Learning Unconventional Policies:
The Dynamic Effects of Forward Guidance

Riccardo Bianchi-Vimercati
Northwestern U.

October 28, 2021

Latest version here

Abstract

I develop a novel *integrated reasoning* model of expectations formation to describe how people learn the effects of unconventional macroeconomic policies. My model of expectations has two key elements. First, people have a limited ability to understand the general-equilibrium effects of a new policy. Second, they revise expectations when observing past errors with a process of learning in real time. To assess the empirical plausibility of the model, I estimate the response of survey-level expectations to identified forward guidance shocks. The time-series properties of people's expectations errors and revisions show that *i)* individuals immediately revise beliefs after announcements; *ii)* forecasts underreact to the shock; and *iii)* beliefs are slowly updated over time. I show that the predictions of my framework are consistent with the evidence, while commonly used alternative models that are special cases of my framework are not. Finally, I embed the model of expectations in a New Keynesian model to assess the importance of learning for the cumulative effects of a "lower-for-longer" interest rate announcement when the economy is at the zero lower bound. In my baseline calibration, learning accounts for 11.5% of the cumulative output response.

1 Introduction

In late 2009, US nominal interest rates hit the Zero Lower Bound (ZLB). Policymakers adopted a variety of new measures in response to that event, collectively referred to as unconventional policies. Many of them are successful at stabilizing the economy by shaping expectations of households, firms, and financial market participants about the future economic effects of the policy intervention. However, the rarity of ZLB episodes and the policies that followed raises the question of how economic actors can form expectations about their effects, without having observed these policies in the past.

In this paper, I develop a new model of expectations formation to describe how people learn about the effects of new policies. My model of expectations has two key elements. First, people have a limited ability to understand the general-equilibrium effects of a new policy. Second, they revise expectations when observing past errors with a process of learning in real time.

I assess the empirical plausibility of the model as follows. To study how beliefs respond to unconventional policies, I estimate the response of expectations of professional forecasters about key macroeconomic variables to identified forward guidance shocks. I employ the external shock series extracted by Jarociński (2021), who relies on high-frequency changes to the price of financial assets in the 30 minutes window around Federal Open Market Committee (FOMC) announcements to identify forward guidance. I use local projection methods as in Jordà (2005) to study how forecast errors and forecast revisions behave in response to the shock. My findings suggest that individuals revise expectations immediately after an unexpected change to the announced path of interest rates, so future effects of new policies are incorporated into forecasts immediately after their announcements. However, forecasters also underestimate the response of key macroeconomic variables in the quarters following the shock, which results in predictable forecast errors. Hence, their ability to anticipate the effects of the policy intervention is imperfect. Finally, the empirical response of forecast revisions, that are systematically different from zero in the quarters following the shock, is consistent with a process of slow adjustment of expectations.

This empirical evidence is at odds with existing expectations formation processes. One is the workhorse rational expectations model, that postulates that individuals do not make systematic mistakes. This prediction is rejected by the fact that forecast errors at initial quarters after the shock are predictable. An alternative class of models that has been proposed for bounded rationality in response to unconventional policies is one where individuals correctly understand the model, but form inferences about future outcomes through an imperfect process of reflection. The level-k thinking model in Farhi and Werning (2019) and the reflective expectations model in García-Schmidt and Woodford (2019) are two examples. In the baseline version of these frameworks, expectations are assumed to be revised only once at the time of the announcement (in "notional" time), but they are not updated at later periods, despite the presence of systematic forecast errors — a counterfactual prediction relative to the evidence on systematic revisions. A distinct approach is to model expectations as a result of updates in real time from past experience, as in adaptive learning models (see the literature review for a detailed discussion). If expectations are updated using *only* past data, a forward guidance announcement does not produce any impact response on beliefs, unlike what is observed in the data.

To understand how aggregate output and inflation respond to an unconventional policy announcement in a setting where individual expectations formation is consistent with the evidence on forecast errors and revisions, I develop a model of bounded rationality called "integrated reasoning" which combines imperfect economic thinking with learning, building on Bianchi-Vimercati et al. (2020). Formally, this equilibrium concept integrates a generalization of the level-k thinking model with a Bayesian learning process. Individuals form imperfect expectations about the mapping between unconventional policy announcements and aggregate variables. They differ in the depth of equilibrium reasoning they perform, which is indexed by their cognitive level k. They also receive noisy signals about realized endogenous outcomes. Level-0 individuals learn about the effects of policy announcements following a Bayesian signal extraction problem. They do not have any structural knowledge of the economy. Their expectations formation process is backward-looking, and is updated only from accumulation of new data over time. Individuals with higher cognitive sophistication levels form beliefs about future changes in macroeconomic variables using a finite deductive procedure about others' behavior that involves k iterations. In this process, they are able to understand the beliefs and actions of less sophisticated individuals, and they correctly infer their equilibrium consequences. They learn from new data indirectly from the behavior of the level-0 reasoners. Since level-k thinkers use structural knowledge about the future behavior of the economy, their beliefs are forward-looking.

The combination of these two reasoning processes has a powerful role in explaining the empirical behavior of forecast errors and revisions in response to a forward guidance shock. I illustrate this role in a stylized New Keynesian (NK) economy with fully rigid wages and prices. The central bank announces a change in the interest rate at a future period T. Households understand that future income will react to the announced interest rate reduction thanks to the forward-looking deductive thinking. Therefore, income expectations are revised on impact. But households have a limited understanding of general-equilibrium forces, since they only perform a finite number of rounds of iterations to form their expectations. As a result, they underestimate the true effects of the policy. Beliefs display the pattern of systematic forecast revision and error on impact that is borne out in the data. In addition, the integrated reasoning model implies a slow process of beliefs revision after the shock. This is due to the fact that level-0 people update expectations by incorporating new information from the signals they observe. Since level-k thinkers use level-0 expectations as the starting point for their deductive reasoning, they also update expectations. Therefore, the model generates slow belief revision and reversion of forecast errors towards zero, both of which are consistent with empirical evidence.

After analyzing the behavior of forecast errors and revisions in the integrated reasoning model, I turn to characterizing its implications for the response of output to a forward guidance announcement. Under plausible parametrizations, the response of output is mitigated at all horizons relative to rational expectations. This result follows from the fact that households do not perfectly anticipate the general-equilibrium effects of the policy announcement. Since they believe income will rise less than under rational expectations, and since current consumption is increasing in expected future income, the response of aggregate demand is mitigated. Hence, similarly to Farhi and Werning (2019) and García-Schmidt and Woodford (2019), this model of bounded rationality does not exhibit a "forward guidance puzzle" — a set of counterfactual implications of forward guidance in standard NK models.[1] However, relative to a benchmark case in which individuals do not update beliefs in real time, integrated reasoning implies a larger output response. The intuition behind this result is that real time learning generates a feedback between realized outcomes and expectations update which strengthens the stimulative power of an announcement as time passes. Upon observing a positive forecast error, households revise income expectations upwards. Because consumption is increasing in expected income, aggregate demand is higher relative to the case in which individuals do not update expectations. Since their ability to understand general-equilibrium effects is limited, they keep making forecast errors over time, and thus revise income expectations upwards, which leads to a larger output response.

To evaluate the quantitative importance of this channel, I turn to the analysis of a more general

[1]See Del Negro et al. (2012) and McKay et al. (2016) for early papers that point out the existence of a forward guidance puzzle.

version of the model where prices and wages are not constant. I assume that wages are subject to Calvo-style frictions, which also implies price stickiness. In this new setting, inflation is an important channel of general-equilibrium effects since it affects both the current and future real interest rates. I use this version of the model to simulate a ZLB recession that lasts for 20 quarters, and look at the effects of a "lower-for-longer" announcement in which the central bank promises to keep interest rates at zero for four additional quarters after the recession is over. I calibrate a parameter that determines the speed of expectation revisions to be consistent with the empirical response of errors and revisions to a forward guidance shock. In this simulation, learning generates a cumulative output response that is 12.6% larger than in a standard level-k thinking model without expectations update; equivalently, it contributes to 11.5% of the overall effect. So, expectations revision is a quantitatively important channel of transmission of unconventional policies. An assessment of the potency of a new policy announcement should keep in consideration the revision of beliefs that builds up over time after the announcement as households, firms, and financial market participants learn about the true effects of the shock. The revision in expectations interacts with the length of the ZLB. In long-lasting recessions, the cumulative effect of forward guidance is larger, despite a smaller initial response. This happens because the revision relates to a larger number of future periods, therefore implying a more powerful response than with shorter recession.

Related literature The belief formation process described in this paper builds on Bianchi-Vimercati et al. (2020), who first introduced a version of integrated reasoning where individuals combine level-k thinking with adaptive learning. Their paper discusses differences and similarities between the two reasoning processes in the context of a general class of linear models, and concludes that this combination is well-suited for fixing individual shortcomings of the two belief formation models. I focus on forward guidance and present novel evidence on the ability of integrated reasoning to match belief revision and forecast error predictability in response to identified shocks. I also propose a different model of learning that, as I argue below, is well-suited for non-stationary environments such as an unconventional monetary policy, and analyze its quantitative implications.

This paper contributes to a growing literature studying the implications of deviations from the rational expectations, full information framework, to study the effectiveness of macroeconomic policies. The departure from rational expectations that I consider follows the approach of modeling expectations formation as limited depth of general-equilibrium knowledge via level-k thinking, which was originally employed by Nagel (1995) and Stahl and Wilson (1995). Farhi and Werning (2019) study the power of monetary policy when individuals are level-k thinkers. García-Schmidt and Woodford (2019) develop a similar belief formation process, called reflective expectations, and use it to analyze forward guidance and interest rate pegs. Both papers find that the power of forward guidance is dampened under these deviations from rational expectations. However, they do not allow for belief revision over time, despite the fact that individuals make systematic forecast errors. As such, they are best suited for the analysis of the impact response of endogenous variables to unconventional policies. Relative to their findings, I look at the dynamic response of output in a model where beliefs are updated over time thanks to the real time learning model. Iovino and Sergeyev (2018) study the effects of central banks' balance sheet policies in a model with level-k thinking and reflective expectations, and briefly discuss the robustness of their results to the presence of expectations updates.

All of the aforementioned papers assume that individuals have common knowledge about exogenous and endogenous variables, but do not fully understand these variables' general-equilibrium implications. An alternative approach, pioneered by Morris and Shin (2002) and Woodford (2003), and followed by Angeletos and Lian (2018), Angeletos and Huo (2021), and Wiederholt (2015), is to assume that individuals have dispersed information.[2] Deviating from common knowledge dampens the equilibrium response of policy announcements about the future, because higher-order beliefs are less anchored to the common prior. Unlike Angeletos and Lian (2018), dispersed information in my model generates a slow process of belief revisions, consistent with the empirical evidence, rather than higher-order forecasting about others' forecasts. The hierarchy of beliefs in my model is generated with rounds of deductive thinking about the policy effects, that are truncated at some finite iteration determined by one's level of sophistication.

This paper builds on a body of work where individuals use past data to improve their understanding of the economy. Following the seminal work by Lucas (1978), many papers have suggested the use of adaptive learning as a way to deviate from rational expectations. Examples include Marcet and Sargent (1989a), Marcet and Sargent (1989b), Evans and Honkapohja (2012), and Eusepi and Preston (2011). Relative to this literature, my model sheds light on a non-stationary environment, in which a policy announcement or shock is only observed once. For this reason, I adopt a learning model that is different from standard econometric least-square learning, and that

[2]A related formulation where dispersed information is endogenous is the rational inattention setting, as in Sims (2003), Mackowiak and Wiederholt (2009), and Maćkowiak and Wiederholt (2015).

is well-suited for analyzing the initial response of expectations to a novel policy shock.[3] Recently, Christiano et al. (2018) explore learning in a non-linear NK model of the ZLB and its implications for fiscal policy. My paper focuses on monetary rather than fiscal policy, and provides empirical evidence on belief revisions.

A large literature has analyzed macroeconomic expectations using survey-level data on forecast errors and revisions. Inertial behavior in average beliefs has been documented (Coibion and Gorodnichenko (2012), Coibion and Gorodnichenko (2015)), together with patterns of overreaction and over-extrapolation of individual expectations (Bordalo et al. (2020), Kohlhas et al. (2019), Kohlhas and Walther (2021), Angeletos et al. (2020)). My findings suggest that beliefs under-react to forward guidance announcements, and are then slowly updated in the quarters following the shock. My methodology is similar to Angeletos et al. (2020), who argue in favor of using impulse responses of errors and revisions to validate different models of expectations. The predictability of forecast errors in the direction of under-reaction following an unconventional policy is also documented in Iovino and Sergeyev (2018) who look at mortgage rate expectations in response to balance sheet interventions. In addition, I look at the response of forecast revisions and find that also a slow adjustment of expectations is needed to match the behavior of forecast errors and revisions.

To my knowledge, this is the first paper that studies the joint behavior of forecast errors and revisions in response to identified forward guidance shocks as a mean of model calibration and validation. Other papers have looked at how monetary policy shocks affect expectations about the state of the economy, in an attempt to infer an "information effect" of monetary policy (Campbell et al. (2012), Nakamura and Steinsson (2018) and Jarociński and Karadi (2020)), or lack thereof (Miranda-Agrippino (2016), Miranda-Agrippino and Ricco (2021), Bauer and Swanson (2020), and Sastry (2021)). The "information effect" of monetary policy is beyond the scope of my paper.[4] However, I document that the identified forward guidance series extracted by Jarociński (2021) that I use for my analysis displays responses of expectations in line with theoretical predictions — a monetary tightening leads to a deterioration of expectations about future economic activity.

Finally, this paper contributes to the line of research that studies the effects of forward guidance.

[3] My learning model is similar to the Bayesian learning set-up used by Ilut and Valchev (2020) and Ilut et al. (2020), where individuals learn about functions.

