

# Benjamin-Ono stability

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## 1 Check the solution

We consider the Benjamin-Ono equation

$$u_t + \mathcal{H}u_{xx} + (u^2)_x = 0, \quad (1)$$

where

$$\mathcal{H}[f(\xi)](x) = \frac{1}{\pi} \int \frac{f(\xi)}{\xi - x} d\xi. \quad (2)$$

Letting  $(z, \tau) = (x - ct, t)$ , so that  $z$  is in the traveling frame, (1) becomes

$$u_\tau - cu_z + \mathcal{H}u_{zz} + (u^2)_z = 0. \quad (3)$$

We look for stationary solutions in this traveling frame

$$-cu_z + \mathcal{H}u_{zz} + (u^2)_z = 0. \quad (4)$$

A 3-parameter family of periodic solutions to (4) is [1] given by

$$u(z; a, k, c) = -\frac{\frac{k^2}{\sqrt{c^2 - 4a - k^2}}}{\sqrt{\frac{c^2 - 4a}{c^2 - 4a - k^2}} - \cos(kz)} - \frac{1}{2} \left( \sqrt{c^2 - 4a} + c \right), \quad (5)$$

where  $c < 0$  and  $k^2 < c^2 - 4a$ . We now verify that this is indeed a solution. We first note that [2]

$$\mathcal{H}\left(\frac{1}{1 - B \cos(D\zeta)}\right) = \operatorname{sgn}(D) \frac{B \sin(D\zeta)}{\sqrt{1 - B^2}(1 - B \cos(D\zeta))}, \quad (6)$$

and our solution may be rewritten

$$u(z; a, k, c) = -\frac{k^2}{\sqrt{c^2 - 4a - k^2}} \sqrt{\frac{c^2 - 4a - k^2}{c^2 - 4a}} \frac{1}{1 - \sqrt{\frac{c^2 - 4a - k^2}{c^2 - 4a}} \cos(kz)} + \frac{1}{2} \left( \sqrt{c^2 - 4a} + c \right) \quad (7)$$

$$= -\frac{k^2}{\sqrt{c^2 - 4a}} \frac{1}{1 - \sqrt{\frac{c^2 - 4a - k^2}{c^2 - 4a}} \cos(kz)} + \frac{1}{2} \left( \sqrt{c^2 - 4a} + c \right) \quad (8)$$

$$= -\frac{k^2}{\alpha} \frac{1}{1 - \frac{\beta}{\alpha} \cos(kz)} + \frac{1}{2} (\alpha + c) \quad (9)$$

$$= -\frac{k^2}{\alpha} \frac{1}{1 - B \cos(kz)} + \frac{1}{2} (\alpha + c). \quad (10)$$

We then note that

$$\mathcal{H}u_{zz} = \mathcal{H} \left( -\partial_z^2 \left( \frac{k^2}{\alpha} \frac{1}{1 - B \cos(k\zeta)} + \frac{1}{2} (\alpha + c) \right) \right) \quad (11)$$

$$= \mathcal{H} \left( -\partial_z^2 \left( \frac{k^2}{\alpha} \frac{1}{1 - B \cos(k\zeta)} \right) \right) \quad (12)$$

$$= -\frac{k^2}{\alpha} \partial_z^2 \mathcal{H} \left( \frac{1}{1 - B \cos(k\zeta)} \right), \quad (13)$$

since two derivatives commute with the Hilbert transform. Therefore

$$\mathcal{H}u_{zz} = -\frac{k^2}{\alpha} \partial_z^2 \operatorname{sgn}(k) \frac{B \sin(k\zeta)}{\sqrt{1 - B^2} (1 - B \cos(k\zeta))} \quad (14)$$

$$= -\frac{k^2}{\alpha} \partial_z^2 \frac{\beta \operatorname{sgn}(k) \sin(k\zeta)}{\alpha \sqrt{1 - \beta^2/\alpha^2} (1 - \beta \cos(k\zeta)/\alpha)} \quad (15)$$

$$= -\partial_z^2 \frac{k^2 \beta \operatorname{sgn}(k) \sin(k\zeta)}{\alpha \sqrt{\alpha^2 - \beta^2} (1 - \beta \cos(k\zeta)/\alpha)}. \quad (16)$$

