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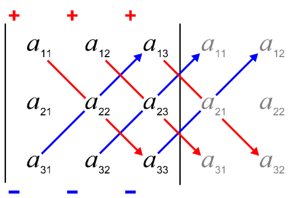
Exercise 2.1 (Spectral decomposition)

(a) Compute the characteristic polynomial and spectral decomposition of the normal matrix

$$A = \begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix}.$$

Hint: The following "rule of Sarrus" might be helpful for calculating the determinant of a 3×3 matrix:



Source: Wikipedia

(b) Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix, and $\{\lambda_1, \dots, \lambda_n\}$ the eigenvalues of A. Show that

$$\mathrm{tr}[A] = \sum_{j=1}^{n} \lambda_j,$$

where $\mathrm{tr}[\cdot]$ denotes the matrix trace (sum of its diagonal entries).

Hint: Start from the spectral decomposition of A, and use that $\mathrm{tr}[AB] = \mathrm{tr}[BA]$ for any square matrices A and B.

$$\det\left[\begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix} - \lambda \mathbb{1}\right] = \begin{vmatrix} -\lambda & 3/5 & 4/5 \\ -3/5 & -\lambda & 0 \\ -4/5 & 0 & -\lambda \end{vmatrix} = -\lambda^3 - \left[+\frac{16}{25}\lambda + \frac{9}{25}\lambda\right] = 0$$

$$-\lambda^3 - \left[\frac{25}{25}\lambda\right] = 0$$

$$-\lambda^3 - \lambda = 0$$

$$\lambda^3 + \lambda = 0$$

$$\lambda(\lambda^2 + 1) = 0$$

eigenvalues: $\lambda = \{0; \pm i\}$

eigenvectors:
$A \cdot v = \lambda \cdot v$
$(A - \lambda) \cdot v = 0$

$$\lambda = 0 \to \begin{pmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \frac{3}{5}x_2 = -\frac{4}{5}x_3$$

$x_2 = -\frac{4}{3}x_3$

$x_1 = 0$

$$v_1 = \begin{pmatrix} 0 \\ -4/3 \\ 1 \end{pmatrix}$$

$$\lambda = i \Rightarrow \begin{pmatrix} -i & 3/5 & 4/5 \\ -3/5 & -i & 0 \\ -4/5 & 0 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad -ix_1 + \frac{3}{5}x_2 + \frac{4}{5}x_3 = 0$$

(1) $-5x_1 + 3x_2 + 4x_3 = 0$

(2) $-3/5\, x_1 - ix_2 = 0 \to x_1 = \frac{-ix_2}{3} \cdot 5$

(3) $-4/5\, x_1 - ix_3 = 0 \to -\frac{4}{\cancel{5}} \cdot \frac{-ix_2}{3}\cancel{5} - ix_3 = 0$

$+4/3\, i\, x_2 - i x_3 = 0$

$\underline{x_3 = 4/3\, x_2 \ (4)}$

$x_2 = 1$

$x_1 = -\frac{5i}{3} \Rightarrow v_2 = \begin{pmatrix} -5i/3 \\ 1 \\ 4/2 \end{pmatrix}$

$x_3 = 4/3$

$$\lambda = -i \Rightarrow \begin{pmatrix} i & 3/5 & 4/5 \\ -3/5 & i & 0 \\ -4/5 & 0 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$ix_1 + 3/5\, x_2 + 4/5\, x_3 = 0$

$-3/5\, x_1 + i x_2 = 0$

$-4/5\, x_1 + x_3 i = 0$

$$v_3 = \begin{pmatrix} -i\frac{5}{3} \\ 1 \\ -i\frac{4}{3} \end{pmatrix}$$

$x_2 = 1 \Rightarrow x_1 = \frac{-i \cdot 5}{3}$

$x_3 = \frac{4}{\cancel{5}} \cdot \frac{\cancel{5}}{3} \cdot -i = -\frac{4}{3}i$

Spectral decomposition.

$$A = P \cdot \Delta \cdot P^T$$

$$P = \begin{pmatrix} 0 & -i5/3 & -i5/3 \\ -4/3 & 1 & 1 \\ 1 & 4/3 & -i4/3 \end{pmatrix} ; \quad \Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i5/3 & -i5/3 \\ -4/3 & 1 & 1 \\ 1 & 4/3 & -i4/3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -4/3 & 1 \\ -i5/3 & 1 & 4/3 \\ -i5/3 & 1 & -i4/3 \end{pmatrix}$$

Ex 2. $\operatorname{tr}[A] = \sum_{j=1}^{n} \lambda_j,$

$$A = P \Delta P^T \Rightarrow \operatorname{tr}[A] = \operatorname{tr}[P \Delta P^+] = \operatorname{tr}[\underbrace{P^+ P}_{\mathbb{1}} \Delta] = \operatorname{tr}[\mathbb{1} \Delta] = \sum_j \lambda_j$$