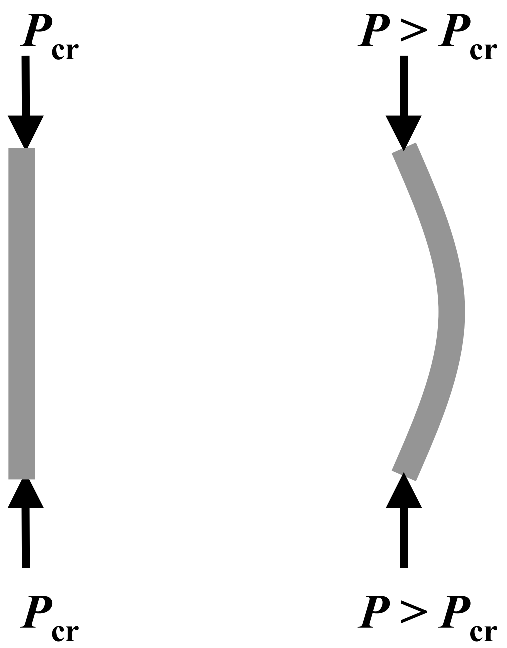
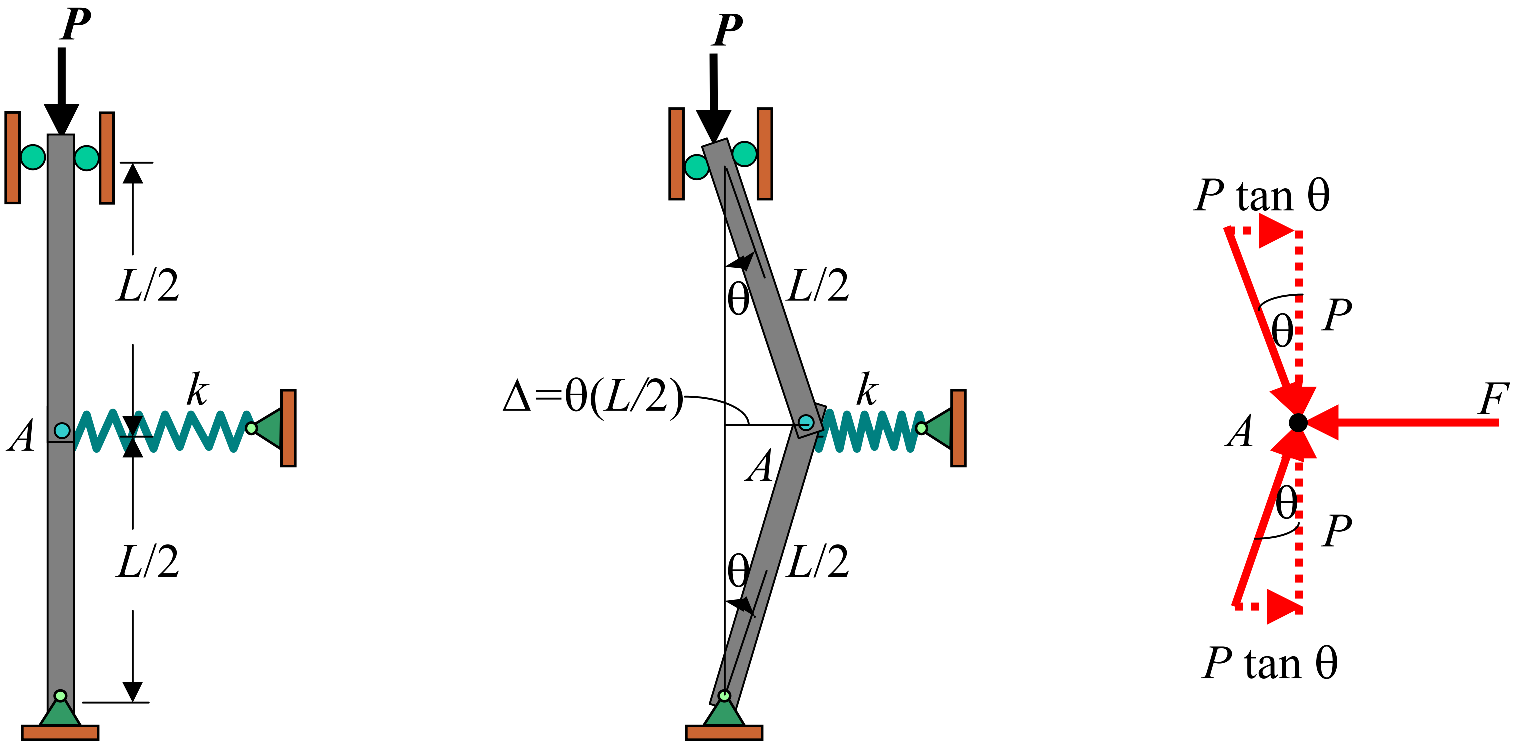
BUCKLING OF COLUMNS

- **Critical Load**
- **Ideal Column with Pin Supports**
- **Columns Having Various Supports**

Critical Load



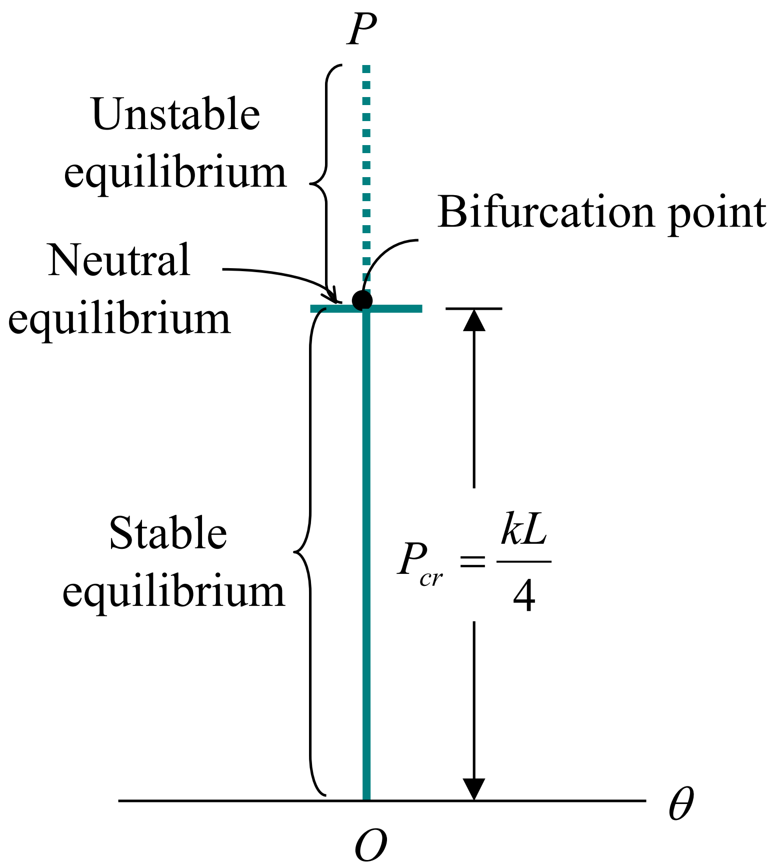


$$\xrightarrow{+} \Sigma F_x = 0: \qquad 2P\tan\theta = F$$

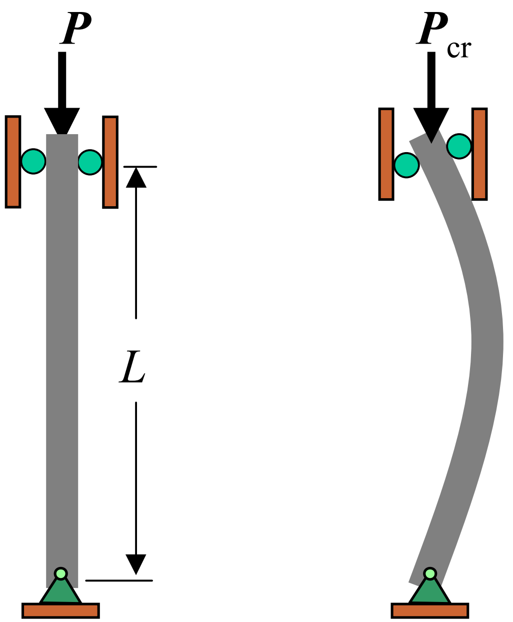
$$2P\tan\theta = k\Delta$$

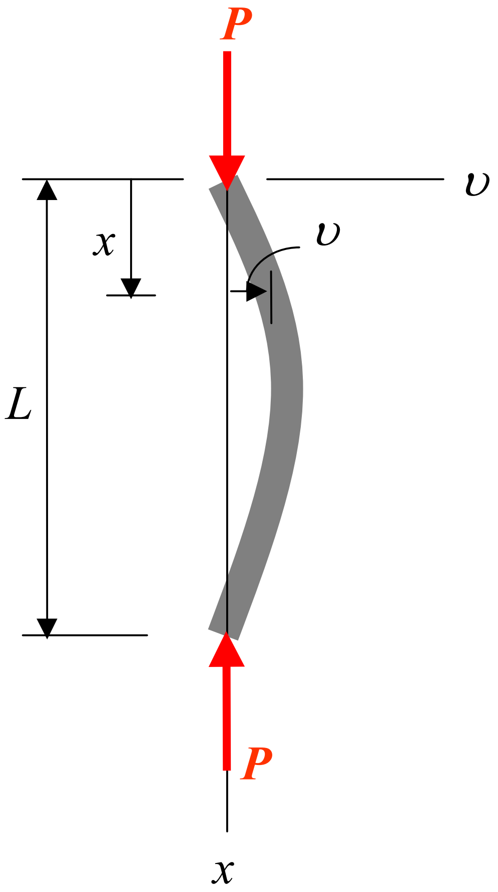
For small θ, $$2P\theta = k(\theta\frac{L}{2})$$

