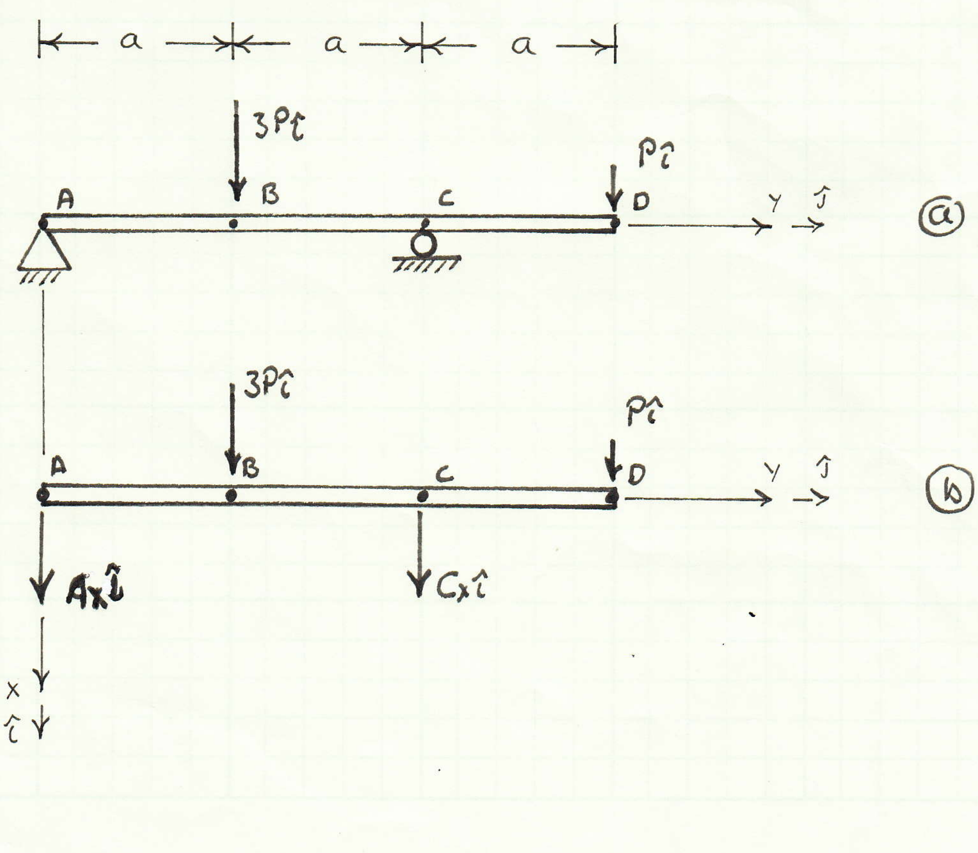
PROBLEM STATEMENT: A HORIZONTAL BEAM OF LENGTH 3a IS SHOWN BELOW. THIS BEAM HAS A PIN SUPPORT AT THE LEFT END OF THE BEAM AND A ROLLER SUPPORT 2a FROM THE LEFT END OF THE BEAM. A 3P LOAD IS APPLIED a FROM THE LEFT END OF THE BEAM AND A LOAD P IS APPLIED AT THE RIGHT END OF THE BEAM. USING FIRST THE DIRECT INTEGRATION METHOD AND THEN USING SINGULARITY FUNCTIONS DETERMINE EXPRESSIONS FOR THE SHEAR FORCE, BENDING MOMENT, CURVATURE, AND DISPLACEMENT IN THIS BEAM. DRAW THE SHEAR FORCE, BENDING MOMENT, CURVATURE, AND DISPLACEMENT DIAGRAMS; LABELING ALL CRITICAL VALUES AND THEIR LOCATIONS.

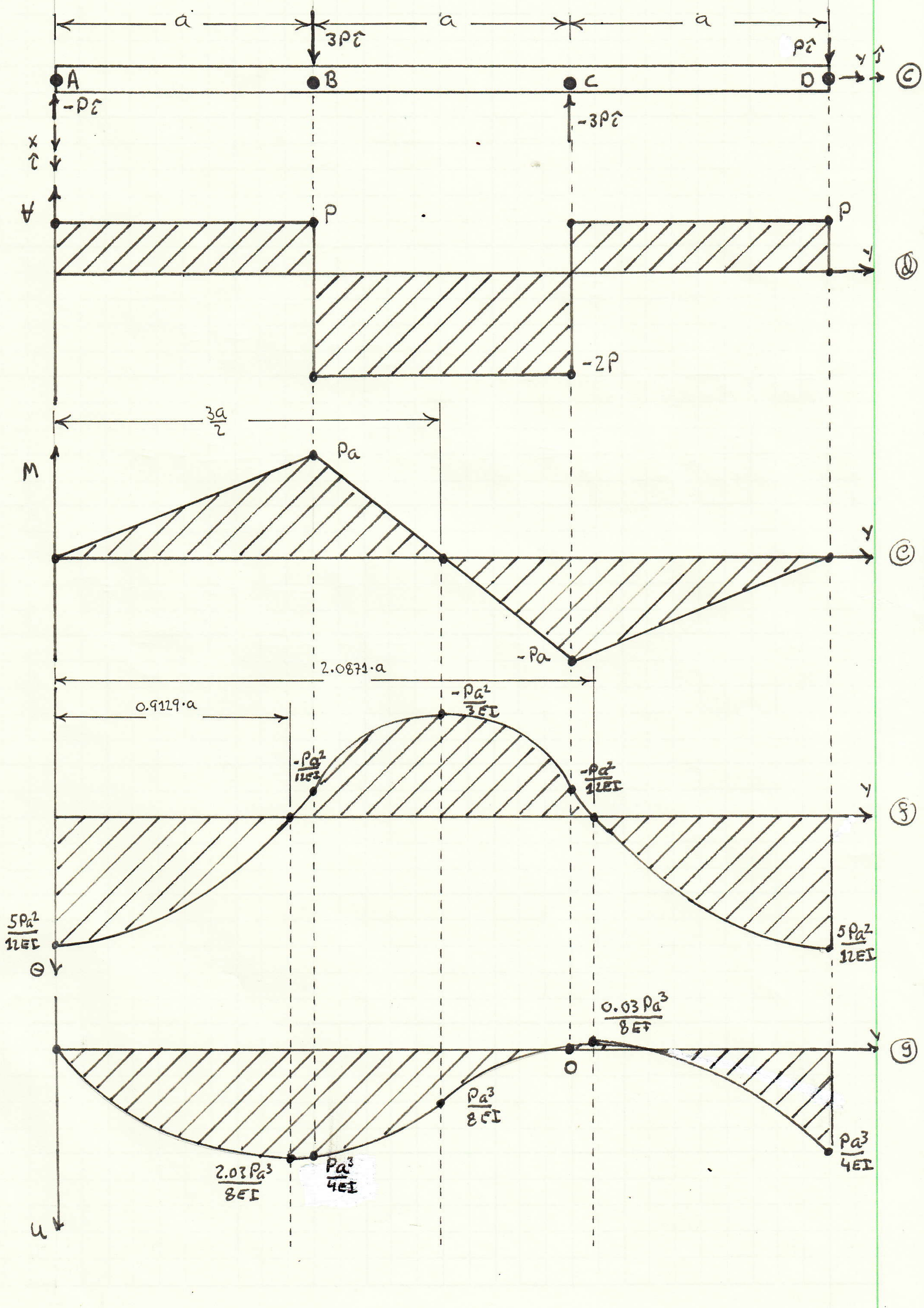
GIVEN:
1. A BEAM OF LENGTH 3a (EI)
2. A TRANSVERSE LOAD OF 3P a FROM THE LEFT END OF THE BEAM
3. A TRANSVERSE LOAD OF P AT THE RIGHT MOST END OF THE BEAM
4. A PIN CONSTRAINT AT THE LEFT MOST END OF THE BEAM
5. A ROLLER SUPPORT 2a FROM THE LEFT END OF THE BEAM

