Hierarchical A Posteriori Error Estimation of Bank–Weiser Type in the FEniCS Project

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  - Properties
- Implementation
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- Perspectives

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Toy problem setting

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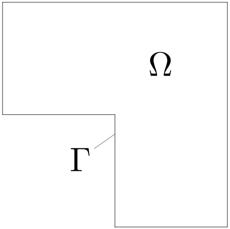
$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma.$$



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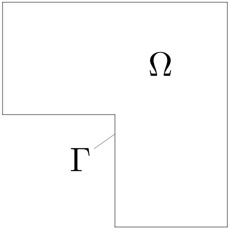
In weak formulation, find u in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega).$$



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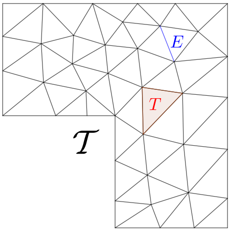

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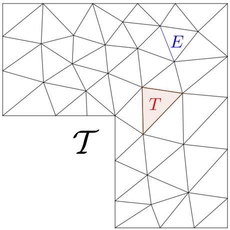

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Goal: estimate $\eta_{\text{err}} = \|\nabla(u_k - u)\|_\Omega$ i.e. find a computable quantity η_{bw} such that $\eta_{\text{bw}} \approx \eta_{\text{err}}$.

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The Bank–Weiser estimator

On a cell T, the Bank–Weiser problem is given by:
find e_T^{bw} in V_T^{bw} such that

$$\int_T \nabla e_T^{\mathrm{bw}} \cdot \nabla v_T^{\mathrm{bw}} = \int_T r_T v_T^{\mathrm{bw}} + \sum_{E \in \partial T} \frac{1}{2} \int_E J_E v_T^{\mathrm{bw}} \qquad \forall v_T^{\mathrm{bw}} \in V_T^{\mathrm{bw}}.$$

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The Bank–Weiser estimator is defined as

$$\eta_{\mathrm{bw}}^2 := \sum_{T \in \mathcal{T}} \eta_{\mathrm{bw},T}^2, \quad \eta_{\mathrm{bw},T} := \|\nabla e_T^{\mathrm{bw}}\|_T.$$

Definition of the Bank–Weiser estimator

The Bank–Weiser estimator

How is V_T^{bw} defined ?

Let $V_T^- \subsetneq V_T^+$ be two finite element spaces and

$$\mathcal{L}_T : V_T^+ \longrightarrow V_T^-,$$

be the local Lagrange interpolation operator,

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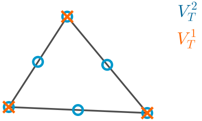

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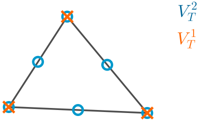

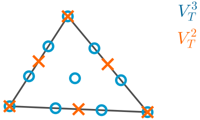

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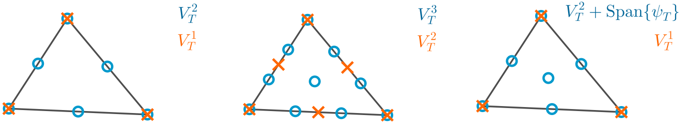

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Idea: we rely on the matrix A_T^+ and vector b_T^+ from

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since V_T^+ is provided by DOLFIN and we look for a matrix N such that:

$$A_T^{\mathrm{bw}} = N^{\mathsf{t}} A_T^+ N, \quad \text{and} \quad b_T^{\mathrm{bw}} = N^{\mathsf{t}} b_T^+.$$

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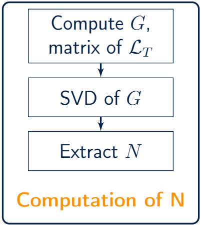

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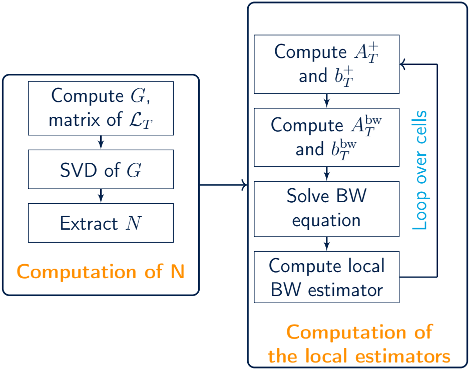

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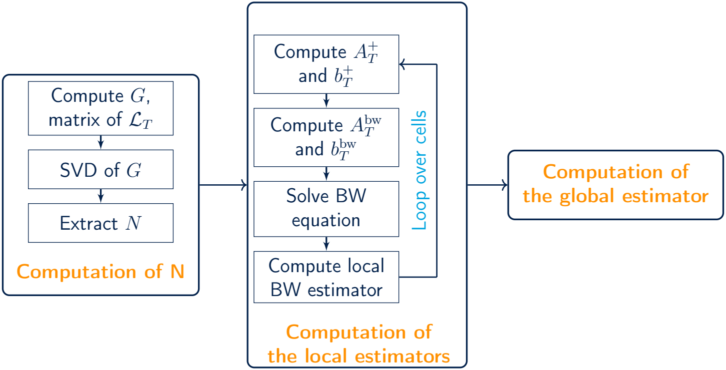

