Learning Unconventional Policies: Forward Guidance with Integrated Reasoning*

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Abstract

I develop an *integrated reasoning* model of expectations formation to describe how individuals learn the effects of novel macroeconomic policies. My model of expectations has two key elements. First, people have a limited ability to understand the general-equilibrium effects of a new policy. Second, they revise their expectations when observing past errors with a process of learning in real time. To assess the empirical plausibility of the model, I estimate the response of survey-level expectations to identified forward guidance shocks. The time-series properties of forecast errors and revisions show that *i)* individuals immediately revise their beliefs after announcements; *ii)* forecasts under-react to the shock; and *iii)* individuals' beliefs are slowly updated over time. In contrast to commonly used alternative models, I show that the predictions of my framework are consistent with this empirical evidence. I estimate parameters to match the response of forecast revisions to the policy shocks. In the estimated model, the endogenous expectations revision due to real-time learning of the effects of a forward guidance announcement accounts for 35% of the cumulative output response.

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1 Introduction

In late 2009, U.S. nominal interest rates hit the zero lower bound (ZLB). Policymakers adopted a variety of new measures in response to that event, collectively referred to as unconventional policies. Many of them are successful at stabilizing the economy by shaping expectations of households, firms, and financial market participants about the future macroeconomic effects of the policy intervention. However, the rarity of ZLB episodes and the policies that followed raises the question of how economic actors can form expectations about their effects, without having observed these policies in the past.

In this paper, I develop a new model of expectations formation to describe how people learn about the effects of novel macroeconomic policies. My model of expectations has two key elements. First, people have a limited ability to understand the general-equilibrium effects of a new policy. Second, they revise expectations when observing past errors with a process of learning in real time.

To assess the empirical plausibility of the model, I estimate how professional forecasters' expectations about key macroeconomic variables respond to identified forward guidance shocks. I employ the shock series constructed by Jarociński (2021), who relies on high-frequency changes to the price of financial assets during the 30-minute windows around Federal Open Market Committee (FOMC) announcements to identify forward guidance. I use local projection methods as in Jordà (2005) to study how forecast errors and forecast revisions behave in response to the shock. My findings suggest that individuals revise expectations immediately after an unexpected change to the announced path of interest rates, so future effects of new policies are incorporated into forecasts immediately after an announcement. However, forecasters also underestimate the response of key macroeconomic variables in the quarters following the shock, which results in predictable forecast errors. Hence, their ability to anticipate the effects of the policy intervention is imperfect. Finally, the empirical response of forecast revisions, that are systematically different from zero in the quarters following the shock, is consistent with a process of slow adjustment of expectations. These results are true for both consensus-level and individual-level forecasts.

This empirical evidence is at odds with expectations formation processes proposed in the literature. One is the workhorse rational-expectations model, which postulates that individuals do not make systematic mistakes. This prediction is rejected by the fact that forecast errors at initial quarters after the shock are predictable. An alternative class of models postulates that forecasters have bounded rationality when predicting the response of the economy to unconventional policies.

In these models, individuals correctly understand the structure of the economy, but form expectations about future outcomes through an imperfect process of reflection. The level-k thinking model in Farhi and Werning (2019) and the reflective expectations model in García-Schmidt and Woodford (2019) are two examples. In the existing formulation of these frameworks, expectations are assumed to be revised only once at the time of the announcement, but they are not updated at later periods, despite the presence of systematic forecast errors—a counterfactual prediction in light of the evidence on systematic revisions. A distinct approach is to model expectations as a result of real-time updates from past experience, as in adaptive learning models (see the literature review for a detailed discussion). Under plausible assumptions, if expectations are updated using *only* past data, the announcement of a novel policy does not produce any impact response on individuals' beliefs, unlike what is observed in the data.

To understand how macroeconomic variables respond to an unconventional policy announcement in a setting where individual expectations formation is consistent with the evidence on forecast errors and revisions, I develop a model of bounded rationality called "integrated reasoning". This model combines imperfect economic thinking with learning. Formally, this equilibrium concept integrates a generalization of the level-k thinking model with a Bayesian learning process. Individuals form imperfect expectations about the mapping between unconventional policy announcements and aggregate variables. They differ in the depth of equilibrium reasoning they perform, which is indexed by their cognitive level k. They also receive noisy signals about realized endogenous outcomes. Level-0 individuals learn about the effects of policy announcements following a Bayesian signal extraction problem. They do not have any structural knowledge of the economy. Their expectations formation process is backward-looking and is updated only from accumulation of new data over time. Individuals with higher cognitive sophistication levels form beliefs about future changes in macroeconomic variables using a finite deductive procedure about others' behavior that involves k iterations. In this process, they are able to understand the beliefs and actions of less sophisticated individuals, and they correctly infer their equilibrium consequences. Since level-k thinkers use structural knowledge about the future behavior of the economy, their beliefs are forward-looking.

The combination of these two reasoning processes has a powerful role in explaining the empirical behavior of forecast errors and revisions in response to a forward guidance shock. I illustrate this role in a stylized New Keynesian (NK) economy with fully rigid wages and prices. The central bank announces a change in the interest rate at a future period T. Households understand that future income will react to the announced interest rate reduction due to their forward-looking deductive thinking process. Therefore, income expectations are revised on impact. But households have a limited understanding of general-equilibrium forces, since they only perform a finite number of rounds of iterations to form their expectations. As a result, they underestimate the true effects of the policy. Impact revision and under-reaction are features that match the empirical evidence. In addition, the integrated reasoning model implies a slow process of beliefs revision after the shock. This is due to the fact that level-0 people update their expectations by incorporating new information from the signals they observe. Since level-k thinkers use level-0 expectations as the starting point for their deductive reasoning, they also revise their expectations. Therefore, the model generates a slow belief revision and reversion of forecast errors towards zero, both of which are consistent with empirical evidence.

I then characterize the implications of integrated reasoning for the response of the economy to a forward guidance announcement. Under plausible parametrizations, the response of output is mitigated at all horizons relative to rational expectations. This result follows from the fact that households do not perfectly anticipate the general-equilibrium effects of the policy announcement. Since they believe income will rise less than under rational expectations, and since current consumption is increasing in expected future income, the response of aggregate demand is mitigated. Hence, similar to Farhi and Werning (2019) and García-Schmidt and Woodford (2019), this model of bounded rationality does not exhibit a "forward guidance puzzle" — a set of counterfactual implications of forward guidance in standard NK models.[1] However, relative to a benchmark case in which individuals do not update beliefs in real time, integrated reasoning implies a larger output response. The intuition behind this result is that real-time learning generates feedback on realized outcomes and updated expectations, which strengthens the stimulative power of an announcement as time passes. Upon observing a positive forecast error, households revise income expectations upwards. Because consumption increases with expected income, aggregate demand is higher relative to the case in which individuals do not revise their expectations. Since their ability to understand general-equilibrium effects is limited, they keep making forecast errors over time, and thus revise income expectations upwards, which leads to a larger output response. Therefore, my model of bounded rationality identifies a "belief revision" channel of transmission of forward guidance.

To evaluate the quantitative importance of this channel, I turn to the analysis of a more general

[1] See Del Negro et al. (2012) and McKay et al. (2016) for information on the existence of a forward guidance puzzle.

version of the model where prices and wages are not constant. I assume that wages are subject to Calvo-style frictions, which also implies price stickiness. In this new setting, inflation is an important channel of general-equilibrium effects since it affects current and future real interest rates. I use this version of the model to simulate a forward guidance announcement that replicates the empirical shift in the yield curve in response to the shocks identified by Jarociński (2021). Using the method of moments, I estimate the parameters that govern the average sophistication level of forecasters and the speed of their expectation revisions. The estimated model is able to match closely the *individual*-level empirical response of forecast revisions to a forward guidance shock. In this exercise, the endogenous expectations revision due to learning of the policy effects over time accounts for 35% of the cumulative output response and 73% of the inflation response. Therefore, expectations revision is a quantitatively important channel of transmission of unconventional policies. An assessment of the potency of a new policy announcement should consider the revision of beliefs that builds up over time after the announcement as households, firms, and financial market participants learn about the true effects of the shock.

Related Literature The belief formation process described in this paper builds on Bianchi-Vimercati et al. (2020), who first introduced a version of integrated reasoning where individuals combine level-k thinking with adaptive learning. This paper discusses differences and similarities between the two reasoning processes in the context of a general class of linear models, and concludes that this combination is well-suited for fixing individual shortcomings of the two belief formation models. I focus on forward guidance and present novel evidence on the ability of integrated reasoning to match belief revision and forecast error predictability in response to identified shocks. I also propose a different model of learning that, as I argue below, is well-suited for non-stationary environments such as an unconventional monetary policy, and analyze its quantitative implications.

This paper contributes to a growing literature studying the implications of deviations from the rational expectations, full information framework, to study the effectiveness of macroeconomic policies. The departure from rational expectations that I consider follows the approach of modeling expectations formation as limited depth of general-equilibrium knowledge via level-k thinking, which was originally employed by Nagel (1995) and Stahl and Wilson (1995). Farhi and Werning (2019) examine the power of monetary policy when individuals are level-k thinkers. García-Schmidt and Woodford (2019) develop a similar belief formation process, called reflective expectations, and use it to analyze forward guidance and interest rate pegs. Both studies find that the power of forward guidance is dampened under these deviations from rational expectations. However, they do not allow for belief revisions over time, despite the fact that individuals make systematic forecast errors. As such, they are best suited for the analysis of the impact response of endogenous variables to unconventional policies. Relative to their findings, I examine the dynamic response of output in a model where beliefs are updated over time thanks to the real-time learning model. Iovino and Sergeyev (2018) study the effects of central banks' balance sheet policies in a model with level-k thinking and reflective expectations, and briefly discuss the robustness of their results to the presence of expectation updates.

All of the aforementioned papers assume that individuals have common knowledge about exogenous and endogenous variables, but do not fully understand these variables' general-equilibrium implications. An alternative approach, pioneered by Morris and Shin (2002) and Woodford (2003), and followed by Angeletos and Lian (2018), Angeletos and Huo (2021), and Wiederholt (2015), is to assume that individuals have dispersed information.[2] Deviating from common knowledge dampens the equilibrium response of policy announcements about the future, because higher-order beliefs are less anchored to the common prior. Unlike Angeletos and Lian (2018), dispersed information in my model generates a slow process of belief revisions, rather than higher-order forecasting about others' forecasts. The hierarchy of beliefs in my model is generated with rounds of deductive thinking about the policy effects that are truncated at some finite iteration determined by one's level of sophistication.

This paper builds on a body of work where individuals use past data to improve their understanding of the economy. Following the seminal work by Lucas (1978), many researchers have suggested the use of adaptive learning as a way to deviate from rational expectations. Examples include Marcet and Sargent (1989a), Marcet and Sargent (1989b), Evans and Honkapohja (2012), and Eusepi and Preston (2011). Relative to this literature, my model sheds light on a non-stationary environment, in which a policy announcement or shock is only observed once. For this reason, I adopt a learning model that is different from standard econometric least-square learning, and that is well-suited for analyzing the initial response of expectations to a novel policy shock.[3] Recently, Christiano et al. (2018) explore learning in a non-linear NK model of the ZLB and its implications for fiscal policy. My paper focuses on monetary rather than fiscal policy, and provides empirical

[2]A related formulation where dispersed information is endogenous is the rational inattention setting, as in Sims (2003), Mackowiak and Wiederholt (2009), and Maćkowiak and Wiederholt (2015).

[3]My learning model is similar to the Bayesian learning set-up used by Ilut and Valchev (2020) and Ilut et al. (2020), where individuals learn about functions.

evidence on belief revisions.

Many researchers have analyzed macroeconomic expectations using survey-level data on forecast errors and revisions. Inertial behavior in average beliefs has been documented (Coibion and Gorodnichenko (2012), Coibion and Gorodnichenko (2015)), together with patterns of over-reaction and over-extrapolation of individual expectations (Bordalo et al. (2020), Kohlhas et al. (2019), Kohlhas and Walther (2021), Angeletos et al. (2020)). My findings suggest that individual beliefs under-react to forward guidance announcements, and are then slowly updated in the quarters following the shock. My methodology is similar to Angeletos et al. (2020), who argue in favor of using impulse responses of errors and revisions to validate different models of expectations. The predictability of forecast errors in the direction of under-reaction following an unconventional policy is also documented by Iovino and Sergeyev (2018), who examine mortgage rate expectations in response to balance sheet interventions. In addition, I examine the response of forecast revisions and also find that a slow adjustment of expectations is needed to match the behavior of forecast errors and revisions.

To my knowledge, this is the first study to investigate the joint behavior of forecast errors and revisions in response to identified forward guidance shocks as a mean of model calibration and validation. Other studies have examined how monetary policy shocks affect expectations about the state of the economy, in an attempt to infer an "information effect" of monetary policy (Campbell et al. (2012), Nakamura and Steinsson (2018) and Jarociński and Karadi (2020)), or lack thereof (Miranda-Agrippino (2016), Miranda-Agrippino and Ricco (2021), Bauer and Swanson (2020), and Sastry (2021)). The "information effect" of monetary policy is beyond the scope of my paper.[4] However, I document that the identified forward guidance series extracted by Jarociński (2021) that I use for my analysis displays responses of expectations in line with theoretical predictions — a monetary tightening leads to a deterioration of expectations about future economic activity.

Finally, this paper contributes to the stream of research on the effects of forward guidance. Many researchers have focused on deviations from the representative agent assumption, using models with incomplete markets and liquidity constraints, such as Werning (2015), McKay et al. (2016), and Hagedorn et al. (2019). As noted by Werning (2015), incomplete markets alone reduce the role of partial-equilibrium effects in favor of greater general-equilibrium effects. However, they

[4] In my model, monetary policy never has an information effect, because the central bank announces a path for interest rates rather than using a rule where interest rates are set as a function of inflation and the output gap. In the latter scenario, communication about future interest rates also provides a signal for the central bank's forecast of future economic conditions.

do not by themselves necessarily imply a solution to the "forward guidance puzzle", and can even make it worse. In contrast, I focus on analyzing the role of expectation formation.

2 Forward Guidance with Integrated Reasoning

In this section, I describe a benchmark model with fully rigid wages. This simple model is used to illustrate the dynamic effects of an announcement about a reduction in future short-term interest rates.

Without loss of generality, wages are equal to one at each period. The representative household preferences over sequences of consumption, C_t, and labor, N_t, are given by:

$$\sum_{s\geq 0} \beta^s \left[u\left(C_{t+s}\right) - v\left(N_{t+s}\right)\right], \tag{2.1}$$

where $u(C) = C^{1-\sigma^{-1}} / \left(1-\sigma^{-1}\right)$ and $v(N) = N_{t+s}^{1+\varphi} / (1+\varphi)$.

Firms Firms are perfectly competitive and maximize profits. I assume that production is linear in labor, $Y_t = N_t$. The solution to the representative firm's problem requires that $P_t = W_t = 1$. Because wages are fully rigid, there is no wage or price inflation.

Monetary policy The monetary policy authority announces a path for the nominal interest rate R_t. As in Farhi and Werning (2019), I look at one-time changes in the nominal interest rate at a future date T. Since there is no inflation, the announcement has a one-to-one impact on real interest rates. In all other periods, the interest rate is equal to the steady state value of β^{-1}. The resulting interest rate path is as follows:

$$R_t = \begin{cases} \beta^{-1} + \Delta R & t = T \\ \beta^{-1} & t \neq T. \end{cases} \tag{2.2}$$

I denote $Z = \{R_s\}_{s\geq 0}$ an arbitrary sequence of current and future interest rates, and $Z^t = \{R_{t+s}\}_{s\geq 0}$ denote the sequence that is announced at time t. In this simple model, the sequence Z^t is the only state variable.

Households The household enters period t with financial assets equal to $B_{i,t}$, which yield a return of R_{t-1}, and receives income $Y_{i,t}$. When solving the dynamic consumption-savings problem, households maximize their perceived expected utility:

$$\max_{C_{i,t+s}} \mathbb{E}_{i,t} \sum_{s\geq 0} \beta^s \left[\frac{C_{i,t+s}^{1-\sigma^{-1}}}{1-\sigma^{-1}} - \frac{N_{t+s}^{1+\varphi}}{1+\varphi} \right],$$

subject to their current and future budget constraints:

$$C_{i,t+s} + B_{i,t+s+1} = Y_{i,t+s} + R_{t+s-1} B_{i,t+s}.$$

Since wages are fully rigid, equilibrium output and labor are demand-determined. Therefore, households are not on their labor supply equation.

Households have rational expectations about the exogenous variable Z^t. As such, they have perfect foresight about the path of interest rates announced by the central bank. This assumption is similar to Farhi and Werning (2019) and García-Schmidt and Woodford (2019), and is justified as a plausible choice for economic scenarios that involve salient policy announcements. The way in which households form beliefs about endogenous variables is described below in more detail.

The rest of this paper focuses on a linearized version of the model. Lower case letters denote log-linear deviations from steady state. The solution to the household's problem in the linearized model implies that $c_{i,t}$ satisfies

$$c_{i,t} = (1-\beta) y_{i,t} + (1-\beta) \sum_{s=1}^{\infty} \beta^s \mathbb{E}_{i,t}[y_{i,t+s}] - \sigma \sum_{s=0}^{\infty} \beta^{s+1} r_{t+s} + b_{i,t}. \tag{2.3}$$

Temporary equilibria I start by defining a temporary equilibrium. This notion of equilibrium takes, as given, a sequence of beliefs $\{\mathbb{E}_{i,t}[y_{t+s}]\}_{s\geq 1}$ for every agent i and every time period t at which the beliefs are formed. A temporary equilibrium is a sequence of allocations which satisfy private optimality for households and firms, and such that markets clear. Let $\overline{\mathbb{E}}_t[y_{t+s}] = \int \mathbb{E}_{i,t}[y_{t+s}] di$ denote the average (consensus) expectations about income. Using Equation (2.3) and imposing the consumption good market clearing condition, output in a temporary equilibrium is given by[5]:

$$y_t = \mathcal{Y}_t\left(z^t\right) = \frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\overline{\mathbb{E}}_t[y_{t+s}] - \sigma\sum_{s=0}^{\infty}\beta^s r_{t+s}. \tag{2.4}$$

Integrated reasoning Households have perfect foresight about the path of future interest rate, but they are limited in their ability to predict the effects of the policy on endogenous variables. I propose a novel expectation formation model, *integrated-reasoning*, a process that combines imperfect structural knowledge of the economy with individuals' updated beliefs based on past data. These two elements are derived from existing models of bounded rationality: level-k thinking and learning, respectively.