[4] In my model, monetary policy never has an information effect, because the central bank announces a path for interest rates rather than using a rule where interest rates are set as a function of inflation and the output gap. In the latter scenario, communication about future interest rates also provides a signal for the central bank's forecast of future economic conditions.

Many papers in the area have focused on deviations from the representative agent assumption, using models with incomplete markets and liquidity constraints, such as Werning (2015), McKay et al. (2016), and Hagedorn et al. (2019). As argued in Werning (2015), incomplete markets alone reduce the role of partial-equilibrium effects in favor of greater general-equilibrium effects. However, they do not by themselves necessarily imply a solution to the "forward guidance puzzle", and can even make it worse. By contrast, I focus on analyzing the role of expectation formation.

2 Forward guidance with integrated reasoning

In this section I describe a benchmark model with fully rigid wages and complete markets. I will use this simple model to illustrate the dynamic effects of an announcement about a reduction in future short-term interest rates.

Without loss of generality, wages are equal to one at each period. The representative household preferences over sequences of consumption, C_t, and labor, N_t, are given by:

$$\sum_{s\geq 0} \beta^s \left[u\left(C_{t+s}\right) - v\left(N_{t+s}\right)\right], \tag{1}$$

where $u\left(C\right) = C^{1-\sigma^{-1}} / \left(1-\sigma^{-1}\right)$ and $v\left(N\right) = N_{t+s}^{1+\varphi} / \left(1+\varphi\right)$.

Firms Firms are perfectly competitive and maximize profits. I assume that production is linear in labor, $Y_t = N_t$. The solution to the representative firm's problem requires that $P_t = W_t = 1$. Because wages are fully rigid, there is no wage or price inflation.

Monetary policy In this section, I assume that the monetary policy authority announces a path for the nominal interest rate R_t. As in Farhi and Werning (2019), I look at one-time changes in the nominal interest rate at a future date T. Since there is no inflation, the announcement has a one-to-one impact on real interest rates. For ease of In all other periods, the interest rate is equal to the steady state value of β^{-1}. So, the resulting interest rate path is:

$$R_t = \begin{cases} \beta^{-1} + \Delta R & t = T \\ \beta^{-1} & t \neq T. \end{cases} \tag{2}$$

For ease of notation, let $Z = \{R_s\}_{s\geq 0}$ denote an arbitrary sequence of current and future interest rates, and $Z^t = \{R_{t+s}\}_{s\geq 0}$ denote the sequence that is announced at time t. In this simple model, the sequence Z^t is the only state variable.

Households and expectations The only exogenous aggregate variable that the household has to forecast is the interest rate. I assume that the household has rational expectations about the path of interest rates. Because the central bank announces the entire path of future interest rates Z^t, this implies perfect foresight with respect to monetary policy shocks. However, households are limited in their ability to predict the effects of such policy on endogenous variables. This assumption is similar to Farhi and Werning (2019) and García-Schmidt and Woodford (2019), and is justified as a plausible choice for economic scenarios that involve salient policy announcements, whose real effects are not obvious because of lack of past data.

In this simple model, the only variable that individuals need to predict in order to make consumption and saving decisions is the level of income. I denote $\mathbb{E}_{i,t}[Y_{t+s}]$ the beliefs formed at time t by household i about the time $t+s$ value of output. I focus on expectation formation processes that can be represented as a time-invariant mapping from the fundamental exogenous state variable Z^t to endogenous outcomes, as in:

$$\mathbb{E}_{i,t}[Y_{t+s}] = \mathcal{Y}_{i,t}(Z^{t+s}), \tag{3}$$

where Z^{t+s} denotes the sequence of all future interest rates that will prevail after time $t+s$. Equation (3) effectively represents the perceived law of motion of output for individual i.

At each period t, the household receives a personal income equal to $Y_{i,t}$. Individual income is the sum of aggregate income and an individual component that is idiosyncratic across households and time and has mean one. For tractability I will assume that the idiosyncratic shock is normal, with variance equal to σ_ν^2:

$$Y_{i,t} = e^{\nu_{i,t}} Y_t,$$

where $\nu_{i,t} \sim \mathcal{N}(0, \sigma_\nu^2)$. This assumption implies that households need to solve a signal extraction problem when forming expectations about the future through a learning process. To do so, they combine their prior beliefs about output with the signal given by their individual income, and form posterior beliefs about current and future aggregate income.

The household enters period t with financial assets equal to $B_{i,t}$, which yield a return of R_{t-1}. When solving the dynamic consumption-savings problem, households maximize their perceived expected utility:

$$\max_{C_{i,t+s}} \mathbb{E}_{i,t} \sum_{s\geq 0} \beta^s \left[\frac{C_{i,t+s}^{1-\sigma^{-1}}}{1-\sigma^{-1}} - \frac{N_{t+s}^{1+\varphi}}{1+\varphi} \right],$$

subject to the current and future budget constraints:

$$C_{i,t+s} + B_{i,t+s+1} = Y_{i,t+s} + R_{t+s-1} B_{i,t+s}.$$

Note that the perceived distribution that households use to form expectations is not necessarily equal to the true distribution. Since wages are fully rigid, equilibrium output and labor are demand-determined. Therefore, households are not on their labor supply equation.

For the rest of the paper, I will focus on a linearized version of the model. Lower case letters denote log-linear deviations from steady state. The solution to the household's problem in the linearized model implies that $c_{i,t}$ satisfies

$$c_{i,t} = (1-\beta) y_{i,t} + (1-\beta) \sum_{s=1}^{\infty} \beta^s \mathbb{E}_{i,t}[y_{t+s}] - \sigma \sum_{s=0}^{\infty} \beta^{s+1} r_{t+s} + b_{i,t}. \tag{4}$$

The aggregate consumption function is equal to

$$c_t = \int c_{i,t} di = \mathcal{C}_t(y_t, z^t),$$

where for ease of notation I omit the dependence on the future path for expected output among the arguments of the aggregate consumption function. The subscript t on the aggregate consumption mapping $\mathcal{C}_t$ captures the fact that the information set is the one available at time t.

I assume that individuals have a correctly specified model: they use the correct aggregate state variables when forming expectations (in this simple version, z^t is the only state variable), and they use a linear model. Section B of the Appendix contains details about the formulation of this model.

Temporary equilibria I start by defining a temporary equilibrium. This notion of equilibrium takes, as given, a sequence of beliefs $\{\mathbb{E}_{i,t}[y_{t+s}]\}_{s\geq 1}$ for every agent i and every time period t at which the beliefs are formed. A temporary equilibrium is a sequence of allocations which satisfy private optimality for households and firms, and such that markets clear. Let $\overline{\mathbb{E}}_t[y_{t+s}] = \int \mathbb{E}_{i,t}[y_{t+s}] di$ denote the average expectations about income. Using equation (4) and imposing market clearing, the temporary equilibrium output is given by

$$y_t = \mathcal{Y}_t\left(z^t\right) = \frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\overline{\mathbb{E}}_t[y_{t+s}] - \sigma\sum_{s=0}^{\infty}\beta^s r_{t+s}. \tag{5}$$

Integrated reasoning The model of belief formation that I use is based on Bianchi-Vimercati et al. (2020). In the *integrated-reasoning equilibrium*, households form expectations using a process that combines imperfect structural knowledge of the economy with belief updates from past data. These two elements are derived from well-known models of bounded rationality: level-k thinking and learning, respectively.

The model features a cognitive hierarchy of individuals indexed by $k \geq 0$. In the spirit of the generalized level-k thinking model introduced by Camerer et al. (2004), each individual believes that there is a distribution of other individuals with lower cognitive abilities. In particular, a level-k individual believes that there is a mass $f_k(j)$ of level-j people, with $j < k$. At each period, every individual with cognitive level higher than one performs a finite deductive reasoning process. The starting point are the beliefs of the level-0 individuals $\left\{\mathbb{E}^0_{i,t}[y_{t+s}]\right\}_{s\geq 1}$ for any t and i, which are exogenous to the generalized level-k thinking process. Given these beliefs, aggregate consumption of level-0 is:

$$\mathcal{C}^0_t\left(y_t, z^t\right) = (1-\beta)y_t + (1-\beta)\sum_{s=1}^{\infty}\beta^s\mathbb{E}^0_t[y_{t+s}] - \sigma\sum_{s=0}^{\infty}\beta^{s+1}r_{t+s}. \tag{6}$$

Level-1 people believe that everyone else is level-0, and they understand the aggregate consumption function (6). So, at each time t, they expect future output to be the result of the following fixed point problem:

$$\mathbb{E}^1_t[y_{t+s}] = \mathcal{C}^0_t\left(\mathbb{E}^1_t[y_{t+s}], z^{t+s}\right), \tag{7}$$

for all s. Note that level-1 expectations are common across households i, because they all form beliefs according to (7).

More generally, level-k individuals have a non-degenerate distribution for individuals over the cognitive hierarchy. So, they believe that output is the solution to:

$$\mathbb{E}^k_t[y_{t+s}] = \sum_{j=1}^{k-1} f_k(j)\mathcal{C}^j_t\left(\mathbb{E}^k_t[y_{t+s}], z^{t+s}\right). \tag{8}$$

To complete the equilibrium notion of integrated reasoning process, I specify an update process for beliefs of level-0 individuals. Because all level-k people have to derive level-0 beliefs in order to form expectations, they also update beliefs over time. In this sense, all agents in this economy combine a learning process from past data with internal reasoning from level-k thinking.

I assume that individuals use Bayesian inference to update their beliefs about the aggregate law of motion. Because central bank announcements consist of infinite paths for nominal interest rates, beliefs are formed as a mapping from the this infinite dimensional object, which includes all future interest rates, to endogenous variables.

Level-0 agents enter time t with a prior about the equilibrium mapping, $\mathcal{Y}^0_{i,t}(z)$. This prior mapping provides a law of motion for output as a function of the state variable z^t. Level-0 people are aware that their individual income $y_{i,t}$ provides a signal for the correct aggregate mapping between interest rates and output $\mathcal{Y}_t\left(z\right)$ at the announced path for interest rates z^t according to:

$$y_{i,t} = \mathcal{Y}_t\left(z^t\right) + \nu_{i,t}, \tag{9}$$

where $\nu_{i,t} \sim \mathcal{N}(0, \sigma^2_\nu)$. So, they update their beliefs about this mapping by solving a Bayesian inference problem. I make the following assumption about time-0 prior beliefs.

Assumption 1 *Time-0 prior beliefs are such that $\mathcal{Y}^0_0(z)$ is a Gaussian Process:*

$$\mathcal{Y}^0_{i,0} \sim \mathcal{GP}\left(\hat{\mathcal{Y}}^0_0, \hat{\Sigma}_0\right) \tag{10}$$

for all i, where $\hat{\mathcal{Y}}^0_0 : \mathbb{R}^\infty \to \mathbb{R}$ is a common mean function and $\hat{\Sigma}_0 : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$ is a common covariance function.

This prior belief has the feature that for any arbitrary pair of inputs z and z', the joint distribution of the resulting function values $\mathcal{Y}^0_{i,0}(z)$ and $\mathcal{Y}^0_{i,0}(z')$ is Gaussian:

$$\begin{bmatrix} \mathcal{Y}^0_{i,0}(z) \\ \mathcal{Y}^0_{i,0}(z') \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \hat{\mathcal{Y}}^0_0\left(z\right) \\ \hat{\mathcal{Y}}^0_0\left(z'\right) \end{bmatrix}, \begin{bmatrix} \hat{\Sigma}_0\left(z,z\right) & \hat{\Sigma}_0\left(z,z'\right) \\ \hat{\Sigma}_0\left(z',z\right) & \hat{\Sigma}_0\left(z',z'\right) \end{bmatrix}\right).$$

The mean function $\hat{\mathcal{Y}}^0_0(z)$ encodes the average behavior of output as a function of an arbitrary sequence of interest rates z according to the individuals' prior beliefs. The prior covariance function $\hat{\Sigma}_0(z,z')$ instead specifies the covariance between the values of output at any pair of interest rate sequences (z,z'):

$$\hat{\Sigma}_0(z,z') = cov_0\left(\mathcal{Y}^0_0\left(z\right), \mathcal{Y}^0_0\left(z'\right)\right).$$

Note that the mean function and the covariance function are assumed to be independent of i, that is, at time 0 all individuals have a common prior. The assumption of Gaussianity allows for a tractable Bayesian inference problem about equilibrium functions.[5]

Given Assumption 1, individual update beliefs in a Bayesian way, combining their prior beliefs $\mathcal{Y}^0_{i,t}(z)$ with the signal provided by (9). The resulting belief update process is a recursion that is similar to a standard Kalman filter.

Lemma 1 *Given time-t prior beliefs $\mathcal{Y}^0_{i,t}(z)$, individual income $Y_{i,t}$, and an announcement for a path of interest rates z^t, beliefs about output are a Gaussian Process $\mathcal{Y}^0_{i,t+1}(z) \sim \mathcal{GP}\left(\hat{\mathcal{Y}}^0_{i,t+1}, \hat{\Sigma}_{t+1}\right)$ where moments evolve according to the following recursion:*

$$\hat{\mathcal{Y}}^0_{i,t+1}(z) = \hat{\mathcal{Y}}^0_{i,t}(z) + \frac{\hat{\Sigma}_t(z, z^t)}{\hat{\Sigma}_t(z^t, z^t) + \sigma_v^2}\left(y_{i,t} - \hat{\mathcal{Y}}^0_{i,t}(z^t)\right) \tag{11}$$

$$\hat{\Sigma}_{t+1}(z, z') = \hat{\Sigma}_t(z, z') - \frac{\hat{\Sigma}_t(z, z^t)\hat{\Sigma}_t(z', z^t)}{\hat{\Sigma}_t(z^t, z^t) + \sigma_v^2}. \tag{12}$$

The covariance matrix $\hat{\Sigma}_t(z, z')$ encodes the correlation in output realization at any arbitrary pair of interest rate announcements (z, z'). Importantly, it determines the informativeness of the signal that individuals receive from their individual income $Y_{i,t}$ through the Kalman gain in equation (11). The higher is the perceived covariance between output under an arbitrary interest rate path z and the observed announcement z^t, the stronger will be the revision in the mean function that describes the equilibrium mapping evaluated at z. Note that the update rule (12) does not depend on the individual signal. So, under assumption 1, the common time-0 prior implies that all individuals hold the same beliefs over the covariance function at each point in time.

