Continuing Education: Finite Element Methods

2D Truss Elements

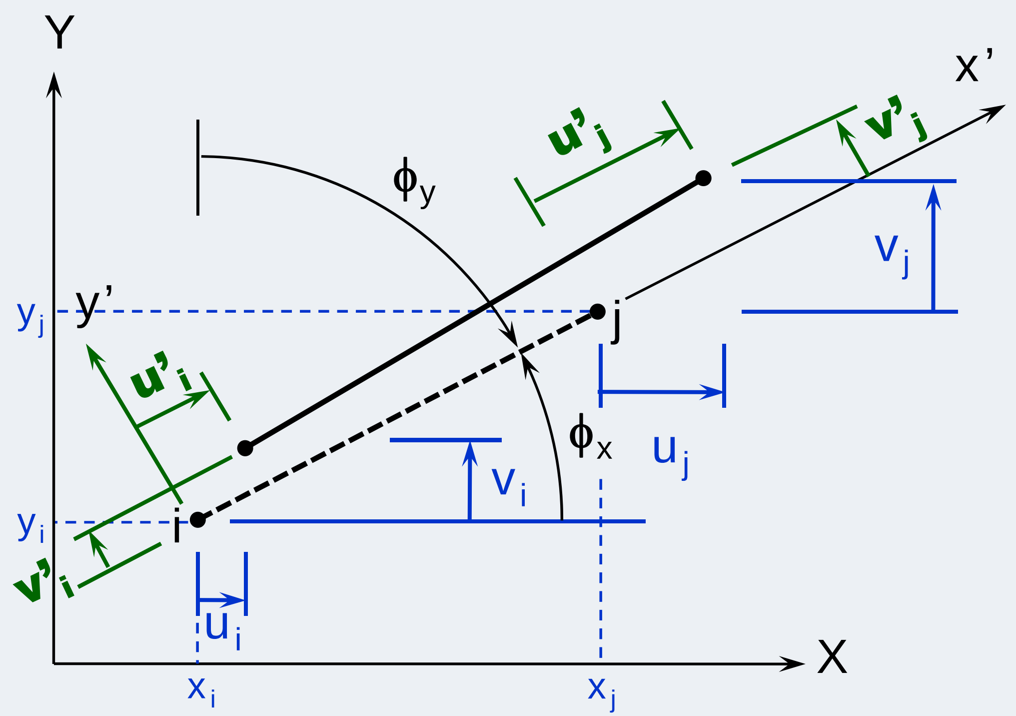
- Direct Stiffness Method

3D Truss Elements

2D Truss Element Requires Transformation Equations

$$\phi_x = \tan^{-1}\left(\frac{y_j - y_i}{x_j - x_i}\right)$$

$$\phi_y = \tan^{-1}\left(\frac{x_j - x_i}{y_j - y_i}\right)$$



u Displacement Transformed from Global to Local System

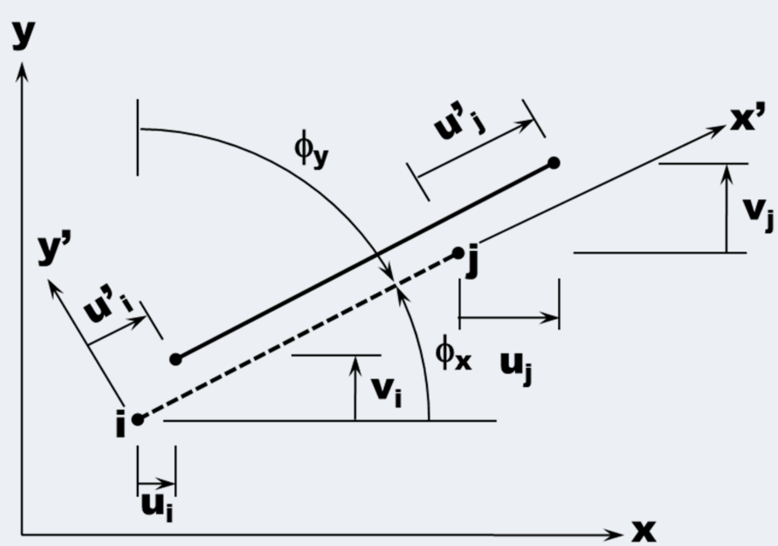
$$u_i' = u_i \cdot \cos\phi_x + v_i \cdot \cos\phi_y$$

