Homework 1

Necessary reading: Chapter 1 until Postulate 4

Exercise 1 - Basic Quantum Mechanics

The state of a quantum system can be written in the bra-ket notation. In this notation, a column vector is written as a Ket, $|a\rangle$. Its conjugate transpose is called a Bra, $\langle a|$:

$$\langle a| = |a\rangle^\dagger .$$

The n-th unit vector is shortened by the Ket

$$|n-1\rangle = \vec{e}_n.$$

Using these unit vectors, the state of a qubit can be written as

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

This can also be represented by means of a column vector

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

In quantum information processing, it is common to sometimes change the measurement basis. One of these common basis is the $\{|+\rangle, |-\rangle\}$ basis, where

$$|+\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

and

$$|-\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right).$$

Task a)

Consider the pure state formulation of quantum mechanics. If we measure in the base

$$\{|0\rangle, |1\rangle\}$$

(often also called the “computational base”), what are the probabilities of getting the measurement outcomes 0 and 1 for a qubit described by $|\Psi\rangle$?

Task b)

Write the state $|\Psi\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis.

Exercice 1.

task a) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

the probability of measuring $0 \to P_0 = |\langle 0|\psi\rangle|^2 = |\alpha|^2$

$1 \to P_1 = |\langle 1|\psi\rangle|^2 = |\beta|^2$

task b)

$|0\rangle = \dfrac{|+\rangle + |-\rangle}{\sqrt{2}} \quad \Rightarrow |\psi\rangle = \dfrac{\alpha}{\sqrt{2}}(|+\rangle + |-\rangle) + \dfrac{\beta}{\sqrt{2}}(|+\rangle - |-\rangle)$

$|1\rangle = \dfrac{|+\rangle - |-\rangle}{\sqrt{2}} \quad = \dfrac{\alpha+\beta}{\sqrt{2}}|+\rangle + \dfrac{\alpha-\beta}{\sqrt{2}}|-\rangle$

task c:

$|\psi\rangle_A = \alpha_A|0\rangle + \beta_A|1\rangle$

$|\psi\rangle_B = \alpha_B|0\rangle + \beta_B|1\rangle$

$$|\psi\rangle_A \otimes |\psi\rangle_B = \alpha_A\cdot\alpha_B|00\rangle + \alpha_A\beta_B|01\rangle + \beta_A\alpha_B|10\rangle + \beta_A\beta_B|11\rangle = \begin{pmatrix}\alpha_A\alpha_B\\ \alpha_A\beta_B\\ \beta_A\alpha_B\\ \beta_A\beta_B\end{pmatrix}$$

$|00\rangle = |0\rangle\otimes|0\rangle$

$e_i \otimes e_j = e_{2i+j+1}$

$$\begin{cases} |00\rangle = e_1 \\ |01\rangle = e_2 \\ |10\rangle = e_3 \\ |11\rangle = e_4 \end{cases} \qquad |\psi\rangle_A \otimes |\psi\rangle_B = \alpha_A\alpha_B\cdot e_1 + \alpha_A\beta_B e_2 + \beta_A\alpha_B e_3 + \beta_A\beta_B e_4$$

task d

$\hat{M}$ – measurement operator

$$\hat{M} = \hat{M}_A \otimes \hat{M}_B = \begin{pmatrix} m_{A00} & m_{A01} \\ m_{A10} & m_{A11}\end{pmatrix} \otimes \begin{pmatrix} m_{B00} & m_{B01} \\ m_{B10} & m_{B11}\end{pmatrix} =$$

probability of (i,j) outcome:

$$\langle i,j|M|i,j\rangle = \langle i|M_A|i\rangle \otimes \langle j|M_B|j\rangle = M_{A_{ii}} \otimes M_{B_{jj}}$$

Task c)

