

PROBLEM 4 THE BEAM ABCDE SHOWN BELOW HAS SIMPLE SUPPORTS AT A,C, AND E; AND A HINGE (OR PIN) AT D. A LOAD OF 4 kN ACTS AT THE END OF THE BRACKET THAT EXTENDS FROM THE BEAM AT B, AND A LOAD OF 2 kN ACTS AT THE MID POINT OF MEMBER DE. DRAW THE SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR THIS BEAM.

GIVEN:

- 1) A 6m member is pin connected to a 2m member
- 2) The 6 m member is simply supported at 0m and 4m
- 3) The 2m member is simply supported 2m from the pin connection
- 4) A 4kN load is applied to the end of a bracket located at 2m, with a horizontal dimension of -1m

ASSUMPTIONS:

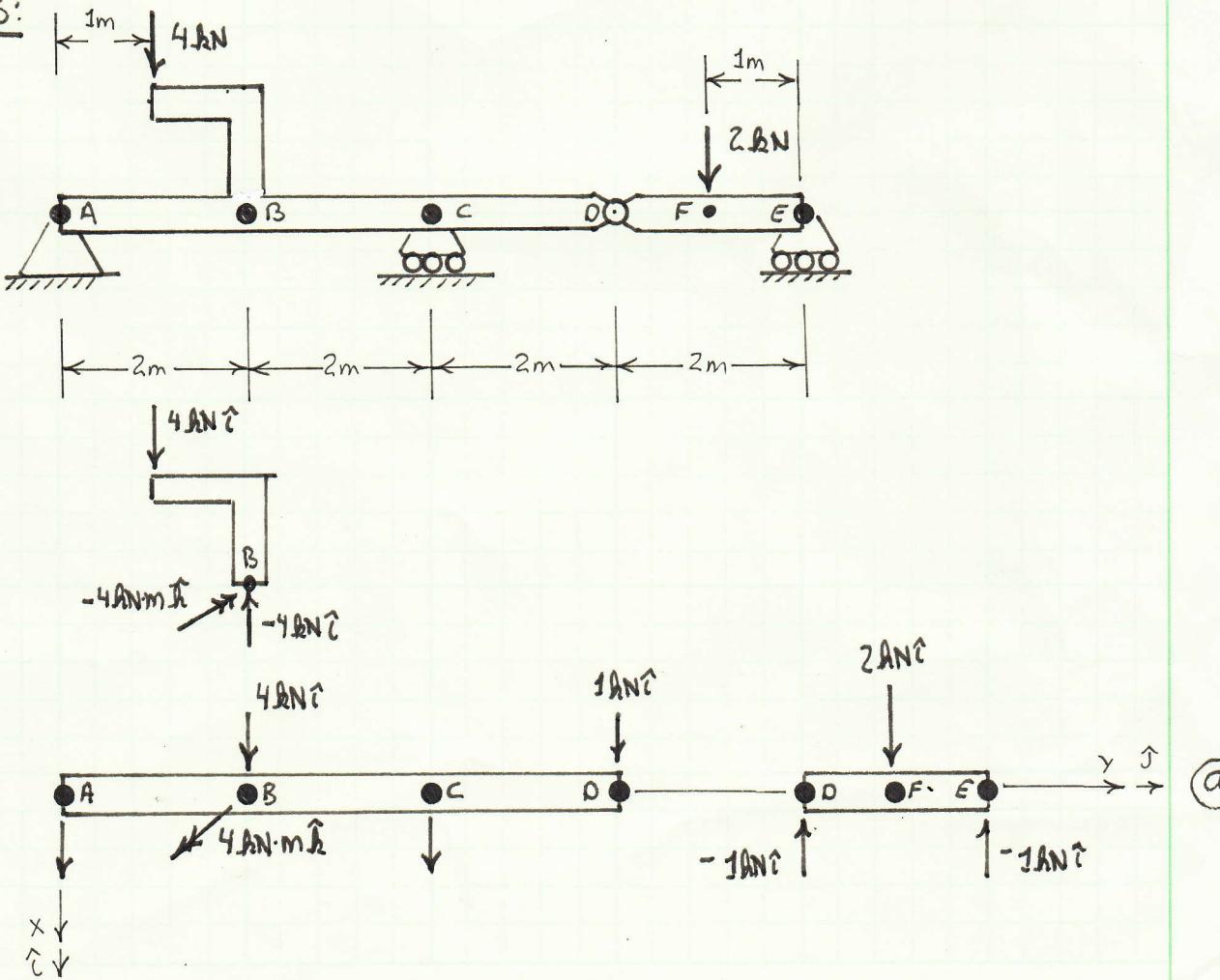
- 1) ALL COMPONENTS STAY OCT STRAIGHT
- 2) LINEAR-ELASTIC DEFORMATION
- 3) SMALL DEFLECTIONS.
- 4) THE PIN JOINT IS FRICTIONLESS.

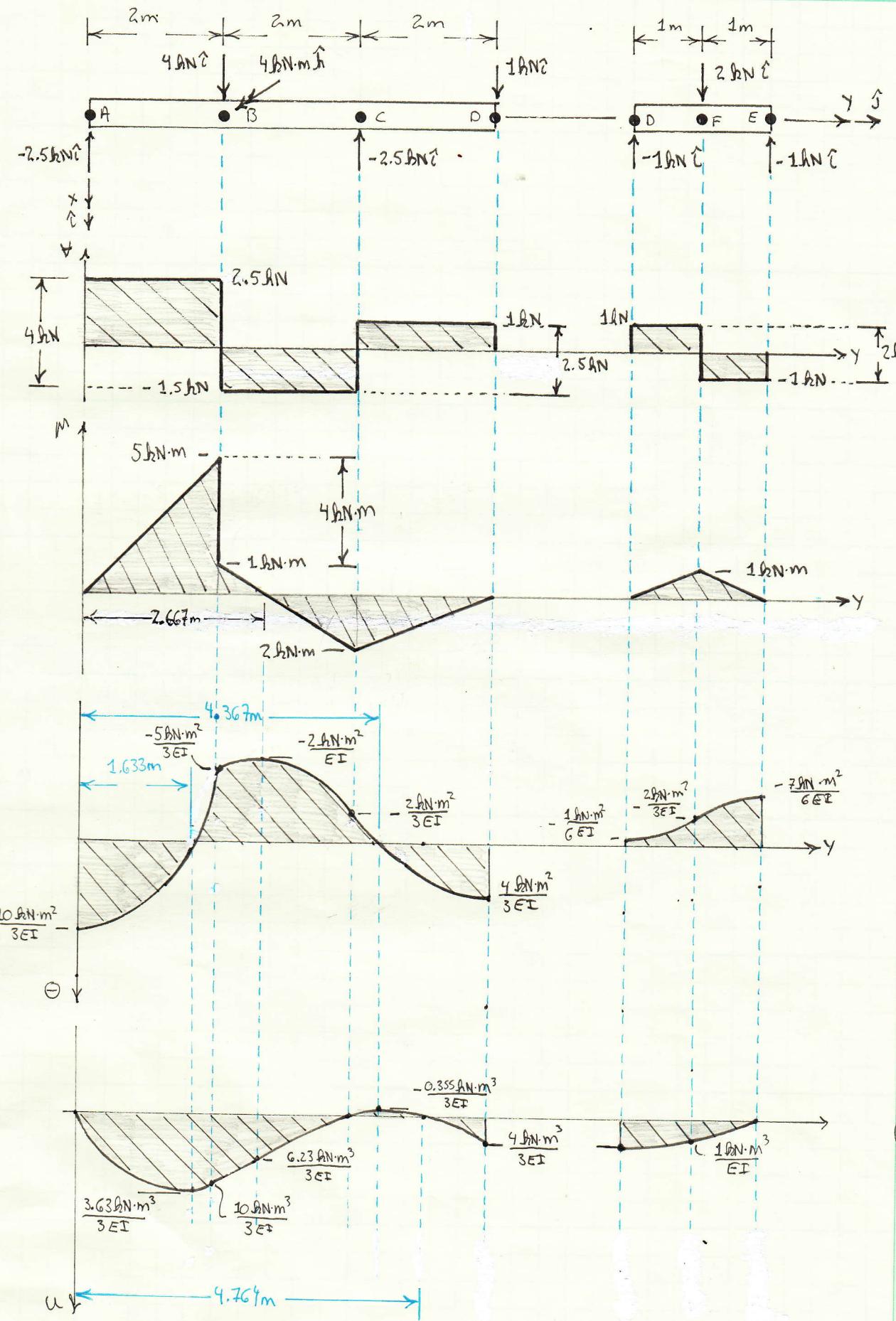
FIND:

1. SHEAR FORCE DIAGRAM
2. BENDING MOMENT DIAGRAM

- Supplemental Questions
3. ELASTIC CURVE CLOUTURE
 4. ELASTIC CURVE REFLECTION

FIGURES:





SOLUTION USING DIRECT INTEGRATION

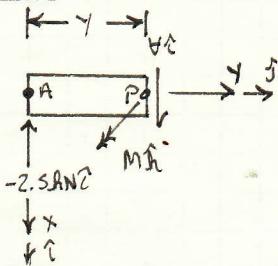
$0 \leq y \leq 2m$

$$\sum F_x = 0 = V - 2.5kN$$

$$\Rightarrow V = 2.5kN \quad (1)$$

$$\sum M_{z \text{ at } p} = 0 = M - 2.5kN \cdot y$$

$$\Rightarrow M = 2.5kN \cdot y \quad (2)$$



(g)

SINCE M IS A FUNCTION OF y , THE VALUES OF M AT THE BOUNDARIES SHOULD BE CALCULATED. FROM (2)

$$M(0) = 0$$

$$M(2m) = 5kN \cdot m$$

$2m \leq y \leq 4m$

$$\sum F_x = 0 = V - 2.5kN + 4kN$$

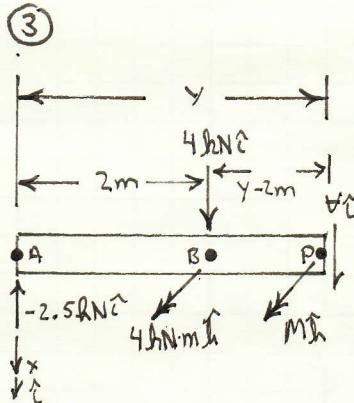
$$\Rightarrow V = -1.5kN \quad (4)$$

$$\sum M_{z \text{ at } p} = 0 = M + 4kN \cdot m$$

$$+ 4kN \cdot (y - 2m) - 2.5kN \cdot y$$

$$\Rightarrow M = -4kN \cdot m - 4kN \cdot y + 8kN \cdot m + 2.5kN \cdot y$$

$$= -1.5kN \cdot y + 4kN \cdot m \quad (5)$$



(h)

Since M IS A FUNCTION OF y , THE VALUES OF M AT THE BOUNDARIES OF THE REGION NEED TO BE CALCULATED. USING (5)

$$M(2m) = -1.5kN \cdot (2m) + 4kN \cdot m = 1kN \cdot m$$

$$M(4m) = -1.5kN \cdot (4m) + 4kN \cdot m = -2kN \cdot m \quad (6)$$

THE TWO REGIONAL BOUNDARY CONDITIONS HAVE OPPOSITE SIGN, INDICATING THAT THE MOMENT GOES TO ZERO IN THIS REGION. THIS POINT IS IMPERANT AND MUST BE DETERMINED. FROM 5, SETTING $M = 0$

$$0 = -1.5kN \cdot y + 4kN \cdot m$$

$$\Rightarrow y = \frac{4kN \cdot m}{1.5kN} = 2.667m \quad (7)$$

4m < y < 6m

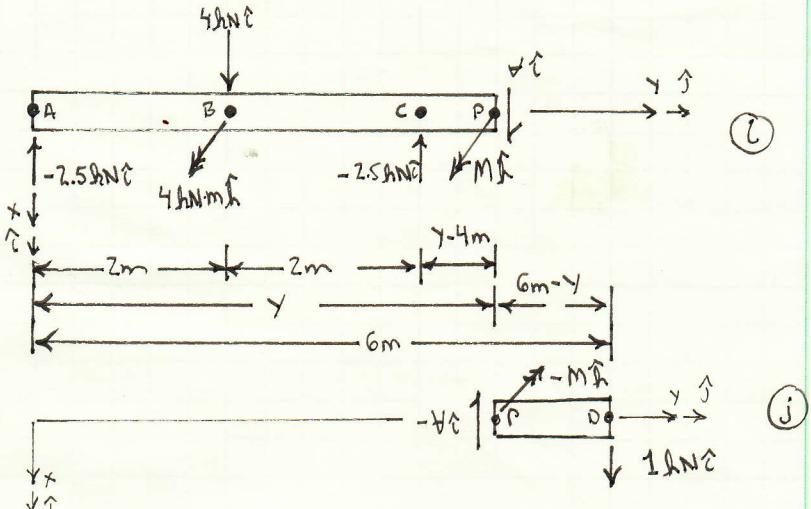
M & V WILL BE DETERMINED
IN THIS REGION USING ①

$$\sum F_x = 0 = -V + 1kN$$

$$\Rightarrow V = 1kN \quad ⑧$$

$$\sum M_{z \text{ at } P} = 0 = -M - (6m - y) \cdot 1kN$$

$$\Rightarrow M = 1kN \cdot y - 6kN \cdot m \quad ⑨$$



SINCE M IS A FUNCTION OF Y
IN THIS REGION, THE VALUE OF M AT THE BOUNDARIES OF THIS REGION
NEEDS TO BE CALCULATED. STARTING WITH ⑨

