PROBLEM 2.11 | RELATIVE TO AN XYZ COORDINATE SYSTEM, THE STATE OF STRESS AT A POINT IS KNOWN TO BE

$$[\sigma] = \begin{bmatrix} -10 & 20 & 30 \\ 20 & 10 & -20 \\ 30 & -20 & 40 \end{bmatrix} \text{MPa} \qquad (1)$$

(a) EVALUATE THE NORMAL AND SHEAR STRESSES ON A SURFACE AT THE POINT WHERE THE SURFACE IS GIVEN BY THE THREE POINTS (1, 0, 0), (0, 2, 0), (0, 0, -1)

(b) DETERMINE THE DIRECTIONAL COSINES FOR THE SHEAR STRESS FOUND IN PART (a) AND IN A ROUGH SKETCH SHOW THE DIRECTION OF THE NORMAL AND SHEAR STRESSES.

GIVEN:

CONSTRAINTS

1. THE STATE OF STRESS AT A POINT IN A STRUCTURE IS DEFINED BY (1)
2. THE PLANE OF INTEREST IS DEFINED BY (1,0,0), (0,2,0), (0,0,-1)

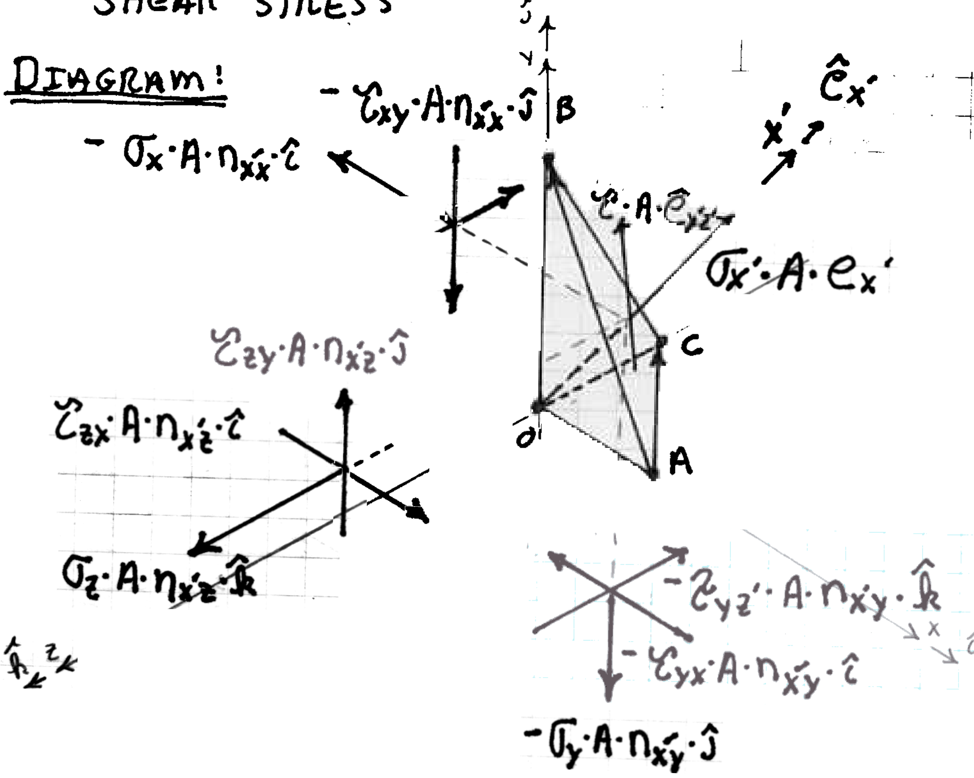
ASSUMPTIONS

1. THE STRESS IS AT A SINGLE POINT.
2. THE STRESS TENSOR IS CARTESIAN.

FIND:

1. CALCULATE THE NORMAL AND SHEAR STRESS ON THE SURFACE
2. DETERMINE THE DIRECTION COSINES FOR THE NORMAL AND SHEAR STRESS

DIAGRAM:



$$A_x = n_{x'x} \cdot A$$
$$A_y = n_{x'y} \cdot A \qquad (A)$$
$$A_z = n_{x'z} \cdot A$$

SOLUTION:

TO FIND THE NORMAL TO THE SURFACE "ABC", THE THREE POINTS

$A: (1,0,0);\ B(0,2,0);\ C(0,0,-1)$

ARE USED TO FORM TWO POSITION VECTORS

$$\vec{r}_{AC} = (0-1)\hat{\imath} + (0-0)\hat{\jmath} + (1-0)\hat{k} = -\hat{\imath} - \hat{k} \qquad (2)$$

$$\vec{r}_{AB} = (0-1)\hat{\imath} + (2-0)\hat{\jmath} + (0-0)\hat{k} = -\hat{\imath} + 2\cdot\hat{\jmath} \qquad (3)$$