The integrated reasoning equilibrium is closed with an actual distribution of agents along the cognitive hierarchy, denoted by $\Phi(k)$. The following definition summarizes this notion of equilibrium.

Definition For each date t and interest rate path Z^t, an *integrated-reasoning equilibrium* is a sequence of allocations $\mathcal{A}_t = \{c_t, y_t\}$ and beliefs $\mathcal{A}^e_t = \{\mathbb{E}^k_{i,t}[y_{t+s}]\}_{i,k,s}$ such that:

1. level-0 agents form expectations according to $\mathcal{Y}^0_{i,t}(Z) \sim \mathcal{GP}\left(\hat{\mathcal{Y}}^0_{i,t}, \hat{\Sigma}_t\right)$, where the mean and variance functions are updated as in (11) and (12), for some initial conditions $(\hat{\mathcal{Y}}^1_0, \hat{\Sigma}_0)$;

2. level-k individuals for $k > 0$ form expectations according to (8);

[5] See Ilut and Valchev (2020) and Ilut et al. (2020) for two different applications.

3. individual consumption solves households' optimization problem as in (4);

4. output solves the market clearing condition:

$$y_t = \mathcal{C}_t \left(y_t, z^t \right) = \sum_{k=1}^{\infty} \Phi(k) \mathcal{C}_t^k \left(y_t, z^t \right). \tag{13}$$

Initial beliefs The initial beliefs of level-0 agents in equation (10) are described by an initial mean function and covariance function. The prior mean function $\hat{\mathcal{Y}}_0^0$ is chosen in line with the literature on bounded rationality where agents are assumed to have a limited understanding of general equilibrium effects. Initial beliefs are specified so that agents believe the economy will remain in steady state after any policy announcement. In my setting, this assumption amounts to setting

$$\hat{\mathcal{Y}}_0^0(z) = 0 \tag{14}$$

for all interest rate announcement z.

I adopt the following specification for the covariance function:[6]

$$\hat{\Sigma}_0 \left(z, z' \right) = \sigma_0^2 \sum_{t \geq 0} \sum_{s \geq 0} w_{t,s} r_t r_s'. \tag{15}$$

The parameter σ_0 measures the amount of initial uncertainty about output realizations. The higher is σ_0, the more uncertain are households' beliefs. Uncertainty also depends on how far each announced interest rate is from the steady state value, as captured by the products $r_t r_s'$. Under this specification, if the central bank announces to keep interest rates at the steady state level, households at time zero believe with certainty that output will remain at the steady state level. Any paths for interest rates that involve deviations from steady state imply some degree of uncertainty about the level of output. The farther any interest rate r_t is from zero, the larger is the uncertainty about output under this path for rates. Finally, the sequence of weights $\{w_{t,s}\}$ captures how deviations of interest rates from steady state at different time periods load on the covariance. To understand the role of this assumption, consider two interest rate paths z and z' with different time periods $T \neq T'$ for interest rate changes of size r. The covariance implied by (15) is:

$$\hat{\Sigma}_0 \left(z, z \right) = \sigma_0^2 w_{T,T'} r^2.$$

[6] In Appendix B, I show how this formulation of the covariance function is a direct result of a primitive assumption on the prior covariance between the sensitivity of output to interest rates at different horizons when beliefs are well-specified, that is, they are linear in the state variable.

The coefficient $w_{T,T'}$ measures how much correlated output realizations are for interest rate changes at different horizons T and T'. So, if $w_{T,T'}$ is large, observing output under the interest rate path z is very informative about possible realizations of output under the interest rate path z'. I will make the following assumptions on $\{w_{t,s}\}$.[7] First, the sequence $\{w_{t,s}\}$ is symmetric, so that $w_{t,s} = w_{s,t}$ for all t, s. Second, $\{w_{t,s}\}$ is weakly decreasing in the distance between t and s, so that interest rate changes at point in time that are far away from each other imply a low level of correlation between the corresponding output realizations. A parsimonious parametrization is to assume that

$$w_{t,s} = \rho^{|t-s|},$$

where $\rho \geq 0$ determines the rate of decay of output covariance for interest rate changes at different time periods.

Special cases The integrated reasoning model described above accommodates standard belief-formation processes as special case. If the economy is only populated by level-0 individuals ($\Phi(0) = 1$), then expectations are only formed through *Bayesian learning*, where beliefs are updated according to (11) and (12), and the market-clearing condition at time t is:

$$y_t = \mathcal{C}_t^0 \left(y_t, z^t \right).$$

If instead σ_0^2 is equal to zero, then the prior distribution is dogmatic, and level-0 agents never update their beliefs over time. This case corresponds to a *generalized level-k thinking* model introduced by Camerer et al. (2004), with beliefs described by (8). The standard level-k thinking model is obtained by assuming that all individuals believe everyone else is one step below them in the cognitive hierarchy, that is, $f_k(k-1) = 1$ for all k, and that all agents are homogeneous in terms of their cognitive abilities, that is, $\Phi(\bar{k}) = 1$ for some $\bar{k}$. Then output is the solution to the following fixed point problem:

$$y_t = \mathcal{C}_t^{\bar{k}} \left(y_t, z^t \right).$$

Finally, a *rational-expectations equilibrium* is a temporary equilibrium in which expectations are consistent with the equilibrium path for these variables: $y_{t+s}^* = \mathbb{E}_t[y_{t+s}]$ for all t and s. As explained below, rational-expectations equilibria can arise as limiting cases of the integrated reasoning framework.

[7] Appendix B shows how this is derived from a primitive assumption in linear expectations model.

Consensus beliefs From equations (11) and (12) it is straightforward to derive average beliefs for level-0 individuals. Average beliefs of level-0 people for the mean output function evolve according to:

$$\hat{\mathcal{Y}}^0_{t+1}(Z) = \hat{\mathcal{Y}}^0_t(Z) + \frac{\hat{\Sigma}_t(Z, Z^t)}{\hat{\Sigma}_t(Z^t, Z^t) + \sigma_v^2}\left(y_t - \hat{\mathcal{Y}}^0_t(Z^t)\right). \tag{16}$$

So, at every t, average level-0 beliefs about output in the future are equal to $\mathbb{E}^0_t\left[y_{t+s}\right] = \hat{\mathcal{Y}}^0_t(z^{t+s})$. For higher k, beliefs are determined according to (8). Consensus beliefs are thus equal to:

$$\overline{\mathbb{E}}_t[y_{t+s}] = \sum_{k\geq 1} \Phi(k)\mathbb{E}^k_t[y_{t+s}]. \tag{17}$$

Convergence Because of the non-ergodicity of the model, the real-time learning process does not yield point-wise convergence of the perceived law of motion to the rational-expectations mapping. However, beliefs about long-run output coincide with long-run rational expectations beliefs if the central bank announces an interest rate path that eventually reverts to the steady-state value.

Lemma 2 *Suppose that the central bank commits at time-0 to the interest rate path in equation (2). Then, consensus beliefs converge to rational expectations for $t \geq T$.*

The intuition for the result is the following. Under assumption 1 with initial mean and variance given by (14) and (15), level-0 people are certain that output will be equal to the steady-state value once the state variable z has reverted to steady state. This property is consistent with the idea that level-0 people have no prior knowledge about the economy, except for the initial steady state, and it implies that beliefs about the long-run are anchored. This belief is self-confirming, since it represents the rational-expectations equilibrium. All individuals with higher level of sophistication understand it, and expect output to be at steady state. So, everyone expects the rational-expectations level of output after T, and beliefs are consistent with realized output. Note that the proof relies on the fact that this model only has future interest rates as an exogenous aggregate state variable. However, it is possible to extend the result to models with backward-looking state variables, provided that they eventually converge back to steady state.

In integrated reasoning, beliefs can also converge because of the structural thinking that individuals perform at each period, as their level of sophistication grows. In the limit case in which everyone is rational and knows that everyone else is rational, the economy is in a rational-expectations equilibrium. This can be obtained as the limiting behavior of a level-k economy in which $f_k(k-1) = 1$ for all k, and $\Phi(\overline{k}) = 1$ for $\overline{k} \to \infty$.

Lemma 3 *Suppose that $f_k(k-1) = 1$ for all k, and $\Phi(\bar{k}) = 1$ for $\bar{k} \to \infty$. Then the economy is in a rational-expectations equilibrium.*

Next, I turn to characterizing the dynamic behavior of output in response to the forward guidance announcement.

2.1 The dynamic effects of forward guidance

In this section, I discuss the dynamic effects of the a forward guidance announcement. The novelty of the expectation model that I propose is that it allows for expectations revision following an unconventional policy announcement within a model of imperfect understanding of general-equilibrium effects – a component of expectations that has been emphasized by the literature on bounded rationality. In most versions of these models, there is no expectation revision process. Importantly, the dynamic response of macro variables to a forward guidance announcement will depend crucially on the feedback between outcomes and expectations that is typical of learning models.

In line with the original formulation of generalized level-k thinking model in Camerer et al. (2004), I make the following assumption about perceived distributions $f_k(\cdot)$ and the actual distribution $\Phi(\cdot)$ of agents along the cognitive hierarchy.

Assumption 2 *The distributions $f_k(\cdot)$ are such that, for any $h < k$,*

$$f_k(h) = \frac{\Phi(h)}{\sum_{s=1}^{k-1} \Phi(s)}$$

Assumption 2 says that agents have an accurate guess about the relative proportions of agents who are doing less thinking than they are. This specification implies that, in the limit, agents doing k and $k-1$ steps of thinking will have the same beliefs, which become arbitrarily close to the actual distribution. Note that, under this specification, a rational-expectations equilibrium is a special case of the generalized level-k thinking model when $\Phi(\infty) = 1$.

The output effect of a forward guidance shock can be decomposed into a partial-equilibrium and a general-equilibrium component. The partial-equilibrium component captures the response of output to the forward guidance announcement, holding initial expectations fixed. This effect reflects intertemporal substitution of consumption due to interest rate changes. The general-equilibrium component takes changes in expectations into account. This effect reflects the general-equilibrium effect of policy changes on expected future income.

Specifically, let

$$y_t^{PE} = \mathcal{C}_0(0, z^t),$$

and

$$y_t^{GE} = \mathcal{C}_t(y_t, z^t) - \mathcal{C}_0(0, z^t).$$

Here $\mathcal{C}_0(0, z^t)$ represents aggregate demand at time t at the announced interest rate path z^t, when expectations are formed at time zero (as indicated by the subscript in $\mathcal{C}_0$), and for time t income kept at the steady state level of zero. $\mathcal{C}_t(y_t, z^t)$ instead is aggregate consumption after accounting for the change in expectations and in income at time t. Clearly, the total response of output is the sum of the partial and general equilibrium components, $y_t = y_t^{PE} + y_t^{GE}$. I will refer to y_t as the time-t elasticity of output.

Time-0 response The analysis of the time-0 extends the results in Farhi and Werning (2019) to the generalized level-k thinking model presented above. It is straightforward to notice that the learning model plays no role in shaping the time-0 response of expected and actual outcomes to the forward guidance announcement.

The interest rate elasticity of output at time-0 is given by:

$$y_0 = \frac{1-\beta}{\beta} \sum_{s=1}^{T-1} \beta^s \overline{\mathbb{E}}_0[y_s] + \sigma \beta^{T-1}, \tag{18}$$

The rational expectations elasticity is equal to:

$$y_0^* = \sigma.$$

The corresponding partial- and general-equilibrium decomposition is such that:

$$y_0^{*,PE} = \sigma \beta^{T-1} \qquad \text{and} \qquad y_0^{*,GE} = \sigma(1 - \beta^{T-1}).$$

The next proposition shows how the interest rate elasticity of output behaves in the generalized level-k thinking model presented above.

Proposition 1 *Suppose assumption 2 holds. The interest rate elasticity of output at time 0 with generalized* *level-k thinking, given by equation* (18)*, is lower than the rational-expectations counterpart:*

$$y_0 \leq y_0^*.$$

The two elasticities coincide in the limiting case where there is a $\bar{k} \to \infty$ *such that* $\Phi(\bar{k}) = 1$.

As a natural extension to the result with standard level-k thinking, the proposition argues that the output response to interest rate announcements is dampened in the presence of bounded rationality. Also in this setting, the result follows from the fact that

$$y_0^{GE} \leq \sigma(1 - \beta^{T-1}),$$

with equality holding only for the limiting case of $\Phi(\infty) = 1$. Under bounded rationality, people fail to fully take into account future general-equilibrium effects of the policy announcement. Because they expect lower incomes in the future, their consumption choices are less responsive to the interest rate change announcement.

Impulse response to an announcement So far, I have only considered the effects of an announcement on impact. As time passes, it is natural to allow for agents to revise their expectations about future incomes after observing new information. The integrated reasoning model provides a set-up where individual expectations are revised over time. All agents combine Bayesian revision of forecasts with structural knowledge of the economy through a finite number of rounds of deductive thinking, except for level-1 agents, who have no understanding of the working of the economy.

Learning leads households to revise their expectations about the effectiveness of monetary policy announcements. Since they do not perfectly observe the realization of output, they have to infer information about the equilibrium mapping between announced paths of interest rates and endogenous outcomes. Consensus beliefs of level-0 agents are updated as in (16).

