Benjamin-Ono stability

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We consider the Benjamin-Ono equation

$$u_t + \mathcal{H}u_{xx} + (u^2)_x = 0, \tag{1}$$

where

$$\mathcal{H}\left[f(\xi)\right](x) = \frac{1}{\pi}\int \frac{f(\xi)}{\xi - x}\,\mathrm{d}\xi.$$

Letting

$$u = \alpha\hat{u}, \qquad \hat{x} = \beta x, \qquad \hat{t} = \gamma t,$$

(1) becomes

$$\gamma\hat{u}_{\hat{t}} + \beta^2\mathcal{H}[\hat{u}_{\hat{x}\hat{x}}] + \alpha\beta(\hat{u}^2)_{\hat{x}} = 0.$$

Letting $(z,\tau) = (x - ct, t)$, so that z is in the traveling frame, (1) becomes

$$u_\tau - cu_z + \mathcal{H}u_{zz} + (u^2)_z = 0.$$

We look for stationary solutions in this traveling frame

$$-cu_z + \mathcal{H}u_{zz} + (u^2)_z = 0. \tag{2}$$

A 3-parameter family of periodic solutions to (2) is [**?**] given by

$$u(z;a,k,c) = -\frac{\frac{k^2}{\sqrt{c^2-4a-k^2}}}{\sqrt{\frac{c^2-4a}{c^2-4a-k^2}} - \cos(kz)} - \frac{1}{2}\left(\sqrt{c^2-4a} + c\right),$$

where $c < 0$ and $k^2 < c^2 - 4a$. We now verify that this is indeed a solution. We first note that [**?**]

$$\mathcal{H}\left(\frac{1}{1 - B\cos(D\zeta)}\right) = \operatorname{sgn}(D)\frac{B\sin(D\zeta)}{\sqrt{1-B^2}(1 - B\cos(D\zeta))},$$

and our solution may be rewritten

$$
\begin{aligned}
u(z;a,k,c) &= -\frac{k^2}{\sqrt{c^2-4a-k^2}}\sqrt{\frac{c^2-4a-k^2}{c^2-4a}}\frac{1}{1-\sqrt{\frac{c^2-4a-k^2}{c^2-4a}}\cos(kz)}+\frac{1}{2}\left(\sqrt{c^2-4a}+c\right)\\
&= -\frac{k^2}{\sqrt{c^2-4a}}\frac{1}{1-\sqrt{\frac{c^2-4a-k^2}{c^2-4a}}\cos(kz)}+\frac{1}{2}\left(\sqrt{c^2-4a}+c\right)\\
&= -\frac{k^2}{\alpha}\frac{1}{1-\frac{\beta}{\alpha}\cos(kz)}+\frac{1}{2}(\alpha+c)\\
&= -\frac{k^2}{\alpha}\frac{1}{1-B\cos(k\zeta)}+\frac{1}{2}(\alpha+c).
\end{aligned}
\tag{3}
$$

We then note that

$$
\begin{aligned}
\mathcal{H}u_{zz} &= \mathcal{H}\left(-\partial_z^2\left(\frac{k^2}{\alpha}\frac{1}{1-B\cos(k\zeta)}+\frac{1}{2}(\alpha+c)\right)\right)\\
&= \mathcal{H}\left(-\partial_z^2\left(\frac{k^2}{\alpha}\frac{1}{1-B\cos(k\zeta)}\right)\right)\\
&= -\frac{k^2}{\alpha}\partial_z^2\mathcal{H}\left(\frac{1}{1-B\cos(k\zeta)}\right),
\end{aligned}
$$

since two derivatives commute with the Hilbert transform. Therefore

$$
\begin{aligned}
\mathcal{H}u_{zz} &= -\frac{k^2}{\alpha}\partial_z^2\,\mathrm{sgn}(k)\frac{B\sin(k\zeta)}{\sqrt{1-B^2}(1-B\cos(k\zeta))}\\
&= -\frac{k^2}{\alpha}\partial_z^2\frac{\beta\,\mathrm{sgn}(k)\sin(k\zeta)}{\alpha\sqrt{1-\beta^2/\alpha^2}(1-\beta\cos(k\zeta)/\alpha)}\\
&= -\partial_z^2\frac{k^2\beta\,\mathrm{sgn}(k)\sin(k\zeta)}{\alpha\sqrt{\alpha^2-\beta^2}(1-\beta\cos(k\zeta)/\alpha)}.
\end{aligned}
$$

But

$$
\alpha^2-\beta^2=c^2-4a-(c^2-4a-k^2)=k^2,
$$

so that

$$
\begin{aligned}
\mathcal{H}u_{zz} &= -\partial_z^2\frac{k^2\beta\,\mathrm{sgn}(k)\sin(k\zeta)}{\alpha k(1-\beta\cos(k\zeta)/\alpha)}\\
&= -\partial_z^2\frac{\beta k\,\mathrm{sgn}(k)\sin(k\zeta)}{\alpha-\beta\cos(k\zeta)}.
\end{aligned}
$$

Then the Mathematica notebook `BronskiAndHurChecks-2.nb` shows that (3) is indeed a solution of (1) and (2).

1 Fourier multiplier of the Hilbert Transform

According to Wikipedia, the Fourier multiplier of $\mathcal{H}$ is $i\,\mathrm{sign}(\omega)$ (note that the sign of the denominator in $\mathcal{H}$ is different in their definition than ours), so that

$$
\mathcal{F}[\mathcal{H}[u]](\omega)=i\,\mathrm{sign}(\omega)\mathcal{F}[u](\omega).
$$