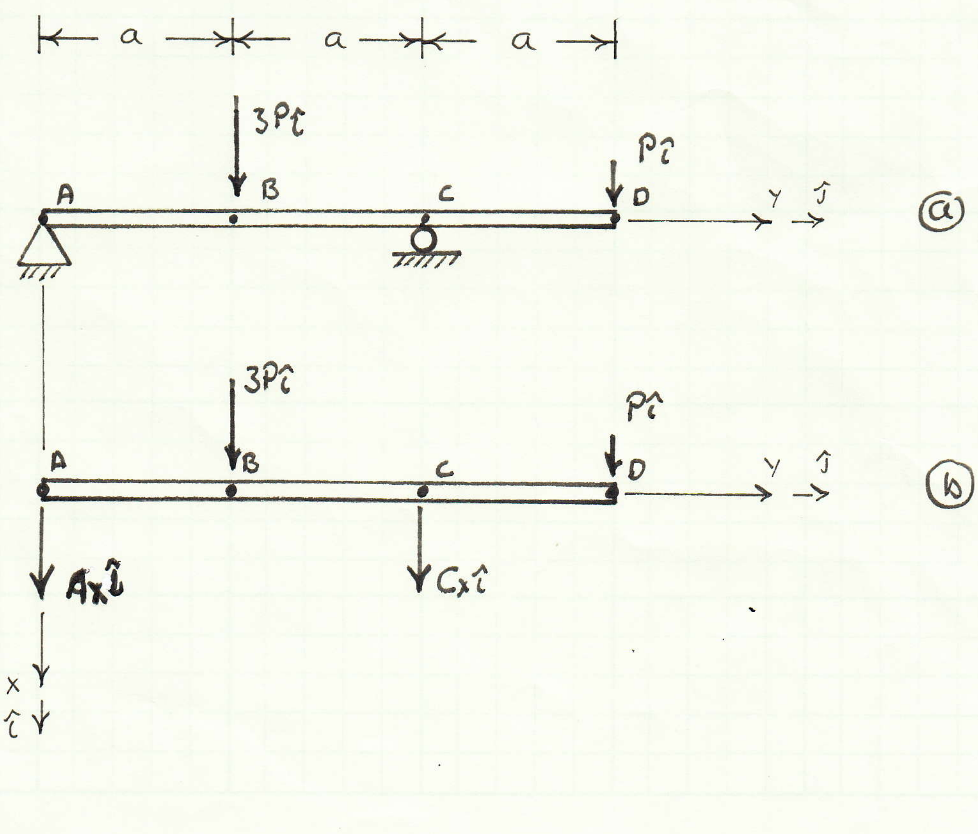
PROBLEM STATEMENT: A HORIZONTAL BEAM OF LENGTH 3a IS SHOWN BELOW. THIS BEAM HAS A PIN SUPPORT AT THE LEFT END OF THE BEAM AND A ROLLER SUPPORT 2a FROM THE LEFT END OF THE BEAM. A 3P LOAD IS APPLIED a FROM THE LEFT END OF THE BEAM AND A LOAD P IS APPLIED AT THE RIGHT END OF THE BEAM. USING FIRST THE DIRECT INTEGRATION METHOD AND THEN USING SINGULARITY FUNCTIONS DETERMINE EXPRESSIONS FOR THE SHEAR FORCE, BENDING MOMENT, CURVATURE, AND DISPLACEMENT IN THIS BEAM. DRAW THE SHEAR FORCE, BENDING MOMENT, CURVATURE, AND DISPLACEMENT DIAGRAMS; LABELING ALL CRITICAL VALUES AND THEIR LOCATIONS.