Individuals with higher levels of cognitive sophistication combine the information from belief updates of other people with their structural knowledge of the economy, and revise themselves their expectations. At each period, they are able to understand how level-0 people have formed a new path of expected income, and reason through the general-equilibrium effects of the belief revision. More sophisticated agents arrive at their updated beliefs after a higher number of iteration of their reasoning. That is, they form more accurate beliefs about the path of output.

At each date, aggregate demand depends positively on expectations about future incomes.

So, the extent of belief revision about output determines the strength of the impulse response of consumption demand. Stronger expectations updates imply a larger effectiveness of policy announcement. This is clear by inspection of the time-t interest rate elasticity of output, which generalizes equation (18) to:

$$y_t = \frac{1-\beta}{\beta} \sum_{s=1}^{T-t-1} \beta^s \overline{\mathbb{E}}_t[y_{t+s}] + \sigma \beta^{T-t-1}. \tag{19}$$

Belief updates about output directly map into revisions of the expected future interest rate elasticity $\overline{\mathbb{E}}_t[y_{t+s}]$, which in turns determines the current elasticity of output. The degree of expectations revision depends on one structural characteristic of the integrated reasoning model. Revisions depend on the informativeness of the signal received by households. This feature is governed by the parameter σ_0^2. If the variance of the time-0 prior is relatively high, individuals consider their signals about income as informative, and revise their expectations relatively more. This result is easy to show for revisions in the first period after the announcement, and can be verified numerically for the other periods. The more expectations are revised, the higher income people expect for the future, the more they react to the policy announcement. Hence, the ratio is a crucial parameter for determining the quantitative importance of learning for the impulse response function of output to a forward guidance announcement.

This result is summarized in the following lemma.

Lemma 4 *Forecast revision at time 1, $\Delta\overline{\mathbb{E}}_1[y_t]$, is increasing in σ_0^2, for all $t \geq 1$, where:*

$$\Delta\overline{\mathbb{E}}_1[y_t] = \sum_{k\geq 1} \Phi(k) \left(\mathbb{E}_1^k[y_t] - \mathbb{E}_0^k[y_t] \right).$$

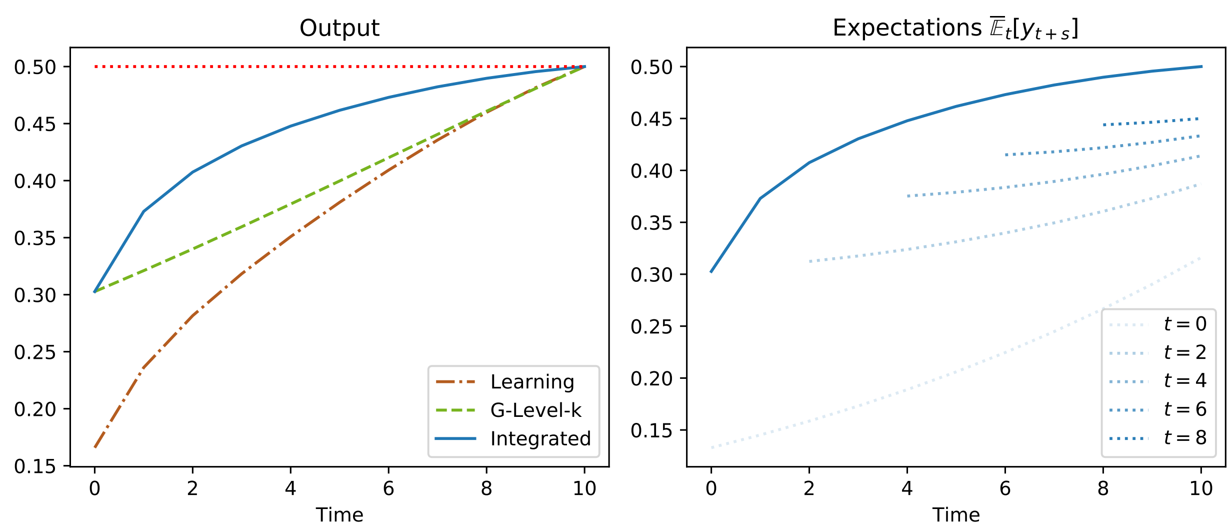
Figure 1 shows output response. Section 4 has details on the calibration. The horizon of the interest rate shock announcement is T equal to 10.

The left panel of the figure displays the response of output under rational expectations as a benchmark case (dotted red line). The three other lines correspond to the following assumptions. The dash-dotted brown line corresponds to the case in which individuals do not have any understanding of the structure of the economy ($\lambda = 0$). The dashed green line shows output under the assumption that households do not revise expectations over time ($\sigma_0^2 = 0$). Finally, the solid blue line is an example of integrated reasoning ($\lambda = 1.5, \sigma_0^2 = 1$).

The integrated reasoning model predicts a mitigated effect of forward guidance at all horizons relative to the case of rational expectations. The intuition for this result is the same as for the time-0 elasticity: due to the limited ability of individuals to understand the future general-equilibrium implications of a policy announcements, their effectiveness is dampened compared to rational expectations. However, the integrated reasoning model predicts that the elasticity is larger than in the two benchmark cases of pure level-k thinking and pure learning. When combining belief revision with strategic thinking about their future effects, individuals anticipate larger income responses, which boosts current demand.

The right panel shows how expectations about future income is revised over time for the case of integrated reasoning. At time 0, consensus expected output is traced by the light-colored dotted blue line labeled as $t = 0$. After observing a forecast error, consensus beliefs about output is revised through a combination of belief revision and strategic thinking. This process gradually leads to revise expected future incomes, as shown by the other dotted blue lines for times $t = 2, 4, 6$ and 8. The graph highlights the importance of expectations revision in determining the overall effect of a forward guidance announcement: the dynamic effects of the policy are shaped by the updated expectations about future incomes that follows from the belief revision model. As is clear from equation (19), the update in expectations about the future effects of the policy, captured by $\overline{\mathbb{E}}_t[y_{t+s}]$, act as an additional stimulus to current demand, and ultimately feeds back into stronger realized output response at each date. By comparison, the generalized level-k thinking model implies that, over time, the path of expected income is constant and equal to the initial path represented by the $t = 0$ light-colored dotted blue line.

Figure 1: Impulse Response Function to a Forward Guidance Announcement ($T = 10$)



In order to assess the efficacy of forward guidance announcement in this setting, it is therefore fundamental to discriminate between the different models nested within the integrated reasoning framework. To do so, in the next section I present results about the response of forecast errors and forecast revisions to the forward guidance announcement.

2.2 Anticipation, forecast errors and forecast revisions

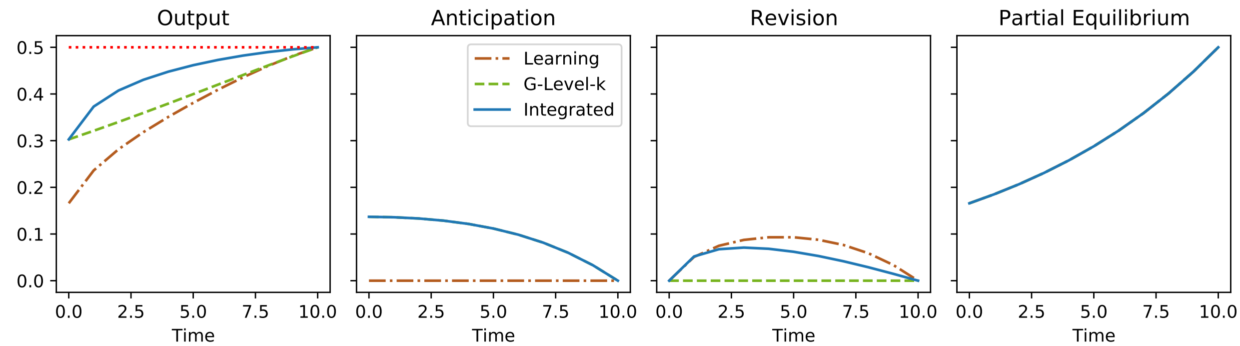
To understand the role of expectations formation, it is useful to further decompose the time-t elasticity in (19). The general-equilibrium component can be separated into an effect that is due to the *anticipation* of future changes to equilibrium income, and an effect that is due to *expectations revision* over time. Output elasticity can be decomposed as follows:

$$y_t = \underbrace{\frac{1-\beta}{\beta}\sum_{s=1}^{T-t-1}\beta^s\left(\overline{\mathbb{E}}_t[y_{t+s}] - \overline{\mathbb{E}}_0[y_{t+s}]\right)}_{belief\ revision} + \underbrace{\frac{1-\beta}{\beta}\sum_{s=1}^{T-t-1}\beta^s\overline{\mathbb{E}}_0[y_{t+s}]}_{anticipation} + \underbrace{\sigma\beta^{T-t-1}}_{partial\ equilibrium} . \tag{20}$$

Here, $\overline{\mathbb{E}}_0[y_{t+s}]$ is the expected elasticity of output in the absence of learning, which is derived from beliefs formed at time 0. The anticipation effect is a consequence of the structural knowledge of the economy that individuals can achieve via generalized level-k thinking at time 0. By construction, it corresponds to the entire general-equilibrium effects of the policy announcement in the benchmark case of generalized level-k thinking. Because households are able to partially work through the implications of the policy announcements, they incorporate higher future incomes in their expectations. This effect is what determines the impact response of output to interest rate announcements above and beyond the partial equilibrium effect. However, in the absence of a feedback between outcomes and new expectations, individuals do not revise their forecasts over time.

The effect due to belief revision is instead a consequence of learning. Upon observing new information, individuals revise their expectations accordingly. A sequence of positive forecast errors, therefore, leads individuals to updated their future income expectations upwards. However, in the absence of any structural knowledge of the economy, the extent of the revision is dampened. Figure 2 shows the decomposition of the impulse response function in equation (20) for the case of a forward guidance announcement about $T = 10$.

Figure 2: Decomposition of Impulse Response Function



The behavior of forecast errors and revisions in response to the announcement provides a tool for validating the expectations model when comparing the predictions to the data. Figure 3 displays forecast errors

$$y_{t+1} - \overline{\mathbb{E}}_t[y_{t+1}]$$

on the left panel, and forecast revisions

$$\overline{\mathbb{E}}_t[y_{t+1}] - \overline{\mathbb{E}}_{t-1}[y_{t+1}]$$

on the right panel.

It is commonly assumed in models of level-k thinking that expectations are formed at time 0 once and for all, and then never revised. This assumption implies that forecast revisions at every horizon are equal to zero. Since the output response to an announcement grows larger over time, another implication of this assumption is that also forecast errors become bigger over time. These implausible features of the level-k thinking model are absent from the integrated reasoning model because individuals learn over time.

Under the assumption of a learning model with no structural knowledge of the economy, instead, expectations are revised in the light of new information. However, since agents have not observed a forward guidance announcement in the past, they do not revise expectations at time 0, when the policy change is communicated.

When individuals also reason strategically about the future, they are better able to predict output response at future dates. As a result, the cumulative belief revision about a specific future period is larger, implying a decrease in forecast errors over time. Hence, the interaction of expectation update and reasoning about the economy is crucial for ensuring that individuals' predictions are closer to the actual realizations over time.

These predictions are summarized in the following Lemma.

Lemma 5 *The following properties of forecast errors and revisions are true:*

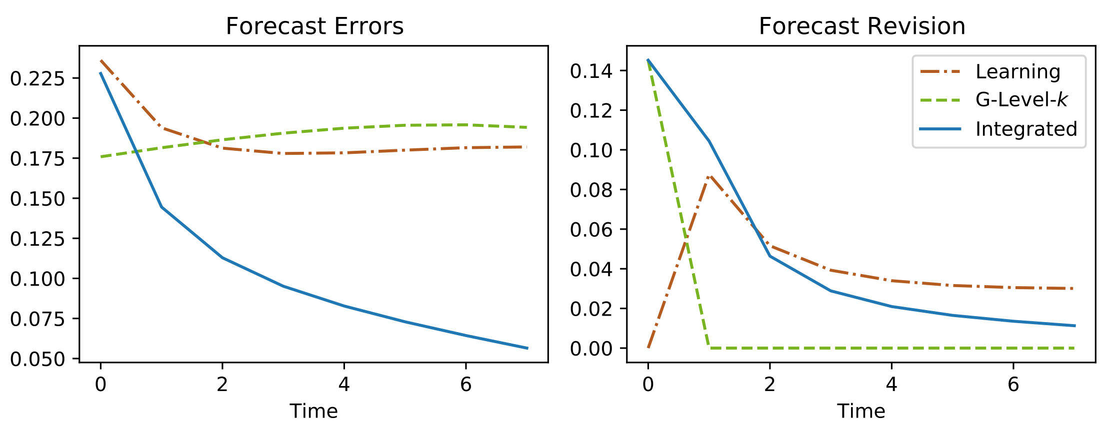
1. *Belief revision at $t = 0$ is equal to zero if and only if $\Phi(0) = 1$.*

2. *Individual expectations underreact to the announcement if and only if $\Phi(\infty) < 1$.*

3. *Beliefs display a process of gradual revision over time if and only if $\sigma_0 > 0$.*

This discussion illustrates the importance of using forecast errors and revisions as a diagnostic tool for selecting the right model of expectations. In this numerical example, the three models produce very starkly different implications for the behavior of errors and revisions in response to a forward guidance announcement. Importantly, the time-0 response alone is not enough to discriminate between a generalized level-k thinking model and a the integrated reasoning model. The impossibility stems from the fact that both models produce the same impact response in terms of outcomes, forecast errors and revisions.