ASSUMPTIONS:
1. THE BEAM IS INITIALLY STRAIGHT
2. THE MATERIAL IS LINEAR ELASTIC
3. ALL DISPLACEMENTS ARE SMALL.

FIND:
1. USING DIRECT INTEGRATION DETERMINE EXPRESSIONS FOR V, M, θ, & u
2. USING SINGULARITY FUNCTIONS DETERMINE EXPRESSIONS FOR V, M, θ, & u
3. DRAW THE V, M, θ, & u DIAGRAMS.

FIGURE:




a
a
a
$3P\hat{\imath}$
$P\hat{\imath}$
A
B
C
D
y $\hat{\jmath}$
$-P\hat{\imath}$
$-3P\hat{\imath}$
x $\hat{\imath}$
(c)
V
P
P
y
-2P
(d)
$\frac{3a}{2}$
M
Pa
y
-Pa
(e)
2.0871·a
0.9129·a
$-\frac{Pa^2}{3EI}$
$-\frac{Pa^2}{12EI}$
$-\frac{Pa^2}{12EI}$
y
$\frac{5Pa^2}{12EI}$
$\frac{5Pa^2}{12EI}$
θ
(f)
$\frac{0.03Pa^3}{8EI}$
y
(g)
0
$\frac{Pa^3}{8EI}$
$\frac{Pa^3}{4EI}$
$\frac{2.03Pa^3}{8EI}$
$\frac{Pa^3}{4EI}$
u

SOLUTION

STARTING WITH THE FREEBODY DIAGRAM IN (b), APPLYING EQUILIBRIUM TO SOLVE FOR THE UNKNOWNS A_x AND C_x

$$\Sigma F_x = 0 = A_x + 3P + C_x + P \Rightarrow \underline{\underline{A_x + C_x = -4P}} \quad (1)$$

$$\Sigma M_{z@A} = 0 = -a\cdot 3P - C_x\cdot 2a - 3a\cdot P$$

$$\Rightarrow 2a\cdot C_x = -6\cdot Pa \Rightarrow \underline{\underline{C_x = -3P}} \quad (2)$$

SUBSTITUTING (2) INTO (1)

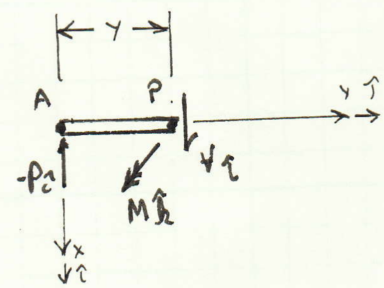
$$A_x - 3P = -4P \Rightarrow \underline{\underline{A_x = -P}} \quad (3)$$

THE ABOVE RESULTS ARE SUMMERIZED IN THE SOLUTION DIAGRAM (c).

EXPRESSIONS FOR THE SHEAR FORCE AND BENDING MOMENT IN EACH SECTION OF THIS BEAM CAN NOW BE DETERMINED.

REGION AB: $0 \le y \le a$

THE FREEBODY DIAGRAM FOR THIS REGION IS SHOWN IN (h)



$$\Sigma F_x = 0 = -P + V$$

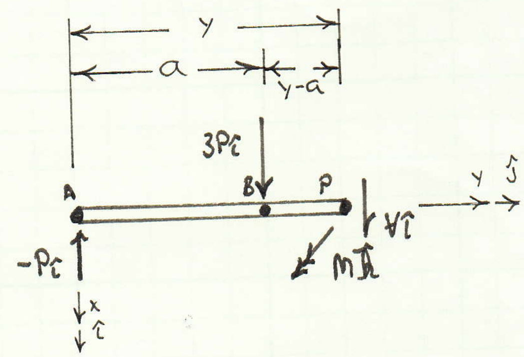
$$\Rightarrow \underline{\underline{V = P}} \quad (4)$$