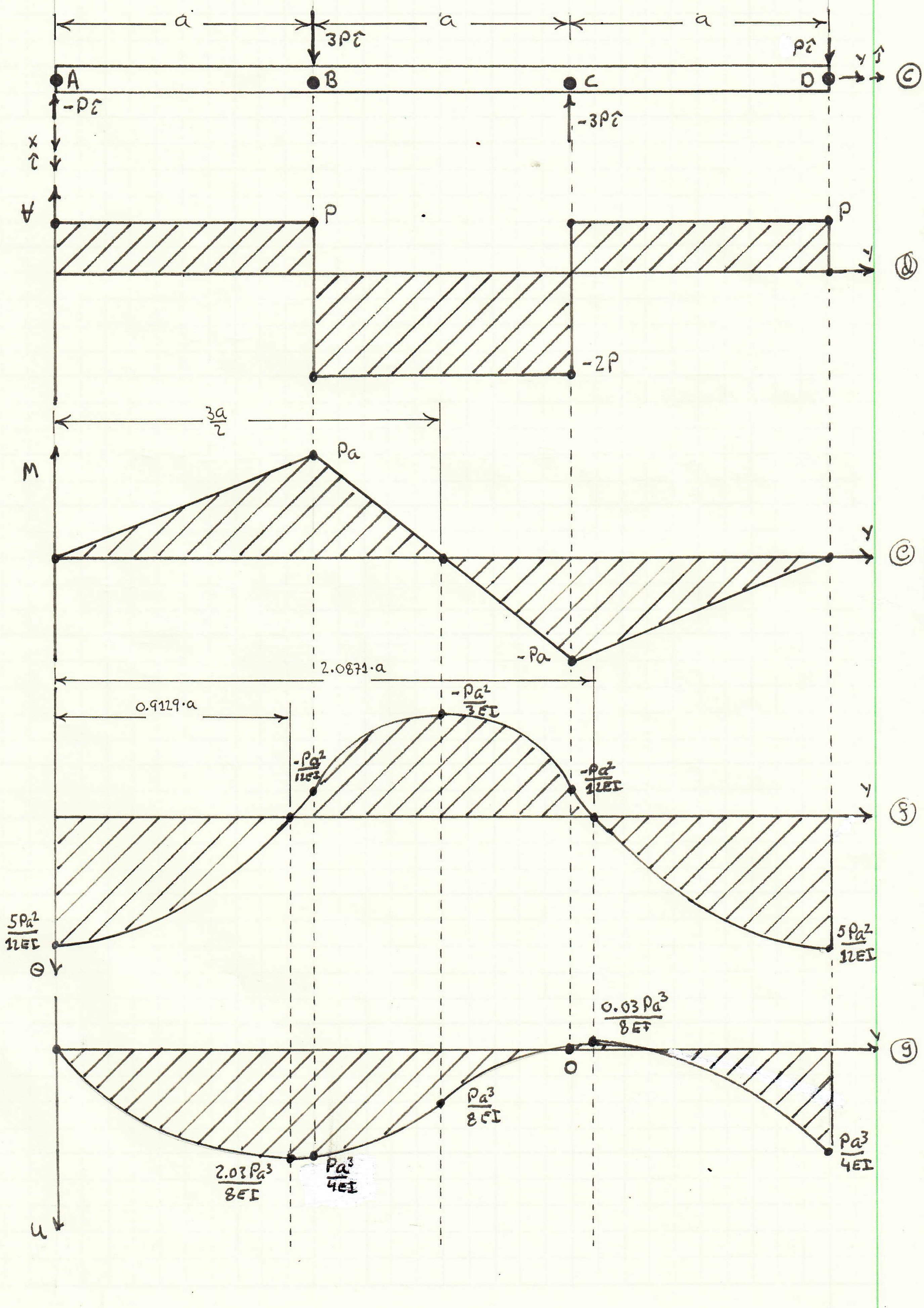
GIVEN:
1. A BEAM OF LENGTH 3a (EI)
2. A TRANSVERSE LOAD OF 3P a FROM THE LEFT END OF THE BEAM
3. A TRANSVERSE LOAD OF P AT THE RIGHT MOST END OF THE BEAM
4. A PIN CONSTRAINT AT THE LEFT MOST END OF THE BEAM
5. A ROLLER SUPPORT 2a FROM THE LEFT END OF THE BEAM

ASSUMPTIONS:
1. THE BEAM IS INITIALLY STRAIGHT
2. THE MATERIAL IS LINEAR ELASTIC
3. ALL DISPLACEMENTS ARE SMALL.

