(1+\nu)}{E} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Strain-Stress Relations

$$\sigma_x = \frac{E}{(1+\nu)\cdot(1-2\cdot\nu)}\left[(1-\nu)\cdot\varepsilon_x + \nu\cdot\left(\varepsilon_y + \varepsilon_z\right)\right] \qquad \boldsymbol{\tau_{yz} = \frac{\gamma_{yz}}{G}}$$

$$\sigma_y = \frac{E}{(1+\nu)\cdot(1-2\cdot\nu)}\left[(1-\nu)\cdot\varepsilon_y + \nu\cdot\left(\varepsilon_x + \varepsilon_z\right)\right] \qquad \boldsymbol{\tau_{xz} = \frac{\gamma_{xz}}{G}}$$

$$\sigma_z = \frac{E}{(1+\nu)\cdot(1-2\cdot\nu)}\left[(1-\nu)\cdot\varepsilon_z + \nu\cdot\left(\varepsilon_x + \varepsilon_y\right)\right] \qquad \boldsymbol{\tau_{xy} = \frac{\gamma_{xy}}{G}}$$

Alternate Form of the Strain-Stress Relations

$$\sigma_x = 2 \cdot G \cdot \varepsilon_x + \lambda \cdot \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)$$

$$\sigma_y = 2 \cdot G \cdot \varepsilon_y + \lambda \cdot \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)$$

$$\sigma_z = 2 \cdot G \cdot \varepsilon_z + \lambda \cdot \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)$$

$$\tau_{xy} = G \cdot \gamma_{xy}$$

$$\tau_{yz} = G \cdot \gamma_{yz}$$

$$\tau_{zx} = G \cdot \gamma_{zx}$$

Shear Modulus

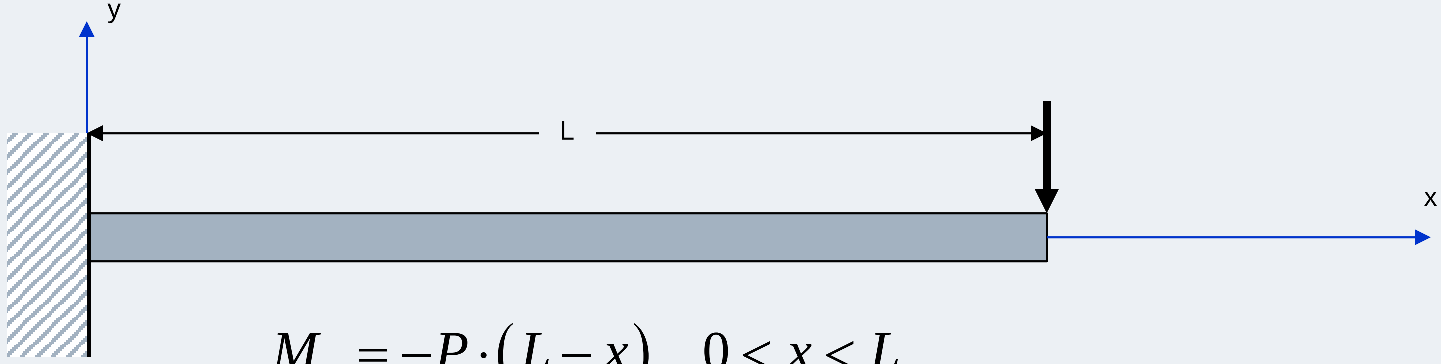
$$G = \frac{E}{2\left(1+\nu\right)}$$

Lame′s

$$\lambda = \frac{\nu \cdot E}{\left(1+\nu\right) \cdot \left(1-2 \cdot \nu\right)}$$

EXAMPLE: Compatibility

For the beam shown, determine the displacement field due to BENDING ONLY. Consider the cross section of the beam to be rectangular and thin so that deflections are not functions of z.