$$M(4m) = 1kN(4m) - 6kN \cdot m = -2kN \cdot m \quad ⑩$$

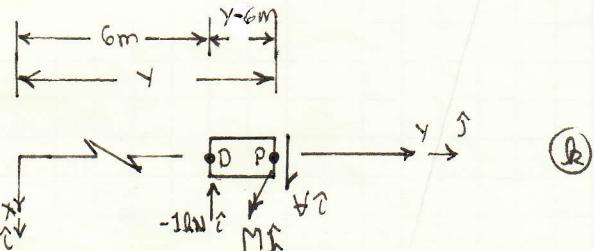
$$M(6m) = 1kN(6m) - 6kN \cdot m = 0$$

NOW THE SECOND BEAM (AFTER THE PIVOT) FROM 6m < y < 8m CAN BE
EVALUATED. THERE ARE TWO REGIONS TO THIS SEGMENT OF THE BEAM.

6m < y < 7m

$$\sum F_x = 0 = V - 1kN$$

$$\Rightarrow V = 1kN \quad ⑪$$



$$\sum M_{z \text{ at } P} = 0 = M - 1kN(y - 6m)$$

$$\Rightarrow M = 1kN \cdot y - 6kN \cdot m \quad ⑫$$

THE VALUES OF M AT THE BOUNDARIES OF THIS REGION NEEDS TO BE DETERMINED.

$$M(6m) = 1kN(6m) - 6kN \cdot m = 0$$

$$M(7m) = 1kN(7m) - 6kN \cdot m = 1kN \cdot m \quad ⑬$$

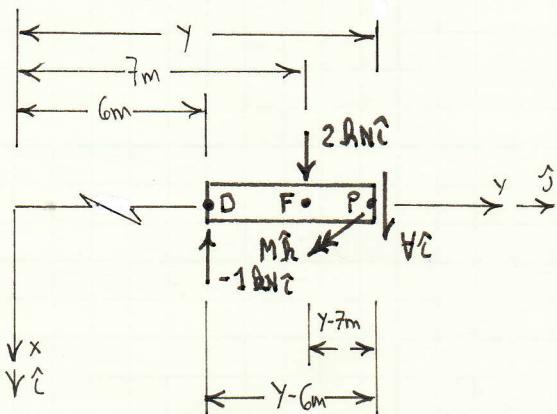
7m < y < 8m

$$\sum F_x = 0 = H - 1\text{ kN} + 2\text{ kN}$$

$$\Rightarrow H = -1\text{ kN} \quad (14)$$

$$\sum M_{zep} = 0 = M + 2\text{ kN}(y-7\text{ m}) - 1\text{ kN}(y-6\text{ m})$$

$$\Rightarrow M = -2\text{ kN}\cdot y + 14\text{ kN}\cdot m + 1\text{ kN}\cdot y - 6\text{ kN}\cdot m \\ = -1\text{ kN}\cdot y + 8\text{ kN}\cdot m \quad (15)$$



THE VALUES OF M AT THE BOUNDARIES OF THIS REGION NEED TO BE CALCULATED, USING (15)

$$M(7\text{ m}) = -1\text{ kN}\cdot(7\text{ m}) + 8\text{ kN}\cdot m = 1\text{ kN}\cdot m$$

$$M(8\text{ m}) = -1\text{ kN}\cdot(8\text{ m}) + 8\text{ kN}\cdot m = 0 \quad (16)$$

EXPRESSIONS FOR THE SHEAR FORCE AND BENDING MOMENT IN EACH REGION OF THE BE HAVE BEEN DETERMINED. THE BENDING MOMENT EXPRESSIONS CAN NOW BE INTEGRATED TO FORM EXPRESSIONS FOR THE SLOPE OF THE ELASTIC CURVE, Θ , AND THE DISPLACEMENT OF THE ELASTIC CURVE, u .

$$\Theta = \int -\frac{M}{EI} dy$$

$$u = \int \Theta \cdot dy$$

DIRECT INTEGRATION IS USED IN EACH REGION, BOUNDARY CONDITIONS ARE USED TO DETERMINE CONSTANTS, AND CONTINUITY CONDITIONS ARE USED TO TIE THE REGIONS TOGETHER.

FOR $y < 2m$ (REGION AB)

STARTING WITH M FROM ②

$$\Theta_{AB}(y) = \int -\frac{M}{EI} dy = \int -\frac{2.5 \text{ kN}}{EI} \cdot y dy = \int -\frac{5 \text{ kN}}{2 \cdot EI} \cdot y \cdot dy$$

$$= -\frac{5 \text{ kN}}{4 \cdot EI} \cdot y^2 + C_1 \quad (17)$$

$$U_{AB}(y) = \int \Theta_{AB}(y) \cdot dy = \int \left[-\frac{5 \text{ kN}}{4 \cdot EI} \cdot y^2 + C_1 \right] dy$$

$$= -\frac{5 \text{ kN}}{12 \cdot EI} \cdot y^3 + C_1 \cdot y + C_2 \quad (18)$$

THE FIRST BOUNDARY CONDITION IS $U_{AB}(0) = 0$, APPLYING THIS TO ⑮

$$U_{AB}(0) = -\frac{5 \text{ kN}}{12 \cdot EI} \cdot (0)^3 + C_1(0) + C_2 = 0 \Rightarrow C_2 = 0$$

⑯ CAN NOW BE REWRITTEN

$$U_{AB}(y) = -\frac{5 \text{ kN}}{12 \cdot EI} \cdot y^3 + C_1 \cdot y \quad (19)$$

THE VALUES OF ⑯ AND ⑰ AT THE END OF THIS REGION, $y = 2m$, NEED TO BE CALCULATED BECAUSE THESE ARE THE CONTINUITY CONDITIONS THAT FORM THE INITIAL CONDITIONS FOR THE NEXT REGION.

$$\textcircled{17} \rightarrow \Theta_{AB}(2m) = -\frac{5 \text{ kN}}{4 \cdot EI} \cdot (2m)^2 + C_1 = -\frac{20 \text{ kN} \cdot m^2}{4 \cdot EI} + C_1$$

$$= -\frac{5 \text{ kN} \cdot m^2}{EI} + C_1 \quad (20)$$

$$\textcircled{19} \rightarrow U_{AB}(2m) = -\frac{5 \text{ kN}}{12 \cdot EI} \cdot (2m)^3 + C_1 \cdot (2m) = -\frac{5 \text{ kN} \cdot 8 \text{ m}^3}{12 \cdot EI} + 2m \cdot C_1$$

$$= -\frac{10 \text{ kN} \cdot m^3}{3 \cdot EI} + 2m \cdot C_1 \quad (21)$$

2m < y < 4m (REGION BC)

