NAME: Solution

PROBLEM 1: A beam of length 4a is shown on the next page. Young's modulus for the beam is E and the moment of inertia for the beam is I. A vertical load of P is applied at point B and a vertical load of –P is applied at point D. A couple of magnitude 2Pa is applied at point C. Points A and E are supported by pin joints.

1a. Determine the reactions at A and E and complete the free body diagram on the next page.

1b. Draw the shear, bending moment, curvature, and displacement diagrams for this beam.

$$\Sigma F_x = 0 = A + P - P + E \Rightarrow A = -E$$

$$\Sigma M_{z/@A} = 0 = -Pa + 2Pa + 3Pa - 4a \cdot E$$

$$E = \frac{4Pa}{4a} = \underline{P}$$

$$A = \underline{-P}$$

THE DEFLECTION AT B

$$u_B = BB' = B'B'' - BB'' = \frac{u_{EA}}{4} - u_{BA}$$

$$u_{EA} = \frac{1}{EA}\left[\frac{1}{2}\cdot Pa \cdot a\frac{2a}{3} - Pa \cdot a \cdot \frac{3}{2}a + Pa \cdot a \cdot \frac{5}{2}a + \frac{1}{2}\cdot Pa \cdot a \cdot \frac{10a}{3}\right]$$

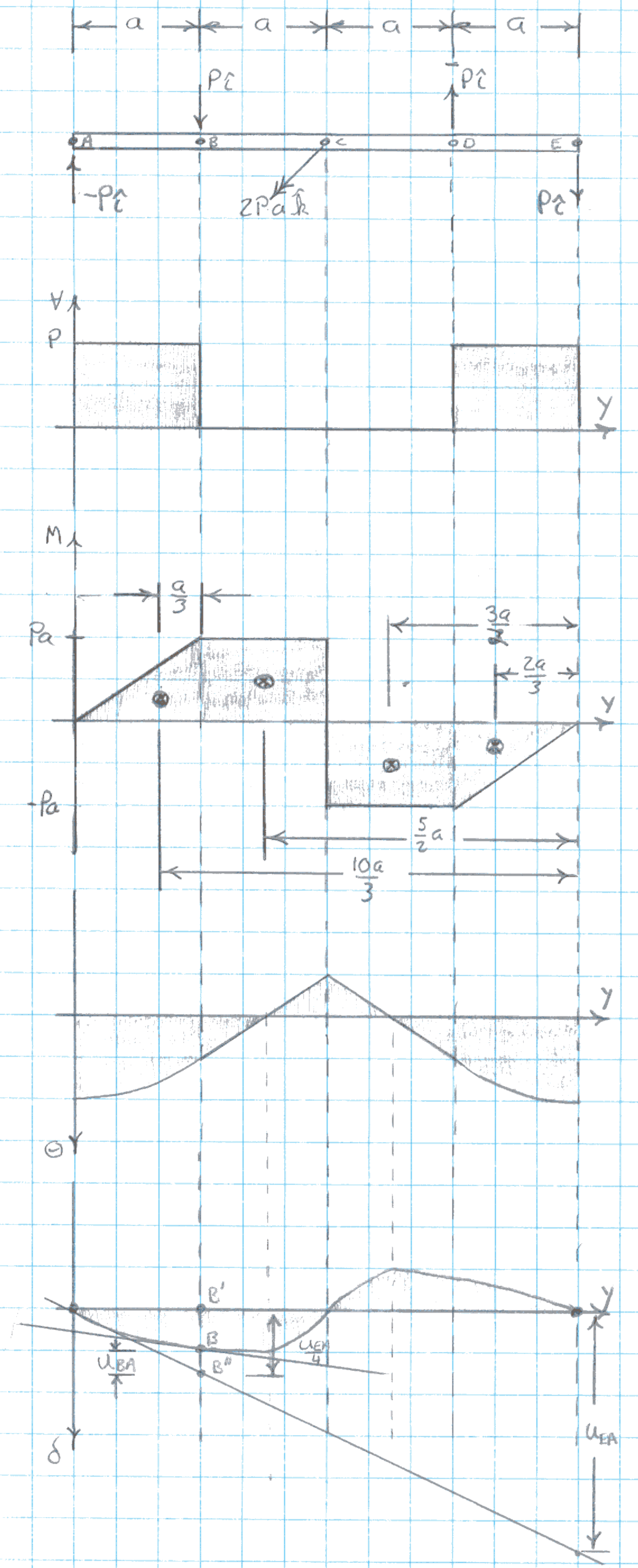
$$= \frac{1}{6EI}\left[-2 \cdot P \cdot a^3 - 9 \cdot P \cdot a^3 + 15 \cdot P a^3 + 10 Pa^3\right]$$