FIND:
1. USING DIRECT INTEGRATION DETERMINE EXPRESSIONS FOR $V, M, \theta, \& u$
2. USING SINGULARITY FUNCTIONS DETERMINE EXPRESSIONS FOR $V, M, \theta, \& u$
3. DRAW THE $V, M, \theta, \& u$ DIAGRAMS.

FIGURE:




a
a
a
3P$\hat{\imath}$
P$\hat{\imath}$
A
B
C
D
-P$\hat{\imath}$
-3P$\hat{\imath}$
x $\hat{\imath}$
y $\hat{\jmath}$
(c)
V
P
P
-2P
y
(d)
$\frac{3a}{2}$
M
Pa
-Pa
y
(e)
2.0871·a
0.9129·a
$-\frac{Pa^2}{3EI}$
$-\frac{Pa^2}{12EI}$
$-\frac{Pa^2}{12EI}$
$\frac{5Pa^2}{12EI}$
$\frac{5Pa^2}{12EI}$
θ
y
(f)
$\frac{0.03Pa^3}{8EI}$
0
$\frac{Pa^3}{8EI}$
$\frac{2.03Pa^3}{8EI}$
$\frac{Pa^3}{4EI}$
$\frac{Pa^3}{4EI}$
u
y
(g)

Consider the same beam using singularity functions. Starting with ⓒ, the load can be expressed as

$$q(y) = -P\langle y-0\rangle_{-1} + 3P\langle y-a\rangle_{-1} - 3P\langle y-2a\rangle_{-1} + P\cdot\langle y-3a\rangle_{-1} \quad (47)$$

$$V = \int -q(y)\cdot dy$$

$$= P\langle y-0\rangle^0 - 3P\langle y-a\rangle^0 + 3P\langle y-2a\rangle^0 - P\langle y-3a\rangle^0 \quad (48)$$

$$M = \int V(y)\cdot dy$$

$$= P\langle y-0\rangle^1 - 3P\langle y-a\rangle^1 + 3P\langle y-2a\rangle^1 - P\langle y-3a\rangle^1 \quad (49)$$

$$\Theta = \int -\frac{M}{EI}\cdot dy$$

$$= -\frac{P}{2EI}\cdot\langle y-0\rangle^2 + \frac{3P}{2EI}\langle y-a\rangle^2 - \frac{3P}{2EI}\langle y-2a\rangle^2 + \frac{P}{2EI}\langle y-3a\rangle^2 + \cancelto{\frac{5Pa^2}{12EI}}{C_1} \quad (50)$$

$$u = \int \Theta\, dy$$

$$= -\frac{P}{6EI}\langle y-0\rangle^3 + \frac{3P}{6EI}\langle y-a\rangle^3 - \frac{3P}{6EI}\langle y-2a\rangle^3 + \frac{P}{6EI}\langle y-3a\rangle^3 + \cancelto{\frac{5Pa^2}{12EI}}{C_1}\cdot y + C_2$$

$$= -\frac{P}{6\cdot EI}\langle y-0\rangle^3 + \frac{P}{2EI}\langle y-a\rangle^3 - \frac{P}{2EI}\langle y-2a\rangle^3 + \frac{P}{6EI}\langle y-3a\rangle^3 + \cancelto{\frac{5Pa^2}{12EI}}{C_1}\cdot y + \cancelto{0}{C_2} \quad (51)$$

There are two constants in the expression for u, and there are two known boundary conditions, $u(0)=0$ and $u(2a)=0$.

$$u(0) = 0 = -\frac{P}{6\cdot EI}(0-0)^3 + C_1\cdot 0 + C_2 \Rightarrow C_2 = 0$$

$$u(2a) = 0 = -\frac{P}{6EI}(2a)^3 + \frac{P}{2EI}(a)^3 - \frac{P}{2EI}(0)^3 + C_1\cdot 2\cdot a + 0$$

$$\Rightarrow 2\cdot a\cdot C_1 = \frac{P\cdot 8a^3}{6EI} - \frac{3\cdot P\cdot a^3}{3\cdot 2EI} = \frac{5\cdot P\cdot a^3}{6EI}$$

$$\Rightarrow C_1 = \frac{5Pa^2}{12EI} \quad (52)$$

(52) CAN NOW BE SUBSTITUTED INTO (50) AND (51)

