DETERMINE THE CRITICAL LOAD FOR THIS BEAM SUBJECTED TO BUCKLING.

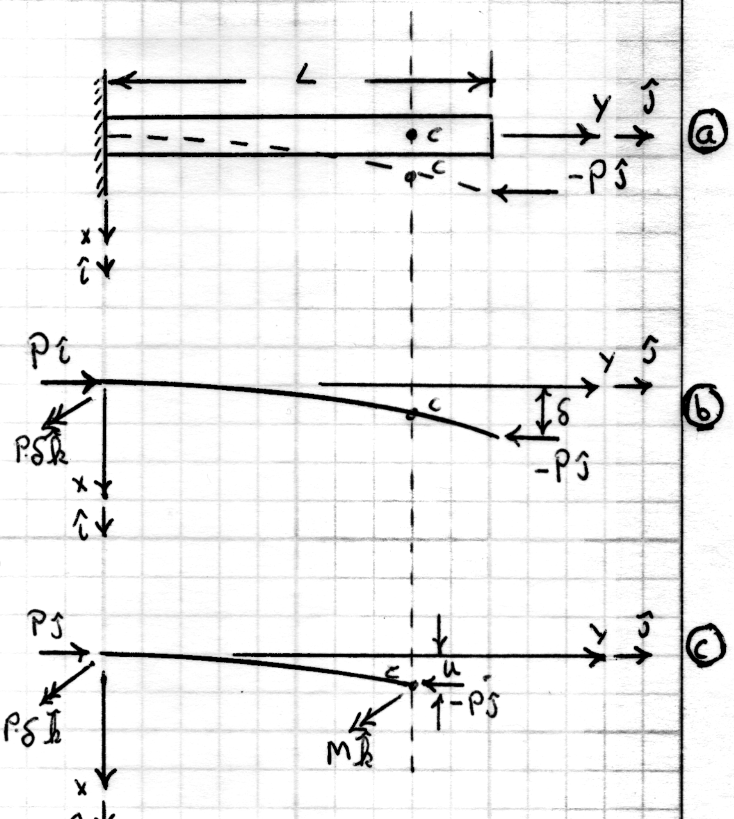


FIGURE (b) WAS DRAWN CONSIDERING THE DEFORMATION THAT RESULTS FROM BUCKLING AND EQUILIBRIUM.

STARTING WITH FIGURE (c). SUMMING MOMENTS ABOUT POINT C

$$\sum M_z|_{@c} = 0 = P \cdot \delta + M - P \cdot u$$

$$M = P(u - \delta)$$

FROM STRENGTH OF MATERIALS, THE INTERNAL BENDING MOMENT IS RELATED TO THE DEFLECTION OF THE ELASTIC CURVE BY

$$\frac{\partial^2 u}{\partial y^2} = -\frac{M}{EI} = \frac{-P(u-\delta)}{EI}$$

$$= \frac{P(\delta - u)}{EI}$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{P}{EI} u = \frac{P}{EI} \delta \qquad (1)$$

THE SOLUTION TO (1) COMES IN THE FOLLOWING FORM

$$u = u_H + u_P \qquad (2)$$

WHERE u_H IS THE HOMOGENIOUS SOLUTION AND u_P IS THE PARTICULAR SOLUTION. STARTING WITH THE HOMOGENIOUS SOLUTION

$$\frac{\partial^2 u_H}{\partial y^2} + \frac{P}{EI} u_H = 0 \qquad (3)$$

ASSUMING THE SOLUTION TO BE OF THE FORM

$$u_H = A_n e^{S_n \cdot y}$$

$$u_H' = \partial u / \partial y = A_n \cdot S_n \cdot e^{S_n \cdot y} \qquad (4)$$

$$u_H'' = \partial^2 u / \partial y^2 = A_n \cdot S_n^2 \cdot e^{S_n \cdot y}$$

SUBSTITUTING THE EQUATIONS IN (4) INTO (3)

$$A_n \cdot S_n^2 \cdot e^{S_n y} + \frac{P}{EI} \cdot A_n \cdot e^{S_n \cdot y} = 0$$

$$A_n \cdot e^{s_n \cdot y} \cdot \left(s_n^2 + \frac{P}{EI}\right) = 0$$

LETTING $A_n = 0$ WOULD RESULT IN THE TRIVIAL SOLUTION, $e^{s_n \cdot y}$ WILL NEVER EQUAL ZERO, THEREFORE

$$s_n^2 + \frac{P}{EI} = 0 \Rightarrow s_n^2 = \frac{-P}{EI} \Rightarrow s_n = \pm\sqrt{\frac{-P}{EI}} \Rightarrow s_n = \pm i\sqrt{\frac{P}{EI}} \qquad (5)$$

SUBSTITUTING (5) INTO (4)

$$u_H = A_0 \cdot e^{i\sqrt{\frac{P}{EI}} \cdot y} + A_1 \cdot e^{-i\sqrt{\frac{P}{EI}} \cdot y}$$