But

$$\alpha^2 - \beta^2 = c^2 - 4a - (c^2 - 4a - k^2) = k^2, \quad (17)$$

so that

$$\mathcal{H}u_{zz} = -\partial_z^2 \frac{k^2 \beta \operatorname{sgn}(k) \sin(k\zeta)}{\alpha k (1 - \beta \cos(k\zeta)/\alpha)} \quad (18)$$

$$= -\partial_z^2 \frac{\beta k \operatorname{sgn}(k) \sin(k\zeta)}{\alpha - \beta \cos(k\zeta)}. \quad (19)$$

Then the Mathematica notebook `BronskiAndHurChecks-2.nb` shows that (10) is indeed a solution of (1) and (4).

## 2 Fourier multiplier of the Hilbert Transform

According to Wikipedia, the Fourier multiplier of  $\mathcal{H}$  is  $i \operatorname{sign}(\omega)$  (note that the sign of the denominator in  $\mathcal{H}$  is different in their definition than ours), so that

$$\mathcal{F}[\mathcal{H}[u]](\omega) = i \operatorname{sign}(\omega) \mathcal{F}[u](\omega). \quad (20)$$

**But Wikipedia blows!** Simon says that with this definition of the Hilbert, transform,

$$\mathcal{F}[\mathcal{H}[u]](\omega) = -i \operatorname{sign}(\omega) \mathcal{F}[u](\omega), \quad (21)$$

which agrees with the plotting sort of checks that Ryan has completed.

### 3 Constant solution

We can find the spectrum for the constant solution analytically, so let's do that as a check for the numerics.

With  $k = 0$ , (5) becomes

$$u(z; a, 0, c) = -\frac{1}{2}(\sqrt{c^2 - 4a} + c) =: A. \quad (22)$$

We linearize about this solution by letting  $u(z, \tau) = A + v(z, \tau)$  where  $|v| \ll 1$ . Plugging into (3) and retaining terms of order  $v$  and lower yields

$$0 = v_\tau - cv_z + \mathcal{H}v_{zz} + \partial_z(A + v)^2 \quad (23)$$

$$= v_\tau - cv_z + \mathcal{H}v_{zz} + 2Av_z. \quad (24)$$

Since the equation is autonomous first-order in  $t$  we let  $v(z, \tau) = e^{\lambda\tau}w(z)$  which yields

$$\lambda w = cw_z - \mathcal{H}w_{zz} - 2Aw_z. \quad (25)$$

Since the coefficients of the above ODE are periodic (since they are constant) with period  $L$ , we use a Floquet decomposition,

$$w(z) = e^{i\mu z} \hat{w}(z), \quad (26)$$

where  $\mu \in [-\pi/L, \pi/L]$  and  $\hat{w}(z + L) = \hat{w}(z)$ . We then use an  $L$ -periodic Fourier series for  $\hat{w}(z)$ :

$$w(z) = e^{i\mu z} \sum_{n \in \mathbb{Z}} \hat{w}_n e^{2\pi i n z / L} = \sum_{n \in \mathbb{Z}} \hat{w}_n e^{(2\pi n / L + \mu) i z}. \quad (27)$$

Plugging this into the ODE (25) yields

$$\lambda_n = ic \left( \frac{2\pi n}{L} + \mu \right) - \left( -i \operatorname{sign} \left( \frac{2\pi n}{L} + \mu \right) \right) \left( i \left( \frac{2\pi n}{L} + \mu \right) \right)^2 - 2iA \left( \frac{2\pi n}{L} + \mu \right) \quad (28)$$

$$= i \left[ \left( \frac{2\pi n}{L} + \mu \right) (2c + \sqrt{c^2 - 4a}) - \operatorname{sgn} \left( \frac{2\pi n}{L} + \mu \right) \left( \frac{2\pi n}{L} + \mu \right)^2 \right]. \quad (29)$$

