Cambridge University

Doctoral Thesis

On the Simulation of Boson Stars in General Relativity

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Contents

Chapter 1

Introduction to Differential Geometry and General Relativity

1.1 Introduction

general gr shit? mention einstein derivation (hilbert derivation a potentially earlier but vacuum), karl schwarzshild (ironicaly meaning black shield), then low mass limit, photon deflection and mercury perihelion around sun. mention cosmology, gravitational waves, more black holes, minkowski. more compact objects.

1.2 Introduction to Compact Objects and Boson Stars

The first non-trivial solution to Einstein's equation found was that of the spherically symmetric, static and asymptotically flat vacuum spacetime by Karl Schwarzschild in 1915. The solution was designed to be used outside a spherically symmetric, non-spinning, body of mass; however it turned out to provide use in describing black holes. This metric was then modified by Tolman, Oppenheimer and Volkov in 1939 to describe the non-vacuum case of a constant density neutron star. This turned out to give an unphysical estimate of $0.7M_\odot$ for the upper limit of neutron star mass due to the equation of state.

The study of compact exotic objects can be traced back to John Wheeler who investigated Geons in 1955 for their potential similarity to elementary particles. Geons are gravito-electromagnetic objects with the name arising from "gravitational electromagnetic entity". In 1968 David Kaup published [] describing what he called "Klein-Gordon Geons", nowadays referred to as boson stars. Importantly, boson stars are a localised complex Klein-Gordon configuration, with the real counterparts being unstable. Many variants such as (Spin 1) proca stars [], electromagnetically charged boson stars and many others have been studied.

Interest in boson stars remains for many reasons. Given the recent discovery of the higgs boson, we know that scalar fields exist in nature and any gravitational wave signals created by compact objects could theoretically be detected with modern gravitational wave interferometers. Secondly, boson stars are a good candidate for dark matter haloes. Boson stars are also useful as a proxy to other compact objects in general relativity; there is a lot of freedom in the construction of different types of boson star and they can be fine tuned to model dense neutron stars for one example. The advantage this would have over simulating a real fluid is that the Klein Gordon equation is linear in the principal part meaning smooth data must always remain smooth; thus avoiding shocks and conserving particle numbers relatively well with less sophisticated numerical schemes.

On a slightly different topic, collisions of boson stars could be a natural method to produce scalar hair around black holes which will be discussed later in more detail.

1.2.1 Conventions

Throughout this thesis physical quantities will be expressed as a dimensionless ratio of the Planck length, time and mass L_{pl}, T_{pl} and M_{pl} respectively; consequently the constants c, G and $\hbar$ evaluate numerically to 1. As an example, Newtons equation of gravity would be recast like

$$F = \frac{GMm}{r^2} \rightarrow \left(\frac{F}{F_{pl}}\right) = \frac{\left(\frac{M}{M_{pl}}\right)\left(\frac{m}{M_{pl}}\right)}{\left(\frac{r}{L_{pl}}\right)^2} \tag{1.2.1}$$

where $F_{pl} = M_{pl}L_{pl}T_{pl}^{-2}$ is the Planck force. $c = G = \hbar = 1$, unless stated otherwise. The metric signature will always be $(-,+,+,+)$.

Tensor fields will be denoted using bold font for index free notation and normal font for the components. The dot product between two vector fields will be written interchangeably as $\boldsymbol{A}\cdot\boldsymbol{B} \leftrightarrow A_\mu B^\mu$ for readability. Additionally, ∇_μ denotes the covariant derivative and ∂_μ is the partial derivative, both with respect to coordinate x^μ.

When considering the ADM decomposition, as in [REF SECT], objects can be associated with both the 3+1 dimensional manifold $\mathcal{M}$ or the 3 dimensional hypersurface Σ. To differentiate here, standard Roman letters such as R represent the object belonging to $\mathcal{M}$ and calligraphic letters such as $\mathcal{R}$ correspond to the projected object belonging to Σ. [MAYBE JUST REMOVE THIS BIT AND MAKE IT OBVIOUS IN THE ACTUAL 3+1 SECTION].