I focus on expectation formation processes that can be represented as a mapping from the fundamental exogenous state variable z^t to endogenous outcomes, as in:

$$\mathbb{E}_{i,t}[y_{t+s}] = \mathcal{Y}_{i,t}(z^{t+s}). \tag{2.5}$$

Equation (2.5) represents the perceived law of motion of output for individual i.

The model features a cognitive hierarchy of individuals indexed by $k \geq 0$. In the spirit of the generalized level-k thinking model introduced by Camerer et al. (2004), each individual believes that there is a distribution of other individuals with lower cognitive abilities. In particular, a level-k individual believes that there is a mass $f_k(j)$ of level-j people, with $j < k$. At each period, every individual with a cognitive level higher than zero performs a finite deductive reasoning process.

The starting point are the beliefs of the level-0 individuals $\left\{\mathbb{E}^0_{i,t}[y_{t+s}]\right\}_{s\geq 1}$ for any t and i, which are exogenous to the generalized level-k thinking process. Given these beliefs, the aggregate consumption of level-0 people is:[6]

$$\mathcal{C}^0_t\left(y_t, z^t\right) = (1-\beta)y_t + (1-\beta)\sum_{s=1}^{\infty}\beta^s\mathbb{E}^0_t[y_{t+s}] - \sigma\sum_{s=0}^{\infty}\beta^{s+1}r_{t+s}, \tag{2.6}$$

where $\mathbb{E}^0_t[y_{t+s}] = \int_0^1 \mathbb{E}_{i,t}[y_{t+s}]di$.

Level-1 people believe that everyone else is level-0, and they understand the aggregate consumption function (2.6). Thus, at each time t, they expect future output to be the result of the

[5]For ease of notation, I omit the dependence on the future path for expected output among the arguments of the temporary equilibrium relation. The subscript t on the mapping $\mathcal{Y}_t$ captures the fact that the information set is the one available at time t.

[6]Notation follows the same convention as for $\mathcal{Y}_t(z)$.

following fixed point problem:

$$\mathbb{E}_t^1[y_{t+s}] = \mathcal{C}_t^0 \left(\mathbb{E}_t^1[y_{t+s}], z^{t+s} \right), \tag{2.7}$$

for all s. Note that level-1 expectations are common across households i, because they all form beliefs according to (2.7). The solution to (2.7) defines a mapping $\mathcal{Y}_t^1(z)$ that describes the perceived law of motion of level-1 people at time t.

More generally, level-k individuals have a non-degenerate distribution for individuals over the cognitive hierarchy described by $f_k(j)$. Therefore, they believe that output is the solution to:

$$\mathbb{E}_t^k[y_{t+s}] = \sum_{j=0}^{k-1} f_k(j) \mathcal{C}_t^j \left(\mathbb{E}_t^k[y_{t+s}], z^{t+s} \right), \tag{2.8}$$

which defines the corresponding perceived law of motion $\mathcal{Y}_t^k(z)$. Aggregate consumption of level-k people is then given by:

$$\mathcal{C}_t^k \left(y_t, z^t \right) = (1-\beta) y_t + (1-\beta) \sum_{s=1}^{\infty} \beta^s \mathbb{E}_t^k[y_{t+s}] - \sigma \sum_{s=0}^{\infty} \beta^{s+1} r_{t+s}. \tag{2.9}$$

To complete the equilibrium notion of integrated reasoning process, I specify an update process for beliefs of level-0 individuals. Because all level-k people have to derive level-0 beliefs in order to form expectations, they also revise their beliefs over time. In this sense, all agents in this economy combine a learning process from past data with internal reasoning from level-k thinking.[7]

Level-0 people use Bayesian inference to update their beliefs about the aggregate law of motion. They observe a signal about aggregate income, which they use to form expectations. The signal is equal to the sum of aggregate income and a white noise process:

$$y_{i,t} = y_t + \nu_{i,t}, \tag{2.10}$$

where $\nu_{i,t} \sim \mathcal{N}(0, \sigma_\nu^2)$. Households understand that $y_{i,t}$ provides a signal for the true aggregate mapping between interest rates and output $\mathcal{Y}_t(z)$. They enter time t with prior beliefs about the equilibrium mapping, $\mathcal{Y}_{i,t}^0(z)$. I make the following assumption about time-0 prior beliefs.

[7] One can imagine that sophisticated agents also revise their beliefs over time. This interesting extension is left for future research.

Assumption 1 *Time-0 prior beliefs are such that individual beliefs $\tilde{\mathcal{Y}}^0_{i,0}(z)$ is a Gaussian Process:*

$$\tilde{\mathcal{Y}}^0_{i,0} \sim \mathcal{GP}\left(\mathcal{Y}^0_0, \Sigma_0\right) \tag{2.11}$$

for all i, where $\tilde{\mathcal{Y}}^0_{i,0}$ denotes the random function representing individual beliefs about output, $\mathcal{Y}^0_0 : \mathbb{R}^\infty \to \mathbb{R}$ is a common mean function and $\Sigma_0 : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$ is a common covariance function.

This prior belief has the feature that for any arbitrary pair of inputs z and z', the joint distribution of the resulting function values $\mathcal{Y}^0_{i,0}(z)$ and $\mathcal{Y}^0_{i,0}(z')$ is Gaussian:

$$\begin{bmatrix} \tilde{\mathcal{Y}}^0_{i,0}(z) \\ \tilde{\mathcal{Y}}^0_{i,0}(z') \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathcal{Y}^0_0(z) \\ \mathcal{Y}^0_0(z') \end{bmatrix}, \begin{bmatrix} \Sigma_0(z,z) & \Sigma_0(z,z') \\ \Sigma_0(z',z) & \Sigma_0(z',z') \end{bmatrix}\right).$$

The mean function $\mathcal{Y}^0_0(z)$ encodes the average behavior of output as a function of an arbitrary sequence of interest rates z according to the individuals' prior beliefs. The prior covariance function $\Sigma_0(z,z')$ instead specifies the covariance between the values of output at any pair of interest rate sequences (z,z'):

$$\Sigma_0(z,z') = cov_0\left(\tilde{\mathcal{Y}}^0_{i,0}(z), \tilde{\mathcal{Y}}^0_{i,0}(z')\right).$$

The mean function and the covariance function are assumed to be independent of i, that is, at time 0 all individuals have a common prior belief. The assumption of Gaussianity allows for a tractable Bayesian inference problem about equilibrium functions.[8] Finally, this specification of the time-0 prior belief implies that level-0 people use the correct state variable, z, to form beliefs about endogenous outcomes.

The integrated reasoning equilibrium is closed with an actual distribution of agents along the cognitive hierarchy, denoted by $\Phi(k)$. The following definition summarizes this notion of equilibrium.

Definition For each date t and interest rate path z^t, an *integrated-reasoning equilibrium* is a sequence of allocations $\mathcal{A}_t = \{c_t, y_t\}$ and beliefs $\mathcal{A}^e_t = \{\mathbb{E}^k_{i,t}[y_{t+s}]\}_{i,k,s}$ such that:

1. Level-0 agents form expectations according to $\tilde{\mathcal{Y}}^0_{i,t}(z) \sim \mathcal{GP}\left(\mathcal{Y}^0_{i,t}, \Sigma_t\right)$, where the mean and variance functions are updated as in (2.13) and (2.14), for some initial conditions $\left(\mathcal{Y}^0_0, \Sigma_0\right)$;

2. Level-k individuals for $k > 0$ form expectations $\mathcal{Y}^k_t(z)$ according to (2.8);

[8]See Ilut and Valchev (2020) and Ilut et al. (2020) for two different applications.

3. Individual consumption solves households' optimization problem as in (2.3);

4. Output solves the market clearing condition:

$$y_t = \mathcal{C}_t\left(y_t, z^t\right) = \sum_{k=0}^{\infty} \Phi(k)\mathcal{C}_t^k\left(y_t, z^t\right). \tag{2.12}$$

2.1 Belief Dynamics

Given Assumption 1, level-0 people update their beliefs in a Bayesian way, combining their prior beliefs $\tilde{\mathcal{Y}}^0_{i,t}(z)$ with the signal provided by Equation (2.10). The resulting belief update process is a recursion that is similar to a standard Kalman filter.

Lemma 1 *Given time-t prior beliefs $\tilde{\mathcal{Y}}^0_{i,t}(z)$, individual income $y_{i,t}$, and an announcement for a path of interest rates z^t, beliefs about output are a Gaussian Process $\tilde{\mathcal{Y}}^0_{i,t+1}(z) \sim \mathcal{GP}\left(\mathcal{Y}^0_{i,t+1}, \Sigma_{t+1}\right)$ where moments evolve according to the following recursion:*

$$\mathcal{Y}^0_{i,t+1}(z) = \mathcal{Y}^0_{i,t}(z) + \frac{\Sigma_t(z, z^t)}{\Sigma_t(z^t, z^t) + \sigma_v^2}\left(y_{i,t} - \mathcal{Y}^0_{i,t}(z^t)\right) \tag{2.13}$$

$$\Sigma_{t+1}(z, z') = \Sigma_t(z, z') - \frac{\Sigma_t(z, z^t)\Sigma_t(z', z^t)}{\Sigma_t(z^t, z^t) + \sigma_v^2}. \tag{2.14}$$

The covariance matrix determines the importance of the signal that individuals receive for expectation revisions through the Kalman gain in Equation (2.13). The higher the perceived covariance between output under an arbitrary interest rate path z and the observed announcement z^t, the stronger will be the revision in the mean function that describes the equilibrium mapping evaluated at z, after observing a forecast error $y_{i,t} - \mathcal{Y}^0_{i,t}(z^t)$.

Initial beliefs The time-0 prior beliefs of level-0 individuals in Equation (2.11) are described by mean and covariance functions. The prior mean function $\mathcal{Y}^0_0$ is such that agents believe the economy will remain in steady state after a policy announcement. In my setting, this assumption amounts to setting

$$\mathcal{Y}^0_0(z) = 0 \tag{2.15}$$

for all interest rate announcements z. This is in line with the literature on bounded rationality where agents are assumed to have a limited understanding of general equilibrium effects (see, for example, Farhi and Werning (2019) and García-Schmidt and Woodford (2019)).

For now, I will make the following minimal assumption about the covariance function:

$$\Sigma_0 \left(z, z^{ss} \right) = 0, \tag{2.16}$$

where $z^{ss} = \{0\}$ is the steady-state sequence of interest rates. In this model without aggregate risk, the steady state represents an initial condition that individuals have observed infinitely many times. So, they know with certainty that, if there are no changes to interest rates, output will stay at steady state.

Beliefs about long-run output coincide with long-run rational expectations beliefs if the central bank announces an interest rate path that eventually reverts to the steady-state value.

Lemma 2 *Suppose that the central bank commits at time-0 to the interest rate path in Equation* (2.2)*. Then, consensus beliefs converge to rational expectations for $t \geq T$.*

The intuition for the result is that individuals' beliefs about the equilibrium mapping when interest rates remain at steady state are never updated. Under assumption 1 with initial mean and variance given by Equations (2.15) and (2.16), level-0 people are certain that output will be equal to the steady-state value once the state variable z has reverted to steady state. This belief is self-confirming, since it represents the rational-expectations equilibrium. All individuals with a higher level of sophistication understand it, and expect output to be at steady state. Thus, everyone expects the rational-expectations level of output after T, and beliefs are consistent with the realized output. Note that the proof relies on the fact that this model only has future interest rates as an exogenous aggregate state variable. However, a similar argument can be made for models with backward-looking state variables, provided that they eventually converge back to steady state.

2.2 The Dynamic Effects of Forward Guidance

In this section, I discuss the dynamic effects of a forward guidance announcement. Importantly, the dynamic response of macro variables to an announcement will depend on the feedback between outcomes and expectations that is typical of learning models.

The output effect of a forward guidance shock can be decomposed into a partial-equilibrium and a general-equilibrium component. The partial-equilibrium component captures the response of output to the forward guidance announcement, holding initial expectations fixed. This effect reflects the intertemporal substitution of consumption due to interest rate changes. The general-equilibrium component takes changes in expectations into account. This effect reflects new forecasts about the response of future income.

Specifically, let

$$y_t^{PE} = \mathcal{C}_0(0, z^t),$$

and

$$y_t^{GE} = \mathcal{C}_t(y_t, z^t) - \mathcal{C}_0(0, z^t).$$

Here $\mathcal{C}_0(0, z^t)$ represents aggregate demand at time t at the announced interest rate path z^t, when expectations are formed at time-0 (as indicated by the subscript in $\mathcal{C}_0$), and for time t income kept at the steady state level of zero. $\mathcal{C}_t(y_t, z^t)$ instead is aggregate consumption after accounting for the change in expectations and in income at time t. Clearly, the total response of output is the sum of the partial and general equilibrium components, $y_t = y_t^{PE} + y_t^{GE}$.

Time-0 response Analysis of the time-0 response extends the results in Farhi and Werning (2019) to the generalized level-k thinking model presented above. Notably, the learning model plays no role in shaping the time-0 response of expected and actual outcomes to the forward guidance announcement.

The response of output at time-0 is given by:[9]

$$y_0 = \frac{1-\beta}{\beta} \sum_{s=1}^{T-1} \beta^s \overline{\mathbb{E}}_0[y_s] + \sigma \beta^{T-1}, \tag{2.17}$$

The rational expectations response is equal to:

$$y_0^* = \sigma.$$

The corresponding partial- and general-equilibrium decomposition is such that:

$$y_0^{*,PE} = \sigma\beta^{T-1} \qquad \text{and} \qquad y_0^{*,GE} = \sigma(1 - \beta^{T-1}).$$

The next Lemma shows how the output response behaves in the generalized level-k thinking model presented above.

[9] The interest rate shock size is normalized to 1.

Lemma 3 *Suppose assumption 2 holds. The response of output at time-0 with generalized level-k thinking, given by Equation (2.17), is lower than the rational-expectations counterpart:*

$$y_0 \leq y_0^*.$$

The two responses coincide in the limiting case where there is a $\bar{k} \to \infty$ such that $\Phi(\bar{k}) = 1$.

As a natural extension to the result with standard level-k thinking, the proposition argues that the output response to interest rate announcements is dampened in the presence of bounded rationality. Also in this setting, the result follows from the fact that:

$$y_0^{GE} \leq \sigma(1 - \beta^{T-1}),$$

with equality holding only for the limiting case of $\Phi(\infty) = 1$. Under bounded rationality, people fail to fully take into account future general-equilibrium effects of the policy announcement. Because they expect lower incomes in the future, their consumption choices are less responsive to the interest rate change announcement.

Dynamic effects In the integrated reasoning model, individual expectations are revised over time. Learning leads households to update their expectations about the effectiveness of monetary policy announcements. Individuals with higher levels of cognitive sophistication combine new information gained from belief updates of other people with their structural knowledge of the economy, and revise themselves their expectations. At each period, they are able to understand how level-0 people have formed a new path of expected income, and reason through the general-equilibrium effects of the belief revision. More sophisticated agents arrive at their updated beliefs after a higher number of iterations of their reasoning. That is, they form more accurate beliefs about the path of output.

At each date, aggregate demand is increasing in expectations about future incomes. So, the extent of belief revision about output determines the strength of the impulse response of consumption demand. Stronger expectation updates imply larger effectiveness of policy announcement. This is clear by inspection of the output response at time-t, which generalizes Equation (2.17) to:

$$y_t = \frac{1-\beta}{\beta} \sum_{s=1}^{T-t-1} \beta^s \overline{\mathbb{E}}_t[y_{t+s}] + \sigma \beta^{T-t-1}. \tag{2.18}$$

Belief updates about output directly map into revisions of the expected future output responses $\overline{\mathbb{E}}_t[y_{t+s}]$, which, in turn, determines the change in output.

For the rest of the paper, I assume that level-0 people use a linear model when forming expectations. The assumption of linearity effectively restricts the class of expectation formation processes to well-specified models, since I am looking at the linear model. Therefore, it allows me to streamline the implications of bounded rationality.

People's beliefs are summarized by a $(T+1) \times 1$ vector $\theta^0_{i,t}$ such that the perceived law of motion for output is:

$$\mathcal{Y}^0_{i,t}(z^{t+s}) = \mathbb{E}^0_{i,t}[y_{t+s}] = \left(\theta^0_{i,t}\right)' z^{t+s}. \tag{2.19}$$

The vector z^t collects announced interest rate deviations from steady-state for a number of periods equal to $T+1$. So, the vector is equal to $z^t = [r_t, r_{t+1}, r_{t+2}, \ldots, r_{t+T}]$. The elements of $\theta^0_{i,t}$ can be interpreted as perceived elasticities of output to interest rate changes at different future dates. In Appendix B, I show that in the linear version of the model I consider, the beliefs of every level-k individual for $k \geq 1$ has a linear representation, which can be written as follows:

$$\mathbb{E}^k_t[y_{t+s}] = \left(\theta^k_t\right)' z^{t+s}.$$

To make the model operational, the covariance function for initial beliefs needs to be fully specified. It is convenient to recast the model in terms of prior beliefs about the vector $\theta^0_{i,t}$. The time-0 prior is:

$$\tilde{\theta}^0_{i,0} \sim \mathcal{N}\left(\theta^0_{i,0}, \Sigma^\theta_0\right), \tag{2.20}$$

where $\tilde{\theta}^0_{i,0}$ is the random vector representing the response of output to changes in interest rates at different time periods, and $\theta^0_{i,0}$ is the mean of the prior distribution. Appendix B shows that the model described by Equations (2.19) and (2.20) is a special case of the one defined by Assumption 1 within the class of linear models.