$$= \frac{14}{6}\frac{Pa^3}{EI} = \frac{7}{3}\frac{Pa^3}{EI} \qquad (1)$$

$$u_{BA} = \frac{\frac{1}{2}\cdot Pa \cdot a \cdot \frac{a}{3}}{EI} = \frac{Pa^3}{EI\,6}$$

$$u_B = \frac{1}{4}\cdot\frac{7}{3}\frac{Pa^3}{EI} - \frac{Pa^3}{6EI} = \frac{7}{12}\frac{Pa^3}{EI} - \frac{2\,Pa^3}{12\,EI}$$

$$= \underline{\underline{\frac{5}{12}\frac{Pa^3}{EI}}}$$



1c. Determine the deflection of the beam at point B.

$$q(y) = -p\langle y-0\rangle_{-1} + p\langle y-a\rangle_{-1} + 2\cdot p\cdot a\cdot\langle y-2\cdot a\rangle_{-2} - p\langle y-3a\rangle_{-1} + p\langle y-4a\rangle_{-1}$$

$$V(y) = p\langle y-0\rangle^0 - p\langle y-a\rangle^0 - 2\cdot p\cdot a\cdot\langle y-2\cdot a\rangle_{-1} + p\langle y-3a\rangle^0 - p\langle y-4a\rangle^0$$

$$M(y) = p\langle y-0\rangle^1 - p\langle y-a\rangle^1 - 2\cdot p\cdot a\langle y-2a\rangle^0 + p\langle y-3a\rangle^1 - p\langle y-4a\rangle^1$$

$$\Theta(y) = \frac{1}{EI}\left[-\frac{p}{2}\langle y-0\rangle^2 + \frac{p}{2}\langle y-a\rangle^2 + 2\cdot p\cdot a\cdot\langle y-2a\rangle^1 - \frac{p}{2}\langle y-3a\rangle^2 + \frac{p}{2}\langle y-4a\rangle^2 + C_1\right] \quad (4)$$

$$v(y) = \frac{1}{EI}\left[-\frac{p}{2\cdot3}\langle y-0\rangle^3 + \frac{p}{2\cdot3}\langle y-a\rangle^3 + \frac{2}{2}p\cdot a\langle y-2a\rangle^2 - \frac{p}{2\cdot3}\langle y-3a\rangle^3 + \frac{p}{2\cdot3}\langle y-4a\rangle^3 + C_1\cdot y + C_2\right] \quad (5)$$

The FIRST BOUNDARY CONDITION IS $v(0)=0$

$$v(0) = 0 = \frac{1}{EI}\left[0 + C_2\right] \Rightarrow \underline{C_2 = 0}$$

THE SECOND BOUNDARY CONDITION IS $v(4a)=0$

$$v(4a) = 0 = \frac{1}{EI}\left[-\frac{p}{6}(4\cdot a)^3 + \frac{p}{6}(3a)^3 + p\cdot a\,(2\cdot a)^2 - \frac{p}{6}(a)^3 + \frac{p}{6}(0) + C_1\cdot 4a\right]$$

$$= \frac{1}{EI}\left[-\frac{64}{6}\cdot p\cdot a^3 + \frac{27}{6}\cdot p\cdot a^3 + \frac{24}{6}p\cdot a^3 - \frac{1}{6}p\,a^3 + C_1\cdot 4a\right]$$

$$= \frac{1}{EI}\left[-\frac{14}{6}\cdot p\cdot a^3 + C_1\cdot 4a\right] \Rightarrow C_1 = +\frac{14}{6}\frac{p\cdot a^3}{4a} = \frac{14}{24}pa^2 = \frac{7}{12}pa^2$$