Finally, unless stated otherwise, Greek indices such as $\{\alpha, \beta, ..., \mu, \nu, ...\}$ label four dimensional tensor components whereas late Latin indices such as $\{i, j, k, ...\}$ label three dimensional tensor components and early Latin indices such as $\{a, b, ...\}$ label two dimensional ones. When the index range is unspecified and unimportant Greek letters will also be used.

make explicit th inner product, dot product, outer product (otimes) and wedge product (for forms, antisymm)

1.3 Differential Geometry

1.3.1 Introduction to Geometry and Manifolds

Everyones first encounter with geometry will cover Pythagoras' theorem; arguably the most famous and useful equation in existance. Pythagoras' equation relates the sidelengths of a right angled triangle, it says that $s^2 = x^2 + y^2$ for a triangle with height y, width x and hypotenuse length s. This can be shown very simply by looking at Fig. 1.1. The area of the partially rotated square is obviously s^2, but we can also calculate it from the the area of the larger square A_{sq} and subtracting four times the area of one of the triangles A_{tr}. Clearly $A_{sq} = (x + y)^2$ and $A_{tr} = \frac{1}{2}xy$, therefore

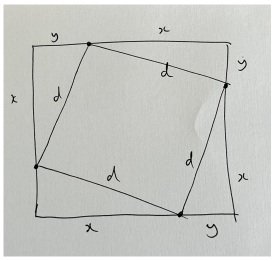

Figure 1.1: Diagram for proof of Pythagoras' theorem.

$$s^2 = (x + y)^2 - 2xy = x^2 + y^2 \tag{1.3.1}$$

and we have proved Pythagoras' theorem. Using an infinitessimally small triangle, we can write $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ and this can be trivially extended to arbitrary dimensions like

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + ... \ . \tag{1.3.2}$$

The infinitessimal form of Pythagoras' theorem is very powerful as it lets us calculate the length of a generic curve by approximating the curve as a collection of infinitessimally small straight lines with length $\mathrm{d}s$. So far we have assumed that space is flat meaning Eq. (1.3.2) is true for all points in space, this is an assumpion we will have to drop if we want to study the curved spaces arising in strong gravity. In the next sections we will explore the generalisation of Pythagoras' equation to curved spaces and use it to measure curve lengths answell as volumes and areas. [REWRITE THIS WHEN THE SECTIONS ARE COMPLETE AND REF]

Differential Geometry (DG) is the extension of calculus, linear algebra and multilinear algebra to general geometries. Einstein's Theory of Relativity is written using the language of DG as it is the natural way to deal with curves, tensor calculus and differential tensor equations in curved spaces. For a basic introduction to DG, we should start with a manifold $\mathcal{M}$ which is an N dimensional space that locally looks like $\mathbb{R}^N$, N dimensional Euclidean space. This is important as at a point $p \in \mathcal{M}$ we can find infinitesimally close neighbouring points $p + \delta p \in \mathcal{M}$. In the following sections we will explore curves, functions, tensors and calculus on manifolds using DG.

1.3.2 Functions, Curves and Tensors on Manifolds

A real scalar function f over $\mathcal{M}$ maps any point $p \in \mathcal{M}$ to a real number, this is denoted as $f : p \to \mathbb{R}$. An important example of a set of scalar functions is the coordinate system ϕ, $\phi : p \to \mathbb{R}^N$, this is normally written x^μ where $\mu \in \{0, 1, ..., N-1\}$ is an index labelling the coordinate. The map ϕ is called a chart, and unlike Euclidean space one chart may not be enough to cover the entire manifold; in this case a set of compatible charts should be smoothly joined, collectively known as an atlas.