STARTING WITH (5)

$$\begin{aligned}\Theta_{BC} &= \int \frac{M}{EI} dy = \int \left[\frac{1.5hN}{EI} y - \frac{4hN \cdot M}{EI} \right] dy = \int \left[\frac{3hN}{2 \cdot EI} y - \frac{4hN \cdot m}{EI} \right] dy \\ &= \frac{3hN}{4 \cdot EI} \cdot y^2 - \frac{4hN \cdot m}{EI} \cdot y + C_3\end{aligned}\quad (22)$$

$$\begin{aligned}U_{BC} &= \int \Theta_{BC} dy = \int \left[\frac{3hN}{4 \cdot EI} \cdot y^2 - \frac{4hN \cdot m}{EI} \cdot y + C_3 \right] dy \\ &= \frac{3hN}{12 \cdot EI} \cdot y^3 - \frac{4hN \cdot m}{2 \cdot EI} \cdot y^2 + C_3 \cdot y + C_4 \\ &= \frac{1hN}{4 \cdot EI} \cdot y^3 - \frac{2hN \cdot m}{EI} \cdot y^2 + C_3 \cdot y + C_4\end{aligned}\quad (23)$$

APPLYING THE CONTINUITY CONDITION IN (20), $\Theta_{AB}(2m) = \Theta_{BC}(2m)$
TO 22

$$\Theta_{AB}(2m) = \Theta_{BC}(2m)$$

$$-\frac{5hN \cdot m^2}{EI} + C_1 = \frac{3hN}{4 \cdot EI} \cdot (2m)^2 - \frac{4hN \cdot m}{EI} \cdot (2m) + C_3$$

$$C_3 = -\frac{5hN \cdot m^2}{4 \cdot EI} + C_1 - \frac{12hN \cdot m^2}{4 \cdot EI} + \frac{48hN \cdot m^2}{4 \cdot EI}$$

$$= C_1$$

SO SUBSTITUTING THIS RESULT INTO (22) & (23)

$$\Theta_{BC} = \frac{3hN}{4 \cdot EI} \cdot y^2 - \frac{4hN \cdot m}{EI} \cdot y + C_1 \quad (24)$$

$$U_{BC} = \frac{1hN}{4 \cdot EI} \cdot y^3 - \frac{2hN \cdot m}{EI} \cdot y^2 + C_1 \cdot y + C_4 \quad (25)$$

APPLYING THE SECOND CONTINUITY CONDITION IN THIS REGION,
 $U_{AB}(2m) = U_{BC}(2m)$, (21) AND (25)

$$U_{AB}(2m) = U_{BC}(2m)$$

$$-\frac{10kN \cdot m^3}{3EI} + 2m \cdot C_1 = \frac{1kN}{4EI} \cdot (2m)^3 - \frac{2kN \cdot m}{EI} \cdot (2m)^2 + 2m \cdot C_1 + C_4$$

$$C_4 = -\frac{10kN \cdot m^3}{3EI} - \frac{3 \cdot 2kN \cdot m^3}{3 \cdot EI} + \frac{3 \cdot 8kN \cdot m^3}{3 \cdot EI} = \frac{8kN \cdot m^3}{3EI}$$

THIS RESULT CAN NOW BE SUBSTITUTED INTO (25)

$$U_{BC} = \frac{1kN}{4EI} \cdot y^3 - \frac{2kN \cdot m}{EI} \cdot y^2 + C_1 \cdot y + \frac{8kN \cdot m^3}{3EI} \quad (26)$$

THE SECOND BOUNDARY CONDITION, $U_{BC}(4m) = 0$, CAN NOW BE APPLIED TO (26) AND C_1 IS DETERMINED.

$$U_{BC}(4m) = 0 = \frac{1kN}{4EI} \cdot (4m)^3 - \frac{2kN \cdot m}{EI} \cdot (4m)^2 + C_1 \cdot (4m) + \frac{8kN \cdot m^3}{3EI}$$

$$\begin{aligned} C_1 &= -\frac{64kN \cdot m^3}{4m \cdot 4 \cdot EI} + \frac{32kN \cdot m^3}{4m \cdot EI} - \frac{8kN \cdot m^3}{4m \cdot 3 \cdot EI} \\ &= -\frac{3 \cdot 4kN \cdot m^3}{3 \cdot EI} + \frac{3 \cdot 8kN \cdot m^3}{3 \cdot EI} - \frac{2 \cdot kN \cdot m^3}{3 \cdot EI} = \frac{10kN \cdot m^2}{3 \cdot EI} \end{aligned} \quad (27)$$

(27) CAN NOW BE SUBSTITUTED INTO (17), (19), (24), AND (26)

$0 < y < 2m$

$$(17) \rightarrow \Theta_{AB} = -\frac{5kN}{4EI} \cdot y^2 + \frac{10kN \cdot m^2}{3EI} \quad (28)$$

$$(19) \rightarrow U_{AB} = -\frac{5kN}{12EI} \cdot y^3 + \frac{10kN \cdot m^2}{3EI} \cdot y \quad (29)$$

$2m < y < 4m$

$$(24) \rightarrow \Theta_{BC} = \frac{3kN}{4EI} \cdot y^2 - \frac{4kN \cdot m}{EI} \cdot y + \frac{10kN \cdot m^2}{3EI} \quad (30)$$

$$(26) \rightarrow U_{BC} = \frac{1kN}{4EI} \cdot y^3 - \frac{2kN \cdot m}{EI} \cdot y^2 + \frac{10kN \cdot m^2}{3 \cdot EI} \cdot y + \frac{8kN \cdot m^3}{3EI} \quad (31)$$

THE VALUES OF THE SLOPE AND DISPLACEMENT OF THE ELASTIC CURVE AT THE BOUNDARIES OF THESE TWO REGIONS CAN NOW BE COMPUTED

$0 < y < 2m$

$$(28) \rightarrow \Theta_{AB}(0) = -\frac{5kN}{4EI} \cdot (0)^2 + \frac{10kN \cdot m^3}{3EI} = \frac{10kN \cdot m^3}{3EI} \quad (32)$$

$$(29) \rightarrow u_{AB}(0) = -\frac{5kN}{12EI} \cdot (0)^3 + \frac{10kN \cdot m^3}{3EI} \cdot (0) = 0 \quad (33)$$

$$(28) \rightarrow \Theta_{AB}(2m) = -\frac{5kN}{4EI} (2m)^2 + \frac{10kN \cdot m^2}{3EI} = -\frac{5kNm^2}{3EI} \quad (34)$$

$$(29) \rightarrow u_{AB}(2m) = \frac{5kN}{12EI} \cdot (2m)^3 + \frac{10kN \cdot m^3}{3EI} (2m) = \frac{10kN \cdot m^3}{3EI} \quad (35)$$