I assume that the average of prior beliefs is $\theta^0_{i,0} = 0$. This implies that the time-0 prior mean is consistent with condition (2.15), which states that individuals expect output to remain at steady state following the interest rate announcement. The prior covariance matrix Σ^θ_0 instead is assumed to have the following structure. The (t,s) element of the matrix is:

$$\left[\Sigma^\theta_0\right]_{t,s} = \sigma^2_0 \rho^{|t-s|} \tag{2.21}$$

The parameter σ_0^2 determines the prior variance of beliefs about the sensitivity of output to interest rate changes at any given horizon. The parameter ρ controls the extent to which information about a shock at some horizon t is informative about the effects of a shock at a different horizon s. The smaller ρ, the faster the correlation between the perceived effect of shocks at different horizons declines with the distance between the time periods.

To understand the implications of this assumption, consider the one-time interest rate change announcement described in Equation (2.2). This structure of the covariance function implies that time-1 forecasts $\{\mathbb{E}^0_{i,1}[y_{1+s}]\}_{s\geq 1}$ are updated with the following Kalman gain[10]:

$$\frac{\Sigma_0(z^s, z^0)}{\Sigma_0(z^0, z^0) + \sigma_v^2} = \frac{\sigma_0^2 \rho^s}{\sigma_0^2 + \sigma_v^2},$$

when forecasting future period $1+s$. Intuitively, the larger the prior variance parameter σ_0^2 relative to signal noise σ_v^2, the more influence forecast errors have on updating beliefs. Moreover, beliefs about output at horizons that are farther away in the future are updated at decaying rate ρ.

In line with the original formulation of the generalized level-k thinking model in Camerer et al. (2004), I make the following assumption about perceived distributions $f_k(\cdot)$ and the actual distribution $\Phi(\cdot)$ of agents along the cognitive hierarchy.

Assumption 2 *The distributions $f_k(\cdot)$ are such that, for any $h < k$,*

$$f_k(h) = \frac{\Phi(h)}{\sum_{s=1}^{k-1} \Phi(s)}$$

Assumption 2 says that individuals have an accurate guess about the relative proportions of less sophisticated people. This specification implies that, in the limit, agents doing k and $k-1$ steps of thinking will have the same beliefs, which become arbitrarily close to the actual distribution. Note that, under this specification, a rational-expectations equilibrium is a special case of the generalized level-k thinking model when $\Phi(\infty) = 1$.

I make the following parametric assumption on $\Phi(\cdot)$: the distribution of cognitive abilities follows a Poisson process with parameter λ, so that $\Phi(k) = \frac{\lambda^k e^{-\lambda}}{k!}$. The average sophistication level in the economy is then equal to λ. A higher level of average sophistication implies that individuals perform, on average, a higher number of rounds of general-equilibrium thinking. The limiting case of $\lambda \to \infty$ is a rational-expectations equilibrium.

[10] The shock size is normalized to 1.

Figure 1 shows the response of output. The horizon of the interest rate shock announcement is T equal to 10. The left panel of the figure displays the response of output under rational expectations as a benchmark case (dotted red line). The three other lines correspond to the following assumptions. The dash-dotted brown line corresponds to the case in which individuals do not have any understanding of the structure of the economy ($\lambda = 0$). The dashed green line shows output under the assumption that households do not revise expectations over time ($\sigma_0^2 = 0$). Finally, the solid blue line is an example of integrated reasoning ($\lambda = 1, \sigma_0^2 = 0.05$).

The integrated reasoning model predicts a mitigated effect of forward guidance at all horizons relative to the case of rational expectations. The intuition for this result is the same as for the time-0 effect, illustrated in Lemma 3: due to the limited ability of individuals to understand the future general-equilibrium implications of policy announcements, their effectiveness is dampened compared to rational expectations. However, the integrated reasoning model predicts that the response is larger than in the two benchmark cases of pure level-k thinking and pure learning. When combining belief revision with strategic thinking about their future effects, individuals anticipate larger income responses, which boosts current demand.

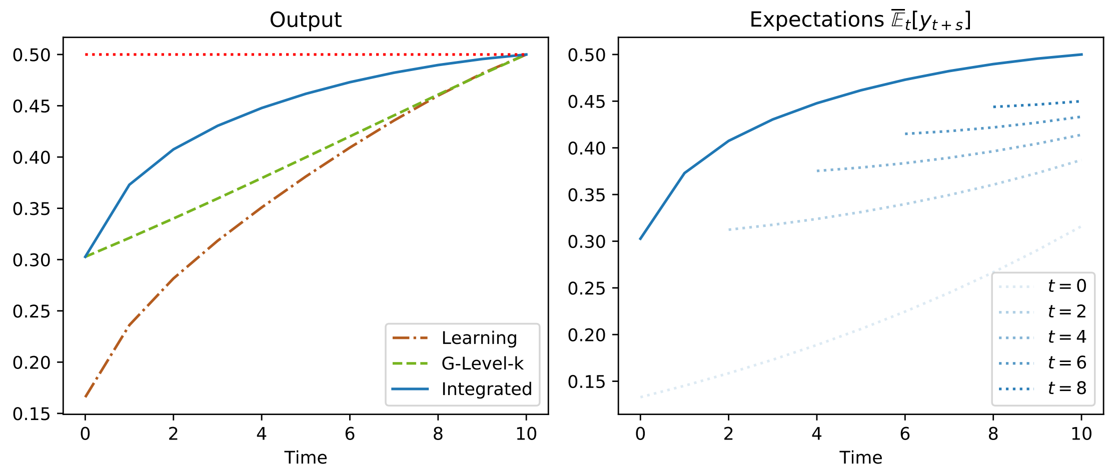
The right panel shows how expectations about future income is revised over time for the case of integrated reasoning. At time-0, consensus expected output is traced by the light-colored dotted blue line labeled as $t = 0$. After observing a forecast error, consensus beliefs about output is revised through a combination of belief revision and strategic thinking. This process gradually leads to revised expected future incomes, as shown by the additional dotted blue lines for times $t = 2, 4, 6$ and 8. The graph highlights the importance of expectations revision in determining the overall effect of a forward guidance announcement: the dynamic effects of the policy are shaped by the updated expectations about future incomes that follows from the belief revision model. As is clear from Equation (2.18), the update in expectations about the future effects of the policy, captured by $\overline{\mathbb{E}}_t[y_{t+s}]$, acts as an additional stimulus to current demand, and ultimately feeds back into a stronger realized output response at each date. By comparison, the generalized level-k thinking model implies that, over time, the path of expected income is constant and equal to the initial path represented by the $t = 0$ light-colored dotted blue line.

2.3 Anticipation, Forecast Errors and Forecast Revisions

To understand the role of expectations formation, it is useful to further decompose the time-t elasticity in Equation (2.18). The general-equilibrium component can be separated into an effect that is

Figure 1: Impulse Response Function to a Forward Guidance Announcement $(T = 10)$

The left panel of the graph shows the response of realized output (equation (2.18)) to a an announcement of an interest rate reduction at time $T = 10$. The figure reports the response normalized by the size of the shock. Parameters that are set to a common value across simulations are: $\beta = 0.7$, $\sigma = 0.5$, $\sigma_v^2 = 0.002$, $\rho = 0.779$. The dotted red line represents rational expectations. The dashed-dotted brown line represents learning $(\lambda = 0, \sigma_0^2 = 0.05)$. The dashed green line represents generalized level-k thinking $(\lambda = 1, \sigma_0^2 = 0)$. The solid blue line represents integrated reasoning $(\lambda = 1, \sigma_0^2 = 0.05)$. The right panel reports realized output with integrated reasoning and average beliefs about the path for output $\overline{\mathbb{E}}_t[y_{t+s}]$ formed at different time periods.



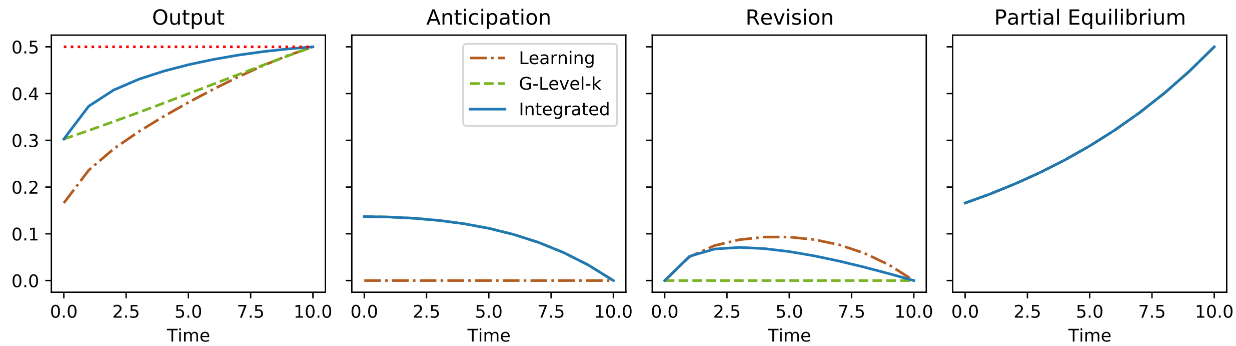
due to the *anticipation* of future changes to equilibrium income, and an effect that is due to *expectations revision* over time. The response of output can be decomposed as follows:

$$y_t = \underbrace{\frac{1-\beta}{\beta}\sum_{s=1}^{T-t-1}\beta^s\left(\overline{\mathbb{E}}_t[y_{t+s}] - \overline{\mathbb{E}}_0[y_{t+s}]\right)}_{belief\ revision} + \underbrace{\frac{1-\beta}{\beta}\sum_{s=1}^{T-t-1}\beta^s\overline{\mathbb{E}}_0[y_{t+s}]}_{anticipation} + \underbrace{\sigma\beta^{T-t-1}}_{partial\ equilibrium}. \tag{2.22}$$

Here, $\overline{\mathbb{E}}_0[y_{t+s}]$ is the expected response of output in the absence of learning, which is derived from beliefs formed at time-0. The anticipation effect is a consequence of the structural knowledge of the economy that individuals can achieve via generalized level-k thinking at time-0. By construction, it corresponds to the entire general-equilibrium effects of the policy announcement in the benchmark case of generalized level-k thinking. Because households are able to partially work through the implications of the policy announcements, they incorporate higher future incomes in their expectations. This effect is what determines the impact response of output to interest rate announcements above and beyond the partial equilibrium effect. However, in the absence of feedback between outcomes and new expectations, individuals do not revise their forecasts over time.

Figure 2: Decomposition of Impulse Response Function

The graph shows the decomposition of output response described by equation (2.22). Parameters that are set to a common value across simulations are: $\beta = 0.7$, $\sigma = 0.5$, $\sigma_v^2 = 0.002$, $\rho = 0.779$. The dotted red line represents rational expectations. The dashed-dotted brown line represents learning ($\lambda = 0, \sigma_0^2 = 0.05$). The dashed green line represents generalized level-k thinking ($\lambda = 1, \sigma_0^2 = 0$). The solid blue line represents integrated reasoning ($\lambda = 1, \sigma_0^2 = 0.05$).



The effect due to belief revision is instead a consequence of learning. Upon observing new information, individuals revise their expectations accordingly. A sequence of positive forecast errors, therefore, leads individuals to revise their future income expectations upwards. However, in the absence of any structural knowledge of the economy, the extent of the revision is dampened. Figure 2 shows the decomposition of the impulse response function in Equation (2.22) for the case of a forward guidance announcement about $T = 10$.

The behavior of forecast errors and revisions in response to the announcement provides a tool to validate the expectations model when comparing the predictions to the data. Figure 3 displays forecast errors on the left panel as:

$$y_{t+1} - \overline{\mathbb{E}}_t[y_{t+1}] \tag{2.23}$$

and forecast revisions on the right panel as:

$$\overline{\mathbb{E}}_t[y_{t+1}] - \overline{\mathbb{E}}_{t-1}[y_{t+1}]. \tag{2.24}$$

It is commonly assumed in models of level-k thinking that expectations are formed at time-0 once and for all, and then never revised. Because of their imperfect ability to understand general-equilibrium effects, individuals make systematic mistakes when forecasting macroeconomic variables. Since they do not revise expectations, these mistakes are persistent over time. In integrated reasoning, instead, people revise expectations in real time and take their forecast errors into account through forecast updates.

The opposite case is where individuals do not have any structural knowledge of the economy, and only update expectations in light of new information. These individuals, who have no prior knowledge about the effects of an unconventional policy, do not revise their expectations about output in response to the policy change announcement. By introducing some degree of deductive thinking, the integrated reasoning model allows for some belief revision on impact.

The following Proposition summarizes the most important features of the behavior of forecast errors and revisions in the integrated reasoning model.

Proposition 1 *Let $fr_{t,s} = \overline{\mathbb{E}}_t[y_{t+s}] - \overline{\mathbb{E}}_{t-1}[y_{t+s}]$ denote average forecast revisions, and $fe_{t,s} = y_{t+s} - \overline{\mathbb{E}}_t[y_{t+s}]$ denote average forecast errors. The following properties are true:*

1. *The average belief revision at $t = 0$ is increasing in the average sophistication level λ, and is equal to zero in the absence of level-k thinking ($\lambda \to 0$):*

$$\frac{\partial fr_{0,s}}{\partial \lambda} \geq 0; \qquad \lim_{\lambda \to 0} fr_{0,s} = 0 \qquad \textit{for all } s \geq 1.$$

2. *The average forecast under-reacts to the announcement at time $t = 0$, and the average forecast error is equal to zero under rational expectations ($\lambda \to \infty$):*

$$fe_{0,0} \geq 0; \qquad \lim_{\lambda \to \infty} fe_{0,0} = 0.$$

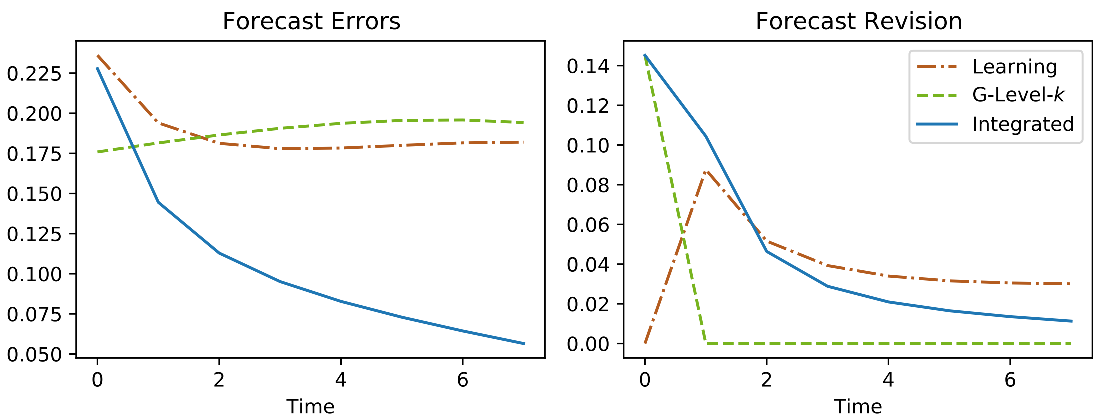
3. *Revisions in average beliefs are different from zero. For $t \geq 1$, revisions are equal to zero in the absence of learning ($\sigma_0^2 \to 0$):*

$$fr_{t,s} \neq 0; \qquad \lim_{\sigma_0^2 \to 0} fr_{t,s} = 0 \qquad \textit{for all } t \geq 1 \textit{ and } s \geq 1.$$

This discussion illustrates the importance of using forecast errors and revisions as a diagnostic tool for selecting the right model of expectations. The properties highlighted in Proposition 1 provide a way to discipline the key parameters of the belief formation using direct survey data. The next section is devoted to this task.

Figure 3: Forecast Errors and Forecast Revisions

The graph shows forecast errors on the left (equation (2.23)) and forecast revisions on the right (equation (2.24)). Parameters that are set to a common value across simulations are: $\beta = 0.7$, $\sigma = 0.5$, $\sigma_v^2 = 0.002$, $\rho = 0.779$. The dotted red line represents rational expectations. The dashed-dotted brown line represents learning ($\lambda = 0, \sigma_0^2 = 0.05$). The dashed green line represents generalized level-k thinking ($\lambda = 1, \sigma_0^2 = 0$). The solid blue line represents integrated reasoning ($\lambda = 1, \sigma_0^2 = 0.05$).



3 Forward Guidance Shocks and Macroeconomic Expectations

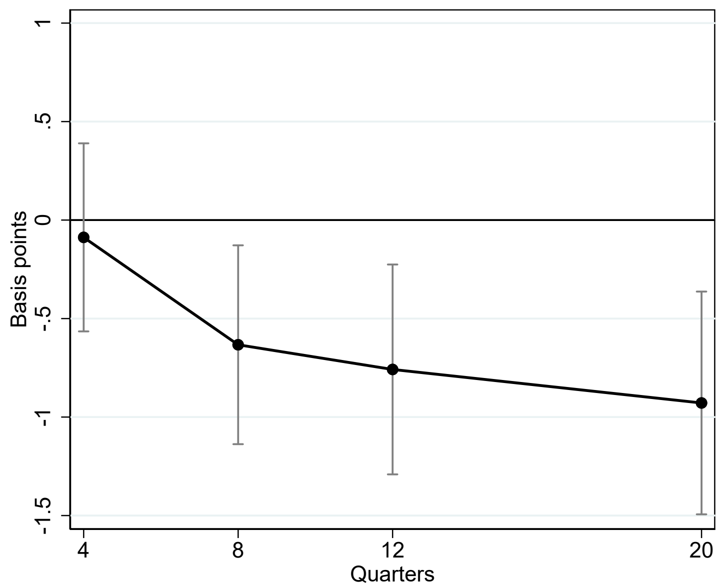
In this section, I use evidence from forward guidance shocks and survey data on expectations about aggregate outcomes to provide empirical estimates of the response of forecast errors and revisions to announcements about future policy rates.