Now that functions have been discussed, the next simplest object we can discuss is a curve, or path, through $\mathcal{M}$. A curve Γ is a set of smoothly connected points $p(\lambda) \in \mathcal{M}$ that smoothly depend on an input parameter $\lambda \in [\lambda_0, \lambda_1]$. This can be expressed in terms of coordinates as $x^\mu(\lambda)$ where $\phi : p(\lambda) \to x^\mu(\lambda)$. Differentiating a function f along Γ with respect to λ gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x^\mu(\lambda)) = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\partial f(x^\mu)}{\partial x^\nu} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \partial_\nu f, \tag{1.3.3}$$

where $\partial_\nu = \partial / \partial x^\nu$ and the Einstein summation convention was invoked, summing over all values of ν. Equation (1.3.3) was derived independantly of the choice of f, therefore we can generally write

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \partial_\nu. \tag{1.3.4}$$

The operator $\mathrm{d}/\mathrm{d}\lambda$ can act on any function f and return a new function $\tilde{f}$ over $\mathcal{M}$, formally this is written as $\mathrm{d}/\mathrm{d}\lambda(f) = \tilde{f}$ where $\tilde{f} : p \in \mathcal{M} \to \mathbb{R}$. We can also think of $\mathrm{d}/\mathrm{d}\lambda$ as a vector $\boldsymbol{X}$ with components $X^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ and basis vectors $\boldsymbol{e}_\mu := \partial_\mu$ taken from Eq. (1.3.4). The vector $\boldsymbol{X}$ can be written as $\boldsymbol{X} = X^\mu \boldsymbol{e}_\mu$ and can act on a general function f over $\mathcal{M}$ as $\boldsymbol{X}(f) = X^\mu \boldsymbol{e}_\mu(f) = X^\mu \partial_\mu f$. Considering the set of all possible curves through a points $p \in \mathcal{M}$, the tangent vector components $\mathrm{d}x^\mu/\mathrm{d}\lambda$ span an N dimensional space with basis $\boldsymbol{e}_\mu = \partial_\mu$; this space is called the tangent space and is denoted as $\mathcal{T}_p(\mathcal{M})$.

The next object to discuss is the co-vector which is defined as a map from vectors to real numbers; this is not the same as the dot product, and doesnt need one to exist [MAKE THIS BETTER]. Similarly to vectors, a co-vector $\boldsymbol{\omega}$ can be expressed as a sum of components ω_μ and basis co-vectors $\boldsymbol{\theta}^\mu$ like $\boldsymbol{\omega} = \omega_\mu \boldsymbol{\theta}^\mu$. Contrary to vectors, co-vector components have downstairs indeces and the basis has upstairs indeces; this choice improves the readability of tensor equations when working with components. The power of co-vectors is that they map a vector to a real number like $\boldsymbol{\omega} : \boldsymbol{X} \to \mathbb{R}$ or $\boldsymbol{\omega}(\boldsymbol{X}) \to \mathbb{R}$. Vectors are equally able to map co-vectors to real numbers like $\boldsymbol{X} : \boldsymbol{\omega} \to \mathbb{R}$. Co-vectors are defined such that $\boldsymbol{\theta}^\mu : \boldsymbol{e}_\nu = \delta^\mu_\nu$ where δ^μ_ν are the components of the Kroneka delta equating to zero unless $\mu = \nu$ in which case they equal unity. The general operation of a co-vector $\boldsymbol{\omega}$ on a vector $\boldsymbol{X}$ is

$$\boldsymbol{\omega} : \boldsymbol{X} = \omega_\mu X^\nu \boldsymbol{\theta}^\mu : \boldsymbol{e}_\nu = \omega_\mu X^\nu \delta^\mu_\nu = \omega_\mu X^\mu \in \mathbb{R}. \tag{1.3.5}$$

This map is linear and identical under reversing the order of operation; $\boldsymbol{\omega} : \boldsymbol{X} = \boldsymbol{X} : \boldsymbol{\omega}$. Similarly to vectors, the set of all possible co-vectors at a point $p \in \mathcal{M}$ span an N-dimensional space called the co-tangent space, written as $\mathcal{T}^*_p(\mathcal{M})$.