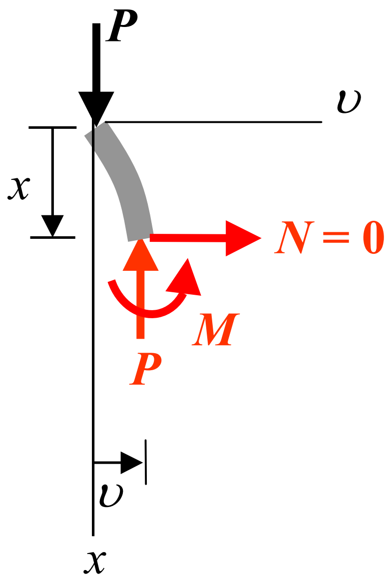
$$\boxed{P_{cr} = \frac{kL}{4}}$$


P
Unstable equilibrium
Bifurcation point
Neutral equilibrium
Stable equilibrium
$P_{cr} = \frac{kL}{4}$
θ
O

Ideal Column with Pin Supports







$$\overset{+}{\curvearrowright} \Sigma M_x = 0\ ;$$

$$P\upsilon + M = 0$$

$$M = -P\upsilon$$

- **Moment-curvature**

$$M = EI\frac{d^2\upsilon}{dx^2} = -P\upsilon$$

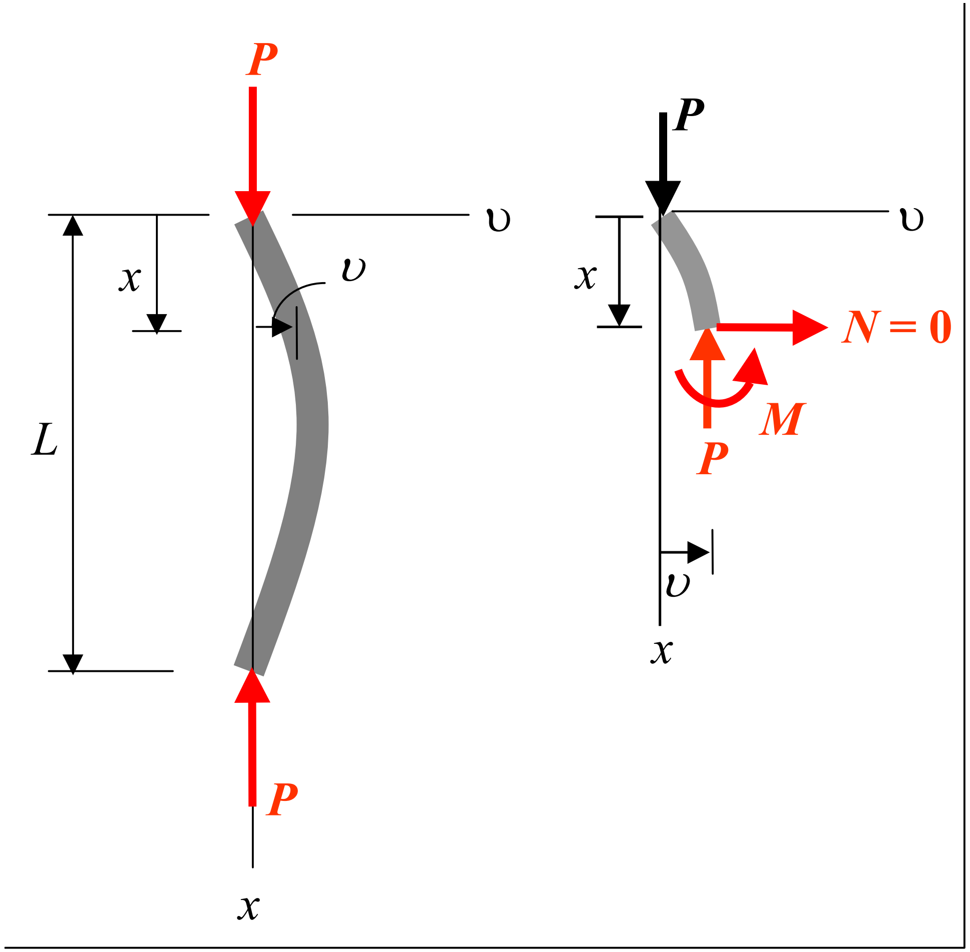
$$EI\frac{d^2\upsilon}{dx^2} + P\upsilon = 0$$

$$\frac{d^2\upsilon}{dx^2} + (\frac{P}{EI})\upsilon = 0$$

$$\frac{d^2\upsilon}{dx^2} + (\sqrt{\frac{P}{EI}})^2\upsilon = 0 \quad ---*$$

$$\upsilon'' + c^2\upsilon = 0$$

$$\upsilon = C_1\sin(\sqrt{\frac{P}{EI}}x) + C_2\cos(\sqrt{\frac{P}{EI}}x)$$



$$\upsilon = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right)$$

- **Boundary condition**

$\Rightarrow x = 0 \quad , \quad \upsilon = 0$

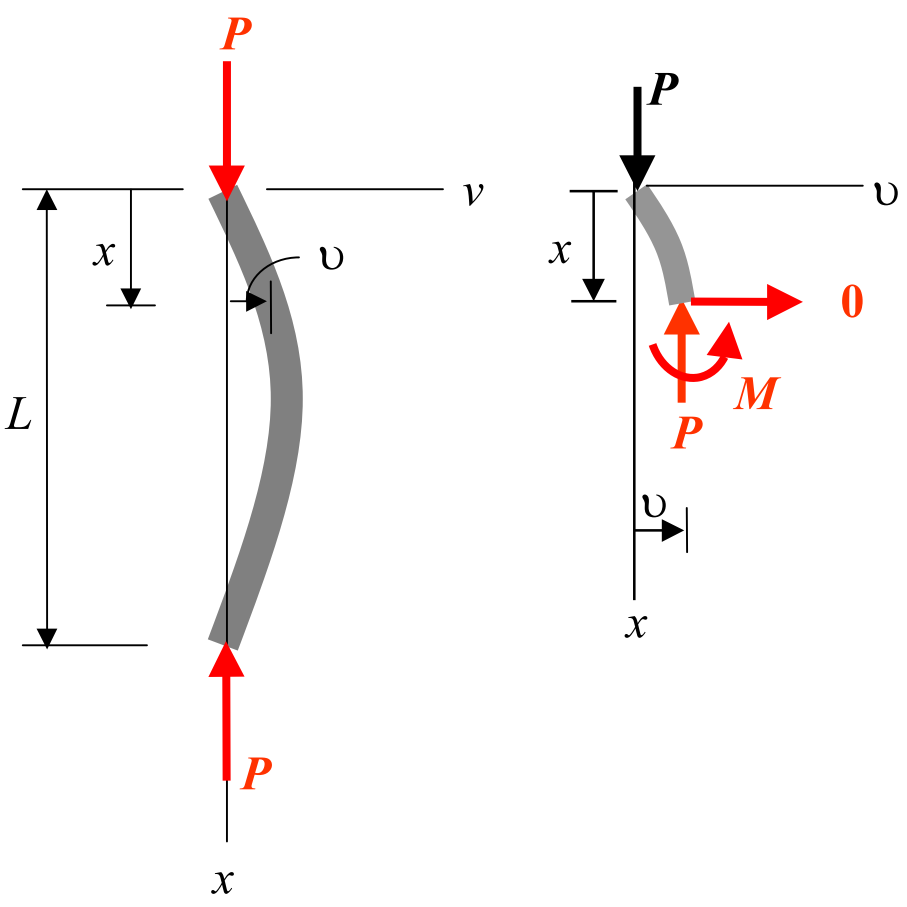
$$0 = C_1(0) + C_2(1), \quad C_2 = 0$$

$\Rightarrow x = L \quad , \quad \upsilon = 0$

$$C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0 \quad , \quad C_1 \neq 0$$

