Cambridge University

Doctoral Thesis

On the Simulation of Boson Stars in General Relativity

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Chapter 1

Introduction to Differential Geometry and General Relativity

1.1 Introduction

1.1.1 Introduction to General Relativity

general gr shit? mention einstein derivation (hilbert derivation a potentially earlier but vacuum), karl schwarzshild (ironicaly meaning black shield), then low mass limit, photon deflection and mercury perihelion around sun. mention cosmology, gravitational waves, more black holes, minkowski. more compact objects.

1.1.2 Introduction to Compact Objects and Boson Stars

The first non-trivial solution to Einstein's equation found was that of the spherically symmetric, static and asymptotically flat vacuum spacetime by Karl Schwarzschild in 1915. The solution was designed to be used outside a spherically symmetric, non-spinning, body of mass; however it turned out to provide use in describing black holes. This metric was then modified by Tolman, Oppenheimer and Volkov in 1939 to describe the non-vacuum case of a constant density neutron star. This turned out to give an unphysical estimate of $0.7M_\odot$ for the upper limit of neutron star mass due to the equation of state.

The study of compact exotic objects can be traced back to John Wheeler who investigated Geons in 1955 for their potential similarity to elementary particles. Geons are gravito-electromagnetic objects with the name arising from "gravitational electromagnetic entity". In 1968 David Kaup published [] describing what he called "Klein-Gordon Geons", nowadays referred to as boson stars. Importantly, boson stars are a localised complex Klein-Gordon configuration, with the real counterparts being unstable. Many variants such as (Spin 1) proca stars [], electromagnetically charged boson stars and many others have been studied.

Interest in boson stars remains for many reasons. Given the recent discovery of the higgs boson, we know that scalar fields exist in nature and any gravitational wave signals created by compact objects could theoretically be detected with modern gravitational wave interferometers. Secondly, boson stars are a good candidate for dark matter haloes. Boson stars are also useful as a proxy to other compact objects in general relativity; there is a lot of freedom in the construction of different types of boson star and they can be fine tuned to model dense neutron stars for one example. The advantage this would have over simulating a real fluid is that the Klein Gordon equation is linear in the principal part meaning smooth data must always remain smooth; thus avoiding shocks and conserving particle numbers relatively well with less sophisticated numerical schemes.

On a slightly different topic, collisions of boson stars could be a natural method to produce scalar hair around black holes which will be discussed later in more detail.

1.1.3 Conventions

Throughout this thesis physical quantities will be expressed as a dimensionless ratio of the Planck length, time and mass L_{pl}, T_{pl} and M_{pl} respectively; consequently the constants c, G and $\hbar$ evaluate numerically to 1. As an example, Newtons equation of gravity would be recast like

$$F = \frac{GMm}{r^2} \rightarrow \left(\frac{F}{F_{pl}}\right) = \frac{\left(\frac{M}{M_{pl}}\right)\left(\frac{m}{M_{pl}}\right)}{\left(\frac{r}{L_{pl}}\right)^2} \tag{1.1.1}$$

where $F_{pl} = M_{pl}L_{pl}T_{pl}^{-2}$ is the Planck force. $c = G = \hbar = 1$, unless stated otherwise. The metric signature will always be $(-,+,+,+)$.

Tensors and tensor fields will be denoted using bold font for index free notation and normal font for the components. The dot product between two vectors or vector fields will be written interchangeably as $\boldsymbol{A} \cdot \boldsymbol{B} \leftrightarrow A_\mu B^\mu$ for readability. Additionally, ∇_μ denotes the covariant derivative and ∂_μ is the partial derivative, both with respect to coordinate x^μ.

When considering the ADM decomposition, as in [REF SECT], objects can be associated with both the 3+1 dimensional manifold $\mathcal{M}$ or the 3 dimensional hypersurface Σ. To differentiate here, standard Roman letters such as R represent the object belonging to $\mathcal{M}$ and calligraphic letters such as $\mathcal{R}$ correspond to the projected object belonging to Σ. [MAYBE JUST REMOVE THIS BIT AND MAKE IT OBVIOUS IN THE ACTUAL 3+1 SECTION].

Finally, unless stated otherwise, Greek indices such as $\{\alpha, \beta, ..., \mu, \nu, ...\}$ label four dimensional tensor components whereas late Latin indices such as $\{i, j, k, ...\}$ label three dimensional tensor components and early Latin indices such as $\{a, b, ...\}$ label two dimensional ones. When the index range is unspecified and unimportant Greek letters will also be used.

EINSTEIN SUMMATION CONV

make explicit th inner product, dot product, outer product (otimes) and wedge product (for forms, antisymm)

Conventions (from q)

Throughout this work the metric has sign $\{-, +, +, +\}$ and physical quantities will be expressed as a dimensionless ratio of the Planck length L_{pl}, time T_{pl} and mass M_{pl} unless stated otherwise; for example Newtons equation of gravity would be written as

$$F = \frac{GMm}{r^2} \quad \rightarrow \quad \left(\frac{F}{F_{pl}}\right) = \frac{\left(\frac{M}{M_{pl}}\right)\left(\frac{m}{M_{pl}}\right)}{\left(\frac{r}{L_{pl}}\right)^2}, \tag{1.1.2}$$

where $F_{pl} = M_{pl} L_{pl} T_{pl}^{-2}$ is the Planck force. Consequently c, G and $\hbar$ take the numerical value of 1. Additionally, tensor fields will be denoted using bold font for index free notation and normal font for the components. The dot product between two vector fields will be written interchangeably as $\boldsymbol{A} \cdot \boldsymbol{B} \leftrightarrow A^\mu B_\mu$ for readability. Additionally, ∇_μ denotes the covariant derivative and ∂_μ is the partial derivative, both with respect to coordinate x^μ. Finally, unless stated otherwise, Greek indices such as $\{\alpha, \beta, ..., \mu, \nu, ...\}$ label four dimensional tensor components whereas late Latin indices such as $\{i, j, k, ...\}$ label three dimensional tensor components and early Latin indices such as $\{a, b, ...\}$ label two dimensional ones.

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1.2 Differential Geometry

1.2.1 Introduction to Geometry and Manifolds

Everyones first encounter with geometry will cover Pythagoras' theorem; arguably the most famous and useful equation in existance. Pythagoras' equation relates the sidelengths of a right angled triangle, it says that $s^2 = x^2 + y^2$ for a triangle with height y, width x and hypotenuse length s. This can be shown very simply by looking at Fig. 1.1. The area of the partially rotated square is s^2, but we can also calculate it from the the area of the larger square A_{sq} and subtracting four times the area of one of the triangles A_{tr}. Given that $A_{sq} = (x + y)^2$ and $A_{tr} = \frac{1}{2}xy$, then

$$s^2 = (x + y)^2 - 2xy = x^2 + y^2, \tag{1.2.1}$$

and we have proved Pythagoras' theorem. Using an infinitessimally small triangle, we can write $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ and this can be trivially extended to arbitrary dimensions like

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + ... \ . \tag{1.2.2}$$

The infinitessimal form of Pythagoras' theorem is very powerful as it can be used to calculate the length of a generic curve by approximating the curve as a collection of infinitessimally small straight lines with length ds. So far we have assumed that space is flat meaning Eq. (1.2.2) is true for all points in space, this is an assumpion we will have to drop if we want to study the curved spaces arising in strong gravity. In the next sections we will explore the generalisation of Pythagoras' equation to curved spaces and use it to measure curve lengths answell as volumes and areas.

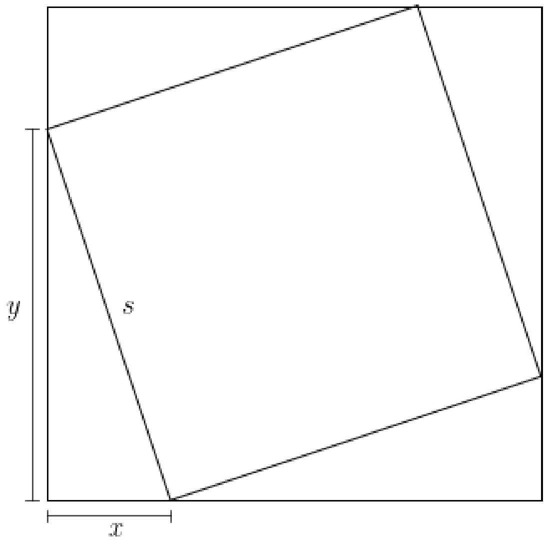


Figure 1.1: Diagram for proof of Pythagoras' theorem.

Differential Geometry (DG) is the extension of calculus, linear algebra and multilinear algebra to general geometries. Einstein's Theory of Relativity is written using the language of DG as it is the natural way to deal with curves, tensor calculus and differential tensor equations in curved spaces. For a basic introduction to DG, we should start with a manifold $\mathcal{M}$ which is an N dimensional space that locally looks like $\mathbb{R}^N$, N dimensional Euclidean space. This is important as at a point $p \in \mathcal{M}$ we can find infinitesimally close neighbouring points $p + \delta p \in \mathcal{M}$. In the following sections we will explore curves, functions, tensors and calculus on manifolds using DG.

1.2.2 Functions, Curves and Tensors on Manifolds

A real scalar function f over $\mathcal{M}$ maps any point $p \in \mathcal{M}$ to a real number, this is denoted as $f : p \to \mathbb{R}$. An important example of a set of scalar functions is the coordinate system ϕ, $\phi : p \to \mathbb{R}^N$, this is normally written x^μ where $\mu \in \{0, 1, ..., N-1\}$ is an index labelling the coordinate. The map ϕ is called a chart, and unlike Euclidean space one chart may not be enough to cover the entire manifold; in this case a set of compatible charts should be smoothly joined, collectively known as an atlas.

Now that functions have been discussed, the next simplest object we can discuss is a curve, or path, through $\mathcal{M}$. A curve Γ is a set of smoothly connected points $p(\lambda) \in \mathcal{M}$ that smoothly depend on an input parameter $\lambda \in [\lambda_0, \lambda_1]$. This can be expressed in terms of coordinates as $x^\mu(\lambda)$ where $\phi : p(\lambda) \to x^\mu(\lambda)$. Differentiating a function f along Γ with respect to λ gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x^\mu(\lambda)) = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\partial f(x^\mu)}{\partial x^\nu} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \partial_\nu f, \tag{1.2.3}$$

where the Einstein summation convention was invoked, summing over all values of ν, and $\partial_\nu = \partial / \partial x^\nu$. Equation (1.2.3) was derived independantly of the choice of f, therefore we can generally write

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \partial_\nu. \tag{1.2.4}$$

The operator $\mathrm{d}/\mathrm{d}\lambda$ can act on any function f and return a new function $\tilde{f}$ over $\mathcal{M}$, this is written as $\mathrm{d}/\mathrm{d}\lambda(f) = \tilde{f}$ where $\tilde{f} : p \to \mathbb{R}$ for $p \in \mathcal{M}$. We can also think of $\mathrm{d}/\mathrm{d}\lambda$ as a vector $\boldsymbol{X}$ with components $X^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ and basis vectors $\boldsymbol{e}_\mu := \partial_\mu$ taken from Eq. (1.2.4). The vector $\boldsymbol{X}$ can be written as $\boldsymbol{X} = X^\mu \boldsymbol{e}_\mu$ and can act on a general function f over $\mathcal{M}$ as $\boldsymbol{X}(f) = X^\mu \boldsymbol{e}_\mu(f) = X^\mu \partial_\mu f$. Considering the set of all possible curves through a points $p \in \mathcal{M}$, the tangent vector components $\mathrm{d}x^\mu/\mathrm{d}\lambda$ span an N dimensional space with basis $\boldsymbol{e}_\mu = \partial_\mu$; this space is called the tangent space and is denoted as $\mathcal{T}_p(\mathcal{M})$ at a point $p \in \mathcal{M}$.

The next object to discuss is the co-vector which is defined as a map from vectors to real numbers; not to be confused with the dot product in section 1.2.3. Similarly to vectors, a co-vector $\boldsymbol{\omega}$ can be expressed as a sum of components ω_μ and basis co-vectors $\boldsymbol{\theta}^\mu$ like $\boldsymbol{\omega} = \omega_\mu \boldsymbol{\theta}^\mu$. Contrary to vectors, co-vector components have downstairs indeces and the basis has upstairs indeces; this choice improves the readability of tensor equations when working with components. A co-vector can map a vector to a real number like $\boldsymbol{\omega} : \boldsymbol{X} \to \mathbb{R}$ or $\boldsymbol{\omega}(\boldsymbol{X}) \to \mathbb{R}$. Vectors are equally able to map co-vectors to real numbers, denoted as $\boldsymbol{X} : \boldsymbol{\omega} \to \mathbb{R}$. Co-vectors are defined such that $\boldsymbol{\theta}^\mu : \boldsymbol{e}_\nu = \delta^\mu_\nu$ where δ^μ_ν are the components of the Kroneka delta equating to zero unless $\mu = \nu$ in which case they equal unity. The operation of a generic co-vector $\boldsymbol{\omega}$ on a generic vector $\boldsymbol{X}$ is

$$\boldsymbol{\omega} : \boldsymbol{X} = \omega_\mu X^\nu \boldsymbol{\theta}^\mu : \boldsymbol{e}_\nu = \omega_\mu X^\nu \delta^\mu_\nu = \omega_\mu X^\mu \in \mathbb{R}. \tag{1.2.5}$$

This map is linear and identical under reversing the order of operation; $\boldsymbol{\omega} : \boldsymbol{X} = \boldsymbol{X} : \boldsymbol{\omega}$. Similarly to vectors, the set of all possible co-vectors at a point $p \in \mathcal{M}$ span an N-dimensional space called the co-tangent space, written as $\mathcal{T}^*_p(\mathcal{M})$.

Multilinear Maps and Tensors

Generalising the previous linear maps between vectors and co-vectors gives the multilinear map. Consider a tensor $\boldsymbol{T}$, this can be expressed in component form like

$$\boldsymbol{T} = T^{\alpha\beta,\dots}_{\mu\nu,\dots} \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta \otimes \dots \otimes \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu \otimes \dots \tag{1.2.6}$$

for an arbitrary number of outer products of vector and co-vector bases. A tensor with m co-vector bases and n vector bases is called an (m, n) tensor and has a rank of $m + n$. Vectors, co-vectors and scalars are $(1, 0)$, $(0, 1)$ and $(0, 0)$ tensors respectively. Tensors can act as multilinear maps between tensors. We have already seen how a vector and co-vector can map each other to a scalar, let's extend this with an example. An $(0, 2)$ tensor, $\boldsymbol{T} = T_{\mu\nu} \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu$ at $p \in \mathcal{M}$, can map two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ to a scalar as shown,

$$\begin{aligned}
\boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}) &= T_{\mu\nu} X^\alpha Y^\beta \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu (\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta), &(1.2.7)\\
&= T_{\mu\nu} X^\alpha Y^\beta (\boldsymbol{\theta}^\mu : \boldsymbol{e}_\alpha)(\boldsymbol{\theta}^\nu : \boldsymbol{e}_\beta), &(1.2.8)\\
&= T_{\mu\nu} X^\alpha Y^\beta \delta^\mu_\alpha \delta^\nu_\beta, &(1.2.9)\\
&= T_{\mu\nu} X^\mu Y^\nu. &(1.2.10)
\end{aligned}$$

The multilinear map can also output generic tensors, for example consider

$$\boldsymbol{T}(\boldsymbol{X}, \star) = T_{\mu\nu} X^\alpha (\boldsymbol{\theta}^\mu : \boldsymbol{e}_\alpha) \boldsymbol{\theta}^\nu = T_{\mu\nu} X^\mu \boldsymbol{\theta}^\nu, \tag{1.2.11}$$

which uses the $(0, 2)$ tensor $\boldsymbol{T}$ to map the vector $\boldsymbol{X}$ to a co-vector $\boldsymbol{W}$ with components $W_\mu = T_{\mu\nu} X^\mu$.

One final example of a mapping is from a single tensor to a lower rank tensor, this is called contraction. To illustrate this, let's take a $(1, 3)$ tensor $\boldsymbol{Z} = Z^\alpha{}_{\mu\nu\rho} \boldsymbol{e}_\alpha \otimes \boldsymbol{\theta}^\mu \otimes \boldsymbol{\theta}^\nu \otimes \boldsymbol{\theta}^\rho$. We can choose to use the basis vector $\boldsymbol{e}_\alpha$ to act on any of the three co-vector bases, choosing $\boldsymbol{\theta}^\mu$ this is

$$Z^\alpha{}_{\mu\nu\rho} (\boldsymbol{e}_\alpha : \boldsymbol{\theta}^\mu) \boldsymbol{\theta}^\nu \otimes \boldsymbol{\theta}^\rho = Z^\mu{}_{\mu\nu\rho} \boldsymbol{\theta}^\nu \otimes \boldsymbol{\theta}^\rho = \tilde{Z}_{\nu\rho} \boldsymbol{\theta}^\nu \otimes \boldsymbol{\theta}^\rho \tag{1.2.12}$$

where $\tilde{Z}_{\nu\rho} = Z^\mu{}_{\mu\nu\rho}$.

1.2.3 The Inner Product and the Metric

To introduce the notion of length on a tanget plane $\mathcal{T}_p(\mathcal{M})$ at a point $p \in \mathcal{M}$ the metric tensor $\boldsymbol{g}$ is introduced. The metric tensor is

$$g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu \tag{1.2.13}$$

where $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu$ represents the inner product (or dot product) on $\mathcal{T}_p(\mathcal{M})$; clearly the metric is symmetric by construction as $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \boldsymbol{e}_\nu \cdot \boldsymbol{e}_\mu$. The inner product can be thought of as a multilinear map,

$$\boldsymbol{g} : (\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}, \tag{1.2.14}$$
$$\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = g_{\mu\nu} Y^\mu X^\nu, \tag{1.2.15}$$

where $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{Y} \in \mathcal{T}_p(\mathcal{M})$ and $\boldsymbol{g} \in \mathcal{T}^*_p(\mathcal{M}) \otimes \mathcal{T}^*_p(\mathcal{M})$. The inner product can also be represented by a second map

$$\boldsymbol{X} : \boldsymbol{Y} \to \mathbb{R}, \tag{1.2.16}$$
$$\boldsymbol{X} \cdot \boldsymbol{Y} = X^\mu Y^\nu \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = X^\mu Y^\nu g_{\mu\nu}, \tag{1.2.17}$$

which is a new mapping. The inner product also gives the length $|\boldsymbol{X}|$ or magnitude of any vector $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$ as,

$$|\boldsymbol{X}|^2 = \boldsymbol{X} \cdot \boldsymbol{X} = g_{\mu\nu} X^\mu X^\nu. \tag{1.2.18}$$

Another way to think of the inner product is that the metric maps a vector $\boldsymbol{X}$ to an *"equivalent"* or *"dual"* co-vector $\boldsymbol{\Xi}$ such that $\boldsymbol{X} : \boldsymbol{\Xi} = X^\mu \Xi_\mu = g_{\mu\nu} X^\mu X^\nu$. In component form $\boldsymbol{\Xi}$ is

$$\Xi_\mu = g_{\mu\nu} X^\nu; \tag{1.2.19}$$

this use of the metric to map a vector to it's corresponding co-vector (and vica versa) is extremely useful. Without loss of information we can write $\Xi_\nu = X_\nu$ to make it obvious that $X_\nu = X^\mu g_{\mu\nu}$ and this convention will be used from now on.

The metric also assigns an inner product and a length measure on the co-tangent plane $\mathcal{T}^*_p(\mathcal{M})$ but instead using the inverse components $g^{\mu\nu} = (g^{-1})_{\mu\nu}$,

$$g^{\mu\nu} = \boldsymbol{\theta}^\mu \cdot \boldsymbol{\theta}^\nu. \tag{1.2.20}$$

Similarly to before the inner product of two co-vectors $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$ is

$$\boldsymbol{\omega} \cdot \boldsymbol{Y} = \omega_\mu \sigma_\nu \boldsymbol{\theta}^\mu \cdot \boldsymbol{\theta}^\nu = \omega_\mu \sigma_\nu g^{\mu\nu} = \omega_\mu \sigma^\mu. \tag{1.2.21}$$

The reason that $g^{\mu\nu}$ must be the inverse matrix to $g_{\mu\nu}$ is as follows. For a vector $x^\mu \boldsymbol{e}_\mu$ and a co-vector $\omega_\mu \boldsymbol{\theta}^\mu$ we would like,

$$\boldsymbol{X} : \boldsymbol{\omega} = \boldsymbol{g}(\boldsymbol{X}, \star) : \boldsymbol{g}^{-1}(\boldsymbol{\omega}, \star), \tag{1.2.22}$$
$$X^\mu \omega_\mu = X_\mu \omega^\mu, \tag{1.2.23}$$
$$= X^\rho g_{\rho\mu} g^{\mu\sigma} \omega_\sigma \tag{1.2.24}$$

which is only true if $g_{\rho\mu} g^{\mu\sigma} = \delta^\sigma_\rho$ which is true by definition if $(g^{-1})_{\mu\nu} = g^{\mu\nu}$.

Not only has the metric provided us with an inner product and a length on tangent planes and cotangent planes, but it has also given a mapping between the two. The metric can raise and lower indeces on general tensors such as

$$T^{\mu\nu\ldots}{}_{\alpha\beta} = g^{\mu\rho} g_{\beta\sigma} T_\rho{}^\nu{}_\alpha{}^{\sigma\ldots}. \tag{1.2.25}$$

1.2.4 Maps Between Manifolds

In section we will be interested in the maps between two manifolds M and N. This has many uses such as pushing and pulling tensors between manifolds, allowing us to calculate a Lie derivative of tensor fields and finding the metric (or any tensor field) on an embedded surface; this very importantly allowed us to perform the 3+1 decomposition [REF] on a spacetime.

Defining a smooth map $\Phi : M \to N$ between manifolds on some coordinate patch and labelling coordinates $x^\mu \in M$ and $y^\mu \in N$. The map $\Phi : x^\mu \to y^\mu$ gives $y^\mu = \Phi^\mu(x^\nu)$, or equivalently $y^\mu(x^\nu)$. Scalar functions must also map trivially $f_N(y^\mu(x^\nu)) = f_M(x^\mu)$ where $f_N \in N$ and $f_M \in M$, thus we will no longer identify which manifold a function is on. The map Φ allows us to push the vector $\boldsymbol{X} \in \mathcal{T}_p(M)$ to $\Phi_*\boldsymbol{X} \in \mathcal{T}_q(N)$, where $q = \Phi(p)$, in a way such that it's action on a function f is the same in either manifold.

$$\boldsymbol{X}(f)\Big|_p = \Phi_*\boldsymbol{X}(f)\Big|_q, \tag{1.2.26}$$

$$X^\mu \frac{\partial f}{\partial x^\mu} = (\Phi_* X)^\nu \frac{\partial f}{\partial y^\nu}, \tag{1.2.27}$$

$$\left(X^\mu \frac{\partial y^\nu}{\partial x^\mu}\right) \frac{\partial f}{\partial y^\nu} = (\Phi_* X)^\nu \frac{\partial f}{\partial y^\nu} \tag{1.2.28}$$

and hence the components of the push-foreward $\Phi_*\boldsymbol{X}$ can be read off,

$$(\Phi_* X)^\mu = \frac{\partial y^\mu}{\partial x^\nu} X^\nu. \tag{1.2.29}$$

Given a co-vector field $\boldsymbol{\omega} \in \mathcal{T}_p^*(N)$ we can pull the field back from $\mathcal{T}_p^*(M) \leftarrow \mathcal{T}_q^*(N)$, denoted $\Phi^*\boldsymbol{\omega}$, by demanding that $\Phi^*\boldsymbol{\omega}(\boldsymbol{X})\big|_p = \boldsymbol{\omega}(\Phi_*\boldsymbol{X})\big|_q$. Evaluating this gives

$$\Phi^*\boldsymbol{\omega}(\boldsymbol{X})\Big|_p = \boldsymbol{\omega}(\Phi_*\boldsymbol{X})\Big|_q, \tag{1.2.30}$$

$$(\Phi^*\omega)_\mu X^\mu = \omega_\nu (\Phi_* X)^\nu, \tag{1.2.31}$$

$$(\Phi^*\omega)_\mu X^\mu = \omega_\nu \frac{\partial y^\nu}{\partial x^\mu} X^\mu, \tag{1.2.32}$$

and the components of pull-back $\Phi_*\boldsymbol{\omega}$ can be read off,

$$(\Phi^*\omega)_\mu = \omega_\nu \frac{\partial y^\nu}{\partial x^\mu}. \tag{1.2.33}$$

Considering an $(0, 2)$ tensor $\boldsymbol{T} \in N$, the pullback $\Phi^*\boldsymbol{T} \in M$ follows simply from demanding that $\boldsymbol{T}(\Phi_*\boldsymbol{X}, \Phi_*\boldsymbol{Y}) = \Phi^*\boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y})$ where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are vector fields on M. The components of the pull-back of $\boldsymbol{T}$ are therefore

$$(\Phi^* T)_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\nu} T_{\rho\sigma}, \tag{1.2.34}$$

and the pull-back of a generic $(0, q)$ tensor and the push-foreward of a generic $(p, 0)$ tensor can be found similarly.