3.1 Data

Forward guidance shocks Forward guidance shocks are announcements of unexpected changes to the path of future short-term interest rates. I employ the series derived in Jarociński (2021), who exploits the leptokurtic nature of the high-frequency financial markets response to FOMC announcements to identify conventional and unconventional monetary policy shocks. The methodology employed in this paper is able to separately identify the effects of a commitment to a future course of policy rates ("Odyssean" forward guidance), and a statement about the future course of policy rates understood as a forecast of the appropriate stance of the policy ("Delphic" forward guidance). Thus, it effectively cleans out possible information effects associated with the future stance of policy rates. The two shocks have similar effects on the high-frequency movements in the yield curve, but opposite effects on the stock market. The forward guidance announcement understood as a commitment to a specific policy stance is the one used in my analysis. Additional

Figure 4: Yield Curve Response to Forward Guidance Shocks

The graph shows the point estimates of $\{\beta^\tau\}$ from equation (3.1), together with their 95% confidence intervals. Standard errors are robust. Daily changes to Treasury yields are obtained from data available on FRED.



details for the estimation are provided in Appendix D.

Figure 4 displays the yield curve's response to a forward guidance shock. The figure shows changes to yields of Treasury bonds with maturities of 1, 2, 3, and 5 years, in response to the forward guidance shock series from Jarociński (2021). The dots correspond to the point estimates of the responses β^τ from the estimating equation

$$r_t^\tau = \alpha^\tau + \beta^\tau u_{FG,t} + \varepsilon_t^\tau, \tag{3.1}$$

where r_t^τ are daily changes of yields with maturities of $\tau = 4, 8, 12$, and 20 quarters, and $u_{FG,t}$ is the forward guidance shock series. The gray bands represent the 95% robust confidence intervals.

Expectations data Survey data on expectations is obtained from the Survey of Professional Forecasters (SPF), a panel survey of about 40 experts from industry, government, and academia.[11] Each quarter, survey participants are asked for point-estimate projections of several macro aggregates. My main analysis focuses on projections about GDP growth, civilian unemployment, and inflation measured as the GDP deflator growth rate. When using measures of consensus forecasts, I show

[11] The data are available at philadelphiafed.org/surveys-and-data/data-files.

results for the median forecast of the object of interest to alleviate concerns about outliers, given the small size of the cross-section.

3.2 Dynamic Response of Macroeconomic Outcomes and Expectations

I first investigate the response of macroeconomic variables and their consensus forecasts to an expansionary forward guidance announcement. To do so, I aggregate the shock at the quarterly level by summing up the monthly shock realizations. I estimate the dynamic response of macro aggregates using Jordà (2005) local projection method with the following specification:

$$x_{t+h} = \alpha_h^x + \beta_h^x u_{FG,t} + (\gamma_h^x)' W_t + \varepsilon_{t+h}^x, \tag{3.2}$$

where $(\beta_h^x)_h$ trace out the dynamic response of the outcome x_{t+h}. The vector W_t collects four lags of the dependent variable and the shock. Finally, $u_{FG,t}$ is one of the forward guidance shock series extracted by Jarociński (2021). The time aggregation produces validly identified monetary policy shocks under the assumption that the shocks are orthogonal to other economic variables in that quarter. Following the recommendation by Montiel Olea and Plagborg-Møller (2021), I compute heteroskedastic-robust standard errors.

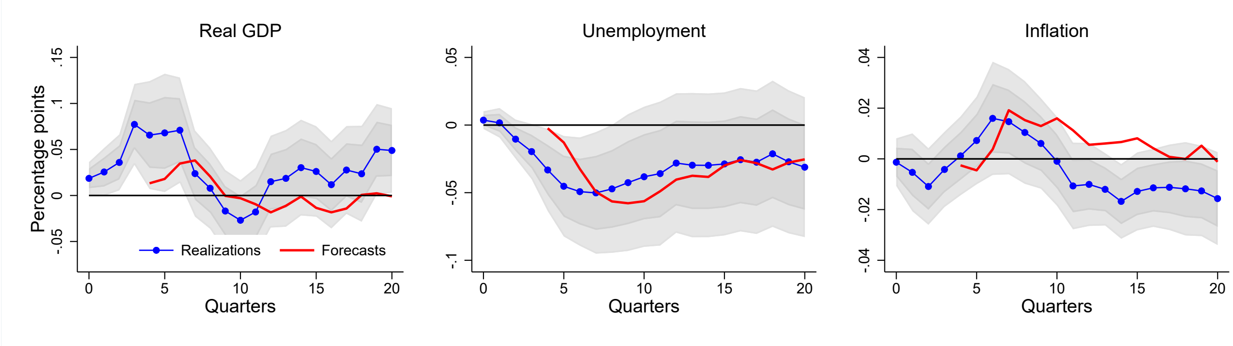
Figure 5 reports the results of estimates of Equation (3.2) for real GDP growth, unemployment, and inflation. The blue lines represent the effects on actual macro realizations, with 68% and 90% confidence intervals, respectively. The shock has expansionary effects on all macro variables. The red lines instead trace out the effect of the shock on the 4-quarter ahead forecasts of the corresponding outcome.

Forward guidance announcements have expansionary effects with peak responses around 6 quarters after the shock. Consensus forecasts also adjust in response to the announcement, displaying an initial under-reaction relative to actual realizations. Expectations track more closely actual realizations after the initial under-reaction, with the exception of inflation, which displays persistently high revisions relative to outcomes.

Proposition 1 illustrates that forecast errors and revisions can be used as a diagnostic tool to select the expectations model that is most consistent with the empirical behavior of expectations. In particular, it implies that the combination of impact revision, under-reaction and slow update of beliefs can only be accounted for by the combination of forward-looking, imperfect structural thinking and slow update of expectations. I employ the same empirical specification from Equa-

Figure 5: Response of Macroeconomic Outcomes to a Forward Guidance Shock

The graph shows the point estimates of $\{\beta_h^x\}$ from equation (3.2) for output growth, unemployment, and inflation (blue lines). The gray-shaded areas are their 68% and 90% confidence intervals. Standard errors are heteroskedasticity robust. The red lines represent the point estimates for macroeconomic forecasts. Both realizations and forecasts data are obtained from the Philadelphia Fed website at philadelphiafed.org/surveys-and-data/data-files.



tion (3.2) using consensus forecast errors and forecast revisions as outcome variables, defined as:

$$x_{t+h+4} - \overline{\mathbb{E}}_{t+h}[x_{t+h+4}], \tag{3.3}$$

and

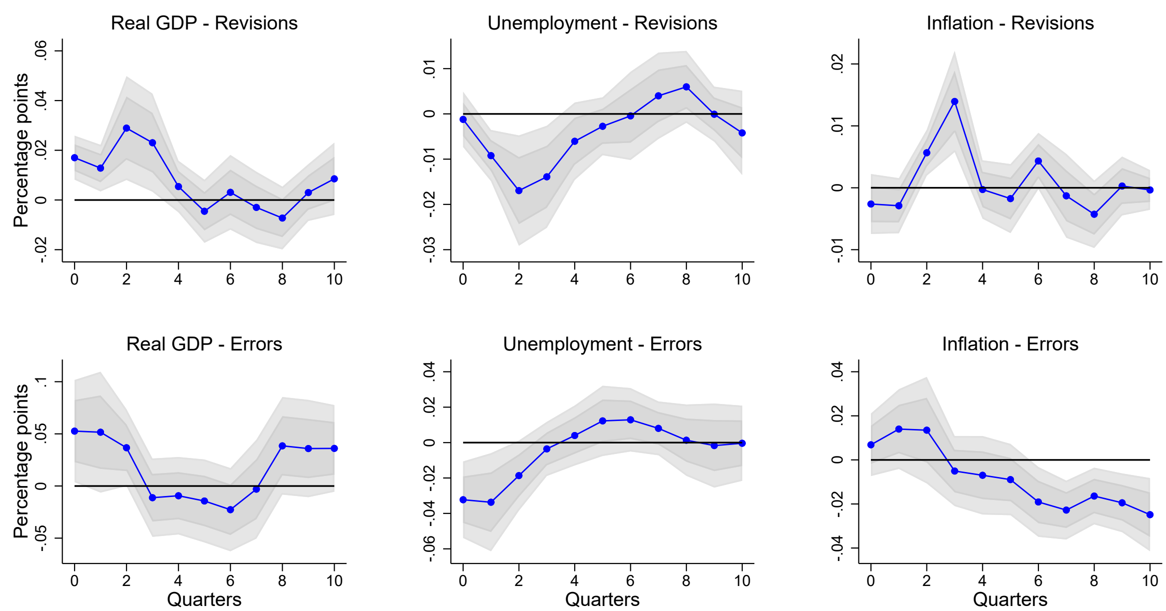
$$\overline{\mathbb{E}}_{t+h}[x_{t+h+4}] - \overline{\mathbb{E}}_{t+h-1}[x_{t+h+4}], \tag{3.4}$$

where $\overline{\mathbb{E}}_t[x_{t+4}]$ denotes the time-t 4-quarter ahead consensus forecasts for variable x_t. Here, the vector of controls include lagged values of actual realizations and forecasts of the variable of interest.

Figure 6 shows the impulse response of forecast errors on the top panel, and of forecast revisions on the bottom panel. The results give an informative picture about the nature of expectation formation and update. Three facts stand out from these results. First, forecast revisions about real GDP growth at time-0 are statistically different from zero. This fact indicates that the effects of future changes to short-term interest rates are incorporated right away into forecasts about future economic activity. There is however little or no evidence of a systematic revision of unemployment and inflation forecasts in the quarter of the announcement. Second, forecasters make systematic forecast errors when predicting the response of key macroeconomic variables to the forward guidance shock. Specifically, they believe that real GDP growth and inflation will be lower than the true realizations, and unemployment will be higher, resulting in positive and negative forecast errors for the initial quarters after the shock. This pattern indicates that average expectations under-react

Figure 6: Response of Forecast Errors and Revisions to a Forward Guidance Shock

The graph shows the point estimates of $\{\beta_h^x\}$ from equation (3.2) for consensus forecast errors (equation (3.3), bottom panels) and consensus forecast revisions (equation (3.4), top panels) for output growth, unemployment, and inflation (blue lines). The gray-shaded areas are their 68% and 90% confidence intervals. Standard errors are heteroskedasticity robust. Both realizations and forecasts data are obtained from the Philadelphia Fed website at philadelphiafed.org/surveys-and-data/data-files.



to the policy announcement. Third, forecast revisions display slow updates that are predictable for a number of quarters after the shock. Errors and revisions gradually revert back to zero at longer horizons.

The fact that forecast errors and forecast revisions are predictable constitute evidence that rejects rational expectations, full information models. Moreover, the evidence is informative for the choice of models that are nested within the integrated reasoning framework, in light of Proposition 1. On the one hand, a model with bounded rationality without belief updates is inconsistent with the evidence that forecast revisions are statistically different from zero at horizons following the time of the shock. On the other hand, a model with learning where agents do not use any structural knowledge is inconsistent with the data in that it implies that individual beliefs do not react on impact to the forward guidance announcements. Therefore, both elements are needed to match the evidence on predictable forecast errors and revisions.

As Bordalo et al. (2020) and Angeletos et al. (2020) point out, it is possible that consensus forecasts *under*-react to news, while individual expectations *over*-react. In the integrated reasoning model, however, individuals under-react to unconventional policy announcements because of their limited ability to forecast general-equilibrium effects. To assess the plausibility of my model, I estimate the effect of forward guidance shocks on individual-level beliefs from the SPF using local projections. Figure 14 from Appendix E reports the impulse response of individual forecast errors and revisions: the behavior of individual-level beliefs is very similar to the response of consensus beliefs. An initial under-reaction is followed by a slow belief update.

To provide additional evidence in support of the prediction of initial under-reaction of belief followed by gradual updates, I estimate the following local projection regression:

$$\mathbb{E}_{i,t+h}[x_{t+4}] - \mathbb{E}_{i,t-1}[x_{t+4}] = \alpha_{i,h}^{fr} + \beta_h^{x,fr} u_{FG,t} + \left(\gamma_h^{fr}\right)' W_t + \varepsilon_{i,t+h}^{fr}, \tag{3.5}$$

for $h = 0, 1, 2, 3$. The left-hand side represents the forecast revision for cumulative output growth, inflation, and unemployment. The forecast horizon $t + 4$ is kept fixed at the fourth quarter after the announcement. The response of the corresponding forecast errors is estimated as:

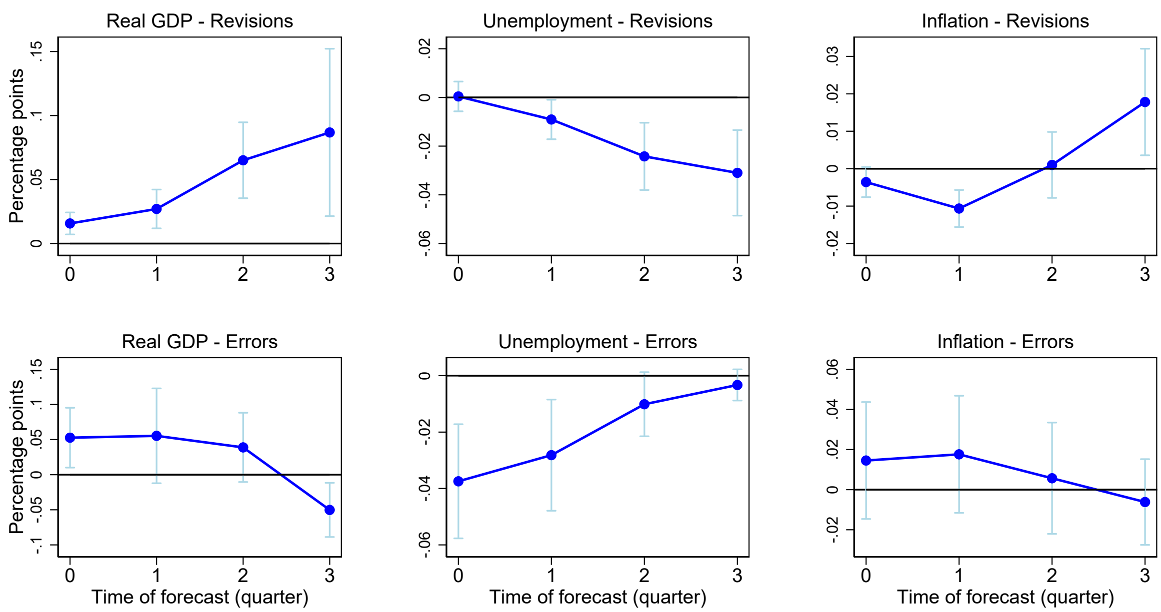
$$x_{t+4} - \mathbb{E}_{i,t+h}[x_{t+4}] = \alpha_{i,h}^{fe} + \beta_h^{x,fe} u_{FG,t} + \left(\gamma_h^{fe}\right)' W_t + \varepsilon_{i,t+h}^{fe}, \tag{3.6}$$

These moments of the data allow me to better isolate the update in expectations (or lack thereof), by eliminating confounding factors arising from the change in forecast horizon. Importantly, in the vector of control variables W_t from Equation (3.6) I include lagged individual-level revisions, to proxy for individual information available to forecasters. Hence, the estimates of $\{\beta_h^{x,fe}\}$ can be interpreted as the forecast errors due to forward guidance shocks, conditional on individual-level information, which is what is needed to test for evidence of under- or over-reaction at the individual level.

The results are shown in Figure 7, with 90% confidence intervals. Beliefs about unemployment most clearly show an under-reaction to forward guidance shocks, with revisions being accompanied by forecast errors of the same sign. Moreover, the size of forecast revisions relative to the pre-shock expectations increases over time, and forecast errors decrease, providing evidence in support of learning. The evidence on output growth and inflation also supports the same mechanism, although forecast errors are somewhat less precisely estimated. Figure 15 in Appendix E.1 shows the same evidence for forecasts about quarter 5 following the shock, and is broadly consistent with these results.

Figure 7: Forecast Revisions for Quarter 4 After Announcement

The graph shows the point estimates of $\{\beta_h^{x,fr}\}$ from equation (3.5) for individual forecast revisions (top panels) and of $\{\beta_h^{x,fe}\}$ from equation (3.6) for individual forecast errors (bottom panels) for output growth, unemployment, and inflation (blue lines). The bars are 90% confidence intervals. Standard errors are heteroskedasticity robust. Both realizations and forecasts data are obtained from the Philadelphia Fed website at philadelphiafed.org/surveys-and-data/data-files.



4 Estimating Integrated Reasoning

In this section, I lay out an extended version of the model in Section 2 to allow for inflation. Inflation is an important general-equilibrium force that pins down the real interest rate for the present and future dates. To do so, I introduce Calvo-style wage rigidities as in Erceg et al. (2000) and Schmitt-Grohé and Uribe (2005), which generates wage and price inflation. Because of wage stickiness, actual employment is demand-determined.

Households The household problem is to maximize lifetime utility (2.1) subject to the current and future budget constraints as follows:

$$P_{t+s}C_{i,t+s} + B_{i,t+s+1} = W_{t+s}N_{i,t+s} + R_{t+s-1}B_{i,t+s}.$$

Monetary policy As before, the central bank announces a path for nominal interest rates. I set the sequence of short-term interest rates announced by the central bank so that the implied shift in the yield curve matches the empirical response displayed in Figure 4. The yields for which data are unavailable are backed out with linear interpolation. This procedure gives a sequence of estimated responses $\{\beta^\tau\}$ for $\tau = 0, 1, \ldots, 21$ which constitutes the empirical counterpart to the time-0 linearized yield curve $\{i_0^\tau\}$ defined as:

$$i_0^\tau = \sum_{t=0}^{\tau} i_t,$$

where $\{i_t\}$ is the announced path for nominal short-term interest rate deviations from steady state. Note that my model has no term premium, so an implicit assumption behind this procedure is that there is no high-frequency change to term-premia in response to a forward guidance shock.