Figure 3: Forecast Errors and Forecast Revisions



3 Expectations data and forward guidance shocks

In this section, I use evidence from forward guidance shocks and survey data on expectations about aggregate outcomes to provide empirical estimates of the response of forecast errors and revisions to announcements about future policy rates. The goal is twofold. First, I show that the response of forecast errors and revisions is qualitatively in line with the predictions of the integrated reasoning model. Second, I discuss what moments of the data are useful to identify the parameters from the integrated reasoning model.

Expectations data Survey data on expectations is obtained from the Survey of Professional Forecasters (SPF), a panel survey of about 40 experts from industry, government, and academia. Each quarter, survey participants are asked for point-estimate projections of several macro aggregates. My main analysis focuses on projections about GDP growth, civilian unemployment, and inflation measured as the GDP deflator growth rate. When using measures of consensus forecasts, I show results for the median forecast of the object of interest to alleviate concerns about outliers, given the small size of the cross-section.

The forecast horizon in my baseline analysis is annual. For GDP and inflation, I compute the annual predicted growth rate from $t-1$ to $t+3$. Realization at $t-1$ are known at the time of the forecast, consistent with the forecasters' information set. Forecast errors and revisions are then computed as:

$$fe_{i,t} = x_{t+3} - \mathbb{E}_{i,t}[x_{t+3}] \qquad\qquad fr_{i,t} = \mathbb{E}_{i,t}[x_{t+3}] - \mathbb{E}_{i,t-1}[x_{t+3}],$$

where x_{t+3} is the variable of interest.

Forward guidance shocks Forward guidance shocks are identified using high-frequency changes to prices of financial instruments in a 30 minutes window within each FOMC announcement. The shock series that I employ is the one derived in Jarociński (2021), who exploits the leptokurtic nature of the financial markets response to FOMC announcements to identify conventional and unconventional monetary policy shocks. The methodology employed in the paper is able to separate the effects of two different forward guidance shocks: a commitment to a future course of policy rates ("Odyssean" forward guidance), and a statement about the future course of policy rates understood as a forecast of the appropriate stance of the policy ("Delphic" forward guidance). The former is particularly suited to be brought to discipline the model I present in Section 2, because it maps into an announcement of a future path of policy rates that the central bank promises to commit to. This series is the one that I will use throughout the analysis. In particular, the identified shocks imply a change in the term structure such that the short term rate is unchanged, while the high-frequency response of the 2-year rate is equal to one basis point.

The identification strategy is based on mutual independence and non-Gaussianity of the shocks, following Lanne and Lütkepohl (2010) and Lanne et al. (2017). The market response to FOMC announcements is studied using the following empirical model:

$$y_t = C' u_t \qquad u_{n,t} \sim i.i.d.\ \mathcal{T}(\nu),$$

where y_t is a vector that collects the first federal funds future, the 2-year and 10-year Treasury yields, and the S&P500 blue chip stock index. The vector u_t represents the underlying structural shocks, that are assumed to be independent and to follow a t-Student distribution with parameter ν. Jarociński (2021) shows how this distribution provides a substantially better fit for high-frequency price changes around FOMC announcements. Additional details for the estimation are provided in Appendix C.

The first three financial prices collected in y_t serve the purpose of identifying the target, path, and large scale asset purchase (LSAP) factors of monetary policy (in line with Swanson (2021)). A strand of literature has focused on understanding the role of the information component in monetary policy announcements, and how to clean pure policy shocks from its effects. The inclusion of the S&P500 is meant to deal with this issue, since stock prices reflect expectations about the future course of the economy.

The two forward guidance shock series delivered by this procedure have similar impacts the yield curve, but opposite effects on the stock market reaction. One is thus interpreted as a commitment to a future path for interest rates, with standard contractionary effects following an increase in interest rates. This shock is associated with an impact reduction in stock prices, and it is the forward guidance shock series that I will use for my analysis. I will denote this series as $u_{FG,t}$. Conversely, the other series is associated with a stock price increase, which can be interpreted as an information shock about future economic conditions. In Appendix C I run local projection regressions as in (21) with expectations data as outcome variables. The impulse responses of expected output growth, unemployment, and inflation are in line with the structural interpretation of the two shocks: a pure policy shock increases output and inflation expectations, and lowers unemployment expectations, while an information shock has the opposite effects. To further alleviate concerns about the potential information component of the announced future course of policy rates, I perform robustness checks in Appendix C.

3.1 Forecast errors and revisions

I estimate the effects of forward guidance shocks on consensus forecast errors and revisions using local projections (LP) as in Jordà (2005) from the following equation at the individual level:

$$z_{i,t+h} = \alpha_h + \beta_h u_{FG,t} + \gamma_h' W_{i,t} + \varepsilon_{i,t+h}, \tag{21}$$

where $(\beta_h)_h$ trace out the dynamic response of the outcome $z_{i,t+h}$. The vector $W_t = (z_{i,t-1}\ u_{FG,t-1})$ collects lagged values of the dependent variable and the shock. Finally, $u_{FG,t}$ is one of the forward guidance shock series extracted by Jarociński (2021). The shocks are aggregated at the quarterly level with a simple sum. The time aggregation produces validly identified monetary policy shocks under the assumption that the shocks are orthogonal to other economic variables in that quarter. Errors are clustered at the quarter level.

Figure 4 shows the impulse response of forecast errors on the left panel, and of forecast revisions on the right panel. The results give an informative picture about the nature of expectations formation and update. First, the effects of the announcement are partly incorporated into forecasts at the time of the shock, as shown by the impact response of revisions. Second, forecast errors are predictable by the forward guidance shock at initial horizons after the shock realization, and they are consistent with underreaction to the shock. Third, forecast revisions display a slow patter of revisions in the quarters following the shock. Errors and revisions gradually revert back to zero at longer horizons.

The fact that forecast errors and forecast revisions are predictable constitute evidence that rejects rational expectations, full information models. This fact alone does not help in telling apart whether both deviations are present at the same time. In settings with rational expectations where agents do not have common knowledge, belief updates are sluggish since updates are an average of past beliefs and past realizations. Hence, belief updates and forecast errors are predictable with past information. Conversely, the lack of rational expectations, even in models with full information, implies by definition that individuals make systematic forecast mistakes.

However, the evidence is informative for the choice of models that nested within the integrated reasoning framework. Lemma 5 provides necessary and sufficient conditions for determining what features of the model are necessary to generate these patterns. On the one hand, a model with bounded rationality without private information, where agents do not update beliefs over time, is inconsistent with the evidence that forecast revisions are statistically different from zero at horizons following the time of the shock. On the other hand, a model with learning where agents do not use any structural knowledge is inconsistent with the data in that it implies *i)* that beliefs do not react on impact to the forward guidance announcements, and *ii)* that forecast errors grow over time. So, both elements are needed to match the evidence on predictable forecast errors and revisions.

Importantly, these responses are derived from individual-level expectations, and as such they are informative about the extent to which individual expectations deviate from the rational-expectations benchmark. Forecast errors support the hypothesis that people underestimate the effects of a forward guidance announcement. This contrasts with the findings in Bordalo et al. (2020), who find evidence of overreaction to in macroeconomic expectations. As I explain later, the key difference is that they look at unconditional correlations between forecast errors and revisions, whereas I analyze their behavior in response to a specific policy shocks.

For completeness, Figure **XXX** in Appendix C also reports the response of consensus beliefs. The response is very similar to the individual-level behavior.

3.2 Calibration of integrated reasoning parameters

The evidence from forecast errors and revisions is also useful for quantitative guidance on the prior variance. The calibration of the prior variance parameter σ_0 is particularly important for the quantitative role of learning in the dynamic response to forward guidance announcements. In practice, the variance determines how fast beliefs are updated after observing forecast errors.

The moment that I am interested in exploiting to calibrate the prior variance is the correlation of residual variations in forecast errors and revisions in response to a forward guidance shock. This helps answering the question: How much are forecasts revised on average after observing a forecast error, conditional on a forward guidance shock?

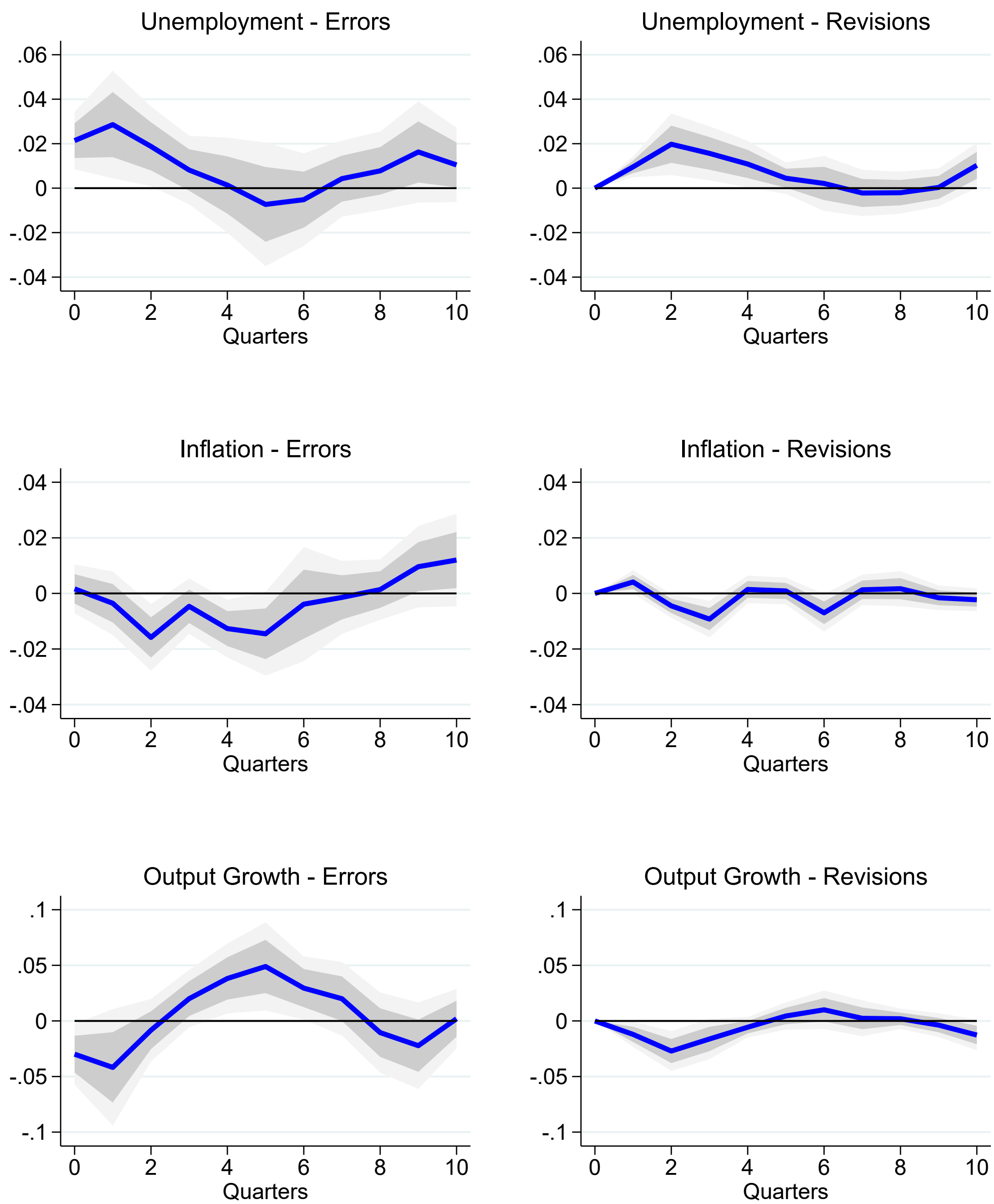
To answer this question, I run the following regression at the individual level:

$$z_{i,t} = \alpha + \beta^z u_{FG,t} + \gamma' W_{i,t} + \varepsilon_{i,t}, \tag{22}$$

separately for forecast errors $fe_{i,t}$ and revisions $fr_{i,t+1}$. That is, $\hat{\beta}^{FE}$ measures the average forecast error in the quarter of the forward guidance shock, and $\hat{\beta}^{FR}$ measures the average forecast revision in the first quarter after the shock. The ratio of the estimated coefficients on forward guidance

Figure 4: Forecasts Errors and Revisions in Response to Forward Guidance Shocks

...



shocks, which I denote by $\hat{g}$, is the quantity of interest:

$$\hat{g} \equiv \frac{\hat{\beta}^{FR}}{\hat{\beta}^{FE}}. \tag{23}$$

Table 1 shows the statistic for unemployment, output growth, and inflation separately. Standard errors are computed with the bootstrap method.

Table 1: Conditional forecast errors and revisions

	Unemployment	Output	Inflation
$\hat{g}$	0.505***	0.564***	0.479*
	(0.0558)	(0.155)	(0.268)
Observations	2567	2449	2340

The results suggest that, in the case of unemployment, output, and inflation, forecast revisions in the quarter following the shock are on average 50%, 56%, and 47% as large as the initial forecast error, respectively. In the calibration exercise, I will target an average response of forecast revisions to forecast errors of 0.5.

The corresponding moment in the model g is given by the response of forecast revisions relative to forecast errors:

$$g = \frac{fr}{fe} = \frac{\overline{\mathbb{E}}_0[y_1] - \overline{\mathbb{E}}_{-1}[y_1]}{y_0 - \overline{\mathbb{E}}_{-1}[y_0]},$$

which depends on σ_0. Intuitively, the higher σ_0, the more individuals trust the signal they receive, the stronger expectations revisions will be. A larger revision in expectations drives the covariance between forecast errors and revisions upward. Figure **XXX** in Appendix C shows values of g for different levels of σ_0.