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0 = \sin(n\pi)$$

$$\sqrt{\frac{P}{EI}}L = n\pi \quad : \quad n = 1,2,3,...$$



$$\sqrt{\frac{P}{EI}}L = n\pi \quad : \quad n = 1,2,3,\ldots$$

- **Critical Load P_{cr}**

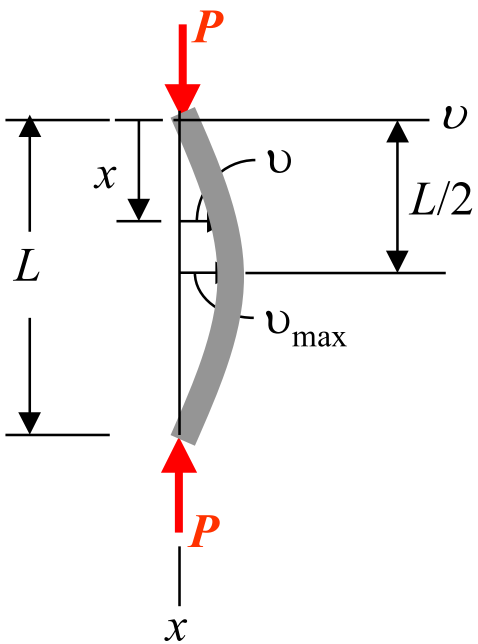
$$\sqrt{\frac{P}{EI}}L = n\pi$$

$$\frac{P}{EI}L^2 = n^2\pi^2$$

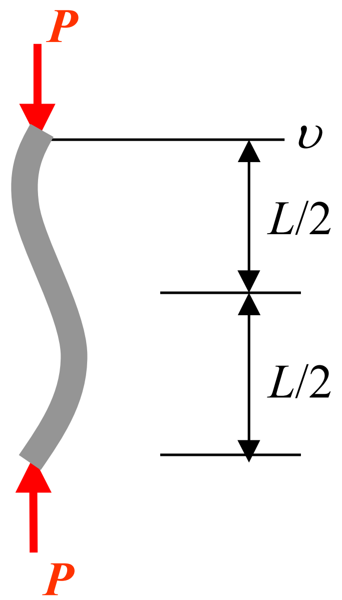
$$P = \frac{n^2\pi^2 EI}{L^2} \quad ------*$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad ------*$$

$$P_{cr} = \frac{n^2 \pi^2 EI}{L^2} \quad , \quad n = 1,2,3,...$$



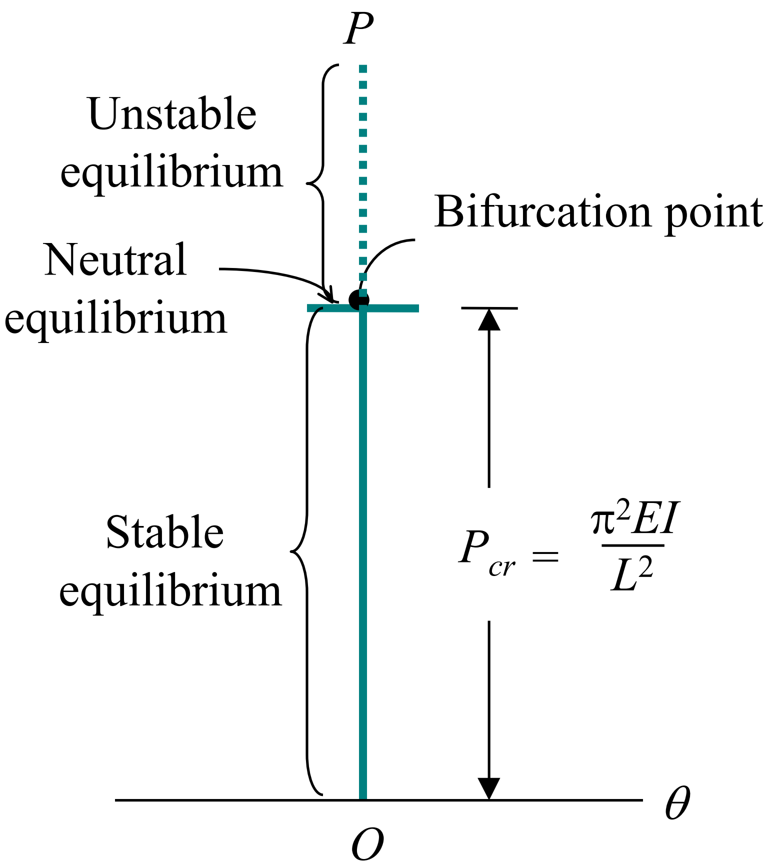
$n = 1$

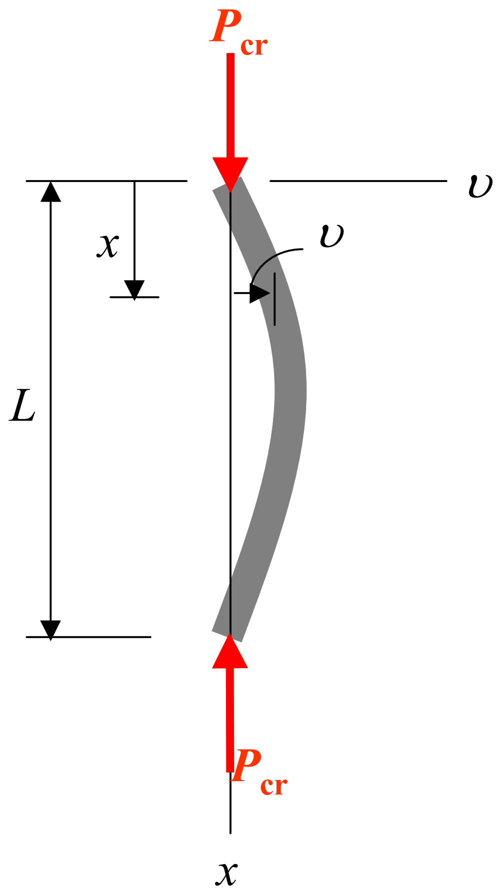


$n = 2$

$$P_{cr} = \frac{1^2 \pi^2 EI}{L^2}$$

$$P_{cr} = \frac{2^2 \pi^2 EI}{L^2}$$


P
Unstable
equilibrium
Bifurcation point
Neutral
equilibrium
Stable
equilibrium
$P_{cr} = \frac{\pi^2 EI}{L^2}$
O
θ


P_{cr}
υ
x
υ
L
P_{cr}
x

- **Critical Stress**

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

$$= \frac{\pi^2 E(Ar^2)}{(KL)^2}$$

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(K\frac{L}{r})^2}$$

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2}}$$

r = radius of gyration

K = effective-length factor, for pin-pin column K = 1

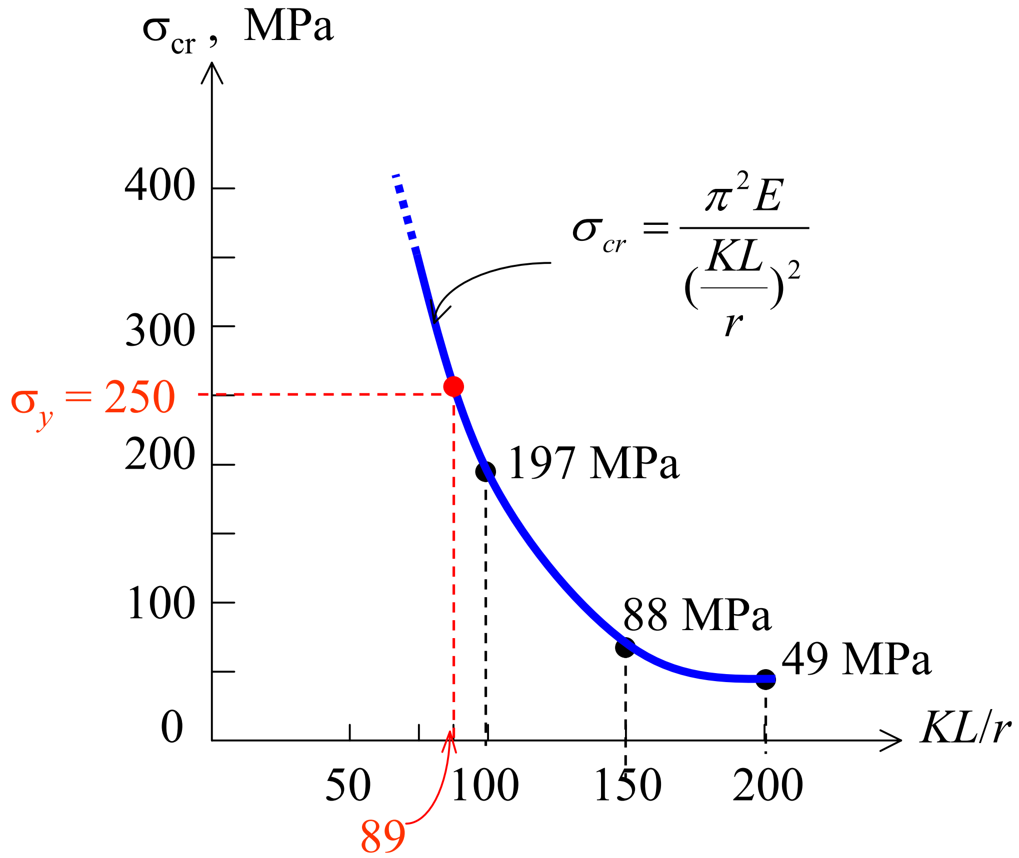
$K\dfrac{L}{r} =$ *effective slenderness ratio*

Structural steel A 36

E **= 200 GPa** σ_y **= 250 MPa**

$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2} = \frac{\pi^2 (200 \times 10^3 MPa)}{(\frac{KL}{r})^2}$$