Final good and labor market There is a continuum of unions in the economy, indexed by $u \in [0, 1]$. Each household has a continuum of workers with differentiated labor skills, and provides $n_{u,t}$ units of type u. The composite labor input N_t is generated using labor varieties according to the technology:

$$N_t = \left[\int_0^1 n_{u,t}^{\frac{\theta-1}{\theta}} du \right]^{\frac{\theta}{\theta-1}},$$

where $\theta > 1$ captures the elasticity of substitution across the labor varieties. The final good technology is linear in composite labor input, so $Y_t = N_t$. The final good producers' maximization problem implies the following labor demand condition for labor variety u:

$$n_{u,t} = \left(\frac{w_{u,t}}{W_t} \right)^{-\theta} N_t, \tag{4.1}$$

where $w_{u,t}$ is the wage for $n_{u,t}$, and the aggregate wage level is:

$$W_t = \left[\int_0^1 w_{u,t}^{1-\theta} du \right]^{\frac{1}{1-\theta}}.$$

Because of the linear production technology, the price of the consumption good is equal to the wage level, $P_t = W_t$.

Unions set wages subject to a Calvo-style friction. At each time, a fraction $1 - \epsilon$ is randomly selected to adjust their wage, $w_{u,t}$. For the other ϵ unions, $w_{u,t} = w_{u,t-1}$. On behalf of households, unions choose wages and labor hours to maximize the expected household's valuation of labor income, subject to the labor demand given by Equation (4.1). The remaining details of the union's problem are in Appendix C.

The linearized economy The model reduces to the familiar two-equations system, which in this case takes the following form:

$$y_t = \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} \beta^s \overline{\mathbb{E}}_t[y_{t+s}] - \sigma \sum_{s=0}^{\infty} \beta^s \left(i_{t+s} - \overline{\mathbb{E}}_t[\pi_{t+s+1}] \right) \tag{4.2}$$

$$\pi_t = \kappa_w \sum_{s=0}^{\infty} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[y_{t+s}] + \frac{1-\epsilon}{\epsilon} \sum_{s=1}^{\infty} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[\pi_{t+s}], \tag{4.3}$$

where $\kappa_w = \frac{(1-\epsilon)(1-\hat{\beta}\epsilon)}{\epsilon} \left(\varphi + \sigma^{-1}\right)$, and $\hat{\beta}$ represents the discount factor of unions. As discussed in the calibration in Section 4.1, I allow for the household's discount factor to differ from the unions' as a reduced-form way to capture market incompleteness on the households, as in Farhi and Werning (2019) and Angeletos et al. (2020). The derivation of these equations is in Appendix C.

As before, households and unions have perfect foresight about the state variable $z^t = \{i_t, i_{t+1}, i_{t+2}, \ldots, i_{t+20}\}$, and they now have to form expectations about the future paths for both income and inflation. Every agent in the economy uses integrated reasoning to form expectations about the two mappings from z^t to output and inflation—$\mathcal{Y}_{i,t}(z^t)$ and $\Pi_{i,t}(z^t)$, represented in the linear case by vectors $\theta^y_{i,t}$ and $\theta^\pi_{i,t}$—and they receive separate signals about their current value, with signal variance governed by $\sigma^2_{v,y}$ and $\sigma^2_{v,\pi}$ respectively. As before, I focus on a linear version of the expectations model for both income and inflation. Prior beliefs are specified according to Equations (2.20) and (2.21), with corresponding parameters $\sigma^2_{0,y}, \sigma^2_{0,\pi}, \rho_y$, and ρ_π.

4.1 Estimation

A two-step procedure to choose parameter values for the model. First, I fix a subset of parameters to values chosen from the literature. Second, I estimate the remaining parameters, which are specific to the integrated reasoning model, with a moment-matching procedure to ensure that the model can replicate the behavior of expectations in response to a forward guidance shock. Table 1 provides a summary of the parametrization.

Fixed parameters The model period is one quarter. The unions' discount factor $\hat{\beta}$ is set to match a 2% annual nominal interest rate. On the household side, I set the time preference parameter

β to match an average marginal propensity to consume of 0.25, an intermediate value compared to the calibration in Angeletos et al. (2020) (MPC = 0.3) and empirical estimates from Fagereng et al. (2021) (contemporaneous MPC = 0.2). This allows for a steeper Keynesian cross than under standard calibrations for representative-agent models, and implies an empirically realistic role of general-equilibrium forces.[12] In Section 5 I discuss the sensitivity of my results to this assumption. Consistent with evidence in Chetty et al. (2011), I set the Frisch elasticity to $\varphi^{-1} = 0.75$. The fraction of unions maintaining a fixed wage is set to $\epsilon = 0.9$. Finally, I set signal noisiness $\sigma^2_{\nu,y}$ and $\sigma^2_{\nu,\pi}$, that determine the steady-state belief dispersion, to match the cross-sectional variance of beliefs about output and inflation over the whole sample period, equal to 0.327 and 0.271, respectively.

Estimated parameters There are five more parameters that need to be estimated: prior variance $\sigma^2_{0,y}$ and $\sigma^2_{0,\pi}$, prior persistence ρ_y and ρ_π, and the average sophistication level λ. I choose them to match the response of forecast revisions to the forward guidance announcement. I use two sets of moments that are influenced by the variance and persistence of forecast revisions.

First, I estimate (3.5) for consensus forecasts as follows:

$$\overline{\mathbb{E}}_{t+h}[x_{t+4}] - \overline{\mathbb{E}}_{t-1}[x_{t+4}] = \alpha_h + \beta^x_h u_{FG,t} + \gamma'_h W_t + \varepsilon_{t+h}, \tag{4.4}$$

for $h = 0, 1, 2, 3$. The dependent variable is the average revision between $t-1$ and $t+h$ for forecasts about $t+4$. The estimates $\hat{\beta}^x_h$ from each regression represent the effects of a forward guidance shock on expectations formed during each of the four quarters before the forecasting horizon. The model-implied forecast revisions about quarter 4 in response to the monetary policy announcement, expressed in annualized terms, are given by:

$$\beta^y_h = \frac{4}{5}\left(\overline{\theta}^y_h - \overline{\theta}^y_{-1}\right)' z^4,$$

and

$$\beta^\pi_h = 4\left(\overline{\theta}^\pi_h - \overline{\theta}^\pi_{-1}\right)' z^4,$$

for $h = 0, 1, 2, 3$. The vector $\overline{\theta}^x_h$ represents consensus beliefs for variable x formed at time h. Since

[12] An equivalent formulation of the model that allows to separately match the discount factor and the average MPC is the one in Del Negro et al. (2012) and Farhi and Werning (2019), where market incompleteness is introduced in the form of occasionally exogenously binding borrowing constraint. In the linearized version of that model, the effective discount rate is $\beta = \hat{\beta}\kappa$, where $\hat{\beta}$ is the household discount rate, and κ is the probability of incurring in a binding collateral constraint.

in steady state, individuals believe there is no change in output following an announcement, we have that $\overline{\theta}^x_{-1} = 0$.

Second, I examine how forecasts of output and inflation are revised in the quarter after the monetary policy announcement. I repeat the exercise for each of the four quarters following the forecast revision. Specifically, I estimate (3.2) using the following dependent variable:

$$\overline{\mathbb{E}}_{t+1}[x_{t+1+h}] - \overline{\mathbb{E}}_t[x_{t+1+h}] = \alpha_h + \delta^x_h u_{FG,t} + \gamma'_h W_t + \varepsilon_{t+h}, \tag{4.5}$$

for $h = 1, 2, 3, 4$. In other words, what is kept fixed now is the time of the forecast revision, which is $t+1$. The forecasting horizon h instead is being changed. The model counterpart of these revisions are:

$$\delta^y_h = \frac{4}{h+1}\left(\overline{\theta}^y_1 - \overline{\theta}^y_0\right)' z^h,$$

and

$$\delta^\pi_h = 4\left(\overline{\theta}^\pi_1 - \overline{\theta}^\pi_0\right)' z^h,$$

for $h = 2, 3, 4, 5$.

These 16 moments are useful to identify the five parameters of interest. First, the average cognitive sophistication parameter λ influences the extent of belief revisions. Importantly, at the time of the shock, beliefs are revised if and only if λ is different from zero. Hence, it would be impossible to match the time-0 response $\{\hat{\beta}^y_0, \hat{\beta}^\pi_0\}$ in a model with $\lambda = 0$, unless the estimated response was exactly equal to zero. Also at subsequent time periods, a higher sophistication level implies larger forecast revisions. Second, in my model, the speed at which expectations are revised over time is influenced by the extent of forecast revisions. Since level-0 individuals update their beliefs in a Bayesian way for $h \geq 1$, the extent of expectations revision depends on the Kalman gain through prior variances $\sigma^2_{0,y}$ and $\sigma^2_{0,\pi}$ and persistence parameters ρ_y and ρ_π. In particular, a model with $\sigma^2_{0,y} = 0$ and $\sigma^2_{0,\pi} = 0$ would be unable to match the responses described by $\{\hat{\delta}^y_h, \hat{\delta}^\pi_h\}_h$, unless the estimates where exactly equal to zero.

Collecting the set of parameters to be estimated in the vector $\Psi = \left(\lambda, \sigma^2_{0,y}, \sigma^2_{0,\pi}, \rho_y, \rho_\pi\right)'$, let $\hat{\mathcal{M}}$ denote the empirical moment described by Equations (4.4) and (4.5), and let $\mathcal{M}(\Psi)$ denote their model-implied counterparts. My estimator $\hat{\Psi}$ solves

$$\min_{\Psi}\left(\mathcal{M}(\Psi) - \hat{\mathcal{M}}\right)'\Sigma^{-1}\left(\mathcal{M}(\Psi) - \hat{\mathcal{M}}\right), \tag{4.6}$$

Table 1: Model Parameters

The table shows calibrated and estimated parameter values of the model. Estimated parameters solve equation (4.6). Standard deviations are computed from the asymptotic variance in equation (4.7).

Parameter	Description	Value	Std. Dev.
	Fixed parameters		
β	Households discount factor	0.750	
$\hat{\beta}$	Unions discount factor	0.995	
φ	Inverse Frisch elasticity	1.333	
ϵ	Wage stickiness	0.900	
$\sigma^2_{\nu,y}$	Signal variance, y	0.327	
$\sigma^2_{\nu,\pi}$	Signal variance, π	0.271	
	Estimated parameters		
λ	Average Level-k	0.076	(0.035)
$\sigma^2_{0,y}$	Prior variance, y	0.066	(0.202)
$\sigma^2_{0,\pi}$	Prior variance, π	0.024	(0.376)
ρ_y	Prior persistence, y	0.594	(0.687)
ρ_π	Prior persistence, π	0.026	(7.397)

where Σ is a diagonal matrix containing the estimated variances of the targeted moments. I also compute an estimator $\hat{\mathbb{V}}$ for the asymptotic covariance matrix of $\hat{\Psi}$:

$$\hat{\mathbb{V}} = \left(\frac{\partial \mathcal{M}}{\partial \Psi} \left(\hat{\Psi}\right)' \Sigma^{-1} \frac{\partial \mathcal{M}}{\partial \Psi} \left(\hat{\Psi}\right) \right)^{-1}, \tag{4.7}$$

where $\frac{\partial \mathcal{M}}{\partial \Psi} \left(\hat{\Psi}\right)$ is the Jacobian of $\mathcal{M}(\Psi)$ evaluated at $\hat{\Psi}$.

Table 1 provides a summary of the estimated parameters. It is worth noting that the estimated value for the average sophistication level implies that approximately 92% of the forecasters are level-0 individuals who do not update their forecasts for output and inflation after a forward guidance announcement.[13] This number is close to the one estimated by Iovino and Sergeyev (2018), who concluded that 86% of the individuals in their sample did not update expectations after quantitative easing intervention by looking at mortgage price data.

Figure 8 shows the point estimates $\{\hat{\beta}^y_h, \hat{\beta}^\pi_h\}^3_{h=0}$, while Figure 9 reports results for $\{\hat{\delta}^y_h, \hat{\delta}^\pi_h\}^5_{h=2}$. Their 90% confidence intervals are included, and the red triangles show the model-implied moments. Overall, the model does a good job at matching the behavior of forecast revisions.

Table 1 reveals that the standard deviation on ρ_π is particularly high. A numerical check shows

[13] An important *caveat* to this conclusion is that, in the model, I simulate a one-time forward guidance shock, whereas the response of expectations in the data is estimated using a time series of shocks. Therefore, it is possible that the empirical response at time-0 is partly due to learning from past shocks. To the extent that this is true, my results overestimate the importance of level-k thinking.

Figure 8: Forecast Revisions for Quarter 4 After Announcement

The graph shows the point estimates of $\{\beta_h^x\}$ from equation (4.4) for consensus forecast revisions of output growth, unemployment, and inflation (blue lines). The bars are 90% confidence intervals. Standard errors are heteroskedasticity robust. Forecasts data are obtained from the Philadelphia Fed website at philadelphiafed.org/surveys-and-data/data-files. The red triangles represent the model-implied moments.

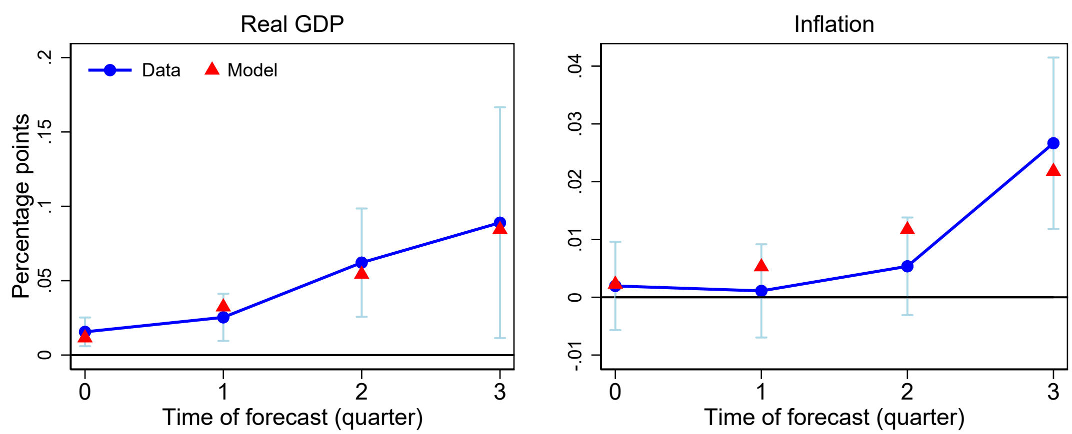


Figure 9: Forecast Revisions One Quarter After Announcement

The graph shows the point estimates of $\{\delta_h^x\}$ from equation (4.5) for consensus forecast revisions of output growth, unemployment, and inflation (blue lines). The bars are 90% confidence intervals. Standard errors are heteroskedasticity robust. Forecasts data are obtained from the Philadelphia Fed website at philadelphiafed.org/surveys-and-data/data-files. The red triangles represent the model-implied moments.

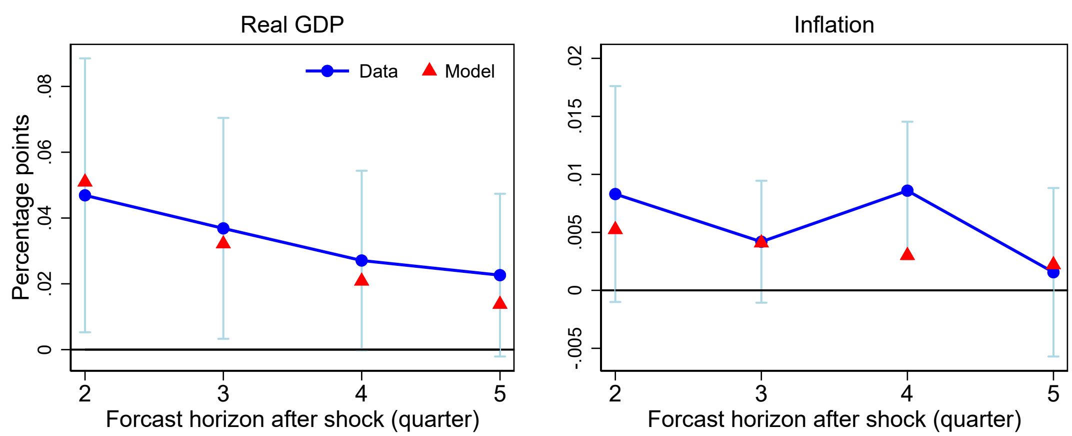
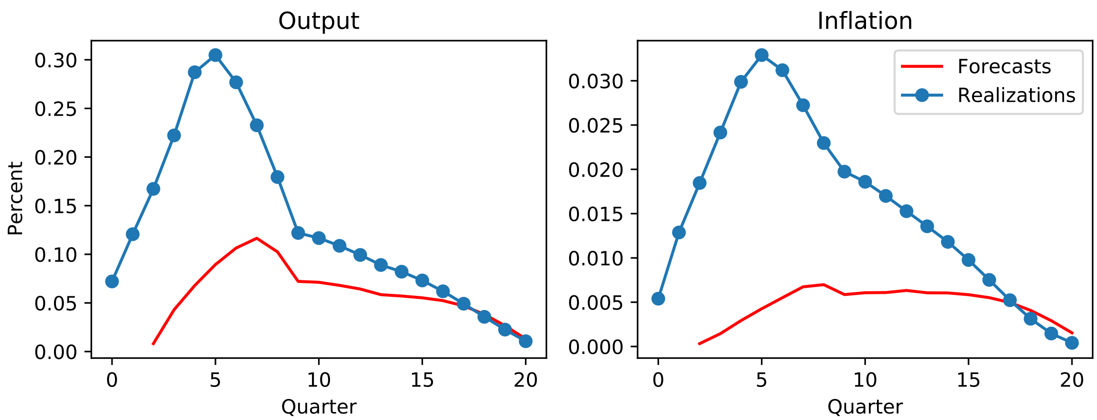


Figure 10: Forward Guidance with Integrated Reasoning

The graph shows the simulated response of output and inflation to a series of short-term nominal interest rates that match the yield curve shift in Figure 4. The blue line represents realizations, the red line reports the two-quarters-ahead expectations. The model parametrization is reported in Table 1.



that the sensitivity of the objective function of the minimization problem for the moment-matching estimation to changes in ρ_π is relatively low. So, the value of ρ_π is weakly identified in the exercise that I consider. In Table 3 in Appendix E.2, I perform robustness checks by fixing ρ_π to different values and estimating the remaining parameters. Importantly, I also check the implications of these alternative estimations for the quantitative assessment described in Section 5, and find negligible differences.

