MER311: Advanced Strength of Materials

Energy Methods

- Work
- Strain Energy
- Castigliano's First Theorem
- Castigliano's Second Theorem

Work

The work performed by a load P over a distance δ Is the area under the P v δ curve

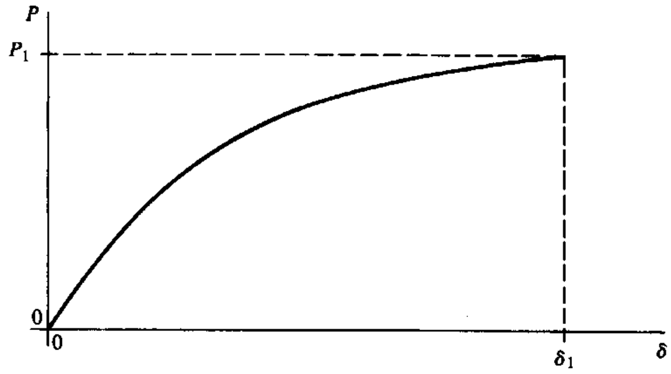
$$W = \int_0^{\delta_1} P \cdot d\delta$$

If P v δ curve is linear

$$W = \tfrac{1}{2} \cdot k \cdot \delta_1^2 = \tfrac{1}{2} \cdot P_1 \cdot \delta_1$$

If P is constant while δ changes

$$W = P_1 \cdot \delta_1$$



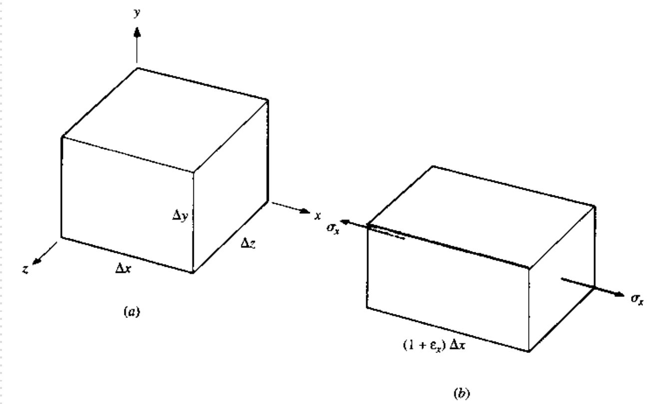
(b)

Strain Energy
Uniaxial Case

If P v δ curve is linear the Total Work on the element

$$W = \int_0^{\delta_1} F_x \cdot d\delta_x$$

$$= \tfrac{1}{2} \cdot F_x \cdot \delta_x = \tfrac{1}{2} \cdot \sigma_x \cdot \varepsilon_x \cdot \left(\Delta x \cdot \Delta y \cdot \Delta z\right)$$



The Work Per Unit Volume

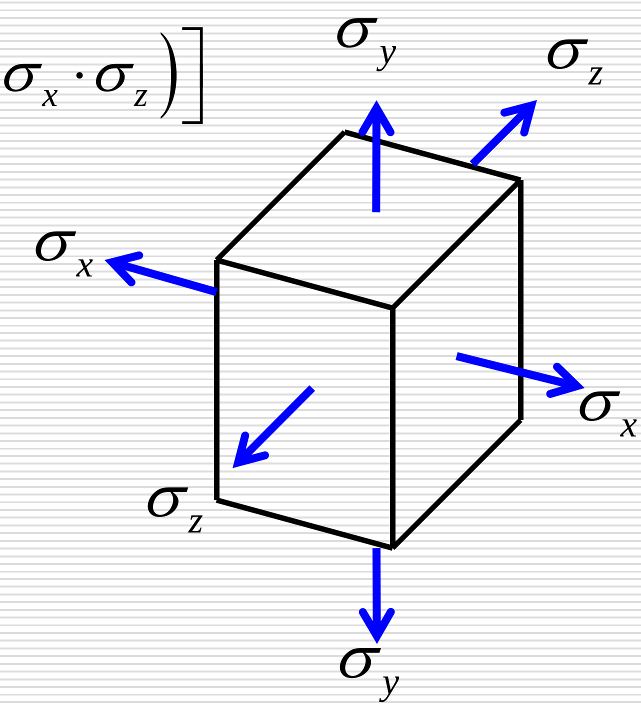
$$w = \tfrac{1}{2} \cdot \sigma_x \cdot \varepsilon_x$$

If the response is perfectly elastic – no energy losses – the Total Work W or Work per unit Volume w increases the Potential Energy of the element U – called the Strain Energy or Strain Energy per unit Volume u.