Suppose we have two qubits, $|\Psi\rangle_A = \alpha_A |0\rangle + \beta_A |1\rangle$ and $|\Psi\rangle_B = \alpha_B |0\rangle + \beta_B |1\rangle$. Expand the Kronecker product $|\Psi\rangle_A \otimes |\Psi\rangle_B$ into the basis $\{|i\rangle \otimes |j\rangle\}_{i,j=0}^{1}$. Then, write down $|\Psi\rangle_A \otimes |\Psi\rangle_B$ using vector notation in the basis $\{e_i \otimes e_j\}_{i,j=0}^{1}$ and finally as a column vector using the map

$$e_i \otimes e_j \to e_{2\cdot i+j+1} \tag{1}$$

from $\mathbb{C}^2 \otimes \mathbb{C}^2$ to $\mathbb{C}^4$ (compare to equation (1.70) in the lecture notes).

Task d)

Assume you are performing measurements in the $\{|0\rangle, |1\rangle\}$ basis using two devices where the first acts on Ψ_A and the second on Ψ_B. Give the joint probability distribution for your measurement outcomes (i, j). Afterwards, calculate the conditional probability for outcomes of your measurements on state Ψ_B given your measurement outcomes from measurements on state Ψ_A. Are these qubits entangled? How can you test for entanglement, given the probabilities for the single qubits and their conditional probabilities?

Exercise 2 - Density Operator

Another possibility to represent a quantum system are density operators or also called density matrices. A density operator can also represent mixed states. From a state vector $|\Psi\rangle$, the corresponding density matrix is given by

$$\hat{\rho} = |\Psi\rangle \langle\Psi| .$$

Task a)

Explain the difference between the density matrices

$$\hat{\rho}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{\rho}_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find orthogonal state vectors $|v\rangle, |w\rangle$ and numbers $\mu, \nu \in [0, 1]$ such that

$$\hat{\rho}_1 = \mu|v\rangle\langle v| + (1-\mu)|w\rangle\langle w|$$
$$\hat{\rho}_2 = \nu|v\rangle\langle v| + (1-\nu)|w\rangle\langle w|$$

Task b)

Assume you have two qubits in two systems, A and B, in a joint state. Let their state be described by the state vector $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$. Write down the corresponding density operator $\hat{\rho}_{AB}$ describing this composite system.

Exercise 2 Density operator (task a)

$\hat{\rho}_1 = 1|v\rangle\langle v| + (1-1)|w\rangle\langle w|$

$\mu = 1$ and $|v\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\hat{\rho}_2 = \frac{1}{2}|v\rangle\langle v| + \frac{1}{2}|w\rangle\langle w|$; $\nu = \frac{1}{2}$ and $|w\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\frac{1}{4}\cdot\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{4}\cdot\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Task b : $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right).$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\cdot\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{1}{2}\cdot\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Task c :

$$tr_A(\hat{\rho}_{AB}) = \sum_{i=0}^{1} \langle i|_A \hat{\rho}_{AB} |i\rangle_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $tr\left[\left(tr_A(\hat{\rho}_{AB})\right)^2\right] = \frac{1}{2} < 1 \Rightarrow$ it is a mixed state

Task c)

What is the partial trace over the qubit A of the density operator from task b) i.e. what is $\mathrm{tr}_A(\hat{\rho}_{AB})$? Is $\mathrm{tr}_A(\hat{\rho}_{AB})$ a pure or mixed quantum state?

Exercise 3 - QuNetSim

Before starting the task, make sure you have configured the environment. It is recommended that you use virtual environment (for example `conda` or `virtualenv`). Then install *QuNetSim* simply by running `pip install qunetsim`.

In this exercise you will learn the basics of the *QuNetSim* framework, like configuring the network, writing and running a protocol, sending and receiving quantum as well as classical messages. Your task is to write two network protocols with the*QuNetSim* framework. One protocol should be written for the *sender* and another for *receiver*. Use the provided template and fill in the missing parts.

Important:

- Only python (`*.py`) files will be accepted. Please do not submit jupyter notebook (`*.ipynb`) files.
- Before submission, make sure that the code is error free.
- Due to the *QuNetSim* package update in progress, there is a slight mismatch in the documentation and version provided by pip. In such cases, the code is given in the comments in the template file. If in doubt, please contact us.