Now that linear maps have been covered, we can generalise to multilinear maps and tensor fields. Consider a tensor field $\boldsymbol{T}$, this can be expressed in component form like

$$\boldsymbol{T} = T^{\alpha\beta,\dots}_{\mu\nu,\dots} \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta \otimes ... \otimes \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu \otimes ... \tag{1.3.6}$$

for an arbitrary number of [MAKE THIS MAKE SNENSE]. A tensor field with m co-vector bases and n vector bases is called an (m, n) tensor field, therefore vectors, co-vectors and scalars are $(1, 0)$, $(0, 1)$ and $(0, 0)$ vectors respectively. Tensors can act as multilinear maps between tensor fields to other tensor fields. We have already seen how a vector and co-vector can map each other to a scalar, extending this we can use an $(0, 2)$ tensor field $\boldsymbol{T} = T_{\mu\nu} \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu$ to map two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ to a scalar like

$$\boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}) = T_{\mu\nu} X^\alpha Y^\beta (\boldsymbol{\theta}^\mu : \boldsymbol{e}_\alpha)(\boldsymbol{\theta}^\nu : \boldsymbol{e}_\beta) = T_{\mu\nu} X^\mu Y^\nu. \tag{1.3.7}$$

As mentioned already, the multilinear map can output generic tensors, for example consider

$$\boldsymbol{T}(\boldsymbol{X},\star) = T_{\mu\nu}X^{\alpha}(\boldsymbol{\theta}^{\mu} : \boldsymbol{e}_{\alpha})\boldsymbol{\theta}^{\nu} = T_{\mu\nu}X^{\mu}\boldsymbol{\theta}^{\nu}, \tag{1.3.8}$$

which uses the $(0,2)$ tensor $\boldsymbol{T}$ to map the vector $\boldsymbol{X}$ to a co-vector $\boldsymbol{W}$ with components $W_{\mu} = T_{\mu\nu}X^{\mu}$. [DOES THE STAR MAKE SENSE?] One final example of a mapping is from a single tensor to a lower rank tensor, this is called contraction. To illustrate this, let's take a $(1,3)$ tensor $\boldsymbol{Z} = Z^{\alpha}{}_{\mu\nu\rho}\boldsymbol{e}_{\alpha}\otimes\boldsymbol{\theta}^{\mu}\otimes\boldsymbol{\theta}^{\nu}\otimes\boldsymbol{\theta}^{\rho}$. We can choose to use the basis vector $\boldsymbol{e}_{\alpha}$ to act on any of the three co-vector bases, choosing $\boldsymbol{\theta}^{\mu}$ this is

$$Z^{\alpha}{}_{\mu\nu\rho}(\boldsymbol{e}_{\alpha} : \boldsymbol{\theta}^{\mu})\boldsymbol{\theta}^{\nu}\otimes\boldsymbol{\theta}^{\rho} = Z^{\mu}{}_{\mu\nu\rho}\boldsymbol{\theta}^{\nu}\otimes\boldsymbol{\theta}^{\rho} = \tilde{Z}_{\nu\rho}\boldsymbol{\theta}^{\nu}\otimes\boldsymbol{\theta}^{\rho} \tag{1.3.9}$$

where $\tilde{Z}_{\nu\rho} = Z^{\mu}{}_{\mu\nu\rho}$.

MAYBE TIDY THE END OF THIS UP NICELY

CHECK WETHER USED A POINT (SINGLE TANGENT SPACE) OR A FIELD HERE. MAYBE USE A POINT HERE AND USE FIELDS ON THE NEXT SECTION?