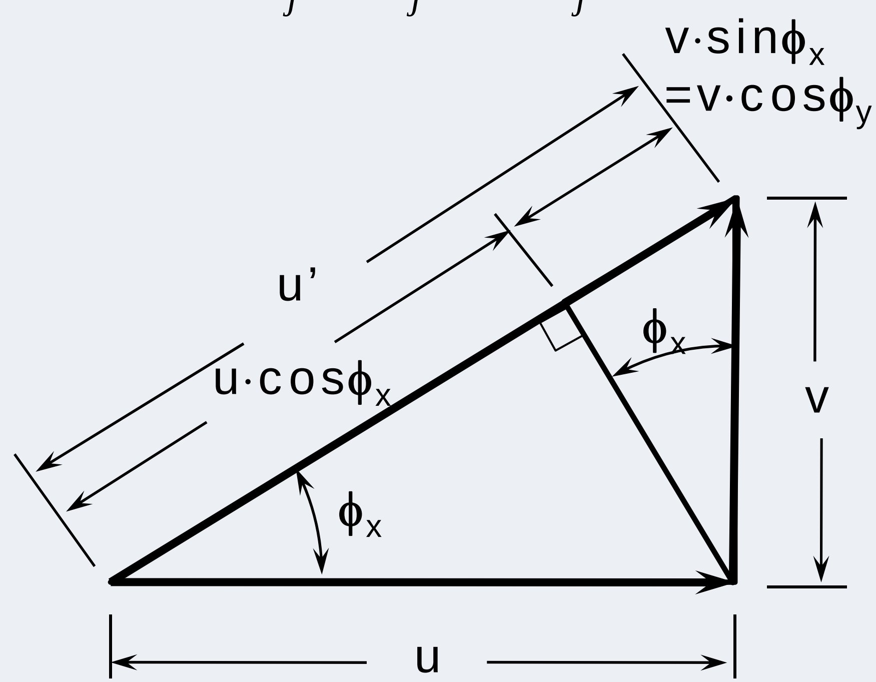
$$u_j' = u_j \cdot \cos\phi_x + v_j \cdot \cos\phi_y$$

$$\Rightarrow \quad l = \cos\phi_x \qquad m = \cos\phi_y \quad \Rightarrow$$

$$u_i' = u_i \cdot l + v_i \cdot m$$

$$u_j' = u_j \cdot l + v_j \cdot m$$





v Displacement Transformed from Global to Local System

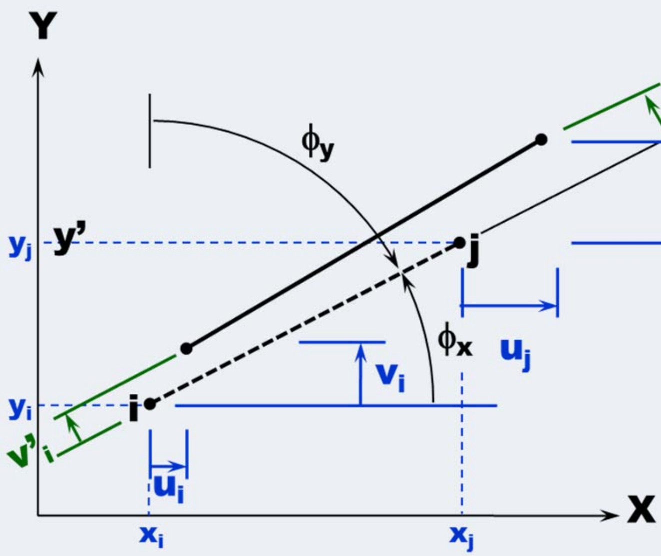
$$v_i' = -u_i \cdot \cos\phi_y + v_i \cdot \cos\phi_x$$

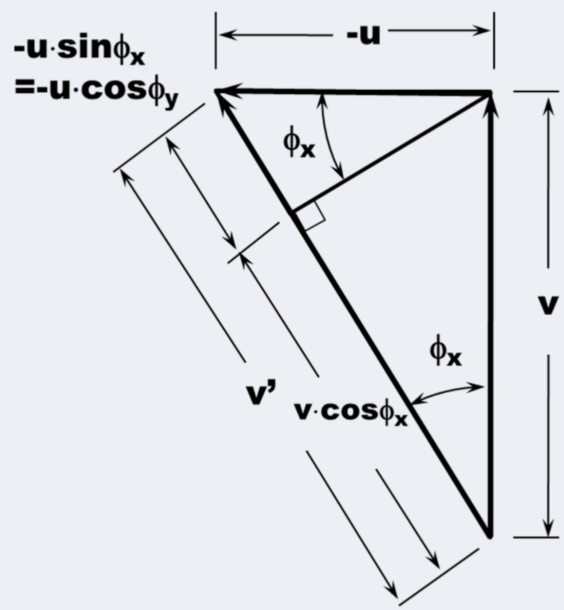
$$v_j' = -u_j \cdot \cos\phi_y + v_j \cdot \cos\phi_x$$

$$\Rightarrow \quad l = \cos\phi_x \qquad m = \cos\phi_y \quad \Rightarrow$$

$$v_i' = v_i \cdot l + u_i \cdot m$$

$$v_j' = v_j \cdot l + u_j \cdot m$$





Transformations from Global to Local System in Matrix Form

$$\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \qquad \begin{Bmatrix} f_{x'i} \\ f_{y'i} \\ f_{x'j} \\ f_{y'j} \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \begin{Bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{Bmatrix}$$

$$\{\mathbf{u}\}_{local} = [\mathbf{T}]\{\mathbf{u}\}_{global} \qquad \{\mathbf{f}\}_{local} = [\mathbf{T}]\{\mathbf{f}\}_{global}$$