KNOWING $e^{i\theta} = \cos\theta + i\cdot\sin\theta$ AND $e^{-i\theta} = \cos\theta - i\cdot\sin\theta$

$$u_H = A_0 \cdot \left(\cos\sqrt{\frac{P}{EI}} \cdot y + i \cdot \sin\sqrt{\frac{P}{EI}} \cdot y\right) + A_1 \cdot \left(\cos\sqrt{\frac{P}{EI}} \cdot y - i \cdot \sin\sqrt{\frac{P}{EI}} \cdot y\right)$$

$$= \underbrace{(A_0 + A_1)}_{C_0} \cos\sqrt{\frac{P}{EI}} \cdot y + \underbrace{(A_0 - A_1) \cdot i}_{C_1} \cdot \sin\sqrt{\frac{P}{EI}} \cdot y$$

$$u_H = C_0 \cdot \cos\sqrt{\frac{P}{EI}} \cdot y + C_1 \cdot \sin\sqrt{\frac{P}{EI}} \cdot y \qquad (6)$$

NOW THE PARTICULAR SOLUTION MUST BE CONSIDERED. THE FORM OF THE PARTICULAR SOLUTION MUST BE THE SAME FORM AS THE RIGHT HAND SIDE OF (1), THEREFORE

$$u_p = C$$

$$u_p' = du_p/dy = 0 \qquad (7)$$

$$u_p'' = d^2u_p/dy^2 = 0$$

SUBSTITUTING THE RESULTS IN (7) INTO (1)

$$\frac{d^2 u_p}{dy^2} + \frac{P}{EI} u_p = \frac{P}{EI}\delta \Rightarrow 0 + \frac{P}{EI} \cdot C = \frac{P}{EI}\delta \Rightarrow C = \delta \qquad (8)$$

SUBSTITUTING THIS RESULT INTO (7)

$$u_p = \delta \qquad (9)$$

SUBSTITUTING (6) AND (9) INTO (2)

$$u = C_0 \cdot \cos\sqrt{\frac{P}{EI}} \cdot y + C_1 \cdot \sin\sqrt{\frac{P}{EI}} \cdot y + \delta \qquad (10)$$

THE TWO CONSTANTS ARE DETERMINED FROM THE BOUNDARY CONDITIONS

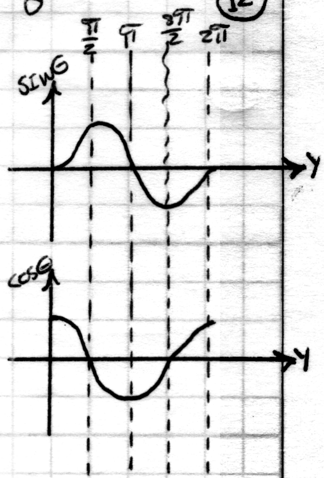
$$@\ y = 0: \quad u(0) = 0, \quad \frac{du}{dy}(0) = 0 \qquad (11)$$

FROM THE FIRST BOUNDARY CONDITION @ $y=0$

$$u(0) = 0 = C_0 \cdot \cos \cancelto{1}{\sqrt{\frac{P}{EI}} \cdot 0} + C_1 \cdot \sin \cancelto{0}{\sqrt{\frac{P}{EI}} \cdot 0} + \delta \Rightarrow C_0 = -\delta \qquad (12)$$

NOW (10) CAN BE WRITTEN

$$u = -\delta \cdot \cos\sqrt{\frac{P}{EI}} \cdot y + C_1 \cdot \sin\sqrt{\frac{P}{EI}} \cdot y + \delta \qquad (13)$$



FROM THE SECOND BOUNDARY CONDITION, $\frac{\partial u(0)}{\partial y} = 0$ @ $y=0$

$$\frac{\partial u(0)}{\partial y} = \delta \cdot \sqrt{\frac{P}{EI}} \cdot \sin \cancelto{0}{\sqrt{\frac{P}{EI}} \cdot 0} + C_1 \cdot \sqrt{\frac{P}{EI}} \cdot \cos \cancelto{1}{\sqrt{\frac{P}{EI}}(0)} = 0$$

$$\Rightarrow C_1 = 0$$

NOW (13) CAN BE WRITTEN

$$u = -\delta \cdot \cos\sqrt{\frac{P}{EI}} \cdot y + \delta = \delta\left(1 - \cos\sqrt{\frac{P}{EI}} \cdot y\right)$$

THE FINAL CONDITION THAT NEEDS TO BE CONSIDERED IS $u(L) = \delta$

$$u(L) = \delta = \delta\left(1 - \cos\sqrt{\frac{P}{EI}} \cdot L\right)$$

THE ONLY WAY THE ABOVE EQUATION IS SATISFIED IN A NON TRIVIAL MANNER IS IF

$$\left(1 - \cos\sqrt{\frac{P}{EI}} \cdot L\right) = 1 \Rightarrow \cos\sqrt{\frac{P}{EI}}\, L = 0 \Rightarrow \sqrt{\frac{P}{EI}} \cdot L = \frac{n \cdot \pi}{2} ;\ n = 1, 3, 5, \ldots$$

THE FIRST STATE THAT SATISFIES THIS CONDITION IS $n=1$

$$\sqrt{\frac{P}{EI}} \cdot L = \frac{\pi}{2} \Rightarrow \boxed{P_{CR} = \frac{\pi^2}{4L^2} EI}$$