It is interesting to compare this moment to existing work on the response of beliefs to macroeconomic shocks. Note that $\hat{g}$ can be interpreted as the conditional projection of forecast errors on forecast revisions, since

$$\hat{g} = \frac{cov(\hat{fr}_{i,t+1}, \hat{fe}_{i,t})}{var(\hat{fe}_{i,t})},$$

where $\hat{fr}_{i,t}$ and $\hat{fe}_{i,t}$ are the projections of $u_{FG,t}$ on $fr_{i,t}$ and $fe_{i,t}$. The moment that I estimate is similar to the conditional counterpart of the regressions estimated in Bordalo et al. (2020) (BGMS), which is obtained as follows:

$$fe_{i,t+1} = \alpha + \beta\, fr_{i,t} + \varepsilon_{i,t} \tag{24}$$

The estimated coefficient from (24), denoted by $\hat{\beta}^{BGMS}$, represents the unconditional relationship between forecast errors and revisions. They find that $\hat{\beta}^{BGMS} < 0$, consistent with overreaction: after a positive (negative) forecast revision, forecast errors tend to be negative (positive), indicating that forecasters overreacted when revising expectations. My estimates of $\hat{g}$, consistently with the dynamic response of errors and revisions discussed above, support the hypothesis that expectations underreact to an unconventional policy shock.

Taken together, the evidence from forecast errors and revisions suggests that the model that I presented in Section 2 is able to replicate the initial revision and underreaction followed by a slow update of expectations. It also provides a simple moment that can be used to pin down the response of expectations to a forward guidance shock.

Other parameters The integrated reasoning model has three more free parameters. First, λ determines the depth of general-equilibrium thinking. I take this parameter from Camerer et al. (2004), who first introduces the cognitive hierarchy model. Like in my framework, individuals' cognitive sophistication follows a Poisson distribution. They estimate λ from individual expectations in an experimental setting, and find $\lambda = 1.5$ as a good fit for the data. This implies that 80% of people are level-2 or below. I choose this parameter for my benchmark analysis. Iovino and Sergeyev (2018) estimate a model of level-k thinking from data on forecast errors in response to large scale asset purchases by the Federal Reserve. They find an estimate for the average sophistication level that implies 99% of people are level-2 or below.[8] In Appendix E, I perform robustness exercises to match this estimate.

The parameter that governs the prior correlation between output realizations after interest rate announcements at different horizons is ρ. An implication of the prior covariance function specified in (15) is that output follows an AR(1) process following a one-time interest rate change announcement in the future, with persistence given by ρ. I set $\rho = 0.87$ to match the estimate of the persistence parameter from an AR(1) process on output. I perform several robustness checks in Appendix E on this parameter, and show that results are not particularly sensitive to this choice.

Finally, the noisiness of individual signals about endogenous variables is determined by σ_v^2. This parameter pins down cross sectional belief dispersion. I separately calibrate one for each endogenous variables that individuals need to forecast (in the model in section 4, output and

[8]However, their model differ from the generalization of level-k thinking that I present here. Their level-k thinkers believe that every one else is just one level of sophistication below, but there is a full distribution of level-k thinkers. This implies that higher k thinkers update beliefs faster than in my model. For this reason, their model requires a lower average level of sophistication than the one presented here would require to match the same empirical moment.

inflation). I set them to match the cross-sectional variance in forecasts of output and inflation, measured as

$$\hat{\varsigma} = \mathbb{E}[(x_{i,t} - \overline{x}_t)^2],$$

where $\overline{x}_t = \int_0^1 x_{i,t} di$, and $x_{i,t}$ is the forecast for output growth or inflation of forecaster i at time t. These estimates are displayed in Table 2. For completeness, I also display the same moment for unemployment. To calibrate σ_v^2, I entirely attribute cross-sectional variation in steady-state beliefs to the idiosyncratic signal that people receive, so that $\sigma_v^2 = \hat{\varsigma}$.

Table 2: Conditional forecast errors and revisions

	Unemployment	Output	Inflation
$\hat{\varsigma}$	0.084	0.349	0.267
Observations	4260	4138	3981

4 "Lower-for-longer" interest rate: the role of learning

Forward guidance has become an indispensable tool for macroeconomic stabilization when nominal interest rates are constrained at zero. A prolonged shock to the natural interest rate that causes the economy to hit the ZLB can cause a large recession in the absence of monetary stimulus. A natural question is how contractionary is such recession under my assumptions about expectations. In this context, I will assess the efficacy of announcing that interest rates will be kept at zero for additional quarters after the recession is over.

To this end, I extend the baseline model along two dimensions. First, I introduce a discount-factor shock that triggers a ZLB episode. As in Eggertsson (2011), Christiano et al. (2011), and Correia et al. (2013), I assume that the steady state discount factor β is perturbed by a shock ξ_t such that:

$$\xi_t = e^{-\chi(T-t)}$$

for $t = 0, 1, \ldots, T$, and $\xi_t = 1$ for $t \geq T$. The size of the shock χ is such that the effective discount rate turns negative, triggering the ZLB for the first T periods. Household preferences are modified to:

$$\sum_{s \geq 0} \beta^s \xi_{t+s} \left[u\left(C_{t+s}\right) - v\left(N_{t+s}\right) \right].$$

Second, I introduce Calvo-style wage rigidities as in Erceg et al. (2000) and Schmitt-Grohé and

Uribe (2005), which generates wage and price inflation. Because of wage stickiness, actual employment is demand determined.

Monetary policy As before, the central bank announces a path for nominal interest rates. During the ZLB period, interest rates are constrained at zero. When implementing forward guidance as a "lower-for-longer" policy, the central bank promises to keep interest rates at zero for an additional τ quarters after the ZLB period is over. Interest rates revert back to steady state after the normalization period. This is given by:

$$R_t = \begin{cases} 1 & t \leq T + \tau \\ \beta^{-1} & t > T + \tau. \end{cases}$$

Final good and labor market There is a continuum of unions in the economy, indexed by $u \in [0, 1]$. Each household has a continuum of workers with differentiated labor skills, and provides $n_{u,t}$ units of type u. The composite labor input N_t is generated using labor varieties according to the technology:

$$N_t = \left[\int_0^1 n_{u,t}^{\frac{\theta-1}{\theta}} du \right]^{\frac{\theta}{\theta-1}},$$

where $\theta > 1$ captures the elasticity of substitution across the labor varieties. The final good technology is linear in composite labor input, so $Y_t = N_t$. The final good producers' maximization problem implies the following labor demand condition for labor variety u:

$$n_{u,t} = \left(\frac{w_{u,t}}{W_t} \right)^{-\theta} N_t, \tag{25}$$

where $w_{u,t}$ is the wage for $n_{u,t}$, and the aggregate wage level is

$$W_t = \left[\int_0^1 w_{u,t}^{1-\theta} du \right]^{\frac{1}{1-\theta}}.$$

Because of the linear production technology, the price of the consumption good is equal to the wage level, $P_t = W_t$.

Unions set wages subject to a Calvo-style friction. At each time, a fraction $1 - \epsilon$ are randomly selected to adjust their wage, $w_{u,t}$. For the other ϵ unions, $w_{u,t} = w_{u,t-1}$. On behalf of households, unions choose wages and labor hours to maximize the expected household's valuation of labor income, subject to the labor demand given by (25). In a symmetric equilibrium, the common reset wage W_t^* chosen by unions is given by:

$$\frac{W_t^*}{P_t} = \frac{\theta}{\theta-1} \frac{\sum_{s\geq 0} \left(\hat{\beta}\epsilon\right)^s \xi_{t+s} \left(\frac{P_{t+s}^e}{P_t}\right)^{\theta} \left(\frac{W_{t+s}^e}{P_{t+s}^e}\right)^{\theta} N_{t+s}^e v'(N_{t+s}^e)}{\sum_{s\geq 0} \left(\hat{\beta}\epsilon\right)^s \xi_{t+s} \left(\frac{P_{t+s}^e}{P_t}\right)^{\theta-1} \left(\frac{W_{t+s}^e}{P_{t+s}^e}\right)^{\theta} N_{t+s}^e u'(C_{t+s}^e)}.$$

The linearized economy The model reduces to the familiar two-equations system, which in this case takes the following form:

$$\begin{aligned} y_t &= \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} \beta^s \overline{\mathbb{E}}_t[y_{t+s}] - \sigma \sum_{s=0}^{\infty} \beta^s \left(\chi_{t+s} + i_{t+s} - \overline{\mathbb{E}}_t[\pi_{t+s+1}]\right) \\ \pi_t &= \kappa_w \sum_{s=0}^{\infty} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[y_{t+s}] + \frac{1-\epsilon}{\epsilon} \sum_{s=1}^{\infty} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[\pi_{t+s}], \end{aligned}$$

where $\kappa_w = \frac{(1-\epsilon)(1-\hat{\beta}\epsilon)}{\epsilon}\left(\varphi + \sigma^{-1}\right)$, and $\hat{\beta}$ represents the discount factor of unions. As discussed in the calibration in section 4.1, I allow for the household's discount factor to differ from the unions' as a reduced-form way to capture market incompleteness on the households, as in Farhi and Werning (2019) and Angeletos et al. (2020). Log-deviations of the nominal interest rate and discount factor shock from steady state are denoted as i_t and χ_t, respectively. The derivation of these equations, together with more details about the model, are in Appendix D.

Compared to the model in Section 2, the relevant state variable z^t now collects both the announced sequence of interest rates and the sequence for the discount factor shock ξ_t. As before, Households and unions have perfect foresight about z^t, and they now have to form expectations about the future paths for both income and inflation. Every agent in the economy uses integrated reasoning to form expectations about the two mappings from z^t to income and inflation—$\mathcal{Y}_{i,t}(z^t)$ and $\Pi_{i,t}(z^t)$—and they receive separate signals about their current value.

Calibration The model period is one quarter, and the baseline calibration is summarized in Table 3. The choice of parameters for the integrated reasoning model is discussed in Section 3.2. The unions' discount factor $\hat{\beta}$ is set to match a 2% annual nominal interest rate. On the household side, following Angeletos et al. (2020), I set the time preference parameter β to match an average marginal propensity to consume of 0.15. This allows for a steeper Keynesian cross than under standard calibrations, and implies a significant role of general equilibrium forces.[9] Consistent

[9] An equivalent formulation of the model that allows to separately match the discount factor and the average MPC is the one in Del Negro et al. (2012) and Farhi and Werning (2019), where market incompleteness is introduced in the

with evidence in Chetty et al. (2011), I set the Frisch elasticity to $\varphi^{-1} = 0.75$. The fraction of unions maintaining a fixed wage, ϵ, and the elasticity of substitution across labor types, θ, are taken from Correia et al. (2013), who set $\epsilon = 0.85$ and $\theta = 3$. Finally, as in McKay et al. (2016), the discount factor shock is calibrated to match an initial output decline of 4%.

Table 3: Model parameters

Parameter	Description	Value
	Integrated reasoning parameters	
λ	Average Level-k	1.5
σ_0^2	Prior variance	1.51
ρ	Prior persistence	0.87
σ_ν^2	Signal variance	0.3
	Other parameters	
β	Households discount factor	0.85
$\hat{\beta}$	Unions discount factor	0.995
φ	Inverse Frisch elasticity	1/0.75
ϵ	Wage stickiness	0.85
θ	Labor types elasticity	3
χ	Discount factor shock	0.0077

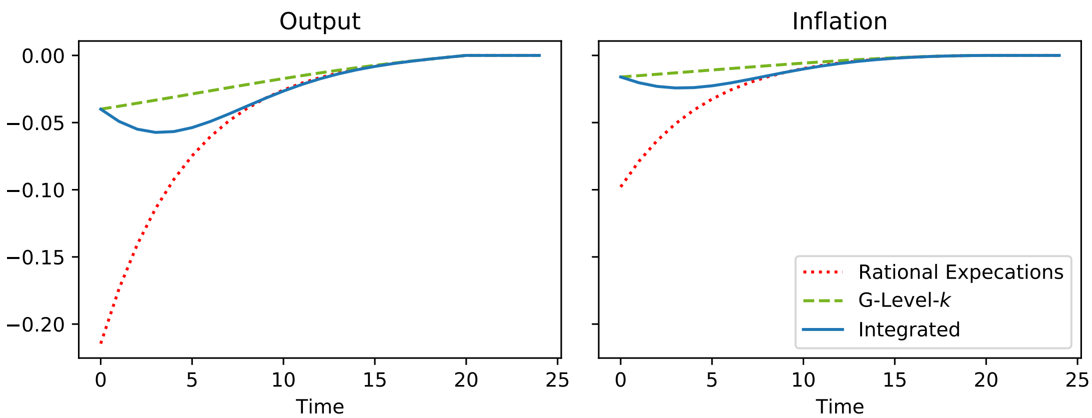
4.1 The ZLB recession across models

I start by considering the effects of a discount factor shock across different models. As in McKay et al. (2016), I simulate a recession that lasts for 20 quarters. In this baseline exercise, the central bank sets $\tau = 0$, that is, interest rates are kept at the ZLB until the recession is over, then they are revert to steady state. Figure 5 shows the dynamic response of output and inflation across three cases: rational expectations, generalized level-k thinking, and integrated reasoning. The effects of the ZLB recession are considerably milder when individuals have bounded rationality relative to the rational expectations case. Because people expect a milder drop in output and inflation relative to rational expectations, the drop in aggregate demand is mitigated. The discount factor shock, calibrated to generate a 4% drop in output on impact with integrated reasoning, implies a 15% drop in output under rational expectations.

form of occasionally exogenously binding borrowing constraint. In the linearized version of that model, the effective discount rate is $\beta = \hat{\beta}\kappa$, where $\hat{\beta}$ is the household discount rate, and κ is the probability of incurring in a binding collateral constraint.

With integrated reasoning, realized forecast errors feed back into forecast revisions, so individuals revise their forecasts in the direction of their initial error. Since the severity of the impact recession is underestimated, level-0 people revise their expectations about future output and inflation downwards. This revision generates a larger recession relative to the level-k economy.

