Benjamin-Ono stability

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1 Check the solution

We consider the Benjamin-Ono equation

$$u_t + \mathcal{H}u_{xx} + (u^2)_x = 0, \tag{1}$$

where

$$\mathcal{H}\left[f(\xi)\right](x) = \frac{1}{\pi}\int \frac{f(\xi)}{\xi - x}\,\mathrm{d}\xi. \tag{2}$$

Letting

$$u = \alpha\hat{u}, \qquad \hat{x} = \beta x, \qquad \hat{t} = \gamma t, \tag{3}$$

(1) becomes

$$\gamma\hat{u}_{\hat{t}} + \beta^2\mathcal{H}[\hat{u}_{\hat{x}\hat{x}}] + \alpha\beta(\hat{u}^2)_{\hat{x}} = 0. \tag{4}$$

Letting $(z,\tau) = (x - ct, t)$, so that z is in the traveling frame, (1) becomes

$$u_\tau - cu_z + \mathcal{H}u_{zz} + (u^2)_z = 0. \tag{5}$$

We look for stationary solutions in this traveling frame

$$-cu_z + \mathcal{H}u_{zz} + (u^2)_z = 0. \tag{6}$$

A 3-parameter family of periodic solutions to (6) is [1] given by

$$u(z;a,k,c) = -\frac{\frac{k^2}{\sqrt{c^2-4a-k^2}}}{\sqrt{\frac{c^2-4a}{c^2-4a-k^2}} - \cos(kz)} - \frac{1}{2}\left(\sqrt{c^2-4a} + c\right), \tag{7}$$

where $c < 0$ and $k^2 < c^2 - 4a$. We now verify that this is indeed a solution. We first note that [2]

$$\mathcal{H}\left(\frac{1}{1 - B\cos(D\zeta)}\right) = \mathrm{sgn}(D)\frac{B\sin(D\zeta)}{\sqrt{1-B^2}(1 - B\cos(D\zeta))}, \tag{8}$$

and our solution may be rewritten

$$u(z;a,k,c) = -\frac{k^2}{\sqrt{c^2-4a-k^2}}\sqrt{\frac{c^2-4a-k^2}{c^2-4a}}\frac{1}{1-\sqrt{\frac{c^2-4a-k^2}{c^2-4a}}\cos(kz)} + \frac{1}{2}\left(\sqrt{c^2-4a}+c\right) \tag{9}$$

$$= -\frac{k^2}{\sqrt{c^2-4a}}\frac{1}{1-\sqrt{\frac{c^2-4a-k^2}{c^2-4a}}\cos(kz)} + \frac{1}{2}\left(\sqrt{c^2-4a}+c\right) \tag{10}$$

$$= -\frac{k^2}{\alpha}\frac{1}{1-\frac{\beta}{\alpha}\cos(kz)} + \frac{1}{2}(\alpha+c) \tag{11}$$

$$= -\frac{k^2}{\alpha}\frac{1}{1-B\cos(k\zeta)} + \frac{1}{2}(\alpha+c). \tag{12}$$

We then note that

$$\mathcal{H}u_{zz} = \mathcal{H}\left(-\partial_z^2\left(\frac{k^2}{\alpha}\frac{1}{1-B\cos(k\zeta)} + \frac{1}{2}(\alpha+c)\right)\right) \tag{13}$$

$$= \mathcal{H}\left(-\partial_z^2\left(\frac{k^2}{\alpha}\frac{1}{1-B\cos(k\zeta)}\right)\right) \tag{14}$$

$$= -\frac{k^2}{\alpha}\partial_z^2\mathcal{H}\left(\frac{1}{1-B\cos(k\zeta)}\right), \tag{15}$$

since two derivatives commute with the Hilbert transform. Therefore

$$\mathcal{H}u_{zz} = -\frac{k^2}{\alpha}\partial_z^2 \operatorname{sgn}(k)\frac{B\sin(k\zeta)}{\sqrt{1-B^2}(1-B\cos(k\zeta))} \tag{16}$$

$$= -\frac{k^2}{\alpha}\partial_z^2\frac{\beta \operatorname{sgn}(k)\sin(k\zeta)}{\alpha\sqrt{1-\beta^2/\alpha^2}(1-\beta\cos(k\zeta)/\alpha)} \tag{17}$$

$$= -\partial_z^2\frac{k^2\beta \operatorname{sgn}(k)\sin(k\zeta)}{\alpha\sqrt{\alpha^2-\beta^2}(1-\beta\cos(k\zeta)/\alpha)}. \tag{18}$$

But

$$\alpha^2-\beta^2 = c^2-4a-(c^2-4a-k^2) = k^2, \tag{19}$$

so that

$$\mathcal{H}u_{zz} = -\partial_z^2\frac{k^2\beta \operatorname{sgn}(k)\sin(k\zeta)}{\alpha k(1-\beta\cos(k\zeta)/\alpha)} \tag{20}$$

$$= -\partial_z^2\frac{\beta k \operatorname{sgn}(k)\sin(k\zeta)}{\alpha-\beta\cos(k\zeta)}. \tag{21}$$

Then the Mathematica notebook `BronskiAndHurChecks-2.nb` shows that (12) is indeed a solution of (1) and (6).