KL/r	σ_{cr} (MPa)
89	250
100	197
125	126
150	88
175	64
200	49
225	39



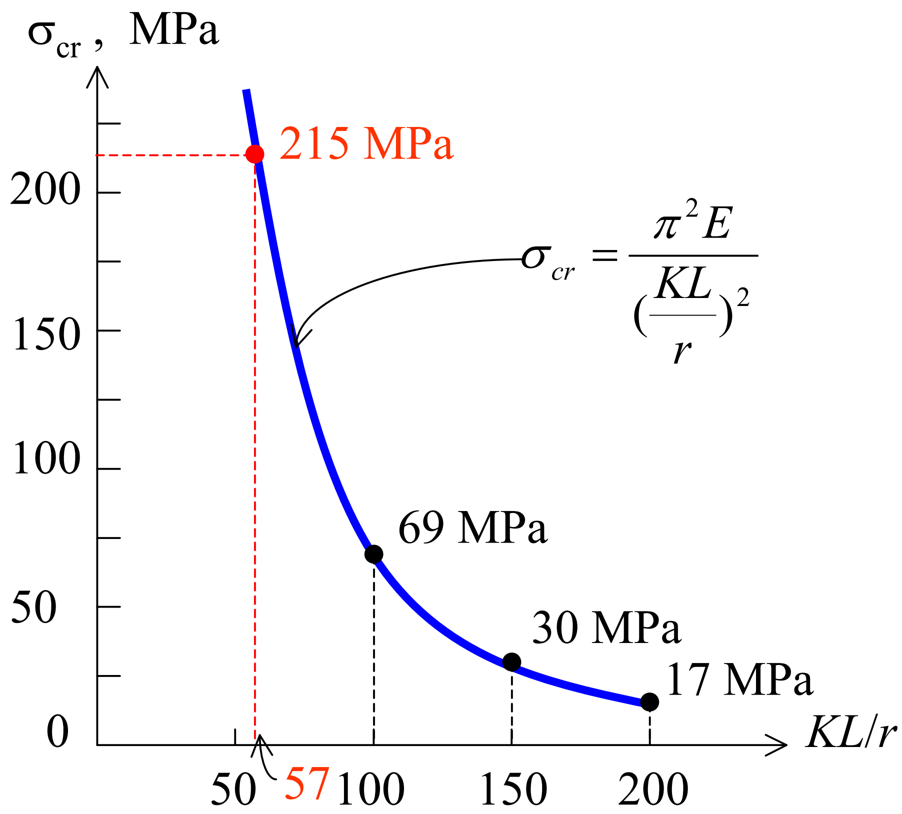
Structural steel
A 36
R 40 (4000 kg/cm²)

Aluminum

E = 70 GPa **σ_y = 215 MPa**

$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2} = \frac{\pi^2 (70 \times 10^3 MPa)}{(\frac{KL}{r})^2}$$

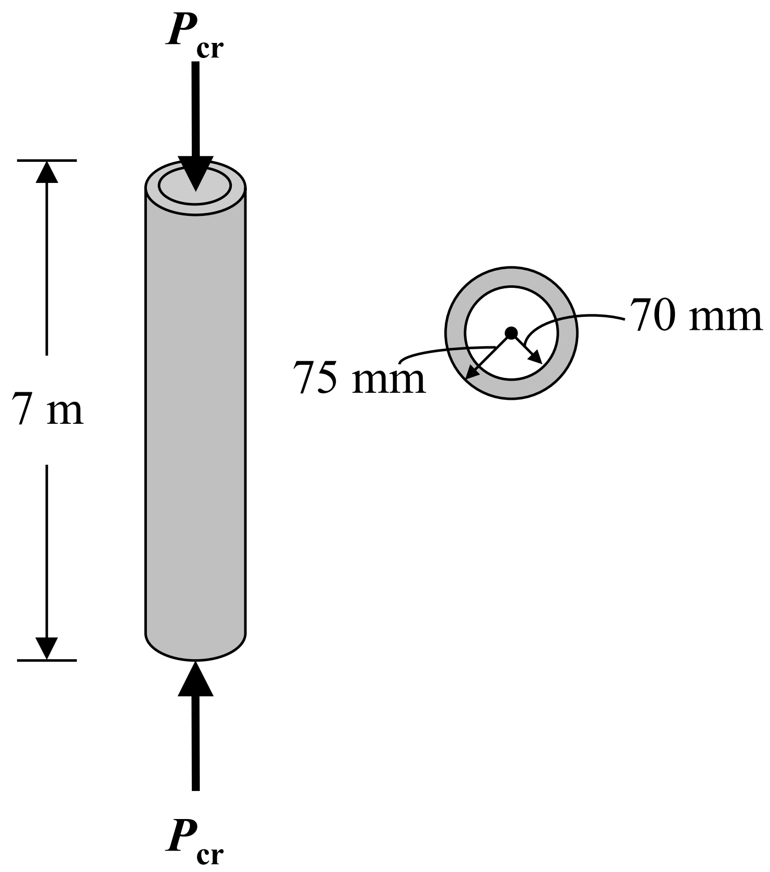
KL/r	σ_{cr} (MPa)
57	215
75	122.8
100	69.1
125	44.2
150	30.7
175	22.6
200	17.3

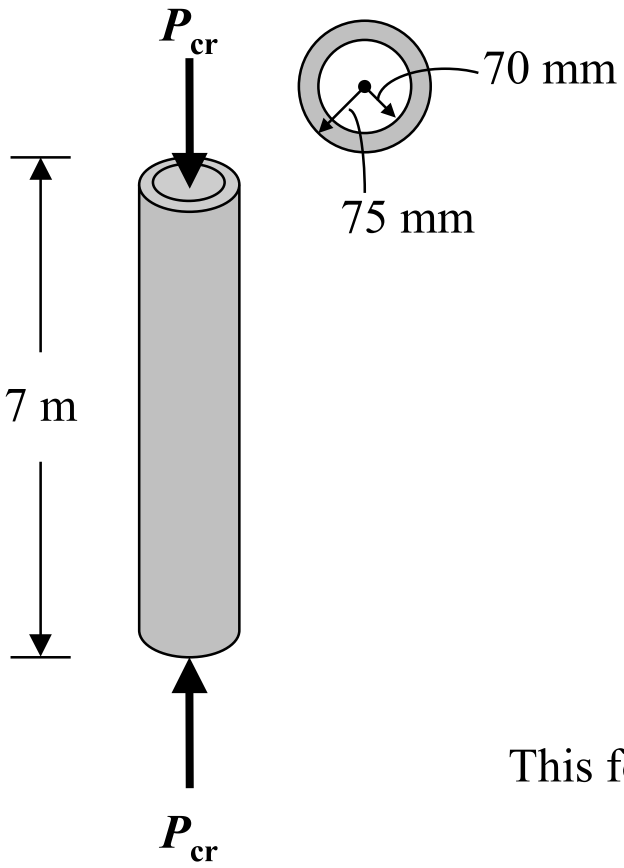


$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2}$$

Example 1

A 7 m long A-36 steel tube having the cross section shown is to be used as a pin-ended column. Determine the maximum allowable axial load the column can support so that it does not buckle or yield. Take the yield stress of 250 MPa





Using Eq. 5 to obtain the critical load with E_{st} = 200 GPa,

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2[(200\times10^6)(\frac{\pi}{4}0.075^4 - \frac{\pi}{4}0.07^4)}{7^2}$$

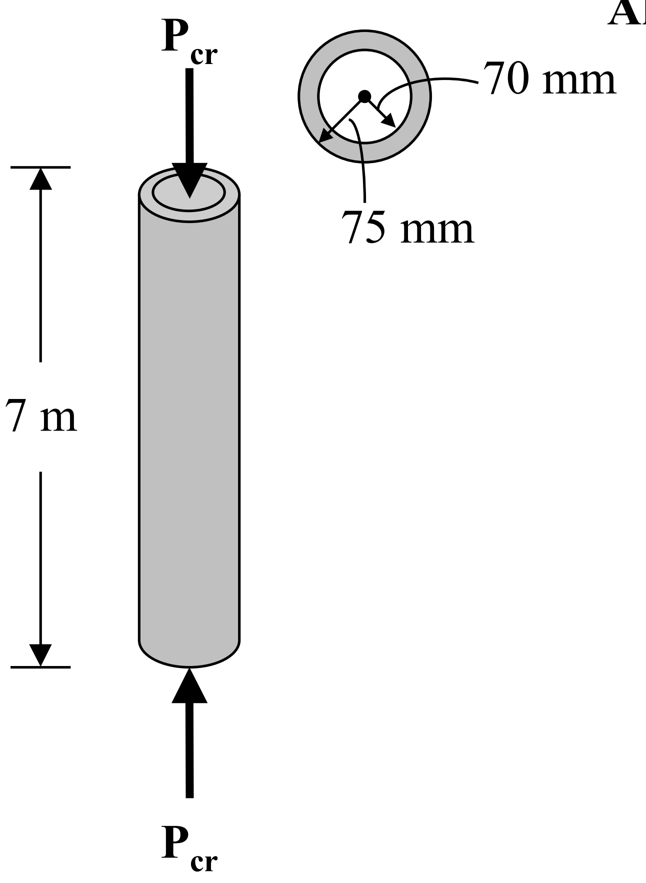
$= 241.4$ kN

This force creates an average compressive stress in the column of

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{241.43}{\pi(0.075)^2 - \pi(0.070)^2}$$

$= 106$ MPa $< \sigma_Y = 250$ MPa O.K

The maximum allowable axial load the column can support is 241.73 kN



Alternate method:

$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2}$$

$$r^2 = \frac{I}{A} \quad = \frac{\pi(.075^4 - .070^4)/4}{\pi(.075^2 - .070^2)} = .002631\, m^2$$

$$r = 79.5\; mm$$

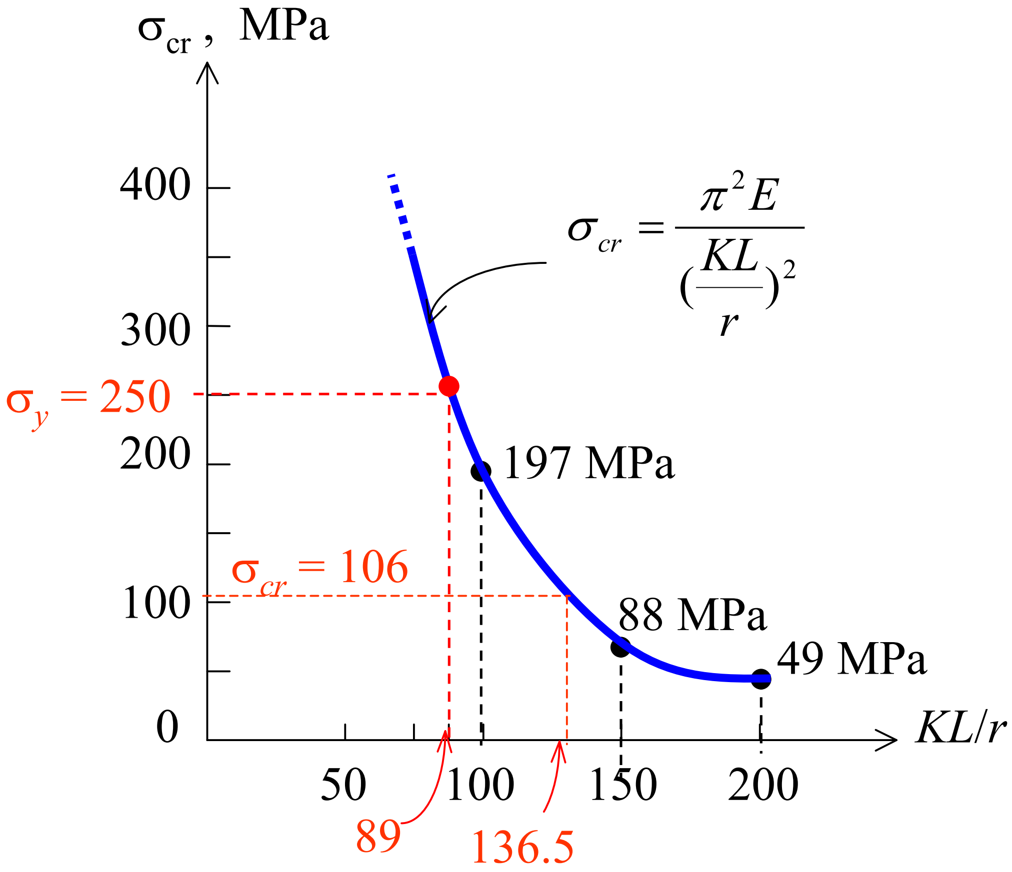
$$\frac{KL}{r} = \frac{(1)(7)}{(0.00795)} = 136.5$$

$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2} \quad = \frac{\pi^2 (200 \times 10^3 MPa)}{(136.5)^2}$$

= 106 MPa $< \sigma_Y$ = 250 MPa O.K

$\boldsymbol{P}_{\mathbf{cr}}$ = $\sigma_{cr} A$ = (106 x 106) π ($.075^2$-$.070^2$)

= 241.7 kN



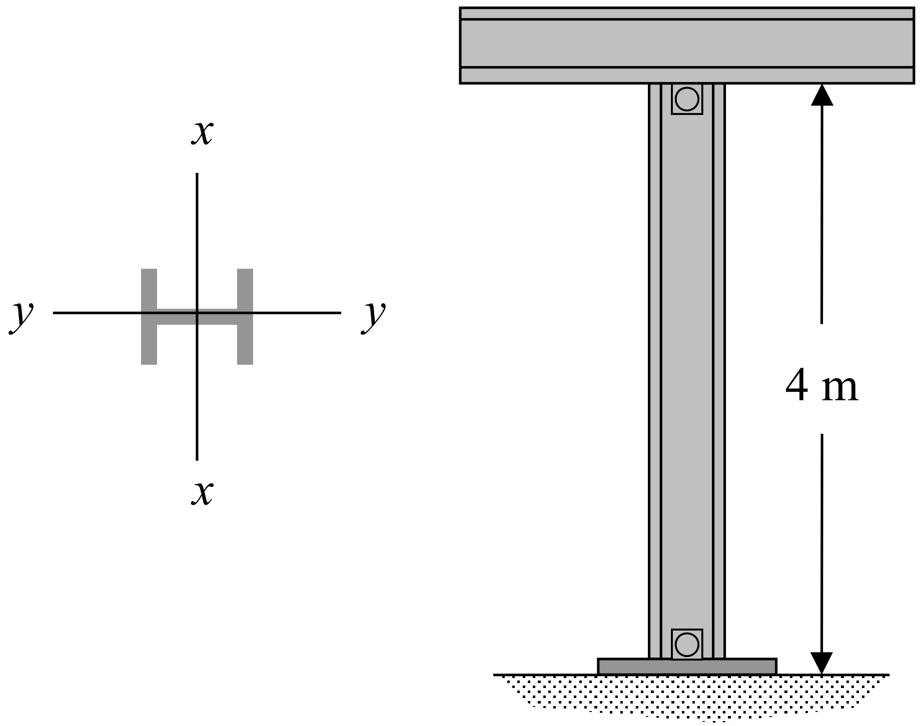
Structural steel

A 36

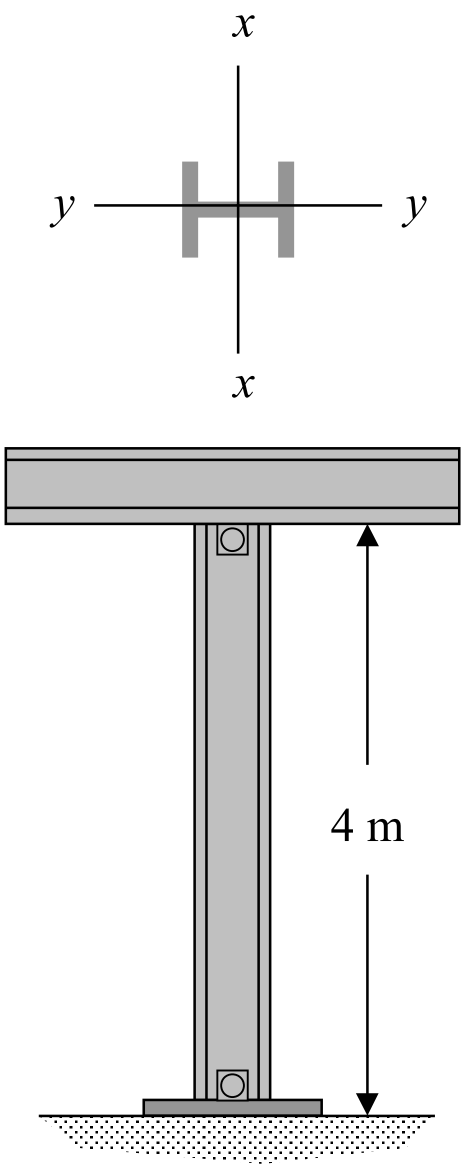
R 40 (4000 kg/cm^2)

Example 2

The A-36 steel W200x46 member show is to be used as a pin-connected column. Determine the largest axial load it can support before it either begins to buckle or the steel yields.



- **Pinned - Pinned Column**



A-36 steel $W200x46$

$A = 5890\ \text{mm}^2$, $I_x = 45.5\text{x}10^6\ \text{mm}^4$, and $I_y = 15.3\text{x}10^6\ \text{mm}^4$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2(200\times10^6)(15.3\times10^{-6})}{4^2}$$

$$= 1887.\ 6\ \text{kN}$$

$$\sigma_{cr} = \frac{P_{cr}}{A}$$

$$= \frac{1887.56}{5890\times10^{-6}}$$

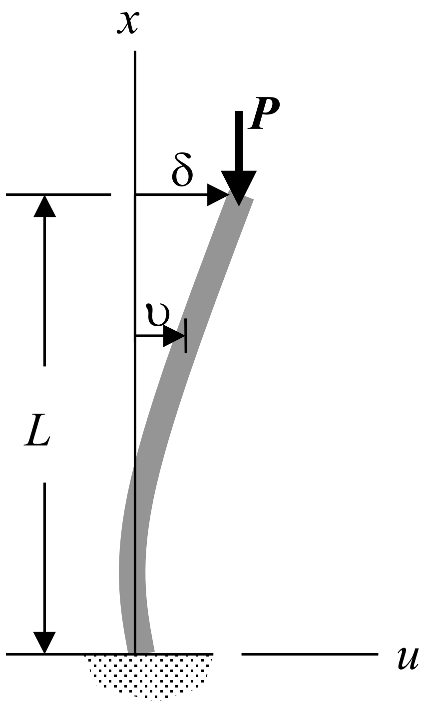
$$= 320.5\ \text{MPa} \quad > \sigma_Y = 250\ \text{MPa}$$

$$P_{allow} = \sigma \bullet A = \sigma_y \bullet A$$

$$= (250\times10^6\ Pa)(5890\times10^{-6}\ m^2)$$

$$= 1472\ kN$$

Columns Having Various Type of Supports



$$EI\frac{d^2\upsilon}{dx^2} = P(\delta - \upsilon)$$

$$\frac{d^2\upsilon}{dx^2} + \frac{P}{EI}\upsilon = \frac{P}{EI}\delta \qquad \text{-----(7)}$$

This equation is non-homogeneous because of the nonzero term on the right side. The solution consists of both a complementary and particular solution, namely,

$$\upsilon = C_1\sin(\sqrt{\frac{P}{EI}}x) + C_2\cos(\sqrt{\frac{P}{EI}}x) + \delta$$

The constants are determined from the boundary conditions. At x = 0, υ = 0, so that C_2 = -δ. Also,

$$\frac{d\upsilon}{dx} = C_1\sqrt{\frac{P}{EI}}\cos(\sqrt{\frac{P}{EI}}x) - C_2\sqrt{\frac{P}{EI}}\sin(\sqrt{\frac{P}{EI}}x)$$

