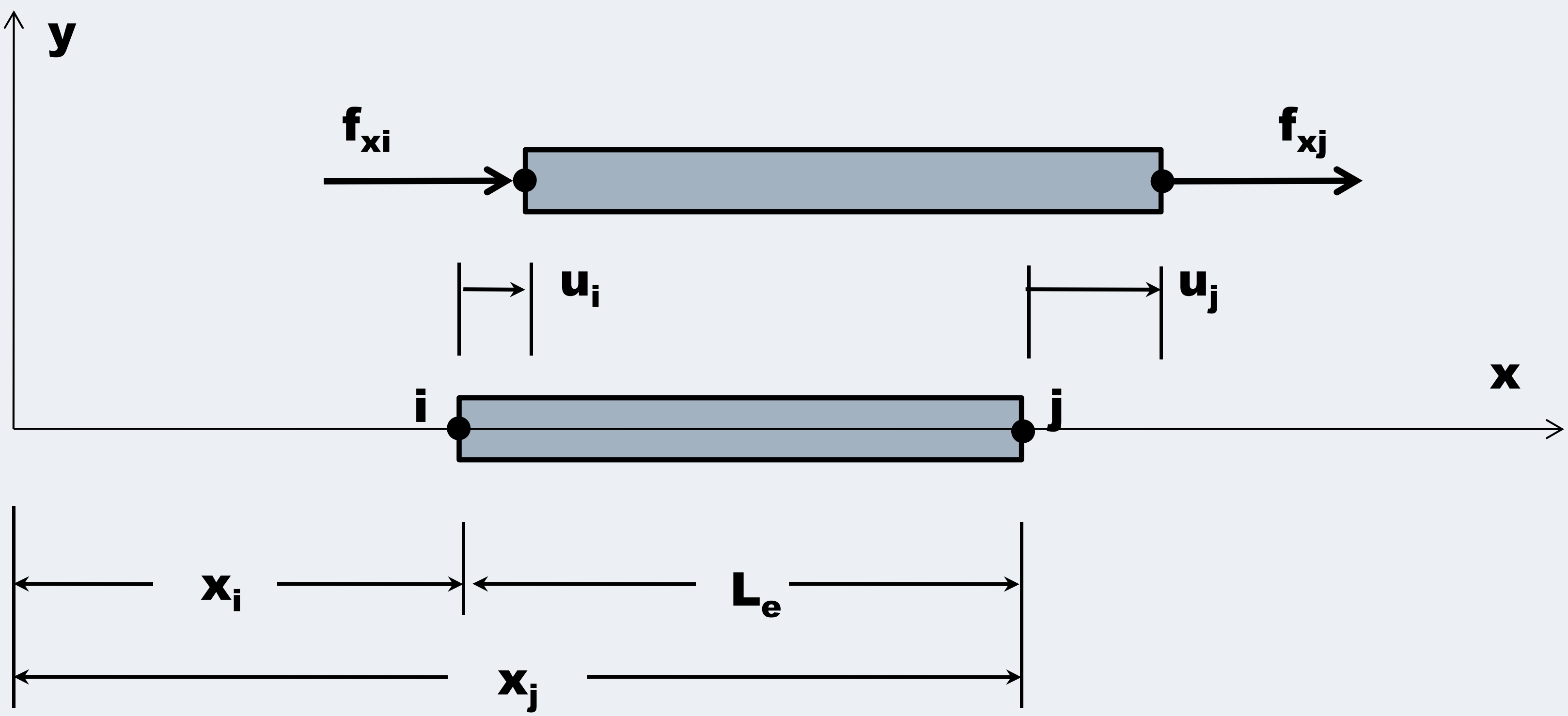
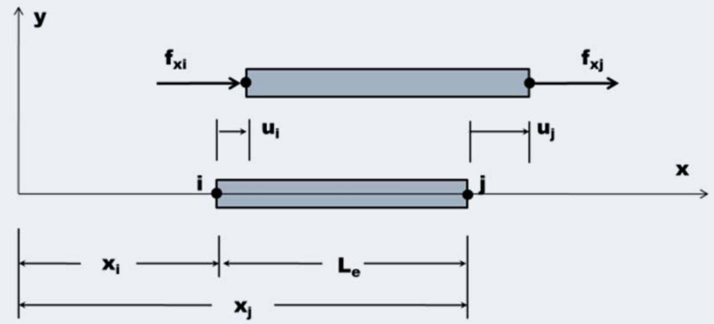
The Finite Element Method