Diffeomorphisms

So far we have only discussed the mapping $\Phi : M \to N$ which is requires a well behaved $\partial y^\nu / \partial x^\mu$. A diffeomorphism is an isomorphism[1] between smooth manifolds $\Phi : M \to N$, meaning M and N have the same number of dimensions. Two infinitessimally close points $\{p, p+\delta p\} \in \mathcal{M}$ map to two infinitessimally close points $(q, q+\delta q) \in \mathcal{N}$ meaning that open sets are preserved. Given that a diffeomorphism is smooth bijective map then it must be invertible with inverse map $\Phi^{-1} : N \to M$, and $y^\nu(x^\mu)$ has a smooth inverse

[1] An isomorphism is a structure preserving bijective map between sets.

$x^\nu(y^\mu)$. When an inverse map Φ^{-1} is defined then the pull-back of $(p,0)$ tensors from N to M along with the push-foreward of $(0,q)$ tensors from M to N is possible. This means it is possible to push or pull generic tensors between M and N in any direction. The tangent spaces associated with $p \in \mathcal{M}$ or $q \in \mathcal{N}$ are therefore also preserved under mapping (aswell as open sets) meaning that local structure on the manifold is preserved under the mapping. Two common examples of diffeomorphisms are coordinate changes and translations.

Projection Mappings

As mentioned, maps between manifolds can be used to determine the metric on an embedded surface. This requires us to consider an m dimensional manifold M with metric $g_{\mu\nu}$ and coordinates x^μ aswell as an embedded n dimensional surface N where $n < m$. We can treat N as a separate n dimensional manifold with metric $h_{\mu\nu}$. As before, we can define a map $\Phi : x^\mu \to y^\mu$

PROJECTIONS HERE I THINK THEN USE IT IN VOLUMES ON MANIFOLDS TO PROJECT THE METRIC.

1.2.5 Lie Derivatives

We have now discussed the necesary formalism to define the Lie derivative. The Lie derivative at a point is the rate of change of a tensor field with respect to a pull-back from a diffeomorphism Φ mapping infinitessimally close points $p, q \in \mathcal{M}$ like $\Phi : p = x^\mu \to q = x^\mu + \epsilon\xi^\mu$ for some vector field ξ. Like any good differential operator, the Lie deriviative $\mathcal{L}_\xi$ along ξ (and $\mathcal{L}_\zeta$ along ζ) should obey,

$$\mathcal{L}_{a\xi+b\zeta}\boldsymbol{T} = a\mathcal{L}_\xi\boldsymbol{T} + b\mathcal{L}_\zeta\boldsymbol{T}, \tag{1.2.35}$$

$$\mathcal{L}_\xi(a\boldsymbol{T} + b\boldsymbol{W}) = a\mathcal{L}_\xi\boldsymbol{T} + b\mathcal{L}_\xi\boldsymbol{W}, \tag{1.2.36}$$

$$\mathcal{L}_\xi(f\boldsymbol{T}) = \boldsymbol{T}\mathcal{L}_\xi f + f\mathcal{L}_\xi\boldsymbol{T}, \tag{1.2.37}$$

for constant $\{a, b\}$, function f and generic tensorial objects of same type $\boldsymbol{T}$ and $\boldsymbol{W}$.

The simplest example is the Lie derivative of a scalar field ϕ, denoted $\mathcal{L}_\xi\phi$ with respect to vector field $\boldsymbol{\xi}$, is

$$\begin{aligned}
\mathcal{L}_\xi\phi &= \lim_{\epsilon\to 0}\left[\frac{\Phi^*\phi|_q - \phi|_p}{\epsilon}\right], &(1.2.38)\\
&= \lim_{\epsilon\to 0}\left[\frac{\phi(x^\mu + \epsilon\xi^\mu) - \phi(x^\mu)}{\epsilon}\right], &(1.2.39)\\
&= \xi^\mu\partial_\mu\phi, &(1.2.40)
\end{aligned}$$

which reduces to the directional derivative of ϕ with respect to $\boldsymbol{\xi}$. Next let's calculate the Lie derivative of a vector field $\boldsymbol{X}$ with respect to vector field $\boldsymbol{\xi}$. Starting with the same definition as Eq. (1.2.38), and using $y^\mu = x^\mu + \epsilon\xi^\mu$, the Lie derivative of $\boldsymbol{X}$ is

$$\begin{aligned}
(\mathcal{L}_\xi X)^\mu &= \lim_{\epsilon\to 0}\left[\frac{(\Phi^* X|_q)^\mu - X|_p{}^\mu}{\epsilon}\right], &(1.2.41)\\
&= \lim_{\epsilon\to 0}\left[\frac{\frac{\partial x^\mu}{\partial y^\nu}X^\nu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho)}{\epsilon}\right], &(1.2.42)\\
&= \lim_{\epsilon\to 0}\left[\frac{(\delta^\mu_\nu - \epsilon\partial_\nu\xi^\mu)X^\nu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho)}{\epsilon}\right], &(1.2.43)\\
&= \lim_{\epsilon\to 0}\left[\frac{-\epsilon\partial_\nu\xi^\mu X^\nu(x^\rho + \epsilon\xi^\rho) + X^\mu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho)}{\epsilon}\right], &(1.2.44)\\
&= \lim_{\epsilon\to 0}\left[\frac{-\epsilon\partial_\nu\xi^\mu X^\nu(x^\rho) + X^\mu(x^\rho + \epsilon\xi^\rho) - X^\mu(x^\rho) + \mathcal{O}(\epsilon^2)}{\epsilon}\right], &(1.2.45)\\
&= \xi^\nu\partial_\nu X^\mu - X^\nu\partial_\nu\xi^\mu. &(1.2.46)
\end{aligned}$$

The Lie derivative for co-vectors and tensors can be derived in the same way, but can be quickly derived from the Liebnitz rule as follows. Define a scalar field ψ, vector field $\boldsymbol{X}$ and co-vector field $\boldsymbol{\omega}$, where $\psi = X^\mu \omega_\mu$, then it follows that

$$\begin{aligned}
\mathcal{L}_\xi \psi &= \xi^\mu \partial_\mu \psi = X^\nu \xi^\mu \partial_\mu \omega_\nu + \omega_\nu \xi^\mu \partial_\mu X^\nu, &(1.2.47)\\
&= \mathcal{L}_\xi (X^\mu \omega_\mu), &(1.2.48)\\
&= \omega_\mu (\mathcal{L}_\xi X)^\mu + X^\mu (\mathcal{L}_\xi \omega)_\mu, &(1.2.49)\\
X^\mu (\mathcal{L}_\xi \omega)_\mu &= X^\nu \xi^\mu \partial_\mu \omega_\nu + \omega_\nu \xi^\mu \partial_\mu X^\nu - \omega_\mu (\mathcal{L}_\xi X)^\mu, &(1.2.50)\\
(\mathcal{L}_\xi \omega)_\mu &= \xi^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu \xi^\nu. &(1.2.51)
\end{aligned}$$

Derivatives of a generic tensor $\boldsymbol{T}$ follows simply, for example

$$(\mathcal{L}_\xi T)^{\alpha\beta...}{}_{\mu\nu...} = \xi^\sigma \partial_\sigma T^{\alpha\beta...}{}_{\mu\nu...} + T^{\alpha\beta...}{}_{\sigma\nu...} \partial_\mu \xi^\sigma + T^{\alpha\beta...}{}_{\mu\sigma...} \partial_\nu \xi^\sigma + ... - T^{\sigma\beta...}{}_{\mu\nu...} \partial_\sigma \xi^\alpha - T^{\alpha\sigma...}{}_{\mu\nu...} \partial_\sigma \xi^\beta - ... \,. \quad (1.2.52)$$

1.2.6 Lengths on Manifolds

The natural entry point for studying curved geometry is to revisit Pythagoras' theorem. For this we need a manifold $\mathcal{M}$ equipped with a metric g, written as $(\mathcal{M}, g)$ for short. The distance $\mathrm{d}s$ between two infinitessimaly close points $p \in \mathcal{M}$ and $p + \delta p \in \mathcal{M}$, with coordinates x^μ and $x^\mu + \mathrm{d}x^\mu$, is given by

$$\mathrm{d}s^2 = \boldsymbol{g}(\mathbf{d}\boldsymbol{x}, \mathbf{d}\boldsymbol{x}) = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \quad (1.2.53)$$

where $g_{\mu\nu}$ are the components of the metric tensor. This is the generalisation of Eq. (1.2.2) to curved space; noteably the line element can now have varying coefficients from $g_{\mu\nu}$ and cross terms such as $\mathrm{d}x\mathrm{d}y$. The special choice of $g_{\mu\nu} = \delta_{\mu\nu}$ gives us flat space, also called Euclidean space, where $\delta_{\mu\nu} = 1$ if $\mu = \nu$ and vanishes otherwise. With the line elemend defined, we can immediately apply it to calculating the length of a general curve in curved space. Consider the curve Γ consisting of a set of smoothly connected points $p(\lambda) \in \mathcal{M}$ smoothly parameterised by λ. We can calculate the length Δs of the curve between $\lambda_1 \geq \lambda \geq \lambda_0$ by

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} g_{\mu\nu} \mathrm{d}\lambda^2, &(1.2.54)\\
\Delta s &= \int_{\lambda_0}^{\lambda_1} \sqrt{\left(\frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} g_{\mu\nu} \right)} \mathrm{d}\lambda. &(1.2.55)
\end{aligned}$$

In the simplified case where λ is one of the coordinates, say ξ, the length Δs becomes,

$$\Delta s = \int_{\xi_0}^{\xi_1} \sqrt{g_{\xi\xi}} \mathrm{d}\xi. \quad (1.2.56)$$

1.2.7 Volumes on Manifolds

The natural extension to measuring the length of a curve is the measurement of volumes; of course we still require a metric $\boldsymbol{g}$ over the manifold $\mathcal{M}$. Let's say that in a coordinate system x^μ we can define the volume V of some subregion M of $\mathcal{M}$ by integrating some weight function $w(x^\mu)$,

$$V = \int_M w(x^\mu) \mathrm{d}x^1 \mathrm{d}x^2 ... \mathrm{d}x^n, \quad (1.2.57)$$

over M. To find $w(\mu)$, start by defining an orthogonal coordinate transformation $x^\mu \to \tilde{x}^\mu$ such that $\tilde{\boldsymbol{g}}$ is diagonal and $\det(\boldsymbol{g}) = \det(\tilde{\boldsymbol{g}})$; this is always possible as $\boldsymbol{g}$ is real and symmetric. In this coordinate system, the volume δV in an infinitessimal cuboid, with i'th sidelength $\delta \tilde{x}^i$, is

$$\delta V = \left(\sqrt{\tilde{g}_{11}} \delta \tilde{x}^1 \right) \left(\sqrt{\tilde{g}_{22}} \delta \tilde{x}^2 \right) ... \left(\sqrt{\tilde{g}_{nn}} \delta \tilde{x}^n \right), \quad (1.2.58)$$

where Eq. (1.2.56) was used to get the length bewteen each $\tilde{x}^i$ and $\tilde{x}^i + \delta\tilde{x}^i$. Given that $\tilde{\boldsymbol{g}}$ is diagonal we know the i'th eigenvalue $\tilde{\lambda}_i = \tilde{g}_{ii}$ and therefore $\det(\tilde{\boldsymbol{g}}) = \prod_i \tilde{g}_{ii}$. Thus the volume δV can be rewritten

$$\delta V = \sqrt{|\det(\tilde{\boldsymbol{g}})|}\delta\tilde{x}^1\delta\tilde{x}^2...\delta\tilde{x}^n, \tag{1.2.59}$$

and the formula for the finite volume M is

$$V = \int_M \sqrt{|\det\{(\tilde{\boldsymbol{g}})\}|}\mathrm{d}\tilde{x}^1\mathrm{d}\tilde{x}^2...\mathrm{d}\tilde{x}^n, \tag{1.2.60}$$

and the form of the weight function in $\tilde{x}^\mu$ coordinates is $w(\tilde{x}^\mu) = \sqrt{|\det(\tilde{\boldsymbol{g}})|}$. We are now free to transform back from $\tilde{x}^\mu \to x^\mu$, and given that the transformation is orthogonal we know that $\det(\boldsymbol{g}) = \det(\tilde{\boldsymbol{g}})$ and the Jacobean matrix $\boldsymbol{J}$ of the coordinate transformation has $\det(\boldsymbol{J}) = 1$, therefore

$$V = \int_M \sqrt{|\det\{(\boldsymbol{g})\}|}\,\mathrm{d}x^1\,\mathrm{d}x^2\,...\,\mathrm{d}x^n\,, \tag{1.2.61}$$

which holds for any non-diagonal, real and symmetric metric $\boldsymbol{g}$. In general we will now denote the deteminant of a metric $\det(\boldsymbol{g})$ with the lower case letter g. When dealing with a pseudo-Riemannian manifold with a negative determinant, such as spacetime, it is more common to see $\sqrt{-g}$ written rather than $\sqrt{|g|}$ giving

$$V = \int_M \sqrt{-g}\,\mathrm{d}x^1\,\mathrm{d}x^2\,...\,\mathrm{d}x^n\,. \tag{1.2.62}$$

Equation (1.2.60) can also be used to find the volume (or area) of a lower dimensional sub-volume. First cover the new sub-volume A with coordinates y^μ, where $\mu \in \{1, 2, ..., m\}$ for $m < n$, and then calculate the metric $\boldsymbol{h}$ which can be done using the pullback of $\boldsymbol{g}$

$$h_{\sigma\rho} = (\Phi^* g)_{\sigma\rho} = \frac{\partial x^\mu}{\partial y^\sigma}\frac{\partial x^\nu}{\partial y^\rho}g_{\mu\nu} \tag{1.2.63}$$

as shown in Section 1.2.4. Using y^μ and h, the volume V_A of A as

$$V_A = \int_A \sqrt{|h|}\,\mathrm{d}y^1\,\mathrm{d}y^2\,...\,\mathrm{d}y^m\,. \tag{1.2.64}$$

NEED TO RE-WRITE THE MAPS SECTION TO HAVE PROJECTIONS AT THE END THEN QUOTE IT FOR THE PROJECTED METRIC HERE INSTEAD OF DERIVING THE PROJECTED METRIC HERE

1.2.8 Geodesics

For a manifold equipped with metric $(\mathcal{M}, g)$ the curve with shortest distance between two points $p, q \in \mathcal{M}$ is called a geodesic. To find the geodesic joining p and q we need to use calculus of variation on the total length Δs from Eq. (1.2.55) of a general curve between two points. Given that the integrand $\mathcal{L}$ of Eq. (1.2.55) is a function like $\mathcal{L}(x^\mu, \dot{x}^\mu)$, where the dot means differentiation by λ, we can use the Eular-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = 0 \tag{1.2.65}$$

to give a differential equation with solution being a geodesic. Applyin the EL equation to the integrand of Eq. (1.2.55) is algebraically messy, it is easier to square the integrand and start from $\mathcal{L}^2$ giving the same solution if $\mathrm{d}\mathcal{L}/\mathrm{d}\lambda = 0$[2],

$$\frac{\partial \mathcal{L}^2}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}^2}{\partial \dot{x}^\alpha} = 0, \tag{1.2.66}$$

$$\frac{\partial}{\partial x^\alpha}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial}{\partial \dot{x}^\alpha}\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) = 0, \tag{1.2.67}$$

$$(\partial_\alpha g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\nu}\dot{x}^\nu\right) = 0, \tag{1.2.68}$$

$$(\partial_\alpha g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - 2\left(\dot{x}^\rho \partial_\rho \left(g_{\alpha\nu}\right)\dot{x}^\nu\right) - 2\ddot{x}^\nu g_{\alpha\nu} = 0. \tag{1.2.69}$$

Rearranging and multiplying by $g^{\alpha\beta}$ gives

$$\ddot{x}^\beta + \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)\dot{x}^\mu \dot{x}^\nu = 0, \tag{1.2.70}$$

$$\ddot{x}^\beta + \Gamma^\beta{}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0, \tag{1.2.71}$$

where $\Gamma^\beta{}_{\mu\nu}$ is the components of the connection-symbol from Eq. (1.3.77). A trivial solution to Eq. (1.2.71) is in flat space using cartesian coordinates where $\Gamma^\beta{}_{\mu\nu} = 0$ and therefore $\ddot{x}^\beta = 0$ so $\dot{x}^\beta$ is a constant; this tells us the shortest distance between two points in flat space is a straight line. In other words, geodesics are straight lines in flat space.

Non-affine Geodesics

The equation of a geodesic given above is true for an affinely parameterised curve. An affine parameter λ is defined so that the length of a curve Δs between two parameter values λ_0 and λ_1 is given by $\Delta s = k(\lambda_1 - \lambda_0)$ for constant k; the arclength along a curve is linearly proportional to the value of the λ. The reason this happened with the calculation above is that $\mathrm{d}\mathcal{L}/\mathrm{d}\lambda = 0$ was assumed.

A non-affine parameter μ could equally be used to describe the curve. Writing $\mu(\lambda)$ the geodesic equation is transformed as shown,

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\lambda^2} + \Gamma^\beta{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0, \tag{1.2.72}$$

$$\left(\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\frac{\mathrm{d}}{\mathrm{d}\mu} + \left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\right)x^\beta + \Gamma^\beta{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\mu}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^2 = 0, \tag{1.2.73}$$

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\mu^2} + \Gamma^\beta{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\mu} = -\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^{-2}\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^\beta}{\mathrm{d}\mu}, \tag{1.2.74}$$

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\mu^2} + \Gamma^\beta{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\mu} = -f(\mu)\frac{\mathrm{d}x^\beta}{\mathrm{d}\mu}, \tag{1.2.75}$$

$$\tag{1.2.76}$$

which is the same as the affine geodesic equation except with an extra non-zero right hand side proportional to $\mathrm{d}x^\beta/\mathrm{d}\mu$ and some function $f(\mu)$.

IS THIS WELL ENOOUGH EXPLAINED, ESPECIALLY THE DLDLAMBDA=0 BIt?

1.3 Tensor Calculus and Curvature

1.3.1 General Covariance and Coordinate transformations

Many laws of physics can be expressed as tensor field equations where a tensor field is the assignment of a tensor to each point in space. This assignment must be smooth as it is to describe physical quantities.

[2] Given that $\mathcal{L}$ is homogeneneus to degree k, $\dot{x}^i\partial\mathcal{L}/\partial\dot{x}^i = k\mathcal{L}$ for constant k, one can show that $\mathrm{d}\mathcal{L}/\mathrm{d}\lambda = 0$ if the Eular-Lagrange equation is assumed.

The power of tensor algebra and tensor calculus is that if a tensor field equation can be written in one coordiante system then must hold (in index form) in all sensible coordinate system. This is a consequence of the tensor transformation law. Looking back, we can write a generic vector $\boldsymbol{X}$ as $X^\mu \boldsymbol{e}_\mu = X^\mu \partial_\mu$ and if we choose a coordinate transformation $x^\mu \to \tilde{x}^\mu$ then we see that in the transformed coordinate system the vector field $\boldsymbol{X}$, written $\tilde{\boldsymbol{X}}$, becomes

$$\tilde{\boldsymbol{X}} = \tilde{X}^\mu \frac{\partial}{\partial \tilde{x}^\mu}, \tag{1.3.1}$$

$$= \tilde{X}^\mu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^\nu}, \tag{1.3.2}$$

$$= X^\nu \frac{\partial}{\partial x^\nu}, \tag{1.3.3}$$

$$= \boldsymbol{X}, \tag{1.3.4}$$

where the components $X^\nu = \tilde{X}^\mu \frac{\partial x^\nu}{\partial \tilde{x}^\mu}$ are required to transform in order to ensure $\boldsymbol{X} = \tilde{\boldsymbol{X}}$. This says that the underlying geometric object (a vector in this case) is independant of the coordinates used to describe them; the tradeoff for this useful property is that the vectors components X^μ have to transform under the tensor transformation law, effectively opposing the transformation of the basis vectors. Working from a co-vector ω we can write it as $\omega_\mu \boldsymbol{\theta}^\mu = \omega_\mu \mathrm{d}x^\mu$ in component-basis form [REF THIS?] and the same coordinate transform gives

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_\mu \mathrm{d}\tilde{x}^\mu, \tag{1.3.5}$$

$$= \tilde{\omega}_\mu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \mathrm{d}x^\nu, \tag{1.3.6}$$

$$= \omega_\nu \mathrm{d}x^\nu, \tag{1.3.7}$$

where the co-vector components transform like $\omega_\nu = \tilde{\omega}_\mu \frac{\partial \tilde{x}^\mu}{\partial x^\nu}$, the opposite way to the vector components. These transformation laws ensure that a scalar field created from the product of a vector field and a co-vector field, like $\boldsymbol{\omega} : \boldsymbol{X}$, is a Lorentz scalar not transforming under coordinate transformations. This can be seen from

$$\tilde{\boldsymbol{\omega}} : \tilde{\boldsymbol{X}} = \tilde{X}^\mu \tilde{\omega}_\mu, \tag{1.3.8}$$

$$= X^\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \omega_\rho, \tag{1.3.9}$$

$$= X^\nu \frac{\partial x^\rho}{\partial x^\nu} \omega_\rho, \tag{1.3.10}$$

$$= X^\nu \delta_\nu^{\ \rho} \omega_\rho, \tag{1.3.11}$$

$$= X^\nu \omega_\nu, \tag{1.3.12}$$

$$= \boldsymbol{\omega} : \boldsymbol{X}. \tag{1.3.13}$$

The general tensor transformation law can be derived easily from chaining multiple of the previous examples together, for example

$$\tilde{T}^{\mu\nu\dots}_{\quad\ \rho\sigma\dots} = T^{\alpha\beta\dots}_{\quad\ \gamma\delta\dots} \left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta}, \dots \times \frac{\partial x^\gamma}{\partial \tilde{x}^\rho} \frac{\partial x^\delta}{\partial \tilde{x}^\sigma}, \dots \right). \tag{1.3.14}$$

Tensor Densities

A tensor density is the generalisation of a tensor field obeying the tensor transformation law in Eq. (1.3.14) to a tensor field multiplied by a power of the determinant of the Jacobean matrix of a coordinate transformation. One important example of a tensor density is the volume element $\sqrt{-g}$, this is a scalar density. This object does not have any indeces so at first glance may pass for a true scalar field. However, when a coordinate transformation $x^\mu \to \tilde{x}^\mu$ is applied we find that $\sqrt{-g} \to \sqrt{-\tilde{g}} \neq \sqrt{-g}$ but for a general scalar field ϕ we find $\phi \to \tilde{\phi} = \phi$; therefore $\sqrt{-g}$ cannot be a scalar field. This can be shown explicitly by look at the the determinant of the metric,

$$\begin{aligned}
\sqrt{-\tilde{g}} &= \sqrt{\det(-\tilde{g}_{\mu\nu})}, && (1.3.15)\\
&= \sqrt{\det\left(-g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\right)}, && (1.3.16)\\
&= \sqrt{-g}\det\left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\right), && (1.3.17)
\end{aligned}$$

and as can be seen, the volume element picks up a factor of the determinant of the Jacobean. This property shows up in multidimensional intergrals, for instance when taking a three-dimensional integral over cartesian coordinates we must replace the $\mathrm{d}x\mathrm{d}y\mathrm{d}z \to \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi r^2\sin(\theta)$ [MAYBE DO THIS LATER AFTER INTEGRATION ON MANIFODS?]