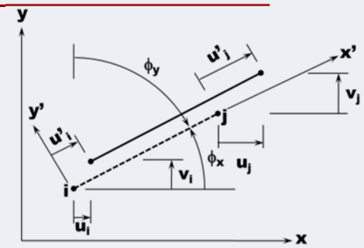
The System Of Equations is Formed In Global Coordinates

$$\{\mathbf{f}\}_{global} = [\mathbf{k}]_{global} \{\mathbf{u}\}_{global}$$

$$[\mathbf{k}]_{global} = \left(\frac{A \cdot E}{L}\right)_e \begin{bmatrix} l^2 & l \cdot m & -l^2 & -l \cdot m \\ l \cdot m & m^2 & -l \cdot m & -m^2 \\ -l^2 & -l \cdot m & l^2 & l \cdot m \\ -l \cdot m & -m^2 & l \cdot m & m^2 \end{bmatrix}_e$$

Forming The System Matrix More Complex Than 1D Case

$$\{\mathbf{f}\}_{global} = [\mathbf{k}]_{global}\{\mathbf{u}\}_{global} \quad (34) \quad \Rightarrow \quad \{\mathbf{f}\}_{system} = [\mathbf{k}]_{system}\{\mathbf{u}\}_{system} \quad (35)$$



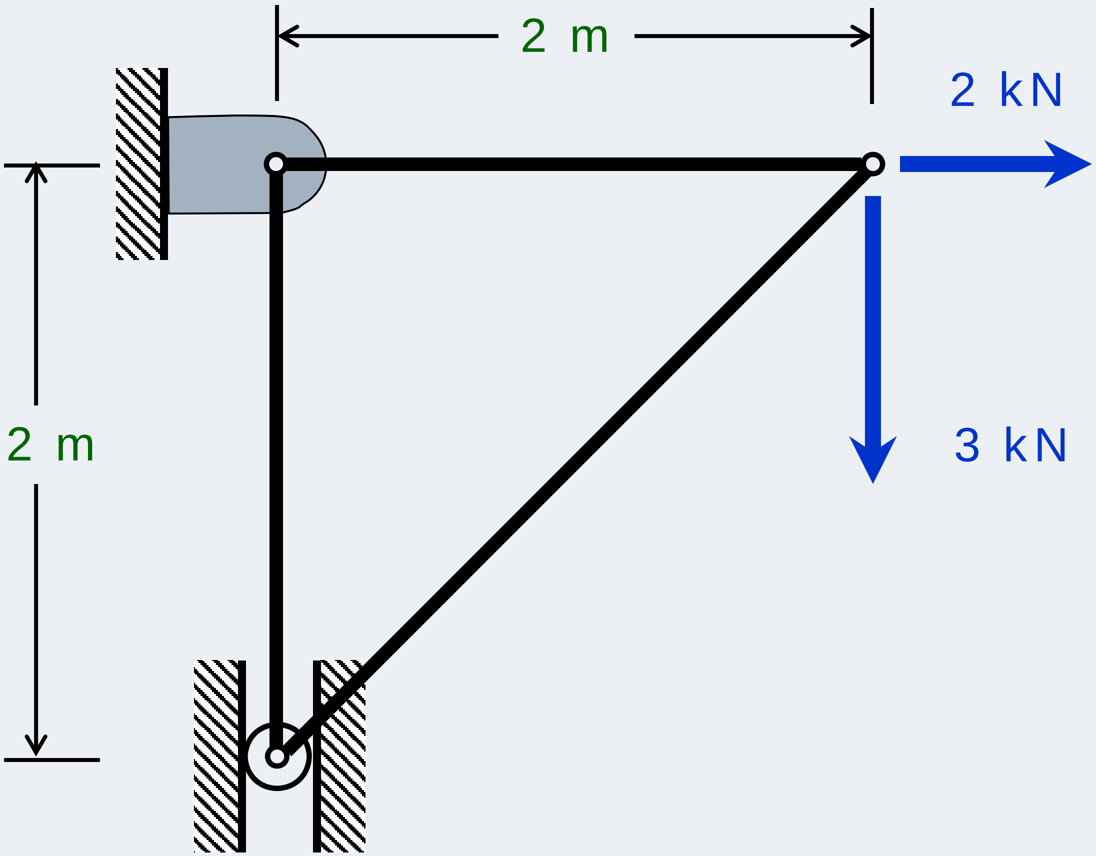
- Nodes do not follow each other in a serial manner
- Two DOF at each node