$$\theta = -\frac{P}{2 \cdot EI}\langle y-0\rangle^2 + \frac{3P}{2 \cdot EI}\langle y-a\rangle^2 - \frac{3P}{2EI}\langle y-2a\rangle^2 + \frac{P}{2 \cdot EI}\langle y-3a\rangle^2 + \frac{5 \cdot Pa^2}{12 \cdot EI} \quad (52)$$

$$u = -\frac{P}{6EI}\langle y-0\rangle^3 + \frac{P}{2EI}\langle y-a\rangle^3 - \frac{P}{2EI}\langle y-2a\rangle^3 + \frac{P}{6EI}\langle y-3a\rangle^3 + \frac{5Pa^2}{12EI} \cdot y \quad (53)$$

(48), (49), (52), AND (53) CAN NOW BE USED TO FIND CRITICAL VALUES OF V, M, θ, AND u IN EACH REGION OF THE BEAM.

REGION AB: $0 \le y \le a$

(48) → $V(0) = P \cdot (0)^0 = 0,\ P$ (54)

(49) → $M(0) = P \cdot (0) = 0$ (55)

(52) → $\theta(0) = -\frac{P}{2EI}(0)^2 + \frac{5Pa^2}{12EI} = \frac{5Pa^2}{12EI}$ (56)

(53) → $u(0) = -\frac{P}{6EI}(0)^3 + \frac{5 \cdot Pa^2}{12 \cdot EI}(0) = 0$ (57)

(48) → $V(a) = P \cdot (a)^0 - 3P(0)^0 = P,\ -2P$ (58)

(49) → $M(a) = P(a) - 3P(0) = Pa$ (59)

(52) → $\theta(a) = \frac{-P(a)^2}{2EI} - \frac{3P(0)^2}{2EI} + \frac{5 \cdot Pa^2}{12EI} = -\frac{Pa^2}{12EI}$ (60)

(53) → $u(a) = -\frac{P(a)^3}{6EI} + \frac{P(0)^3}{2EI} + \frac{5 \cdot Pa^2 \cdot (a)}{12 \cdot EI} = \frac{3Pa^3}{12 \cdot EI} = \frac{Pa^3}{4EI}$ (61)

THE CHANGE IN SIGN BETWEEN $\theta(0)$ (56) AND $\theta(a)$ (60) INDICATES THE VALUE OF θ GOES TO ZERO IN THIS REGION WHICH MEANS ~~EE~~ u WILL BE A MAX OR MIN IN THIS REGION. THIS LOCATION MUST BE FOUND. SETTING (52) EQUAL TO ZERO FOR THE REGION $0 \le y \le a$

(52) $\theta = 0 = -\frac{\cancel{P} \cdot y^2}{2\cancel{EI}} + \frac{5 \cdot \cancel{P}a^2}{12\cancel{EI}} \Rightarrow y = \pm\sqrt{\frac{5a^2}{6}} = -0.9129 \cdot a,\ \underline{0.9129 \cdot a}$ (62)

ONLY THE SECOND ROOT EXISTS IN THE REGION BEING CONSIDERED. THE VALUE OF u AT THIS ROOT IS

$$u(0.9129 \cdot a) = -\frac{P \cdot (0.9129 \cdot a)^3}{6 \cdot EI} + \frac{5 \cdot Pa^2 \cdot (0.9129 \cdot a)}{12EI} = \frac{0.2536 \cdot Pa^3}{EI} \quad (63)$$

REGION BC: $a \leq y \leq 2a$

(58) → $V(a) = P, -2P$

(59) → $M(a) = Pa$

(60) → $\theta(a) = -\dfrac{Pa^2}{12EI}$

(61) → $u(a) = \dfrac{Pa^3}{4EI}$

(48) → $V(2a) = P\cdot(2a)^0 - 3P(a)^0 + 3P(0)^0 = -2P, \; P$ (64)

(49) → $M(2a) = P\cdot(2a) - 3P\cdot(a) + 3P(0) = -Pa$ (65)