1.3.3 The Inner Product and the Metric

To introduce the notion of length on a tanget plane $\mathcal{T}_p(\mathcal{M})$ at a point p the metric tensor $\boldsymbol{g}$ is introduced. The metric tensor is

$$g_{\mu\nu} = \boldsymbol{e}_{\mu}\cdot\boldsymbol{e}_{\nu} \tag{1.3.10}$$

where $\boldsymbol{e}_{\mu}\cdot\boldsymbol{e}_{\nu}$ represents the inner product on $\mathcal{T}_p(\mathcal{M})$; clearly the metric is symmetric by construction. The inner product can be thought of as a multilinear map,

$$\boldsymbol{g} : (\boldsymbol{X},\boldsymbol{Y}) \to \mathbb{R}, \tag{1.3.11}$$

$$\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y}) = g_{\mu\nu}Y^{\mu}X^{\nu}, \tag{1.3.12}$$

where $\boldsymbol{X}\in\mathcal{T}_p(\mathcal{M})$, $\boldsymbol{Y}\in\mathcal{T}_p(\mathcal{M})$ and $\boldsymbol{g}\in\mathcal{T}^*_p(\mathcal{M})\otimes\mathcal{T}^*_p(\mathcal{M})$. The inner product can also be represented by a second map

$$\boldsymbol{X} : \boldsymbol{Y} \to \mathbb{R}, \tag{1.3.13}$$

$$\boldsymbol{X}\cdot\boldsymbol{Y} = X^{\mu}Y^{\nu}\boldsymbol{e}_{\mu}\cdot\boldsymbol{e}_{\nu} = X^{\mu}Y^{\nu}g_{\mu\nu}, \tag{1.3.14}$$

which is a new mapping at this point, before we needed a vector and a co-vector to map to $\mathbb{R}$. Another way to think of the inner product is that the metric maps a vector $\boldsymbol{X}$ to a co-vector $\boldsymbol{\Xi}$ such that $\boldsymbol{X} : \boldsymbol{\Xi} = X^{\mu}\Xi_{\mu} = g_{\mu\nu}X^{\mu}X^{\nu}$. In component form $\boldsymbol{\Xi}$ is

$$\Xi_{\mu} = g_{\mu\nu}X^{\nu}; \tag{1.3.15}$$

this use of the metric to map a vector to it's corresponding co-vector (and vica versa) is extremely useful. Without loss of information we can write $\Xi_{\nu} = X_{\nu}$ to make it obvious that $X_{\nu} = X^{\mu}g_{\mu\nu}$ and this convention will be used from now on.

The metric also assigns an inner product and a length measure on the co-tangent plane $\mathcal{T}^*_p(\mathcal{M})$ but instead using the inverse components $g^{\mu\nu} = (g^{-1})_{\mu\nu}$,

$$g^{\mu\nu} = \boldsymbol{\theta}^{\mu}\cdot\boldsymbol{\theta}^{\nu}. \tag{1.3.16}$$

Similarly to before the inner product of two co-vectors $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$ is

$$\boldsymbol{\omega}\cdot\boldsymbol{Y} = \omega_{\mu}\sigma_{\nu}\boldsymbol{\theta}^{\mu}\cdot\boldsymbol{\theta}^{\nu} = \omega_{\mu}\sigma_{\nu}g^{\mu\nu} = \omega_{\mu}\sigma^{\mu}. \tag{1.3.17}$$

The reason that $g^{\mu\nu}$ must be the inverse matrix to $g_{\mu\nu}$ (a property not seen in most tensors) is that we want

$$\delta^{\nu}_{\mu} = \boldsymbol{e}_{\mu} : \boldsymbol{\theta}^{\nu}, \tag{1.3.18}$$
$$= (g_{\alpha\mu}\boldsymbol{\theta}^{\alpha}) : (g^{\beta\nu}\boldsymbol{e}_{\beta}), \tag{1.3.19}$$
$$= g_{\alpha\mu}g^{\beta\nu}\boldsymbol{\theta}^{\alpha} : \boldsymbol{e}_{\beta}, \tag{1.3.20}$$
$$= g_{\alpha\mu}g^{\beta\nu}\delta^{\alpha}_{\beta}, \tag{1.3.21}$$
$$\delta^{\nu}_{\mu} = g_{\alpha\mu}g^{\alpha\nu}, \tag{1.3.22}$$

where the final line is only true if $g^{\mu\nu}$ are the components of the inverse metric.