$2m < y < 4m$

$$(30) \rightarrow \Theta_{BC}(2m) = \frac{3kN}{4EI} \cdot (2m)^2 - \frac{4kN \cdot m}{EI} \cdot (2m) + \frac{10kN \cdot m^2}{3EI} = -\frac{5kNm^2}{3EI} \quad (36)$$

$$(31) \rightarrow u_{BC}(2m) = \frac{1kN}{4EI} \cdot (2m)^3 - \frac{2kN \cdot m}{EI} \cdot (2m)^2 + \frac{10kN \cdot m^2}{3EI} \cdot (2m) + \frac{8kN \cdot m^3}{3EI} \\ = \frac{10kNm^3}{3EI} \quad (37)$$

$$(30) \rightarrow \Theta_{BC}(4m) = \frac{3kN}{4EI} \cdot (4m)^2 - \frac{4kN \cdot m}{EI} \cdot (4m) + \frac{10kN \cdot m^2}{3EI} = -\frac{2kNm^2}{3EI} \quad (38)$$

$$(31) \rightarrow u_{BC}(4m) = \frac{1kN}{4EI} \cdot (4m)^3 - \frac{2kN \cdot m}{EI} \cdot (4m)^2 + \frac{10kN \cdot m^2}{3EI} \cdot (4m) + \frac{8kN \cdot m^3}{3EI} = 0 \quad (39)$$

(38) AND (39) ARE CONTINUITY CONDITIONS THAT ARE USED IN ITS INITIAL CONDITIONS FOR THE NEXT REGION. ALSO NEEDED

THE BOUNDARY VALUES FOR THE SLOPE OF THE ELASTIC CURVE SHOWS A CHANGE IN SIGN BETWEEN $0 < y < 2m$, REGION AB.

$$(32) \rightarrow \Theta_{AB}(0) = \frac{10 \text{ kN}\cdot\text{m}^3}{3EI}$$

$$(34) \rightarrow \Theta_{AB}(2m) = \frac{5 \text{ kN}\cdot\text{m}^3}{3EI}$$

THE LOCATION OF WHERE Θ_{AB} IS ZERO IS THE LOCATION OF WHERE U_{AB} IS A MAX/MIN. THE LOCATION OF THIS POINT IS FOUND BY SETTING (28) EQUAL TO 0 AND SOLVING FOR Y

$$(28) \rightarrow 0 = -\frac{5 \text{ kN}}{4EI} \cdot y^2 + \frac{10 \text{ kN}\cdot\text{m}^2}{3EI}$$

$$y^2 = \frac{\frac{10 \text{ kN}\cdot\text{m}^2}{3EI} \cdot \frac{4EI}{5 \text{ kN}}}{8 \text{ kN}} = \frac{8}{3} \text{ m}^2 \Rightarrow y = \pm \sqrt{\frac{8}{3} \text{ m}^2} = \pm 1.633 \text{ m}$$

SINCE THE NEGATIVE VALUE IS OUTSIDE THE DOMAIN OF THIS REGION, ONLY THE POSITIVE RESULT IS A VALID SOLUTION.

$$Y_{AB, \text{ext}} = \underline{1.633 \text{ m}}$$

(40)

THE VALUE OF U_{AB} AT THIS POINT CAN NOW BE COMPUTED USING (29)

$$(29) \rightarrow U_{AB}(1.633 \text{ m}) = -\frac{5 \text{ kN}}{12EI} \cdot (1.633 \text{ m})^3 + \frac{10 \text{ kN}\cdot\text{m}^2}{3EI} \cdot (1.633 \text{ m}) \\ = \underline{\underline{\frac{3.629 \text{ kN}\cdot\text{m}^3}{EI}}}$$

(41)

FROM THE MOMENT DIAGRAM (2), THE SLOPE OF THE ELASTIC CURVE Θ IS MAX/MIN IN REGION BC AT $y = 2.667 \text{ m}$. THE VALUE OF Θ_{BC} AT THIS LOCATION IS COMPUTED USING (30)

$$(30) \rightarrow \Theta_{BC} = \frac{3 \text{ kN}}{4EI} \cdot (2.667 \text{ m})^2 - \frac{4 \text{ kN}\cdot\text{m}}{EI} \cdot (2.667 \text{ m}) + \frac{10 \text{ kN}\cdot\text{m}^2}{3EI} = \underline{\underline{-\frac{2.00 \text{ kN}\cdot\text{m}^2}{EI}}}$$

(42)

THE FINAL SEGMENT OF THIS BEAM, REGION CD, FROM $4 \text{ m} < y < 6 \text{ m}$ CAN NOW BE EVALUATED.

$$4m \leq y \leq 6m$$

STARTING WITH THE EXPRESSION FOR THE MOMENT IN THIS REGION, (38)

$$\begin{aligned}\Theta_{CD} &= \int \frac{-M}{EI} dy = \int \left[-\frac{18N \cdot y + 6N \cdot m}{EI} \right] dy \\ &= -\frac{18N \cdot y^2}{2EI} + \frac{6N \cdot m \cdot y}{EI} + C_5\end{aligned}\quad (43)$$

THE CONTINUITY CONDITION (38) IS NOW USED TO EVALUATE C_5

$$\begin{aligned}\Theta_{BC}(4m) &= \Theta_{CD}(4m) \\ -\frac{2N \cdot m^2}{3 \cdot EI} &= -\frac{18N}{2EI} \cdot (4m)^2 + \frac{6N \cdot m}{EI} \cdot (4m) + C_5 \\ \Rightarrow C_5 &= -\frac{2N \cdot m^2}{3 \cdot EI} + \frac{18N \cdot 16m^2}{3 \cdot 2 \cdot EI} - \frac{24N \cdot m^2}{6 \cdot EI} = -\frac{100N \cdot m^2}{6EI} - \frac{50N \cdot m^2}{3EI}\end{aligned}$$

SUBSTITUTING THIS RESULT INTO (43)

$$\underline{\Theta_{CD} = -\frac{18N}{2EI} \cdot y^2 + \frac{6N \cdot m}{EI} \cdot y - \frac{50N \cdot m^2}{3 \cdot EI}}$$

THE DISPLACEMENT IN THIS REGION CAN NOW BE FORMULATED

$$\begin{aligned}u_{CD} &= \int \Theta_{CD} dy = \int \left[-\frac{18N}{2EI} \cdot y^2 + \frac{6N \cdot m}{EI} \cdot y - \frac{50N \cdot m^2}{3 \cdot EI} \right] dy \\ &= -\frac{18N}{6EI} \cdot y^3 + \frac{3N \cdot m}{EI} \cdot y^2 - \frac{50N \cdot m^2}{3 \cdot EI} \cdot y + C_6\end{aligned}\quad (45)$$