Simplification to help organize the problem

$$\{\mathbf{u}\}_{system} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix} \qquad \{\mathbf{f}\}_{system} = \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ \vdots \\ f_{xn} \\ f_{yn} \end{Bmatrix}$$

- For a given Element, always define
  - The lower element as the i node
  - The higher element as the j node
- After the element equations are formulated with respect to the Element DOF, then expand with respect to the System DOF

Example



Determine the nodal deflections, nodal forces, and stress in each of the elements of the truss. Ignore buckling. Each member has a cross sectional area of 80 mm^2. The members have a modulus of 200 Gpa.

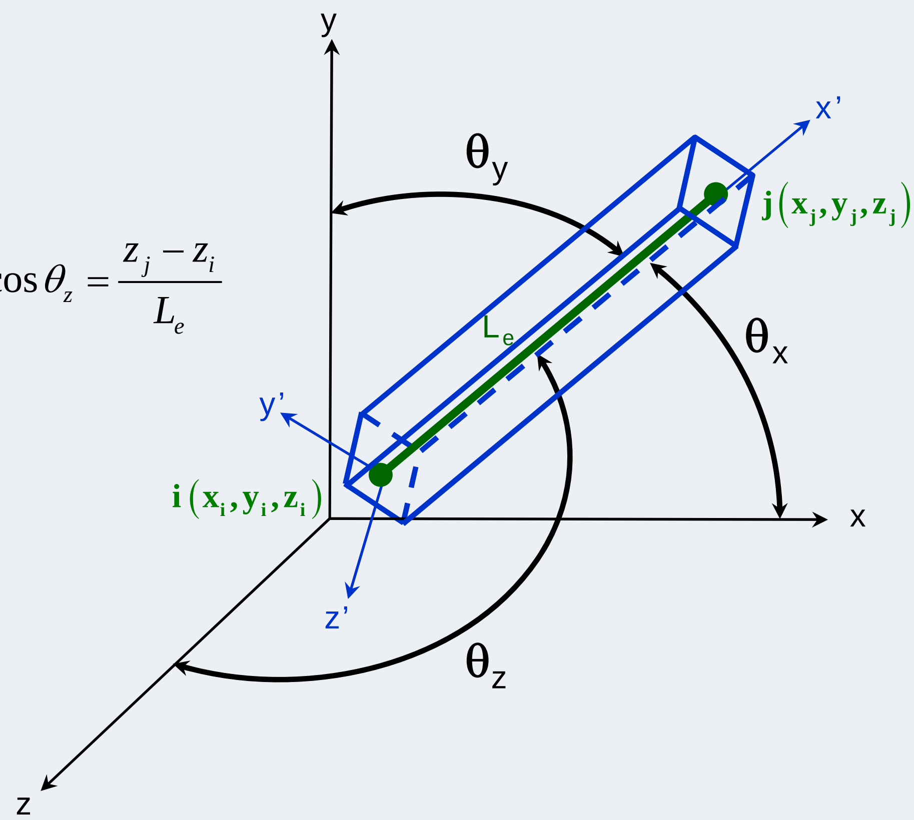
Transformations for the 3D Truss Element

Length of the Element

$$L_e = \left[\left(x_j - x_i \right)^2 + \left(y_j - y_i \right)^2 + \left(z_j - z_i \right)^2 \right]^{1/2}$$

Defining the direction cosines

$$l = \cos\theta_x = \frac{x_j - x_i}{L_e}, \qquad m = \cos\theta_y = \frac{y_j - y_i}{L_e}, \qquad n = \cos\theta_z = \frac{z_j - z_i}{L_e}$$



The Transformation Equations are given by

$$u_i' = u_i \cdot l + v_i \cdot m + w_i \cdot n$$

$$u_j' = u_j \cdot l + v_j \cdot m + w_j \cdot n$$

$$\Rightarrow \quad \begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

$$\{\mathbf{u}\}_{local} = [\mathbf{T}]\{\mathbf{u}\}_{global} \quad (36)$$

The Local Stiffness Matrix Can Now Be Transformed

$$[\mathbf{k}]_{global} = [\mathbf{T}]^T \{\mathbf{k}\}_{local} [\mathbf{T}]$$

$$[\mathbf{k}]_{global} = \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \left(\frac{A \cdot E}{L}\right)_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

$$= \left(\frac{A \cdot E}{L}\right)_e \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix}$$