$$\Sigma M_{z@P} = 0 = M - P_y$$

$$\Rightarrow \underline{\underline{M = P\cdot y}} \quad (5)$$

REGION BC: $a \le y \le 2a$

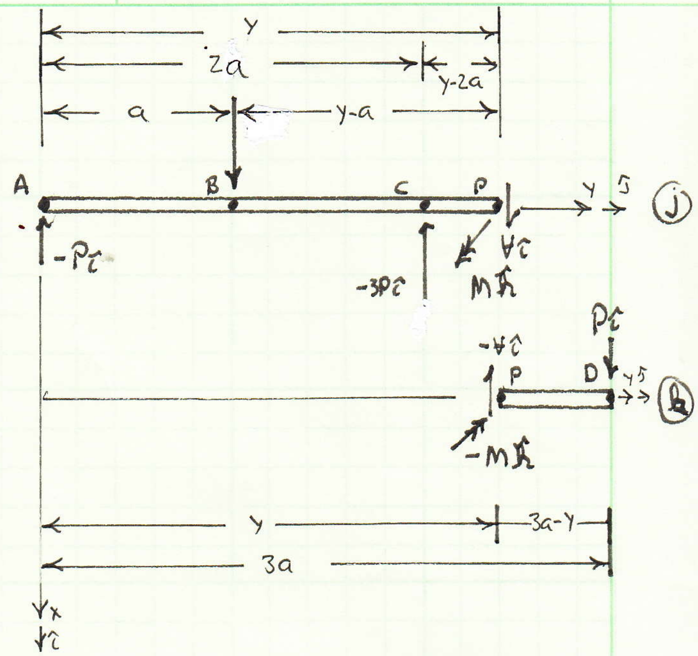
THE FREEBODY DIAGRAM FOR THIS REGION IS SHOWN IN (i)



$$\Sigma F_x = 0 = V + 3P - P \Rightarrow \underline{\underline{V = -2P}} \quad (6)$$

$$\Sigma M_{z@P} = 0 = M + 3P\cdot(y-a) - y\cdot P$$

$$M = -3P\cdot y + 3Pa + y\cdot P = \underline{\underline{-2P\cdot y + 3Pa}} \quad (7)$$



REGION CD : $2a \leq y \leq 3a$

EITHER THE FREEBODY DIAGRAM IN (j) OR (k) CAN BE USED TO DETERMINE THE INTERNAL SHEAR AND BENDING MOMENT IN THIS REGION OF THE BEAM. (k) WILL BE USED HERE

$$\Sigma F_x = 0 = -V + P \Rightarrow \underline{V = P} \quad (8)$$

$$\Sigma M_{z@P} = 0 = -M - P\cdot(3a - y)$$

$$\Rightarrow \underline{M = -3a\cdot P + P_y} \quad (9)$$

(5), (7), & (9) CAN NOW BE USED TO DEVELOP EXPRESSIONS FOR THE CURVATURE θ ~~AA~~ AND DEFLECTION u OF THE ELASTIC CURVE.

REGION AB: $0 \leq y \leq a$

STARTING WITH (5)

$$\theta_{AB} = \int \frac{-M}{EI}\,dy = \int -\frac{P\cdot y}{EI}\cdot dy = -\frac{P\cdot y^2}{2\cdot EI} + C_1 \quad (10)$$

$$u_{AB} = \int \theta_{AB}\,dy = \int \left[-\frac{P\cdot y^2}{2\cdot EI} + C_1\right]\cdot dy = -\frac{P\cdot y^3}{6\cdot EI} + C_1\cdot y + C_2 \quad (11)$$

IN THIS REGION THERE IS ONE BOUNDARY CONDITION, $u_{AB}(0) = 0$

$$(11) \rightarrow \quad u_{AB}(0) = 0 = -\frac{P\cdot(0)^3}{6\cdot EI} + C_1\cdot(0) + C_2 \quad \Rightarrow \quad \underline{C_2 = 0}$$

(11) CAN NOW BE REWRITTEN

$$u_{AB} = -\frac{P\cdot y^3}{6\cdot EI} + C_1\cdot y \quad (12)$$

(10) AND (11) ARE NOW USED TO CREATE THE CONTINUITY CONDITIONS THAT FORM THE INITIAL CONDITIONS FOR THE NEXT REGION.

$$(10) \rightarrow \quad \theta_{AB}(a) = -\frac{P\cdot a^2}{2\cdot EI} + C_1 \quad (13)$$

$$(12) \rightarrow \quad u_{AB}(a) = -\frac{P\cdot a^3}{6\cdot EI} + C_1\cdot a \quad (14)$$

REGION BC: $a \le y \le 2a$

STARTING WITH (7)

$$\Theta_{BC} = \int \frac{-M}{IE}\cdot dy = \int -\left[\frac{-2\cdot P\cdot y}{IE} + \frac{3Pa}{IE}\right]\cdot dy = \frac{2\cdot P\cdot y^2}{2EI} - \frac{3Pa}{EI}y + C_3$$

$$= \frac{P\cdot y^2}{EI} - \frac{3Pa}{EI}\cdot y + C_3 \quad (15)$$

$$u_{BC} = \int \Theta_{BC}\cdot dy = \int \left[\frac{P\cdot y^2}{EI} - \frac{3Pa}{EI}\cdot y + C_3\right]\cdot dy$$

$$= \frac{P\cdot y^3}{3\cdot EI} - \frac{3\cdot P\cdot a}{2EI}\cdot y^2 + C_3\cdot y + C_4 \quad (16)$$

THE TWO CONTINUITY CONDITIONS (13) AND (14) NOW NEED TO BE EQUATED TO (15) AND (16) AT THE START OF THIS REGION.

$$\Theta_{AB}(a) = \Theta_{BC}(a)$$

(13) & (15) $\rightarrow$ $$-\frac{P\cdot a^2}{2\cdot EI} + C_1 = \frac{P\cdot a^2}{EI} - \frac{3P\cdot a^2}{EI} + C_3$$

$$C_3 = -\frac{P\cdot a^2}{2\cdot EI} + C_1 + \frac{2\cdot 2Pa^2}{2\cdot EI} = \frac{3\cdot P\cdot a^2}{2\cdot EI} + C_1 \quad (17)$$

(17) CAN NOW BE BACK SUBSTITUTED INTO (15) AND (16)

(15) $\rightarrow$ $$\Theta_{BC} = \frac{P\cdot y^2}{EI} - \frac{3Pa}{EI}\cdot y + \frac{3\cdot P\cdot a^2}{2\cdot EI} + C_1 \quad (18)$$