THEREFORE

$$v(y) = \frac{1}{EI}\left[-\frac{p}{6}\langle y-0\rangle^3 + \frac{p}{6}\langle y-a\rangle^3 + p\cdot a\cdot\langle y-2a\rangle^2 - \frac{p}{6}\langle y-3a\rangle^3 + \frac{p}{6}\langle y-4a\rangle^3 + \frac{7}{12}pa^2 y\right]$$

$$v(a) = \frac{1}{EI}\left[-\frac{p}{6}a^3 + \frac{7}{12}pa^3\right] = \boxed{\frac{5}{12}\cdot\frac{p\cdot a^3}{EI}}$$

NOW DETERMINING THE LOCATION OF THE MAXIMUM DEFLECTION SETTING (4) EQUAL TO 0
FROM THE DIAGRAMS IT IS CLEAR THAT THE MAXIMUM WILL FALL BETWEEN a AND $2a$.
THEREFORE

$$0 = \frac{1}{EI}\left[-\frac{\cancel{p}}{\cancel{2}}(y)^2 + \frac{\cancel{p}}{\cancel{2}}\cancel{(0)} + \frac{7}{\cancel{12}_{6}}\cancel{p}\cdot a^2\right] \Rightarrow y^2 = \frac{7}{6}a^2 \Rightarrow \boxed{y = 1.080\,a}$$

$$v(1.08\,a) = \frac{1}{EI}\left[-\frac{p}{6}(1.08a)^3 + \frac{p}{6}(1.08a-a)^3 + \frac{7}{12}\cdot p\cdot a^2(1.08a)\right]$$

$$= \frac{p\cdot a^3}{EI}\cdot 0.4201 = \boxed{0.4201\,\frac{p\cdot a^3}{EI}}$$

BONUS

PROBLEM 2: A machined steel shaft has a diameter of 1.0 inches. This shaft is subjected to a purely reversing moment of 1000 lb-in and a constant torque of 1200 lb-in. The ultimate strength of the steel is 100,000 lb/in^2 and the yield strength is 80,000 lb/in^2.

2a. Construct the modified Goodman diagram for the material in the as manufactured shaft.

$$\sigma_a = \frac{M \cdot C}{I} = \frac{1000\text{ lb-in} \cdot 0.5\text{in}}{\frac{\pi \cdot (1\text{in})^4}{64}} = 10.18\,(10^3)\,\frac{\text{lb}}{\text{in}^2} = \underline{\underline{10.18\text{ ksi}}}$$

$$\frac{T \cdot r}{J} = \frac{1200\text{ lb-in} \cdot 0.5\text{in}}{\frac{\pi\,(1\text{in})^4}{32}} = 6.111\,(10^3)\,\frac{\text{lb}}{\text{in}^2} = 6.111\text{ ksi}$$

