Tutorial 1:

a) $(U_F U_F^\dagger)_{kl} = \sum_j u_{kj} u_{lj}^*$

$= \sum_j \frac{1}{N} e^{2\pi i (kj - lj)/N}$

$= \frac{1}{N} \sum_j e^{2\pi j i (k-l)/N}$

$\xrightarrow{k=l} = \frac{1}{N} \cdot N = 1$

$\xrightarrow{k \neq l} = \frac{1}{N} \cdot \frac{1 - e^{2\pi i (k-l)}}{1 - e^{2\pi i (k-l)/N}} = \frac{1}{N} \cdot \frac{0}{\underbrace{1 - e^{2\pi i (k-l)/N}}_{\neq 1}} = \underline{0}$

b) $(U_F^\dagger)_{kj} = \frac{1}{\sqrt{N}} e^{-2\pi i jk/N}$

c) $(F(x) \cdot F(y))_j = \left(\left(\sum_k \frac{1}{\sqrt{N}} e^{2\pi i jk/N} x_k\right) \cdot \left(\sum_{k'} \frac{1}{\sqrt{N}} e^{2\pi i jk'/N} \cdot y_{k'}\right)\right)$

$= \frac{1}{N} \sum_k \sum_{k'} e^{2\pi i j(k+k')/N} x_k y_{k'}$

$(\sqrt{N} F^{-1}(F(x) \cdot F(y)))_j$

$= \sqrt{N} \cdot \sum_l \frac{1}{\sqrt{N}} e^{-2\pi i l j/N} \cdot \frac{1}{N} \sum_k \sum_{k'} e^{2\pi i l (k+k')/N} x_k y_{k'}$

$= \frac{1}{N} \sum_l \sum_k \sum_{k'} e^{\frac{2\pi i}{N} l (k+k'-j)} \cdot x_k y_{k'}$

$= \sum_k \sum_{k'} x_k y_{k'} \cdot \delta_{k', j-k}$

$= \sum_k x_k \, y_{(j-k) \bmod N}$

(1) faster than matrix representation $> O(n)$ — $O(n)$

Ex. 1.1: a) $\int_0^1 e^{2\pi i n t} dt = \left[\frac{1}{2\pi i n} e^{2\pi i n t}\right]_0^1 = \frac{1}{2\pi i n}\left(e^{2\pi i n} - 1\right)$

$= \frac{i}{2\pi n}\left(1 - e^{2\pi i n}\right)$

$\xrightarrow{n=0} = \frac{i(-2\pi i e^{2\pi i 0})}{2\pi} = \frac{2\pi}{2\pi} = 1$ //

$\xrightarrow{n \neq 0} \frac{1}{2\pi i n}(1-1) = 0$ //

$\int_0^1 \sum_{k'} y_{k'} e^{-2\pi i k' t} \cdot e^{2\pi i k t} dt$

$= \int_0^1 \sum_{k'} y_{k'} e^{2\pi i (k-k')t} dt$

$= \sum_{k'} y_{k'} \int_0^1 e^{2\pi i (k-k')t} dt$

$= \sum_{k'} y_{k'} \delta_{k-k', 0}$

$= \sum_{k'} y_{k'} \delta_{k,k'}$

$= \underline{\underline{y_k}}$

d) $y_0 = \frac{1}{4}$

$\Big|_{k \text{ odd}} y_k = -\frac{1}{(\pi k)^2}$

$f(0) = 0 = \sum_{k \in \mathbb{Z}} y_k$

$= y_0 + 2 \cdot \sum_{\substack{k=1 \\ k \text{ odd}}} -\frac{1}{(\pi k)^2}$

$\frac{y_0}{2} = \sum_{\substack{k=1 \\ k \text{ odd}}} \frac{1}{(\pi k)^2}$

$\frac{y_0 \pi^2}{2} = \sum_{\substack{k=1 \\ k \text{ odd}}} \frac{1}{k^2}$

$\frac{\pi^2}{8} = \sum_{\substack{k=1 \\ \text{odd}}} \frac{1}{k^2}$ //

b) $z_k = \int_0^1 (f * g)(t) \, e^{2\pi i k t} dt$

$= \int_0^1 \left(\int_0^1 f(s) g(t-s) ds\right) e^{2\pi i k t} dt$

$= \int_0^1 f(s) \left(\int_0^1 g(t-s) e^{2\pi i k t} dt\right) ds$

$= \int_0^1 f(s) \left(\int_0^1 g(t-s) e^{2\pi i k (t-s)} dt\right) \cdot e^{2\pi i k s} ds$

$= \int_0^1 f(s) \left(\int_0^1 g(t) e^{2\pi i k t} dt\right) e^{2\pi i k s} ds$

$= \int_0^1 f(s) \cdot \tilde{y}_k \cdot e^{2\pi i k s} ds$

$= \tilde{y}_k \int_0^1 f(s) \cdot e^{2\pi i k s} ds$

$= \underline{\tilde{y}_k \cdot y_k}$

c) $y_k = x_k \cdot x_k$ — for g

$x_k = \int_0^1 g(t) \, e^{2\pi i k t} dt \longrightarrow y_k = \begin{cases} 1/4 & ; k=0 \\ -\frac{1}{(\pi k)^2} & ; k = 2n+1 \\ 0 & ; k \neq 0 ; k = 2n \end{cases}$

$= \int_0^{1/2} e^{2\pi i k t} dt$

$= \left[\frac{1}{2\pi i k} e^{2\pi i k t}\right]_0^{1/2}$

$= \frac{1}{2\pi i k}\left[e^{\pi i k} - 1\right]$

$\xrightarrow{k=2n;\, k \neq 0} = 0$

$\xrightarrow{k=2n+1} = \frac{-2}{2\pi i k} = \frac{i}{\pi k}$

$\xrightarrow{k=0} = \frac{\pi i}{2\pi i} = \frac{1}{2}$

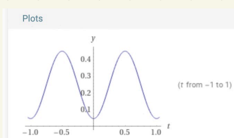
$\rightarrow S_{1,f}(t) = y_{-1} e^{+2\pi i t} + y_0 + y_1 e^{-2\pi i t}$

$= -\frac{1}{\pi^2} e^{2\pi i t} + \frac{1}{4} - \frac{1}{\pi^2} e^{-2\pi i t}$

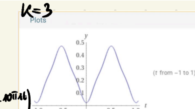
$\rightarrow S_{3,f}(t) = \frac{1}{4} - \frac{1}{\pi^2}\left(e^{2\pi i t} + e^{-2\pi i t}\right) - \frac{1}{9\pi^2}\left(e^{6\pi i t} + e^{-6\pi i t}\right)$

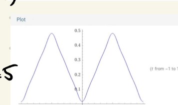
$\rightarrow S_{5,f}(t) = \frac{1}{4} - \frac{1}{\pi^2}\left(e^{2\pi i t} + e^{-2\pi i t}\right) - \frac{1}{9\pi^2}\left(e^{6\pi i t} + e^{-6\pi i t}\right) - \frac{1}{25\pi^2}\left(e^{10\pi i t} + e^{-10\pi i t}\right)$

$K=1$



$K=3$





$K=5$