(16) $\rightarrow$ $$u_{BC} = \frac{P\cdot y^3}{3\cdot EI} - \frac{3\cdot P\cdot a}{2\cdot EI}\cdot y^2 + \left[\frac{3\cdot P\cdot a^2}{2EI} + C_1\right]\cdot y + C_4 \quad (19)$$

$$u_{AB}(a) = u_{BC}(a)$$

(14) & (19) $\rightarrow$ $$-\frac{P\cdot a^3}{6\cdot EI} + C_1\cdot a = \frac{P\cdot a^3}{3\cdot EI} - \frac{3Pa^3}{2EI} + \frac{3\cdot P\cdot a^3}{2EI} + C_1\cdot a + C_4$$

$$\Rightarrow C_4 = -\frac{Pa^3}{6EI} + \cancel{C_1\cdot a} - \frac{2\cdot Pa^3}{2\cdot 3EI} + \frac{3\cdot 3Pa^3}{3\cdot 2EI} - \frac{3\cdot 3Pa^3}{3\cdot 2EI} - \cancel{C_1\cdot a} = -\frac{3\cdot Pa^3}{6EI}$$

$$= -\frac{Pa^3}{2EI} \quad (20)$$

(20) IS NOW BACK SUBSTITUTED INTO (19)

$$(19) \rightarrow \quad u_{BC} = \frac{P \cdot y^3}{3 \cdot EI} - \frac{3 \cdot Pa}{2 \cdot EI} \cdot y^2 + \frac{3\,Pa^2}{2\,EI} \cdot y + C_1 \cdot y - \frac{P \cdot a^3}{2 \cdot EI} \quad (21)$$

THE BOUNDARY CONDITION IN THIS REGION IS $u_{BC}(2a)=0$, THEREFORE

$$(21) \rightarrow \quad u_{BC}(2a) = \frac{P \cdot (2a)^3}{3EI} - \frac{3 \cdot Pa \cdot (2a)^2}{2EI} + \frac{3 \cdot Pa^2 \cdot (2a)}{2EI} + C_1 \cdot (2a) - \frac{Pa^3}{2EI} = 0$$

$$2a \cdot C_1 = -\frac{P \cdot (2a)^3}{3 \cdot EI} + \frac{3 \cdot Pa \cdot (2a)^2}{2EI} - \frac{3\,Pa^2 \cdot (2a)}{2EI} + \frac{Pa^3}{2EI}$$

$$C_1 = -\frac{P \cdot (2a)^2}{3EI} + \frac{3\,Pa \cdot (2a)}{2EI} - \frac{3\,Pa^2}{2EI} + \frac{Pa^2}{4EI}$$

$$= -\frac{4 \cdot a^2 \cdot P}{3EI} + \frac{3 \cdot Pa^2}{EI} - \frac{3 \cdot Pa^2}{2 \cdot EI} + \frac{Pa^2}{4EI} = \frac{5 \cdot Pa^2}{12 \cdot EI} \quad (22)$$

(22) CAN NOW BE BACK SUBSTITUTED INTO (10), (12), (18), & (21)

<u>REGION AB: $0 \le y \le a$</u>

$$(10) \rightarrow \quad \underline{\underline{\theta_{AB} = -\frac{P \cdot y^2}{2 \cdot EI} + \frac{5 \cdot Pa^2}{12 \cdot EI}}} \quad (23)$$

$$(12) \rightarrow \quad \underline{\underline{u_{AB} = -\frac{P \cdot y^3}{6 \cdot EI} + \frac{5 \cdot Pa^2}{12EI} \cdot y}} \quad (24)$$

<u>REGION BC: $a \le y \le 2a$</u>

$$(18) \rightarrow \quad \theta_{BC} = \frac{P \cdot y^2}{EI} - \frac{3 \cdot Pa}{EI} \cdot y + \frac{3 \cdot Pa^2}{2 \cdot EI} + \frac{5 \cdot Pa^2}{12EI} = \underline{\underline{\frac{P \cdot y^2}{EI} - \frac{3\,Pa}{EI} \cdot y + \frac{23\,Pa^2}{12\,EI}}} \quad (25)$$

$$(21) \rightarrow \quad u_{BC} = \frac{P \cdot y^3}{3EI} - \frac{3 \cdot Pa}{2EI} \cdot y^2 + \frac{3\,Pa^2}{2EI} y + \frac{5 \cdot Pa^2}{12EI} \cdot y - \frac{P \cdot a^3}{2 \cdot EI}$$

$$= \underline{\underline{\frac{P \cdot y^3}{3 \cdot EI} - \frac{3 \cdot Pa}{2EI} \cdot y^2 + \frac{23 \cdot Pa^2}{12} \cdot y - \frac{Pa^3}{2EI}}} \quad (26)$$

(23) – (26) CAN NOW BE USED TO CALCULATE CRITICAL VALUES OF θ AND u IN THESE TWO REGIONS.

REGION AB: $0 \leq y \leq a$

(23) → $$\theta_{AB}(0) = -\frac{P\cdot(0)^2}{2\cdot EI} + \frac{5\cdot Pa^2}{12EI} = \frac{5\cdot Pa^2}{12EI} \quad (27)$$

(24) → $$u_{AB}(0) = -\frac{P\cdot(0)^3}{6EI} + \frac{5\cdot Pa^2}{12EI}\cdot(0) = 0 \quad (28)$$

(23) → $$\theta_{AB}(a) = -\frac{P\cdot a^2}{2EI} + \frac{5\cdot Pa^2}{12\cdot EI} = -\frac{Pa^2}{12EI} \quad (29)$$

(24) → $$u_{AB}(a) = -\frac{P\cdot a^3}{6EI} + \frac{5\cdot Pa^3}{12EI} = \frac{3\cdot P\cdot a^3}{12\cdot EI} = \frac{P\cdot a^3}{4\cdot EI} \quad (30)$$

SINCE THERE IS A SIGN CHANGE IN θ_{AB} BETWEEN $\theta_{AB}(0)$ (27) AND $\theta_{AB}(a)$ (29), THE FUNCTION θ_{AB} MUST GO TO ZERO IN THIS REGION. THE LOCATION OF THE ROOTS OF θ_{AB} IN THIS REGION ARE IMPORTANT BECAUSE THE ARE LOCATIONS OF MAX/MINS IN u_{AB}.

$$\theta_{AB} = 0 = -\frac{\cancel{P}\cdot y^2}{\cancel{2EI}} + \frac{5\cdot \cancel{P}a^2}{\underset{6}{\cancel{12\cdot EI}}}$$

$$\Rightarrow y^2 = \frac{5}{6}a^2$$

$$\Rightarrow y = \pm\sqrt{\frac{5}{6}}\cdot a = -0.9129\cdot a\ ,\ \underline{\underline{0.9129\cdot a}}$$