A tensor density $\boldsymbol{\mathcal{T}}$ of weight w can be written in the form,

$$\boldsymbol{\mathcal{T}} = \sqrt{-g}^{\,w}\boldsymbol{T}, \tag{1.3.18}$$

where $\boldsymbol{T}$ is a tensor obeying the tensor transformation law. It should be noted that a tensor density of weight zero is a regular tensors and the weight of a tensor has nothing to do with the rank of the tensor.

Lie Derivatives of Tensor Densities

To calculate the Lie derivative of a tensor density, first the Lie derivative of $\sqrt{-g}$ should be calculated, and in order to calculate the Lie derivative of the volume element a preliminary result is needed. Following the definition of a Lie derivative in section 1.2.5 and setting $y^\mu = x^\mu + \epsilon\xi^\mu$, the determinant of the Jacobean matrix is,

$$\begin{aligned}
\det\left(\frac{\partial y^\mu}{\partial x^\rho}\right) &= \det\left(\delta^\mu_\rho + \epsilon\frac{\partial \xi^\mu}{\partial x^\rho}\right), && (1.3.19)\\
&= \det\begin{pmatrix}
1+\epsilon\frac{\partial\xi^1}{\partial x^1} & \epsilon\frac{\partial\xi^1}{\partial x^2} & \epsilon\frac{\partial\xi^1}{\partial x^3} & \epsilon\frac{\partial\xi^1}{\partial x^4}\\
\epsilon\frac{\partial\xi^2}{\partial x^1} & 1+\epsilon\frac{\partial\xi^2}{\partial x^2} & \epsilon\frac{\partial\xi^2}{\partial x^3} & \epsilon\frac{\partial\xi^2}{\partial x^4}\\
\epsilon\frac{\partial\xi^3}{\partial x^1} & \epsilon\frac{\partial\xi^3}{\partial x^2} & 1+\epsilon\frac{\partial\xi^3}{\partial x^3} & \epsilon\frac{\partial\xi^3}{\partial x^4}\\
\epsilon\frac{\partial\xi^4}{\partial x^1} & \epsilon\frac{\partial\xi^4}{\partial x^2} & \epsilon\frac{\partial\xi^4}{\partial x^3} & 1+\epsilon\frac{\partial\xi^4}{\partial x^4}
\end{pmatrix}, && (1.3.20)\\
&= \left(1+\epsilon\sum_i\frac{\partial\xi^i}{\partial x^i} + \mathcal{O}(\epsilon^2)\right), && (1.3.21)\\
&= 1+\epsilon\partial_\mu\xi^\mu + \mathcal{O}(\epsilon^2), && (1.3.22)
\end{aligned}$$

where four dimensions was used for clarity, but the calculation works exactly the same in any number of dimensions. Using this result and the definition of a Lie derivative, $\mathcal{L}_\xi\sqrt{-g}$ evaluates to

$$
\begin{aligned}
\mathcal{L}_\xi\sqrt{-g} &= \lim_{\epsilon\to 0}\left(\frac{\Phi^*\sqrt{-\tilde{g}}\Big|_q - \sqrt{-g}\Big|_p}{\epsilon}\right), & (1.3.23)\\
&= \lim_{\epsilon\to 0}\left(\frac{\sqrt{-\det\left(g_{\mu\nu}(y^\alpha)\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma}\right)} - \sqrt{-g}(x^\alpha)}{\epsilon}\right), & (1.3.24)\\
&= \lim_{\epsilon\to 0}\left(\frac{\sqrt{-g}(y^\alpha)\det\left(\frac{\partial y^\mu}{\partial x^\rho}\right) - \sqrt{-g}(x^\alpha)}{\epsilon}\right), & (1.3.25)\\
&= \lim_{\epsilon\to 0}\left(\frac{\sqrt{-g}(y^\alpha)(1+\epsilon\partial_\mu\xi^\mu + \mathcal{O}(\epsilon^2)) - \sqrt{-g}(x^\alpha)}{\epsilon}\right), & (1.3.26)\\
&= \lim_{\epsilon\to 0}\left(\frac{[\sqrt{-g}(x^\alpha) + \epsilon\xi^\mu\partial_\mu\sqrt{-g}(x^\alpha)](1+\epsilon\partial_\mu\xi^\mu + \mathcal{O}(\epsilon^2)) - \sqrt{-g}(x^\alpha)}{\epsilon}\right), & (1.3.27)\\
&= \lim_{\epsilon\to 0}\left(\frac{\epsilon\xi^\mu\partial_\mu\sqrt{-g}(x^\alpha) + \epsilon\sqrt{-g}(x^\alpha)\partial_\mu\xi^\mu + \mathcal{O}(\epsilon^2))}{\epsilon}\right), & (1.3.28)\\
\mathcal{L}_\xi\sqrt{-g} &= \xi^\mu\partial_\mu\sqrt{-g} + \sqrt{-g}\partial_\mu\xi^\mu. & (1.3.29)
\end{aligned}
$$

Given that $\mathcal{L}_\xi\sqrt{-g}$ has been calculated, it is straightforeward to calculate the Lie derivarive of a tensor density $\boldsymbol{\mathcal{T}} = \sqrt{-g}^w\boldsymbol{T}$ of weight w, where $\boldsymbol{T}$ is a regular tensor, using the Liebnitz rule in Eq. (1.2.37). The Lie derivative is,

$$
\mathcal{L}_\xi\boldsymbol{\mathcal{T}} = \sqrt{-g}^w\left(\mathcal{L}_\xi\boldsymbol{T} + w\boldsymbol{T}\left(\frac{1}{\sqrt{-g}}\xi^\mu\partial_\mu\sqrt{-g} + \partial_\mu\xi^\mu\right)\right), \tag{1.3.30}
$$

where the Leibnitz rule of differentiation has been used. As would be expected, setting $w = 0$ returns the regular Lie derivative of a tensor. This can also be written as,

$$
\mathcal{L}_\xi\boldsymbol{\mathcal{T}} = \widetilde{\mathcal{L}}_\xi\boldsymbol{\mathcal{T}} + w\boldsymbol{\mathcal{T}}\partial_\mu\xi^\mu, \tag{1.3.31}
$$

where $\widetilde{\mathcal{L}}_\xi$ is the differential operator equivalent to the Lie derivate of the same tensor but with weight zero.

CHECK THIS IS CORRECT AND USED CORRECTLY LATER IN THE CCZ4 EQUATIONS SECTION

1.3.2 The Covariant Derivative

There are many types of derivative on a manifold, all related to each other, that we care about in the context of General Relativity. The simplest is the partial derivative, denoted

$$
\partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1.3.32}
$$

which works much the same as always. Two other derivatives are the exterior derivative [MAYBE REF THIS] and the Lie derivative which are discussed in [REF] and section 1.2.5 respectively.

The purpose of the covariant derivative, denoted ∇_μ, is to generalise the partial derivative to curved spaces (or curvilinear coordinates). The covariant derivative exactly reduces to the partial derivative (∂_μ) in flat space with cartesian coordinates. As suggested by the name, the covariant derivative of an object is covariant; it obeys the tensor transformation law in section 1.3.1. The covariant derivative uses a vector field $\boldsymbol{X}$ to map a (p, q) tensor field $\boldsymbol{T}$ to a (p, q) tensor field $\boldsymbol{\nabla_X T}$. Requiring the covariant derivative of a tensor to return another tensor may sound pedantic but it allows the writing of physical differential equations that are covariant, i.e. that hold in all coordiante systems. Encoding the laws of physics with tensor differential equations is explored in more detail in Section [REF!].

To be the analogue of the partial derivative, three properties of the covariant derivative are required,

$$\boldsymbol{\nabla}_{f\boldsymbol{X}+g\boldsymbol{Y}}\boldsymbol{T} = f\boldsymbol{\nabla_X T} + g\boldsymbol{\nabla_Y T}, \tag{1.3.33}$$
$$\boldsymbol{\nabla_X}(a\boldsymbol{T} + b\boldsymbol{W}) = a\boldsymbol{\nabla_X T} + b\boldsymbol{\nabla_X W}, \tag{1.3.34}$$
$$\boldsymbol{\nabla_X}(f\boldsymbol{T}) = f\boldsymbol{\nabla_X T} + \boldsymbol{T}\boldsymbol{\nabla_X} f, \tag{1.3.35}$$

where a and b are constants, f and g are functions and $\boldsymbol{T}$ and $\boldsymbol{W}$ are tensors of equal type. These are exactly the same as the conditions imposed on Lie derivatives in section 1.2.5. From now on the coraviant derivative $\boldsymbol{\nabla}_{\boldsymbol{e}_\mu}$ with respect to a basis vector $\boldsymbol{e}_\mu$ will be written as ∇_μ.

Lets start by finding the covariant derivative of a of a scalar field φ. The partial derivative $\partial_\mu \varphi$ obeys the tensor transformation law for a co-vector,

$$\frac{\partial}{\partial \tilde{x}^\mu}\tilde{\varphi} = \frac{\partial}{\partial \tilde{x}^\mu}\varphi, \tag{1.3.36}$$
$$= \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\nu}\varphi, \tag{1.3.37}$$

and therefore the $\nabla_\mu \varphi = \partial_\mu \varphi$. Note that $\varphi = \tilde{\varphi}$ for any point p as a scalar remains unchanged in a coordinate transformation. Complications arise when taking the partial derivative of any other higher rank tensor; let's demonstrate this with a vector $\boldsymbol{X}$.

$$\frac{\partial}{\partial \tilde{x}^\mu}\tilde{X}^\alpha = \frac{\partial}{\partial \tilde{x}^\mu}\left(\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}X^\beta\right), \tag{1.3.38}$$
$$= \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\nu}\left(\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}X^\beta\right), \tag{1.3.39}$$
$$= \underbrace{\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{\partial x^\nu}{\partial \tilde{x}^\mu}}_{\text{Tensor transformation law}}\frac{\partial}{\partial x^\nu}X^\beta + X^\beta\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\nu}\left(\frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\right), \tag{1.3.40}$$

and as can be seen, only the first term on the right hand side should exist if the components $\partial_\mu X^\alpha$ were to obey the tensor transformation law.

The problem with performing differentiation on tensors is that it requires the comparison of tensors at two different (infinitessimaly close) tangent spaces. Lie derivatives circumvented this problem by comparing two two tangent planes with a pullback defined by a diffeomorphism. Another way of overcoming this problem is to consider how the coordinate basis vectors change over the manifold, not just the components. Defining the covariant derivative of the basis vector as

$$\boldsymbol{\nabla}_{\boldsymbol{e}_\rho}\boldsymbol{e}_\nu = \nabla_\rho \boldsymbol{e}_\nu := \Gamma^\mu{}_{\nu\rho}\boldsymbol{e}_\mu \tag{1.3.41}$$

where $\Gamma^\mu{}_{\nu\rho}$ is called the connection due to it defining a connection between neighbouring tangent space. The connection can be used to get the covariant derivative of the vector field $\boldsymbol{X} = X^\rho \boldsymbol{e}_\rho$,

$$\nabla_\rho(X^\nu \boldsymbol{e}_\nu) = (\partial_\rho X^\nu)\boldsymbol{e}_\nu + X^\nu(\nabla_\rho \boldsymbol{e}_\nu), \tag{1.3.42}$$
$$= (\partial_\rho X^\nu)\boldsymbol{e}_\nu + X^\nu \Gamma^\mu{}_{\nu\rho}\boldsymbol{e}_\mu, \tag{1.3.43}$$
$$= (\partial_\rho X^\mu + \Gamma^\mu{}_{\nu\rho}X^\nu)\boldsymbol{e}_\mu. \tag{1.3.44}$$

Note that on the first line above we used $\nabla_\rho X^\nu = \partial_\rho X^\nu$ as the X^ν are being treated as a set of scalar function coefficients multiplying the basis vectors $\boldsymbol{e}_\mu$. Strictly we should write the covariant derivative of $\boldsymbol{X}$ as

$$\boldsymbol{\nabla X} = (\nabla X)_\sigma{}^\mu \boldsymbol{e}_\mu \otimes \boldsymbol{\theta}^\sigma \tag{1.3.45}$$

but for convenience the coefficients $(\nabla X)_\sigma{}^\mu$ are usually denoted as,

$$\nabla_\sigma X^\mu = \partial_\sigma X^\mu + \Gamma^\mu{}_{\nu\sigma} X^\nu. \tag{1.3.46}$$

This is a slight abuse of notation as $\nabla_\sigma X^\mu$ might be understood as the covariant derivate of the components X^μ, but really it denotes the component $\boldsymbol{\theta}^\sigma \cdot (\boldsymbol{\nabla X}) \cdot \boldsymbol{e}_\mu$ where $\boldsymbol{\nabla X}$ is given in Eq. (1.3.45).

Given that the covariant derivative of a scalar reduces to the partial derivative we can see that

$$\nabla_\rho(\boldsymbol{e}_\mu : \boldsymbol{\theta}^\nu) = 0, \tag{1.3.47}$$

and using the Liebnitz rule wae see that

$$\begin{aligned}
\nabla_\rho(\boldsymbol{e}_\nu : \boldsymbol{\theta}^\mu) &= (\nabla_\rho \boldsymbol{e}_\nu) : \boldsymbol{\theta}^\mu + \boldsymbol{e}_\nu : (\nabla_\rho \boldsymbol{\theta}^\mu), && (1.3.48)\\
&= (\Gamma^\sigma{}_{\nu\rho} \boldsymbol{e}_\sigma) : \boldsymbol{\theta}^\mu + \boldsymbol{e}_\nu : (\nabla_\rho \boldsymbol{\theta}^\mu), && (1.3.49)\\
&= \Gamma^\mu{}_{\nu\rho} + \boldsymbol{e}_\nu : (\nabla_\rho \boldsymbol{\theta}^\mu), && (1.3.50)\\
\boldsymbol{e}_\nu : (\nabla_\rho \boldsymbol{\theta}^\mu) &= -\Gamma^\mu{}_{\nu\rho} && (1.3.51)
\end{aligned}$$

therefore we must have

$$\nabla_\rho \boldsymbol{\theta}^\mu = -\Gamma^\mu{}_{\nu\rho} \boldsymbol{\theta}^\nu. \tag{1.3.52}$$

In an identical way to before, we might ask what is the covariant derivative of a co-vector $\boldsymbol{\omega} = \omega_\alpha \boldsymbol{\theta}^\alpha$. The covariant derivative $\boldsymbol{\nabla\omega}$ can be found like

$$\begin{aligned}
\nabla_\sigma \boldsymbol{\omega} &= \nabla_\sigma(\omega_\alpha \boldsymbol{\theta}^\alpha), && (1.3.53)\\
&= \partial_\sigma(\omega_\alpha)\boldsymbol{\theta}^\alpha + \omega_\alpha \nabla_\sigma \boldsymbol{\theta}^\alpha, && (1.3.54)\\
&= \partial_\sigma(\omega_\alpha)\boldsymbol{\theta}^\alpha - \omega_\alpha \Gamma^\alpha{}_{\nu\sigma} \boldsymbol{\theta}^\nu, && (1.3.55)\\
&= (\partial_\sigma \omega_\alpha - \omega_\nu \Gamma^\nu{}_{\alpha\sigma})\boldsymbol{\theta}^\alpha. && (1.3.56)
\end{aligned}$$

Again, we used $\nabla_\sigma \omega_\alpha = \partial_\sigma \omega_\alpha$ as the components ω_α are scalar coefficients of the basis co-vectors $\boldsymbol{\theta}^\alpha$. Similarly to earlier, from now on the components $(\nabla\omega)_{\sigma\alpha}$ are written as $\nabla_\sigma \omega_\alpha$ even though this is a mild abuse of notation.

The covariant derivative of a general tensor can be found by following the simple rule of adding a connection symbol term for each index, for example

$$\nabla_\mu T^{\alpha\beta\ldots}_{\lambda\nu\ldots} = \partial_\mu T^{\alpha\beta\ldots}_{\lambda\nu\ldots} + \Gamma^\alpha{}_{\sigma\mu} T^{\sigma\beta\ldots}_{\lambda\nu\ldots} + \Gamma^\beta{}_{\sigma\mu} T^{\alpha\sigma\ldots}_{\lambda\nu\ldots} + \ldots - \Gamma^\sigma{}_{\lambda\mu} T^{\alpha\beta\ldots}_{\sigma\nu\ldots} - \Gamma^\sigma{}_{\nu\mu} T^{\alpha\beta\ldots}_{\lambda\sigma\ldots} - \ldots \tag{1.3.57}$$

COV DERIV OF TENSOR DENSITIES - MAYBE TENSOR DENSITIES ARE THEIR OWN SECTION. COULD BE QUICK, I ASSUME NABLA MU ROOTG =0

1.3.3 The Connection

In flat space we are used to the idea that the partial derivative commutes, i.e. $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, and this is trivially true in curved space too. However, the covariant derivative does not generally commute, $\nabla_\mu \nabla_\nu \neq \nabla_\nu \nabla_\mu$. Applying $\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$ to a scalar field φ gives,

$$\begin{aligned}
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\varphi &= \nabla_\mu \nabla_\nu \varphi - \nabla_\nu \nabla_\mu \varphi, && (1.3.58)\\
&= \nabla_\mu \partial_\nu \varphi - \nabla_\nu \partial_\mu \varphi, && (1.3.59)\\
&= (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\varphi - \Gamma^\sigma{}_{\nu\mu} \partial_\sigma \varphi + \Gamma^\sigma{}_{\mu\nu} \partial_\sigma \varphi, && (1.3.60)\\
&= (\Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu})\partial_\sigma \varphi, && (1.3.61)
\end{aligned}$$

where we used the fact that $\nabla_\mu \varphi = \partial_\mu \varphi$ for a scalar field. This non-commutativity of derivatives on a scalar field is known as torsion.

Torsion

If a connection is torsion free then $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\varphi = 0$ which implies $\Gamma^\sigma{}_{\nu\mu} = \Gamma^\sigma{}_{\mu\nu}$. This leads to two important tensor identities; first the antisymetric derivative of a co-vector,

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\nu - \underbrace{(\Gamma^\sigma{}_{\nu\mu} - \Gamma^\sigma{}_{\mu\nu})}_{=0} A_\sigma, \tag{1.3.62}$$

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\nu, \tag{1.3.63}$$

and the second identity,

$$\boldsymbol{\nabla_X Y} - \boldsymbol{\nabla_Y X} = (X^\mu \nabla_\mu Y^\nu - Y^\mu \nabla_\mu X^\nu)\boldsymbol{e}_\nu, \tag{1.3.64}$$

$$= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu + \Gamma^\nu{}_{\sigma\mu} X^\mu Y^\sigma - \Gamma^\nu{}_{\sigma\mu} Y^\mu X^\sigma)\boldsymbol{e}_\nu, \tag{1.3.65}$$

$$= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu + (\Gamma^\nu{}_{\sigma\mu} - \Gamma^\nu{}_{\mu\sigma}) X^\mu Y^\sigma)\boldsymbol{e}_\nu, \tag{1.3.66}$$

$$= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\boldsymbol{e}_\nu, \tag{1.3.67}$$

$$\boldsymbol{\nabla_X Y} - \boldsymbol{\nabla_Y X} = [\boldsymbol{X}, \boldsymbol{Y}]. \tag{1.3.68}$$

The commutator bracket $[\boldsymbol{X}, \boldsymbol{Y}]$ of two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined by

$$[X^\mu \partial_\mu, Y^\nu \partial_\nu] = X^\mu \partial_\mu (Y^\nu \partial_\nu) - Y^\nu \partial_\nu (X^\mu \partial_\mu), \tag{1.3.69}$$

$$= X^\mu \partial_\mu Y^\nu - Y^\nu \partial_\nu (X^\mu) \partial_\mu + X^\mu Y^\nu \partial_\mu \partial_\nu - Y^\nu X^\mu \partial_\nu \partial_\mu, \tag{1.3.70}$$

$$= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu, \tag{1.3.71}$$

where the basis vector $\boldsymbol{e}_\mu$ has been written as ∂_μ with the intention of acting on a function f over the manifold $\mathcal{M}$.

Metric Compatibility

Another useful property that can be imposed on the connection is metric compatibility; this is $\nabla_\mu g_{\rho\sigma} = 0$, where $\boldsymbol{g}$ is the metric tensor, which immediately tells us $\nabla_\mu g^{\alpha\beta} = 0$ as

$$\nabla_\mu \delta^\alpha_\rho = 0 \tag{1.3.72}$$

$$= \nabla_\mu (g^{\alpha\nu} g_{\nu\rho}), \tag{1.3.73}$$

$$= g_{\nu\rho} \nabla_\mu g^{\alpha\nu} + g^{\alpha\nu} \underbrace{\nabla_\mu g_{\nu\rho}}_{=0}, \tag{1.3.74}$$

which implies that $\nabla_\mu g^{\alpha\nu} = 0$. Demanding metric compatibility may seem a little arbitrary but turns out to have many nice algebraic properties such as the raising and lowering of indeces with the metric commuting with the covariant derivatives,

$$\nabla_\mu T^{\alpha\beta\dots} = \nabla_\mu g^{\alpha\rho} T_\rho{}^{\beta\dots} = g^{\alpha\rho} \nabla_\mu T_\rho{}^{\beta\dots} \tag{1.3.75}$$

and the derivative of a vector's length,

$$\nabla_\alpha (X^\mu X_\mu) = \nabla_\alpha (g_{\mu\nu} X^\mu X^\nu) = 2X^\mu \nabla_\alpha X_\mu = 2X_\mu \nabla_\alpha X^\mu. \tag{1.3.76}$$

The Levi Civita-Connection

A connection that obeys Eqs. (1.3.33,1.3.34 & 1.3.35), is both torsion-free and metric-compatible, can be demanded; this connection is called the Levi-Civita connection. The Levi-Civita connection will always be assumed from now and leads to a unique choice of connection coefficients,

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\nu\mu}), \tag{1.3.77}$$

which are also called Christoffel symbols of the second kind. It is also common to see the connection coefficients with the lowered index

$$\Gamma_{\sigma\mu\nu} = \frac{1}{2}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\nu\mu}), \tag{1.3.78}$$

this is also called a Christoffel symbol of the first kind. It is very important to note that even though the connection symbols may look like a tensor they are not a tensor. This can easily be seen from applying the tensor transformation law to the Christoffel symbol of the first kind,

$$\begin{aligned}
2\tilde{\Gamma}_{\sigma\mu\nu} &= \tilde{\partial}_\mu \tilde{g}_{\sigma\nu} + \tilde{\partial}_\nu \tilde{g}_{\mu\sigma} - \tilde{\partial}_\sigma \tilde{g}_{\nu\mu}, && (1.3.79)\\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\partial_\alpha\left(\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu} g_{\beta\gamma}\right) + \frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\partial_\gamma\left(\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} g_{\alpha\beta}\right) - \frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\partial_\beta\left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu} g_{\gamma\alpha}\right), && (1.3.80)\\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\left(\partial_\alpha g_{\beta\gamma} + \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\gamma\alpha}\right)\\
&\qquad + \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} g_{\beta\gamma}\partial_\alpha\left(\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\right) + \frac{\partial x^\gamma}{\partial \tilde{x}^\nu} g_{\alpha\beta}\partial_\gamma\left(\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\right) - \frac{\partial x^\beta}{\partial \tilde{x}^\sigma} g_{\gamma\alpha}\partial_\beta\left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\right), && (1.3.81)\\
&= 2\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\Gamma_{\beta\alpha\gamma}\\
&\qquad + \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} g_{\beta\gamma}\partial_\alpha\left(\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\right) + \frac{\partial x^\gamma}{\partial \tilde{x}^\nu} g_{\alpha\beta}\partial_\gamma\left(\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\right) - \frac{\partial x^\beta}{\partial \tilde{x}^\sigma} g_{\gamma\alpha}\partial_\beta\left(\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\right), && (1.3.82)\\
&= 2\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\sigma}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\Gamma_{\beta\alpha\gamma} + \Xi_{\sigma\mu\nu}. && (1.3.83)\\
& && (1.3.84)
\end{aligned}$$

As can be seen above, if $\Xi_{\sigma\mu\nu} = 0$ then the $\Gamma_{\sigma\mu\nu}$ would transform as an $(0,3)$ tensor, however this is not the case and the existance of nonzero $\Xi_{\sigma\mu\nu}$ means the Christoffel symbol of first kind is not a tensor but instead a symbol. The non-tensor nature of the Christoffel symbol of first kind is sufficient to prove that the Christoffel symbol of second kind is also not a tensor.