$$M_z = -P \cdot (L - x) \quad 0 < x < L$$

SOLUTION: Basic Mechanics

Given the Bending Moment in the Beam

$$M_z = -P \cdot (L - x) \quad 0 < x < L$$

The Stress Field from Strength of Materials

$$\sigma_x = -\frac{M_z \cdot y}{I_z} = \frac{P}{I_z} \cdot y \cdot (L - x)$$

$$\sigma_y = 0$$

$$\tau_{xy} = 0$$

Strains can not be computed using Hooke's Law

$$\varepsilon_x = \frac{1}{E} \cdot (\sigma_x - \nu \cdot \sigma_y) = \frac{P}{E \cdot I_z} \cdot y \cdot (L - x)$$

$$\varepsilon_y = \frac{1}{E} \cdot (\sigma_y - \nu \cdot \sigma_x) = -\frac{\nu \cdot P}{E \cdot I_z} \cdot y \cdot (L - x)$$

$$\gamma_{xy} = \frac{2 \cdot (1 + \nu)}{E} \cdot \tau_{xy} = 0$$

SOLUTION: Strain-Displacement Equations

$$\varepsilon_x = \frac{1}{E}\cdot\left(\sigma_x - \nu\cdot\sigma_y\right) = \frac{P}{E\cdot I_z}\cdot y\cdot(L-x)$$

$$\varepsilon_y = \frac{1}{E}\cdot\left(\sigma_y - \nu\cdot\sigma_x\right) = -\frac{\nu\cdot P}{E\cdot I_z}\cdot y\cdot(L-x)$$

$$\gamma_{xy} = \frac{2\cdot(1+\nu)}{E}\cdot\tau_{xy} = 0$$

Strain-Displacement Relationships

$$\frac{\partial u}{\partial x} = \varepsilon_x = \frac{P}{E\cdot I_z}\cdot y\cdot(L-x) \quad \Rightarrow \quad u = \frac{P}{2\cdot E\cdot I_z}\cdot x\cdot y\cdot(2\cdot L - x) + f(y)$$

$$\frac{\partial v}{\partial y} = \varepsilon_y - \frac{\nu\cdot P}{E\cdot I_z}\cdot y\cdot(L-x) \quad \Rightarrow \quad v = -\frac{\nu\cdot P}{2\cdot E\cdot I_z}\cdot y^2\cdot(L-x) + g(y)x$$

SOLUTION: Shear Strain-Displacement

$$u = \frac{P}{2 \cdot E \cdot I_z} \cdot x \cdot y \cdot (2 \cdot L - x) + f(y)$$

$$v = -\frac{\nu \cdot P}{2 \cdot E \cdot I_z} \cdot y^2 \cdot (L - x) + g(x)$$

Shear Strain-Displacement Relationships

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\nu \cdot P}{2 \cdot E \cdot I_z} \cdot y^2 + \frac{\partial g(x)}{\partial x} + \frac{P}{2 \cdot E \cdot I_z} \cdot x \cdot (2 \cdot L - x) + \frac{\partial f(y)}{\partial y}$$

Given the shear stress is zero

$$\gamma_{xy} = 0 = \frac{\nu \cdot P}{2 \cdot E \cdot I_z} \cdot y^2 + \frac{\partial g(x)}{\partial x} + \frac{P}{2 \cdot E \cdot I_z} \cdot x \cdot (2 \cdot L - x) + \frac{\partial f(y)}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g(x)}{\partial x} + \frac{P}{2 \cdot E \cdot I_z} \cdot x \cdot (2 \cdot L - x) = -\frac{\partial f(y)}{\partial y} - \frac{\nu \cdot P}{2 \cdot E \cdot I_z} \cdot y^2$$

SOLUTION: Integrating the Result

$$\frac{\partial g(x)}{\partial x}+\frac{P}{2\cdot E\cdot I_z}\cdot x\cdot(2\cdot L-x)=-\frac{\partial f(y)}{\partial y}-\frac{\nu\cdot P}{2\cdot E\cdot I_z}\cdot y^2=C_1$$

$$\frac{\partial g(x)}{\partial x}+\frac{P}{2\cdot E\cdot I_z}\cdot x\cdot(2\cdot L-x)=C_1 \quad\Rightarrow\quad g(x)=-\frac{P}{6\cdot E\cdot I_z}\cdot x^2\cdot(3\cdot L-x)+C_1\cdot x+C_2$$

$$-\frac{\partial f(y)}{\partial y}-\frac{\nu\cdot P}{2\cdot E\cdot I_z}\cdot y^2=C_1 \quad\Rightarrow\quad f(y)=-\frac{\nu\cdot P\cdot y^3}{6\cdot E\cdot I_z}-C_1\cdot y+C_3$$