ONLY THE POSITIVE ROOT IS IN THE DOMAIN OF THE REGION.

$$\theta_{AB}(0.9129\cdot a) = -\frac{P\cdot(0.9129\cdot a)^2}{2\cdot EI} + \frac{5\cdot Pa^2}{12\cdot EI} = 0$$

$$u(0.9129\cdot a) = -\frac{P\cdot(0.9129\cdot a)^3}{6\cdot EI} + \frac{5\cdot Pa^2}{12EI}\cdot(0.9129\cdot a)$$

$$= 0.2536\cdot\frac{Pa^3}{EI} \quad (31)$$

REGION BC: $a \le y \le 2a$

$$\Theta_{BC}(a) = \frac{P\cdot a^2}{EI} - \frac{3P\cdot a^2}{EI} + \frac{23\cdot Pa^2}{12EI} = -\frac{Pa^2}{12\cdot EI} \quad (32)$$

$$u_{BC}(a) = \frac{P\cdot a^3}{3\cdot EI} - \cancel{\frac{3\cdot Pa^3}{2EI}} + \cancel{\frac{3Pa^3}{2EI}} + \frac{5Pa^3}{12\cdot EI} - \frac{P\cdot a^3}{2EI}$$

$$= \frac{3\cdot Pa^3}{12\cdot EI} = \frac{Pa^3}{4EI} \quad (33)$$

$$\Theta_{BC}(2a) = \frac{P\cdot(2a)^2}{EI} - \frac{3\cdot Pa}{EI}\cdot(2a) + \frac{23\cdot Pa^2}{12EI} = -\frac{Pa^2}{12EI} \quad (33)$$

$$u_{BC}(2a) = \frac{P\cdot(2a)^3}{3EI} - \frac{3\cdot Pa}{2EI}\cdot(2a)^2 + \frac{3Pa^2}{2\cdot EI}(2a) + \frac{5}{12}\frac{Pa^2}{EI}(2a) - \frac{Pa^3}{2EI}$$

$$= 0 \quad (34)$$

EXPRESSIONS FOR Θ AND u IN THE FINAL REGION OF THE BEAM CAN NOW BE DETERMINED. (33) AND (34) ARE CONTINUITY CONDITIONS FOR THIS REGION

REGION CD: $2a \le y \le 3a$

STARTING WITH (9)

$$\Theta_{CD} = -\int \frac{M}{EI}dy = \int -\left[-\frac{3a\cdot P}{EI} + \frac{P\cdot y}{EI}\right]dy = \int\left[\frac{3\cdot a\cdot P}{EI} - \frac{P\cdot y}{EI}\right]dy$$

$$= \frac{3\cdot a\cdot P}{EI}\cdot y - \frac{P\cdot y^2}{2\cdot EI} + C_5 \quad (35)$$

USING THE CONTINUITY CONDITION (33)

$$\Theta_{BC}(2a) = \Theta_{CD}(2a)$$

$$-\frac{P\cdot a^2}{12\cdot EI} = \frac{3\cdot Pa}{EI}(2a) - \frac{P\cdot(2a)^2}{2EI} + C_5$$

$$\Rightarrow C_5 = -\frac{P\cdot a^2}{12EI} - \frac{6\cdot Pa^2}{EI} + \frac{2Pa^2}{EI} = -\frac{49Pa^2}{12EI} \quad (36)$$

(36) INTO (35)

$$\Theta_{CD} = -\frac{P\cdot y^2}{2\cdot EI} + \frac{3\cdot Pa}{EI}\cdot y - \frac{49\,Pa^2}{12\,EI} \qquad (37)$$

CONTINUING

$$u_{CD} = \int \Theta_{CD}\cdot dy = \int\left[-\frac{P\cdot y^2}{2\cdot EI} + \frac{3\,Pa}{EI}\cdot y - \frac{49\,Pa^2}{12\,EI}\right]dy$$

$$= -\frac{P\cdot y^3}{6\cdot EI} + \frac{3\cdot Pa}{2\,EI}\cdot y^2 - \frac{49\cdot Pa^2}{12\,EI}\cdot y + C_6 \qquad (38)$$

THE CONSTANT IS DETERMINED FROM THE CONTINUITY CONDITION (34)

$$u_{BC}(2a) = u_{CD}(2a)$$

$$0 = -\frac{P\cdot(2a)^3}{6\cdot EI} + \frac{3\cdot Pa\cdot(2a)^2}{2\cdot EI} - \frac{49\cdot Pa^2\cdot(2a)}{12\,EI} + C_6$$

$$\Rightarrow C_6 = \frac{8\cdot Pa^3}{6\,EI} - \frac{12\cdot Pa^3}{2\,EI} + \frac{98\cdot Pa^3}{12\,EI}$$

$$= \frac{8}{6}\frac{Pa^3}{EI} - \frac{36}{6}\frac{Pa^3}{EI} + \frac{49}{6}\frac{Pa^3}{EI} = \frac{21}{6}\frac{Pa^3}{EI} \qquad (39)$$

SUBSTITUTING (39) INTO (38)

$$(38) \rightarrow \quad u_{CD} = -\frac{P\cdot y^3}{6\cdot EI} + \frac{3\cdot Pa}{2\,EI}\cdot y^2 - \frac{49\cdot Pa^2}{12\,EI}y + \frac{21}{6}\frac{Pa^3}{EI} \qquad (40)$$

(37) AND (40) CAN NOW BE USED TO CALCULATE CRITICAL VALUES OF Θ_{CD} AND u_{CD} IN THIS REGION

<u>REGION CD: $2a \leq y \leq 3a$</u>

$$(37) \rightarrow \quad \Theta_{CD}(2a) = -\frac{P\cdot(2a)^2}{2\cdot EI} + \frac{3\cdot Pa}{EI}\cdot(2a) - \frac{49\,Pa^2}{12\,EI} = -\frac{Pa^2}{12\,EI} \qquad (41)$$

$$(40) \rightarrow \quad u_{CD}(2a) = -\frac{6\cdot(2a)^3}{6\cdot EI} + \frac{3\cdot Pa}{2\cdot EI}(2a)^2 - \frac{49\cdot Pa^2\cdot(2a)}{12\,EI} + \frac{21\,Pa^3}{6\,EI}$$

$$= 0 \qquad (42)$$

(37) →
$$\theta_{CD}(3a) = -\frac{P\cdot(3a)^2}{2\cdot EI} + \frac{3Pa}{EI}\cdot(3a) - \frac{49\cdot Pa^2}{12EI} = \frac{5Pa^2}{12EI} \quad (43)$$