- Overview of Technique
- Sample Element Library
- Errors Associated with the technique
- 1D Truss Element
  - Direct Stiffness Method

One-Dimensional Truss Element: Direct Stiffness Development



A Displacement Field is Assumed, u(x), nodal par.



- **Simplest Function for Displacement Field**
  - **One dimensional element**
  - **Two nodes**

$$u(x) = a_1 + a_2 \cdot x \quad (1)$$

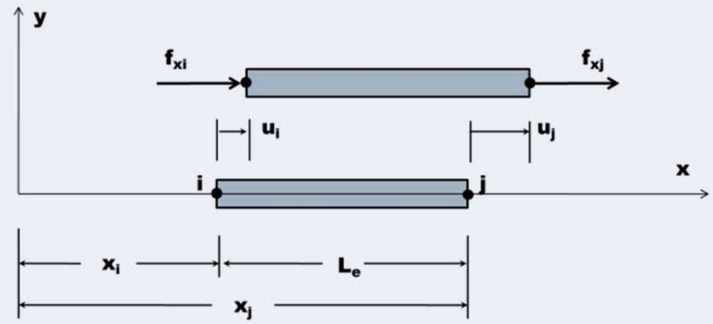
- **At the two nodal points u(x) can be written.** $u_i = u(x_i)$ & $u_j = u(x_j)$

$$u_i = a_1 + a_2 \cdot x_i \qquad u_j = a_1 + a_2 \cdot x_j$$

$$a_1 = \frac{u_i \cdot x_j - u_j \cdot x_i}{x_j - x_i} \qquad a_2 = \frac{u_j - u_i}{x_j - x_i} \qquad L_e = x_j - x_i$$

$$\boxed{a_1 = \frac{u_i \cdot x_j - u_j \cdot x_i}{L_e}} \quad (2) \qquad \boxed{a_2 = \frac{u_j - u_i}{L_e}} \quad (3)$$

The Displacement Functions are in Terms of u_i and u_j



$$u(x) = a_1 + a_2 \cdot x \qquad L_e = x_j - x_i$$

Substituting 2 and 3 into 1

$$u(x) = \frac{u_i \cdot x_j - u_j \cdot x_i}{L_e} + \frac{u_j - u_i}{L_e} \cdot x$$

$$= \frac{1}{L_e}\left[\left(u_i \cdot x_j - u_j \cdot x_i\right) + \left(u_j - u_i\right) \cdot x\right]$$

$$u(x) = \frac{x_j - x}{L_e} \cdot u_i + \frac{x - x_i}{L_e} \cdot u_j \boxed{= N_i(x) \cdot u_i + N_j(x) \cdot u_j} \qquad 4$$

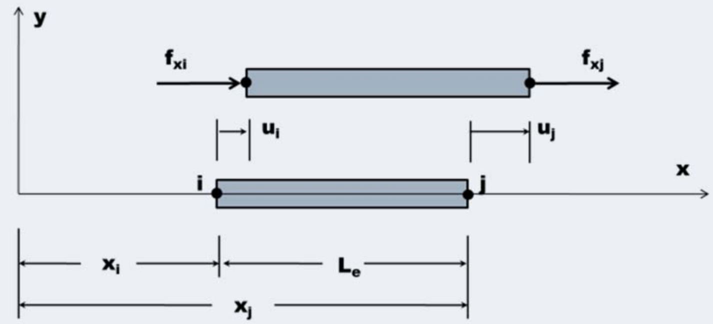
$$N_i(x) = \frac{x_j - x}{L_e} \qquad N_j(x) = \frac{x - x_i}{L_e}$$