Strain Energy
Additional Normal Stress

$$u = \frac{\sigma_x \cdot \varepsilon_x}{2} + \frac{\sigma_y \cdot \varepsilon_y}{2} + \frac{\sigma_z \cdot \varepsilon_z}{2}$$

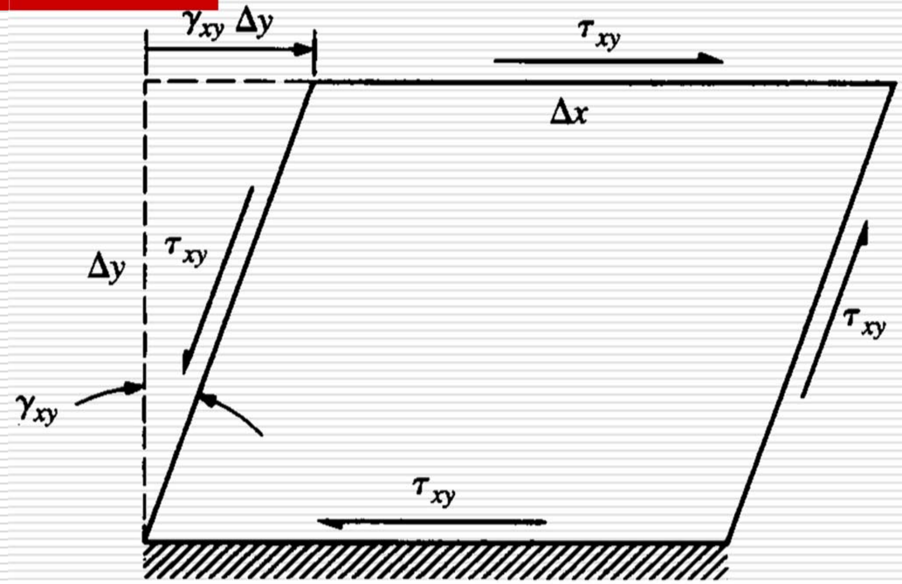
$$= \frac{1}{2 \cdot E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2 \cdot \nu\left(\sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_x \cdot \sigma_z\right)\right]$$



Strain Energy
Shear Stress

$$u = \frac{1}{2} \cdot \gamma_{xy} \cdot \tau_{xy}$$

$$= \frac{1+\nu}{E} \cdot \tau_{xy}^2$$



General Case

$$u = \frac{1}{2 \cdot E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2 \cdot \nu\left(\sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_x \cdot \sigma_z\right) + 2(1+\nu) \cdot \left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right]$$

$$= \frac{1}{2 \cdot E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2 \cdot \nu\left(\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3 + \sigma_1 \cdot \sigma_3\right)\right]$$

