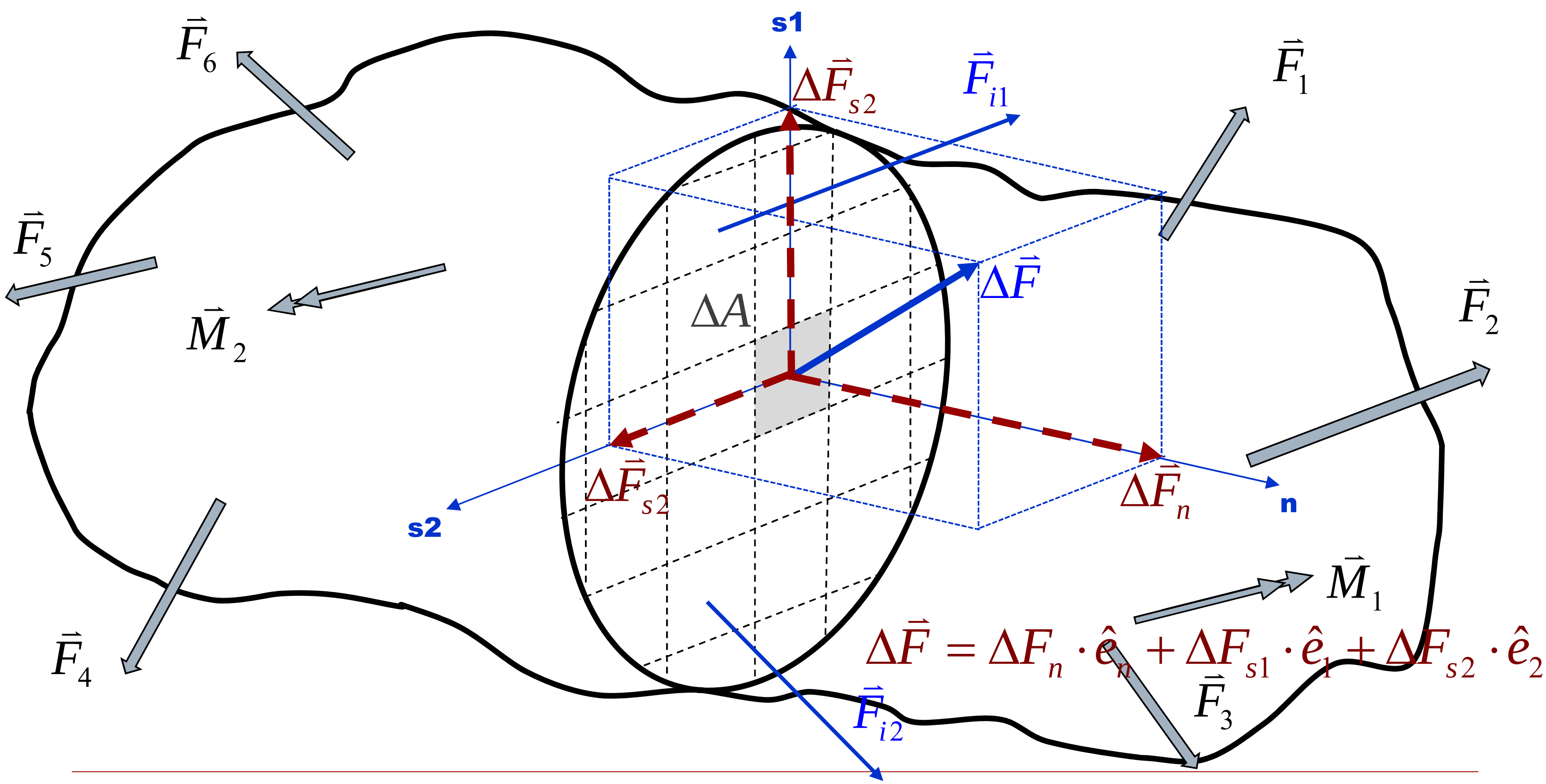
MER311: Advanced Strength of Materials

LECTURE OUTLINE

- State of Stress
- Stress Tensor
- Equilibrium

Body and Surface Loads Result In An Internal Force Distribution