2 Fourier multiplier of the Hilbert Transform

According to Wikipedia, the Fourier multiplier of $\mathcal{H}$ is $i\operatorname{sign}(\omega)$ (note that the sign of the denominator in $\mathcal{H}$ is different in their definition than ours), so that

$$\mathcal{F}[\mathcal{H}[u]](\omega) = i\operatorname{sign}(\omega)\mathcal{F}[u](\omega). \tag{22}$$

But Wikipedia blows! Simon says that with this definition of the Hilbert, transform,

$$\mathcal{F}[\mathcal{H}[u]](\omega) = -i\,\mathrm{sign}(\omega)\mathcal{F}[u](\omega), \tag{23}$$

which agrees with the plotting sort of checks that Ryan has completed.

3 Constant solution

We can find the spectrum for the constant solution analytically, so let's do that as a check for the numerics.

With $k = 0$, (7) becomes

$$u(z; a, 0, c) = -\frac{1}{2}(\sqrt{c^2 - 4a} + c) =: A. \tag{24}$$

We linearize about this solution by letting $u(z, \tau) = A + v(z, \tau)$ where $|v| \ll 1$. Plugging into (5) and retaining terms of order v and lower yields

$$\begin{aligned} 0 &= v_\tau - cv_z + \mathcal{H}v_{zz} + \partial_z(A + v)^2 &(25)\\ &= v_\tau - cv_z + \mathcal{H}v_{zz} + 2Av_z. &(26)\end{aligned}$$

Since the equation is autonomous first-order in t we let $v(z, \tau) = e^{\lambda\tau}w(z)$ which yields

$$\lambda w = cw_z - \mathcal{H}w_{zz} - 2Aw_z. \tag{27}$$

Since the coefficients of the above ODE are periodic (since they are constant) with period L, we use a Floquet decomposition,

$$w(z) = e^{i\mu z}\hat{w}(z), \tag{28}$$

where $\mu \in [-\pi/L, \pi/L)$ and $\hat{w}(z + L) = \hat{w}(z)$. We then use an L-periodic Fourier series for $\hat{w}(z)$:

$$w(z) = e^{i\mu z}\sum_{n\in\mathbb{Z}} \hat{w}_n e^{2\pi i n z/L} = \sum_{n\in\mathbb{Z}} \hat{w}_n e^{(2\pi n/L+\mu)iz}. \tag{29}$$

Plugging this into the ODE (27) yields

$$\begin{aligned} \lambda_n &= ic\left(\frac{2\pi n}{L} + \mu\right) - \left(-i\,\mathrm{sign}\left(\frac{2\pi n}{L} + \mu\right)\right)\left(i\left(\frac{2\pi n}{L} + \mu\right)\right)^2 - 2iA\left(\frac{2\pi n}{L} + \mu\right) &(30)\\ &= i\left[\left(\frac{2\pi n}{L} + \mu\right)(2c + \sqrt{c^2 - 4a}) - \mathrm{sgn}\left(\frac{2\pi n}{L} + \mu\right)\left(\frac{2\pi n}{L} + \mu\right)^2\right]. &(31)\end{aligned}$$

Note that

$$\mu + \frac{2\pi n}{L} \in \left[\frac{\pi}{L}(2n - 1), \frac{\pi}{L}(2n + 1)\right). \tag{32}$$

Also,

$$2c + \sqrt{c^2 - 4a} < 0 \quad \Leftrightarrow \quad c^2 > -\frac{4}{3}a, \tag{33}$$

which is always satisfied for $a > 0$. $\lambda_n > 0$ only if $2\pi n/L + \mu < 0$, the expression becomes

$$\lambda_{n<0} = i\left(\frac{2\pi n}{L} + \mu\right)\left[2c + \sqrt{c^2 - 4a} + \frac{2\pi n}{L} + \mu\right], \tag{34}$$

which is maximized when $2\pi n/L + \mu$ is minimized, or when

$$\frac{2\pi n}{L} + \mu = \frac{\pi}{L}(1 - 2N), \tag{35}$$

where N is the number of Fourier modes used for the truncation. Therefore

$$\lambda_{max} = i\frac{\pi}{L}(1 - 2N)\left[2c + \sqrt{c^2 - 4a} + \frac{\pi}{L}(1 - 2N)\right]. \tag{36}$$

References

[1] J. C. Bronski, V. M. Hur, and M. A. Johnson, *Modulational instability in equations of kdv type*, in New approaches to nonlinear waves, Springer, 2016, pp. 83–133.

[2] H. Ono, *Algebraic solitary waves in stratified fluids*, Journal of the Physical Society of Japan, 39 (1975), pp. 1082–1091.