(40) →
$$u_{CD}(3a) = -\frac{P\cdot(3a)^3}{6\cdot EI} + \frac{3\cdot Pa(3a)^2}{2EI} - \frac{49\cdot Pa^2\cdot(3a)}{12EI} + \frac{21\,Pa^3}{6EI}$$
$$= \frac{3Pa^3}{12EI} = \frac{Pa^3}{4EI} \quad (44)$$

SINCE THERE IS A CHANGE OF SIGN IN $\theta_{CD}(2a)$ (41) AND $\theta_{CD}(3a)$ (43), θ_{CD} BECOMES 0 IN THIS REGION WHICH INDICATES A MAX/MIN IN u. THE LOCATION WHERE $\theta_{CD}=0$ IS FOUND BY SETTING (37) EQUAL TO 0 AND SOLVING FOR Y

(37) →
$$\theta_{CD} = 0 = -\frac{\cancel{P}\cdot Y^2}{2\cancel{EI}} + \frac{3\cancel{P}a}{\cancel{EI}}\cdot Y - \frac{49\cdot\cancel{P}a^2}{12\cancel{EI}}$$
$$0 = \frac{1}{2}\cdot Y^2 - 3\cdot Ya + \frac{49}{12}a^2$$
$$0 = Y^2 - 6Ya + \frac{49}{6}\cdot a^2 = \underbrace{Y^2 - 6a\cdot Y + \left(\frac{6a}{2}\right)^2}_{(Y-3a)^2} - \left(\frac{-6a}{2}\right)^2 + \frac{49}{6}a^2$$
$$0 = (Y-3a)^2 - \frac{36}{4}a^2 + \frac{49}{6}a^2 = (Y-3a)^2 - \frac{20}{24}a^2$$
$$(Y-3a)^2 = \frac{5}{6}a^2$$
$$\Rightarrow \quad Y = 3a \pm \sqrt{\frac{5}{6}}\cdot a = 3a \pm 0.9129\cdot a$$
$$= \underline{\underline{2.087\cdot a}},\ 3.9129\cdot a \quad (45)$$

THE SECOND VALUE IS OUTSIDE THE DOMAIN OF THE PROBLEM, SO THE ROOT OF THIS EQUATION IS THE FIRST. THE VALUE OF u_{CD} AT THIS POINT MUST BE CALCULATED USING (40)

(40) →
$$u_{CD}(2.087\cdot a) = -\frac{P\cdot(2.087\cdot a)^3}{6\cdot EI} + \frac{3Pa\cdot(2.087a)^2}{2EI} - \frac{49\cdot Pa^2(2.087a)}{12EI} + \frac{21}{6}\frac{Pa^3}{EI}$$
$$= -0.00358\cdot\frac{Pa^3}{EI} \quad (46)$$

THE RESULTS OF THE CALCULATIONS OF θ AND u IN EACH REGION OF THIS BEAM ARE SUMMARIZED IN (f) AND (g).

CONSIDER THE SAME BEAM USING SINGULARITY FUNCTIONS. STARTING WITH ©, THE LOAD CAN BE EXPRESSED AS

$$q(y) = -P\langle y-0\rangle_{-1} + 3P\langle y-a\rangle_{-1} - 3P\langle y-2a\rangle_{-1} + P\cdot\langle y-3a\rangle_{-1} \qquad (47)$$

$$V = \int -q(y)\cdot dy$$

$$= P\langle y-0\rangle^{0} - 3P\langle y-a\rangle^{0} + 3P\langle y-2a\rangle^{0} - P\langle y-3a\rangle^{0} \qquad (48)$$

$$M = \int V(y)\cdot dy$$

$$= P\langle y-0\rangle^{1} - 3P\langle y-a\rangle^{1} + 3P\langle y-2a\rangle^{1} - P\langle y-3a\rangle^{1} \qquad (49)$$

$$\theta = \int -\frac{M}{EI}\cdot dy$$

$$= -\frac{P}{2EI}\cdot\langle y-0\rangle^{2} + \frac{3P}{2EI}\langle y-a\rangle^{2} - \frac{3P}{2EI}\langle y-2a\rangle^{2} + \frac{P}{2EI}\langle y-3a\rangle^{2} + \overset{\frac{5Pa^2}{12EI}}{C_1} \qquad (50)$$

$$u = \int \theta\, dy$$

$$= -\frac{P}{6EI}\langle y-0\rangle^{3} + \frac{3P}{6EI}\langle y-a\rangle^{3} - \frac{3P}{6EI}\langle y-2a\rangle^{3} + \frac{P}{6EI}\langle y-3a\rangle^{3} + \overset{\frac{5Pa^2}{12EI}}{C_1}\cdot y + C_2$$

$$= -\frac{P}{6\cdot EI}\langle y-0\rangle^{3} + \frac{P}{2EI}\langle y-a\rangle^{3} - \frac{P}{2EI}\langle y-2a\rangle^{3} + \frac{P}{6EI}\langle y-3a\rangle^{3} + \overset{\frac{5Pa^2}{12EI}}{C_1}\cdot y + \overset{0}{C_2} \qquad (51)$$

THERE ARE TWO CONSTANTS IN THE EXPRESSION FOR u, AND THERE ARE TWO KNOWN BOUNDARY CONDITIONS, $u(0)=0$ AND $u(2a)=0$.