Body and Surface Loads Over An Area Define Stress On A Surface

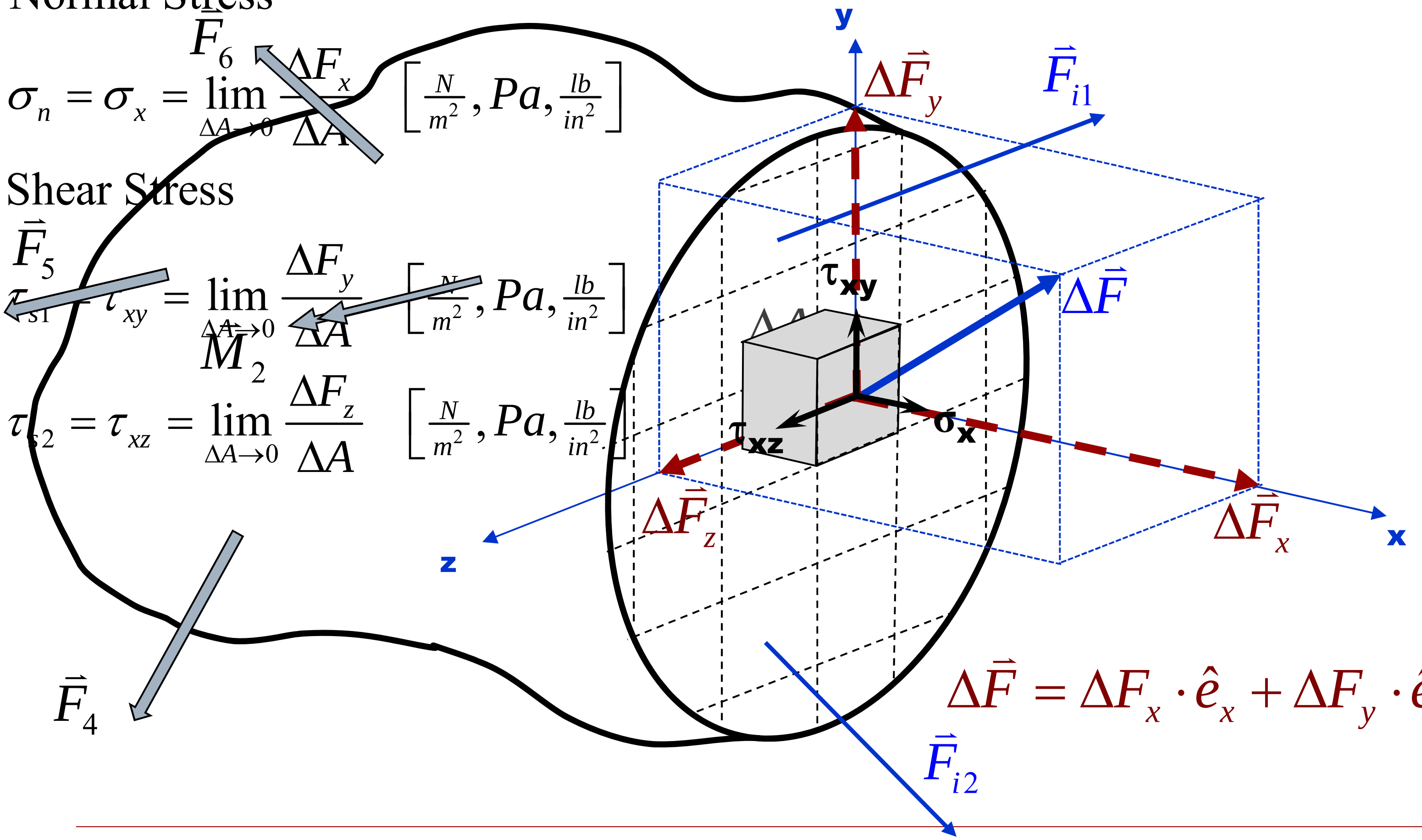
Normal Stress

$$\sigma_n = \sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} \quad \left[\frac{N}{m^2}, Pa, \frac{lb}{in^2}\right]$$

