Ex. 1.2

ⓐ.

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix} = \begin{pmatrix} 2\cdot 4 - i\cdot i - 3\cdot 5 \\ 3\cdot 4 + 0\cdot i - 1\cdot 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix} = \begin{pmatrix} -10+42i & 8 \\ 15+6i-12 & -12 \end{pmatrix} = \begin{pmatrix} -10+42i & 8 \\ 3+6i & -12 \end{pmatrix}$$

ⓑ. $A\cdot A^{+} \neq A^{+}\cdot A \Rightarrow A$ is not normal.

$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ $\quad A\cdot A^{+} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ $\quad A\cdot A^{+} \neq A^{+}A \Rightarrow \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ not normal

$A^{+} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ $\quad A^{+}\cdot A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

ⓒ. unitary $\Rightarrow U\cdot U^{+} = \mathbb{1}$

$$\begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta)+\sin^2(\theta) & -i\cos(\theta)\sin(\theta) + i\sin(\theta)\cdot\cos(\theta) \\ +i\sin(\theta)\cos(\theta) - i\sin(\theta)\cos(\theta) & \cos^2(\theta)+\sin^2(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \square$$

ⓓ. (i) def of det: $\boxed{\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j\sigma(j)}}$

$$\det(A^*) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a^*_{j,\sigma(j)}$$

$$\left.\begin{aligned} \det(A)^* &= \left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cdot \prod_{j=1}^{n} a_{j\sigma(j)}\right)^* \\ \operatorname{sgn}(\sigma)^* &= \operatorname{sgn}(\sigma) \end{aligned}\right\} \Rightarrow \det(A)^* = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a^*_{j\sigma(j)} = \det(A^*)$$

(ii) U - unitary $\Rightarrow U\cdot U^{+} = \mathbb{1}$

$$|\det(U)| = \mathbb{1} = \det(U\cdot U^{+}) = \det(U)\cdot\det(U^{+}) = \det(U)\cdot\det(U)^{+} = \det(U)\cdot\det(U)^{*} = |\det(U)|$$