General Load Deflection Curve Castigliano's First Theorem

Work done by moving Q

$$W = \int Q \cdot d\Delta$$

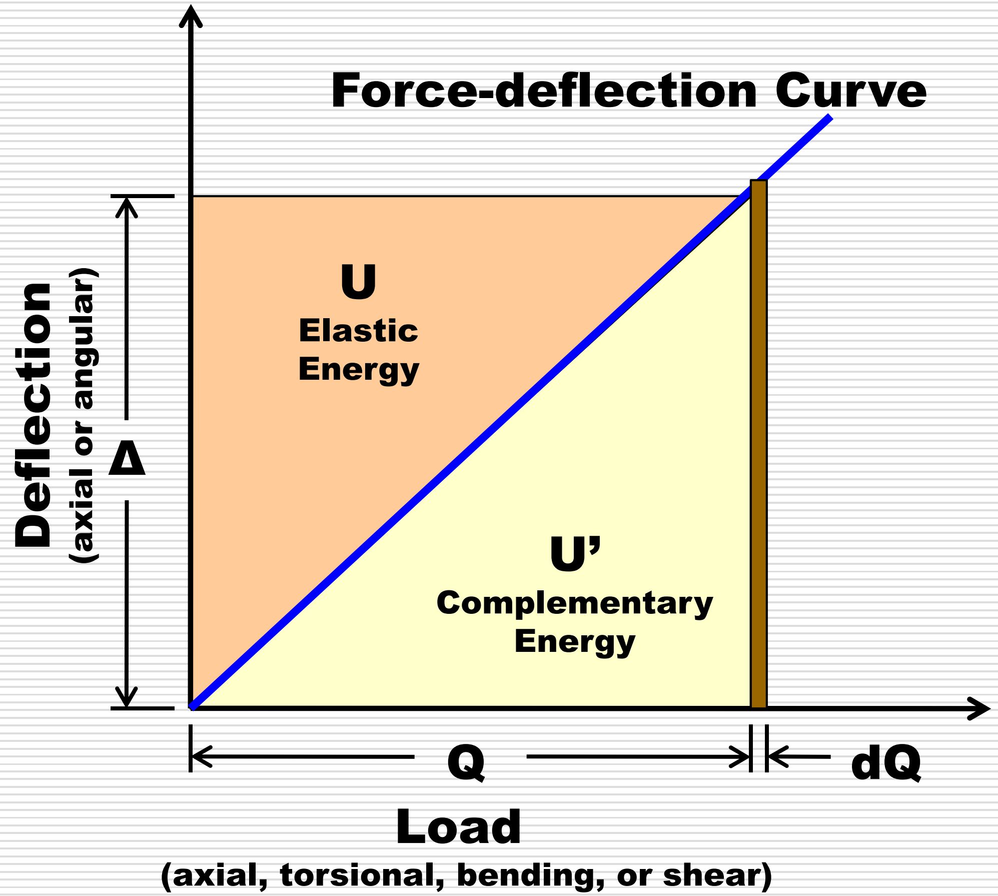
For a linear system, the stored energy is equal to deflection times average force

$$U' = U = \frac{Q \cdot \Delta}{2}$$

Additional energy associated with incremental load dQ

$$dU' = dU = \Delta \cdot dQ$$

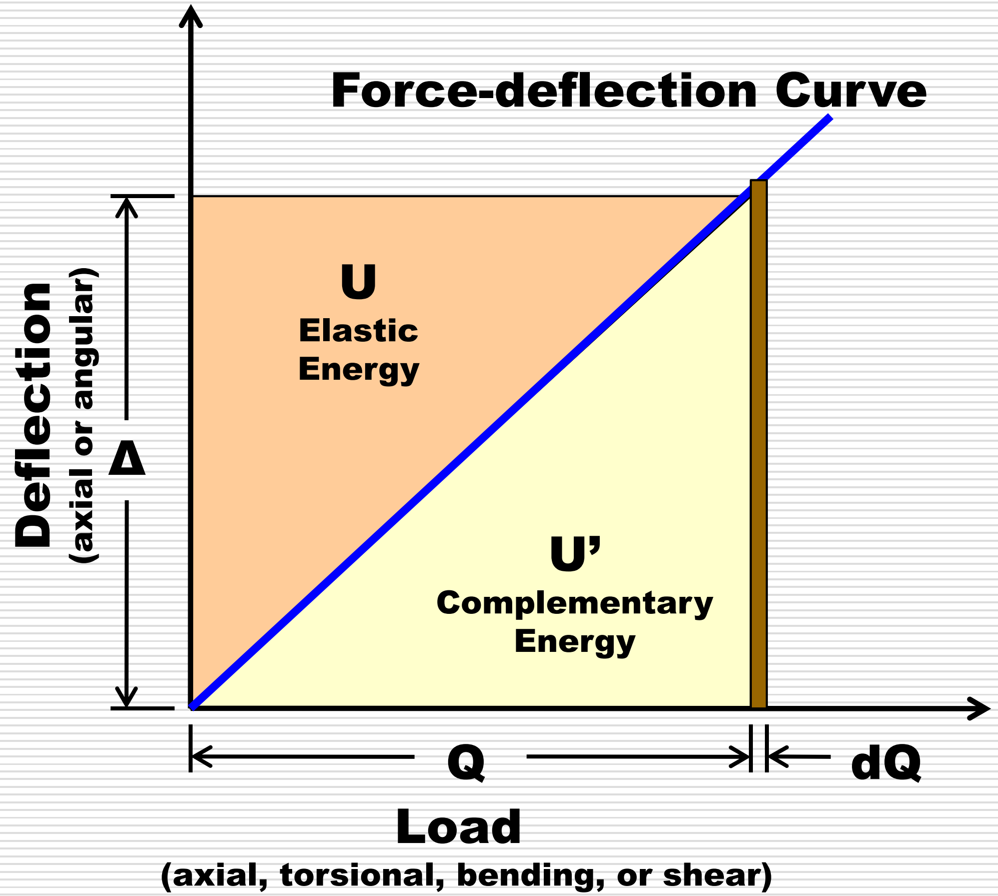
$$\frac{dU}{dQ} = \frac{\Delta \cdot dQ}{dQ} = \Delta$$