Shape Functions or Interpolation Functions

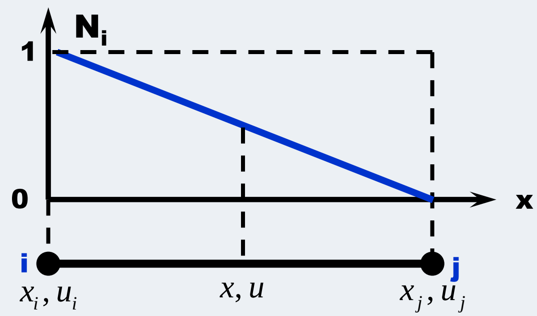
Shape Functions Critical To Element Development

$$u(x) = \frac{x_j - x}{L_e} \cdot u_i + \frac{x - x_i}{L_e} \cdot u_j$$

$$\boxed{= N_i(x) \cdot u_i + N_j(x) \cdot u_j}$$ 4



$$N_i(x) = \frac{x_j - x}{L_e}$$

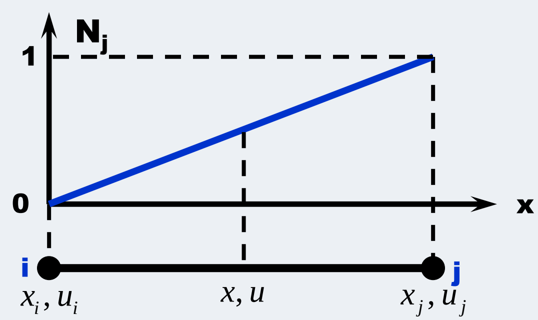


$$x = x_i \Rightarrow N_i = 1$$

$$x = x_j \Rightarrow N_i = 0$$

$$\boxed{N_i + N_j = 1 \text{ for all values of x}}$$

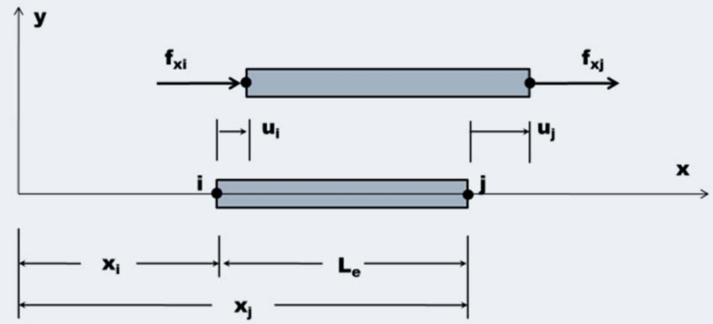
$$N_j(x) = \frac{x - x_i}{L_e}$$



$$x = x_i \Rightarrow N_j = 0$$

$$x = x_j \Rightarrow N_i = 1$$

From Elasticity's Strain-Displacement Relationship ε_x



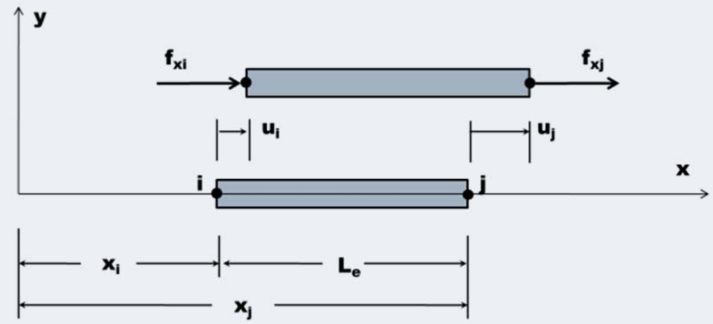
$$\varepsilon(x) = \partial u / \partial x$$ **Substituting in 4**

$$= \frac{\partial}{\partial x}\left[\frac{x_j - x}{L_e}\cdot u_i + \frac{x - x_i}{L_e}\cdot u_j\right]$$

$$= \frac{-u_i + u_j}{L_e} = \frac{1}{L_e}\left(u_j - u_i\right) = \underbrace{\frac{1}{L_e}\{-1 \quad 1\}}_{\text{[B] matrix}}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e \quad \textbf{5}$$