At x = 0, $d\upsilon/dx$ = 0, so that C_1 = 0. The deflection curve is therefore

$$\upsilon = \delta[1 - \cos(\sqrt{\frac{P}{EI}}x)] \qquad \text{-----(8)}$$

Since the deflection at the top of the column is δ, that is, at $x = L$, $\upsilon = \delta$, we require

$$\delta = \delta[1 - \cos(\sqrt{\frac{P}{EI}}x)] \qquad \text{-->} \qquad \delta \cos(\sqrt{\frac{P}{EI}}L) = 0$$

Since $\delta \neq 0$,

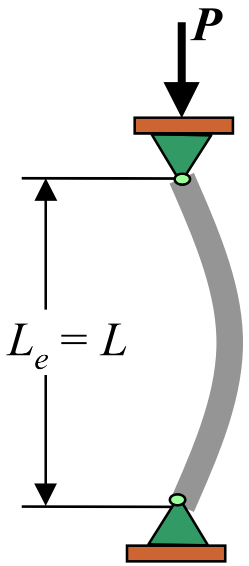
$$\cos(\sqrt{\frac{P}{EI}}L) = 0 \qquad or \qquad \cos(\sqrt{\frac{P}{EI}}L) = \cos(\frac{n\pi}{2})$$

The smallest critical load occurs when $n = 1$, so that

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad \text{-----(9)}$$

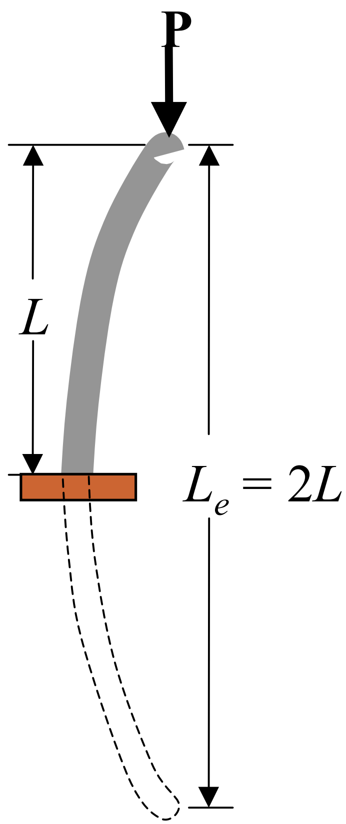
- **Effective Length (L_e)**

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} \quad or \quad \sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2}$$



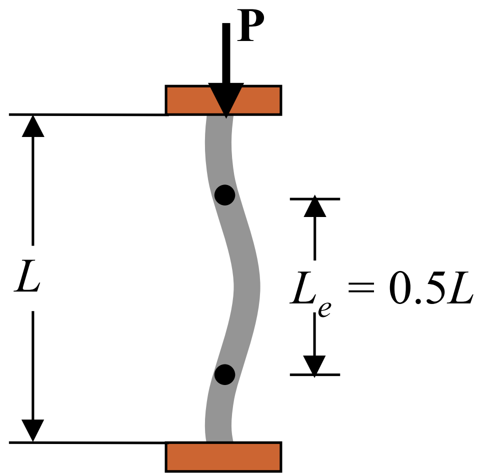
Pinned -pinned ends

$K = 1$



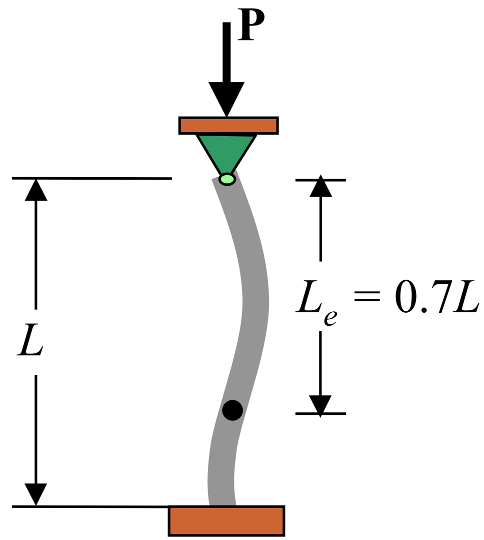
Fixed - free ends

$K = 2$



Fixed fixed ends

$K = 0.5$



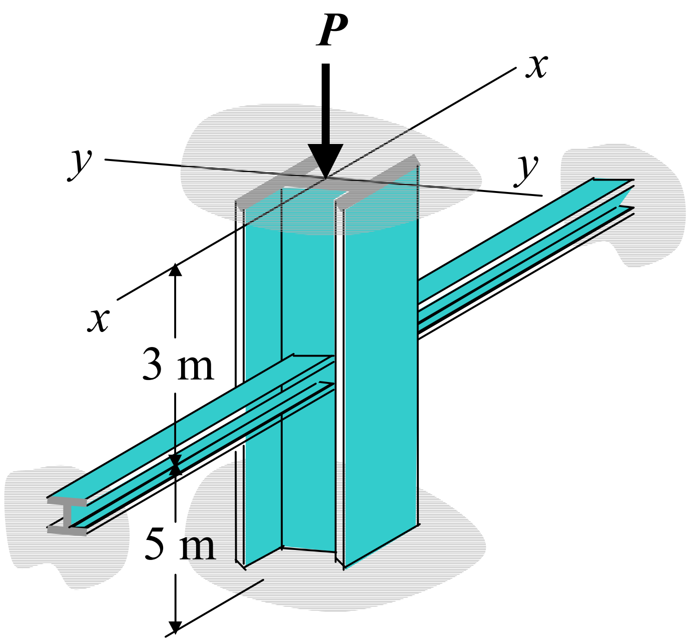
Fixed - Pinned ends

$K = 0.7$

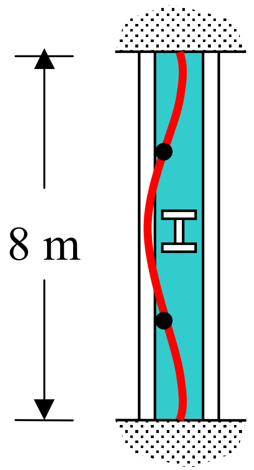
Note : K = ***effective-length factor***

Example 3

A W 150x24 (A=3060 mm^2, I_x = 13.4x10^6 mm^4, I_y = 1.83x10^6 mm^4) steel column is 8 m long and is fixed at its ends as shown. Its load-carrying capacity is increased by bracing it about the y-y (weak) axis using struts that are assumed to be pin-connected to its mid-height. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress. Take E = 200 GPa and σ_y = 250 MPa

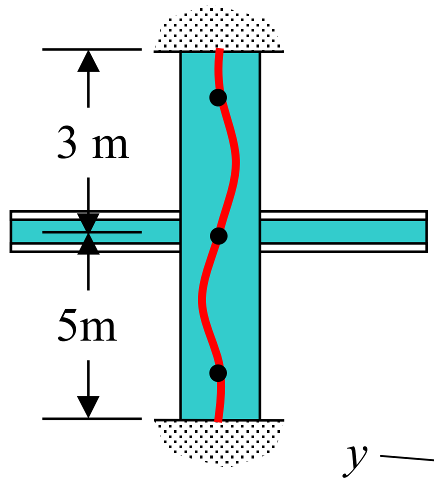


$E = 200$ GPa , $\sigma_y = 414$ MPa $\quad$ *W* 150x24 A = 3060 mm^2



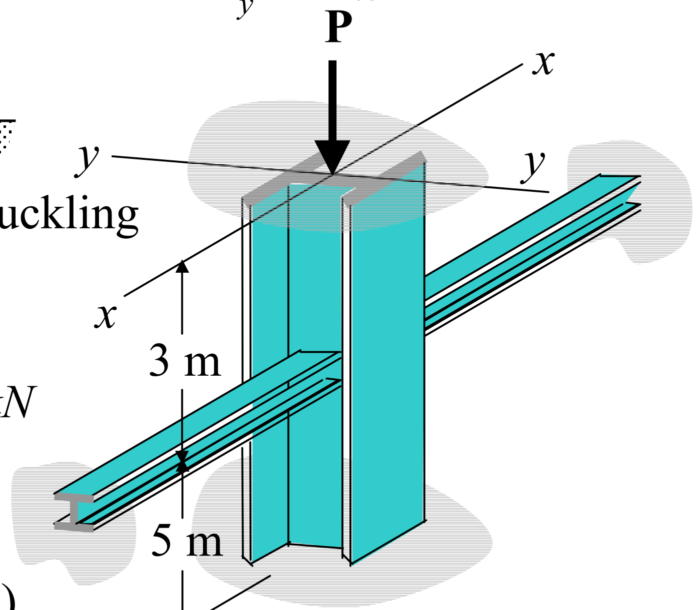
$I_x = 13.4 \times 10^6$ mm^4
Fixed (top)
Fixed (bottom)
$K_x = 0.5$
$r_x = 66.2$ mm

x-x axis buckling



$I_y = 1.83 \times 10^6$ mm^4
Pinned (top)
Fixed (bottom)
$K_y = 0.7$
$r_y = 24.5$ mm

y-y axis buckling



- **Yield Stress (σ_y)**

$$P_Y = \sigma_y A = (250 \times 10^6\, Pa)(3060 \times 10^{-6}\, m^2) = 765\ kN$$

- **Bucking *x-x* axis**

$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^{\ 2}} = \frac{\pi^2 (200 \times 10^6\, kPa)(13.4 \times 10^{-6}\, m^4)}{(0.5 \times 8\ m)^2} = 1653\ kN$$

- **Bucking *y-y* axis**

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^{\ 2}} = \frac{\pi^2 (200 \times 10^6\, kPa)(1.83 \times 10^{-6}\, m^4)}{(0.7 \times 5\ m)^2} = 294.9\ kN$$