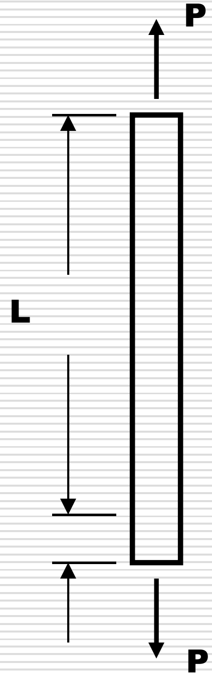
Castigliano's Second Theorem

When a body is elastically deflected by any system of loads, the deflection at any point *q* and in any direction *a* is equal to a partial derivative of strain energy (with the system of loads acting) with respect to a load at *q* acting in direction *a*.

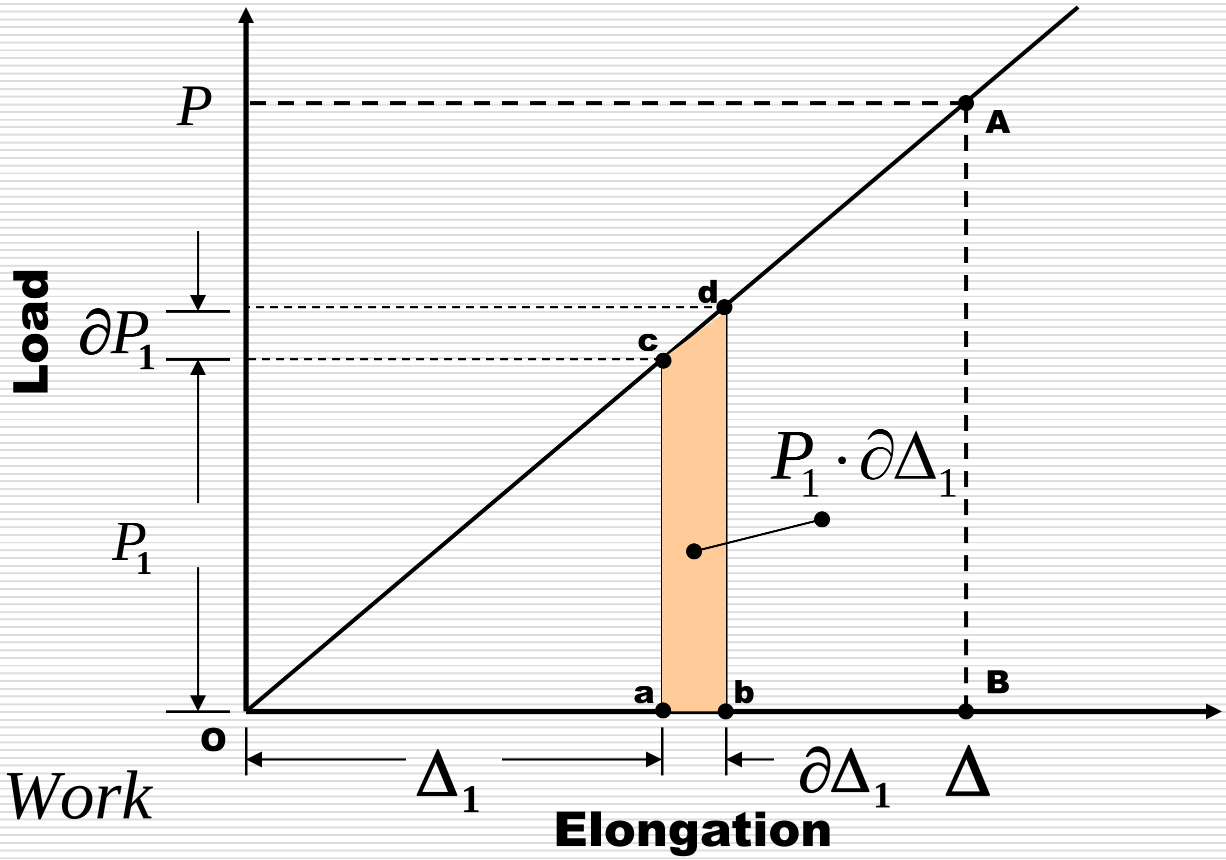
$$\Delta = \frac{dU}{dQ}$$



Elastic Strain Energy in Tension



Δ



$$U = \frac{P \cdot \Delta}{2} \equiv Total\ Work$$

Gradual Loading of a Bar