Not only has the metric provided us with an inner product and a length on tangent planes and cotangent planes, but it has also given a mapping between the two. The metric can raise and lower indeces on general tensors such as

$$T^{\mu\nu\ldots}{}_{\alpha\beta} = g^{\mu\rho}g_{\beta\sigma}T_{\rho}{}^{\nu}{}_{\alpha}{}^{\sigma\ldots.}, \tag{1.3.23}$$

for example.

FIX THE ABUSE OF NOTATION IN THE INVERSE METRIC PROOF BIT

1.3.4 Maps Between Manifolds

In section we will be interested in the maps between two manifolds M and N. This has many uses such as pushing and pulling tensors between manifolds, allowing us to calculate a Lie derivative and finding the metric (or any tensor) on an embedded surface; this very importantly allowed us to perform the 3+1 decomposition [REF] on a spacetime.

Let's start by defining a smooth map $\Phi : M \to N$ between manifolds on some coordinate patch. Labelling coordinates $x^{\mu} \in M$ and $y^{\mu} \in N$ the map $\Phi : x^{\mu} \to y^{\mu}$ the map gives $y^{\mu} = \Phi^{\mu}(x^{\nu})$, or equivalently $y^{\mu}(x^{\nu})$. Scalar functions must also map trivially $f_N(y^{\mu}(x^{\nu})) = f_M(x^{\mu})$ where $f_N \in N$ and $f_M \in M$, thus we will no longer identify which manifold a function is on. The map Φ allows us to push the vector $\boldsymbol{X} \in \mathcal{T}_p(M)$ to $\Phi_*\boldsymbol{X} \in \mathcal{T}_q(N)$, where $q = \Phi(p)$, in a way such that it's action on a function f is the same in either manifold.

$$\boldsymbol{X}(f)\Big|_p = \Phi_*\boldsymbol{X}(f)\Big|_q, \tag{1.3.24}$$
$$X^{\mu}\frac{\partial f}{\partial x^{\mu}} = (\Phi_*X)^{\nu}\frac{\partial f}{\partial y^{\nu}}, \tag{1.3.25}$$
$$\left(X^{\mu}\frac{\partial y^{\nu}}{\partial x^{\mu}}\right)\frac{\partial f}{\partial y^{\nu}} = (\Phi_*X)^{\nu}\frac{\partial f}{\partial y^{\nu}} \tag{1.3.26}$$

and hence the components of the push-foreward $\Phi_*\boldsymbol{X}$ can be read off,

$$(\Phi_*X)^{\mu} = \frac{\partial y^{\mu}}{\partial x^{\nu}}X^{\nu}. \tag{1.3.27}$$

Given a co-vector field $\boldsymbol{\omega} \in \mathcal{T}^*_p(N)$ we can pull the field back from $\mathcal{T}^*_p(M) \leftarrow \mathcal{T}^*_q(N)$, denoted $\Phi^*\boldsymbol{\omega}$, by demanding that $\Phi^*\boldsymbol{\omega}(\boldsymbol{X})\big|_p = \boldsymbol{\omega}(\Phi_*\boldsymbol{X})\big|_q$. Evaluating this gives

$$\Phi^*\boldsymbol{\omega}(\boldsymbol{X})\Big|_p = \boldsymbol{\omega}(\Phi_*\boldsymbol{X})\Big|_q, \tag{1.3.28}$$
$$(\Phi^*\omega)_{\mu}X^{\mu} = \omega_{\nu}(\Phi_*X)^{\nu}, \tag{1.3.29}$$
$$(\Phi^*\omega)_{\mu}X^{\mu} = \omega_{\nu}\frac{\partial y^{\nu}}{\partial x^{\mu}}X^{\mu}, \tag{1.3.30}$$

and the components of pull-back $\Phi_*\boldsymbol{\omega}$ can be read off,

$$(\Phi^*\omega)_\mu = \omega_\nu \frac{\partial y^\nu}{\partial x^\mu}. \tag{1.3.31}$$

Considering an $(0,2)$ tensor $\boldsymbol{T} \in N$, the pullback $\Phi^*\boldsymbol{T} \in M$ follows simply from demanding that $\boldsymbol{T}(\Phi_*\boldsymbol{X}, \Phi_*\boldsymbol{Y}) = \Phi^*\boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y})$ where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are vector fields on M. The components of the pull-back of $\boldsymbol{T}$ are therefore

$$(\Phi^*T)_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} T_{\rho\sigma}, \tag{1.3.32}$$

and the pull-back of a generic $(0,q)$ tensor and the push-foreward of a generic $(p,0)$ tensor can be found similarly.

