PROBLEM 1 a. GIVEN THE EXPRESSIONS FOR STRAIN IN TERMS OF STRESS

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \quad (1)$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \quad (2)$$

$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (3)$$

DERIVE THE EXPRESSIONS FOR STRESS IN TERMS OF STRAIN.

b. DERIVE THE EXPRESSIONS FOR THE STRAIN IN TERMS OF STRESS FOR THE CASE OF PLANE STRESS AND PLANE STRAIN.

c. DERIVE THE EXPRESSION FOR THE STRESS IN TERMS OF STRAIN FOR THE CASE OF PLANE STRESS AND PLANE STRAIN

GIVEN:
CONSTRAINTS
1. $\sigma_x, \sigma_y, \sigma_z, E, \nu$
ASSUMPTIONS
1. THE MATERIAL IS LINEAR · ELASTIC

FIND:
1. $\sigma_x(\epsilon_x, \epsilon_y, \epsilon_z)$
2. $\sigma_y(\epsilon_x, \epsilon_y, \epsilon_z)$
3. $\sigma_z(\epsilon_x, \epsilon_y, \epsilon_z)$

SOLUTION:

$$(1) \rightarrow E\cdot\epsilon_x = \sigma_x - \nu(\sigma_y + \sigma_z) \Rightarrow \sigma_x = E\cdot\epsilon_x + \nu(\sigma_y + \sigma_z) \quad (4)$$

$$(2) \Rightarrow E\cdot\epsilon_y = \sigma_y - \nu(\sigma_x + \sigma_z) \Rightarrow \sigma_y = E\cdot\epsilon_y + \nu(\sigma_x + \sigma_z) \quad (5)$$

$$(3) \Rightarrow E\cdot\epsilon_z = \sigma_z - \nu(\sigma_x + \sigma_y) \Rightarrow \sigma_z = E\cdot\epsilon_z + \nu(\sigma_x + \sigma_y) \quad (6)$$

$$(6)(5) \rightarrow (4) \Rightarrow \sigma_x = E\cdot\epsilon_x + \nu\left[(E\cdot\epsilon_y + \nu\cdot\sigma_x + \nu\sigma_z) + (E\cdot\epsilon_z + \nu\cdot\sigma_x + \nu\sigma_y)\right]$$

$$= E\epsilon_x + \nu\left[E\cdot\epsilon_y + E\cdot\epsilon_z + 2\cdot\nu\cdot\sigma_x + \nu(\sigma_y + \sigma_z)\right]$$

$$\sigma_x\cdot(1 - 2\cdot\nu^2) = E\cdot\epsilon_x + \nu\left[E\cdot\epsilon_y + E\cdot\epsilon_z + \nu(\sigma_y + \sigma_z)\right] \quad (7)$$

FROM (4)

$$\nu(\sigma_y + \sigma_z) = \sigma_x - E\cdot\epsilon_x \quad (8)$$

SUBSTITUTING (8) INTO (7)

$$\sigma_x \cdot (1-2\cdot\nu^2) = E\cdot\epsilon_x + \nu[E\cdot\epsilon_y + E\cdot\epsilon_z + \sigma_x - E\cdot\epsilon_x]$$

$$\sigma_x \cdot (1-2\nu^2-\nu) = E\cdot\epsilon_x - E\cdot\nu\cdot\epsilon_x + E\cdot\nu(\epsilon_y+\epsilon_z)$$

$$\frac{(1-\nu-2\nu^2)}{E}\sigma_x = (1-\nu)\epsilon_x + \nu(\epsilon_y+\epsilon_z)$$

$$\underline{\underline{\sigma_x = \frac{E}{(1-\nu-2\nu^2)}\cdot\left[(1-\nu)\cdot\epsilon_x + \nu\cdot(\epsilon_y+\epsilon_z)\right]}} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\cdot\epsilon_x + \nu\cdot(\epsilon_y+\epsilon_z)\right]$$

TO FIND $\sigma_y(\epsilon_x, \epsilon_y, \epsilon_z)$, (4) AND (6) ARE SUBSTITUTED INTO (5)

$$\sigma_y = E\cdot\epsilon_y + \nu[E\cdot\epsilon_x + \nu\sigma_y + \nu\sigma_z + E\cdot\epsilon_z + \nu\cdot\sigma_x + \nu\cdot\sigma_y]$$

$$= E\cdot\epsilon_y + \nu[E\cdot\epsilon_x + E\cdot\epsilon_z + 2\cdot\nu\sigma_y + \nu\cdot\sigma_z + \nu\cdot\sigma_x]$$

$$\sigma_y(1-2\cdot\nu^2) = E\cdot\epsilon_y + \nu[E\cdot\epsilon_x + E\cdot\epsilon_z + \nu(\sigma_z+\sigma_x)] \qquad (9)$$