THE CONTINUITY CONDITION (39) IS NOW USED TO EVALUATE C_6

$$0 = -\frac{18N}{6EI} \cdot (4m)^3 + \frac{3N \cdot m}{EI} \cdot (4m)^2 - \frac{50N \cdot m^2}{3 \cdot EI} \cdot (4m) + C_6$$

$$C_6 = \frac{18N}{6EI} \cdot (4m)^3 - \frac{3N \cdot m}{EI} \cdot (4m)^2 + \frac{50N \cdot m^2}{3 \cdot EI} \cdot (4m) = \frac{88N \cdot m^3}{3EI}$$

SUBSTITUTING THIS RESULT INTO (45)

$$\underline{u_{CD} = -\frac{18N}{6EI} \cdot y^3 + \frac{3N \cdot m}{EI} \cdot y^2 - \frac{50N \cdot m^2}{3 \cdot EI} \cdot y + \frac{88N \cdot m^3}{3EI}}$$

VALUES FOR Θ AND U AT THE BOUNDARIES OF THIS REGION CAN NOW BE CALCULATED USING (44) AND (46)

$$(44) \rightarrow \Theta_{co}(4m) = -\frac{1}{2EI} \cdot (4m)^2 + \frac{6\text{ kN} \cdot \text{m}}{EI} \cdot (4m) - \frac{50\text{ kN} \cdot \text{m}^2}{3EI} = -\frac{4\text{ kN} \cdot \text{m}^2}{6EI} = -\frac{2\text{ kN} \cdot \text{m}^2}{3EI} \quad (47)$$

$$(44) \rightarrow \Theta_{co}(6m) = -\frac{1}{2EI} \cdot (6m)^2 + \frac{6\text{ kN} \cdot \text{m}}{EI} \cdot (6m) - \frac{50\text{ kN} \cdot \text{m}^2}{3EI} = \frac{8\text{ kN} \cdot \text{m}^2}{6EI} = \frac{4\text{ kN} \cdot \text{m}^2}{3EI} \quad (48)$$

THE CHANGE IN SIGN BETWEEN (47) & (48) INDICATES A ROOT EXISTS IN THIS REGION THAT WILL BE THE LOCATION OF A MAX/MIN U .

$$(44) \rightarrow \Theta_{co}(y) = 0 = -\frac{1}{2EI} \cdot y^2 + \frac{6\text{ kN} \cdot \text{m}}{EI} \cdot y - \frac{50\text{ kN} \cdot \text{m}^2}{3EI}$$

$$\Rightarrow 0 = y^2 - 12m \cdot y + \frac{100}{3} \cdot m^2 = y^2 - 12m \cdot y + (-6m)^2 - (-6m)^2 + \frac{100}{3} m^2$$

$$\Rightarrow (y - 6m)^2 = \frac{36m^2 \cdot 3}{3} - \frac{100m^2}{3} = \frac{108m^2}{3} - \frac{100m^2}{3} = \frac{8m^2}{3}$$

$$\Rightarrow y = 6m \pm \sqrt{\frac{8}{3} m^2} = 7.633m, \underline{4.367m} \quad (49)$$

ONLY THE 2nd VALUE FOR y ($= 4.367m$) IS IN THE REGION BEING CONSIDERED. NOW THE DISPLACEMENTS OF THE BEAM AT POINTS OF INTEREST CAN BE CALCULATED.

$$(46) \rightarrow U_{co}(4m) = -\frac{1}{6EI} \cdot (4m)^3 + \frac{3\text{ kN} \cdot \text{m}}{EI} \cdot (4m)^2 - \frac{50\text{ kN} \cdot \text{m}^2}{3EI} \cdot (4m) + \frac{88\text{ kN} \cdot \text{m}^3}{3EI} = 0 \quad (50)$$

$$(46) \rightarrow U_{co}(6m) = -\frac{1}{6EI} (6m)^3 + \frac{3\text{ kN} \cdot \text{m}}{EI} \cdot (6m)^2 - \frac{50\text{ kN} \cdot \text{m}^2}{3EI} \cdot (6m) + \frac{88\text{ kN} \cdot \text{m}^3}{3EI} = \frac{8\text{ kN} \cdot \text{m}^3}{6EI} = \frac{4\text{ kN} \cdot \text{m}^3}{3EI} \quad (51)$$

BECAUSE A PIN EXISTS IN THE BEAM AT $y = 6m$, THE SLOPES OF THE TWO SEGMENTS OF THE BEAM AT THIS POINT ARE NOT CONSTRAINED TO BE EQUAL. THIS MEANS THAT (48) IS NOT A CONTINUITY CONDITION. THE PIN AT 6m DOES RESTRICT THE DEFLECTION AT $y = 6m$, SO (51) IS A CONTINUITY CONDITION.

THE MAXIMUM U IN THIS SEGMENT IS CALCULATED BY SUBSTITUTING (47) INTO (46)

$$(46) \rightarrow U_c(4.367m) = \frac{1}{6EI} \cdot (4.367m)^3 + \frac{3\text{ kN} \cdot \text{m}}{EI} \cdot (4.367m)^2 - \frac{50\text{ kN} \cdot \text{m}^2}{3EI} \cdot (4.367m) + \frac{88\text{ kN} \cdot \text{m}^3}{3EI}$$

$$= -\frac{0.7093\text{ kN} \cdot \text{m}^3}{6EI} = -\frac{0.1182\text{ kN} \cdot \text{m}^3}{EI} \quad (52)$$

THE SECOND SEGMENT OF THE BEAM, DEF, CAN NOW BE EVALUATED. THIS SEGMENT OF THE BEAM HAS TWO REGIONS, DF FROM $6m \leq y \leq 7m$ AND FE FROM $7m \leq y \leq 8m$.

REGION DF: $6m \leq y \leq 7m$

STARTING WITH THE EXPRESSION FOR THE MOMENT IN THIS REGION, (12)

$$\Theta_{DF} = \int -\frac{M}{EI} dy = \int \left[\frac{1kN}{EI} \cdot y - \frac{6kN \cdot m}{EI} \right] dy = \int \left[-\frac{1kN}{EI} \cdot y + \frac{6kN \cdot m}{EI} \right] dy \\ = -\frac{1kN}{2EI} \cdot y^2 + \frac{6kN \cdot m}{EI} \cdot y + C_7 \quad (53)$$

$$U_{DF} = \int \Theta_{DF} \cdot dy = \int \left[-\frac{1kN}{2EI} \cdot y^2 + \frac{6kN \cdot m}{EI} \cdot y + C_7 \right] dy \\ = -\frac{1kN}{6 \cdot EI} y^3 + \frac{6kN \cdot m}{2 \cdot EI} \cdot y^2 + C_7 \cdot y + C_8 = -\frac{1kN}{6 \cdot EI} y^3 + \frac{3kN \cdot m}{EI} \cdot y^2 + C_7 \cdot y + C_8$$