- From Basic Strength of Materials

$$\Delta = \frac{P \cdot L}{A \cdot E} \quad \Rightarrow \quad P = \frac{\Delta \cdot A \cdot E}{L}$$

- Work During Loading (Strain Energy)

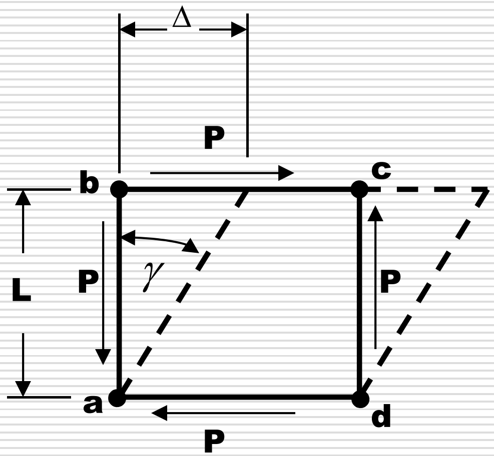
$$U = \frac{P \cdot \Delta}{2} = \frac{P^2 \cdot L}{2 \cdot A \cdot E} = \frac{A \cdot E \cdot \Delta^2}{2 \cdot L}$$

- Strain Energy per Unit Volume

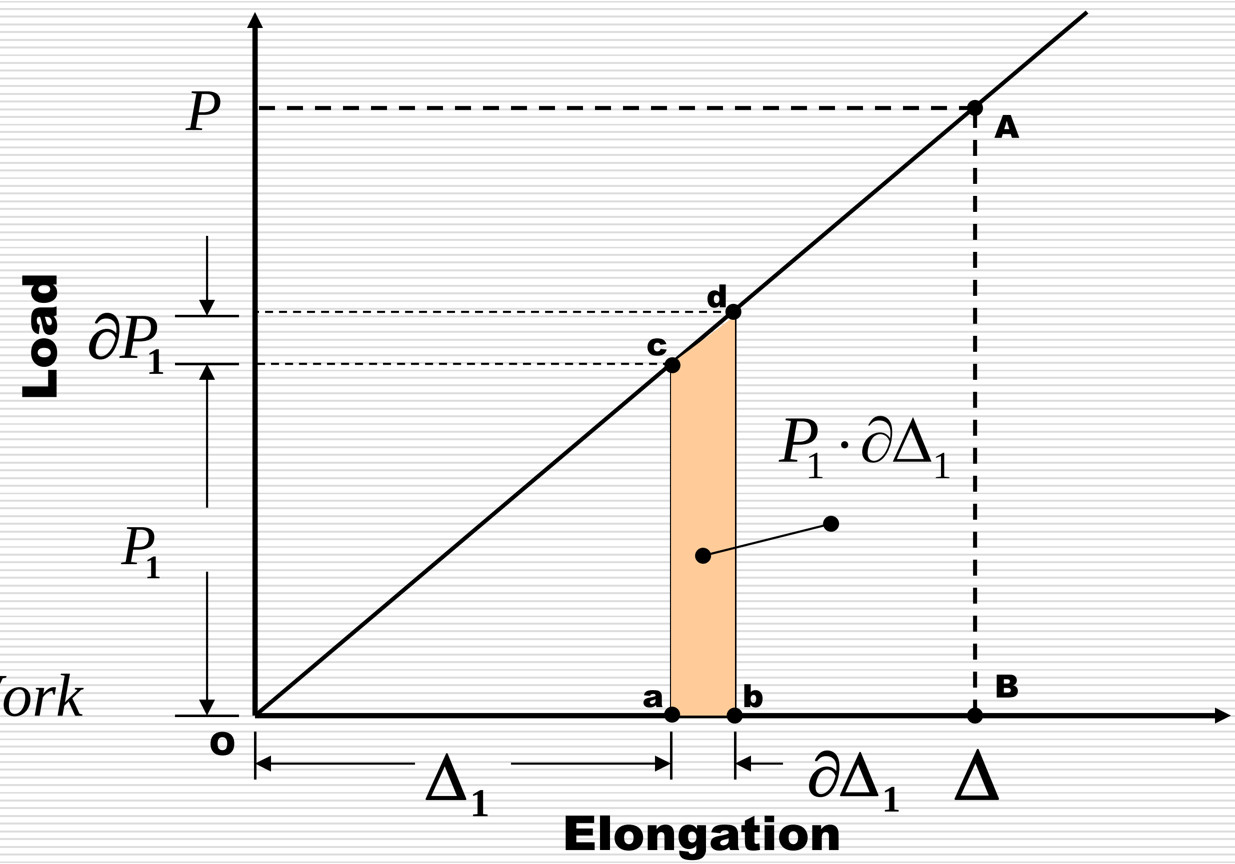
$$w = \frac{U}{A \cdot L} = \frac{\sigma^2}{2 \cdot E} = \frac{E \cdot \varepsilon^2}{2}$$