Shear Stress

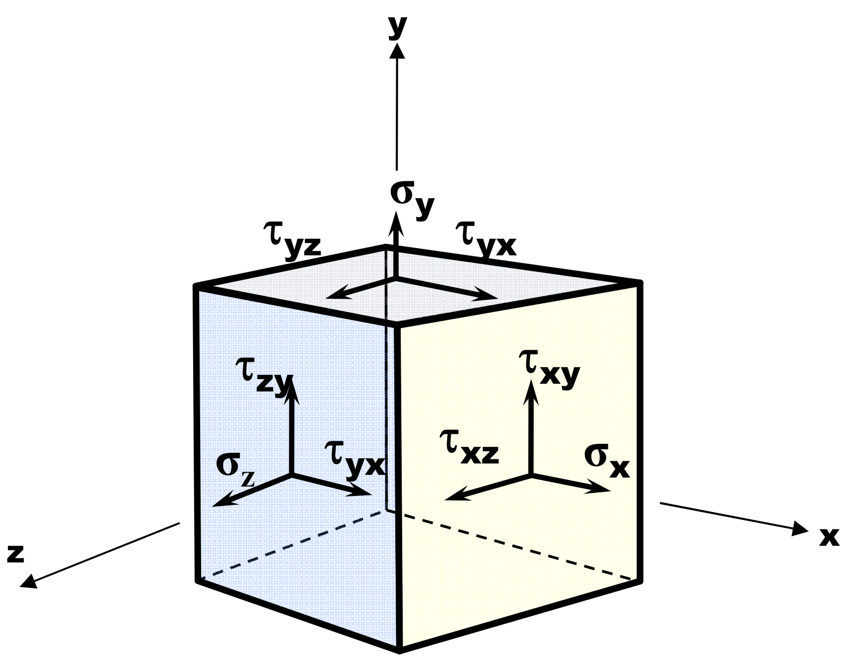
$$\tau_{s1} = \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \quad \left[\frac{N}{m^2}, Pa, \frac{lb}{in^2}\right]$$

$$\tau_{s2} = \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A} \quad \left[\frac{N}{m^2}, Pa, \frac{lb}{in^2}\right]$$

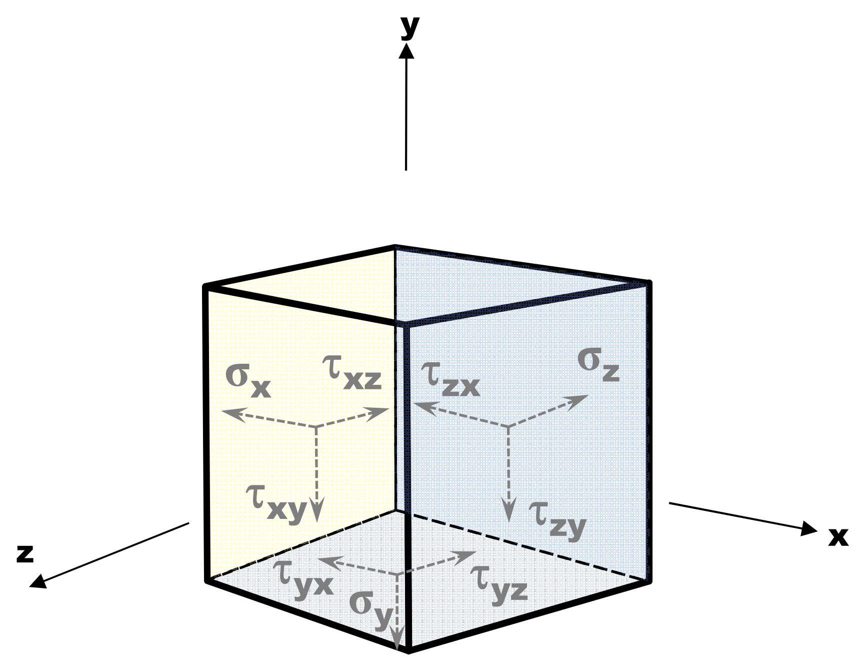


$$\Delta\vec{F} = \Delta F_x \cdot \hat{e}_x + \Delta F_y \cdot \hat{e}_y + \Delta F_z \cdot \hat{e}_z$$

Stress at a Point
Shown in the Tensile (+) Direction



Surfaces with a Positive Directed Area Normal



Surfaces with a Negative Directed Area Normal

Stress Tensor Can Be Expressed In Multiple Ways

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \sigma_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \tau_{ij}$$