Note that

$$\mu + \frac{2\pi n}{L} \in \left[ \frac{\pi}{L}(2n - 1), \frac{\pi}{L}(2n + 1) \right). \quad (30)$$

Also,

$$2c + \sqrt{c^2 - 4a} < 0 \quad \Leftrightarrow \quad c^2 > -\frac{4}{3}a, \quad (31)$$

which is always satisfied for  $a > 0$ .  $\lambda_n > 0$  only if  $2\pi n/L + \mu < 0$ , the expression becomes

$$\lambda_{n < 0} = i \left( \frac{2\pi n}{L} + \mu \right) \left[ 2c + \sqrt{c^2 - 4a} + \frac{2\pi n}{L} + \mu \right], \quad (32)$$

which is maximized when  $2\pi n/L + \mu$  is minimized, or when

$$\frac{2\pi n}{L} + \mu = -\frac{\pi}{L}(1 + 2N), \quad (33)$$

where  $N$  is the number of Fourier modes used for the truncation. Therefore

$$\lambda_{max} = -i\frac{\pi}{L}(1 + 2N) \left[ 2c + \sqrt{c^2 - 4a} - \frac{\pi}{L}(1 + 2N) \right]. \quad (34)$$

## 4 Fixing the period

Letting

$$u = \epsilon \hat{u}, \quad \hat{z} = \beta z, \quad \hat{\tau} = \gamma \tau, \quad c = \delta \hat{c} \quad (35)$$

(3) becomes

$$0 = \gamma \hat{u}_{\hat{\tau}} - \delta \beta \hat{c} \hat{u}_{\hat{z}} + \beta^2 \mathcal{H}[\hat{u}_{\hat{z}\hat{z}}] + \epsilon \beta (\hat{u}^2)_{\hat{z}} \quad (36)$$

$$\Rightarrow 0 = \hat{u}_{\hat{\tau}} - \frac{\delta \beta}{\gamma} \hat{c} \hat{u}_{\hat{z}} + \frac{\beta^2}{\gamma} \mathcal{H}[\hat{u}_{\hat{z}\hat{z}}] + \frac{\epsilon \beta}{\gamma} (\hat{u}^2)_{\hat{z}}. \quad (37)$$

In order for this to match (3) in the traveling frame,

$$u = \beta \hat{u}, \quad \hat{z} = \beta z, \quad \hat{\tau} = \beta^2 t, \quad c = \beta \hat{c}. \quad (38)$$

Since we want  $kz = \hat{z}$ , we take  $\beta = k$ . Therefore,

$$u = k \hat{u}, \quad \hat{z} = kz, \quad \hat{\tau} = k^2 t, \quad c = k \hat{c}, \quad (39)$$

and

$$\hat{u}_{\hat{\tau}} - \hat{c} \hat{u}_{\hat{z}} + \mathcal{H} \hat{u}_{\hat{z}\hat{z}} + (\hat{u}^2)_{\hat{z}} = 0. \quad (40)$$

The solution (10) becomes

$$\hat{u} = -\frac{k}{\hat{\alpha}} \frac{1}{1 - \hat{B} \cos(\hat{z})} + \frac{1}{2}(\hat{\alpha} + k \hat{c}), \quad (41)$$

where

$$\hat{\alpha} = \sqrt{k^2 \hat{c}^2 - 4a}, \quad \hat{B} = \sqrt{\frac{k^2 \hat{c}^2 - 4a - k^2}{k^2 \hat{c}^2 - 4a}}. \quad (42)$$

The mathematica notebook “BronskiAndHurChecks-2.nb” verifies this solution.

## References

- [1] J. C. BRONSKI, V. M. HUR, AND M. A. JOHNSON, *Modulational instability in equations of kdv type*, in New approaches to nonlinear waves, Springer, 2016, pp. 83–133.
- [2] H. ONO, *Algebraic solitary waves in stratified fluids*, Journal of the Physical Society of Japan, 39 (1975), pp. 1082–1091.