THE CONTINUITY CONDITION $U_{CD}(6m)$ (51) IS SET EQUAL TO $U_{DF}(6m)$ (54)

$$U_{CD}(6m) = U_{DF}(6m)$$

$$\frac{4kN \cdot m^3}{3 \cdot EI} = -\frac{1kN}{6EI} \cdot (6m)^3 + \frac{3kN \cdot m}{EI} \cdot (6m)^2 + C_7 \cdot (6m) + C_8$$

$$C_8 = \frac{4kN \cdot m^3}{3 \cdot EI} + \frac{1kN}{6EI} \cdot (6m)^3 - \frac{3kN \cdot m}{EI} \cdot (6m)^2 - C_7 \cdot (6m) \\ = -\frac{212kN \cdot m^3}{3 \cdot EI} + 6m \cdot C_7$$

THIS RESULT CAN NOW BE BACK SUBSTITUTED INTO (54)

$$U_{DF} = -\frac{1kN}{6EI} \cdot y^3 + \frac{3kN \cdot m}{EI} \cdot y^2 + C_7 \cdot y - \frac{212kN \cdot m^3}{3 \cdot EI} - 6m \cdot C_7 \quad (54)$$

THE VALUES FOR Θ_{DF} (53) AND U_{DF} (54) AT $y = 7m$ ARE CONTINUITY CONDITIONS FOR THE REGION FE, $7m \leq y \leq 8m$.

$$(53) \rightarrow \Theta_{DF}(7m) = -\frac{1kN}{2EI} \cdot (7m)^2 + \frac{6kN \cdot m}{2EI} \cdot (7m) + C_7 = \frac{35kN \cdot m^2}{2 \cdot EI} + C_7 \quad (55)$$

$$(54) \rightarrow U_{DF}(7m) = -\frac{1kN}{6EI} \cdot (7m)^3 + \frac{3kN \cdot m}{2EI} \cdot (7m)^2 + C_7 \cdot (7m) - \frac{212kN \cdot m^3}{3 \cdot EI} - 6m \cdot C_7 \\ = \frac{115kN \cdot m^3}{6 \cdot EI} + C_7 \quad (56)$$

REGION FE: $7m \leq y \leq 8m$

STARTING WITH THE EXPRESSION FOR M IN THIS REGION

$$\Theta_{FE} = \int \frac{M}{EI} dy = \int -\left[\frac{1kN}{EI} \cdot y + \frac{8kN \cdot m}{EI} \right] dy = \int \left[\frac{1kN}{EI} \cdot y - \frac{8kN \cdot m}{EI} \right] dy \\ = \frac{1kN}{2 \cdot EI} \cdot y^2 - \frac{8kN \cdot m}{EI} \cdot y + C_9 \quad (57)$$

USING (55) AND THE CONTINUITY CONDITION $\Theta_{DF}(7m) = \Theta_{FE}(7m)$

$$\frac{35kN \cdot m^2}{2 \cdot EI} + C_7 = \frac{1kN}{2 \cdot EI} \cdot (7m)^2 - \frac{8kN \cdot m}{EI} \cdot 7m + C_9 \\ \Rightarrow C_9 = \frac{35kN \cdot m^2}{2 \cdot EI} + C_7 - \frac{49kN \cdot m^2}{2 \cdot EI} + \frac{56kN \cdot m^2}{EI} = \frac{49kN \cdot m^2}{EI} + C_7 \quad (58)$$

(58) INTO (57)

$$\Theta_{FE} = \frac{1kN}{2 \cdot EI} \cdot y^2 - \frac{8kN \cdot m}{EI} \cdot y + \frac{49kN \cdot m^2}{EI} + C_7 \quad (59)$$

$$U_{FE} = \int \Theta_{FE} \cdot dy = \int \left[\frac{1kN}{2 \cdot EI} \cdot y^2 - \frac{8kN \cdot m}{EI} \cdot y + \frac{49kN \cdot m^2}{EI} + C_7 \right] dy \\ = \frac{1kN}{6 \cdot EI} \cdot y^3 - \frac{4kN \cdot m}{EI} \cdot y^2 + \frac{49kN \cdot m^2}{EI} \cdot y + C_7 \cdot y + C_{10} \quad (60)$$

USING (56) AND THE CONTINUITY CONDITION $U_{DF}(7m) = U_{FE}(7m)$

$$\frac{115kN \cdot m^3}{6 \cdot EI} + C_7 = \frac{1kN}{6 \cdot EI} \cdot (7m)^3 - \frac{4kN \cdot m}{EI} \cdot (7m)^2 + \frac{49kN \cdot m^2}{EI} \cdot (7m) + C_7 \cdot 7m + C_{10} \\ C_{10} = \frac{115kN \cdot m^3}{6 \cdot EI} + C_7 - \frac{1kN}{6 \cdot EI} \cdot (7m)^3 + \frac{4kN \cdot m}{6 \cdot EI} \cdot (7m)^2 - \frac{49kN \cdot m^2}{6 \cdot EI} \cdot 7m - 7m \cdot C_7 \\ = - \frac{185kN \cdot m^3}{6 \cdot EI} - 6m \cdot C_7 \quad (61)$$

BACK SUBSTITUTING (61) INTO (60)

$$U_{FE} = \frac{1kN}{6 \cdot EI} \cdot y^3 - \frac{4kN \cdot m}{EI} \cdot y^2 + \frac{49kN \cdot m^2}{EI} \cdot y - \frac{185kN \cdot m^3}{6 \cdot EI} + C_7 \cdot y - 6m \cdot C_7 \quad (62) \\ = \frac{1kN}{6 \cdot EI} \cdot y^3 - \frac{4kN \cdot m}{EI} \cdot y^2 + \frac{49kN \cdot m^2}{EI} \cdot y - \frac{185kN \cdot m^3}{6 \cdot EI} + C_7(y - 6m)$$

THE FINAL BOUNDARY CONDITION CAN NOW BE APPLIED TO (62) AND AN EXPRESSION FOR C_7 DETERMINED

$$(62) \rightarrow U_{FE}(8m) = 0 = \frac{1\text{ kN}}{6EI} \cdot (8m)^3 - \frac{4\text{ kN}\cdot\text{m}}{EI} \cdot (8m)^2 + \frac{49\text{ kN}\cdot\text{m}^2}{EI} \cdot (8m) - \frac{185\text{ kN}\cdot\text{m}^3}{EI} + C_7 \cdot 2m$$

$$\Rightarrow 2m \cdot C_7 = -\frac{1\text{ kN} \cdot (8m)^3}{6EI} + \frac{4\text{ kN}\cdot\text{m} \cdot (8m)^2}{EI} - \frac{49\text{ kN}\cdot\text{m}^2 \cdot (8m)}{EI} + \frac{185\text{ kN}\cdot\text{m}^3}{EI} = -\frac{218\text{ kN}\cdot\text{m}^3}{6EI}$$

$$\Rightarrow C_7 = -\frac{109\text{ kN}\cdot\text{m}^2}{6 \cdot EI} \quad (63)$$

(63) IS BACK SUBSTITUTED INTO (53), (54), (55), AND (62) SO FINAL EXPRESSIONS FOR Θ AND U IN REGIONS DF AND FE CAN BE WRITTEN.