$$\sigma = {}^{P}\!/_{A} \quad \varepsilon = {}^{\Delta}\!/_{L}$$

Elastic Strain Energy in Shear



$$U = \frac{P \cdot \Delta}{2} \equiv Total\ Work$$



Pure Shear

- **From Basic Strength of Materials**

$$\frac{\delta}{L} = \gamma = \frac{\tau}{G} = \frac{P}{A \cdot G}$$

- **Work During Loading (Strain Energy)**

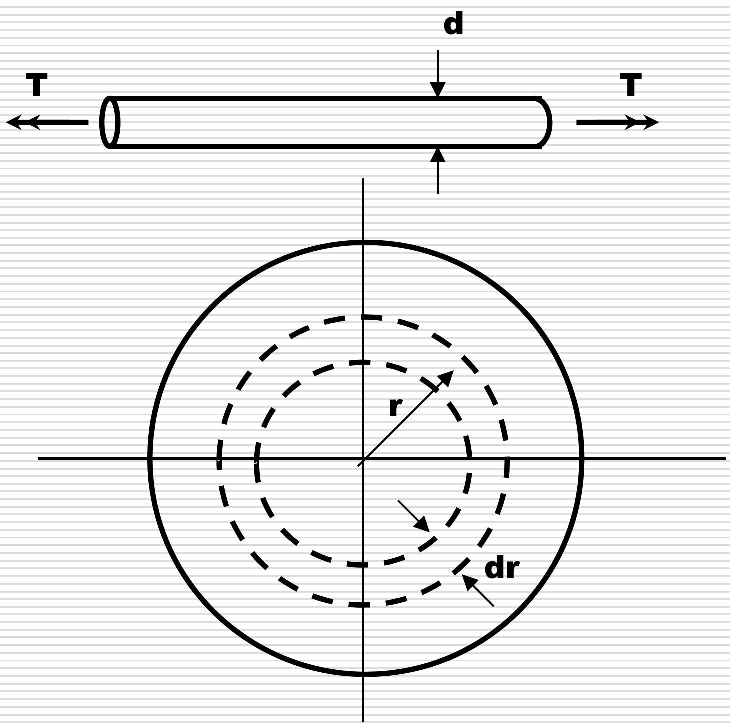
$$U = \frac{P \cdot \Delta}{2} = \frac{P^2 \cdot L}{2 \cdot A \cdot G} = \frac{A \cdot G \cdot \Delta^2}{2 \cdot L}$$