$$u(0) = 0 = -\frac{P}{6\cdot EI}(0-0)^{3} + C_1\cdot 0 + C_2 \Rightarrow C_2 = 0$$

$$u(2a) = 0 = -\frac{P}{6EI}(2a)^{3} + \frac{P}{2EI}(a)^{3} - \frac{P}{2EI}(0)^{3} + C_1\cdot 2\cdot a + 0$$

$$\Rightarrow 2\cdot a\cdot C_1 = \frac{P\cdot 8a^{3}}{6EI} - \frac{3\cdot P\cdot a^{3}}{3\cdot 2EI} = \frac{5\cdot P\cdot a^{3}}{6EI}$$

$$\Rightarrow C_1 = \frac{5Pa^{2}}{12EI} \qquad (52)$$

(52) CAN NOW BE SUBSTITUTED INTO (50) AND (51)

$$\Theta = -\frac{P}{2\cdot EI}\langle y-0\rangle^2 + \frac{3P}{2\cdot EI}\langle y-a\rangle^2 - \frac{3P}{2EI}\langle y-2a\rangle^2 + \frac{P}{2\cdot EI}\langle y-3a\rangle^2 + \frac{5\cdot Pa^2}{12\cdot EI} \quad (52)$$

$$u = -\frac{P}{6EI}\langle y-0\rangle^3 + \frac{P}{2EI}\langle y-a\rangle^3 - \frac{P}{2EI}\langle y-2a\rangle^3 + \frac{P}{6EI}\langle y-3a\rangle^3 + \frac{5Pa^2}{12EI}\cdot y \quad (53)$$

(48), (49), (52), AND (53) CAN NOW BE USED TO FIND CRITICAL VALUES OF V, M, Θ, AND u IN EACH REGION OF THE BEAM.

REGION AB: $0 \le y \le a$

(48) → $V(0) = P\cdot(0)^0 = 0,\ P$ (54)

(49) → $M(0) = P\cdot(0) = 0$ (55)

(52) → $\Theta(0) = -\frac{P}{2EI}(0)^2 + \frac{5Pa^2}{12EI} = \frac{5Pa^2}{12EI}$ (56)

(53) → $u(0) = -\frac{P}{6EI}(0)^3 + \frac{5\cdot Pa^2}{12\cdot EI}(0) = 0$ (57)

(48) → $V(a) = P\cdot(a)^0 - 3P(0)^0 = P,\ -2P$ (58)

(49) → $M(a) = P(a) - 3P(0) = Pa$ (59)

(52) → $\Theta(a) = -\frac{P(a)^2}{2EI} - \frac{3P(0)^2}{2EI} + \frac{5\cdot Pa^2}{12EI} = -\frac{Pa^2}{12EI}$ (60)

(53) → $u(a) = -\frac{P(a)^3}{6EI} + \frac{P(0)^3}{2EI} + \frac{5\cdot Pa^2\cdot(a)}{12\cdot EI} = \frac{3Pa^3}{12\cdot EI} = \frac{Pa^3}{4EI}$ (61)

THE CHANGE IN SIGN BETWEEN $\Theta(0)$ (56) AND $\Theta(a)$ (60) INDICATES THE VALUE OF Θ GOES TO ZERO IN THIS REGION WHICH MEANS u WILL BE A MAX OR MIN IN THIS REGION. THIS LOCATION MUST BE FOUND. SETTING (52) EQUAL TO ZERO FOR THE REGION $0 \le y \le a$

(52) $$\Theta = 0 = -\frac{\cancel{P}\cdot y^2}{2\cancel{EI}} + \frac{5\cdot \cancel{P}a^2}{12\cancel{EI}} \Rightarrow y = \pm\sqrt{\frac{5a^2}{6}} = -0.9129\cdot a,\ \underline{\underline{0.9129\cdot a}} \quad (62)$$

ONLY THE SECOND ROOT EXISTS IN THE REGION BEING CONSIDERED. THE VALUE OF u AT THIS ROOT IS

$$u(0.9129\cdot a) = -\frac{P\cdot(0.9129\cdot a)^3}{6\cdot EI} + \frac{5\cdot Pa^2\cdot(0.9129\cdot a)}{12EI} = \frac{0.2536\cdot Pa^3}{EI} \quad (63)$$

REGION BC: $a \le y \le 2a$

(58) → $V(a) = P, -2P$

(59) → $M(a) = Pa$

(60) → $\Theta(a) = -\frac{Pa^2}{12EI}$

(61) → $u(a) = \frac{Pa^3}{4EI}$

(48) → $V(2a) = P\cdot(2a)^0 - 3P(a)^0 + 3P(0)^0 = -2P,\ P$ (64)

(49) → $M(2a) = P\cdot(2a) - 3P\cdot(a) + 3P(0) = -Pa$ (65)

(52) → $\Theta(2a) = -\frac{P\cdot(2a)^2}{2\cdot EI} + \frac{3P\cdot(a)^2}{2EI} - \frac{3P\cdot(0)^2}{2EI} + \frac{5Pa^2}{12EI} = -\frac{Pa^2}{12EI}$ (66)

(53) → $u(2a) = -\frac{P\cdot(2a)^3}{6EI} + \frac{P\cdot(a)^3}{2EI} - \frac{P\cdot(0)^3}{2EI} + \frac{5\cdot Pa^2\cdot(2a)}{12EI} = 0$ (67)

THERE IS A SIGN CHANGE BETWEEN M(a) (59) AND M(2a) (65) THAT INDICATES THAT M GOES TO ZERO IN THIS REGION, WHICH MEANS THAT Θ IS A MAX/MIN AT THE POINT M GOES TO ZERO. FINDING WHERE M GOES TO ZERO IN THIS REGION IS ACCOMPLISHED BY SETTING M=0 AND FINDING Y.