5 Belief Updates and Unconventional Policy Announcements

Having estimated the model to match micro data on macroeconomics expectations, I turn to a quantitative analysis of the transmission of a forward guidance announcement.

Figure 10 reports the model-implied response of output and inflation (blue line). The model predicts a peak effect on output and inflation around quarter 5, with output expanding as much as 30 basis points above steady state and quarter-or-quarter inflation raising to just above 3 basis points.[14]

The red lines represent the response of two-quarter-ahead forecasts. Expectations substantially under-react initially, while forecasts get closer to actual realizations as individuals update their beliefs after observing the policy effects in real time.

[14] Remember that the calibrated shock implies a reduction in the 2-year Treasury yield of around 0.6 basis points.

The role of learning A natural question is how important is the role of expectations update in determining the overall effect of a monetary policy announcement. To answer this question, I compute the decomposition described in Equation (2.22) for the calibrated model.

Figure 11 highlights the contribution of each term of the decomposition for the effects of forward guidance. The dash-dotted, brown lines show the partial-equilibrium effect of the policy, a counterfactual where expectations about future income and inflation are kept constant at the steady state value of zero. The hump-shaped response is due to the promised path of nominal interest rates, which decline fast in the first 4 quarters before moving towards normalization. The dashed, green lines instead reports the sum of partial equilibrium and anticipation effects. This corresponds to the generalized level-k model, where the learning parameters $\sigma_{0,y}$ and $\sigma_{0,\pi}$ are set to zero. Because the estimated value of the average sophistication level λ is low, expectation revision due to structural thinking is very modest. As a result, the stimulative power of the announcement that is due to higher forecasts of future income and inflation is limited.

The blue, solid line, shows the impulse response with the estimated parameters of the integrated reasoning model, accounting for the effects of belief revisions. Since individuals make positive forecast errors, they revise their expectations about future income and inflation. These additional general-equilibrium effects stimulate consumption, since people expect to have higher incomes in the future, and the real rate is lower because of higher expected inflation. They also put upwards pressure on current inflation through the Phillips curve, as price-setters expect higher future inflation rates.

My estimated model implies that the role of expectations update for the transmission of forward guidance is substantial. Figure 12 reports the decomposition expressed in percentage terms. The effects that are due to belief revision account for the majority of the inflation response, and between 25% and 50% of the output response. As a percentage of the cumulative effects on output and inflation, belief revision accounts for 35% and 72%, respectively, of the response. One has to read these numbers with the following caveat in mind. The component labeled as belief revision is the result of allowing level-0 individuals to update their forecasts in real time. People with higher sophistication levels add structural, general-equilibrium thinking to form their expectations. Thus, this component is really determined by the interaction between level-k thinking and learning.

One important implication of my findings is that unconventional policies act partially through the endogenous expectations update generated by the effects of the policy itself. However, one has to keep in mind that, in this exercise, individuals are perfectly aware of the policy intervention. In a

Figure 11: Forward Guidance Effects Decomposition

The graph shows the simulated response of output and inflation to a series of short-term nominal interest rates that match the yield curve shift in Figure 4. The graph reports the decomposition from equation (2.22). The model parametrization is reported in Table 1.

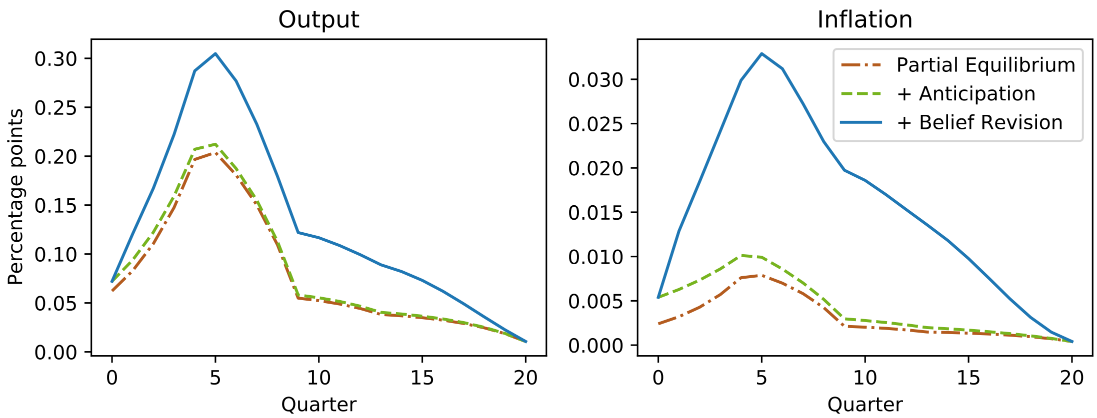


Figure 12: Forward Guidance Effects Decomposition - Percentage

The graph shows the simulated response of output and inflation to a series of short-term nominal interest rates that match the yield curve shift in Figure 4. The graph reports the decomposition from equation (2.22) in percentage terms. The model parametrization is reported in Table 1.

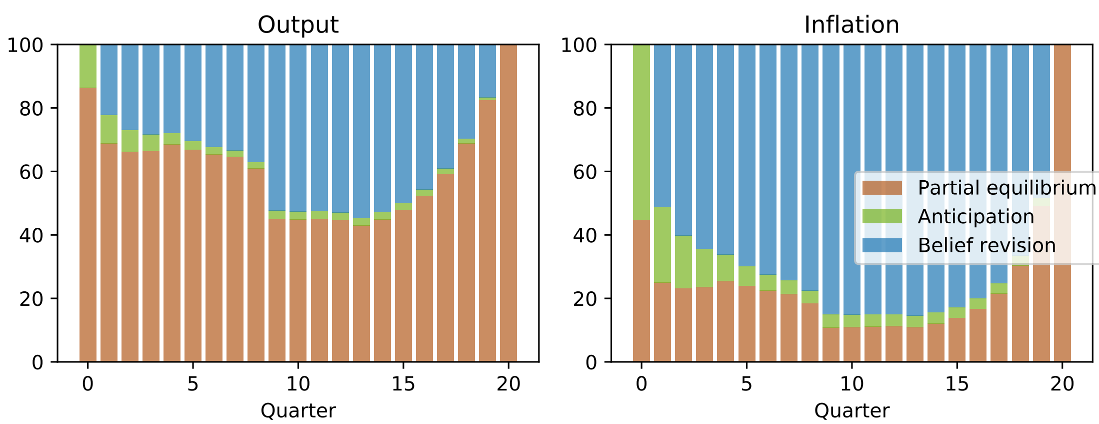
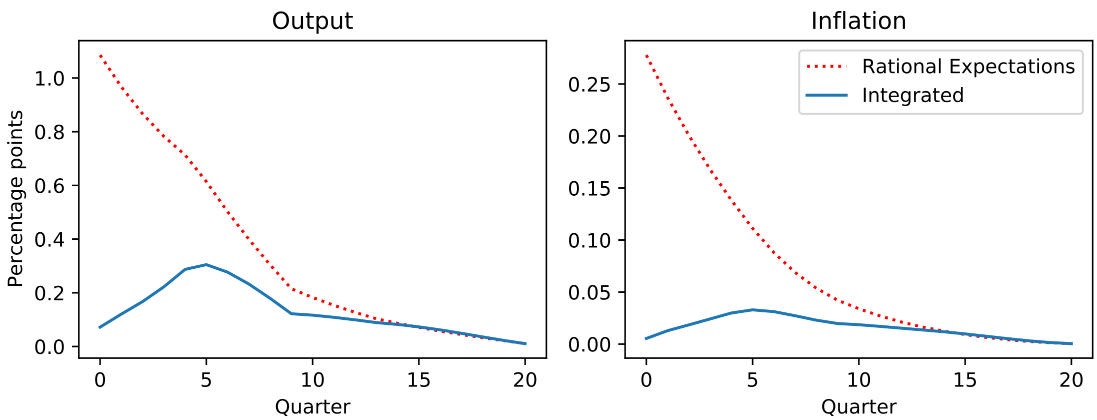


Figure 13: Rational Expectations and Integrated Reasoning

The graph shows the simulated response of output and inflation to a series of short-term nominal interest rates that match the yield curve shift in Figure 4. The model parametrization for the blue line (integrated reasoning) is reported in Table 1. The red line represents rational expectations ($\lambda \to \infty$).



sticky information version of this model, instead, part of the expectations revision would be due to agents who become aware of the policy announcement. So, the extent of expectations revision due to learning about the policy effects, rather than the policy itself, would be quantitatively smaller.

It is worth emphasizing that, while integrated reasoning implies an amplification of real effects relative to generalized level-k thinking, the model still predicts very strong dampening effects compared to rational expectations. Figure 13 compares the rational expectations response to integrated reasoning. This result, together with impulse response functions that are similar to the data, implies that my estimated model is not subject to a "forward guidance puzzle".

Partial-equilibrium effects Farhi and Werning (2019) analyze the interaction between market incompleteness and bounded rationality for generating a dampened impact response of forward guidance announcements relative to rational expectations. The main insight is that market incompleteness tilts the transmission mechanism away from partial-equilibrium and towards general-equilibrium effects. Precisely the latter are under-estimated by boundedly rational individuals.

Partial-equilibrium effects of the proposed policy path also play an important role in estimating the integrated reasoning model parameters, and ultimately determining the quantitative role of expectations revisions. The reason is that, to match the empirically observed forecast revisions, a model with relatively powerful (weak) partial-equilibrium effects requires relatively mild (strong) endogenous revisions of expectations via either structural thinking or belief updates over time.

Table 2: MPCs and Expectations Revision

The table compares the estimated value of λ and the contribution of belief revision to the cumulative effect on output and inflation in the baseline calibration (first column) and with alternative calibrations (second and third columns).

	Baseline	**Alternative calibrations**	
	$MPC = 0.25$	$MPC = 0.20$	$MPC = 0.30$
λ	0.076	0.065	0.087
$\mathcal{D}^y$	35.35%	29.03%	41.48%
$\mathcal{D}^\pi$	72.74%	66.54%	77.60%

To illustrate this point, I repeat the estimation for two different values of the household discount factor β, corresponding to different average marginal propensities to consume. In a high MPC calibration, the partial-equilibrium effects of forward guidance are muted. The first column of Table 2 reports results under the baseline calibration in which $MPC = 0.25$. It shows the estimated value of the average sophistication level λ and the contribution of real-time belief revisions to the cumulative response of output and inflation, defined as:

$$\mathcal{D}^x = 1 - \frac{x^{GLk}}{x^{IR}},$$

for variable x. Here, x^{GLk} is the cumulative response with generalized level-k thinking (absent any belief revision), and x^{IR} is the cumulative response with integrated reasoning.

Columns 2 and 3 show results under two alternative calibrations: low MPC, chosen to match the contemporaneous MPC estimated in Fagereng et al. (2021), equal to 0.2, and high MPC, as in the calibration used by Angeletos et al. (2020), where it is set to 0.3. In the low MPC calibration, forward guidance has powerful partial equilibrium effects. As a result, parameter estimation leads to a lower sophistication level, and a lower overall importance of the belief update channel for the transmission of the shock. The opposite is true for the alternative scenario of high MPC. Overall, the qualitative assessment of the importance of belief revisions in the transmission of this unconventional policy remains valid also with these alternative calibrations.

6 Conclusions

I develop a novel model of bounded rationality called *integrated reasoning* to study the effects of unconventional policies. The model combines forward-looking imperfect reasoning in the form of generalized level-k thinking with slow belief updates due to Bayesian learning. I assess the ability of the model to replicate empirical estimates of the effects of a specific unconventional policy—forward guidance—on macroeconomic expectations. The empirical analysis uncovers three facts about forecast errors and revisions in response to forward guidance announcements. First, individuals' expectations are revised at the time of the announcement. Second, forecasters make systematic mistakes in the direction of under-reaction at the time of the shock and in the subsequent quarters. Third, individuals slowly revise their expectations over time.

My model nests three special cases: rational expectations, pure level-k thinking, and pure learning. I show that none of these three models are able to replicate all the empirical facts at the same time. Rational expectations is rejected by the predictability of forecast errors and revisions. Pure level-k thinking is rejected by slow revision of expectations. Pure learning is rejected by the initial response of expectations to the announcement. Therefore, integrated reasoning more closely reproduces the response of expectations to unconventional policies.

I estimate the new parameters of my expectations formation model by matching the empirical response of forecast revisions to a forward guidance shock. I find a low level of sophistication about general-equilibrium thinking, with parameter estimates implying that 92% of the forecasters in my data do not revise their expectations in response to a forward guidance shock.

I embed my expectations model into a small-scale New Keynesian model to assess the quantitative importance of the belief update channel for the transmission of forward guidance announcements. My model predicts that 35% of the cumulative output response and 72% of the cumulative inflation response are due to expectation updates.

While the results presented here are specific to the case of forward guidance, I believe that they can help shed light on other novel stabilization policies. For many such policies, including Quantitative Easing and the extent of fiscal policy interventions in response to Covid-19, there is little evidence from past episodes that can be used to understand their effects. My model provides an empirically grounded model of expectations about their effects.

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Appendix

A Proofs

Proof of Lemma 1 The proof follows by applying Bayesian updating techniques. Suppose $\mathcal{Y}^0_{i,t} \sim \mathcal{GP}\left(\hat{\mathcal{Y}}^0_{i,t}, \hat{\Sigma}_t\right)$, which is true by assumption for $t = 0$. Let the signal about $\mathcal{Y}_t(z)$ be given by (2.10). Then for any z and z'

$$\begin{bmatrix} \mathcal{Y}^0_{i,t}(z) \\ \mathcal{Y}^0_{i,t}(z') \\ Y_{i,t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \hat{\mathcal{Y}}^0_{i,t}(z) \\ \hat{\mathcal{Y}}^0_{i,t}(z') \\ \hat{\mathcal{Y}}^0_{i,t}(z^t) \end{bmatrix}, \begin{bmatrix} \hat{\Sigma}_t(z,z) & \hat{\Sigma}_t(z,z') & \hat{\Sigma}_t\left(z,z^t\right) \\ \hat{\Sigma}_t(z',z) & \hat{\Sigma}_t(z',z') & \hat{\Sigma}_t\left(z',z^t\right) \\ \hat{\Sigma}_t\left(z^t,z\right) & \hat{\Sigma}_t\left(z^t,z'\right) & \hat{\Sigma}_t\left(z^t,z^t\right) + \sigma^2_v \end{bmatrix}\right).$$

By standard properties of multivariate Gaussian distributions, it follows that the distribution of $\mathcal{Y}^0_{i,t+1} \equiv \mathcal{Y}^0_i | Y_{i,t}$ is given by:

$$\begin{bmatrix} \mathcal{Y}^0_{i,t+1}(z) \\ \mathcal{Y}^0_{i,t+1}(z') \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \hat{\mathcal{Y}}^0_{i,t+1}(z) \\ \hat{\mathcal{Y}}^0_{i,t+1}(z') \end{bmatrix}, \begin{bmatrix} \hat{\Sigma}_{t+1}(z,z) & \hat{\Sigma}_{t+1}(z,z') \\ \hat{\Sigma}_{t+1}(z',z) & \hat{\Sigma}_{t+1}(z',z') \end{bmatrix}\right),$$

where

$$\hat{\mathcal{Y}}^0_{i,t+1}(z) = \hat{\mathcal{Y}}^0_{i,t}(z) + \frac{\hat{\Sigma}_t(z,z^t)}{\hat{\Sigma}_t(z^t,z^t) + \sigma^2_v}\left(Y_{i,t} - \hat{\mathcal{Y}}^0_{i,t}(z^t)\right)$$

and

$$\hat{\Sigma}_{t+1}(z,z') = \hat{\Sigma}_t(z,z') - \frac{\hat{\Sigma}_t(z,z^t)\hat{\Sigma}_t(z',z^t)}{\hat{\Sigma}_t(z^t,z^t) + \sigma^2_v}.$$

□

Proof of Lemma 2 Let $z^{ss} = \{0,0,0,\dots\}$ denote the path of steady-state current and future interest rates. Under the proposed interest rate path, there is a T such that, for all $t \geq T$, $z^t = z^{ss}$.

First, note that $\hat{\Sigma}_t(z^{ss}, z) = 0$ for all t and for any arbitrary path z. The proof is by induction. For $t = 0$, it holds by construction of $\hat{\Sigma}_0$. Suppose now that $\hat{\Sigma}_t(z^{ss}, z) = 0$ for any z. Then it also holds that $\hat{\Sigma}_t\left(z^{ss}, z^t\right) = 0$. From the updating scheme in (2.14), it holds that:

$$\hat{\Sigma}_{t+1}(z^{ss}, z) = \hat{\Sigma}_t(z^{ss}, z) - \frac{\hat{\Sigma}_t(z^{ss},z^t)\hat{\Sigma}_t(z,z^t)}{\hat{\Sigma}_t(z^t,z^t) + \sigma^2_v} = 0.$$

From the update rule for the mean output function in (2.13), this result implies that expectations about output at the steady state path for interest rate z^{ss} are never updated:

$$\hat{\mathcal{Y}}_t^0\left(z^{ss}\right) = \hat{\mathcal{Y}}_0^0\left(z^{ss}\right) = 0,$$

for all t. As a consequence, at each date $t \geq T$, after the policy rate has reverted to steady state, consensus beliefs of level-1 agents is such that output will remain constant and equal to the steady-state level of one:

$$\mathbb{E}_t^0[y_{t+s}] = \hat{\mathcal{Y}}_t^0\left(z^{ss}\right) = 0,$$

for all s. Since $C_t^0(1) = 1$, then level-2 expectations are also such that $\mathbb{E}_t^2[y_{t+s}] = 1$ for all s. Iteratively, this holds for every k, so that consensus beliefs are:

$$\overline{\mathbb{E}}_t[y_{t+s}] = \sum_{k\geq 1} \Phi(k)\mathbb{E}_t^k[y_{t+s}] = 0.$$

From the temporary equilibrium relation in (2.4), the resulting output level is $y_t = 0$. So, beliefs about output under the proposed policy eventually converge to the rational-expectations beliefs.