NOTE

Structural steel A 36

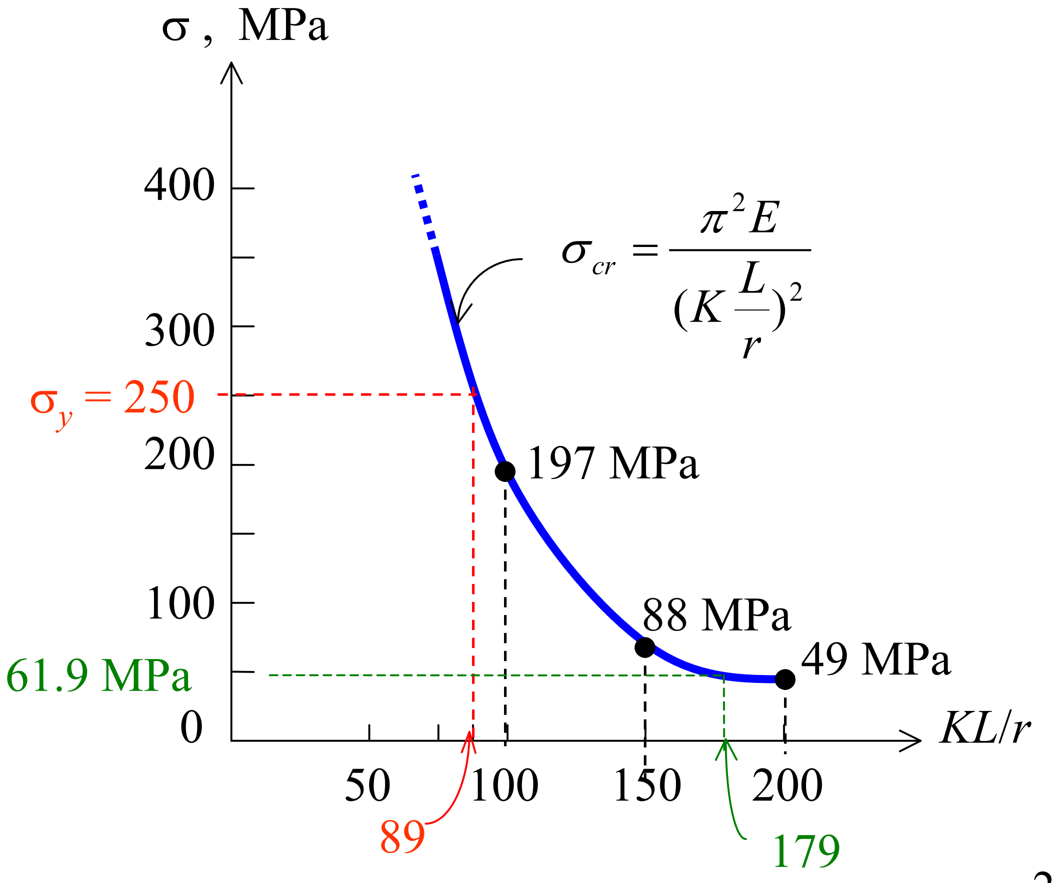
$E = 200$ GPa

$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2} = \frac{\pi^2 (200\times10^3 MPa)}{(\frac{KL}{r})^2}$$

KL/r	σ_{cr} (MPa)
89	250
100	197
125	126
150	88
175	64
200	49
225	39

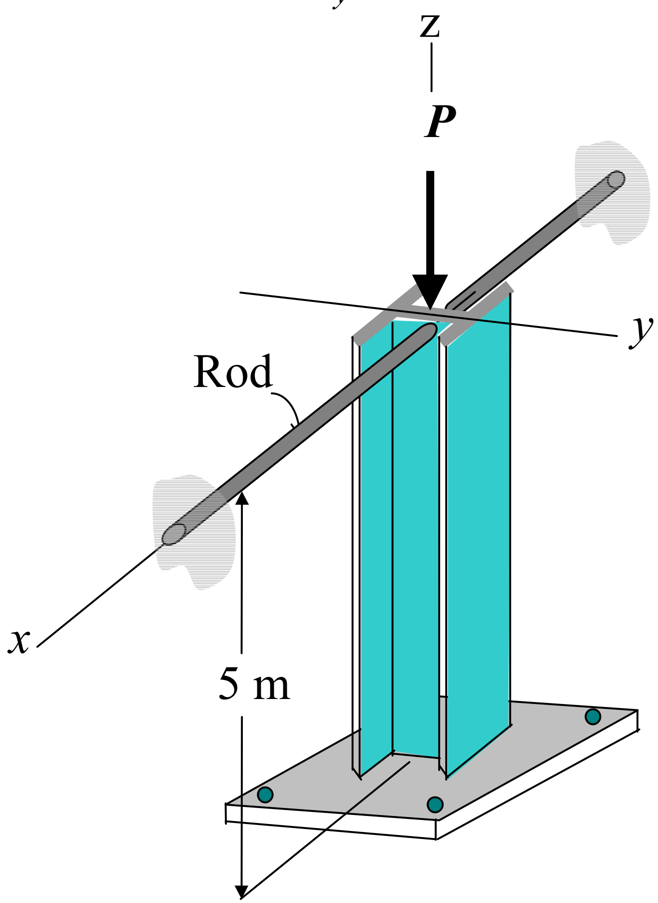
$$(\frac{KL}{r})_x = \frac{(0.5)(8\times10^3)}{66.2} = 60.42$$

$$(\frac{KL}{r})_y = \frac{(0.7)(5\times10^3)}{24.5} = 178.6$$ <--- buckling occurs

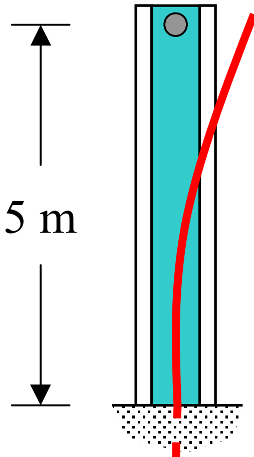


Example 4

The aluminum column is fixed at its bottom and is braced at its top by two rods so as to prevent movement at the top along the x axis, If it is assumed to be fixed at its base, determine the largest allowable load P that can be applied. Use a factor of safety for buckling of $F.S.$ = 3.0. Take E_{al} = 70 GPa, σ_y = 215 MPa, A = 7.5(10^{-3}) m^2, I_x = 61.3(10^{-6}) m^4, I_y = 23.2(10^{-6}) m^4.



E_{al} = 70 GPa, σ_y = 215 MPa, $A = 7.5(10^{-3})$ m^2



$I_x = 61.3(10^{-6})$ m^4

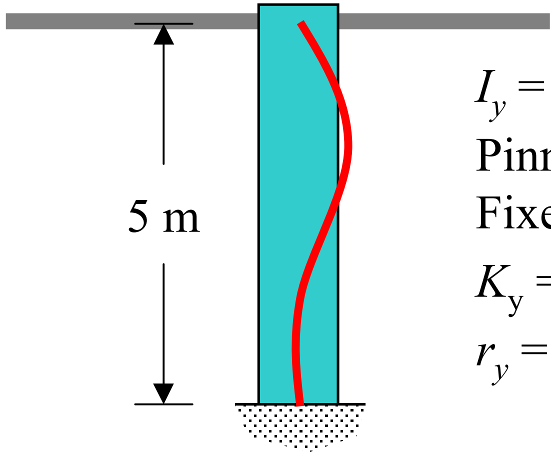
Free (top)

Fixed (bottom)

$K_x = 2$

r_x = 90 mm

x-x axis buckling



$I_y = 23.2(10^{-6})$ m^4

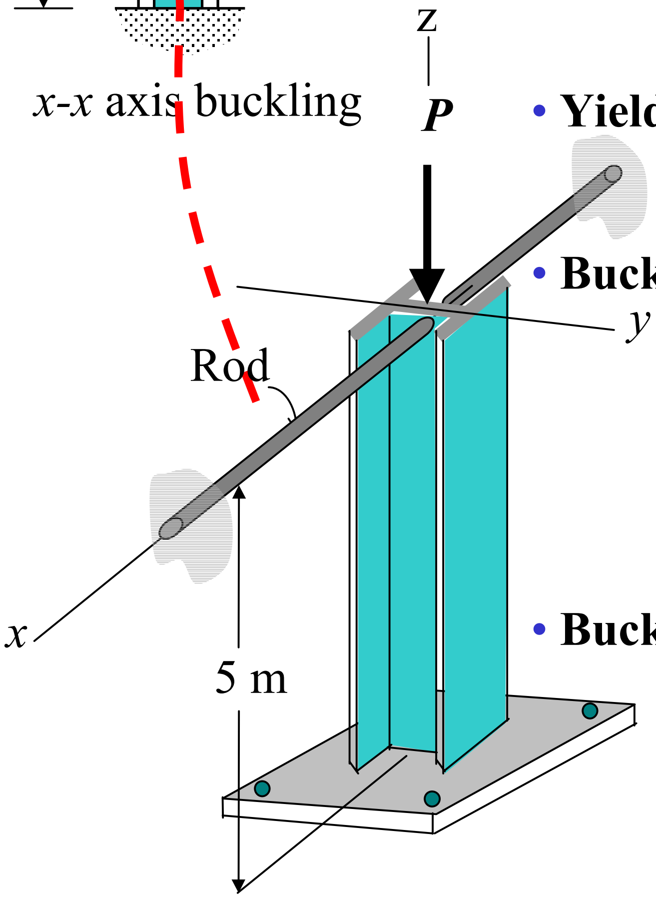
Pinned (top)