(49) → $M = 0 = P\cdot y - 3P(y-a) = P\cdot y - 3P\cdot y + 3Pa = -2P\cdot y + 3Pa$

$\Rightarrow y = \frac{3}{2}\cdot a$ (68)

(52) → $\Theta\left(\frac{3a}{2}\right) = -\frac{P}{2EI}\left(\frac{3a}{2}\right)^2 + \frac{3P}{2EI}\cdot\left(\frac{a}{2}\right)^2 + \frac{5Pa^2}{12EI}$

$= -\frac{9}{8}\frac{Pa^2}{EI} + \frac{3}{8}\frac{Pa^2}{EI} + \frac{5}{12}\frac{Pa^2}{EI} = -\frac{16}{48}\frac{Pa^2}{EI} = -\frac{Pa^2}{3EI}$ (69)

(53) → $u\left(\frac{3a}{2}\right) = -\frac{P}{6EI}\left(\frac{3a}{2}\right)^3 + \frac{P}{2EI}\left(\frac{a}{2}\right)^3 + \frac{5Pa^2}{12EI}\cdot\left(\frac{3a}{2}\right)$

$= -\frac{9\cdot Pa^3}{16EI} + \frac{Pa^3}{16EI} + \frac{15Pa^3}{24EI} = \frac{6}{48}\frac{Pa^3}{EI} = \frac{Pa^3}{8EI}$ (70)

REGION CD: $2a \le y \le 3a$

(64) → $V(2a) = -2P,\ P$

(65) → $M(2a) = -Pa$

(66) → $\theta(2a) = -\frac{Pa^2}{12EI}$

(67) → $u(2a) = 0$

(48) → $V(3a) = P(3a)^0 - 3P(2a)^0 + 3P(a)^0 - P(0)^0 = P,\ 0$ (71)

(49) → $M(3a) = P(3a) - 3P(2a) + 3P(a) - P(0) = 0$

(52) → $\theta(3a) = -\frac{P}{2EI}(3a)^2 + \frac{3P}{2EI}(2a)^2 - \frac{3P}{2EI}(a)^2 + \frac{P}{2EI}(0)^2 + \frac{5Pa^2}{12EI}$

$$= -\frac{9 \cdot Pa^2}{2 \cdot EI} + \frac{12 \cdot Pa^2}{2 \cdot EI} - \frac{3 \cdot Pa^2}{2 \cdot EI} + \frac{5 \cdot Pa^2}{12 \cdot EI} = \frac{5Pa^2}{12EI} \quad (72)$$

(53) → $u(3a) = -\frac{P}{6EI}(3a)^3 + \frac{P}{2 \cdot EI}(2a)^3 - \frac{P}{2EI}(a)^3 + \frac{P}{6EI}(0)^3 + \frac{5Pa^2}{12EI}(3a)$

$$= -\frac{27Pa^3}{6EI} + \frac{8Pa^3}{2EI} - \frac{Pa^3}{2EI} + \frac{15Pa^3}{12EI} = \frac{3Pa^3}{12EI} = \frac{Pa^3}{4EI} \quad (73)$$

THERE IS A SIGN CHANGE BETWEEN $\theta(2a)$ (66) AND $\theta(3a)$ (72) INDICATING THAT θ GOES TO ZERO IN THIS REGION WHICH MEANS THAT u WILL BE A MAX/MIN AT THE LOCATION WHERE $\theta = 0$. TO FIND THIS LOCATION THE EXPRESSION OF θ IS SET TO ZERO, STARTING WITH (52)

(52) → $\theta = 0 = -\frac{P \cdot y^2}{2EI} + \frac{3P}{2 \cdot EI}(y-a)^2 - \frac{3P}{2EI}(y-2a)^2 + \frac{5Pa^2}{12EI}$

$$= -\frac{Py^2}{2EI} + \frac{3P}{2EI}(y^2 - 2ay + a^2) - \frac{3P}{2EI}(y^2 - 4a \cdot y + 4a^2) + \frac{5Pa^2}{12EI}$$

$$= -\frac{P \cdot y^2}{2EI} + \frac{3P \cdot y^2}{2EI} - \frac{3Pay}{EI} + \frac{3Pa^2}{2EI} - \frac{3Py^2}{2EI} + \frac{12 \cdot Pay}{2EI} - \frac{12Pa^2}{2EI} + \frac{5Pa^2}{12EI}$$

$$= -\frac{\cancel{P} \cdot y^2}{2\cancel{EI}} + \frac{3 \cdot \cancel{P}a}{\cancel{EI}} \cdot y - \frac{49\cancel{P}a^2}{12\cancel{EI}} = \frac{y^2}{2} - 3 \cdot ya + \frac{49}{12} \cdot a^2$$

$$= y^2 - 6 \cdot ya + \frac{98}{12}a^2 = y^2 - 6 \cdot ya + \frac{49}{6}a^2$$

$$0 = y^2 - 6a\cdot y + \frac{49}{6}a^2 = \underbrace{y^2 - 6a\cdot y + (-3a)^2}_{(y-3a)^2} - (-3a)^2 + \frac{49}{6}\cdot a^2$$

$$(y-3a)^2 = 9a^2 - \frac{49}{6}\cdot a^2 = \frac{5}{6}\cdot a^2$$

$$y - 3a = \pm\sqrt{\frac{5}{6}a^2} \Rightarrow y = 3a \pm \sqrt{\frac{5}{6}}\cdot a = 3a \pm 0.9129\cdot a$$

$$= \underline{\underline{2.0871\cdot a}},\ 3.9129a \qquad (74)$$

ONLY THE FIRST ROOT IS IN THE DOMAIN OF THE BEAM. u AT THIS LOCATION IS A MAX/MIN AND NEEDS TO BE CALCULATED

$$u(2.0871\cdot a) = -\frac{P}{6EI}(2.0871a)^3 + \frac{P}{2EI}(1.0871\cdot a)^3 - \frac{P}{2EI}(0.0871)^3 + \frac{5Pa^2}{12EI}(2.0871\cdot a)$$

$$= -0.00358\cdot\frac{P\cdot a^3}{EI} \qquad (75)$$

THE CRITICAL VALUES OF THE BEAM'S V, M, θ, AND u FUNCTIONS ARE SUMMERIZED IN FIGURES d, e, f, & g.

Summary:

THE DIRECT INTEGRATION APPROACH (PAGES 3-10) AND THE SINGULARITY FUNCTION APPROACH (Pages 11-15) YIELD EXACTLY THE SAME RESULT. ONE ADVANTAGE OF THE SINGULARITY FUNCTION APPROACH IS THAT CONTINUITY CONDITIONS DO NOT HAVE TO BE COMPUTED BETWEEN REGIONS OF THE BEAM. THIS SIGNIFICANTLY REDUCES THE COMPLEXITY OF THE CALCULATIONS AND THE OPPORTUNITY FOR ERRORS.

BECAUSE OF THE CHOICE OF COORDINATE SYSTEMS, THE CALCULATIONS CAN BE EASILY CHECKED BY VISUAL INSPECTION.