(52) → $\theta(2a) = -\dfrac{P\cdot(2a)^2}{2\cdot EI} + \dfrac{3P\cdot(a)^2}{2EI} - \dfrac{3P\cdot(0)^2}{2EI} + \dfrac{5Pa^2}{12EI} = -\dfrac{Pa^2}{12EI}$ (66)

(53) → $u(2a) = -\dfrac{P\cdot(2a)^3}{6EI} + \dfrac{P\cdot(a)^3}{2EI} - \dfrac{P\cdot(0)^3}{2EI} + \dfrac{5\cdot Pa^2\cdot(2a)}{12EI} = 0$ (67)

THERE IS A SIGN CHANGE BETWEEN M(a) (59) AND M(2a) (65) THAT INDICATES THAT M GOES TO ZERO IN THIS REGION, WHICH MEANS THAT θ IS A MAX/MIN AT THE POINT M GOES TO ZERO. FINDING WHERE M GOES TO ZERO IN THIS REGION IS ACCOMPLISHED BY SETTING M=0 AND FINDING Y.

(49) → $M = 0 = P\cdot y - 3P(y-a) = P\cdot y - 3P\cdot y + 3Pa = -2P\cdot y + 3Pa$

$\Rightarrow y = \frac{3}{2}\cdot a$ (68)

(52) → $\theta(\frac{3a}{2}) = -\dfrac{P}{2EI}\left(\dfrac{3a}{2}\right)^2 + \dfrac{3P}{2EI}\cdot\left(\dfrac{a}{2}\right)^2 + \dfrac{5Pa^2}{12EI}$

$= -\dfrac{9}{8}\dfrac{Pa^2}{EI} + \dfrac{3}{8}\dfrac{Pa^2}{EI} + \dfrac{5}{12}\dfrac{Pa^2}{EI} = -\dfrac{16}{48}\dfrac{Pa^2}{EI} = -\dfrac{Pa^2}{3EI}$ (69)

(53) → $u(\frac{3a}{2}) = -\dfrac{P}{6EI}\left(\dfrac{3a}{2}\right)^3 + \dfrac{P}{2EI}\left(\dfrac{a}{2}\right)^3 + \dfrac{5Pa^2}{12EI}\cdot\left(\dfrac{3a}{2}\right)$

$= -\dfrac{9\cdot Pa^3}{16EI} + \dfrac{Pa^3}{16EI} + \dfrac{15Pa^3}{24EI} = \dfrac{6}{48}\dfrac{Pa^3}{EI} = \dfrac{Pa^3}{8EI}$ (70)

REGION CD: $2a \le y \le 3a$

(64) $\rightarrow$ $V(2a) = -2P,\ P$

(65) $\rightarrow$ $M(2a) = -Pa$

(66) $\rightarrow$ $\theta(2a) = -\dfrac{Pa^2}{12EI}$

(67) $\rightarrow$ $u(2a) = 0$

(48) $\rightarrow$ $V(3a) = P(3a)^0 - 3P(2a)^0 + 3P(a)^0 - P(0)^0 = P,\ 0$ (71)

(49) $\rightarrow$ $M(3a) = P(3a) - 3P(2a) + 3P(a) - P(0) = 0$

(52) $\rightarrow$
$$\theta(3a) = -\frac{P}{2EI}(3a)^2 + \frac{3P}{2EI}(2a)^2 - \frac{3P}{2EI}(a)^2 + \frac{P}{2EI}(0)^2 + \frac{5Pa^2}{12EI}$$
$$= -\frac{9\cdot Pa^2}{2\cdot EI} + \frac{12\cdot Pa^2}{2\cdot EI} - \frac{3\cdot Pa^2}{2\cdot EI} + \frac{5\cdot Pa^2}{12\cdot EI} = \frac{5Pa^2}{12EI} \quad (72)$$

(53) $\rightarrow$
$$u(3a) = -\frac{P}{6EI}(3a)^3 + \frac{P}{2\cdot EI}(2a)^3 - \frac{P}{2EI}(a)^3 + \frac{P}{6EI}(0)^3 + \frac{5Pa^2}{12EI}(3a)$$
$$= -\frac{27Pa^3}{6EI} + \frac{8Pa^3}{2EI} - \frac{Pa^3}{2EI} + \frac{15Pa^3}{12EI} = \frac{3Pa^3}{12EI} = \frac{Pa^3}{4EI} \quad (73)$$