Fixed (bottom)

$K_y = 0.7$

r_y = 50 mm

y-y axis buckling



- **Yield Stress (σ_y)**

$$P_Y = \sigma_y A = (215\times10^6\,Pa)(7.5\times10^{-3}\,m^2) = 1612\;kN$$

- **Bucking *x-x* axis**

$$(P_{cr})_x = \frac{\pi^2 EI_x}{(KL)_x^2} = \frac{\pi^2(70\times10^6\,kPa)(61.3\times10^{-6}\,m^4)}{(2\times5\,m)^2} = 425\;kN$$

$$P_{allow} = \frac{P_{cr}}{F.S} = \frac{425\;kN}{3} = 141\;kN$$

- **Bucking *y-y* axis**

$$(P_{cr})_y = \frac{\pi^2 EI_y}{(KL)_y^2} = \frac{\pi^2(70\times10^6\,kPa)(23.2\times10^{-6}\,m^4)}{(0.7\times5\,m)^2} = 1308\;kN$$

$$P_{allow} = \frac{P_{cr}}{F.S} = \frac{1308\;kN}{3} = 436\;kN$$

NOTE

Aluminum

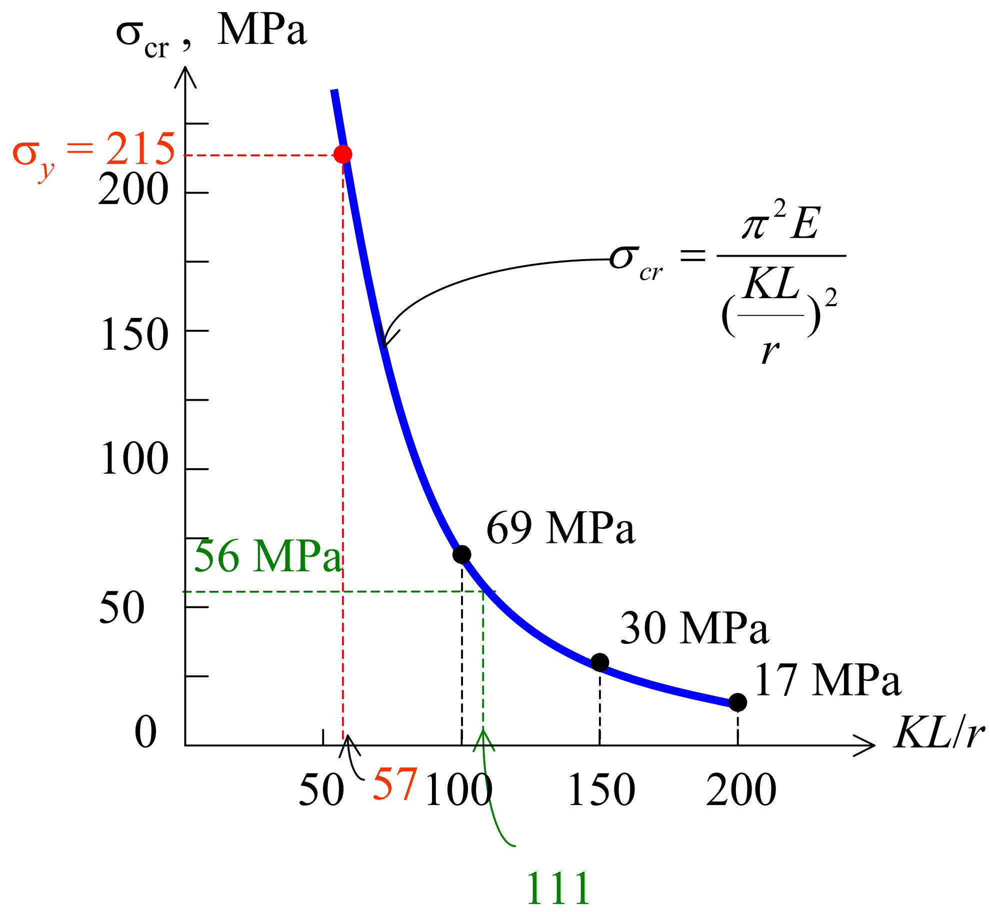
E **= 70 Gpa** $\quad$ σ_y **= 215 MPa**

$$\sigma_{cr} = \frac{\pi^2 E}{(\frac{KL}{r})^2} = \frac{\pi^2 (70\times10^3\,MPa)}{(\frac{KL}{r})^2}$$

KL/r	σ_{cr} (MPa)
57	215
75	122.8
100	69.1
125	44.2
150	30.7
175	22.6
200	17.3

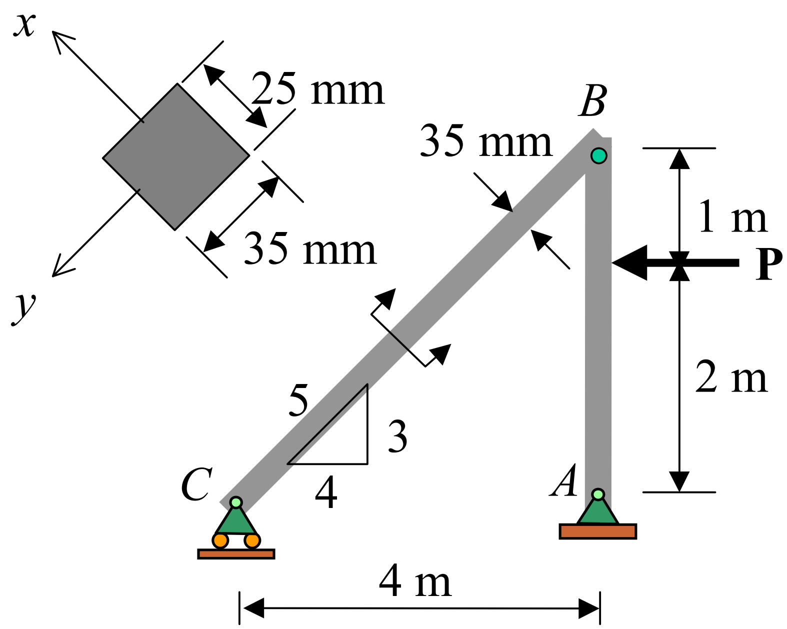
$$(\frac{KL}{r})_y = \frac{0.7\times5\times10^3\ mm}{50\ mm} = 70$$

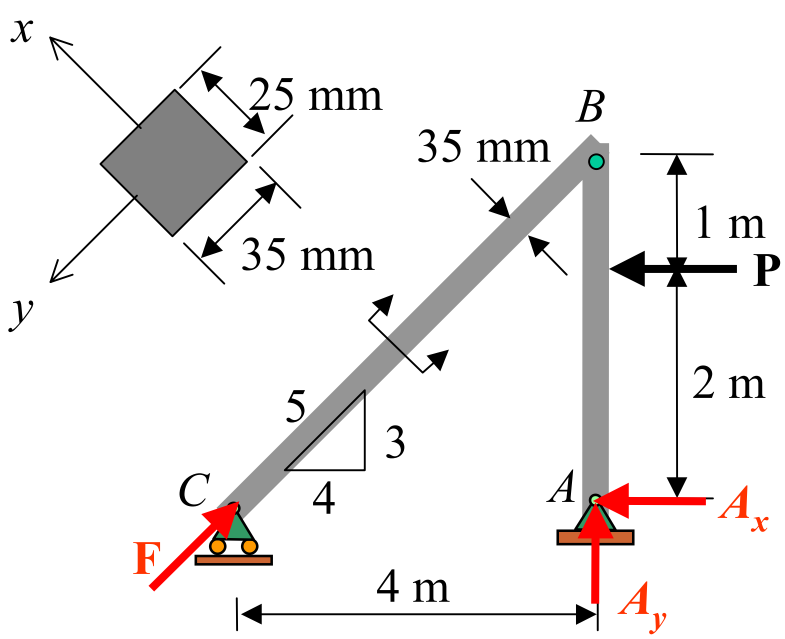
$$(\frac{KL}{r})_x = \frac{2\times5\times10^3\ mm}{90\ mm} = 111.1$$ <---occur buckling



Example 5

Determine the maximum load **_P_** the column can support before it either begins to buckle or the steel yields. Assume that member *BC* is pinned at its end for the *x*-*x* axis and fixed for *y*-**y** axis buckling. Take E = 200 GPa, σ_y = 250 MPa.





$$\circlearrowright^{+} \Sigma M_A = 0: \quad -(\frac{3}{5}F)(4) + P(2) = 0$$

$$F = \frac{5}{6}P \quad ---*$$

$$F_Y = \frac{5}{6}P_Y = \sigma_Y A = (250\times10^3)(.025\times.035) = 218.8\,kN$$

$$P_Y = 262.5\ kN$$

- $$(\frac{KL}{r})_x = \frac{1(5)}{[\frac{1}{12}\frac{(0.025)(0.035)^3}{(0.025)(0.035)}]^{1/2}} = 495$$

- $$(\frac{KL}{r})_y = \frac{0.5(5)}{[\frac{1}{12}\frac{(0.035)(0.025)^3}{(0.035)(0.025)}]^{1/2}} = 346$$

- $$F = \sigma_{cr}A = \frac{\pi^2 E}{(\frac{KL}{r})^2}A$$

$$\frac{5}{6}P = \frac{\pi^2(200\times10^9)}{(495)^2}(0.025)(0.035)$$

$$P = 8.46\text{ kN} < 262.5\text{ kN}$$