TAKING THE CROSS PRODUCT OF (2) AND (3) YIELDS A VECTOR PERPENDICULAR TO THE SURFACE NORMAL

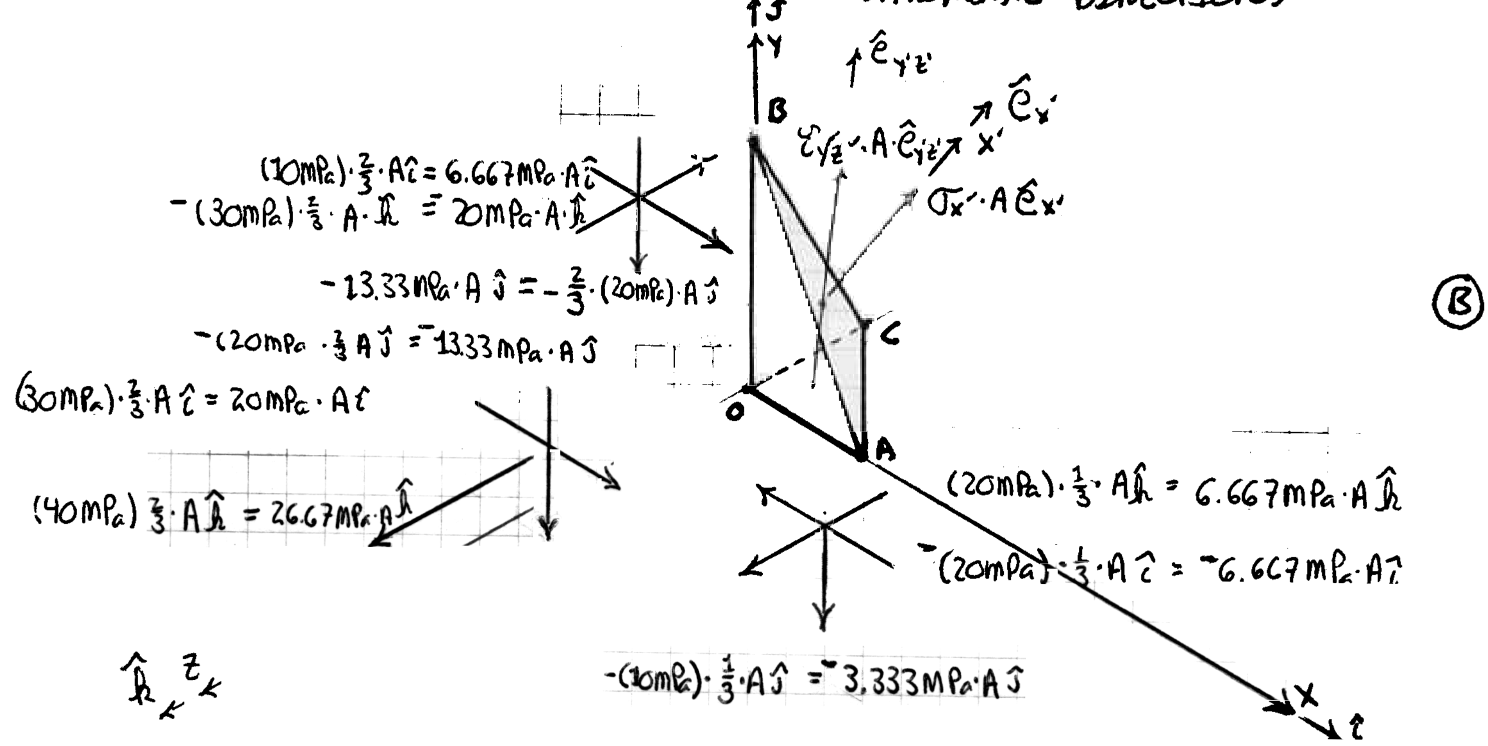
$$\vec{r}_{x'} = \vec{r}_{AC} \times \vec{r}_{AB} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & -1 \\ -1 & 2 & 0 \end{vmatrix} = 2\cdot\hat{\imath} + \hat{\jmath} - 2\cdot\hat{k}$$

$$= \sqrt{9}\left(\frac{2}{\sqrt{9}}\cdot\hat{\imath} - \frac{1}{\sqrt{9}}\hat{\jmath} - \frac{2}{\sqrt{9}}\hat{k}\right) = 3\left(\frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}\right) = r_{x'}\cdot\hat{e}_{x'}$$

THE UNIT VECTOR $\hat{e}_{x'}$ DEFINES THE DIRECTION OF THE SURFACE NORMAL

$$\boxed{\hat{e}_{x'} = \frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}} = n_{x'x}\,\hat{\imath} + n_{x'y}\,\hat{\jmath} - n_{x'z}\,\hat{k} \qquad (4)$$