So far we have only discussed the mapping $\Phi : M \to N$, which is required to have a well behaved $\partial y^\nu/\partial x^\mu$. In the case $y^\nu(x^\mu)$ has a smooth inverse $x^\nu(y^\mu)$, and therefore a well behaved $\partial x^\nu/\partial y^\mu$, we have a smooth inverse map $\Phi^{-1} : N \to M$ and the mapping Φ is a diffeomorphism; this requires M and N have the same number of dimensions. When Φ is a diffeomorphism we can then also define the pull-back of $(p,0)$ tensors from N to M along with the push-foreward of $(0,q)$ tensors from M to N, therefore we can push-pull generic tensors between M and N. Two common examples of diffeomorphisms are coordinate changes and translations.

As mentioned, maps between manifolds can be used to determine the metric on an embedded surface. This requires us to consider an m dimensional manifold M with metric $g_{\mu\nu}$ and coordinates x^μ aswell as an embedded n dimensional surface N where $n < m$. We can treat N as a separate n dimensional manifold with metric $h_{\mu\nu}$. As before, we can define a map $\Phi : x^\mu \to y^\mu$

[MENTION SOMEWHERE HERE THAT MAPPINGS ARE USED FOR LIE DERIVS TOO]

ARE WE ALLOWED TO TALK AOBUT THE METRIC HERE?

CHECK WETHER WE SHOULD STICK TO MAPS OF SAME DIMENSION OR NOT? I GUESS SAME DIMENSINO IS REQUIRED FOR REVERSIBILITY AND HENCE DIFFEOMORPHISM. WE SHOULD ADD PROJECTIONS HERE I THINK THEN USE IT IN VOLUMES ON MANIFOLDS TO PROJECT THE METRIC.

CHECK THE P AND Q ASSIGNMENT IN THE EQS

1.3.5 Lie Derivatives

We have now discussed the necesary formalism to define the Lie derivative. The Lie derivative at a point is the rate of change of a tensor field with respect to a pull-back from a diffeomorphism Φ mapping infinitessimally close points $p, q \in \mathcal{M}$ like $\Phi : p = x^\mu \to q = x^\mu + \epsilon\xi^\mu$ for some vector field ξ. The simplest example is the Lie derivative of a scalar field ϕ, denoted $\mathcal{L}_\xi\phi$ with respect to vector field $\boldsymbol{\xi}$, is

$$\mathcal{L}_\xi\phi = \lim_{\epsilon\to 0}\left[\frac{\Phi^*\phi|_q - \phi|_p}{\epsilon}\right], \tag{1.3.33}$$

$$= \lim_{\epsilon\to 0}\left[\frac{\phi(x^\mu + \epsilon\xi^\mu) - \phi(x^\mu)}{\epsilon}\right], \tag{1.3.34}$$

$$= \xi^\mu\partial_\mu\phi, \tag{1.3.35}$$

which reduces to the directional derivative of ϕ with respect to $\boldsymbol{\xi}$. Next let's calculate the Lie derivative of a vector field $\boldsymbol{X}$ with respect to vector field $\boldsymbol{\xi}$. Starting with the same definition as Eq. (1.3.36), and using $y^\mu = x^\mu + \epsilon\xi^\mu$, the Lie derivative of $\boldsymbol{X}$ is

$$(\mathcal{L}_\xi X)^\mu = \lim_{\epsilon\to 0}\left[\frac{(\Phi^* X|_q)^\mu - X|_p{}^\mu}{\epsilon}\right], \tag{1.3.36}$$