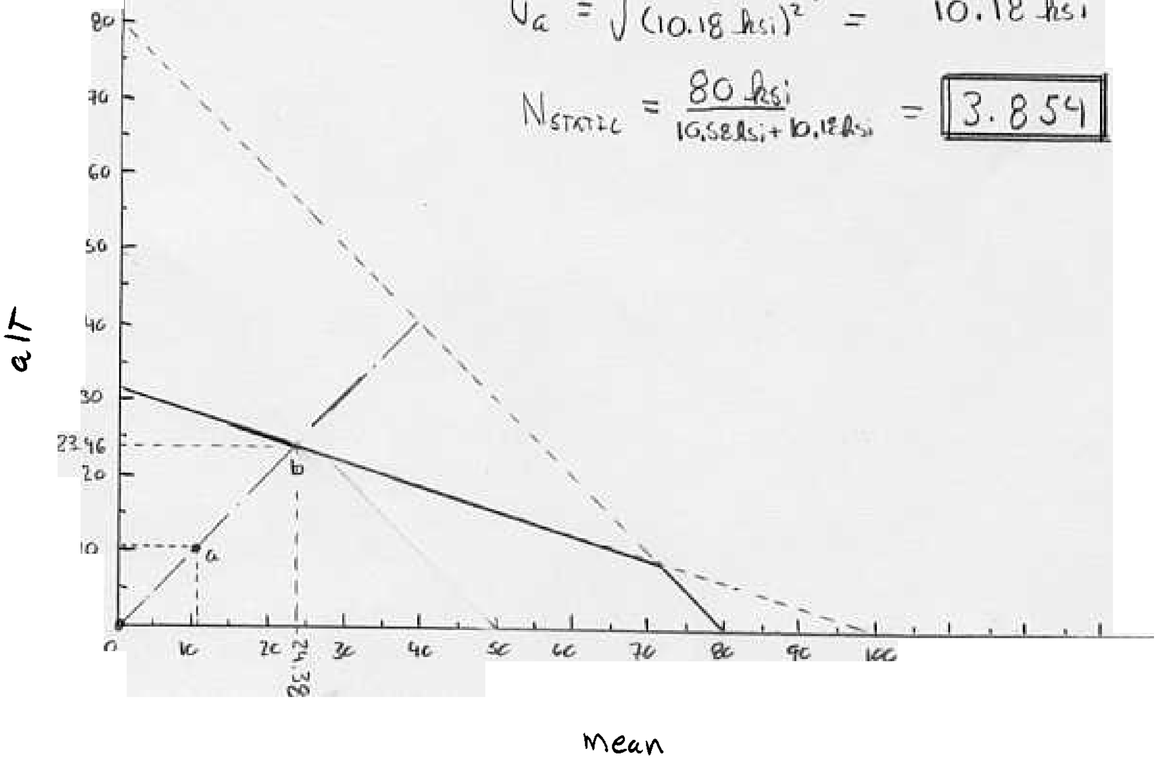
$$S_e' = 0.5\,S_{UT} = 0.5 \cdot 100{,}000\,\frac{\text{lb}}{\text{in}^2} = 50{,}000\,\frac{\text{lb}}{\text{in}^2}$$

$$0.73 \cdot 0.85 \cdot 1 \cdot 50\,000\,\frac{\text{lb}}{\text{in}^2} = 31.02\,(10^3)\,\frac{\text{lb}}{\text{in}^2} = \underline{\underline{31.02\text{ ksi}}}$$

$$\sigma_m' = \sqrt{3 \cdot (6.111\text{ ksi})^2} = 10.58\text{ ksi}$$

$$\sigma_a' = \sqrt{(10.18\text{ ksi})^2} = 10.18\text{ ksi}$$

$$N_{STATIC} = \frac{80\text{ ksi}}{10.58\text{ksi} + 10.18\text{ksi}} = \boxed{3.854}$$



$$N_{fatigue} = \frac{\overline{Ob}}{\overline{Oa}}$$

$$= -\frac{31.02\,ksi}{100\,ksi}\sigma_{m,G} + 31.02\,ks = 0.3102\cdot\sigma_{m,G} + 3\ \ 02\,ksi$$

$$\frac{10.18\,ksi}{10.58\,ksi}\cdot\sigma_{m,G} = 0.9622\cdot\sigma_{m,G}$$

$$0.9622\cdot\sigma_m = -0.3102\cdot\sigma_m + 31.02\,ksi$$

$$\sigma_m = \frac{31.02\,ksi}{0.9622+0.3102} = 24.38\,ksi$$

$$\sigma_a = 23.46\,ksi$$

$$N_{fatigue} = \frac{\sqrt{(23.46)^2+(24.38)^2}}{\sqrt{(10.18)^2+(10.58)^2}} = \boxed{2.304}$$