NOW (A) IS DRAWN WITH THE STRESSES IN APPROPRIATE DIRECTIONS



(B)

CONSIDERING THE EQUILIBRIUM OF ELEMENT "ABCO"

$$\Sigma \vec{F} = \vec{0} = \sigma_{x'} \cdot A \cdot \hat{e}_{x'} + \tau_{y'z'} \cdot A \cdot \hat{e}_{y'z'} + (6.667\,\text{MPa} + 20\,\text{MPa} - 6.667\,\text{MPa}) \cdot A \cdot \hat{i}$$
$$+ A \cdot (-13.33\,\text{MPa} - 13.33\,\text{MPa} - 3.333\,\text{MPa})\hat{j} + (-20\,\text{MPa} + 26.67\,\text{MPa} + 6.67\,\text{MPa}) A \cdot \hat{k}$$

$$= \sigma_{x'} \cdot \hat{e}_{x'} + \tau_{y'z'} \cdot \hat{e}_{y'z'} + 20\,\text{MPa}\,\hat{i} - 30\,\text{MPa}\,\hat{j} + 13.34\,\text{MPa}\,\hat{k}$$

$$\sigma_{x'} \hat{e}_{x'} + \tau_{y'z'} \cdot \hat{e}_{y'z'} = -20\,\text{MPa}\,\hat{i} + 30\,\text{MPa}\,\hat{j} - 13.34\,\text{MPa}\,\hat{k} \qquad (5)$$

DOTTING (5) WITH THE UNIT VECTOR NORMAL TO THE SURFACE $\hat{e}_{x'}$, (4)

$$(\sigma_{x'} \cdot \hat{e}_{x'} + \tau_{y'z'} \cdot \hat{e}_{y'z'}) \circ \hat{e}_{x'} = (-20\,\text{MPa}\,\hat{i} + 30\,\text{MPa}\,\hat{j} + 13.34\,\text{MPa}\,\hat{k}) \circ (\tfrac{2}{3}\hat{i} + \tfrac{1}{3}\hat{j} - \tfrac{2}{3}\hat{k})$$

$$\sigma_{x'} \cdot \underbrace{\hat{e}_{x'} \circ \hat{e}_{x'}}_{1} + \tau_{y'z'} \cdot \underbrace{\hat{e}_{y'z'} \circ \hat{e}_{x'}}_{0} = (-20\,\text{MPa})(\tfrac{2}{3}) + (30\,\text{MPa})(+\tfrac{1}{3}) + (-13.34\,\text{MPa})(-\tfrac{2}{3})$$

$$\sigma_{x'} = 5.556\,\text{MPa} = \boxed{5.56\,\text{MPa}}$$

NOW (5) CAN BE SOLVED FOR $\tau_{y'z'} \cdot \hat{e}_{y'z'}$

$$\tau_{y'z'} \cdot \hat{e}_{y'z'} = (-20\,\text{MPa} \cdot \hat{i} + 30\,\text{MPa} \cdot \hat{j} - 13.34\,\text{MPa}\,\hat{k}) - \sigma_{x'} \cdot \hat{e}_{x'}$$
$$= (-20\,\text{MPa}\,\hat{i} + 30\,\text{MPa}\,\hat{j} - 13.34\,\text{MPa}\,\hat{k}) - 5.556\,\text{MPa}\,(\tfrac{2}{3}\hat{i} + \tfrac{1}{3}\hat{j} - \tfrac{2}{3}\hat{k})$$
$$= (-20\,\text{MPa}\,\hat{i} + 30\,\text{MPa}\,\hat{j} - 13.34\,\text{MPa}\,\hat{k}) - 3.704\,\text{MPa}\,\hat{i} - 1.852\,\text{MPa}\,\hat{j} + \cancel{4.07}\ 3.704\,\text{MPa}\,\hat{k}$$
$$= -23.70\,\text{MPa}\,\hat{i} + 28.15\,\text{MPa}\,\hat{j} - 9.636\,\text{MPa}\,\hat{k}$$
$$= 38.04\,\text{MPa} \cdot (-.6230\hat{i} + .7400\hat{j} - 2.533\hat{k})$$

$$\boxed{\tau_{y'z'} = 38.0\,\text{MPa}}$$

$$\boxed{\hat{e}_{y'z'} = -.6230\hat{i} + .7400\hat{j} - 2.533\hat{k}}$$

SUMMARY:

THE SOLUTION IS OBTAINED THROUGH A CAREFUL APPLICATION OF EQUILIBRIUM OF THE FORCES THAT RESULT FROM INTEGRATING THE STRESSES OVER THE SURFACES THAT THEY ACT UPON.