THERE IS A SIGN CHANGE BETWEEN $\theta(2a)$ (66) AND $\theta(3a)$ (72) INDICATING THAT θ GOES TO ZERO IN THIS REGION WHICH MEANS THAT u WILL BE A MAX/MIN AT THE LOCATION WHERE $\theta=0$. TO FIND THIS LOCATION THE EXPRESSION OF θ IS SET TO ZERO. STARTING WITH (52)

(52) $\rightarrow$
$$\theta = 0 = -\frac{P\cdot y^2}{2EI} + \frac{3P}{2\cdot EI}(y-a)^2 - \frac{3P}{2EI}(y-2a)^2 + \frac{5Pa^2}{12EI}$$
$$= -\frac{Py^2}{2EI} + \frac{3P}{2EI}(y^2 - 2ay + a^2) - \frac{3P}{2EI}(y^2 - 4a\cdot y + 4a^2) + \frac{5Pa^2}{12EI}$$
$$= -\frac{P\cdot y^2}{2EI} + \frac{3P\cdot y^2}{2EI} - \frac{3Pay}{EI} + \frac{3Pa^2}{2EI} - \frac{3Py^2}{2EI} + \frac{12\cdot Pay}{2EI} - \frac{12Pa^2}{2EI} + \frac{5Pa^2}{12EI}$$
$$= -\frac{\cancel{P}\cdot y^2}{2\cancel{EI}} + \frac{3\cdot \cancel{P}a}{\cancel{EI}}\cdot y - \frac{49\cancel{P}a^2}{12\cancel{EI}} = \frac{y^2}{2} - 3\cdot ya + \frac{49}{12}\cdot a^2$$
$$= y^2 - 6\cdot ya + \frac{98}{12}a^2 = y^2 - 6\cdot ya + \frac{49}{6}a^2$$

$$0 = y^2 - 6a\cdot y + \frac{49}{6}a^2 = \underbrace{y^2 - 6a\cdot y + (-3a)^2}_{(y-3a)^2} - (-3a)^2 + \frac{49}{6}\cdot a^2$$

$$(y-3a)^2 = 9a^2 - \frac{49}{6}\cdot a^2 = \frac{5}{6}\cdot a^2$$

$$y - 3a = \pm\sqrt{\frac{5}{6}a^2} \Rightarrow y = 3a \pm \sqrt{\frac{5}{6}}\cdot a = 3a \pm 0.9129\cdot a$$

$$= \underline{\underline{2.0871\cdot a}},\ 3.9129a \qquad (74)$$

ONLY THE FIRST ROOT IS IN THE DOMAIN OF THE BEAM. u AT THIS LOCATION IS A MAX/MIN AND NEEDS TO BE CALCULATED

$$u(2.0871\cdot a) = -\frac{P}{6EI}(2.0871a)^3 + \frac{P}{2EI}(1.0871\cdot a)^3 - \frac{P}{2EI}(0.0871)^3 + \frac{5Pa^2}{12EI}(2.0871\cdot a)$$

$$= -0.00358\cdot\frac{P\cdot a^3}{EI} \qquad (75)$$

THE CRITICAL VALUES OF THE BEAM'S V, M, θ, AND u FUNCTIONS ARE SUMMERIZED IN FIGURES d, e, f, & g.

SUMMARY:

THE DIRECT INTEGRATION APPROACH (PAGES 3-10) AND THE SINGULARITY FUNCTION APPROACH (Pages 11-15) YIELD EXACTLY THE SAME RESULT. ONE ADVANTAGE OF THE SINGULARITY FUNCTION APPROACH IS THAT CONTINUITY CONDITIONS DO NOT HAVE TO BE COMPUTED BETWEEN REGIONS OF THE BEAM. THIS SIGNIFICANTLY REDUCES THE COMPLEXITY OF THE CALCULATIONS AND THE OPPORTUNITY FOR ERRORS.

BECAUSE OF THE CHOICE OF COORDINATE SYSTEMS, THE CALCULATIONS CAN BE EASILY CHECKED BY VISUAL INSPECTION.