FROM (5)

$$\nu(\sigma_x+\sigma_z) = \sigma_y - E\cdot\epsilon_y \qquad (10)$$

SUBSTITUTING (10) INTO (9)

$$\sigma_y(1-2\cdot\nu^2) = E\cdot\epsilon_y + \nu[E\cdot\epsilon_x + E\cdot\epsilon_z + \sigma_y - E\cdot\epsilon_y]$$

$$\frac{\sigma_y(1-\nu-2\nu^2)}{E} = (1-\nu)\epsilon_y + \nu(\epsilon_x+\epsilon_z)$$

$$\underline{\underline{\sigma_y = \frac{E}{(1-\nu-2\nu^2)}\cdot\left[(1-\nu)\cdot\epsilon_y + \nu\cdot(\epsilon_x+\epsilon_z)\right]}} = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_x + \nu(\epsilon_y+\epsilon_z)\right]$$

TO FIND $\sigma_z(\epsilon_x, \epsilon_y, \epsilon_z)$, (4) AND (5) ARE SUBSTITUTED INTO (6)

$$\sigma_z = E\cdot\epsilon_z + \nu(E\cdot\epsilon_x + \nu\cdot\sigma_y + \nu\cdot\sigma_z + E\cdot\epsilon_y + \nu\cdot\sigma_x + \nu\sigma_z)$$

$$= E\cdot\epsilon_z + \nu(E\cdot\epsilon_x + E\cdot\epsilon_y + 2\nu\sigma_z + \nu\sigma_y + \nu\sigma_x)$$

$$\sigma_z(1-2\cdot\nu^2) = E\cdot\epsilon_z + \nu(E\cdot\epsilon_x + E\cdot\epsilon_y + \nu(\sigma_y+\sigma_x)) \qquad (11)$$

FROM (6)

$$\nu(\sigma_x+\sigma_y) = \sigma_z - E\cdot\epsilon_z \qquad (12)$$

SUBSTITUTING (12) INTO (11)

$$\sigma_z(1-2\cdot\nu^2) = E\cdot\epsilon_z + \nu\left(E\cdot\epsilon_x + E\cdot\epsilon_y + \sigma_z - E\cdot\epsilon_z\right)$$

$$\sigma_z(1-\nu-2\cdot\nu^2) = E\cdot\epsilon_z(1-\nu) + E\cdot\nu(\epsilon_x+\epsilon_y)$$

$$\sigma_z = \frac{E}{(1-\nu-2\nu^2)}\left[\epsilon_z(1-\nu)+\nu(\epsilon_x+\epsilon_y)\right] = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_z+\nu(\epsilon_x+\epsilon_y)\right]$$

IN SUMMARY

$$\boxed{\begin{array}{ll} \epsilon_x = \frac{1}{E}\left[\sigma_x-\nu(\sigma_y+\sigma_z)\right] \ (1) & \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_x+\nu\cdot(\epsilon_y+\epsilon_z)\right] \\ \epsilon_y = \frac{1}{E}\left[\sigma_y-\nu(\sigma_x+\sigma_z)\right] \ (2) & \sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_y+\nu\cdot(\epsilon_x+\epsilon_z)\right] \\ \epsilon_z = \frac{1}{E}\left[\sigma_z-\nu(\sigma_x+\sigma_y)\right] \ (3) & \sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_z+\nu(\epsilon_x+\epsilon_y)\right] \end{array}}$$

(13) (14) (15)

THE CASE OF "PLANE STRESS" THE THIRD DIMENSION IS VERY SMALL (OR THIN). AS A RESULT IT IS ASSUMED THAT (GIVEN "Z" IS THE THIRD DIMENSION)

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \qquad (16)$$

SUBSTITUTING (16) INTO (1)-(3)

$$(1)\rightarrow \boxed{\begin{array}{l} \epsilon_x = \frac{1}{E}\left[\sigma_x-\nu\cdot\sigma_y\right] \quad (17) \\ \epsilon_y = \frac{1}{E}\left[\sigma_y-\nu\cdot\sigma_x\right] \quad (18) \end{array}}$$