Lie Derivatives in the Levi Civita Connection

The covariant derivative ∇, described in section 1.3.2 can replace all partial derivatives in a Lie derivative due to the connection symbols $\left(\Gamma^{\mu}{}_{\rho\sigma}\right)$ cancelling out.

A special example of a Lie derivative is of the metric tensor, $\boldsymbol{g}$, giving

$$\begin{aligned}
(\mathcal{L}_\xi g)_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho, && (1.3.85)\\
&= \xi^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu}\nabla_\mu \xi^\rho + g_{\mu\rho}\nabla_\nu \xi^\rho, && (1.3.86)\\
& && (1.3.87)
\end{aligned}$$

where $\nabla_\rho g_{\mu\nu} = 0$ is assumed from metric compatibility, described in section 1.3.3. In the case the Lie derivative vanishes we get Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{1.3.88}$$

and a vector field $\boldsymbol{\xi}$ satisfying Killing's equation is called a Killing vector.

MAYBE DERIVE THE FORM OF BIG GAMMA?

Normal Coordinates

Consider the set of all affinely parameterised geodesics, parameterised with λ, passing through the point p on a manifold $\mathcal{M}$. At p, each geodesic has a tangent vector $X^\mu\big|_p$. For all geodesics, define p to be the origin with $\lambda = 0$ and $x^\mu = 0$. Following a geodesic associated with $X^\mu\big|_p$ to a parameter value of λ will map to a new point q close to p for small enough λ. Normal coordinates $x^\mu(\lambda)$ at $q(\lambda)$ are defined such that $x^\mu(\lambda) = \lambda X^\mu\big|_p$. Given that $X^\mu\big|_p$ is constant, the geodesic equation, from Eq. (1.2.71), becomes

$$\frac{\mathrm{d}^2 x^\mu(\lambda)}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\mu\nu}\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu(\lambda)}{\mathrm{d}\lambda} = \Gamma^\alpha{}_{\mu\nu}X^\mu\big|_p X^\nu\big|_p = 0. \tag{1.3.89}$$

Using the Levi-civita connection, the connection symbol must be symmetric in lower two indeces and this implies $\Gamma^\alpha{}_{\mu\nu} = 0$ as the geodesic equation must hold for generic $X^\mu\big|_p$. From the definition of the covariant derivative,

$$\partial_\alpha g_{\mu\nu} = \nabla_\alpha g_{\mu\nu} + \Gamma^\beta{}_{\alpha\nu}g_{\mu\beta} + \Gamma^\beta{}_{\alpha\mu}g_{\beta\nu}, \tag{1.3.90}$$

which must vanish in normal coordintes as $\Gamma^\alpha{}_{\mu\nu} = 0$ and the Levi-Civita connection demands $\nabla_\alpha g_{\mu\nu} = 0$. Therefore, it is possible to construct a coordinate system that at one point p both $\partial_\alpha g_{\mu\nu} = 0$ and $\Gamma^\alpha{}_{\mu\nu} = 0$; importantly $\partial_\alpha\partial_\beta g_{\mu\nu} \neq 0$. As well as the metric's first derivative vanishing at p, it is also possible to demand that $g_{\mu\nu}\big|_p = \eta_{\mu\nu}$, the Minkowski metric. This can be done with a set of ${}^{(j)}X^\mu\big|_p$, one for each normal coordinate x^j, that satisfy

$$\boldsymbol{g}({}^{(j)}\boldsymbol{X}\big|_p, {}^{(k)}\boldsymbol{X}\big|_p) = \eta_{jk}. \tag{1.3.91}$$

1.3.4 Curvature Tensors

We have already seen that with the Levi-Civita connection, the commuted derivative of a scalar field vanishes. But taking the commuted derivative of a vector field $\boldsymbol{X}$ gives,

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)X^\sigma = (\partial_\mu\nabla_\nu - \partial_\nu\nabla_\mu)X^\sigma + (\Gamma^\sigma{}_{\mu\rho}\nabla_\nu - \Gamma^\sigma{}_{\nu\rho}\nabla_\mu)X^\rho - \underbrace{(\Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu})}_{=0}\nabla_\rho X^\sigma, \tag{1.3.92}$$

$$= \underbrace{(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)}_{=0}X^\sigma + (\partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho})X^\rho + (\Gamma^\sigma{}_{\mu\rho}\nabla_\nu - \Gamma^\sigma{}_{\nu\rho}\nabla_\mu)X^\rho, \tag{1.3.93}$$

$$= (\partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho})X^\rho + (\Gamma^\sigma{}_{\mu\rho}\partial_\nu - \Gamma^\sigma{}_{\nu\rho}\partial_\mu)X^\rho + (\Gamma^\sigma{}_{\mu\rho}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\sigma{}_{\nu\rho}\Gamma^\rho{}_{\mu\lambda})X^\lambda, \tag{1.3.94}$$

$$= X^\rho(\partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho}) + (\Gamma^\sigma{}_{\mu\rho}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\sigma{}_{\nu\rho}\Gamma^\rho{}_{\mu\lambda})X^\lambda, \tag{1.3.95}$$

where one should note that whenever a term appears after a derivative here it is to be differentiated, even if it is outside a bracket. We can introduce the Riemann tensor here from

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)X^\sigma = R^\sigma{}_{\rho\mu\nu}X^\rho, \tag{1.3.96}$$

and setting $\boldsymbol{X} = \boldsymbol{e}_\rho$, the coordinate basis vector associated with the x^ρ coordinate, the Riemann tensor can be written as

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\rho}. \tag{1.3.97}$$

Symmetries of the Riemann Tensor

Now we will discuss the symmetries of the Riemann tensor. Firstly, from the definition of the Riemann tensor, it follows that $R^\sigma{}_{\rho\mu\nu} = -R^\sigma{}_{\rho\nu\mu}$; this can be written succinctly as $R^\sigma{}_{\rho[\mu\nu]} = 0$. The next symmetry of the Riemann tensor will prove easy to derive using normal coordinates (described in [REF]) at a point p where $\Gamma = 0$ (but $\partial\Gamma \neq 0$) to get

$$R^\sigma{}_{\rho\mu\nu}\big|_p = \partial_\mu\Gamma^\sigma{}_{\nu\rho} - \partial_\nu\Gamma^\sigma{}_{\mu\rho}, \tag{1.3.98}$$

and it is simple to show that $R^{\sigma}_{[\rho\mu\nu]} = 0$ as

$$R^{\sigma}{}_{[\rho\mu\nu]}\big|_p = \partial_{[\mu}\Gamma^{\sigma}{}_{\nu\rho]} - \partial_{[\nu}\Gamma^{\sigma}{}_{\mu\rho]} = 0, \tag{1.3.99}$$

as the antisymmetrisation of any connection symbol like $\Gamma^{\sigma}{}_{[\nu\rho]} = 0$. Given that the tensor equation $R^{\sigma}_{[\rho\mu\nu]} = 0$ is true at p in normal coordinates then it is true in any coordinate system; ontop of this the point p was arbitrary so therefore $R^{\sigma}_{[\rho\mu\nu]} = 0$ holds globally.

The next symmetry of the Riemann tensor is $R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$. We can prove this again using normal coordiantes at a point p; here derivatives of the metric and it's inverse vanish, but second derivatives do not. The proof of the symmetry is as follows,

$$R_{\sigma\rho\mu\nu}\big|_p = g_{\lambda\sigma}\partial_\mu\Gamma^{\lambda}{}_{\nu\rho} - g_{\lambda\sigma}\partial_\nu\Gamma^{\lambda}{}_{\mu\rho}, \tag{1.3.100}$$

$$= \partial_\mu g_{\lambda\sigma}\Gamma^{\lambda}{}_{\nu\rho} - \partial_\nu g_{\lambda\sigma}\Gamma^{\lambda}{}_{\mu\rho}, \tag{1.3.101}$$

$$= \partial_\mu \Gamma_{\sigma\nu\rho} - \partial_\nu \Gamma_{\sigma\mu\rho}, \tag{1.3.102}$$

$$= \frac{1}{2}\left(\partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\mu\partial_\sigma g_{\rho\nu} + \partial_\nu\partial_\sigma g_{\rho\mu} - \partial_\nu\partial_\rho g_{\sigma\mu}\right), \tag{1.3.103}$$

and it is a simple to show that this final expression doesn't change under swapping indeces $\sigma \leftrightarrow \mu$ and $\rho \leftrightarrow \nu$.

The final symmetry of the Riemann tensor is the Bianchi identity, $\nabla_{[\lambda}R_{\sigma\rho]\mu\nu}=0$. Using normal coordinates at a point p, we can write

$$\nabla_\lambda R_{\sigma\rho\mu\nu}\big|_p = \partial_\lambda R_{\sigma\rho\mu\nu}\big|_p \tag{1.3.104}$$

as all the christoffel symbols generated by the covariant derivative cancel and therefore

$$2\nabla_\lambda R_{\sigma\rho\mu\nu}\big|_p = \partial_\lambda\partial_\mu\partial_\rho g_{\sigma\nu} - \partial_\lambda\partial_\mu\partial_\sigma g_{\rho\nu} + \partial_\lambda\partial_\nu\partial_\sigma g_{\rho\mu} - \partial_\lambda\partial_\nu\partial_\rho g_{\sigma\mu}. \tag{1.3.105}$$

Antisymmetrising over λ, ρ and σ makes each term vanish as the triple partial derivates always contain two of the antisymmetrised indeces and must vanish.

To summarise, we have the following symmetries of the Riemann tensor;

$$R_{\sigma\rho[\mu\nu]} = 0, \tag{1.3.106}$$

$$R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}, \tag{1.3.107}$$

$$\nabla_{[\lambda}R_{\sigma\rho]\mu\nu} = 0. \tag{1.3.108}$$

The first two of these can be used together to give another useful relation $R_{[\sigma\rho]\mu\nu} = 0$.

Contractions of the Riemann Tensor

Now that we have explored the Riemann tensor, it is time to introduce the Ricci tensor and Ricci scalar. The Ricci tensor $R_{\mu\nu}$ is simply defined by the unique, non-zero self contraction (or trace) of the Riemann tensor,

$$R_{\rho\mu} := R^{\mu}{}_{\rho\mu\nu} = R_{\sigma\rho\mu\nu}g^{\sigma\mu}. \tag{1.3.109}$$

Contracting the the Riemann tensor with $g^{\mu\nu}$ or $g^{\sigma\rho}$ would give zero due to the antisymmetries of those indeces in the tensor. Any other contractions, such as with $g^{\rho\mu}$ can be shown to be exactly the same (upto a minus sign) as contracting with $g^{\sigma\mu}$ using the symmetries of the Riemann tensor. The symmetries of the Riemann tensor guarentee that the Ricci tensor itself is symmetric. We can contract the Ricci tensor with itself (the same as taking the trace with the metric) to give us the Ricci scalar R,

$$R = g^{\rho\nu}R_{\rho\nu}. \tag{1.3.110}$$

We can also take the trace of the Bianchi identity in Eq. (1.3.108) which gives us

$$g^{\lambda\mu}g^{\rho\nu}(\nabla_\lambda R_{\sigma\rho\mu\nu} + \nabla_\rho R_{\lambda\sigma\mu\nu} + \nabla_\sigma R_{\rho\lambda\mu\nu}) = 0, \tag{1.3.111}$$
$$\nabla^\mu R_{\sigma\mu} + \nabla^\nu R_{\sigma\nu} - \nabla_\sigma R = 0, \tag{1.3.112}$$
$$\nabla^\mu R_{\mu\sigma} - \frac{1}{2}\nabla_\sigma R = 0. \tag{1.3.113}$$

Using the contracted Bianchi identity and $\boldsymbol{\nabla g} = 0$ from metric compatibility, we can define the (symmetric) Einstein tensor $G_{\mu\nu}$,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{1.3.114}$$
$$\nabla^\mu G_{\mu\nu} = \nabla^\mu R_{\mu\nu} - \frac{1}{2}\left(g_{\mu\nu}\nabla^\mu R + \nabla^\mu(g_{\mu\nu})R\right), \tag{1.3.115}$$
$$= \nabla^\mu R_{\mu\nu} - \frac{1}{2}\nabla_\nu R, \tag{1.3.116}$$
$$= 0. \tag{1.3.117}$$

The Einstein tensor $G_{\mu\nu}$ therefore has a vanishing divergence $\nabla_\mu G^{\mu\nu} = 0$.

One more useful contraction of the Riemann tensor is with a second Riemann tensor which gives the Kretschmann scalar k,

$$k := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{1.3.118}$$

Both k and R are scalar fields so they are a coordiante invariant curvature measure; however section 1.4.6 will have more use of the Kretschmann scalar.

NORMAL COORDS SOMEWHERE, MAKES EXPANSIONS AND SOME DERIVATIONS EASIER, USE NORMAL COORDS FOR SYMMETRIES OF RIEMANN

MAYBE ADD THE [] AND () SYMMETRY DEFINITION IN PRE-DEFINITONS? ALSO CHECK THIS WITH THE DIFF FORMS SECTION

DEGREES OF FREEDOM

1.3.5 The Divergence Theorem

There is a generalisation of the divergence theorem to non-flat spaces, using differential geometry, which will be extremely useful in section [REF]. First, we will have to find a convenient form of the divergence $\nabla_\mu X^\mu$ of a vector $\boldsymbol{X}$. Expanding the covariant deriative gives

$$\nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^\mu{}_{\mu\nu}X^\nu, \tag{1.3.119}$$
$$= \partial_\mu X^\mu + \frac{1}{2}g^{\mu\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)X^\nu, \tag{1.3.120}$$
$$= \partial_\mu X^\mu + \frac{1}{2}g^{\mu\rho}\partial_\nu g_{\mu\rho}X^\nu. \tag{1.3.121}$$

To simplify any furthar, we need to prove a matrix identity for a symmetric real matrix $\boldsymbol{M}$ with determinant M,

$$M^{-1}\partial_\mu M = M^{-1}_{ij}\partial_\mu M_{ij} = \mathrm{Tr}(\boldsymbol{M}^{-1}\partial_\mu \boldsymbol{M}). \tag{1.3.122}$$

Simplifying the left hand side in terms of the eigenvalues λ_i of $\boldsymbol{M}$. Given that $M = \det\{\boldsymbol{M}\} = \prod_i \lambda_i$ we can easily write

$$
\begin{aligned}
M^{-1}\partial_\mu M &= \partial_\mu \ln(|M|), && (1.3.123)\\
&= \partial_\mu \ln\left(\left|\prod_i \lambda_i\right|\right), && (1.3.124)\\
&= \partial_\mu \sum_i \ln(|\lambda_i|), && (1.3.125)\\
&= \sum_i \lambda_k^{-1}\partial_\mu \lambda_k. && (1.3.126)
\end{aligned}
$$

Now we will show that the right hand side of Eq. (1.3.122) also equals this. To do this we start by decomposing $\boldsymbol{M}$ into a diagonal matrix $\boldsymbol{D}$ like

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{O}^{-1}\boldsymbol{D}\boldsymbol{O}, && (1.3.127)\\
\boldsymbol{M}^{-1} &= \boldsymbol{O}^{-1}\boldsymbol{D}^{-1}\boldsymbol{O}, && (1.3.128)
\end{aligned}
$$

then using the fact that $\mathrm{Tr}(\boldsymbol{AB}....\boldsymbol{CD}) = \mathrm{Tr}(\boldsymbol{DAB}....\boldsymbol{C})$ for matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ... and $\boldsymbol{D}$,

$$
\begin{aligned}
\mathrm{Tr}(\boldsymbol{M}^{-1}\partial_\mu \boldsymbol{M}) &= \mathrm{Tr}(\boldsymbol{O}^{-1}\boldsymbol{D}^{-1}\boldsymbol{O}\partial_\mu(\boldsymbol{O}^{-1}\boldsymbol{D}\boldsymbol{O})), && (1.3.129)\\
&= \mathrm{Tr}(\boldsymbol{D}^{-1}\partial_\mu \boldsymbol{D}) + \mathrm{Tr}(\boldsymbol{O}^{-1}\partial_\mu \boldsymbol{O}) + \mathrm{Tr}(\boldsymbol{O}\partial_\mu \boldsymbol{O}^{-1}), && (1.3.130)\\
&= \mathrm{Tr}(\boldsymbol{D}^{-1}\partial_\mu \boldsymbol{D}) + \mathrm{Tr}(\underbrace{\partial_\mu(\boldsymbol{O}^{-1}\boldsymbol{O})}_{=0}), && (1.3.131)\\
&= \mathrm{Tr}(\boldsymbol{D}^{-1}\partial_\mu \boldsymbol{D}). && (1.3.132)
\end{aligned}
$$

Given that $\boldsymbol{D}$ is the diagonal matrix composed of the eigenvalues λ_i then it follow that,

$$
\begin{aligned}
\boldsymbol{D} &= \mathrm{Diag}\{\lambda_1, \lambda_2, ..., \lambda_n\}, && (1.3.133)\\
\partial_\mu \boldsymbol{D} &= \mathrm{Diag}\{\partial_\mu\lambda_1, \partial_\mu\lambda_2, ..., \partial_\mu\lambda_n\}, && (1.3.134)\\
\boldsymbol{D}^{-1} &= \mathrm{Diag}\{\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_n^{-1}\}, && (1.3.135)
\end{aligned}
$$

then finally $\mathrm{Tr}(\boldsymbol{D}^{-1}\partial_\mu \boldsymbol{D})$ can be evaluated in terms of the λ_i as follows,

$$
\begin{aligned}
\mathrm{Tr}(\boldsymbol{D}^{-1}\partial_\mu \boldsymbol{D}) &= \sum_{ij} D_{ij}^{-1}\partial_\mu D_{ij}, && (1.3.136)\\
&= \sum_i D_{ii}^{-1}\partial_\mu D_{ii}, && (1.3.137)\\
&= \sum_i \lambda_i^{-1}\partial_\mu \lambda_i, && (1.3.138)
\end{aligned}
$$

which proves that Eq. (1.3.122) is true. Applying Eq. (1.3.122) to the metric $\boldsymbol{g}$ gives,

$$
g^{\mu\rho}\partial_\nu g_{\mu\rho} = g^{-1}\partial_\nu g, \tag{1.3.139}
$$

and the covariant divergence $\nabla_\mu X^\mu$ of a vector $\boldsymbol{X}$ simplifies to

$$
\begin{aligned}
\nabla_\mu X^\mu &= \partial_\mu X^\mu + \frac{1}{2g}\partial_\mu g, && (1.3.140)\\
&= \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}X^\mu\right), && (1.3.141)
\end{aligned}
$$

which in the standard pseudo-Riemannian spacetime is often written

$$
\boldsymbol{\nabla}\cdot\boldsymbol{X} = \nabla_\mu X^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}X^\mu). \tag{1.3.142}
$$

This equation will have much use later in section [REF] where the divergence of a vector will be integrated over a finite n-dimensional volume M in $\mathcal{M}$. In order to do this, the divergence theorem of curved space is used, this is

$$\int_M \boldsymbol{\nabla} \cdot \boldsymbol{X} \sqrt{|^{(n)}g|}\, \mathrm{d}^n x = \sum_i \int_{\partial M_i} \boldsymbol{s}^{(i)} \cdot \boldsymbol{X} \sqrt{|^{(n-1)}g^{(i)}|}\, \mathrm{d}^{n-1}x, \tag{1.3.143}$$

where the surface of M is divided into a set of $(n-1)$-dimensional surfaces ∂M_i; for each of these surfaces there is an $(n-1)$-dimensional metric $^{(n-1)}\boldsymbol{g}^{(i)}$ with determinant $^{(n-1)}g^{(i)}$ and unit normal vector $\boldsymbol{s}^{(i)}$ with $^{(n-1)}\boldsymbol{g}^{(i)}(\boldsymbol{s}^{(i)}, \boldsymbol{s}^{(i)}) = \pm 1$. When dealing with a pseudo-Riemannian manifold, we need to take care of the direction of $\boldsymbol{s}^{(i)}$; in the case that $\boldsymbol{s}^{(i)}$ is timelike it should be in-directed, and if it's spacelike then it should be out-directed.

CHECK THE LAST STATEMENT

MAYBE DERIVE DIV THEOREM, OTHERWISE REFERENCE IT

1.4 Relativity

1.4.1 Special Relativity

In the nineteenth century, it was widely beleived that the universe was permeated by an invisible luminous aether. It was thought that light travels at a fixed speed through this aether; the aether could be thought of as a universal rest frame. A consequnece of this is that moving towards/away from a light source would cause the wavelength of light to get shorter/longer in a similar fashion to the doppler effect. With that line of thought one could measure the earths speed through this rest frame by setting up an interferometer experiment. An interferometer sends a light beam through a splitter, dividing the beam into two perpendicular paths. The two beams then reach a seperate mirror and are reflected to a half-mirror that recombines the two beams after they have travelled identical distance. If the time taken for the two beams to complete their identical length journeys differs then an interference pattern will be seen at a detector placed after the half-mirror due to the phase change between the two beams. In 1887 Michelson and Morley conducted their famous interferometer experiment to measure the earth's velocity through the aether. They expected to see interfernece when the different beams made differnet angles with the velocity through the aether. However, no matter which orientation they put their experiment there was no interference pattern. This implied that light moved at exactly the same speed in any direction; this is only possible if earth is in the rest frame of the aether, but this cannot possibly be true as the planet accelerates round a circular path around a sun that is following a bigger circular path and so on. This famous result demonstrated that the speed of light was constant in all inertial frames - a result that defied newtonian mechanics. This was the first and biggest hint that a new theory of dynamics was needed.

In 1905, Einstein published "On the Electrodynamics of Moving Bodies" which contained a description of Special Relativity (SR) [CHECK THIS]. SR is essentially the idea that the laws (excluding gravity) of physics are the same in any inertial (non-accelerating) rest frame - the consequence of this is that you cannot measure the velocity of your own rest frame as no frame is special. One big problem with SR is that it does not properly describe gravity and thus describes an infinite vaccum universe; this is a problem that will be adressed in the next sections. SR alone contains many interesting results such as time dilation, length contraction and the inclusion of time into the metric.