□

Proof of Lemma 3 The temporary equilibrium at time 0, following equation (2.4), is given by:

$$y_0 = \frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\overline{\mathbb{E}}_0[y_s] - \sigma\beta^{T-1}.$$

where

$$\overline{\mathbb{E}}_0[y_s] = \sum_{k\geq 1}\Phi(k)\mathbb{E}_0^k[y_s]$$

I will now show that

$$\mathbb{E}_0^k[y_s] < \sigma$$

for all k, so that $\overline{\mathbb{E}}_0[y_s] < \sigma$. Under assumption 2, the level-k response $\mathbb{E}_0^k[y_s]$ is a weighted average of the level-$k-1$ response and the output response that would happen if everyone was level-$k-1$, which I denote as δ_s^{k-1}:

$$\mathbb{E}_0^k[y_s] = (1-\gamma_k)\mathbb{E}_0^{k-1}[y_s] + \gamma_k\delta_s^{k-1}$$

where $\gamma_k = \frac{\Phi(k-1)}{\sum_{j=1}^{k-1}\Phi(j)}$, with $\sum_{j=1}^{0}\Phi(j) = 1$. The proof is by induction. First note that $\mathbb{E}_0^1[y_s] = 0 < \sigma$

for all s. Then:

$$\delta_s^1 = \frac{1-\beta}{\beta}\sum_{t=1}^{T-s-1}\beta^t \mathbb{E}_0^1[y_t] + \sigma\beta^{T-s-1}$$
$$= \sigma\beta^{T-s-1} < \sigma$$

for all s. Hence,

$$\mathbb{E}_0^2[y_s] = 0 = \delta_s^1 < \sigma$$

for all s. Now suppose that $\mathbb{E}_0^{k-1}[y_s] < \sigma$ for all s. Then:

$$\delta_s^{k-1} = \frac{1-\beta}{\beta}\sum_{t=1}^{T-s-1}\beta^t \mathbb{E}_0^{k-1}[y_t] + \sigma\beta^{T-s-1}$$
$$< \frac{1-\beta}{\beta}\sum_{t=1}^{T-s-1}\beta^t \sigma + \sigma\beta^{T-s-1} = \sigma$$

for all s. Hence,

$$\mathbb{E}_0^k[y_s] = (1-\gamma_k)\mathbb{E}_0^{k-1}[y_s] + \gamma_k\delta_s^{k-1} < \sigma.$$

Finally, suppose that $\Phi(k^*) = 1$ for some k^*. Then $\gamma_k = 1$ for all k, and the model is a standard level-k thinking model as in Farhi and Werning (2019). The limiting case of $k^* \to \infty$ is such that $y_0 = y_0^*$. $\square$

Proof of Proposition 1 In Appendix B, I show that, in the linear version of the expectations model, beliefs along the cognitive hierarchy are updated according to:

$$\theta_{k,t} = \theta_{k-1,t} + f_k(k-1)(T(\theta_{k-1,t}) - \theta_{k-1,t}). \tag{A.1}$$

This fact will be used for proving the three results of Proposition 1.

1. The time-0 revision is equal to $fr_{0,s} = \overline{\mathbb{E}}_0[y_s]$. When $\lambda \to 0$, all individuals in the economy are level-0, so $\overline{\mathbb{E}}_0[y_s] = \mathbb{E}_0^0[y_s]$. By assumption on initial conditions, expressed in equation (2.15), the beliefs of level-0 people at time-0 are such that $\mathbb{E}_0^0[y_s] = 0$ for all $s \geq 1$.

   To see that $\frac{\partial fr_{0,s}}{\partial \lambda} \geq 0$, I will show that $\mathbb{E}_0^k[y_s] \geq \mathbb{E}_0^{k-1}[y_s]$. Once this result is established, it follows that increasing λ puts a higher weight on more sophisticated individuals, so the average forecast about income is higher. The proof is by induction. Note that for every k,

$$\mathbb{E}_0^k[y_s] \geq \mathbb{E}_0^{k-1}[y_s] \iff \left(\theta_0^k\right)' z^s \geq \left(\theta_0^{k-1}\right)' z^s \iff \left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right)' z^s \geq 0,$$

where the last step follows from the update rule in (A.1). For $k = 1$, this condition is verified since:

$$T(\theta_0^0)'z^s = \left(A\theta_0^0 + b\right)' z^s = b'z^s = \sigma\beta^{T-s} \geq 0,$$

where the last equality follows from the fact that z^s is a vector with a -1 entry in position s, and all other elements are equal to zero.

Now suppose that $\left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right)' z^s \geq 0$. The vector z^s selects the $T - s$ column of the matrix that pre-multiplies it, and changes sign to all of the elements. This implies that the s-th element $\left[T(\theta_0^{k-1}) - \theta_0^{k-1}\right]_s$ is negative. Note that:

$$\begin{aligned}
T(\theta_0^k) - \theta_0^k &= (A - I)\theta_0^k + b \\
&= (A - I)\theta_0^{k-1} + b + f_k(k-1)(A - I)\left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right) \\
&= \underbrace{(I + f_k(k-1)(A - I))}_{D_k}\left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right).
\end{aligned}$$

Then $\left(T(\theta_0^k) - \theta_0^k\right)' z^s = \left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right)' D_k' z^s$. Since D is a lower-triangular matrix with $1 - f_k(k-1)$ as diagonal elements, and $f_k(k-1)(1-\beta)\beta^{t+s-2}$ as (s,t)-element of the lower triangular section of the matrix, all elements are positive. Hence, all elements of $D_k' z^s$ are negative. By assumption, all elements of $\left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right)$ are negative, so $\left(T(\theta_0^k) - \theta_0^k\right)' z^s \geq 0$.

2. The forecast error at time 0 is equal to:

$$\begin{aligned}
fe_{0,0} &= y_0 - \overline{\mathbb{E}}_0[y_0] \\
&= \left(T(\overline{\theta}_0) - \overline{\theta}_0\right)' z^s.
\end{aligned}$$

The previous part of the proposition established that $\left(T(\theta_0^{k-1}) - \theta_0^{k-1}\right)' z^s \geq 0$ for all k. Then, it follows that this also holds for the average, so $\left(T(\overline{\theta}_0) - \overline{\theta}_0\right)' z^s \geq 0$.

3. The forecast revision for level-k thinkers at time t is equal to:

$$\begin{aligned}\mathbb{E}_t^k[y_{t+s}] - \mathbb{E}_{t-1}^k[y_{t+s}] &= (\theta_{k,t} - \theta_{k,t-1})' z^{t+s} \\ &= (\theta_{k-1,t} - \theta_{k-1,t-1})' (I + f_k(k-1)(A-I))' z^{t+s} \\ &= (\theta_{k-1,t} - \theta_{k-1,t-1})' D_k' z^{t+s}\end{aligned}$$

By the same argument in the proof of point 1, if the forecast revisions are different from zero for level-$k-1$ thinkers, they are also different from zero for level-k. Furthermore, if level-$k-1$ thinkers make a positive (negative) forecast error, level-k thinkers also make a positive (negative) forecast error.

The forecast revision for level-0 individuals is given by equation (2.13). Their forecast error $y_t - \mathcal{Y}_t^0(z^t)$ is non-zero since they do not have rational expectations. So, provided that the Kalman gain is different from zero, their forecast revision is non-zero. Taking the limit for $\sigma_0^2 \to 0$, the Kalman gain is zero, so level-0 individuals and people with higher sophistication levels do not update their forecasts. □

B The Linear Model of Expectations

I will describe the linear model fo expectations in the setup described in Section 4. The temporary equilibrium is described by:

$$\begin{aligned}y_t &= \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} \beta^s \hat{\mathbb{E}}_t[y_{t+s}] - \sigma \sum_{s=0}^{\infty} \beta^s \left(i_{t+s} - \hat{\mathbb{E}}_t[\pi_{t+s+1}]\right) \\ \pi_t &= \kappa_w \sum_{s=0}^{\infty} \left(\hat{\beta}\epsilon\right)^s \hat{\mathbb{E}}_t[y_{t+s}] + \frac{1-\epsilon}{\epsilon} \sum_{s=1}^{\infty} \left(\hat{\beta}\epsilon\right)^s \hat{\mathbb{E}}_t[\pi_{t+s}],\end{aligned}$$

for some expectations operator $\hat{\mathbb{E}}$.

Level-0 agents form expectations according to:

$$\mathbb{E}_{i,t}^0[y_{t+s}] = \left(\theta_{i,t}^{y,0}\right)' z^{t+s},$$

and

$$\mathbb{E}_{i,t}^0[\pi_{t+s}] = \left(\theta_{i,t}^{\pi,0}\right)' z^{t+s},$$

I will first show that, if the expectations of level-0 people are linear in the state vector z^t, also level-k people beliefs can be represented as linear beliefs. Level-1 individuals expect output and inflation to be determined according to:

$$
\begin{aligned}
y_t &= \frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\left(\theta_t^{y,0}\right)' z^{t+s} - \sigma\sum_{s=0}^{\infty}\beta^s\left(i_{t+s} - \left(\theta_t^{\pi,0}\right)' z^{t+s+1}\right) \\
\pi_t &= \kappa_w\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{y,0}\right)' z^{t+s} + \frac{1-\epsilon}{\epsilon}\sum_{s=1}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{\pi,0}\right)' z^{t+s}.
\end{aligned}
$$

Denoting by Γ the matrix such that $z^{t+s} = \Gamma^s z^t$ and α the vector such that $i_{t+s} = \alpha' z^{t+s}$, these equations can be written as:

$$
\begin{aligned}
y_t &= \left(\frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\left(\theta_t^{y,0}\right)'\Gamma^s - \sigma\sum_{s=0}^{\infty}\beta^s\left(\alpha\Gamma^s - \left(\theta_t^{\pi,0}\right)'\Gamma^{s+1}\right)\right) z^t = T^y\left(\theta_t^{y,0},\theta_t^{\pi,0}\right)' z^t \\
\pi_t &= \left(\kappa_w\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{y,0}\right)'\Gamma^s + \frac{1-\epsilon}{\epsilon}\sum_{s=1}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{\pi,0}\right)'\Gamma^s\right) z^t = T^\pi\left(\theta_t^{y,0},\theta_t^{\pi,0}\right)' z^t.
\end{aligned}
$$

Here, T^y and T^π represent the temporary equilibrium mapping for output and inflation. These equations define the perceived law of motions for level-1 people as:

$$
\mathbb{E}_t^1[y_{t+s}] = T^y\left(\theta_t^{y,0},\theta_t^{\pi,0}\right)' z^{t+s} = \left(\theta_t^{y,1}\right)' z^{t+s}
$$

and

$$
\mathbb{E}_t^1[\pi_{t+s}] = T^\pi\left(\theta_t^{y,0},\theta_t^{\pi,0}\right)' z^{t+s} = \left(\theta_t^{\pi,1}\right)' z^{t+s}.
$$

Now by induction, suppose that $\mathbb{E}_t^{k-1}[y_{t+s}] = \left(\theta_t^{y,k-1}\right)' z^{t+s}$ and $\mathbb{E}_t^{k-1}[\pi_{t+s}] = \left(\theta_t^{\pi,k-1}\right)' z^{t+s}$. Then level-$k$ people form beliefs according to:

$$
\begin{aligned}
y_t &= \sum_{j=0}^{k-1} f_k(j)\left[\frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\left(\theta_t^{y,j}\right)' z^{t+s} - \sigma\sum_{s=0}^{\infty}\beta^s\left(i_{t+s} - \left(\theta_t^{\pi,j}\right)' z^{t+s+1}\right)\right] \\
\pi_t &= \sum_{j=0}^{k-1} f_k(j)\left[\kappa_w\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{y,j}\right)' z^{t+s} + \frac{1-\epsilon}{\epsilon}\sum_{s=1}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{\pi,j}\right)' z^{t+s}\right].
\end{aligned}
$$

Note that under Assumption 2 we have that $f_k(j) = (1 - f_{k-1}(j))$ for $j < k-1$. It follows that

$$y_t = f_k(k-1)\left[\frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\left(\theta_t^{k-1,j}\right)' z^{t+s} - \sigma\sum_{s=0}^{\infty}\beta^s\left(i_{t+s} - \left(\theta_t^{\pi,k-1}\right)' z^{t+s+1}\right)\right] +$$
$$(1 - f_k(k-1))\sum_{j=0}^{k-2} f_{k-1}(j)\left[\frac{1-\beta}{\beta}\sum_{s=1}^{\infty}\beta^s\left(\theta_t^{y,j}\right)' z^{t+s} - \sigma\sum_{s=0}^{\infty}\beta^s\left(i_{t+s} - \left(\theta_t^{\pi,j}\right)' z^{t+s+1}\right)\right]$$
$$\pi_t = f_k(k-1)\left[\kappa_w\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{y,k-1}\right)' z^{t+s} + \frac{1-\epsilon}{\epsilon}\sum_{s=1}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{\pi,k-1}\right)' z^{t+s}\right] +$$
$$(1 - f_k(k-1))\sum_{j=0}^{k-2} f_{k-1}(j)\left[\kappa_w\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{y,j}\right)' z^{t+s} + \frac{1-\epsilon}{\epsilon}\sum_{s=1}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left(\theta_t^{\pi,j}\right)' z^{t+s}\right],$$

which implies

$$\mathbb{E}_t^k[y_{t+s}] = \left(f_k(k-1)T^y\left(\theta_t^{y,k-1}, \theta_t^{\pi,k-1}\right) + (1 - f_k(k-1))\,\theta_t^{y,k-1}\right)' z^{t+s} = \left(\theta_t^{y,k}\right)' z^{t+s}$$
$$\mathbb{E}_t^k[\pi_{t+s}] = \left(f_k(k-1)T^\pi\left(\theta_t^{y,k-1}, \theta_t^{\pi,k-1}\right) + (1 - f_k(k-1))\,\theta_t^{\pi,k-1}\right)' z^{t+s} = \left(\theta_t^{\pi,k}\right)' z^{t+s}.$$

Equilibrium in this model can be written as a function of average beliefs $\overline{\theta}_t^y$ and $\overline{\theta}_t^\pi$ as a mapping:

$$y_t = T^y\left(\overline{\theta}_t^y, \overline{\theta}_t^\pi\right)' z^t \tag{B.1}$$
$$\pi_t = T^\pi\left(\overline{\theta}_t^y, \overline{\theta}_t^\pi\right)' z^t \tag{B.2}$$

Level-0 expectations are updated as follows. The time-0 prior for variable x is given by equation (2.20), reported here:

$$\tilde{\theta}_{i,0}^{x,0} \sim \mathcal{N}\left(\theta_{i,0}^{x,0}, \Sigma_0^{\theta,x}\right)$$

Individuals observe signals about income $y_{i,t}$ and inflation $\pi_{i,t}$:

$$y_{i,t} = y_t + \eta_{i,t}^y \qquad \pi_{i,t} = \pi_t + \eta_{i,t}^\pi \tag{B.3}$$

I assume that the signal is linear in the state vector, so that:

$$\eta_{i,t}^y = \left(\nu_{i,t}^y\right)' z^t \qquad \eta_{i,t}^\pi = \left(\nu_{i,t}^\pi\right)' z^t, \tag{B.4}$$

where $\nu_{i,t}^y$ and $\nu_{i,t}^\pi$ are white noise multivariate processes with covariance matrix equal to $\sigma_{\nu,y}^2 I$ and

$\sigma^2_{\nu,\pi} I$ respectively.

Combining the prior (2.20), the signals (B.3) and noise structure (B.4), and the equilibrium mappings in equations (B.1) and (B.2), the belief update process for the individual mean and variance of the belief distribution is described by:

$$\left(\Sigma^{\theta,x}_{t+1}\right)^{-1} = \left(\Sigma^{\theta,x}_{t}\right)^{-1} + \frac{z^t z^{t\prime}}{\sigma^2_{\nu,x} z^{t\prime} z^t}$$

and

$$\theta^{x,0}_{i,t+1} = \theta^{x,0}_{i,t} + \Sigma^{\theta,x}_{t+1} \frac{z^t z^{t\prime}}{\sigma^2_{\nu,x} z^{t\prime} z^t} \left(T^x \left(\overline{\theta}^y_t, \overline{\theta}^\pi_t \right) - \theta^{x,0}_{i,t} \right) + \Sigma^{\theta,x}_{t+1} \frac{z^t}{\sigma^2_{\nu,x} z^{t\prime} z^t} \nu^x_{i,t}$$

Consensus beliefs for level-0 individuals $\theta^{x,0}_t$, evolves according to:

$$\theta^{x,0}_{t+1} = \theta^{x,0}_{t} + \Sigma^{\theta,x}_{t+1} \frac{z^t z^{t\prime}}{\sigma^2_{\nu,x} z^{t\prime} z^t} \left(T^x \left(\overline{\theta}^y_t, \overline{\theta}^\pi_t \right) - \theta^{x,0}_{t} \right)$$

It is therefore immediate to see that this model is a special case of the belief update process described by (2.13) and (2.14), where the mean functions for output and inflation are given by

$$\mathcal{Y}^0_{i,t}(z) = \left(\theta^{y,0}_{i,t}\right)' z$$

and

$$\Pi^0_{i,t}(z) = \left(\theta^{\pi,0}_{i,t}\right)' z,$$

and the covariance functions are given by

$$\Sigma^y_t(z, z') = \sum_j \sum_\ell \left[\Sigma^{\theta,y}_t\right]_{j,\ell} z_j z'_\ell$$

and

$$\Sigma^\pi_t(z, z') = \sum_j \sum_\ell \left[\Sigma^{\theta,\pi}_t\right]_{j,\ell} z_j z'_\ell.$$

C Details on the Sticky Wage Model

Households The linearized Euler equation and budget constraint of the household are given by:

$$c_{i,t} = c_{i,t+1} - \sigma \left(i_t - \mathbb{E}_{i,t}[\pi_{t+1}] \right)$$

and

$$\sum_{s=0}^{\infty} \beta^s \left(c_{i,t+s} - \left(\mathbb{E}_{i,t}[w_{t+s}] - \mathbb{E}_{i,t}[p_{t+s}]\right) - \mathbb{E}_{i,t}[n_{t+s}]\right) = \tilde{b}_{i,t},$$

where $\tilde{b}_{i,t} = B_{i,t}/Y$.