PLANE STRESS

SINCE THE "Z" DIMENSION IS SMALL TO BEGIN WITH, CHANGES IN THE "Z" DIMENSION WILL BE EXTREMELY SMALL AND ARE ASSUMED ZERO. THERE ARE MANY ASSUMPTIONS THAT ARE ASSOCIATED WITH PLANE STRESS THE RESULT IN (17) AND (18) BEING APPROXIMATIONS. WHAT THIS IS IMPORTANT IS TO FIND AN EXPRESSION FOR σ_x AND σ_y IN TERMS OF ϵ_x AND ϵ_y, (17) AND (18) NEED TO BE SOLVED SIMULTANEOUSLY. STARTING FROM (13)-(15) REQUIRES AN ASSORTMENT OF ASSUMPTIONS

STARTING WITH (17)

$$E \cdot \epsilon_x = \sigma_x - \nu \cdot \sigma_y \tag{19}$$

FROM (18)

$$E \cdot \epsilon_y = \sigma_y - \nu \cdot \sigma_x$$

$$\sigma_y = E \cdot \epsilon_y + \nu \cdot \sigma_x \tag{20}$$

SUBSTITUTING (20) INTO (19)

$$E \cdot \epsilon_x = \sigma_x - \nu(E \cdot \epsilon_y + \nu \cdot \sigma_x) = \sigma_x - \nu \cdot E \cdot \epsilon_y - \nu^2 \cdot \sigma_x$$

$$E \cdot \epsilon_x + \nu \cdot E \cdot \epsilon_y = \sigma_x (1 - \nu^2) \Rightarrow \underline{\underline{\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu \epsilon_y)}} \tag{21}$$

NOW STARTING WITH (18)

$$E \cdot \epsilon_y = \sigma_y - \nu \cdot \sigma_x \tag{20}$$

FROM (17)

$$E \cdot \epsilon_x = \sigma_x - \nu \cdot \sigma_y \Rightarrow \sigma_x = E \cdot \epsilon_x + \nu \cdot \sigma_y$$

SUBSTITUTING THIS RESULT INTO THE ABOVE EQUATION

$$E \cdot \epsilon_y = \sigma_y - \nu(E \cdot \epsilon_x + \nu \cdot \sigma_y) \Rightarrow \sigma_y (1 - \nu^2) = E(\epsilon_y + \nu \cdot \epsilon_x)$$

$$\underline{\underline{\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu \cdot \epsilon_x)}}$$

IN SUMMARY

$$\boxed{\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu \cdot \epsilon_y) \quad (22) \\ \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu \cdot \epsilon_x) \quad (23) \end{aligned}}$$

PLANE STRESS

NOW LET'S CONSIDER THE CASE OF "PLANE STRAIN." FOR THE PLANE STRAIN CONDITION THE THIRD DIMENSION (Z DIRECTION HERE) OF THE STRUCTURE IS CONSIDERED TO BE VERY THICK OR CONSTRAINED FROM DEFORMATION. THIS RESULTS IN

$$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \tag{24}$$

SUBSTITUTING (24) INTO (3)

$$\cancel{\epsilon_z}^{\,0} = \frac{1}{E}\left[\sigma_z - \nu(\sigma_y + \sigma_x)\right]$$

$$0 = \sigma_z - \nu(\sigma_x + \sigma_y) \Rightarrow \sigma_z = \nu(\sigma_x + \sigma_y) \qquad (25)$$

SUBSTITUTING (25) INTO (1)

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \nu\cdot\sigma_x + \nu\sigma_y)\right] = \frac{1}{E}\left[\sigma_x - \nu\sigma_y - \nu^2\cdot\sigma_x - \nu^2\sigma_y\right]$$

$$= \frac{1}{E}\left[\sigma_x(1-\nu^2) - \nu\cdot\sigma_y(1+\nu)\right] = \frac{1}{E}\left[\sigma_x(1+\nu)(1-\nu) - \nu\cdot\sigma_y(1+\nu)\right]$$

$$= \frac{1+\nu}{E}\left[\sigma_x(1-\nu) - \nu\sigma_y\right] = \frac{1+\nu}{E}\left[\sigma_x(1-\nu) - \frac{(1-\nu)}{(1-\nu)}\cdot\nu\sigma_y\right]$$

$$= \frac{(1+\nu)(1-\nu)}{E}\left[\sigma_x - \frac{\nu}{1-\nu}\cdot\sigma_y\right] = \frac{(1-\nu^2)}{E}\left[\sigma_x - \frac{\nu}{1-\nu}\cdot\sigma_y\right] \qquad (26)$$