Minkowski Space

In Newtonian physics the metric of flat space, using cartesian coordiantes, is

$$g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.4.1}$$

with the line element

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{1.4.2}$$

In SR time is promoted to a dimension and the metric is over space and time (spacetime). The spacetime metric has a single negative eigenvalue and hence negative determinant. In cartesian coordinates, the flat spacetime metric is,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.4.3}$$

and line element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{1.4.4}$$

where c is the speed of light which will eb set to unity again. The line element is often written $\mathrm{d}s^2 = -\mathrm{d}\tau^2$ where τ is the *proper time* experienced by an observer. Having a metric over spacetime is a non-intuitive concept step where time is included into geometry and paths through space and time (spacetime) can have negative length. This flat spacetime is very important and is called the Minkowski spacetime.

Worldlines and Causality

Consider a particle moving through the rest frame with some world-line (path through spacetime) described by $x^\mu(\tau)$. Without loss of generality the motion can be restricted to the x axis. The world-line of this particle is given by $x^\mu = \{t, \int_{t_0}^t v(t')\mathrm{d}t' + v(t_0), 0, 0\}$. The infinitessimal form of this description is $\mathrm{d}x^\mu = \{\mathrm{d}t, v\,\mathrm{d}t, 0, 0\}$ and the proper time associated with the interval is

$$\mathrm{d}\tau^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = (1 - v^2)\mathrm{d}t^2. \tag{1.4.5}$$

This equation says that if $v < 1$, where it should be remembered that the speed of light here is $c = 1$, then proper time τ and coordinate time t both flow in the same direction. Curves like this have $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu < 0$ and are called *time-like*. If $v = 1$, which is only possible for a massless particle, then the path is called *light-like*. For $v > 1$, the path is unphysical as nothing can travel faster than the speed of light, it would correspond to an observer travelling backwards in time. A curve x^μ with $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu > 0$ is instead called a *space-like* curve and cannot be tranversed by matter or information.

MAYBEW REWRITE THIS JUST FROM A DELTA T DELTA X PERSPECTIVE AND THEN USE IT TO DESCRIBE WORLDLINES AFTER?? MAYBE JUST DO THIS IN THE gr SECTION AS THERES NO REASON NOT TO

MAYBE DO NOW THAT WORLDLINES ARE GEODESICS?

1.4.2 Physics in Special Relativity

BOOSTS?

Many physical theories can be written using tensor calculus on this flat spacetime; the correct description of electromagnetism was itself a large reason for the development of special relativity.

The Wave Equation

One very simple example is the wave equation governing the dynamics of a scalar field ϕ,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(x^i, t) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(x^i, t) = 0, \tag{1.4.6}$$

using cartesian coordinates and c as the speed of light which we will set to one. It should be noted that this equation only holds in one coordinate system, changing coordinate system would change the equation of motion. In SR, using the language of tensor calculus, we can write this as

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0, \tag{1.4.7}$$

which is not only much simpler but is also true in any inertial frame (using Cartesian coordinates) as is one of the postulates of special relativity.

Electromagnetism

Electromagnetism can also be cast in a very succinct form using this field theory notation. Traditionally Maxwell's equations of electromagnetism are written as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{1.4.8}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.4.9}$$

$$\nabla \times \boldsymbol{E} + \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = 0, \tag{1.4.10}$$

$$\nabla \times \boldsymbol{B} - \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} = \mu_0\boldsymbol{J} \tag{1.4.11}$$

for charge density ρ, current density $\boldsymbol{J}$. Note these differential equations are not written using the language of differential geometry and tensor calculus. In SR, the current density and charge is promoted to a single 4-vector $j^\mu = \{\rho, j^i\}$ and the 6 degrees of freedom of the electromagnetic field are encoded in the components $F_{\mu\nu}$ of an antisymmetric tensor $\boldsymbol{F}$. The four electromagnetic potentials ϕ and A_i, defined by $E_i = -\partial_i\phi - \partial_t A_i$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, are also combined into one 4-vector $A_\mu = \{-\phi, A_i\}$. In cartesian coordinates, the Electromagnetic tensor $\boldsymbol{F}$ is,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{1.4.12}$$

Transforming A_μ with the electromagnetic gauge transformation like $A_\mu \to A_\mu + \partial_\mu f$, for some scalar field f, leaves the physically measureable field $F_{\mu\nu}$ unchanged,

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu) - \partial_\nu(A_\mu + \partial_\mu), \tag{1.4.13}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{\partial_\mu\partial_\nu f - \partial_\nu\partial_\mu f}_{=0}, \tag{1.4.14}$$

$$= F_{\mu\nu}. \tag{1.4.15}$$

In SR the Maxwell Eqs. (1.4.8) and (1.4.11) can be identically represented as

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \tag{1.4.16}$$

and the other two Maxwell Eqs. (1.4.9) and (1.4.10) are identically true from computing $\partial_{[\mu}F_{\alpha\beta]}$,

$$\partial_{[\mu}F_{\alpha\beta]} = \partial_\mu\partial_\alpha A_\beta - \partial_\mu\partial_\beta A_\alpha + \partial_\alpha\partial_\beta A_\mu - \partial_\alpha\partial_\mu A_\beta\partial_\beta\partial_\mu A_\alpha - \partial_\beta\partial_\alpha A_\mu, \tag{1.4.17}$$

$$\partial_{[\mu}F_{\alpha\beta]} = 0. \tag{1.4.18}$$

talk about postulates of SR.

Boosts

stress tensors?

quick conservation law

1.4.3 Physics in Curved Space

It is a commonly known result of newtonian physics that in a rotating reference frame, with angular velocity ω, a test particle will experience three fictitious forces; the centrifugal force that grows with distance from the origin, the coriolis force that depends on the velocity of the test particle, and the Euler force depending on $\partial_t \omega$. From the point of view of an observer in the rotating frame a test particle would appear to accelerate which is a violation of Newton's first law if there is no external force. The curved path followed by the particle in the rotating reference frame is of course a constant speed straight line in the non-rotating frame, and therefore the particle follows a geodesic in the inertial frame. It is possible (but very involved) to do a coordinate transformation from the rest frame to the inertial frame and find the metric $\tilde{\eta}_{\mu\nu}$ of the rotating frame; it would then be possible to compute the geodesics in the rotating frame using Eq. (1.5.12).

In a similar way to the fictitious forces arising in rotating frames, the gravitational force can be described as a fictitious force. In the absense of rotation, the path of a particle moving without external force is seen to accelerate towards massive bodies. In other words, matter density deflects particle paths towards itself. This is a property also possesed by curved spaces; the path followed by a free particle on a non-flat spacetime can bend. MENTION MATTER IS SIMILAR TO CURVATURE THEN. The brilliant insight by Einstein was that upgrading the flat spacetime of special relativity to a curved spacetime would give a relativistic theory of gravity; this theory is general relativity. In the same way that fictitious forces arise from departing from an inertial frame with rotation, the gravitational force is a fictitious force arising from departing from an inertial frame; the difference is that an inertial frame now must follow a geodesic. This explains why an observer standing on earth experiences a reaction force, the frame of teh observer is not following a geodesic, or not free falling, therefore to counteract the fictitious gravitational force generated the ground must push back with a reaction force.

Given that POSTULATE says the local laws of physics in a free falling frame are indistinguishable from special relativity, any equation of motions that we want to hold in general relativity must agree with special relativity in the low curvature limit. As a general rule, if there is a law of physics expressed as a differential equation on Minkowski space that we want to use in curved space, we must replace all partial derivatives of fields with co-variant derivatives of tensor fields; this process is called the minimal coupling approach MAYBE MAKE THIS MORE OBVIOUS WITH LAGRANGEANS?. An easy example is the wave equation from Eq. (1.4.7),

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0 \to g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0 \tag{1.4.19}$$

where we had to replace $\partial_\mu \to \nabla_\mu$ and replace the Minkowski metric $\boldsymbol{\eta}$ with the curved space metric $g_{\mu\nu}$ like $\boldsymbol{\eta} \to \boldsymbol{g}$. Writing the laws of physics as tensor equations is extremely useful as they must hold for over all points in a curved spacetime in any coordinate system; the power of these equations is that they can be written without reference to an explicit coordinate system. If a coordinate system is then picked, assuming knowledge of $g_{\mu\nu}$, then the wave equation becomes

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi, \tag{1.4.20}$$

$$= g^{\mu\nu}\partial_\mu\partial_\nu\phi - g^{\mu\nu}\Gamma^\rho{}_{\mu\nu}\partial_\rho\phi, \tag{1.4.21}$$

in terms of partial derivatives. There is the small question of uniqueness here, what is to stop us adding arbitrary amounts and types of terms that vanish in the no curvature limit. For a simple example one is free to choose the wave equation to be

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + f(x^\mu)R^n = 0, \tag{1.4.22}$$

for constant n, function f and the Ricci scalar R discussed in REF. This equation certainly returns the regular wave equation in the low curvature limit where $R \to 0$. The general rule is to keep things simple and terms such as R which are proportional to second order derivaties of the metric are thought to be less dominant that terms such as ${\Gamma^{\mu}}_{\rho\sigma}$ which are proportional to first derivatives of the metric. Navigating this minefield of which terms to include in the laws of physics leads to the topic of modified gravity REF, the next simplest theories beyond general relativity.

As another quick example the Maxwell equations are very simple to extend to curved space

maybe principle of minimal coupling to gravity is easier to understand with lagrangeans?

do postulates of GR and SR better?

decide how to deal with geodesic / path in the early parts of this section

1.4.4 The Stress-Energy-Momentum Tensor

At the heart of field theory in physics is the stress-energy-momentum tensor $\boldsymbol{T}$, also called the energy-momentum tensor or stress tensor for short. Roughly speaking, component T^{00} is energy density, components $T^{0i} = T^{i0}$ contain energy flux or momentum density and components $T^{ij} = T^{ji}$ contain momentum fluxes. The diagonal part of T^{ij} can also be though of as containing pressure and the off-diagonal terms containing shear stress. In flat space, the conservation of energy and momentum can be written as

$$\partial_\mu T^{\mu\nu} = 0, \tag{1.4.23}$$

in the absence of external forces. Equation (1.4.23) is also called the continuity equation and can be split into two sets of familiar equations,

$$\partial_0 T^{00} = -\partial_i T^{i0}, \tag{1.4.24}$$

$$\partial_0 T^{0j} = -\partial_i T^{ij}, \tag{1.4.25}$$

where the first equation states *"The rate of change of energy density is equal and opposite to the divergence of energy flux density"* and the second equation states *"The rate of change of momentum density is equal and opposite to the divergence of momentum flux density"*.

The stress tensor is also very useful in curved space and the continuity equation becomes,

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + {\Gamma^{\mu}}_{\mu\rho} T^{\rho\nu} + {\Gamma^{\nu}}_{\mu\rho} T^{\mu\rho} = 0, \tag{1.4.26}$$

which can be rewtitten using Eq. (1.3.142) as

$$\partial_\mu(\sqrt{-g}T^{\mu\nu}) = -\sqrt{-g}{\Gamma^{\nu}}_{\mu\rho} T^{\mu\rho}, \tag{1.4.27}$$

$$\partial_\mu(\mathcal{T}^{\mu\nu}) = -{\Gamma^{\nu}}_{\mu\rho} \mathcal{T}^{\mu\rho}, \tag{1.4.28}$$

where the second equation writes the stress tensor as a tensor density $\boldsymbol{\mathcal{T}} = \sqrt{-g}\boldsymbol{T}$ making the equation resemble the flat space continuity equation more closely. The take away message is that traditional continuity of energy and momentum no longer hold for curved spaces. This point will be revisited in section REF where a differnet type of variable is defined to try recover traditional continuity.

1.4.5 The Einstein Equation

Building on the vague notion of matter causing spacetime curvature it would be helpful to have a mathematical law saying how much curvature is caused by a matter distribution. The first guess that Einstein arrived at was to write $R_{\mu\nu} = kT_{\mu\nu}$ for some constant k. The problem is that given $\nabla_\mu T^{\mu\nu} = 0$ is the generic equation of continuity for matter, it would imply $\nabla_\mu R^{\mu\nu}$ vanishes which is not generally true. As shown in Eq. (1.3.116), the Einstein tensor $G_{\mu\nu}$ does satisfy $\nabla_\mu G^{\mu\nu} = 0$; the next simplest guess at a physical law for spacetime curvature would be $G_{\mu\nu} = kT_{\mu\nu}$. Remarkebly this turns out to be correct and has succesfully described all gravitational physics to date.

SOME EARLY EXPERIMENTAL STUFF LIKE PRECESSION, GRAVITATIONAL TIEM DILATION, LIGHT RAY DEFLECTION, RED SHIFT FOR ATOMIC SPECTRA.

At the core of General Relativity is the Einstein Equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.4.29}$$

also called the Einstein Field Equations. This equation relates spacetime curvature, encoded in the Einstein tensor $G_{\mu\nu}$, to the matter distribution described by the stress tensor $T_{\mu\nu}$. To be able to solve this equation in the presence of matter a curvature dependant equation of motion to dictate how matter moves is required. This leads nicely to Wheeler's insightful one line summary of general relativity:

"Spacetime tells matter how to move; matter tells spacetime how to curve."

This desceptively simple equation can describe an infinite amount of vastly diverse spacetime geometries including regular flat Minkowski space, black holes, stars, planets, gravitational waves and even the entire universe; to properly describe the entire universe a small modification has to be made to this equation as shown in REF.

General Relativity in Vacuum

In the case of a vacuum spacetime with $T_{\mu\nu} = 0$, or one where the matter distribution is supposed to be so small that it does not cause a spacetime curvature backreaction, the Einstein equation simplifies greatly to

$$G_{\mu\nu} = 0. \tag{1.4.30}$$

Taking the trace of the above equation and the definition of $G_{\mu\nu}$ in Eq. (REF), we can see that

$$g^{\mu\nu} G_{\mu\nu} = 0 = g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = R(1 - \frac{D}{2}), \tag{1.4.31}$$

where D is the number of spacetime dimensions. Clearly for $D \neq 2$ a vanishing einstein implies a vanishing Ricci scalar R; if the Ricci scalar vanishes then $G_{\mu\nu} = R_{\mu\nu}$ and the Einstein equation in vacuum simplifies to

$$R_{\mu\nu} = 0. \tag{1.4.32}$$

In the special case of $D = 2$ it can be shown that $R_{\mu\nu} = R g_{\mu\nu}/2$ [MAYBE REF EARLIER SECTION ON RIEMANN TENSOR] and the Einstein tensor $G_{\mu\nu} \equiv 0$ which renders Einstein's equation nonsensical.

1.4.6 Black Holes

As it turns out, vacuum General Relativity can describe more than just Minkowski space. Arguably the most important family of solutions to Einstein's equations in vacuum are black holes. The first black hole solution was found by Karl Schwarzschild, whos surname fittingly means "*black shield*" in German. This solution, known as the Schwarzschild solution, was discovered in 1916 with the intention of computing the spacetime curvature in the vacuum about a spherically symmetric mass such as stars and planets. The solution for $g_{\mu\nu}$ was assumed to take the following ansatz,

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi^2\right) \tag{1.4.33}$$

which is manifestly spherically symmetric and static; in the case $A(r) = B(r) = 1$ the solution is exactly Minkowski space in spherical polar coordinates. Schwarzschild saught a solution of this form that at large radius as $r \to \infty$ the functions $A \to 1$ and $B \to 1$, this means at large radius the solution tends to flat space. By solving $R_{\mu\nu} = 0$ from REF, Schwarzschild found a solution of the form

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2Gm}{rc^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi^2\right), \tag{1.4.34}$$

known as the Schwarzschild solution. The constants G and c are included for completeness, but are equal to one in Planck units. The Schwarzschild solution describes the spacetime about a non-spinning sphere of mass m and as $r \to \infty$ or $m \to 0$ we do approach the vacuum Minkowski spacetime as desired.

Polar-Areal Coordinates

The type of coordinates used in the Schwarzschild solution are called polar-areal coordinates; these are coordiantes that satisfy $g_{\theta\theta} = r^2$, $g_{\phi\phi} = \sin^2(\theta) r^2$ and otherwise $g_{\mu\phi} = g_{\mu\theta} = 0$ CHECK THIS. Polar areal coordinates return a length of $2\pi r_0$ when integrating the length of a complete circle at fixed radius r_0. This can be calculated using Eq. (1.2.56) and integrating with respect to ϕ around a circle while keeping $t = t_0$, $r = r_0$ and $\theta = \pi/2$ constant,

$$\int_0^{2\pi} \sqrt{g_{\phi\phi}}\Big|_{t=t_0, r=r_0, \theta=\frac{\pi}{2}} \mathrm{d}\phi = r_0 \sin\left(\frac{\pi}{2}\right) \int_0^{2\pi} \mathrm{d}\phi = 2\pi r_0, \tag{1.4.35}$$

which is the same result at in regular flat space. Using Eq. (1.2.64) the surface area of a sphere with $t = t_0$, $r = r_0$ can be calculated like

$$\int_0^{\pi} \left[\int_0^{2\pi} \sqrt{g_{\phi\phi} g_{\theta\theta}}\Big|_{t=t_0, r=r_0} \mathrm{d}\phi\right] \mathrm{d}\theta = r_0^2 \int_0^{\pi} \left[\int_0^{2\pi} \mathrm{d}\phi\right] \sin^2(\theta)\, \mathrm{d}\theta = 4\pi r_0^2, \tag{1.4.36}$$

where $\sqrt{g_{\phi\phi} g_{\theta\theta}}$ is the determinant of the metric on the two-dimensional surface defined by $t = t_0$ and $r = r_0$. Both the circumference of a circle and the area of a sphere are the same as would be in flat space, this property comes from the metric ansatz in Eq. (1.4.33).

Coordinate Singularities and Physical Singularities

There is obviously some kind problem at $r = 2mG/c^2$, known as the Schwarzschild radius, as g_{rr} diverges here. For any planet or star observed, the radius of the object would be much larger than the Schwarzschild metric and so the solution should be seen as invalid inside the object as it has been derived in vacuum. For a valid solution to Einstein's equations inside a spherically symmetric star or planet a non-zero stress tensor must be assumed, this is explored in more detail in REF.

What Schwarzschild didn't realise before his unfortunately early death was that his solution could infact be trusted down to radii well inside the schwarzschild radius $r_s = 2m$. This solution describes the eternal, non-spinning black hole of mass m called the Schwarzschild black hole. The problem at radius $r = 2m$ (where we have now set $c = G = 1$) is due to the choice of coordinates and is not physically problematic. An easy way to show this is to compute a curvature scalar for the spacetime and show that it is smooth at $r = 2m$. The first choice of curvature scalar would be the Ricci scalar, but this was assumed to be zero in vaccum so is not useful here. Another curvature scalar is the Kretschmann scalar k, defined in Eq. (1.3.118), which does not generally vanish in vacuum. Vaccum general relativity asserts that $R_{\mu\nu} = 0$, while that guarentees that $R = 0$ it does not guarentee that $R_{\mu\nu\rho\sigma} = 0$. Calculating the Kretschmann scalar for the Schwarzschild metric gives

$$k_{sc} = \frac{48m^2}{r^6}. \tag{1.4.37}$$

As can be seen, k is continuous and infinitely differentiable at $r = 2m$ but as $r \to 0$, $k \to \infty$ and there is a real coordinate independant singularity called a physical singularity. Scalar curvature invariants are useful as if they diverge in one coordinate system then they must diverge in all coordinate systems as they do not transform under coordinate transformations. It should be noted that $\sqrt{-g}$ is a scalar density and not a true scalar so cannot be used as a scalar curvature invariant.

To remove the coordinate singularity at $r = 2m$, new coordinate singularities can be introduced. One example is to use ingoing-Eddington-Finkelstein coordinates $\{v, s, \theta, \phi\}$ defined by

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \left(1 - \frac{2m}{r}\right), \tag{1.4.38}$$

$$v = t + s, \tag{1.4.39}$$

which transforms the line element to

$$g_{\mu\nu}x^\mu x^\nu = -\left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\left(\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi^2\right), \tag{1.4.40}$$

and as can be seen the metric no longer diverges at $r = 2m$. Being careful to notice that now the metric is not diagonal, the metric determinant can be calculated, giving $\sqrt{-g} = r^2\sin(\theta)$. Given that the metric is finite at $r = 2m$ and the metric determinant is non-zero, the metric inverse is guarenteed to be well behaved at $r = 2m$ aswell. Therefore it has been demonstrated that the singularity at $r = 2m$ in Schwarzschild polar-areal coordinates is a coordinate singularity that vanishes when usingingoing-Eddington-Finkelstein coordinates; therefore there is no physical singularity.

Throughout this section we have ignored the fact that the inverse metric also diverges as $\theta \to 0$ or $\theta \to \pi$, this is a coordinate singularity that is present in flat space (which is equivalent to the Schwarzshild spacetime with $m = 0$). This coordinate singularity arises in flat space due to the azimuthal angle ϕ being undefined at $\theta = 0$ and $\theta = \pi$. There are no physical singularities in flat space, as you might expect, and this coordinate singularity vanishes when using cartesian coordinates.

Isotropic Coordinates

A coordinate system that will be very useful later on in REF is the isotropic coordinate system. Isotropic coordinates are have the line element,

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\Omega(r)^2\mathrm{d}t^2 + \Psi^2(r)\left(\underbrace{\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right)}_{\text{flat space}}\right) \tag{1.4.41}$$

where the flat space line element is multiplied by a scale factor $\Psi^2(r)$. The reason these coordinates are called isotropic coordinates is that each spatial direction is equivalent when using Cartesian coordinates, as can be seen by writing the line element,

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\Omega(r)^2\mathrm{d}t^2 + \Psi^2(r)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{1.4.42}$$

where $r^2 = x^2 + y^2 + z^2$. The Schwarzschild black hole solution can be expressed in isotropic coordinates aswell,

$$g_{\mu\nu}\mathrm{d}x^\mu x^\nu = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2\mathrm{d}t^2 + \left(1 + \frac{m}{2r}\right)^4\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right)\right) \tag{1.4.43}$$

$$= -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2\mathrm{d}t^2 + \left(1 + \frac{m}{2r}\right)^4\mathrm{d}s_E^2, \tag{1.4.44}$$

where $\mathrm{d}s_E^2$ is the three-dimensional Euclidean space. As $r \to \infty$ the line element reduces to the Minkowski space one. The radius $r = m/2$ has the same coordinate singularity that was seen in polar areal coordinates and at first glance one might think that there is a physical singularity at $r = 0$. However, if a new radial coordiante ξ is used, where $r = \frac{m^2}{4\xi}$ for some constant, the line element becomes

$$g_{\mu\nu}\mathrm{d}x^\mu x^\nu = -\left(\frac{1 - \frac{m}{2\xi}}{1 + \frac{m}{2\xi}}\right)^2\mathrm{d}t^2 + \left(1 + \frac{m}{2\xi}\right)^4\left(\mathrm{d}\xi^2 + \xi^2(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2)\right). \tag{1.4.45}$$

This is a remarkeable result, inverting the radial coordinate about $r = 2/m$ has returned an exactly identical metric. Given that at $r = \infty$ we have flat space this implies that at $\xi = \infty$ (or $r = 0$) there is another separate flat space, not a physical singularity might have been thought. This second flat space as $\xi \to \infty$ is often called the alternate (PARALLEL MAYBE) universe and they are joined by the intraversable Einstein-Rosen bridge at $r = 2/m$. The reason that the physical singularity at $r = 0$ does not appear in isotropic coordinates is desceptively simple; in isotropic coordinates $r = 0$ does not correspond to the same point on the manifold as $r = 0$ does in the polar areal gauge for any value of t, θ or ϕ. Infact, the physical singularity of the black hole is outside of the patch on the manifold covered by isotropic coordinates. The idea of pathes is covered later EXPAND THIS WITH PENROSE MAYBE/

As described in section REF, spherically symmetric non-vacuum solutions to Einstein's equations can be found using isotropic coordinates; these solutions represent stars, planets and exotic balls of matter. The isotropic gauge is used when superposing these solutions later on in sections REF. EXPAND THIS

Penrose Diagrams and the Causal Structure of the Schwarzschild Black Hole

pics/scbh_penrose.png

Figure 1.2: Diagram for proof of Penrose diagram of a Schwarzschild diagram.