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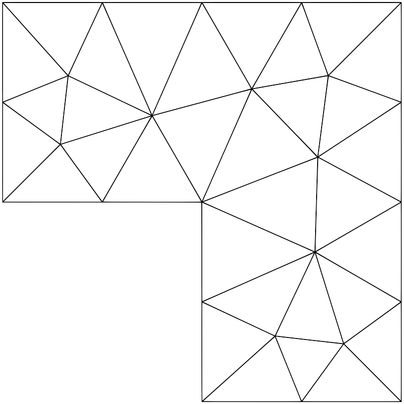

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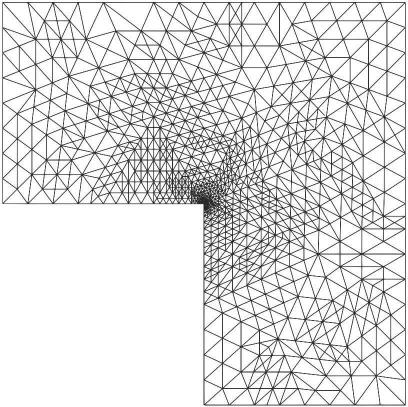

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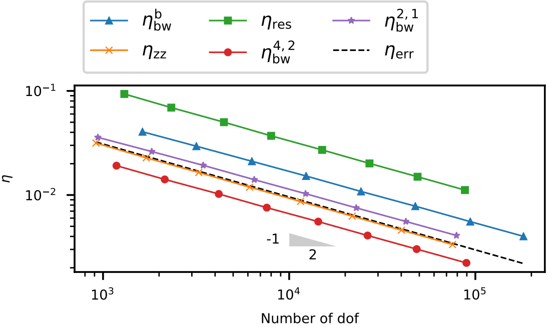


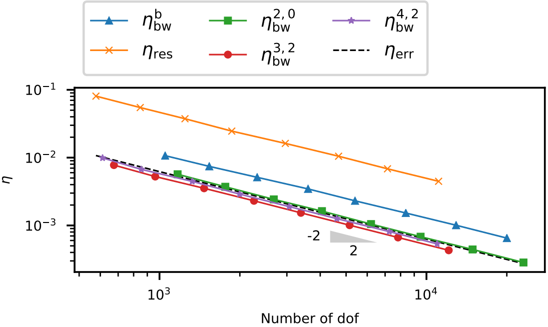
Notation	V_T^+	V_T^-
$\eta_{\mathrm{bw}}^{k_+,k_-}$	$V_T^{k_+}$	$V_T^{k_-}$
η_{bw}^{b}	$V_T^2 + \text{bubble}$	V_T^1

Implementation

Numerical results

Adaptive finite elements for a Poisson problem:

$-\Delta u = 0$ in Ω, $\quad u = u_D$ on Γ. $\quad$ Quadratic finite elements.



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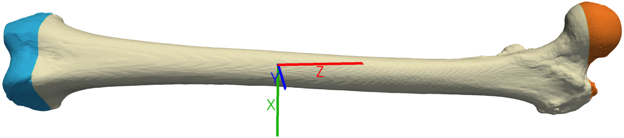
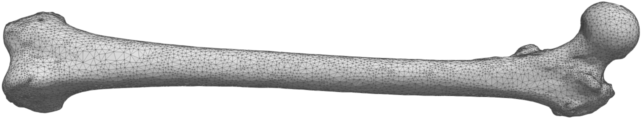
GO AFEM for a linear elasticity problem:

we used a technique from [Khan et al., 2019] to compute the estimators.

The goal functional is defined by $J(\mathbf{u_2}, p_1) := \int_\Gamma \mathbf{u_2} \cdot \mathbf{n} c$.





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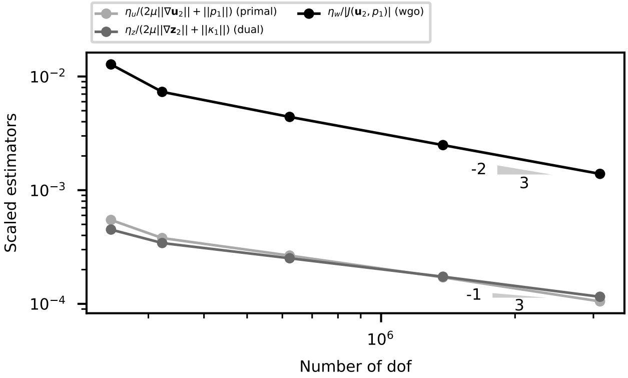

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- Definition of the Bank–Weiser estimator
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- Investigate performance of Bank–Weiser estimators for error estimation in L^2 norm.

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Thank you for your attention!

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