SUBSTITUTING (25) INTO (2)

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \nu\sigma_x + \nu\sigma_y)\right]$$

$$= \frac{1}{E}\left[\sigma_y - \nu\cdot\sigma_x - \nu^2\cdot\sigma_x - \nu^2\sigma_y\right] = \frac{1}{E}\left[\sigma_y(1-\nu^2) - \nu\cdot\sigma_x(1+\nu)\right]$$

$$= \frac{1+\nu}{E}\left[(1-\nu)\cdot\sigma_y - \nu\sigma_x\right] = \frac{1+\nu}{E}\left[(1-\nu)\cdot\sigma_y - \nu\cdot\frac{(1-\nu)}{(1-\nu)}\cdot\sigma_x\right]$$

$$= \frac{(1+\nu)(1-\nu)}{E}\left[\sigma_y - \frac{\nu}{(1-\nu)}\cdot\sigma_x\right] = \frac{(1-\nu^2)}{E}\left[\sigma_y - \frac{\nu}{(1+\nu)}\cdot\sigma_x\right] \qquad (27)$$

IN SUMMARY

$$\boxed{\begin{aligned}\epsilon_x &= \frac{1-\nu^2}{E}\cdot\left[\sigma_x - \frac{\nu}{1-\nu}\cdot\sigma_y\right]\\ \epsilon_y &= \frac{1-\nu^2}{E}\cdot\left[\sigma_y - \frac{\nu}{1-\nu}\cdot\sigma_x\right]\end{aligned}}$$

PLANE STRAIN

SUBSTITUTING (24) INTO (13)

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_x + \nu\cdot(\epsilon_y + \cancel{\epsilon_z}^{\,0})\right] = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\epsilon_x + \nu\cdot\epsilon_y\right)$$

$$\sigma_x = \frac{E\cdot(1-\nu)}{(1+\nu)(1-2\nu)}\left[\epsilon_x + \frac{\nu}{1-\nu}\epsilon_y\right] \qquad (28)$$

SUBSTITUTE (24) INTO (14)

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_y + \nu\cdot(\epsilon_x + \cancel{\epsilon_z}^{\,0})\right] = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\epsilon_y + \nu\epsilon_x\right]$$

$$= \frac{(1-\nu)\cdot E}{(1+\nu)\cdot(1-2\nu)}\cdot\left[\epsilon_y + \frac{\nu}{1-\nu}\cdot\epsilon_x\right] \qquad (29)$$

SUBSTITUTING (24) INTO (15)

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\cdot\left[(1-\nu)\cdot\cancel{\epsilon_z}^{\,0} + \nu\cdot(\epsilon_x + \epsilon_y)\right] = \frac{\nu\cdot E}{(1+\nu)\cdot(1-2\nu)}\cdot(\epsilon_x + \epsilon_y) \qquad (30)$$

IN SUMMARY

$$\boxed{\begin{aligned}
\sigma_x &= \frac{(1-\nu)}{(1+\nu)\cdot(1-2\nu)}\cdot E\cdot\left[\epsilon_x + \frac{\nu}{1-\nu}\cdot\epsilon_y\right]\\
\sigma_y &= \frac{(1-\nu)}{(1+\nu)\cdot(1-2\nu)}\cdot E\cdot\left[\epsilon_y + \frac{\nu}{1-\nu}\cdot\epsilon_x\right]\\
\sigma_z &= \frac{\nu}{(1+\nu)\cdot(1-2\nu)}\cdot E\cdot(\sigma_x + \sigma_y)
\end{aligned}}$$

PLANE STRAIN

SUMMARY:

FROM THE DEVELOPMENT IT IS CLEAR THAT ~~THAT~~ THE STATE OF PLANE STRESS IS AN APPROXIMATION AND THE STATE OF PLANE STRAIN IS AN EXACT SOLUTION. ALSO NOTE THAT EVEN THOUGH THERE IS NO STRAIN IN THE THIRD DIMENSION (Z) FOR THE CASE OF PLANE STRAIN, THERE IS STILL STRESS IN THE THIRD DIMENSION.