Dilatational and Distortional Stress Tensors

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix}$$

Dilatational Tensor **Distortional Tensor**

where $\sigma_m = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)$

EXAMPLE:
Stress Tensor

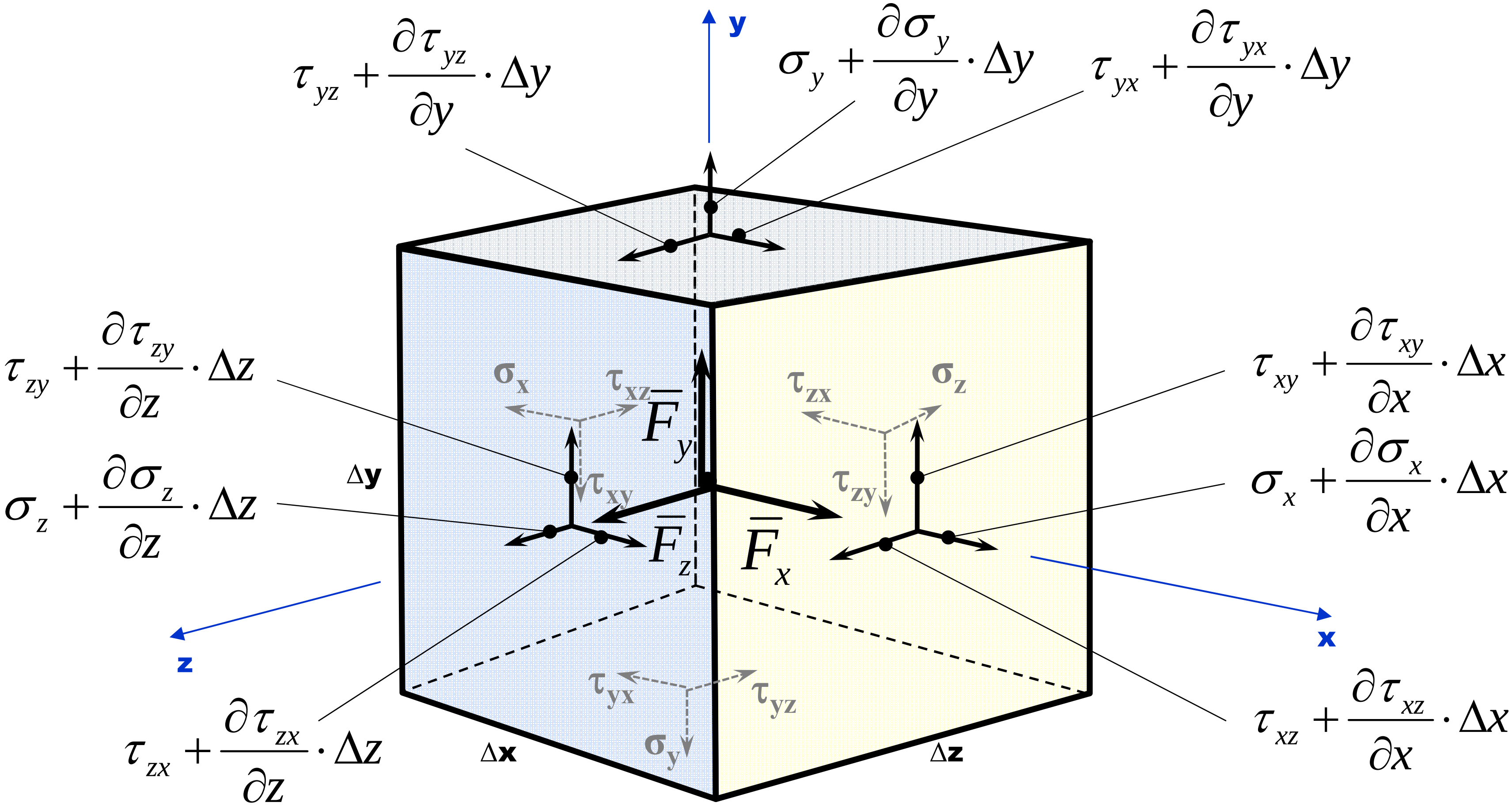
At a specified point in a body, the state of stress with respect to a Cartesian coordinate system is,

$$[\sigma] = \begin{bmatrix} 12 & 6 & 9 \\ 6 & 10 & 3 \\ 9 & 3 & 14 \end{bmatrix} MPa$$

Determine the Dilatation and Distortion stress tensors

$$[\sigma] = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} MPa + \begin{bmatrix} 0 & 6 & 9 \\ 6 & -2 & 3 \\ 9 & 3 & 2 \end{bmatrix} MPa$$

Moment Equilibrium in an Element with Finite Dimensions (about x)