Figure 5: Zero Lower Bound Recession



4.2 The effects of forward guidance: "lower-for-longer"

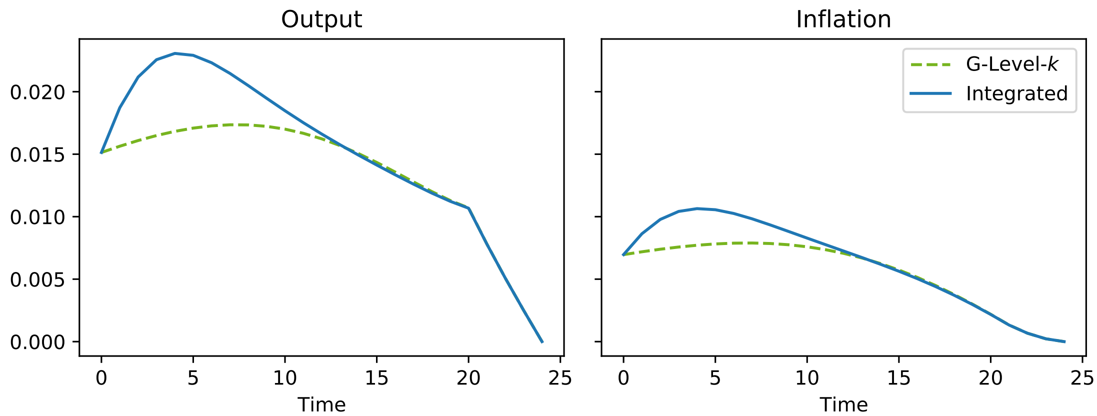
What are the effects of announcing that interest rates will be kept at zero after the ZLB recession is over? In this section, I simulate the response of output and inflation when the central bank promises to keep interest rates at zero for an additional $\tau = 2$ quarters. The promise to generate an expansion in the future is partially anticipated thanks to structural thinking upon announcement. This anticipation implies an immediate response in output. But the dynamic effects of the shock are largely influenced by the strength and direction of expectations revision induced by the policy. The announcement not only stimulates output on impact, but it also implies a reduction in the initial forecast error, since the time 0 prior of level-0 individuals is unaffected by the announcement, but output drops by a lower amount. The smaller initial forecast error implies a weaker downward revision of expectations in the periods after the announcement, which contribute to stimulating consumption.

The impact of the announcement can be computed as follows. Let x_t^0 denote the response of variable x at time t for the benchmark case of ZLB recession with $\tau = 0$, and let x_t^τ denote the same variable under the lower-for-longer policy announcement with $\tau = 4$. The effects of the policy are given by:

$$\Delta x_t = x_t^\tau - x_t^0. \tag{26}$$

Figure 6 shows the response of output and inflation—Δy_t and $\Delta \pi_t$—for generalized level-k thinking and for integrated reasoning. The time-0 response is the same in the two cases, but when expectations are allowed to be revised over time, the initial lower forecast error implies relatively more optimistic expectations about the future, which feed back into stronger aggregate demand. The same effect also drives the inflation response, which feeds back into consumption via the optimal consumption function. The cumulative output effect of forward guidance with integrated reasoning for output and inflation is 38% and 15.8% respectively, corresponding to annualized responses of 1.5% and 0.07%.[10] These responses also imply that integrated reasoning generates a response that is 12.7% and 14.6% larger, respectively, than the generalized level-k thinking model.

Figure 6: Dynamic Effects of Lower for Longer



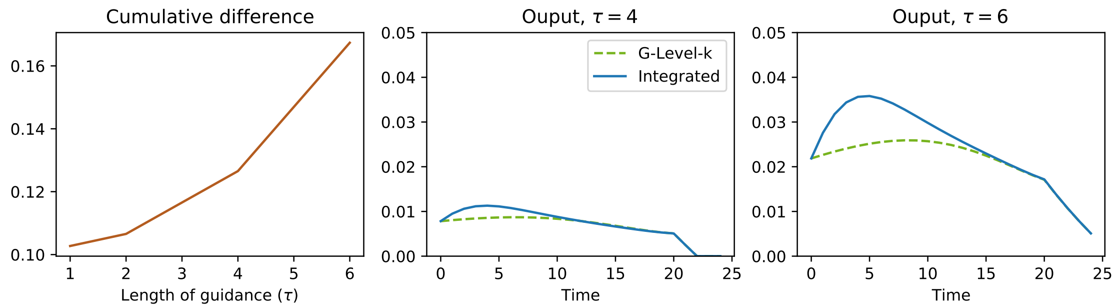
Length of forward guidance and ZLB The interaction between initial forecast error and subsequent revisions described above implies that the benefits of the initial response compounds over time because of belief updates. This result is illustrated when considering the extension of the lower-for-longer policy for more than 2 quarters. The left panel of figure 7 displays the cumulative difference in output effects for integrated reasoning relative to level-k thinking:

$$\mathcal{D} = \frac{\sum_{t\geq 0} \Delta y_t^{IR}}{\sum_{t\geq 0} \Delta y_t^{GLk}} - 1. \tag{27}$$

With longer guidance, the larger impact effect also generates larger revisions, and contributes to stimulating output through the learning channel in subsequent periods.

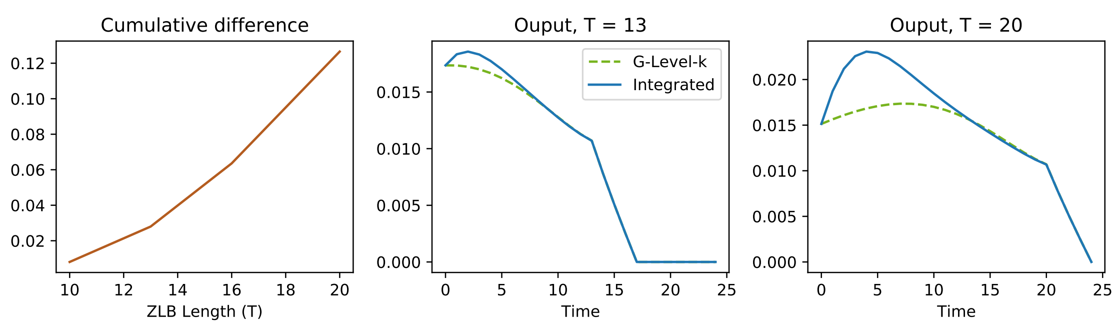
[10] By comparison, the same policy under rational expectations (not shown here) would imply a cumulative response of 278% and 128%, respectively.

Figure 7: Longer Guidance



A natural question is how sensitive the results are to the length of the ZLB period. There are two forces at play. On the one hand, when the recession is long, the initial response of the announcement is mitigated because the policy will be implemented far away in the future. On the other hand, the longer recessionary period implies income and inflation and expected to respond in to the announcement for longer, so the initial revision in beliefs generates a relatively larger revision in discounted income and future real rates. The left panel of figure 8 shows that, as the recession is expected to last longer, forward guidance is more powerful. This happens despite the fact that the impact response of forward guidance in longer recession is smaller, since the policy is announced to happen far away in the future. The reason for this result is that the forecast revision pertains to a larger number of future periods when recessions are expected to last longer. So, the longer the recession, the more periods are affected by the positive revision of output and inflation expectations, despite the smaller initial forecast error.

Figure 8: Longer ZLB Episodes



GE strength and learning As argued in Farhi and Werning (2019), the interaction of bounded rationality and the strength of general-equilibrium (GE) forces is particularly important for understanding the effects of forward guidance. The interaction between GE feedback and learning is therefore an important channel to consider. In this model, there are two sources of GE amplification, controlled by the steepness of the Keynesian cross and the Phillips curve, respectively. The first is determined by the marginal propensity to consume, which in my model can be varied by changing the household's discount factor. The second instead depends on the degree of price stickiness. Interestingly, their interaction with learning delivers *opposite* predictions. In particular, the cumulative difference $\mathcal{D}$ as defined in (27) increases with the steepness of the Phillips curve, but decreases with the marginal propensity to consume, as shown in Figure **XXX**. To see why, note that stronger GE effects from the demand side imply a lower time-0 effect of the forward guidance announcement, because . A steeper Phillips curve, instead,

5 Conclusions

I develop a novel model of bounded rationality called *integrated reasoning* to study the effects of unconventional policies. The model combines forward looking imperfect reasoning in the form of generalized level-k thinking with slow belief update due to Bayesian learning. I assess the ability of the model to replicate empirical estimates of the effects of a specific unconventional policy—forward guidance—on macroeconomic expectations. The empirical analysis uncovers three facts about forecast errors and revisions in response to forward guidance announcements. First, expectations are revised at the time of the announcement. Second, forecasters make systematic mistakes in the direction of underreaction at the time of the shock and in the subsequent quarters. Third, they slowly revise expectations over time.

My model nests three special cases: rational expectations, pure level-k thinking, and pure learning. I show that none of these three models are able to replicate all the empirical facts at the same time. Rational expectations is rejected by the predictability of forecast errors and revisions. Pure level-k thinking is rejected by slow revision of expectations. Pure learning is rejected by the initial response of expectations to the announcement. Therefore, integrated reasoning more closely reproduces the response of expectations to unconventional policies.

In the final section, I embed my expectations model into a small-scale New Keynesian model and simulate a ZLB recession that lasts for 20 quarters. I calibrate the model using my empirical evidence on forecast revision. I find that the cumulative output response to a lower-for-longer announcement, in which the central bank promises to keep interest rates at zero for an additional 2 quarters after the ZLB is over, is 19.5%, or 0.9% per year. Around 11.5% of this effect is due to the revision of expectation. I further show how the response role of learning depends on the length of the promised guidance, and on the length of the ZLB. My analysis suggests that the power of forward guidance compounds over time in the presence of learning, because its transmission depends on belief revisions as individuals learn its effects. Hence, additional periods of promised lower interest rates deliver a larger cumulative response relative to standard level-k thinking because of the more pronounced initial revision. Moreover, in more severe recessions that are expected to last longer, the benefits of an initial successful response are projected farther in the future, so the effectiveness of the policy is enhanced.

While the results presented here are specific to the case of forward guidance, I believe that they can help shedding lights on a number of other novel stabilization policies. For many such policies, including Quantitative Easing and the extent of fiscal policy intervention in response to COVID, there is little evidence from past episodes that can be used to understand their effects. My model provides an empirically grounded model of expectations about their effects. I leave further analysis of other macroeconomic policies with integrated reasoning for future.

References

Angeletos, G.-M. and Huo, Z. (2021). Myopia and anchoring. *American Economic Review*, 111(4):1166–1200.

Angeletos, G.-M., Huo, Z., and Sastry, K. A. (2020). Imperfect macroeconomic expectations: Evidence and theory. Technical report, National Bureau of Economic Research.

Angeletos, G.-M. and Lian, C. (2018). Forward guidance without common knowledge. *American Economic Review*, 108(9):2477–2512.

Bauer, M. D. and Swanson, E. T. (2020). The fed's response to economic news explains the" fed information effect". Technical report, National Bureau of Economic Research.

Bianchi-Vimercati, R., Eichenbaum, M., and Guerreiro, J. (2020). Introspection, extrospection, and expectations: A note on learning in macroeconomic models. Technical report, Mimeo.

Bordalo, P., Gennaioli, N., Ma, Y., and Shleifer, A. (2020). Overreaction in macroeconomic expectations. *American Economic Review*, 110(9):2748–82.

Camerer, C. F., Ho, T.-H., and Chong, J.-K. (2004). A cognitive hierarchy model of games. *The Quarterly Journal of Economics*, 119(3):861–898.

Campbell, J. R., Evans, C. L., Fisher, J. D., Justiniano, A., Calomiris, C. W., and Woodford, M. (2012). Macroeconomic effects of federal reserve forward guidance [with comments and discussion]. *Brookings papers on economic activity*, pages 1–80.

Chetty, R., Guren, A., Manoli, D., and Weber, A. (2011). Are micro and macro labor supply elasticities consistent? a review of evidence on the intensive and extensive margins. *American Economic Review*, 101(3):471–75.

Christiano, L., Eichenbaum, M., and Rebelo, S. (2011). When is the government spending multiplier large? *Journal of Political Economy*, 119(1):78–121.

Christiano, L., Eichenbaum, M. S., and Johannsen, B. K. (2018). Does the new keynesian model have a uniqueness problem? Technical report, National Bureau of Economic Research.

Coibion, O. and Gorodnichenko, Y. (2012). What can survey forecasts tell us about information rigidities? *Journal of Political Economy*, 120(1):116–159.

Coibion, O. and Gorodnichenko, Y. (2015). Information rigidity and the expectations formation process: A simple framework and new facts. *American Economic Review*, 105(8):2644–78.

Correia, I., Farhi, E., Nicolini, J. P., and Teles, P. (2013). Unconventional fiscal policy at the zero bound. *American Economic Review*, 103(4):1172–1211.

Del Negro, M., Giannoni, M. P., and Patterson, C. (2012). The forward guidance puzzle. *FRB of New York Staff Report*, (574).

Eggertsson, G. B. (2011). What fiscal policy is effective at zero interest rates? *NBER Macroeconomics Annual*, 25(1):59–112.

Erceg, C. J., Henderson, D. W., and Levin, A. T. (2000). Optimal monetary policy with staggered wage and price contracts. *Journal of monetary Economics*, 46(2):281–313.

Eusepi, S. and Preston, B. (2011). Expectations, learning, and business cycle fluctuations. *American Economic Review*, 101(6):2844–72.

Evans, G. W. and Honkapohja, S. (2012). *Learning and expectations in macroeconomics*. Princeton University Press.

Farhi, E. and Werning, I. (2019). Monetary policy, bounded rationality, and incomplete markets. *American Economic Review*, 109(11):3887–3928.