$$[\mathbf{B}]_e = \frac{1}{L_e}\{-1 \quad 1\}$$

From Elasticity's Constitutive Relationship σ_x



$$\sigma(x) = E_e \cdot \varepsilon_x$$

$E_e \equiv$ Modulus of Elasticity of the Element

Substituting in 5

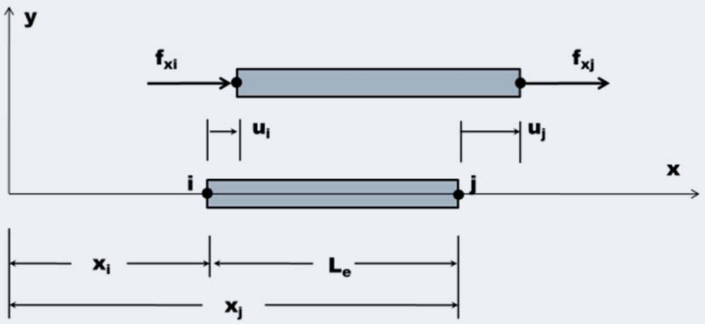
$$(\boldsymbol{\sigma}_x)_e = \frac{E_e}{L_e} \cdot \{-1 \quad 1\} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e$$

$$= \left(\frac{E}{L}\right)_e \cdot \{-1 \quad 1\} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e \qquad 6$$

Stress Multiplied by Area gives Nodal Forces

$$f_{xi} = -\left(\sigma_x \cdot A\right)_e \qquad f_{xj} = \left(\sigma_x \cdot A\right)_e$$

The sign comes from the Convention that on a negative surface a tensile stress (and the force associated with it) will be directed in the negative coordinate direction



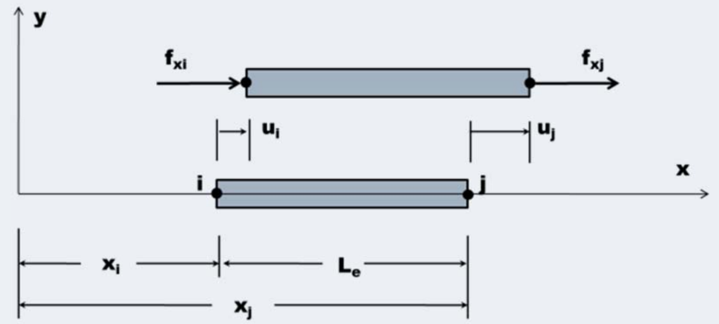
Substituting in 6

$$\begin{Bmatrix} f_{xi} \\ f_j \end{Bmatrix}_e = A_e \cdot \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cdot (\sigma_x)_e = A_e \cdot \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cdot \frac{E_e}{L_e} \cdot \begin{Bmatrix} -1 & 1 \end{Bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e$$

$$= \frac{A_e \cdot E_e}{L_e} \cdot \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{Bmatrix} -1 & 1 \end{Bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e \qquad = \frac{A_e \cdot E_e}{L_e} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e \qquad 7$$

The Stiffness Matrix Relates Forces and Displacements

$$\boxed{\begin{Bmatrix} f_{xi} \\ f_j \end{Bmatrix}_e = \underbrace{\frac{A_e \cdot E_e}{L_e} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_e} \quad 7$$



$[k]_e \equiv$ Stiffness Matrix

$$\{\mathbf{k}\}_e = \frac{A_e \cdot E_e}{L_e} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left(\frac{A \cdot E}{L}\right)_e \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k_e \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_e & -k_e \\ -k_e & k_e \end{bmatrix} \quad 8$$

A Summary of the Direct Stiffness Steps

1. Define
   a. the displacements and loads associated with the degrees of freedom of the nodes
   b. the geometry and material properties of the elements

2. Assume a deflection field of the element
   a. as many unknowns as degrees of freedom

3. Determine the unknown constants in terms of nodal position and displacements

4. Differentiate the displacement field equations to obtain the strain as a function of nodal displacements

5. Substitute the stress-strain relations to write the stress as a function of nodal Displacement