REGION DF: $6m \leq y \leq 7m$

$$(53) \rightarrow \Theta_{DF} = -\frac{1\text{ kN}}{2EI} \cdot y^2 + \frac{6\text{ kN}\cdot\text{m}}{EI} \cdot y - \frac{109\text{ kN}\cdot\text{m}^2}{6EI} \quad (64)$$

$$(54) \rightarrow U_{DF} = -\frac{1\text{ kN}}{6EI} \cdot y^3 + \frac{3\text{ kN}\cdot\text{m}}{EI} \cdot y^2 - \frac{109\text{ kN}\cdot\text{m}^2}{6EI} (y-6m) - \frac{212\text{ kN}\cdot\text{m}^3}{3EI}$$

$$= -\frac{1\text{ kN}}{6EI} \cdot y^3 + \frac{3\text{ kN}\cdot\text{m}}{EI} \cdot y^2 - \frac{109\text{ kN}\cdot\text{m}^2}{6EI} \cdot y + \frac{115\text{ kN}\cdot\text{m}^3}{3EI} \quad (65)$$

REGION FE: $7m \leq y \leq 8m$

$$(55) \rightarrow \Theta_{FE} = \frac{1\text{ kN}}{2EI} \cdot y^2 - \frac{8\text{ kN}\cdot\text{m}}{EI} \cdot y + \frac{49\text{ kN}\cdot\text{m}^2}{EI} - \frac{109\text{ kN}\cdot\text{m}^2}{6EI}$$

$$= \frac{1\text{ kN}}{2EI} \cdot y^2 - \frac{8\text{ kN}\cdot\text{m}}{EI} \cdot y + \frac{185\text{ kN}\cdot\text{m}^2}{6EI} \quad (66)$$

$$(62) \rightarrow U_{FE} = \frac{1\text{ kN}}{6EI} \cdot y^3 - \frac{4\text{ kN}\cdot\text{m}}{EI} \cdot y^2 + \frac{49\text{ kN}\cdot\text{m}^2}{EI} \cdot y - \frac{185\text{ kN}\cdot\text{m}^3}{EI} - \frac{109\text{ kN}\cdot\text{m}^2}{6EI} (y-6m)$$

$$= \frac{1\text{ kN}}{6EI} \cdot y^3 - \frac{4\text{ kN}\cdot\text{m}}{EI} \cdot y^2 + \frac{185\text{ kN}\cdot\text{m}^2}{6EI} \cdot y - \frac{76\text{ kN}\cdot\text{m}^3}{EI} \quad (67)$$

THE VALUES FOR (64) - (67) AT THE ENDS AND CRITICAL POINTS IN THESE REGIONS NOW CAN BE CALCULATED.

REGION DF: $6m \leq y \leq 7m$

$$(64) \rightarrow \Theta_{DF}(6m) = -\frac{1}{2EI} \cdot (6m)^2 + \frac{6kN \cdot m}{EI} \cdot (6m) - \frac{109kN \cdot m^2}{6EI} = -\frac{1}{6EI} kN \cdot m^2 \quad (68)$$

$$(65) \rightarrow U_{DF}(6m) = -\frac{1}{6EI} \cdot (6m)^3 + \frac{3kN \cdot m}{EI} \cdot (6m)^2 - \frac{109kN \cdot m^2}{6EI} \cdot (6m) + \frac{115kN \cdot m^3}{3EI} = \frac{4kN \cdot m^3}{3EI} \quad (69)$$

$$(64) \rightarrow \Theta_{DF}(7m) = -\frac{1}{2EI} \cdot (7m)^2 + \frac{6kN \cdot m}{EI} \cdot (7m) - \frac{109kN \cdot m^2}{6EI} = -\frac{2}{3EI} kN \cdot m^2 \quad (70)$$

$$(65) \rightarrow U_{DF}(7m) = -\frac{1}{6EI} \cdot (7m)^3 + \frac{3kN \cdot m}{EI} \cdot (7m)^2 - \frac{109kN \cdot m^2}{6EI} \cdot (7m) + \frac{115kN \cdot m^3}{3EI} = \frac{1kN \cdot m^3}{EI} \quad (71)$$

REGION FE: $7m \leq y \leq 8m$

$$(66) \rightarrow \Theta_{FE}(7m) = \frac{1}{2EI} \cdot (7m)^2 - \frac{8kN \cdot m}{EI} \cdot (7m) + \frac{185kN \cdot m^2}{6EI} = -\frac{2}{3EI} kN \cdot m^2 \quad (72)$$

$$(67) \rightarrow U_{FE}(7m) = \frac{1}{6EI} \cdot (7m)^3 - \frac{4kN \cdot m}{EI} \cdot (7m)^2 + \frac{185kN \cdot m^2}{6EI} \cdot (7m) - \frac{76kN \cdot m^3}{EI} = \frac{1}{EI} kN \cdot m^3 \quad (73)$$

$$(66) \rightarrow \Theta_{FE}(8m) = \frac{1}{2EI} \cdot (8m)^2 - \frac{8kN \cdot m}{EI} \cdot (8m) + \frac{185kN \cdot m^2}{6EI} = -\frac{7}{6EI} kN \cdot m^2 \quad (74)$$

$$(67) \rightarrow U_{FE}(8m) = \frac{1}{6EI} \cdot (8m)^3 - \frac{4kN \cdot m}{EI} \cdot (8m)^2 + \frac{185kN \cdot m^2}{6EI} \cdot (8m) - \frac{76kN \cdot m^3}{EI} = 0 \quad (75)$$

THE VALUES IN (68) - (75) ARE SUMMARIZED ON FIGURES (c) AND (f).

DIRECT INTEGRATION SUMMARY

THE DIRECT INTEGRATION APPROACH TO THE DETERMINATION OF Θ AND U FOR THIS BEAM REQUIRES THE USE OF BOUNDARY CONDITIONS AND CONTINUITY CONDITION TO SOLVE FOR ALL THE CONSTANTS CREATED. THE MOST INTERESTING CONTINUITY CONDITION IS THE PIN JOINT BECAUSE IT DOES NOT CREATE A CONTINUITY CONDITION FOR THE DISPLACEMENT, BUT NOT FOR THE CURVATURE BECAUSE THE PIN DOES NOT SUPPORT A MOMENT TRANSFER.