García-Schmidt, M. and Woodford, M. (2019). Are low interest rates deflationary? a paradox of perfect-foresight analysis. *American Economic Review*, 109(1):86–120.

Hagedorn, M., Luo, J., Manovskii, I., and Mitman, K. (2019). Forward guidance. *Journal of Monetary Economics*, 102:1–23.

Ilut, C., Valchev, R., and Vincent, N. (2020). Paralyzed by fear: Rigid and discrete pricing under demand uncertainty. *Econometrica*, 88(5):1899–1938.

Ilut, C. L. and Valchev, R. (2020). Economic agents as imperfect problem solvers. Technical report, National Bureau of Economic Research.

Iovino, L. and Sergeyev, D. (2018). Central bank balance sheet policies without rational expectations.

Jarociński, M. (2021). Estimation of the fed's conventional and unconventional policy shocks and their effects. Technical report, Working Paper.

Jarociński, M. and Karadi, P. (2020). Deconstructing monetary policy surprises—the role of information shocks. *American Economic Journal: Macroeconomics*, 12(2):1–43.

Jordà, Ò. (2005). Estimation and inference of impulse responses by local projections. *American economic review*, 95(1):161–182.

Kohlhas, A., Broer, T., et al. (2019). Forecaster (mis-) behavior. In *2019 Meeting Papers*, number 1171. Society for Economic Dynamics.

Kohlhas, A. N. and Walther, A. (2021). Asymmetric attention. *American Economic Review*, 111(9):2879–2925.

Lanne, M. and Lütkepohl, H. (2010). Structural vector autoregressions with nonnormal residuals. *Journal of Business & Economic Statistics*, 28(1):159–168.

Lanne, M., Meitz, M., and Saikkonen, P. (2017). Identification and estimation of non-gaussian structural vector autoregressions. *Journal of Econometrics*, 196(2):288–304.

Lucas, R. E. (1978). Asset prices in an exchange economy. *Econometrica*, pages 1429–1445.

Mackowiak, B. and Wiederholt, M. (2009). Optimal sticky prices under rational inattention. *American Economic Review*, 99(3):769–803.

Maćkowiak, B. and Wiederholt, M. (2015). Business cycle dynamics under rational inattention. *The Review of Economic Studies*, 82(4):1502–1532.

Marcet, A. and Sargent, T. J. (1989a). Convergence of least-squares learning in environments with hidden state variables and private information. *Journal of Political Economy*, 97(6):1306–1322.

Marcet, A. and Sargent, T. J. (1989b). Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic theory*, 48(2):337–368.

McKay, A., Nakamura, E., and Steinsson, J. (2016). The power of forward guidance revisited. *American Economic Review*, 106(10):3133–58.

Miranda-Agrippino, S. (2016). Unsurprising shocks: information, premia, and the monetary transmission.

Miranda-Agrippino, S. and Ricco, G. (2021). The transmission of monetary policy shocks. *American Economic Journal: Macroeconomics*, 13(3):74–107.

Morris, S. and Shin, H. S. (2002). Social value of public information. *american economic review*, 92(5):1521–1534.

Nagel, R. (1995). Unraveling in guessing games: An experimental study. *The American Economic Review*, 85(5):1313–1326.

Nakamura, E. and Steinsson, J. (2018). High-frequency identification of monetary non-neutrality: the information effect. *The Quarterly Journal of Economics*, 133(3):1283–1330.

Sastry, K. A. (2021). Disagreement about monetary policy. Technical report.

Schmitt-Grohé, S. and Uribe, M. (2005). Optimal fiscal and monetary policy in a medium-scale macroeconomic model. *NBER Macroeconomics Annual*, 20:383–425.

Sims, C. A. (2003). Implications of rational inattention. *Journal of Monetary Economics*, 50(3):665–690.

Stahl, D. O. and Wilson, P. W. (1995). On players' models of other players: Theory and experimental evidence. *Games and Economic Behavior*, 10(1):218–254.

Swanson, E. T. (2021). Measuring the effects of federal reserve forward guidance and asset purchases on financial markets. *Journal of Monetary Economics*, 118:32–53.

Werning, I. (2015). Incomplete markets and aggregate demand. Technical report, National Bureau of Economic Research.

Wiederholt, M. (2015). Empirical properties of inflation expectations and the zero lower bound. *manuscript, Goethe University*.

Woodford, M. (2003). Imperfect common knowledge and the effects of monetary policy. *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, pages 25–58.

A Proofs

Proof of Lemma 1 The proof follows by applying Bayesian updating techniques. Suppose $\mathcal{Y}^0_{i,t} \sim \mathcal{GP}\left(\hat{\mathcal{Y}}^0_{i,t}, \hat{\Sigma}_t\right)$, which is true by assumption for $t = 0$. Let the signal about $\mathcal{Y}_t(z)$ be given by (9). Then for any z and z'

$$\begin{bmatrix} \mathcal{Y}^0_{i,t}(z) \\ \mathcal{Y}^0_{i,t}(z') \\ Y_{i,t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \hat{\mathcal{Y}}^0_{i,t}(z) \\ \hat{\mathcal{Y}}^0_{i,t}(z') \\ \hat{\mathcal{Y}}^0_{i,t}(z^t) \end{bmatrix}, \begin{bmatrix} \hat{\Sigma}_t(z,z) & \hat{\Sigma}_t(z,z') & \hat{\Sigma}_t(z,z^t) \\ \hat{\Sigma}_t(z',z) & \hat{\Sigma}_t(z',z') & \hat{\Sigma}_t(z',z^t) \\ \hat{\Sigma}_t(z^t,z) & \hat{\Sigma}_t(z^t,z') & \hat{\Sigma}_t(z^t,z^t) + \sigma_v^2 \end{bmatrix}\right).$$

By standard properties of multivariate Gaussian distributions, it follows that the distribution of $\mathcal{Y}^0_{i,t+1} \equiv \mathcal{Y}^0_i | Y_{i,t}$ is given by:

$$\begin{bmatrix} \mathcal{Y}^0_{i,t+1}(z) \\ \mathcal{Y}^0_{i,t+1}(z') \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \hat{\mathcal{Y}}^0_{i,t+1}(z) \\ \hat{\mathcal{Y}}^0_{i,t+1}(z') \end{bmatrix}, \begin{bmatrix} \hat{\Sigma}_{t+1}(z,z) & \hat{\Sigma}_{t+1}(z,z') \\ \hat{\Sigma}_{t+1}(z',z) & \hat{\Sigma}_{t+1}(z',z') \end{bmatrix}\right),$$

where

$$\hat{\mathcal{Y}}^0_{i,t+1}(z) = \hat{\mathcal{Y}}^0_{i,t}(z) + \frac{\hat{\Sigma}_t(z,z^t)}{\hat{\Sigma}_t(z^t,z^t) + \sigma_v^2}\left(Y_{i,t} - \hat{\mathcal{Y}}^0_{i,t}(z^t)\right)$$

and

$$\hat{\Sigma}_{t+1}(z,z') = \hat{\Sigma}_t(z,z') - \frac{\hat{\Sigma}_t(z,z^t)\hat{\Sigma}_t(z',z^t)}{\hat{\Sigma}_t(z^t,z^t) + \sigma_v^2}.$$

□

Proof of Lemma 2 Let $z^{ss} = \{0, 0, 0, \dots\}$ denote the path of steady-state current and future interest rates. Under the proposed interest rate path, there is a T such that, for all $t \geq T$, $z^t = z^{ss}$.

First, note that $\hat{\Sigma}_t(z^{ss}, z) = 0$ for all t and for any arbitrary path z. The proof is by induction. For $t = 0$, it holds by construction of $\hat{\Sigma}_0$ in (15). Suppose now that $\hat{\Sigma}_t(z^{ss}, z) = 0$ for any z. Then it also holds that $\hat{\Sigma}_t\left(z^{ss}, z^t\right) = 0$. From the updating scheme in (12), it holds that:

$$\hat{\Sigma}_{t+1}(z^{ss}, z) = \hat{\Sigma}_t(z^{ss}, z) - \frac{\hat{\Sigma}_t(z^{ss},z^t)\hat{\Sigma}_t(z,z^t)}{\hat{\Sigma}_t(z^t,z^t) + \sigma_v^2} = 0.$$

From the update rule for the mean output function in (16), this result implies that expectations about output at the steady state path for interest rate z^{ss} are never updated:

$$\hat{\mathcal{Y}}^0_t(z^{ss}) = \hat{\mathcal{Y}}^0_0(z^{ss}) = 0,$$

for all t. As a consequence, at each date $t \geq T$, after the policy rate has reverted to steady state, consensus beliefs of level-1 agents is such that output will remain constant and equal to the steady-state level of one:

$$\mathbb{E}_t^0[y_{t+s}] = \hat{\mathcal{Y}}_t^0(z^{ss}) = 0,$$

for all s. Since $C_t^0(1) = 1$, then level-2 expectations are also such that $\mathbb{E}_t^2[y_{t+s}] = 1$ for all s. Iteratively, this holds for every k, so that, following equation (17), consensus beliefs are:

$$\overline{\mathbb{E}}_t[y_{t+s}] = \sum_{k\geq 1} \Phi(k)\mathbb{E}_t^k[y_{t+s}] = 0.$$

From the temporary equilibrium relation in (5), the resulting output level is $y_t = 0$. So, beliefs about output under the proposed policy eventually converge to the rational-expectations beliefs.

□

Proof of Lemma 3

Proof of Proposition 1 To obtain equation (18), note that $\mathbb{E}_0^k[Y_t] = 0$ for $t \geq T$ and for all k. Using this fact and taking differences with respect to steady state, from equation (5) one gets:

$$\Delta Y_0 = \frac{(1-\beta)\sum_{s=1}^{T-1}\beta^s\overline{\mathbb{E}}_0[\Delta Y_s]}{\beta - \beta^T + (1+\beta\Delta R)^{\sigma-1}} + \frac{(1+\beta\Delta R)^{-1}\beta^T - (1+\beta\Delta R)^{\sigma-1}\beta^T}{\beta - \beta^T + (1+\beta\Delta R)^{\sigma-1}}.$$

This equation implies that

$$\epsilon_0 = \frac{1-\beta}{\beta}\sum_{s=1}^{T-1}\beta^s\overline{\epsilon}_{0,s} + \sigma\beta^{T-1}.$$

To see how the average elasticity compares to the rational-expectations counterpart, note that

$$\overline{\epsilon}_{0,s} = \sum_{k\geq 1}\Phi(k)\epsilon_{0,s}^k,$$

where

$$\epsilon_{0,s}^k = \lim_{\Delta R\to\infty} -\beta^{-1}\frac{\mathbb{E}_0^k[\Delta Y_s]}{\Delta R}.$$

I will now show that

$$\epsilon_{0,s}^k < \sigma$$

for all k, so that $\overline{\epsilon}_{0,s} < \sigma$. Under assumption 2, the level-k elasticity $\epsilon_{0,s}^k$ is a weighted average of the level-$k-1$ elasticity and the output elasticity that would happen if everyone was level-$k-1$, δ_s^{k-1}:

$$\epsilon_{0,s}^k = (1-\gamma_k)\epsilon_{0,s}^{k-1} + \gamma_k \delta_s^{k-1}$$

where $\gamma_k = \frac{\Phi(k-1)}{\sum_{j=1}^{k-1}\Phi(j)}$, with $\sum_{j=1}^{0}\Phi(j) = 1$. The proof is by induction. First note that $\epsilon_{0,s}^1 = 0 < \sigma$ for all s. Then:

$$\begin{aligned} \delta_s^1 &= \frac{1-\beta}{\beta}\sum_{t=1}^{T-s-1}\beta^t \epsilon_{0,t}^1 + \sigma\beta^{T-s-1} \\ &= \sigma\beta^{T-s-1} < \sigma \end{aligned}$$

for all s. Hence,

$$\epsilon_{0,s}^2 = \delta_s^1 < \sigma$$

for all s. Now suppose that $\epsilon_{0,s}^{k-1} < \sigma$ for all s. Then:

$$\begin{aligned} \delta_s^{k-1} &= \frac{1-\beta}{\beta}\sum_{t=1}^{T-s-1}\beta^t \epsilon_{0,t}^{k-1} + \sigma\beta^{T-s-1} \\ &< \frac{1-\beta}{\beta}\sum_{t=1}^{T-s-1}\beta^t \sigma + \sigma\beta^{T-s-1} = \sigma \end{aligned}$$

for all s. Hence,

$$\epsilon_{0,s}^k = (1-\gamma_k)\epsilon_{0,s}^{k-1} + \gamma_k\delta_s^{k-1} < \sigma.$$

Finally, suppose that $\Phi(k^*) = 1$ for some k^*. Then $\gamma_k = 1$ for all k, and the model is a standard level-k thinking model as in Farhi and Werning (2019). The limiting case of $k^* \to \infty$ is such that $\epsilon_0 = \epsilon_0^*$. □

Proof of Lemma 4 It is sufficient to show that $\Delta\mathbb{E}_1^1[Y_t] = \mathbb{E}_1^1[Y_t] - \mathbb{E}_0^1[Y_t]$ is increasing in σ_0^2/σ_v^2. By induction, this also holds for any $k \geq 2$, and thus also for consensus beliefs. Note that:

$$\begin{aligned} \Delta\mathbb{E}_1^1[Y_t] &= \frac{\Sigma_0(Z^t, Z^0)}{\Sigma_0(Z^0, Z^0) + \sigma_v^2}\left(Y_0 - \hat{\mathcal{Y}}_0^1(Z^0)\right) \\ &= \frac{\rho^t}{1+\sigma_v^2/\sigma_0^2} Y_0, \end{aligned}$$

which is increasing in σ_0^2/σ_v^2. □

B The Linear Model of Expectations

C Forward Guidance Shocks

D Details on the Sticky Wage Model

E Robustness on Calibration