Spinning, Charged Black Holes and Higher Dimensions

do polar areal, then transform to isotropic for use later. then talk about other types of black holes. talk about the horizon maybe and causal disconnect

The subject of black holes in General Relativity is huge.

refernce kerr don't bother writing it

grav time dilation - maye put this in the general GR bit, along with precession mercuries orbit?

ref higher dimensions maybe

BIRKOFFS THEOREM

NEED A PROPER BH SOLUTION SECTION IN TEH NR SECTION THAT HAS ALL THE NUMERICAL APPROACHES

1.4.7 The Cosmological Constant

A discussion of general relativity is incomplete without discussing the cosmological constant Λ. At a geometric level, the cosmological constant encodes the homogeneous spacetime curvature in the absense of matter, and indeed setting $\Lambda \to 0$ (in vacuum) returns asymptotically flat vacuum general relativity. The cosmological constant is added into Einstein's equation, Eq. (1.4.29), with a term like $\Lambda g_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.4.46}$$

Each term still has a zero-divergence as $\nabla_\mu g_{\alpha\beta} = 0$ due to the Levi-Civita connection defined in section 1.3.3. The Einstein equation in vacuum becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.4.47}$$

and the traced becomes

$$R = \frac{2D}{D-2}\Lambda \tag{1.4.48}$$

for D spacetime dimensions. This quite nicely shows us that in the limit $\Lambda = 0$ the normal vacuum GR solution $R = 0$ is returned. In the case $\Lambda \neq 0$ it describes a homogeneous curved universe with constant non-zero R.

Friedmann–Lemaître–Robertson–Walker Spacetime

To model the entire universe more realistically, an isotropic uniform fluid as added. A uniform matter distribution is often done with a perfect fluid with stress tensor

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}, \tag{1.4.49}$$

where ρ and P are the rest frame density and pressure. The fluid is taken to be at rest with $u^i = 0$ and $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u}) = -1$. The line element takes the following ansatz

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1.4.50}$$

where $a(t)$ is the scale factor of the universe and the spacelike metric $\boldsymbol{\gamma}$ is given by

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \begin{cases} \mathrm{d}r^2 + \sin^2(r)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right) & , \text{ (closed)} \\ \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right) & , \text{ (flat)} \\ \mathrm{d}r^2 + \sinh^2(r)\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right) & , \text{ (hyperbolic)} \end{cases} \tag{1.4.51}$$

for either uniformly curved hyperbolic space, uniformly curved closed space or flat space. This spacetime is called the Friedmann–Lemaître–Robertson–Walker spacetime and the trace of Eintein's equation in four dimensions becomes,

$$R = 4\Lambda - 8\pi T, \tag{1.4.52}$$

where planck units are used and $T = T_{\mu\nu}g^{\mu\nu}$. It should be noted that R and T are constant over the entire spatial domain of the spacetime for a moment of time. This form of Einstein's equation makes it abundantly clear that the cosmological constant has the same effect on curvature as a uniform matter distribution. In the case $4\Lambda = 8\pi T$, Einstein's equation is once again $R = 0$ and can be solved with the flat metric of Minkowski (but now it is not in vacuum unless $T = \Lambda = 0$). If $8\pi T > 4\Lambda$, then the spatial hypersurface of the universe is closed and finite sized. Finally, if $4\Lambda > 8\pi T$ then the spatial hypersurface of the universe is open or hyperbolic; as a radial geodesics is followed, the amount of space grows faster than the r^2 as expected in flat space.

Putting the ansatz for the metric and stress tensor into the Einstein equation returns an ODE for $a(t)$. Along with an equation of state, $P(\rho)$, the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ returns an ODE for $\rho(t)$. The two solutions $\rho(t)$ and $a(t)$, along with knowledge of whether the spatial time-slices are flat, closed or hyperbolic, fully specify the spacetime. These solutions cover many topics such as the big bang, inflation, universe expansion, bouncing cosmologies (ending with a big crunch) and also minkowksi space given the correct conditions.

1.4.8 The Lagrangean Formulation of General Relativity

A common procedure in theoretical physics is to encapsulate the solution space in an action functional S,

$$S = \int \mathcal{L}\sqrt{-g}\,\mathrm{d}x^4. \tag{1.4.53}$$

Using the calculus of variation on this action returns differential equations governing the system. As found by Hilbert [REF] the following lagrangean density $\mathcal{L} = R$, equal to the Ricci scalar, returns the vacuum Einstein equations under varying with respect to $g^{\mu\nu}$.

$$\begin{aligned}
\delta S &= \int \left[\sqrt{-g}(\delta R) + R(\delta\sqrt{-g})\right]\,\mathrm{d}x^4 && (1.4.54)\\
&= \int \left[\sqrt{-g}\delta(g^{\mu\nu}R_{\mu\nu}) - \frac{1}{2}\sqrt{-g}g_{\mu\nu}R(\delta g^{\mu\nu})\right]\,\mathrm{d}x^4 && (1.4.55)\\
&= \int \left[\sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu}) + \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}\right]\,\mathrm{d}x^4 && (1.4.56)
\end{aligned}$$

where we used Eq. (1.3.139) to vary $\sqrt{-g}$. Remembering that the difference of two christoffel symbols such as,

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu}|_{g^{\mu\nu}+\delta g^{\mu\nu}} - \Gamma^{\lambda}{}_{\mu\nu}|_{g^{\mu\nu}} \tag{1.4.57}$$

is a tensor, and using normal coordinates [WRITE UP NORMAL COORDS] the left hand term of Eq. (1.4.56) becomes

$$\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\left(\partial_\lambda\delta\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\delta\Gamma^{\lambda}{}_{\mu\lambda}\right), && (1.4.58)\\
&= g^{\mu\nu}\left(\nabla_\lambda\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\lambda}{}_{\mu\lambda}\right), && (1.4.59)\\
&= \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}{}_{\mu\nu}\right), && (1.4.60)\\
&= \nabla_\lambda X^\lambda. && (1.4.61)
\end{aligned}$$

The ability to transform the partial derivatives into covariant derivatives comes from using normal coordinates. Putting everything together, δS becomes

$$\begin{aligned}
\delta S &= \int \left[\nabla_\mu X^\mu + \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\right]\sqrt{-g}\,\mathrm{d}x^4, && (1.4.62)\\
&= \int_B X^\mu \hat{s}_\mu\sqrt{|^{(3)}g|}\mathrm{d}x^3 + \int\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\sqrt{-g}\,\mathrm{d}x^4, && (1.4.63)\\
& && (1.4.64)
\end{aligned}$$

where the integral over B represents the surface integral over the boundary of our spacetime with metric ${}^{(3)}g_{ij}$; on B $\delta g^{\mu\nu} \to 0$ and therefore $X^\mu \to 0$. Setting $\mathrm{d}S = 0$ implies

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{1.4.65}$$

which is the vacuum Einstein equation.

Non-Vacuum Spacetimes

Matter is often added into a spacetime at the level of the lagrangean with a term $\frac{16\pi G}{c^4}\mathcal{L}_m$. The cosmological constant can be added in exactly the same way with $\mathcal{L}_\Lambda$. The total lagrangean becomes,

$$S = \int\left(R + \frac{16\pi G}{c^4}\mathcal{L}_m + \mathcal{L}_\Lambda\right)\sqrt{-g}\,\mathrm{d}x^4. \tag{1.4.66}$$

As we have already seen earlier in this section, the variation of R with respect to the inverse metric components $g^{\mu\nu}$ returns the vacuum Einstein equation; adding the two new terms from $\mathcal{L}_m$ and $\mathcal{L}_\Lambda$, setting $\delta S = 0$ gives different equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{16\pi G}{c^4}\frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}} + \frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_\Lambda\sqrt{-g})}{\delta g^{\mu\nu}} = 0. \tag{1.4.67}$$

Comparing the $\mathcal{L}_\Lambda$ term to the Einstein equation with cosmological constant in Eq. (1.4.46) we must have

$$g_{\mu\nu}\Lambda = \frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_\Lambda\sqrt{-g})}{\delta g^{\mu\nu}}, \tag{1.4.68}$$

$$= \mathcal{L}_\Lambda\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} + \frac{\delta(\mathcal{L}_\Lambda)}{\delta g^{\mu\nu}}, \tag{1.4.69}$$

$$= -\frac{1}{2}g_{\mu\nu}\mathcal{L}_\Lambda + \frac{\delta(\mathcal{L}_\Lambda)}{\delta g^{\mu\nu}}, \tag{1.4.70}$$

$$\tag{1.4.71}$$

which is solved by $\mathcal{L}_\Lambda = -2\Lambda$. Comparing the matter term instead returns another definition of the stress tensor,

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}}, \tag{1.4.72}$$

$$= -2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m. \tag{1.4.73}$$

Collecting these results, the full lagrangean is,

$$\mathcal{L} = R + \frac{16\pi G}{c^4}\mathcal{L}_m - 2\Lambda, \tag{1.4.74}$$

or in Planck units,

$$\mathcal{L} = R + 16\pi\mathcal{L}_m - 2\Lambda. \tag{1.4.75}$$

The form of $\mathcal{L}_m$ is problem specific, depending on the type of matter. The equation of motion of the matter, described by a set of fields ϕ_i, and their partial derivatives with respect to x^μ is,

$$\sqrt{-g}\frac{\delta\mathcal{L}}{\delta\phi_i} - \partial_\mu\left(\sqrt{-g}\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\right) = 0. \tag{1.4.76}$$

If the ϕ_i are scalar fields then the equation of motion simplifies, using Eq. (1.3.142), to

$$\frac{\delta\mathcal{L}}{\delta\phi_i} - \nabla_\mu\left(\frac{\delta\mathcal{L}}{\delta\nabla_\mu\phi_i}\right) = 0. \tag{1.4.77}$$

Modified Theories of Gravity

CAN HELP WITH MORE THEORETICAL THEORIES, NEXT EXPANSIONS OF R SUCH AS MODIIFED GRAV OR CHERNS SIEMANS OR HORNDENSKI.

[ADD SOME MOTIVATION OF WHY WE LIKE LAGRANGEANS, EASY TO QUNATIZE OR ADD MORE MATTER RO INTERACTIONS OF MATTER? note hilbert did einstein eq before einstein

1.5 Stuff

1.5.1 notes

maybe put the field theory bit straght into the curved space bit and dont bother with SR as much?

prove the diff of christoffels is a tensor?

NEwman penrose formalism?

GRChombo section?

einstein summation conventions? and upstiars/downstairs incedes in conventions

check tensor vs tensor field

maybe work coordinate transformation into the text earlier with ther introduction of vectors? maybe split the long diff geom section (fucntions curves and tensors) into two sections, functions curves, coords trans, vectors part 1 then covectors, tensors and tensor transformation part 2, forms part 3 then differentiation then calculuas?

mention SR and the fact that GR is based on the equivalence principles (weak and strong?) somewhere .

maybe make SR as a solution to the Einstein eqation a different section to just SR.

decide when to use SR and GR acronyms

connection/christoffel and capitalise or not?

talk about newman penrose scalars for GW extraction? nah

check capital letters

define trace

3+1 stress tensor is wrong for klein gordon

adm metric is wring in 4th term report

add a sections with intro reading such as books alcubbiere baumgarte, adn GR stuff like sean caroll, tong, harvey ..

use invisible (with star) subsubsections to break up longer sections

1.5.2 Differential Forms

A differential p-form is an antisymetric $(0, p)$ tensor; the tensor components (e.g. $A_{\alpha\beta\ldots\zeta}$) vanish if there is a repeating index, equal 1 for an even permutation of indeces (like 1,2,3,...,N) and -1 for an odd permutation (such as 2,1,3,4,...,N). Note that a 0-form is a function, a 1-form is a co-vector and for an N-dimensional manifold the N-form is unique and the p-forms with $p > N$ vanish.

When considering differential forms, the conventional covector basis is often changed from $\boldsymbol{\theta}^\mu$ to $\mathrm{d}x^\mu$ which will lend itself nicely to integrating p-forms on manifolds later. This convention means we can write the metric as $\boldsymbol{g} = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$; the same can be done for any tensor. Note that the metric cannot be a 2-form as it is not a symmetric tensor, infact the metric is a symmetric tensor.

Differential forms have their own unique differential operator called the exterior derivative, it acts on a p-form and returns a $p+1$-form. For a p-form $\boldsymbol{A} = A_\mu \mathrm{d}x^\mu$, the exterior derivative is given by

$$(\mathrm{d}A)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}, \tag{1.5.1}$$

where the square brackets mean the antisymmetric [check as i might need epsilons here]. This derivative guarentees to return a tensor without the need for a connection or metric on the manifold, unlike the partial derivative that we will see next. One common example of an exterior derivative is of a 1-form, say $\boldsymbol{A}$, giving

$$(\mathrm{d}A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.5.2}$$

which returns a 2-form. To be a 2-form the tensor must be antisymmetric under swapping indeces like $(\mathrm{d}A)_{\mu\nu} = -(\mathrm{d}A)_{\nu\mu}$, which is trivially true, and must transform like a tensor. Performing a coordinate transformation on $(\mathrm{d}A)_{\mu\nu}$ we see it transforms like

$$\begin{aligned}
\frac{\partial}{\partial \tilde{x}^\mu}\tilde{A}_\nu - \frac{\partial}{\partial \tilde{x}^\nu}\tilde{A}_\mu &= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}A_\sigma\right) - \frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial x^\rho}{\partial \tilde{x}^\mu}A_\rho\right), && (1.5.3)\\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\left(\frac{\partial}{\partial x^\rho}A_\sigma - \frac{\partial}{\partial x^\sigma}A_\rho\right) + \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\right)A_\sigma - \frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial x^\rho}{\partial \tilde{x}^\mu}\right)A_\rho, && (1.5.4)\\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\left(\frac{\partial}{\partial x^\rho}A_\sigma - \frac{\partial}{\partial x^\sigma}A_\rho\right) + \left(\frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} - \frac{\partial^2 x^\rho}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu}\right)A_\rho, && (1.5.5)\\
&= \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\left(\frac{\partial}{\partial x^\rho}A_\sigma - \frac{\partial}{\partial x^\sigma}A_\rho\right), && (1.5.6)
\end{aligned}$$

which is exactly the tensor transformation law.

COVER HER THAT DD GIVES ZERO? HODGE ? REWRITE DIFF EQNS WITH FORM NOTATION AND HODGE THEORY? PROVE

1.5.3 Parallel Transport

Parallel transport is the procedure of moving tensors along a smooth curve in a manifold. The curve Γ can be expressed parametrically as $x^\mu(\tau)$ with respect to a parameter τ, this automatically gives us the tangent vector $\boldsymbol{X}$ to Γ like

$$X^\mu(\tau) = \frac{\partial x^\mu(\tau)}{\partial \tau} = \dot{x}^\mu. \tag{1.5.7}$$

Parallel transporting a tensor $\boldsymbol{T}$ along Γ requires that $\boldsymbol{\nabla_X T}$ must be satisfied along Γ where $\boldsymbol{\nabla_X} = x^\mu \nabla_\mu$.

MAYBE DO THE FAMOUS VECTOR ON A SPHERE EXAMPLE

Having discussed parallel transport, we can re-derive geodesics. Take a vector field $\boldsymbol{X}$ and parallel transport is along it's own integral curves, therefore $\boldsymbol{X}$ must satisfy $\boldsymbol{\nabla_X X} = 0$. With a little algebra we can show that this describes a geodesic,

$$\begin{aligned}
x^\mu \nabla_\mu X^\nu &= 0, && (1.5.8)\\
x^\mu \nabla_\mu X^\nu &= X^\mu \partial_\mu X^\nu + X^\mu \Gamma^\nu{}_{\rho\mu} X^\rho, && (1.5.9)\\
&= \frac{\partial x^\mu(\tau)}{\partial \tau}\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu(\tau)}{\partial \tau} + \Gamma^\nu{}_{\rho\mu}\frac{\partial x^\mu(\tau)}{\partial \tau}\frac{\partial x^\rho(\tau)}{\partial \tau}, && (1.5.10)\\
&= \frac{\partial}{\partial \tau}\frac{\partial x^\nu(\tau)}{\partial \tau} + \Gamma^\nu{}_{\rho\mu}\frac{\partial x^\mu(\tau)}{\partial \tau}\frac{\partial x^\rho(\tau)}{\partial \tau}, && (1.5.11)\\
&= \ddot{x}^\nu + \Gamma^\nu{}_{\rho\mu}\dot{x}^\rho \dot{x}^\mu && (1.5.12)
\end{aligned}$$

and we have re-derived Eq. (1.2.71). Again this aligns with our intuition of a geodesic in flat space. We already remarked in Section 1.2.6 that $\ddot{x}^\mu = 0$ describes a straight line in flat space (when using cartesian coordinates) and now we can add another interpretation; in flat space a geodesic (which is a straight line) can be created by transporting a vector along it's own direction.

1.5.4 Low Curvature Limit of General Relativity

An important use of General Relativity is it's use in the low curvature limit. The simplest example of this would be Special Relativity; if General Relativity i[DO] reproduce Special Relativity in the limit of vanishing curvature then it is. We work with the assumption of a vacuum spacetime with no Cosmological constant and seek solutions to the Einstein equation with metric $g_{\mu\nu} = \eta_{\mu\nu}$ where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5.13}$$

in Cartesian coordinates. As seen before, in Eq.[REF]REF the Einstein equation in vacuum simplfies to just a vanishing Ricci tensor,

$$\begin{aligned} R_{\mu\nu} &= 0, && (1.5.14) \\ &= \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\rho\sigma}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\rho\nu}. && (1.5.15) \end{aligned}$$

Given that the Connection symbols $\Gamma^\mu{}_{\nu\rho}$ vanish everywhere for the metric components $\eta_{\mu\nu}$ then Eq. (1.5.14) is trivially satisfied and we have proved that Special Relativity is the zero-curvature limit of GR.

Relaxing the condition $g_{\mu\nu} = \eta_{\mu\nu}$ to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the components $h_{\mu\nu} \ll 1$, we can create a vacuum spacetime consisting of small curvature fluctuations; this turns out to describe gravitational waves. Ignoring terms of order $\mathcal{O}(h^2)$ we can see that

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu}, && (1.5.16) \\ h^{\rho\sigma} &= \eta^{\mu\rho}\eta^{\nu\sigma}h_{\mu\nu}, && (1.5.17) \\ g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho &= \eta^{\mu\nu}\eta_{\nu\rho} + \eta^{\mu\nu}h_{\nu\rho} - h^{\mu\nu}\eta_{\nu\rho} + \mathcal{O}(h^2), && (1.5.18) \\ &= \delta^\mu_\rho + \eta^{\mu\nu}h_{\nu\rho} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\eta_{\nu\rho}, && (1.5.19) \\ &= \delta^\mu_\rho + \underbrace{\eta^{\mu\nu}h_{\nu\rho} - \eta^{\mu\alpha}h_{\alpha\rho}}_{=0}. && (1.5.20) \end{aligned}$$

Note that we raise/lower the indeces of $h_{\mu\nu}/h^{\mu\nu}$ with $\boldsymbol{\eta}$ and not $\boldsymbol{g}$ THIS IS WRONG. Looking at the connection symbols and Ricci tensor while ignoring $\mathcal{O}(h^2)$ terms we get

$$\begin{aligned} \Gamma^\rho{}_{\mu\nu} &= \frac{1}{2}\left(\eta^{\rho\sigma} - h^{\rho\sigma}\right)\left(\partial_\mu(\eta_{\sigma\nu} + h_{\sigma\nu}) + \partial_\nu(\eta_{\mu\sigma} + h_{\mu\sigma}) - \partial_\sigma(\eta_{\mu\nu} + h_{\mu\nu})\right), && (1.5.21) \\ &= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + \mathcal{O}(h^2), && (1.5.22) \\ R_{\mu\nu} &= \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \mathcal{O}(h^2), && (1.5.23) \\ &= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\rho\partial_\mu h_{\sigma\nu} - \partial_\rho\partial_\sigma h_{\mu\nu} - \partial_\nu\partial_\mu h_{\sigma\rho} + \partial_\sigma\partial_\mu h_{\rho\mu}\right) + \mathcal{O}(h^2), && (1.5.24) \\ R &= (\eta^{\mu\nu} - h^{\mu\nu})R_{\mu\nu}, && (1.5.25) \\ &= \eta^{\mu\nu}\eta^{\rho\sigma}(\partial_\mu\partial_\rho h_{\sigma\nu} - \partial_\sigma\partial_\rho h_{\mu\nu}) + \mathcal{O}(h^2). && (1.5.26) \end{aligned}$$

To simplify the solving of Einsteins eqn [FIX THIS] we can consider coordinate transformation like $x^\mu \to x^\mu + \zeta^\mu$ where $\zeta^\mu \ll 1$. It can easily be shown that to first order in ζ^μ and $h_{\mu\nu}$ that $h_{\mu\nu}$ transforms like

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu \tag{1.5.27}$$

where raising/lowering the indeces of $\boldsymbol{\zeta}$ is done with $\boldsymbol{\eta}$ as terms of order $\mathcal{O}(h)\mathcal{O}(\zeta)$ are ignored.

Note [TALK ABOUT G PLUS H RATHER THAN ETA PLUS H FOR BH RINGDOWN AND MAYBE BS STABILITY? THIS IS MORE PERTURBATION THEORY THOUGH]

[MENTION : GW'S, PRECESSION, LIGHT DEFLECTION, TIME DILATION AROUND EARTH/-SUN, MAYBE MENTION INTERSTELLAR'S TIME DILATION? ARE THESE REALLY LOW ENERGY LIMIT? MENTION POST NEWTONIAN OTHER THINGS?]

Chapter 2

Numerical Relativity and Boson Stars

MAYBE PUT SOME STUFF HERE ABOUT NUMERICAL ALGORITHMS LIKE RK4 EULAR STEP LEAPFROG AND DO ERROR GROWTH RATES AND COURANT

2.1 Numerical Relativity

2.1.1 Spacetime Foliation

Einstein's equation is a classical field equation which governs the dynamics of physical objects and spacetime curvature.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.1.1}$$

The above version is fully covariant, agnostic of the definition of time, and many solutions are known analytically, for instance Black Hole geometries. When the system of interest becomes more complicated, such as the case of orbiting objects which will be discussed later, finding an analytic expression becomes impossible. For low energy dynamics, Newtonian theory, Post-Newtonian theory and perturbation theory can make more progress; however this report will focus on the highly nonlinear regime where Numerical relativity is truly the only hope to solve Einstein's equations. To do this it is common to split spacetime into 3+1 dimensions, evolving a 3 dimensional manifold (maybe with matter) on a computer along the final 4th dimension. To do this we need to define a suitable hypersurface $\Sigma \in \mathcal{M}$. This is usually done by demanding the hypersurface Σ_t be the set of points $p \in \mathcal{M}$ where some scalar function $f : \mathcal{M} \mapsto \mathbb{R}$ satisfies $f(p) = t$. This hypersurface should be a Cauchy surface, intersecting all causal curves only once, or a partial Cauchy surface which intersects all causal curves at most once. Generally we will choose a partial Cauchy surface due to the finite memory of computers, however by picking certain compactified coordinates it is possible to use a Cauchy surface [REF]. A foliation $\mathcal{F}$ is then the union of a set of Σ_t for some range of the parameter t,

$$\mathcal{F} = \cup_t(\Sigma_t) \subseteq \mathcal{M}. \tag{2.1.2}$$

This means we should be careful to pick a parameter t such that the foliation is not self intersecting for the parameter range that covers the region of $\mathcal{M}$ that we are interested in simulating. Fortunately the time coordinate in a suitable coordinate system works in all cases covered by this report; it also gives the physical interpretation of Σ_t being an instance of time. Now we should define unit normal vector n to Σ_t,

$$n^\mu = -\frac{\nabla^\mu t}{\sqrt{|g_{\mu\nu}\nabla^\mu t \nabla^\nu t|}} \quad \& \quad n_\mu = -\frac{dt_\mu}{\sqrt{|g_{\mu\nu}\nabla^\mu t \nabla^\nu t|}}. \tag{2.1.3}$$

For simplicity we define the lapse function α to be

$$\alpha := \frac{1}{\sqrt{|g_{\mu\nu}\nabla^\mu t \nabla^\nu t|}}. \tag{2.1.4}$$

giving us $n_\mu = -\alpha \mathrm{d}t_\mu$ as well as the normal evolution vector $m_\mu = \alpha n_\mu$. Defining two infinitessimaly close points $(p, q) \in (\Sigma_t, \Sigma_{t'})$ where $q^\mu = p^\mu + m^\mu \delta t$ we see,

$$t(q) = t(p^\mu + m^\mu \delta t) = t(p) + \frac{\mathrm{d}t}{\mathrm{d}x^\mu} m^\mu \delta t = t(p) + \mathrm{d}t_\mu m^\mu \delta t = t(p) + \delta t, \tag{2.1.5}$$

showing that m^μ points between neighbouring hypersurfaces; therefore when creating evolution equations we should care about Lie derivates along m^μ, $\mathcal{L}_m$, rather than $\mathcal{L}_n$.

