MER 532: Composite Materials
Review of Matrix Algebra

- Square Matrices
- Matrix Addition
- Matrix Multiplication
- Matrix Transpose
- Determinants
- Cofactor Matrix
- Matrix Inversion
- Eigenvalues and Eigenvectors

Matrix Algebra

$$u = a_{11} \cdot x + a_{12} \cdot y + a_{13} \cdot z$$

$$v = a_{21} \cdot x + a_{22} \cdot y + a_{23} \cdot z$$

$$w = a_{31} \cdot x + a_{32} \cdot y + a_{33} \cdot z$$

$$\{\delta\} = [A] \cdot \{s\} \qquad \delta_i = A_{ij} \cdot s_j$$

$$\{\delta\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \qquad [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \{s\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

Types of Square Matrices

Diagonal Matrix: $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

Identity Matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Symmetric Matrix: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ sym & & a_{33} \end{bmatrix}$ $\quad a_{ij} = a_{ji}$

Matrix Addition

$$[A]+[B]=[C]$$

$$c_{ij}=a_{ij}+b_{ij}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 6 & -3 & 2 \end{bmatrix}+\begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}=\begin{bmatrix} 2 & -2 & 0 \\ 7 & -3 & 1 \end{bmatrix}$$

Scalar Multiplication

$$s \cdot [A] = [s \cdot a_{ij}]$$

$$3 \cdot \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 0 \\ 6 & -3 & 3 \\ 0 & 6 & -6 \end{bmatrix}$$

Matrix Multiplication

- Number of columns of the of the first matrix must equal the number of rows of the second

$$[C]=[A]\cdot[B]$$

$$c_{ij} = a_{ik} \cdot b_{kj}$$

- Example

$$\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 1 \\ -7 & -9 & 2 \end{bmatrix}$$

Matrix Multiplication of Square Matrices

- In general

$$[A]\cdot[B]\neq[B]\cdot[A]$$

- Pre- and postmultiplicatoin of the identity matrix

$$[I]\cdot[A]=[A]\cdot[I]=[A]$$

- Associative Law

$$[A]\cdot\left([B]\cdot[C]\right)=\left([A]\cdot[B]\right)\cdot[C]$$

Matrix Transpose

- Interchanging the rows and columns of a matrix

$$[A] = a_{ij} \quad [A]^T = a_{ji}$$

- Example

$$\begin{bmatrix} 2 & 5 & -4 \\ -3 & 7 & -9 \end{bmatrix}^T = \begin{bmatrix} 2 & -3 \\ 5 & 7 \\ -4 & -9 \end{bmatrix}$$

- Transpose of the products

$$\left([A]\cdot[B]\cdot[C]\right)^T = [C]^T \cdot [B]^T \cdot [A]^T$$

Determinant of a Matrix

- $|A|$ is the determinant of an n by n square matrix $[A]$
  - method of cofactors eventually reduces to a 2 by 2 determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

- For an n by n matrix $[A]$
  - select any row i or column j

$$|A| = \sum_{j=1}^{n} a_{ij} \cdot \tilde{a}_{ij} \qquad \text{selecting row i}$$

$$|A| = \sum_{i=1}^{n} a_{ij} \cdot \tilde{a}_{ij} \qquad \text{selecting column j}$$

  - $\tilde{a}_{ij}$ is the cofactor of a_{ij}

Cofactor Matrix

- The cofactor matrix $\left[\tilde{A}\right]$ is the same order of $\left[A\right]$
- Each term in $\left[\tilde{A}\right]$ is given by

  $\tilde{a}_{ij} = (-1)^{i+j} \cdot m_{ij}$

  - m is the **minor** of the matrix a_{ij} and is the determinate of the (n-1)$\cdot$(n-1) matrix obtained by eliminating row i and column j of a_{ij}
- Example

$$\left[A\right] = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \qquad \left[\tilde{A}\right] = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 2 & 6 \\ -2 & -1 & 6 \end{bmatrix}$$

Matrix Inversion

- Consider the linear equations

$$\{\delta\} = [A] \cdot \{s\}$$

- If $\{\delta\}$ and $[A]$ are known, $\{s\}$ can be found

$$[A]^{-1} \cdot \{\delta\} = [A]^{-1} \cdot [A] \cdot \{s\} = [I] \cdot \{s\} = \{s\}$$

- The inverse of $[A]$, $[A]^{-1}$ is

$$[A]^{-1} = \frac{\left[\tilde{A}\right]^T}{|A|}$$

  - $\left[\tilde{A}\right]^T$ is the transpose of the cofactor or **adjoint matrix**

Special Cases of Matrix Inversion

- The inverse of $[A]$, $[A]^{-1}$ is

$$[A]^{-1} = \frac{\left[\tilde{A}\right]^T}{|A|}$$

- If the determinate of the matrix is zero it is refered to as **singular**
  - The inverse does not exist
  - This typically means that the equations are not **independent**

- The inverse of an orthogonal transformation matrix
  is simply the transpose of the transformation matrix

$$\{V'\} = [T] \cdot \{V\} \qquad \{V\} = [T]^T \cdot \{V'\}$$

Eigenvalues and Eigenvectors

- The Eigen value problem is of the form, $[A]$ is an n by n square matrix

$$[A]\cdot\{s\} = \lambda\cdot[I]\cdot\{s\}$$

$$([A]-\lambda\cdot[I])\cdot\{s\} = 0$$

- To avoid the trivial solution, $\{s\} = 0,$ $[A]-\lambda\cdot[I]$ is forced to be singular

$$\left\|[A]-\lambda\cdot[I]\right\| = 0$$