Substituting the Result into the Displacement Expressions

$$u=\frac{P}{2\cdot E\cdot I_z}\cdot x\cdot y\cdot(2\cdot L-x)+f(y)=\frac{P}{2\cdot E\cdot I_z}\cdot x\cdot y\cdot(2\cdot L-x)-\frac{\nu\cdot P\cdot y^3}{6\cdot E\cdot I_z}-C_1\cdot y+C_3$$

$$v=-\frac{\nu\cdot P}{2\cdot E\cdot I_z}\cdot y^2\cdot(L-x)+g(x)=-\frac{\nu\cdot P}{2\cdot E\cdot I_z}\cdot y^2\cdot(L-x)-\frac{P}{6\cdot E\cdot I_z}\cdot x^2\cdot(3\cdot L-x)+C_1\cdot x+C_2$$

Anisotropic Materials

- ISOTROPIC - material properties are the same in all directions
- ANISOTROPIC - material properties change with direction
- HOMOGENEOUS - material of uniform composition throughout and whose properties are constant at every point
- HETEROGENEOUS - material uniformity within a body consisting of dissimilar constituents separately identifiable

Stress-Strain Relations

Stiffness

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{63}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{64}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{65}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{54} & C_{66}\end{bmatrix}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix}$$

Compliance

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{bmatrix}\cdot\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix}$$

Symmetry of the Stiffness Matrix

- Elastic Potential/Strain Energy Density
  - Incremental work per unit volume
  - $dW=\sigma_i d\varepsilon_i$
- Using the Stress-Strain Relations
  - $dW=C_{ij}\varepsilon_j d\varepsilon_j$
- Work per Unit Volume
  - $W=1/2\ C_{ij}\varepsilon_i\varepsilon_j$
- $dW/d\varepsilon_i=C_{ij}\varepsilon_j$ **or** $dW^2/d\varepsilon_i\, d\varepsilon_j=C_{ij}$ **thus** $C_{ij}=C_{ji}$

Stiffness and Compliance down from 36 to 21 Constants

Stiffness

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{63} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{64} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Compliance

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

One Plane of Elastic Symmetry

Monoclinic
13 Independent Constants

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{63} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{Bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Three Planes of Elastic Symmetry

Orthotropic Body
9 Independent Constants

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Relationship Between S and C

$$C_{11} = \frac{S_{22} \cdot S_{33} - S_{23}^2}{S} \quad ; \quad C_{44} = \frac{1}{S_{44}} \quad ; \quad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{S}$$

$$C_{22} = \frac{S_{33} \cdot S_{11} - S_{13}^2}{S} \quad ; \quad C_{55} = \frac{1}{S_{55}} \quad ; \quad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{S}$$

$$C_{33} = \frac{S_{11} \cdot S_{22} - S_{12}^2}{S} \quad ; \quad C_{66} = \frac{1}{S_{66}} \quad ; \quad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{S}$$

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13}$$

One Plane in which the Mechanical Properties are Equal

Transversely Isotropic
6 Independent Constants

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\left(C_{11} - C_{12}\right) \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}
$$

Material Properties Equal in all Directions

Isotropic
2 Independent Constants

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

Matrix Form of Stress-Strain Relations

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Matrix Form of Stress-Strain Relations

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\cdot(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\cdot(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\cdot(1+\nu)}{E} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Example 1

Given the following state of stress, determine the state of strain. E=200Gpa, υ=0.3

$$[\sigma]=\begin{bmatrix} \mathbf{12} & \mathbf{6} & \mathbf{9} \\ \mathbf{6} & \mathbf{10} & \mathbf{3} \\ \mathbf{9} & \mathbf{3} & \mathbf{14} \end{bmatrix} MPa$$

Example 2

Given the following state of strain, determine the state of stress. E=200GPa, υ=0.3

$$[\varepsilon]=\begin{bmatrix} 3 & -1 & 2 \\ -1 & 0 & -4 \\ 2 & -4 & 5 \end{bmatrix}\times 10^{-4}$$