2.1.2 The 3+1 Decomposition

With the notion of a spacetime foliation we should define how to project tensors onto Σ_t; clearly scalars need no projecting. Splitting a vector $X^\mu \boldsymbol{e}_\mu = X^\mu_\parallel \boldsymbol{e}_\mu + X^\mu_\perp \boldsymbol{e}_\mu$ into components tangent or normal to Σ_t

we define the orthogonal projector $\perp^\mu_\nu$ and parallel projector $-n^\mu n_\nu$,

$$X^\mu_\parallel = [\delta^\mu_\nu + n^\mu n_\nu] X^\mu = \perp^\mu_\nu X^\nu, \tag{2.1.6}$$

$$X^\mu_\perp = -n^\mu n_\nu X^\nu. \tag{2.1.7}$$

Considering scalars such as $\phi = w_\mu X^\mu$ or $\psi = T^{\mu\nu} w_\mu w_\nu$, and remembering scalars don't vary under projection, it is simple to show that any tensor T can be projected by contracting a projection operator $\perp$ on any free index,

$$T_\parallel{}^{ij\ldots}{}_{kl\ldots} = \mathcal{T}^{ij\ldots}{}_{kl\ldots} = \perp^i_\mu \perp^j_\nu \perp^\rho_k \perp^\sigma_l \cdots T^{\mu\nu\ldots}{}_{\rho\sigma\ldots}. \tag{2.1.8}$$

We can find the 3-metric $\gamma_{\mu\nu}$ of Σ_t by projecting $g_{\mu\nu}$,

$$\gamma_{ij} = \perp^\mu_i \perp^\nu_j g_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad \to \quad \gamma^i_j = \perp^i_j, \tag{2.1.9}$$

and we find it is equivalent to the projector $\perp$; this had to be the case as $\perp_{ij}\, \mathrm{d}x^i \mathrm{d}x^j$ gives the line element along Σ_t. With this machinery we can define the extrinsic curvature tensor $\mathcal{K}_{ij}$ representing curvature due to the choice of spacetime foliation; it could be nonzero for certain foliations of Minkowski space. It is not the same as the 3-Ricci tensor $\mathcal{R}_{ij}$ which is due to genuine spacetime curvature of $\mathcal{M}$ regardless of foliation. The extrinsic curvature tensor is defined the following way,

$$\mathcal{K}_{ij} = \mathcal{K}_{ji} := -\perp^\mu_i \perp^\nu_j \nabla_\mu n_\nu = -\perp^\mu_i \nabla_\mu n_j = -\nabla_i n_j - n_i a_j, \tag{2.1.10}$$

$$\mathcal{K} = \mathcal{K}^i_i = -\boldsymbol{\nabla} \cdot \boldsymbol{n}, \tag{2.1.11}$$

where $a_i = \boldsymbol{n} \cdot \boldsymbol{\nabla} n_i$ is called the Eularian acceleration; it should be noted $\mathcal{K}_{ij}$ is symmetric. It can also be shown to take the following form,

$$\mathcal{K}_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij} = -\frac{1}{2\alpha}\mathcal{L}_m \gamma_{ij}, \tag{2.1.12}$$

which gives the intuitive explanation of $\mathcal{K}_{ij}$ being the rate of change of the 3-metric γ_{ij} with respect to the foliation. The next object we should discuss is the projected covariant 3-derivative $\mathcal{D}_i$. This is the covariant derivative belonging to Σ_t and hence it's arguments should be tensors belonging to Σ_t. This means we can define it as so,

$$\mathcal{T}^{ij\ldots}{}_{kl\ldots} = \perp^i_\mu \perp^j_\nu \perp^\rho_k \perp^\sigma_l \cdots T^{\mu\nu\ldots}{}_{\rho\sigma\ldots}, \tag{2.1.13}$$

$$\mathcal{D}_m \mathcal{T}^{ij\ldots}{}_{kl\ldots} := \perp^\mu_m \perp^i_\mu \perp^j_\nu \perp^\rho_k \perp^\sigma_l \cdots \nabla_\mu \mathcal{T}^{\mu\nu\ldots}{}_{\rho\sigma\ldots}. \tag{2.1.14}$$

A simple example is the derivative of a vector $X^i \boldsymbol{e}_i \in \mathcal{T}(\Sigma_t)$,

$$\mathcal{D}_i X^j = \partial_i X^j + \Upsilon^i{}_{jk} X^k, \tag{2.1.15}$$

$$\Upsilon^i{}_{jk} = \frac{1}{2}\gamma^{il}\left[\partial_j \gamma_{lk} + \partial_k \gamma_{jk} - \partial_l \gamma_{jk}\right], \tag{2.1.16}$$

where $\Upsilon^i{}_{jk}$ is the Christoffel symbol of Σ_t. Another useful example is a^μ which can be equated to,

$$a_\mu = \boldsymbol{n} \cdot \boldsymbol{\nabla} n_\mu = \mathcal{D}_\mu \ln \alpha = \frac{1}{\alpha}\mathcal{D}_\mu \alpha, \tag{2.1.17}$$

and allows us to evaluate the Lie derivative of the projector $\perp^i_j$,

$$\mathcal{L}_m \perp^i_j = \alpha n^k \nabla_k \perp^i_j + \perp^i_k \nabla_j \alpha n^k - \perp^k_j \nabla_k \alpha m^i, \tag{2.1.18}$$

$$= \alpha n^k \nabla_k \left[n^i n_j\right] + \alpha \nabla_j n^i - \left[\alpha K^i_j + n^i \mathcal{D}_j \alpha\right], \tag{2.1.19}$$

$$= 0. \tag{2.1.20}$$

The result $\mathcal{L}_m \perp^i_j = 0$ is incredibly important, it tells us that the projector commutes with $\mathcal{L}_m$ and as a result any tensor $\boldsymbol{T}$ which when projected onto Σ_t, written $\boldsymbol{\mathcal{T}}$, satisfies

$$\mathcal{L}_m \mathcal{T}^{ij\ldots}{}_{kl\ldots} = \perp^i_\mu \perp^j_\nu \perp^\rho_k \perp^\sigma_l \mathcal{L}_m T^{\mu\nu\ldots}{}_{\rho\sigma\ldots}. \tag{2.1.21}$$

In other words, evolving a projected tensor along integral curves of m leaves the tensor parallel to Σ_t.

2.1.3 Gauss, Codazzi and Ricci Equations

The projections of the 4-dimensional curvature tensors into a combination of 3-dimensional curvature tensors and $\boldsymbol{\mathcal{K}}$ is very useful as it captures all the degrees of freedom of the 4-dimensional Riemann tensor in terms of variables on Σ_t. This property is crucial when numerically simulating a single time slice Σ_t as we only have access to variables on Σ_t.

The Gauss Equations

From the definition of the Riemann tensor in section 1.3.4 we know,

$$[\mathcal{D}_i\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_i]v^k = \mathcal{R}^k{}_{mij}v^m, \tag{2.1.22}$$

$$[\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha]v^\gamma = R^\gamma{}_{\lambda\alpha\beta}v^\lambda, \tag{2.1.23}$$

where the vector $v^m = \perp^m_\rho v^\rho$ is tangent to Σ_t. Expanding the $\mathcal{D}$'s in terms of ∇'s gives,

$$\mathcal{D}_i\mathcal{D}_j v^k = \perp^\mu_i \perp^\sigma_j \perp^k_\xi \nabla_\mu(\perp^\nu_\sigma \perp^\xi_\rho \nabla_\nu v^\rho), \tag{2.1.24}$$

and using the following properties; impotence of projections $\perp^i_\mu \perp^\mu_j = \perp^i_j$, null projection of orthogonal vectors $\perp(\boldsymbol{n}) = 0$, metric compatibility $\nabla_\mu \perp^i_j = n_j\nabla_\mu n^i + n^i\nabla_\mu n_j$ and Eq. (2.1.10) for $\mathcal{K}_{ij}$ we obtain the Gauss relation,

$$\perp^\mu_i \perp^\nu_j \perp^k_\rho \perp^\sigma_l R^\rho{}_{\sigma\mu\nu} = \mathcal{R}^k{}_{lij} + \mathcal{K}^k_i\mathcal{K}_{lj} - \mathcal{K}^k_j\mathcal{K}_{il}. \tag{2.1.25}$$

Contracting over i, k above and relabelling indices we get the contracted Gauss relation,

$$\perp^\mu_i \perp^\nu_j R_{\mu\nu} + \gamma_{i\mu}n^\nu \perp^\rho_j n^\sigma R^\mu{}_{\nu\rho\sigma} = \mathcal{R}_{ij} + \mathcal{K}\mathcal{K}_{ij} - \mathcal{K}^k_j\mathcal{K}_{ik}. \tag{2.1.26}$$

Contracting again and realising $R_{\mu\nu\rho\sigma}n^\mu n^\nu n^\rho n^\sigma = 0$ from antisymmetry in indeces 0 and 1 or 2 and 3 in the Riemann tensor, gives the scalar Gauss equation,

$$R + 2R_{\mu\nu}n^\mu n^\nu = \mathcal{R} + \mathcal{K}^2 - \mathcal{K}_{ij}\mathcal{K}^{ij}. \tag{2.1.27}$$

The Codazzi Equations

The Codazzi relations are derived from a different start point. Instead of projecting the Riemann tensor fully onto Σ_t with projection operators $\perp$ and a spacelike vector $\boldsymbol{v}$, it is now projected with a timelike vector $\boldsymbol{n}$,

$$[\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha]n^\gamma = R^\gamma{}_{\lambda\alpha\beta}n^\lambda, \tag{2.1.28}$$

and again projecting to Σ_t with three projection operators. The following relations are used

$$\nabla_j n^k = -\mathcal{K}^k{}_j - a^k n_j, \tag{2.1.29}$$

$$\perp^i_\mu \perp^j_\nu \perp^\rho_k \nabla_i\nabla_j n^k = -\mathcal{D}_i\mathcal{K}^k{}_j + a^k\mathcal{K}_{ij}, \tag{2.1.30}$$

which lead immediately to the Codazzi relation,

$$\perp^\mu_i \perp^\nu_j \perp^k_\rho n^\sigma R^\rho{}_{\sigma\mu\nu} = \mathcal{D}_j\mathcal{K}^k{}_i - \mathcal{D}_i\mathcal{K}^k{}_j, \tag{2.1.31}$$

and the contracted Codazzi relation,

$$\perp^\mu_i n^\nu R_{\mu\nu} = \mathcal{D}_i\mathcal{K} - \mathcal{D}_\mu\mathcal{K}^\mu{}_i. \tag{2.1.32}$$

The Ricci Equation

Finally we turn our attention to the Ricci equation, the projection of the Riemann tensor twice onto Σ_t and twice contracting with $\boldsymbol{n}$. This is done by projecting Eq (2.1.28) with two projectors $\perp$ and one timelike $\boldsymbol{n}$. If we contract with n^γ then the antisymmetry in the first two Riemann tensor indeces would identically gie to zero, therefore the unique choice (upto a minus sign) is to project with n^β,

$$R_{\gamma\lambda\alpha\beta}n^\lambda n^\beta = n^\beta \left[\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right] n_\gamma, \tag{2.1.33}$$

and project the remaining free indeces with $\perp$ like,

$$\perp_i^\gamma \perp_j^\alpha R_{\gamma\lambda\alpha\beta}n^\lambda n^\beta = \perp_i^\gamma \perp_j^\alpha n^\beta \left[\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right] n_\gamma. \tag{2.1.34}$$

Rearanging Eqs. (2.1.10) and (2.1.17),

$$\nabla_\sigma n_\mu = -\mathcal{K}_{\mu\sigma} - n_\sigma \mathcal{D}_\mu \ln(\alpha), \tag{2.1.35}$$

which can be used to expand the right hand side of Eq. (2.1.36),

$$\begin{aligned}
\perp_i^\gamma \perp_j^\alpha R_{\gamma\lambda\alpha\beta}n^\lambda n^\beta =& \perp_i^\gamma \perp_j^\alpha n^\beta \left[-\nabla_\alpha \mathcal{K}_{\gamma\beta} + \nabla_\beta \mathcal{K}_{\gamma\alpha}\right] && (2.1.36)\\
&+ \perp_i^\gamma \perp_j^\alpha n^\beta \left[-\nabla_\alpha(n_\beta \mathcal{D}_\gamma \ln(\alpha)) + \nabla_\beta(n_\alpha \mathcal{D}_\gamma \ln(\alpha))\right],\\
=& \perp_i^\gamma \perp_j^\alpha n^\beta n^\sigma \nabla_\sigma \mathcal{K}_{\gamma\alpha} + \perp_\gamma^i \perp_j^\alpha \mathcal{K}_{\gamma\beta}\nabla_\alpha n^\beta && (2.1.37)\\
&+ \perp_i^\gamma \perp_j^\alpha n^\beta \left[-n_\beta \nabla_\alpha(\mathcal{D}_\gamma \ln(\alpha)) + n_\alpha \nabla_\beta \mathcal{D}_\gamma \ln(\alpha) + (\mathcal{D}_\gamma \ln(\alpha))\nabla_\beta n_\alpha\right],\\
=& \perp_i^\gamma \perp_j^\alpha n^\beta n^\sigma \nabla_\sigma \mathcal{K}_{\gamma\alpha} - \mathcal{K}_{ik}\mathcal{K}^k{}_j && (2.1.38)\\
&+ \mathcal{D}_j \mathcal{D}_i \ln(\alpha) + \mathcal{D}_i \ln(\alpha) \mathcal{D}_j \ln(\alpha),\\
=& \perp_i^\gamma \perp_j^\alpha n^\beta n^\sigma \nabla_\sigma \mathcal{K}_{\gamma\alpha} - \mathcal{K}_{ik}\mathcal{K}^k{}_j + \frac{1}{\alpha}\mathcal{D}_j \mathcal{D}_i \alpha, && (2.1.39)
\end{aligned}$$

where $\boldsymbol{n}^2 = -1$, $\perp_i^\mu n_\mu = 0$, $\mathcal{D}_\alpha \ln \alpha = n^\beta \nabla_\beta n_\alpha$ from Eq. (2.1.17) and $n^\beta \nabla_\alpha n_\beta = 0$ were used. This expression can be simplified by calculating the Lie derivative of $\boldsymbol{\mathcal{K}}$,

$$\begin{aligned}
\mathcal{L}_m \mathcal{K}_{ij} =& \perp_i^\mu \perp_j^\nu \mathcal{L}_m \mathcal{K}_{\mu\nu}, && (2.1.40)\\
=& \alpha \perp_i^\mu \perp_j^\nu \mathcal{L}_n \mathcal{K}_{\mu\nu}, && (2.1.41)\\
=& \alpha \perp_i^\mu \perp_j^\nu \left[\alpha \boldsymbol{n}\cdot\boldsymbol{\nabla}\mathcal{K}_{\mu\nu} + 2\mathcal{K}_{k(\nu}\nabla_{\mu)}n^k\right], && (2.1.42)\\
=& \alpha \perp_i^\mu \perp_j^\nu n^\sigma \nabla_\sigma \mathcal{K}_{\mu\nu} - 2\alpha \mathcal{K}_{ik}\mathcal{K}^k{}_j, && (2.1.43)
\end{aligned}$$

where the results $\mathcal{L}_m \boldsymbol{\mathcal{K}} = \alpha \mathcal{L}_n \boldsymbol{\mathcal{K}}$ from Eq. (2.1.12), $\mathcal{L}_m \mathcal{K}_{\alpha\beta} \in \Sigma_t$ from Eq. (2.1.21) and Eq. (2.1.10) were used. Putting everything together we arrive at the Ricci equation,

$$\perp_i^\mu \perp_j^\nu n^\rho n^\sigma R_{\mu\rho\nu\sigma} = \frac{1}{\alpha}\mathcal{L}_m \mathcal{K}_{ij} + \frac{1}{\alpha}\mathcal{D}_j \mathcal{D}_i \alpha + \mathcal{K}_{ik}\mathcal{K}^k{}_j. \tag{2.1.44}$$

This is the final contraction of the Riemann tensor that can be made with $\perp$ and $\boldsymbol{n}$ as any projections with three or more contractions with $\boldsymbol{n}$ would identically give zero due to the symmetries of the Riemann tensor.

2.1.4 Decomposition of Einstein's Equation

To evolve General Relativity numerically we must project the Einstein Equation into 3+1 dimensions. Relations between three and four dimensional geometric objects have been derived above and will be used to decompose the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ from the left hand side of Eq. (2.1.1). The second component, for simulating non-vacumm spacetimes, is the 3+1 decomposition of the Stress tensor

T_{ab}. We contract twice with $\boldsymbol{n}$, then once with $\boldsymbol{n}$ while projecting onto Σ_t and finally twice projecting onto Σ_t to get an energy, momentum and stress-like split,

$$\mathcal{E} = \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n}) = T_{\mu\nu} n^\mu n^\nu, \tag{2.1.45}$$
$$\mathcal{S}_i = -\boldsymbol{T}(\boldsymbol{n}, \cdot) = - \perp_i^\mu n^\nu T_{\mu\nu}, \tag{2.1.46}$$
$$\mathcal{S}_{ij} = \perp_i^\mu \perp_j^\nu T_{\mu\nu}, \tag{2.1.47}$$

and by construction,

$$T_{\mu\nu} = \mathcal{E} n_\mu n_\nu + \mathcal{S}_\mu n_\nu + \mathcal{S}_\nu n_\mu + \mathcal{S}_{\mu\nu}. \tag{2.1.48}$$

With this and the Gauss-Codazzi equations of section 2.1.3 we can project the Einstein equation. Lets first look at the scalar equation,

$$G_{\mu\nu} n^\mu n^\nu = R_{\mu\nu} n^\mu n^\nu + \frac{1}{2} R = 8\pi\mathcal{E}, \tag{2.1.49}$$

and equating the geometric terms to the scalar Gauss equation we get the Hamiltonian constraint, $\mathcal{H} = 0$,

$$\mathcal{H} = \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} - \mathcal{K}^2 - \mathcal{R} + 16\pi\mathcal{E} = 0. \tag{2.1.50}$$

Now looking at the once projected part we see,

$$\perp_i^\mu n^\nu G_{\mu\nu} = \perp_i^\mu n^\nu R_{\mu\nu} = -8\pi\mathcal{S}_i, \tag{2.1.51}$$

and substituting the geometric terms for the contracted Codazzi relation we get the momentum constraint, $\mathcal{P}_i = 0$,

$$\mathcal{P}_i = \mathcal{D}_i\mathcal{K} - \mathcal{D}_j\mathcal{K}_i^j + 8\pi\mathcal{S}_i = 0. \tag{2.1.52}$$

Finally, the space-space projection gives the 6 evolution PDE's. This time start with the trace reversed Einstein Equation

$$R_{\mu\nu} = 8\pi\left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right], \tag{2.1.53}$$
$$\perp_i^\mu \perp_j^\nu R_{\mu\nu} = 8\pi\left[\mathcal{S}_{ij} - \frac{1}{2}(\mathcal{S} - \mathcal{E})\gamma_{ij}\right], \tag{2.1.54}$$

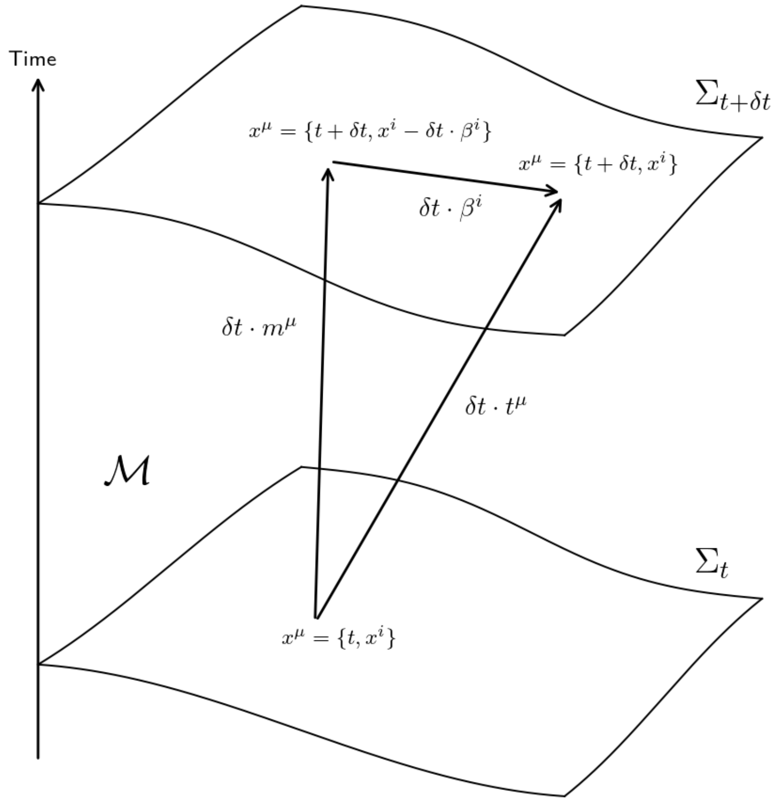
where we used $T = [\gamma^{\mu\nu} - n^\mu n^\nu] T_{\mu\nu} = \mathcal{S} - \mathcal{E}$. Project the Ricci tensor, we use the contracted Gauss equation but replace the term with $R^\mu{}_{\nu\rho\sigma}$ with the Ricci equation in Eq. (2.1.44). Rearranging gives a normal evolution for the extrinsic curvature,

$$\mathcal{L}_m\mathcal{K}_{ij} = -\mathcal{D}_j\mathcal{D}_i\alpha + \alpha\left[\mathcal{R}_{ij} + \mathcal{K}\mathcal{K}_{ij} - 2\mathcal{K}_i^k\mathcal{K}_{kj} + 4\pi\left[\gamma_{ij}\left[\mathcal{S} - \mathcal{E}\right] - 2\mathcal{S}_{ij}\right]\right]. \tag{2.1.55}$$

Along with the definition of $\mathcal{K}_{ij}$ in Eq. (2.1.12),

$$\mathcal{L}_m\gamma_{ij} = -2\alpha\mathcal{K}_{ij}, \tag{2.1.56}$$

this gives the normal evolution equations for γ_{ij} and $\mathcal{K}_{ij}$. In 4 or n spacetime dimensions the normal evolution equations contain 6 or $\frac{n^2-n}{2}$ differential equations, the Hamiltonian constraint is a single differential equation, the momentum constraint contains 3 or $n-1$ differential equations and the Einstein equation contains 10 or $\frac{n^2+n}{2}$ differential equations In four spacetime dimensions this corresponds to 6 evolution equations and 4 constraint equations over the surface Σ_t.