Combining them, we have:

$$c_{i,t} = \tilde{b}_{i,t} + (1-\beta) \sum_{s=0}^{\infty} \beta^s \left(\mathbb{E}_{i,t}[w_{t+s}] - \mathbb{E}_{i,t}[p_{t+s}] + \mathbb{E}_{i,t}[n_{t+s}]\right) - \sigma\beta \sum_{s=0}^{\infty} \beta^s \left(\mathbb{E}_{i,t}[i_{t+s} - \pi_{t+s+1}]\right)$$

Using the fact that $P_t = W_t$ and $Y_t = N_t$, this reduces to:

$$c_{i,t} = \tilde{b}_{i,t} + (1-\beta) \sum_{s=0}^{\infty} \beta^s \mathbb{E}_{i,t}[y_{t+s}] - \sigma\beta \sum_{s=0}^{\infty} \beta^s \left(\mathbb{E}_{i,t}[i_{t+s} - \pi_{t+s+1}]\right)$$

The corresponding temporary equilibrium is:

$$y_t = \frac{1-\beta}{\beta} \sum_{s=1}^{\infty} \beta^s \mathbb{E}_{i,t}[y_{t+s}] - \sigma \sum_{s=0}^{\infty} \beta^s \left(\mathbb{E}_{i,t}[i_{t+s} - \pi_{t+s+1}]\right)$$

which gives equation (4.2).

Unions The derivation of the union problem is identical to Bianchi-Vimercati et al. (2021). The problem of a union that gets to reset its wage is

$$\max_{w_{u,t}, \{\tilde{n}_{u,t+s}\}} \mathbb{E}_{u,t} \sum_{s\geq 0} \left(\hat{\beta}\epsilon\right)^s \left\{ u'\left(C_{t+s}\right) \frac{1-\tau^n_{t+s}}{1+\tau^c_{t+s}} \frac{w_{u,t}\tilde{n}_{u,t+s}}{P_{t+s}} - v'\left(L_{t+s}\right) \tilde{n}_{u,t+s} \right\}$$

subject to the constraint

$$\tilde{n}_{u,t+s} = \left(\frac{w_{u,t}}{W_{t+s}}\right)^{-\theta} N_{t+s}.$$

Because every union represents an infinitesimal number of workers in each household, the union does not directly affect aggregate consumption, C_t, hours worked by the household, L_t, the composite labor input, N_t, aggregate wages, W_t, and prices, P_t. As discussed in the main text, we assume that the union has rational expectations with respect to the exogenous variables, but is boundedly rational with respect to future endogenous variables.

The optimal reset wage $W_{u,t}^*$ solves the following first order condition:

$$\mathbb{E}_{u,t}\sum_{s\geq 0}\left(\hat{\beta}\epsilon\right)^s\left\{-\left(\theta-1\right)u'\left(C_{t+s}\right)\frac{1-\tau_{t+s}^n}{1+\tau_{t+s}^c}\frac{W_{u,t}^*\left(\frac{W_{u,t}^*}{W_{t+s}}\right)^{-\theta}N_{t+s}}{P_{t+s}}+\theta v'\left(L_{t+s}\right)\left(\frac{W_{u,t}^*}{W_{t+s}}\right)^{-\theta}N_{t+s}\right\}=0$$

which can be equivalently written as follows:

$$\frac{W_{u,t}^*}{P_t}=\frac{\theta}{\theta-1}\frac{\mathbb{E}_{u,t}\sum_{s\geq 0}\left(\hat{\beta}\epsilon\right)^s\left(\frac{P_{t+s}}{P_t}\right)^{\theta}\left(\frac{W_{t+s}}{P_{t+s}}\right)^{\theta}N_{t+s}v'(N_{t+s})}{\mathbb{E}_{u,t}\sum_{s\geq 0}\left(\hat{\beta}\epsilon\right)^s\left(\frac{P_{t+s}}{P_t}\right)^{\theta-1}\left(\frac{W_{t+s}}{P_{t+s}}\right)^{\theta}N_{t+s}u'(C_{t+s})}.$$

Aggregate wage is given by

$$W_t=\int_0^1\left[\lambda W_{t-1}^{1-\theta}+\left(1-\lambda\right)\left(W_{u,t}^*\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}du,$$

and total hours worked by the household are

$$L_t=\mu_t N_t,$$

where

$$\mu_t=\int_0^1\left(\frac{w_{u,t}}{W_t}\right)^{-\theta}du=\lambda\mu_{t-1}\left(\frac{W_{t-1}}{W_t}\right)^{-\theta}+\left(1-\lambda\right)\int_0^1\left(\frac{W_{u,t}^*}{W_t}\right)^{-\theta}du$$

with $\mu_{-1}=1$.

Log-linearizing the reset wage condition, one gets:

$$w_{u,t}^*-p_t=\left(1-\hat{\beta}\lambda\right)\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^s\left\{\varphi\mathbb{E}_{u,t}[n_{t+s}]+\sigma^{-1}\mathbb{E}_{u,t}[c_{t+s}]\right\}+\sum_{s=1}^{\infty}\left(\hat{\beta}\epsilon\right)^s\mathbb{E}_{u,t}[p_{t+s}]-\sum_{s=0}^{\infty}\left(\hat{\beta}\epsilon\right)^{s+1}\mathbb{E}_{u,t}[p_{t+s}],$$

or equivalently

$$w_{u,t}^*-w_t=\left(1-\hat{\beta}\epsilon\right)\sum_{s\geq 0}\left(\hat{\beta}\epsilon\right)^s\left\{\varphi\mathbb{E}_{u,t}[n_{t+s}]+\sigma^{-1}\mathbb{E}_{u,t}[c_{t+s}]\right\}+\sum_{s\geq 1}\left(\hat{\beta}\epsilon\right)^s\mathbb{E}_{u,t}[\pi_{t+s}^w]$$

where $\pi_t^w=w_t-w_{t-1}$ is wage inflation, and where I have used the fact that $p_t=w_t$.

The log-linearized aggregate wage condition is

$$w_t=\epsilon w_{t-1}+\left(1-\epsilon\right)w_t^*\quad\Longleftrightarrow\quad w_t^*-w_t=\frac{\epsilon}{1-\epsilon}\pi_t^w,$$

where $w_t^* = \int_0^1 w_{u,t}^* du$. We can replace this equation in the reset wage condition after aggregating to obtain an expression for wage inflation:

$$\begin{aligned}
\pi_t^w &= \frac{(1-\epsilon)\left(1-\hat{\beta}\epsilon\right)}{\epsilon} \sum_{s\geq 0} \left(\hat{\beta}\epsilon\right)^s \left\{\varphi\overline{\mathbb{E}}_t[n_{t+s}] + \sigma^{-1}\overline{\mathbb{E}}_t[c_{t+s}]\right\} + \frac{1-\epsilon}{\epsilon} \sum_{s\geq 1} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[\pi_{t+s}^w] \\
&= \frac{(1-\epsilon)\left(1-\hat{\beta}\epsilon\right)}{\epsilon} \left(\varphi + \sigma^{-1}\right) \sum_{s\geq 0} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[y_{t+s}] + \frac{1-\epsilon}{\epsilon} \sum_{s\geq 1} \left(\hat{\beta}\epsilon\right)^s \overline{\mathbb{E}}_t[\pi_{t+s}^w]
\end{aligned}$$

Since $\pi_t = \pi_t^w$, this also gives equation (4.3).

D Forward Guidance Shocks

The forward guidance shock series is taken from Jarociński (2021). The paper uses high-frequency changes in financial asset prices around FOMC announcements to derive high-frequency shocks. The identification strategy is based on mutual independence and non-Gaussianity of the shocks, following Lanne and Lütkepohl (2010) and Lanne et al. (2017). The market response to FOMC announcements is studied using the following empirical model:

$$y_t = C' u_t \qquad u_{n,t} \sim i.i.d.\ \mathcal{T}(\nu),$$

where y_t is a vector that collects the first federal funds future, the 2-year and 10-year Treasury yields, and the S&P500 blue chip stock index. The vector u_t represents the underlying structural shocks, that are assumed to be independent and to follow a t-Student distribution with parameter ν. Jarociński (2021) shows how this distribution provides a substantially better fit for high-frequency price changes around FOMC announcements.

The first three financial prices collected in y_t serve the purpose of identifying the target, path, and large scale asset purchase (LSAP) factors of monetary policy (in line with Swanson (2021)). A strand of literature has focused on understanding the role of the information component in monetary policy announcements, and how to clean pure policy shocks from its effects. The inclusion of the S&P500 is meant to deal with this issue, since stock prices reflect expectations about the future course of the economy.

The model is estimated using maximum likelihood, which allows to recover ν and the matrix C. The structural shock series are then built as $u_t = C^{-1} y_t$. One of the structural factor represents standard monetary policy shocks, that affects federal fund future prices but has no impact on the yield curve and implies a negative shock to the S&P500. One factor resembles QE policies in that it affects the very long end of the yield curve while inducing an increase in stock prices. The remaining two forward guidance shock series have similar impacts the yield curve (no effects on short-term rates, increase in 2- and 10-year yields), but opposite effects on the stock market reaction. One is thus interpreted as a commitment to a future path for interest rates, with standard contractionary effects following an increase in interest rates. This shock is associated with an impact reduction in stock prices, and it is the forward guidance shock series that I will use for my analysis.

E Additional Empirical Results

E.1 Individual-level forecast errors and revisions

Figure 14: Response of Individual Forecast Errors and Revisions to a Forward Guidance Shock

The figure shows the response of individual-level forecast errors (top panel) and forecast revisions (bottom panel), with 68% and 90% confidence interval. Estimates come from the following local projection regressions:

$$x_{t+h+4} - \mathbb{E}_{i,t+h}[x_{t+h+4}] = \alpha_{i,h} + \beta_h^x u_{FG,t} + \gamma_h' W_t + \varepsilon_{i,t+h},$$

for forecast errors, and

$$\mathbb{E}_{i,t+h}[x_{t+h+4}] - \mathbb{E}_{i,t+h-1}[x_{t+h+4}] = \alpha_{i,h} + \beta_h^x u_{FG,t} + \gamma_h' W_t + \varepsilon_{i,t+h},$$

for forecast revisions, for $h = 0, \dots, 10$.

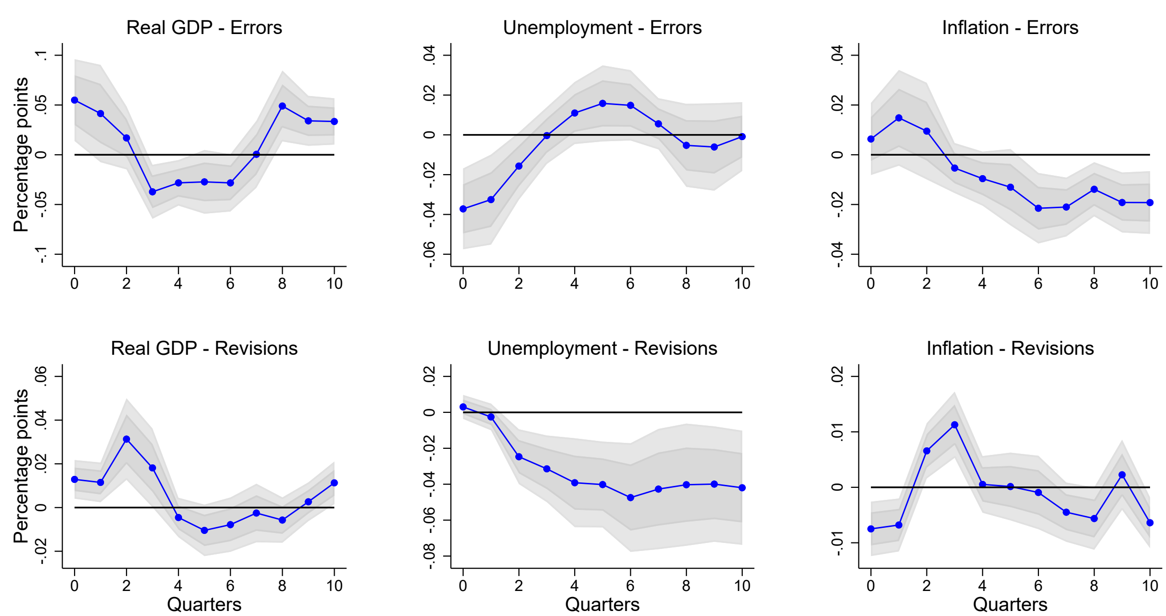


Figure 15: Forecast Revisions for Quarter 5 After Announcement

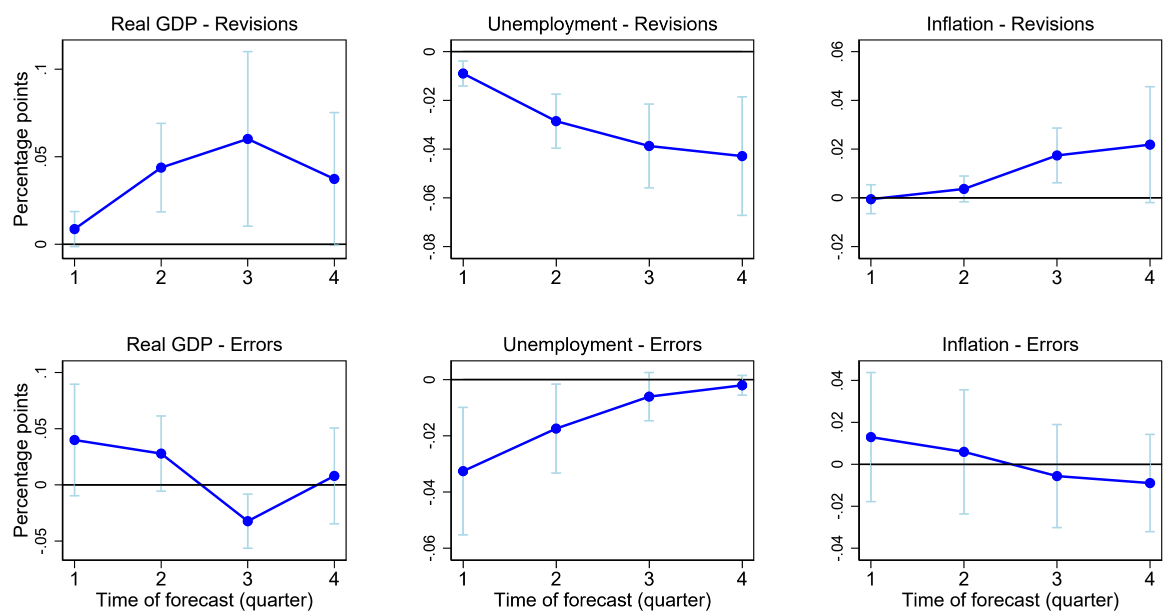
The figure reports errors and revisions about quarter 5 after the shock, as a function of the time of the forecast, with 90% confidence intervals. Estimates come from the following local projection regressions:

$$\mathbb{E}_{i,t+h}[x_{t+5}] - \mathbb{E}_{i,t-1}[x_{t+5}] = \alpha_{i,h} + \beta_h^x u_{FG,t} + \gamma_h' W_t + \varepsilon_{i,t+h},$$

for forecast revisions, and

$$x_{t+5} - \mathbb{E}_{i,t+h}[x_{t+5}] = \alpha_{i,h} + \beta_h^x u_{FG,t} + \gamma_h' W_t + \varepsilon_{i,t+h},$$

for forecast errors, for $h = 1, 2, 3, 4$.



E.2 Calibration of ρ_π

Table 3 performs a robustness exercise for the estimation step. I fix ρ_π equal to three different values (0.01, 0.25 and 0.5), and re-estimate the three remaining parameters from the Baysian learning model, $\sigma_{0,y}^2$, ρ_y, and $\sigma_{0,\pi}^2$. The average sophistication level λ and signal noisiness $\sigma_{v,y}^2$ $\sigma_{v,\pi}^2$, are estimated separately from these parameters by using moments that in my model are not affected by $\sigma_{0,y}^2$, ρ_y, and $\sigma_{0,\pi}^2$, as described in the paper. The Table reports these new estimates with corresponding standard deviations. The last column shows results from the full estimation for reference.

The table also reports the results from the quantitative exercise in Section 5. The last two rows show the contribution of belief revision to the response of output and inflation, defined as:

$$\mathcal{D}^x = 1 - \frac{x^{GLk}}{x^{IR}},$$

where x^{GLk} is the response of variable x in the generalized level-k thinking model, and x^{IR} is the response with integrated reasoning.

Table 3: Robustness with Calibrated Values for ρ_π

The table reports a robustness exercises with calibrated values of ρ_π. It shows the re-estimated values of λ, $\sigma^2_{0,y}$, ρ_y, and $\sigma^2_{0,\pi}$, and the contribution of belief revision to the cumulative effects on output and inflation.

	Robustness			**Baseline**
	$\rho_\pi = 0.010$	$\rho_\pi = 0.250$	$\rho_\pi = 0.500$	$\rho_\pi = 0.026$
λ	0.076	0.076	0.076	0.076
	(0.034)	(0.034)	(0.035)	(0.035)
$\sigma^2_{0,y}$	0.066	0.066	0.067	0.066
	(0.202)	(0.203)	(0.206)	(0.202)
$\sigma^2_{0,\pi}$	0.025	0.015	0.008	0.024
	(0.018)	(0.010)	(0.005)	(0.376)
ρ_y	0.594	0.594	0.592	0.594
	(0.687)	(0.687)	(0.686)	(0.687)
$\mathcal{D}^y$	35.36	35.22	35.04	35.35
$\mathcal{D}^\pi$	72.75	72.60	72.39	72.74