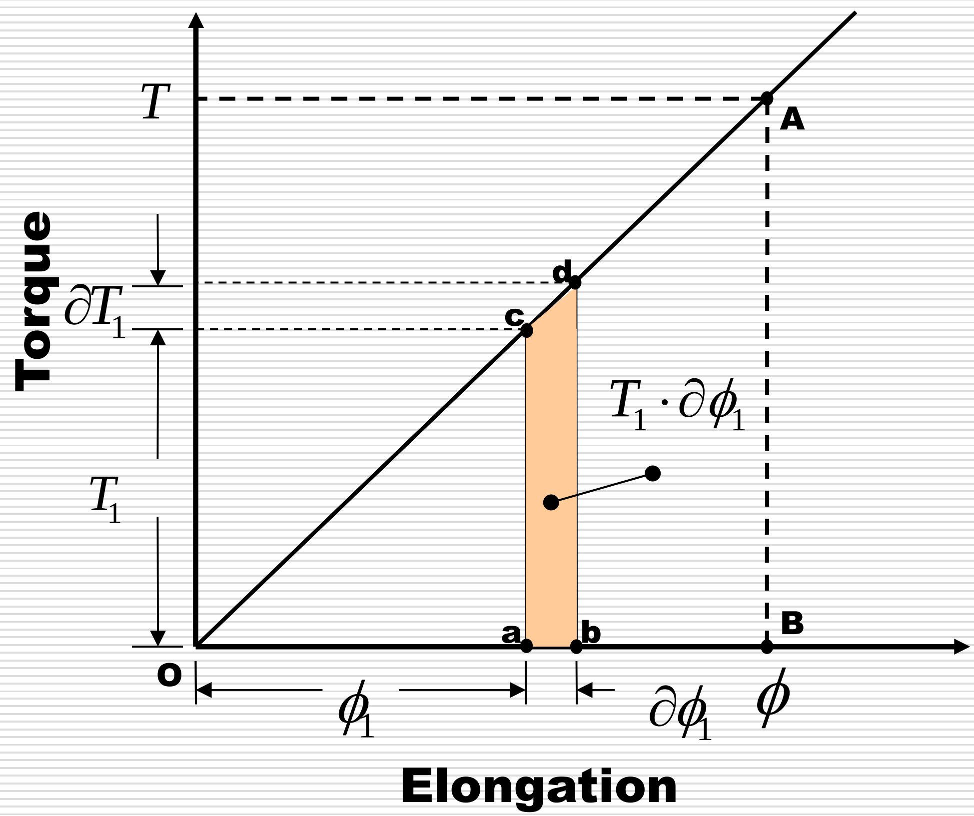
- **Strain Energy per Unit Volume**

$$w = \frac{U}{A \cdot L} = \frac{\tau^2}{2 \cdot G} = \frac{G \cdot \gamma^2}{2}$$

$$\tau = {}^{P}\!/_{A} \quad \gamma = {}^{\Delta}\!/_{L}$$

Elastic Strain Energy in Torsion





$$U = \frac{P \cdot \Delta}{2} \equiv Total\ Work$$

Torsion

- From Basic Strength of Materials

$$\phi = \frac{T \cdot L}{J \cdot G} \quad \tau = \frac{T \cdot r}{J} = \tau_{\max} \cdot \left(\frac{2 \cdot r}{d} \right)$$

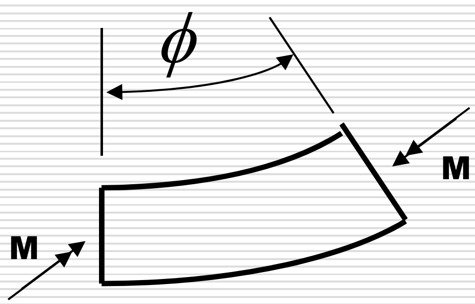
- Work During Loading (Strain Energy)

$$U = \tfrac{1}{2} \cdot \frac{\pi \cdot d^2 \cdot L}{4} \cdot \frac{\tau_{\max}^2}{2 \cdot G}$$