$$= \lim_{\epsilon\to 0}\left[\frac{\frac{\partial x^\mu}{\partial y^\nu}X^\nu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho)}{\epsilon}\right], \tag{1.3.37}$$

$$= \lim_{\epsilon\to 0}\left[\frac{(\delta^\mu_\nu - \epsilon\partial_\nu\xi^\mu)X^\nu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho)}{\epsilon}\right], \tag{1.3.38}$$

$$= \lim_{\epsilon\to 0}\left[\frac{-\epsilon\partial_\nu\xi^\mu X^\nu(x^\rho + \epsilon\xi^\rho) + X^\mu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho)}{\epsilon}\right], \tag{1.3.39}$$

$$= \lim_{\epsilon\to 0}\left[\frac{-\epsilon\partial_\nu\xi^\mu X^\nu(x^\rho) + X^\mu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho) + \mathcal{O}(\epsilon^2)}{\epsilon}\right], \tag{1.3.40}$$

$$= \xi^\nu\partial_\nu X^\mu - X^\nu\partial_\nu\xi^\mu. \tag{1.3.41}$$

The Lie derivative for co-vectors and tensors can be derived in the same way, but can be quickly derived from the Liebnitz rule as follows. Define a scalar field ψ, vector field $\boldsymbol{X}$ and co-vector field $\boldsymbol{\omega}$, where $\psi = X^\mu\omega_\mu$, then it follows that

$$\mathcal{L}_\xi\psi = \xi^\mu\partial_\mu\psi = X^\nu\xi^\mu\partial_\mu\omega_\nu + \omega_\nu\xi^\mu\partial_\mu X^\nu, \tag{1.3.42}$$

$$= \mathcal{L}_\xi(X^\mu\omega_\mu), \tag{1.3.43}$$

$$= \omega_\mu(\mathcal{L}_\xi X)^\mu + X^\mu(\mathcal{L}_\xi\omega)_\mu, \tag{1.3.44}$$

$$X^\mu(\mathcal{L}_\xi\omega)_\mu = X^\nu\xi^\mu\partial_\mu\omega_\nu + \omega_\nu\xi^\mu\partial_\mu X^\nu - \omega_\mu(\mathcal{L}_\xi X)^\mu, \tag{1.3.45}$$

$$(\mathcal{L}_\xi\omega)_\mu = \xi^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu\xi^\nu. \tag{1.3.46}$$

As it turns out, any Lie derivative can replace all partial derivatives with covariant derivatives ∇ due to the connection/Christoffel symbols cancelling.

A special example of a Lie derivative is of the metric tensor, $\boldsymbol{g}$, giving

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu\xi^\rho + g_{\mu\rho}\partial_\nu\xi^\rho, \tag{1.3.47}$$

$$= \xi^\rho\underbrace{\nabla_\rho g_{\mu\nu}}_{=0} + g_{\rho\nu}\nabla_\mu\xi^\rho + g_{\mu\rho}\nabla_\nu\xi^\rho, \tag{1.3.48}$$

$$\tag{1.3.49}$$

where $\nabla_\rho g_{\mu\nu} = 0$ is assumed from metric compatibility in [REF]. In the case the Lie derivative vanishes we get Killing's equation

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0 \tag{1.3.50}$$

and a vector field $\boldsymbol{\xi}$ satisfying Killing's equation is called a Killing vector. [MENTION IS MENAS CONSERVATINO OF ENERGY-MOM HERE?]

special case lei metric

LIE DERIV OF TENSOR DENSITIES?

TALK ABT DERIV REQUIRING LINEARITY, ASSOCIATVITY AND LEIBNITZ

JUSTIFY THE CHRISTOFFEL SYMBOLS CANCELLING?

WE HAVE NOT YET MET HE COVARIANT DERIVATIVE, MAYBE FIND A BETTER WAY OF PUTTING THAT BIT HERE? OR MAYBE JUST SAY, WE CAN REPLACE THE DERIVS WITH COV DERIVS THAT WILL BE INTRODUCED LATER