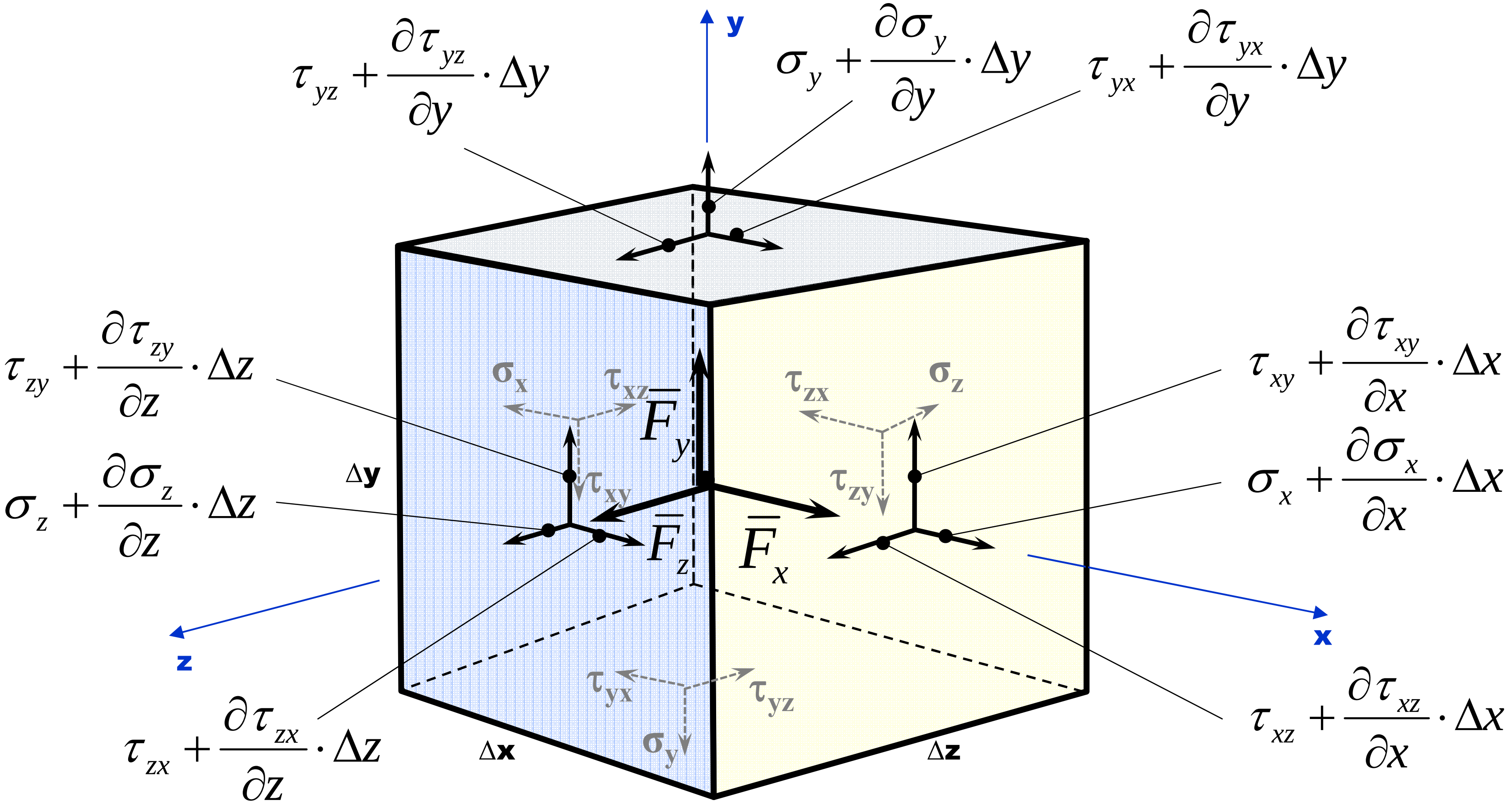
Relationship of Shear Stress on Perpendicular Surfaces

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{xz} = \tau_{zx}$$

Force Equilibrium in an Element with Finite Dimensions (in x)



Elastic Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \overline{F}_x = \mathbf{0}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{F}_y = \mathbf{0}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{F}_z = \mathbf{0}$$

NOTE: $\overline{F}_x$, $\overline{F}_y$, and $\overline{F}_z$ have units of [force/volume]

The Stress Tensor Rewritten Accounting for Symmetry

$$[\sigma]=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix}$$

$$=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix}=\sigma_{ij}$$

$$=\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}=\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \tau_{yy} & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \tau_{zz}\end{bmatrix}=\tau_{ij}$$

$$\Rightarrow \{\sigma\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xz}\\ \tau_{yz}\\ \tau_{xy}\end{Bmatrix}$$

EXAMPLE:
3D Equilibrium

The stress field within an elastic structural member is expressed as follows:

$$\sigma_x = -x^3 + y^2,\ \ \tau_{xy} = 5z + 2y^2,\ \ \tau_{xz} = xz^3 + x^2 y$$
$$\sigma_y = 2x^3 + .5y^2,\ \tau_{yz} = 0,\ \ \sigma_z = 4y^2 - z^3$$

Determine the body force distribution required for equilibrium.

EXAMPLE: Equilibrium Equations

Determine whether the following two dimensional stress field is possible within an elastic structural member. Assume that the body forces are negligible

$$\sigma_{ij} = \begin{bmatrix} -\frac{3}{2} \cdot x^2 \cdot y^2 & x \cdot y^3 \\ x \cdot y^3 & -\frac{1}{4} \cdot y^4 \end{bmatrix}$$