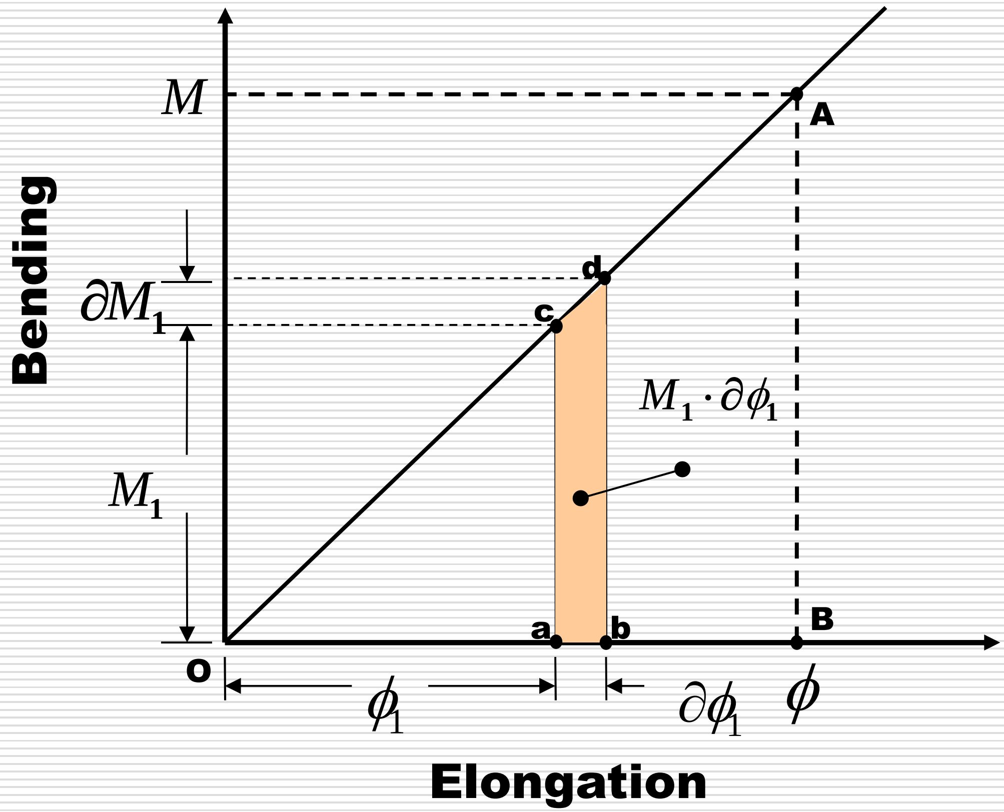
- Strain Energy per Unit Volume

$$w = \frac{\tau^2}{2 \cdot G} = \frac{2 \cdot \tau_{\max}^2 \cdot r^2}{G \cdot d^2}$$

Elastic Strain Energy in Bending



$$U = \frac{P \cdot \Delta}{2} \equiv Total\ Work$$



Bending

- From Basic Strength of Materials

$$\phi = \frac{M \cdot L}{I \cdot E} \quad \tau = \frac{M \cdot c}{I}$$

- Work During Loading (Strain Energy)

$$U = \frac{M \cdot \phi}{2} = \frac{M^2 \cdot L}{2 \cdot E \cdot I} = \frac{\phi^2 \cdot E \cdot I}{2 \cdot L}$$

- Strain Energy per Unit Volume

$$w = \frac{\tau^2}{2 \cdot G} = \frac{2 \cdot \tau_{\max}^2 \cdot r^2}{G \cdot d^2}$$

Summary of Energy Equations

Load Type	Factors Involved	General Equations	For Const Factors
Axial	P,E,A	$U = \int_0^L \frac{P^2}{2 \cdot E \cdot A} \cdot dx$	$U = \frac{P^2 \cdot L}{2 \cdot A \cdot E}$
Bending	M,E,I	$U = \int_0^L \frac{M^2}{2 \cdot E \cdot A} \cdot dx$	$U = \frac{M^2 \cdot L}{2 \cdot A \cdot E}$
Torsion	T,G,k	$U = \int_0^L \frac{T^2}{2 \cdot G \cdot k} \cdot dx$	$U = \frac{T^2 \cdot L}{2 \cdot G \cdot k}$
Tran. Shear	V,G,A	$U = \int_0^L \frac{3 \cdot V^2}{5 \cdot G \cdot A} \cdot dx$	$U = \frac{3 \cdot V^2 \cdot L}{5 \cdot A \cdot G}$

Summary of Deflection Equations

Load Type	Factors Involved	General Equations	For Const Factors
Axial	P,E,A	$\Delta = \int_0^L \frac{P \cdot \left(\partial P / \partial Q \right)}{E \cdot A} \cdot dx$	$\Delta = \frac{P \cdot L}{A \cdot E}$
Bending	M,E,I	$\Delta = \int_0^L \frac{M \cdot \left(\partial M / \partial Q \right)}{E \cdot A} \cdot dx$	$\Delta = \frac{M \cdot L}{A \cdot E}$
Torsion	T,G,k	$\Delta = \int_0^L \frac{T \cdot \left(\partial T / \partial Q \right)}{G \cdot k} \cdot dx$	$\Delta = \frac{T \cdot L}{G \cdot k}$
Tran. Shear	V,G,A	$\Delta = \int_0^L \frac{6 \cdot V \cdot \left(\partial V / \partial Q \right)}{5 \cdot G \cdot A} \cdot dx$	$\Delta = \frac{6 \cdot V \cdot L}{5 \cdot A \cdot G}$