2.1.5 Foliation Adapted Coordinates

Picking coordinates in GR introduces a large gauge freedom which allows us to use a level set of the time coordinate $x^0 = t$ to define our foliation hypersurfaces Σ_t. The other three coordinates x^i for $i \in [1, 2, 3]$ can be used to span each hypersurface Σ_t however we define. It is conventional to split the normal evolution vector m^μ into time t^μ and space parts β^μ,

$$m^\mu = t^\mu - \beta^\mu = (\partial_0)^\mu - \beta^i (\partial_i)^\mu, \tag{2.1.57}$$

$$m^\mu = \left(1, -\beta^1, -\beta^2, -\beta^3\right). \tag{2.1.58}$$

We can view t^μ as the (not necessarily causal) worldline for a simulation gridpoint, hence we would like to evolve our PDE's along t^μ on a computer. Equations (2.1.57) and (2.1.58) along with the definition of $\boldsymbol{m}$, $\boldsymbol{m} = \alpha \boldsymbol{n}$, specify n^μ and n_μ,

$$n^\mu = \frac{1}{\alpha}\left(1, -\beta^1, -\beta^2, -\beta^3\right), \tag{2.1.59}$$

$$n_\mu = -\alpha\,(1, 0, 0, 0)\,. \tag{2.1.60}$$

The decomposed metric can be calculated, using the property that β is tangent to Σ_t and orthogonal to m,

$$g_{00} = \boldsymbol{g}(\boldsymbol{\partial}_0, \boldsymbol{\partial}_0) = \boldsymbol{g}(\boldsymbol{m} + \beta^i \boldsymbol{\partial}_i, \boldsymbol{m} + \beta^j \boldsymbol{\partial}_j) = \boldsymbol{g}(\boldsymbol{m}, \boldsymbol{m}) + \beta^i \beta_j \langle \boldsymbol{\partial}_i, \mathbf{d}\boldsymbol{x}^j \rangle = -\alpha^2 + \beta^i \beta_i, \tag{2.1.61}$$

$$g_{0i} = \boldsymbol{g}(\boldsymbol{\partial}_0, \boldsymbol{\partial}_i) = \boldsymbol{g}(\boldsymbol{m} + \beta^j \boldsymbol{\partial}_j, \boldsymbol{\partial}_i) = \beta^j \boldsymbol{g}(\boldsymbol{\partial}_j, \boldsymbol{\partial}_i) = \beta_i, \tag{2.1.62}$$

$$g_{ij} = \boldsymbol{g}(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j) = \boldsymbol{\gamma}(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j) = \gamma_{ij}. \tag{2.1.63}$$

This is commonly called the $3+1$ ADM metric [REF] and α, β^i are referred to the lapse and shift vector in this context. The line element and metric are commonly written as,

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}\left[\mathrm{d}x^i + \beta^i \mathrm{d}t\right]\left[\mathrm{d}x^j + \beta^j \mathrm{d}t\right], \tag{2.1.64}$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i\beta_i & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}, \tag{2.1.65}$$

$$g^{\mu\nu} = \frac{1}{\alpha^2}\begin{pmatrix} -1 & \beta^i \\ \beta^j & \alpha^2\gamma^{ij} - \beta^i\beta^j, \end{pmatrix} \tag{2.1.66}$$

and using Cramers rule for metric determinant,

$$g^{00} = \frac{\det\{\gamma_{ij}\}}{\det\{g_{\mu\nu}\}}, \tag{2.1.67}$$

we get the important relationship,

$$\sqrt{-g} = \alpha\sqrt{\gamma}, \tag{2.1.68}$$

where g and γ are the determinants of $g_{\mu\nu}$ and γ_{ij} respectively.

2.1.6 ADM Equations

Now that we have some coordinates suitable for the spacetime foliation we can find the Arnowitt-Deser-Misner (ADM) evolution equations for $\mathcal{K}_{ij}$ and γ_{ij}. First expand the Lie derivative $\mathcal{L}_m$ using

$$t^\mu = (1,0,0,0) \to \mathcal{L}_t T^{\mu\nu\ldots}{}_{\rho\sigma\ldots} = \partial_t T^{\mu\nu\ldots}{}_{\rho\sigma\ldots} \tag{2.1.69}$$

$$\mathcal{L}_m = \mathcal{L}_t - \mathcal{L}_\beta = \partial_t - \mathcal{L}_\beta \tag{2.1.70}$$

and the ADM equations can be written by expanding the $\mathcal{L}_m$ in the normal evolution equations in section 2.1.4 for $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\gamma}$. The ADM equations are,

$$\partial_t \mathcal{K}_{ij} = \mathcal{L}_\beta \mathcal{K}_{ij} - \mathcal{D}_j\mathcal{D}_i\alpha + \alpha\left[\mathcal{R}_{ij} + \mathcal{K}\mathcal{K}_{ij} - 2\mathcal{K}_i^k\mathcal{K}_{kj} + 4\pi\left[\gamma_{ij}\left[\mathcal{S} - \mathcal{E}\right] - 2\mathcal{S}_{ij}\right]\right], \tag{2.1.71}$$

$$\partial_t \gamma_{ij} = \mathcal{L}_\beta \gamma_{ij} - 2\alpha\mathcal{K}_{ij}. \tag{2.1.72}$$

Unfortunately, these PDE's turn out to be ill-posed [REF]; this means that the time evolution of these equations does not depens smoothly on the initial data. The ill-posedness of the ADM equations is resolved in the next section.

2.1.7 BSSN

To tackle the ill-posedness of the ADM equations in section 2.1.6 the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formalism [REF] is introduced. The first step in BSSN is to decompose the 3-metric into the conformal metric $\tilde{\gamma}_{ij}$ and the conformal factor χ,

$$\tilde{\gamma}_{ij} = \chi\gamma_{ij}, \tag{2.1.73}$$

$$\det\{\tilde{\gamma}_{ij}\} = \tilde{\gamma} = \chi^3\gamma = 1, \tag{2.1.74}$$

with the above being the convention used in GRChombo described in [REF]. Other conventions include factors such as $\tilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij}$ or $e^{-\phi}\gamma_{ij}$. Along with this the extrinsic curvature $\mathcal{K}_{ij}$ is modified to be trace free,

$$\tilde{A}_{ij} = \chi\left[\mathcal{K}_{ij} - \frac{1}{3}\mathcal{K}\gamma_{ij}\right], \tag{2.1.75}$$

so that $\tilde{A}_{ij}\gamma^{ij} = 0$. During an evolution the conditions $\operatorname{tr}\tilde{A}_{ij} = 0$ and $\tilde{\gamma} = 1$ are enforced which are observed to improve numerical stability; however it is unclear why beyond heuristic arguments. The definition of $\chi = \gamma^{-1/3}$ is good for black hole simulations where $\gamma \to \infty$ but $\chi \to 0$; for example the isotropic schwarzschild metric has

$$\gamma = \left[1 + \frac{M}{2r}\right]^{12}, \tag{2.1.76}$$

$$\chi = \left[\frac{r}{\frac{M}{2} + r}\right]^{4}. \tag{2.1.77}$$

The next step is to introduce the conformal connection functions as auxiliary variables,

$$\tilde{\Upsilon}^i = \tilde{\gamma}^{jk}\tilde{\Upsilon}^i{}_{jk} = -\partial_i\tilde{\gamma}^{ij}, \tag{2.1.78}$$

$$\tilde{\Upsilon}^i{}_{jk} = \frac{1}{2}\tilde{\gamma}^{il}\left[\partial_j\tilde{\gamma}_{kl} + \partial_k\tilde{\gamma}_{lj} - \partial_l\tilde{\gamma}_{jk}\right] = \Upsilon^i{}_{jk} + \left[\delta^i_j\partial_k + \delta^i_k\partial_j - \gamma^{il}\gamma_{jk}\partial_l\right]\ln\sqrt{\chi}. \tag{2.1.79}$$

This reduces the set of vacuum evolution variables to $\{\chi, \tilde{\gamma}_{ij}, \mathcal{K}, \tilde{\mathcal{A}}_{ij}, \tilde{\Upsilon}^i\}$. It is conventional to use $-\partial_i\tilde{\gamma}^{ij}$ to evaluate the conformal connection coefficients when they appear in the RHS of an equation, but $\partial_j\tilde{\Upsilon}^i$ is calculated by differentiating the evolution variable $\tilde{\Upsilon}^i$. One final necessity, not included in the CCZ4 formulation discussed later, is to add multiples of the constraint equations (in section 2.1.4) to the evolution equations to change the characteristic matrix and improve stability.

The BSSN formalism is not the only way to find a well-posed set of evolution equations for general relativity. Another strongly hyperbolic formalism is the generalised harmonic gauge [?] with,

$$\Box x^\mu = H^\mu, \tag{2.1.80}$$

for some functions H^μ.

2.1.8 Z4 Formalism

The Z4 formalism [?] generalises the Einstein equation to include an unphysical field Z_μ, along with damping terms parameterised by κ_1, κ_2,

$$R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu - \kappa_1\left[n_\mu Z_\nu + n_\nu Z_\mu - [1+\kappa_2]g_{\mu\nu}n^\alpha Z_\alpha\right] = 8\pi G\left[T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right]. \tag{2.1.81}$$

Of course regular General Relativity is returned setting $Z_\mu = 0$. It is shown in [] that achieving $Z_\mu = 0$ whilst dynamically evolving Z_μ is equivalent to solving the constraints. Z_μ is subjected to a wave equation, transporting constraint violation off the computational domain. It can be shown that the system is driven to $Z_\mu = 0$ for $k_1 > 0$ and $k_2 < -1$. It is much cheaper to evolve the variables Z_μ, driven to zero, than do perform four elliptic solves for the constraints $\{\mathcal{H}, \mathcal{M}^i\}$ on each timestep.

2.1.9 CCZ4

Joining BSSN with Z4 gives the CCZ4 formalism. The additional modifications,

$$\Theta = -n \cdot Z = -\alpha Z^0, \tag{2.1.82}$$

$$\hat{\Upsilon}^i = \tilde{\Upsilon}^i + \frac{2\gamma^{ij}Z_j}{\chi}, \tag{2.1.83}$$

are made, leaving us with the following set of vacuum evolution variables $\{\chi, \tilde{\gamma}_{ij}, \mathcal{K}, \tilde{\mathcal{A}}_{ij}, \hat{\Upsilon}^i, \Theta\}$. Notably the pair of variables $\mathcal{R}_{ij} + \mathcal{D}_{(i}Z_{j)}$, and traced version $\mathcal{R} + \boldsymbol{\mathcal{D}} \cdot \boldsymbol{Z}$, always appear together; separately they ruin strong hyperbolically but together they do not. The evolution equations can now be found in the CCZ4 scheme by doing a 3+1 decomposition of the Z4 modified Einstein equation in Eq. (2.1.81) and following the BSSN formalism. To illustrate this, the equation of motion for χ is derived. Using Eqs. (1.3.122) and (2.1.56) with $\chi^{-3} = \gamma$,

$$\mathcal{L}_m \gamma = \gamma \gamma^{ij} \mathcal{L}_m \gamma_{ij} = -2\gamma\alpha\gamma^{ij}\mathcal{K}_{ij} = -2\gamma\alpha\mathcal{K}. \tag{2.1.84}$$

This can be used to simplify the Lie derivative of χ,

$$\begin{aligned}
\mathcal{L}_m \chi &= \mathcal{L}_{\partial_t}\chi - \mathcal{L}_\beta \chi, &(2.1.85)\\
&= (\partial_t)^i \partial_i \chi + \omega\chi\partial_i(\partial_t)^i - \beta^i\partial_i\chi - \omega\chi\partial_i\beta^i, &(2.1.86)\\
&= \partial_t\chi - \beta^i\partial_i\chi + \frac{2}{3}\chi\partial_i\beta^i, &(2.1.87)\\
\mathcal{L}_m \chi &= \mathcal{L}_m \gamma^{-\frac{1}{3}}, &(2.1.88)\\
&= -\frac{1}{3}\gamma^{-\frac{4}{3}}\mathcal{L}_m\gamma, &(2.1.89)\\
&= \frac{2}{3}\gamma^{-\frac{1}{3}}\alpha\mathcal{K}, &(2.1.90)\\
&= \frac{2}{3}\chi\gamma\alpha\mathcal{K}, &(2.1.91)
\end{aligned}$$

where $\omega = -2/3$ is the weight of χ. The equation for the lie derivative for a tensor density $\boldsymbol{\mathcal{T}}$ of weight w,

$$\mathcal{L}_\xi \boldsymbol{\mathcal{T}} = \widetilde{\mathcal{L}}_\xi \boldsymbol{\mathcal{T}} + w\boldsymbol{\mathcal{T}}\partial_\mu\xi^\mu, \tag{2.1.92}$$

given in Eq. 1.3.31 was used where $\widetilde{\mathcal{L}}_X$ represents the differential operator that is equivalent to $\mathcal{L}_X$ if it didn't act on a tensor density. Finally, rearanging gives the equation of motion for χ,

$$\partial_t\chi = \beta^k\partial_k\chi + \frac{2\chi}{3}\left[\alpha\mathcal{K} - \partial_k\beta^k\right]. \tag{2.1.93}$$

A similar process returns the CCZ4 equations but care should be taken to include the Z4 terms where they are needed. The complete list of CCZ4 equations used in simulations with GRChombo [REF] are given below.

$$\begin{aligned}
\partial_t\chi &= \beta^k\partial_k\chi + \frac{2\chi}{3}\left[\alpha\mathcal{K} - \partial_k\beta^k\right] &(2.1.94)\\
\partial_t\tilde{\gamma}_{ij} &= \beta^k\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{kj}\partial_i\beta^k + \tilde{\gamma}_{ik}\partial_j\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k - 2\alpha\tilde{\mathcal{A}}_{ij} &(2.1.95)\\
\partial_t\mathcal{K} &= \beta^k\partial_k\mathcal{K} + \alpha\left[\mathcal{R} + 2\boldsymbol{\mathcal{D}}\cdot\boldsymbol{Z} + \mathcal{K}\left[\mathcal{K} - 2\Theta\right]\right] - 3\alpha\kappa_1\left[1+\kappa_2\right]\Theta\\
&\quad - \chi\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\alpha + 4\pi G\alpha\left[\mathcal{S} - 3\mathcal{E}\right] &(2.1.96)\\
\partial_t\tilde{\mathcal{A}}_{ij} &= \beta^k\partial_k\tilde{\mathcal{A}}_{ij} + \chi\left[\alpha\left[\mathcal{R}_{ij} + 2\mathcal{D}_{(i}Z_{j)} - 8\pi G\mathcal{S}_{ij}\right] - \mathcal{D}_i\mathcal{D}_j\alpha\right]^{TF}\\
&\quad + \tilde{\mathcal{A}}_{ij}\left[\alpha\left[\mathcal{K} - 2\Theta\right] - \frac{2}{3}\mathcal{K}^2\right] + 2\tilde{\mathcal{A}}_{k(i}\partial_{j)}\beta^k - 2\alpha\tilde{\gamma}^{kl}\tilde{\mathcal{A}}_{ik}\tilde{\mathcal{A}}_{lj} &(2.1.97)\\
\partial_t\Theta &= \beta^k\partial_k\Theta + \frac{1}{2}\alpha\left[\mathcal{R} + 2\boldsymbol{\mathcal{D}}\cdot\boldsymbol{Z} - \tilde{\mathcal{A}}_{kl}\tilde{\mathcal{A}}^{kl} + \frac{2}{3}\mathcal{K}^2 - 2\Theta\mathcal{K}\right] - \kappa_1\alpha\Theta\left[2+\kappa_2\right] - Z^k\partial_k\alpha - 8\pi G\alpha\mathcal{E}\\
&&(2.1.98)\\
\partial_t\hat{\Upsilon}^i &= \beta^k\partial_k\hat{\Upsilon}^i + \frac{2}{3}\left[\partial_k\beta^k\left[\tilde{\Upsilon}^i + 2\kappa_3\frac{Z^i}{\chi}\right] - 2\alpha\mathcal{K}\frac{Z^j}{\chi}\right] - 2\alpha\kappa_1\frac{Z^i}{\chi}\\
&\quad + 2\tilde{\gamma}^{ij}\left[\alpha\partial_j\Theta - \Theta\partial_j\alpha\right] - 2\tilde{\mathcal{A}}^{ij}\partial_j\alpha - \alpha\left[\frac{4}{3}\tilde{\gamma}^{ij}\partial_j\mathcal{K} + 3\tilde{\mathcal{A}}^{ij}\frac{\partial_j\chi}{\chi}\right]\\
&\quad - \left[\tilde{\Upsilon}^j + 2\kappa_3\frac{Z^j}{\chi}\right]\partial_j\beta^i + 2\alpha\tilde{\Upsilon}^i{}_{jk}\tilde{\mathcal{A}}^{jk} + \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_k\partial_j\beta^k - 16\pi G\alpha\tilde{\gamma}^{ij}\mathcal{S}_j &(2.1.99)\\
\partial_t\varphi &= \beta^k\partial_k\varphi - \alpha\Pi &(2.1.100)\\
\partial_t\Pi &= \beta^k\partial_k\Pi - \chi\tilde{\gamma}^{ij}\partial_i\varphi\partial_j\alpha + \alpha\left[\chi\tilde{\Upsilon}^k\partial_k\varphi + \frac{1}{2}\tilde{\gamma}^{lk}\partial_k\chi\partial_l\varphi - \chi\tilde{\gamma}^{ij}\partial_i\partial_j\varphi + \mathcal{K}\Pi + V'\varphi\right] &(2.1.101)
\end{aligned}$$

Note that Eqs. (2.1.100) and (2.1.101) are for the 3+1 Klein-Gordon equation which will be derived later in section 2.2.3. Also to be noted, in the CCZ4 equations there is an additional parameter κ_3 premultiplying terms in the evolution of $\hat{\Upsilon}^i$ which experimentally were found to ruin numerical stability for black hole simulations []. Setting $\kappa_3 < 1$ stabilises the simulation but at the cost of covariance. Later on it was realised that setting $\kappa_3 = 1$ and $\alpha\kappa_1 \rightarrow \kappa_1$ retains covariance as well as numerical stability [REF].

The CCZ4 scheme proves useful in my simulations for a few reason. Firstly, any initial data that does not satisfy the constraints will not do so along evolution either when using BSSN. Given that superposition of solutions in GR does not generally give a new solution, but does approximate one for separated compact objects, all the simulated binaries considered in the report will have non constraint satisfying initial data. The use of the CCZ4 scheme will also help simulations satisfy the constraints even if they initially satisfy them; one reason being that finite resolution imposes some small deviation from the continuum solution. More importantly, the use of adaptive mesh refinement AMR introduces large interpolation errors into the simulation at the boundary of the different grid resolution levels. Finally, the Sommerfeld boundary conditions used are inexact in GR and will introduce small errors at the boundary that ruin constraint satisfaction. In all above cases, the CCZ4 system forces the evolution towards constraint satisfaction, despite the numerical errors and approximations.

2.1.10 Gauge Conditions

In GR, the lapse α and shift β^i are freely specifiable being gauge variables, however they must be chosen carefully along with a suitable initial Cauchy surface Σ_{t_0} for sensible foliations and numerical accuracy. Figure 2.1 (left) shows how a poor initial Cauchy surface could extend to the singularity for ingoing Eddington-Finkelstein coordinates; likely causing a simulation crash. In this work Σ_{t_0} is chosen to be in the isotropic gauge as in figure 2.1 (right); not only does this allow trivial swapping between spherical polar and Cartesian (used in the simulation) coordinates but also provides an initial Cauchy surface free of singularities and easy to compute. For a poor choice of lapse function, even a well chosen Σ_{t_0} can advance to the singularity in finite simulation time.

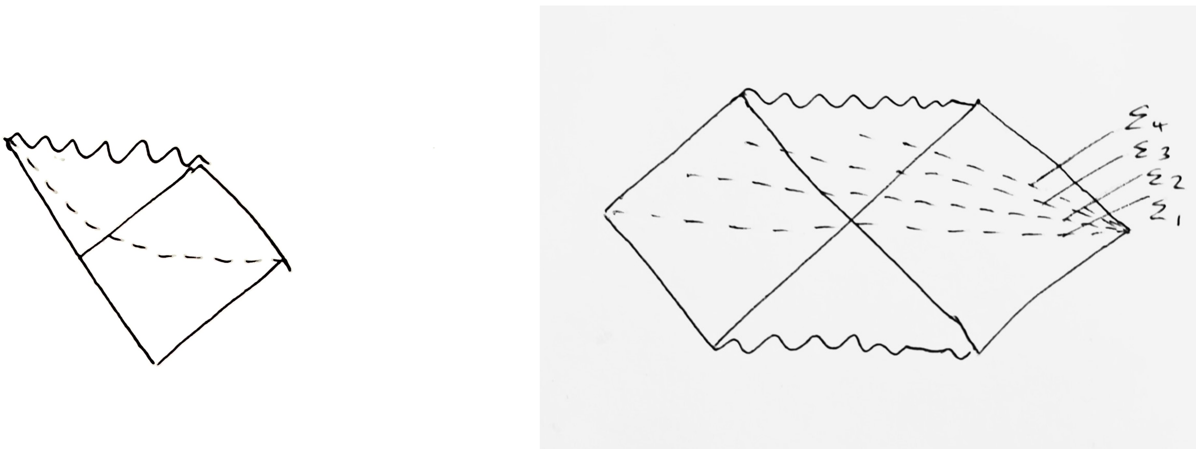


Figure 2.1: Penrose Diagrams, Σ_{t_0} dashed, Left: Ingoing Eddington-Finkelstein Coordinates, Right: Isotropic Coordinates.

Lapse Gauge Conditions

The simplest lapse choice would be to enforce $\alpha = 1$, called geodesic slicing, with the hypersurface following intergral curves of n^μ; given that geodesics can converge this can lead to hypersurface self-intersection which breaks the definition of a Cauchy surface and the simulation will likely fail. Another problem is that a black hole singularity can be reached in finite simulation time. Geodesic slicing can be modified to the maximal slicing condition which keeps the volume element $\sqrt{-g}$ constant along geodesics. This means as $\gamma \to \infty$ nearing a singularity $\alpha \to 0$ from Eq. (2.1.68), causing the hypersurface to advance more slowly before a singularity is reached. This property is called singularity avoiding and is crucial for numerical stability (unless using excision?). Maximal slicing can be implemented by forcing $\mathcal{K} = \partial_t \mathcal{K} = 0$ $\forall t$ which requires a slow elliptic solve for α at each timestep. Instead α is promoted to an evolution variable and is evolved along with every other simulation variable. To do this we can pick an algebraic slicing condition of the Bona-Masso type,

$$\mathcal{L}_m \alpha = \partial_t \alpha - \beta^i \partial_i \alpha = -\alpha^2 f(\alpha) \mathcal{K}. \tag{2.1.102}$$

This is used with $f = 2\alpha^{-1}$ in GRChombo [REF] giving,

$$\mathcal{L}_m \alpha = -2\alpha \mathcal{K}, \tag{2.1.103}$$

and using gaussian normal coordinates $\beta^i = 0$ it reduces to,

$$\alpha = 1 + \ln \gamma, \tag{2.1.104}$$

which is called 1+log slicing; this is very common in Numerical Relativity codes. In practice 1+log slicing is strongly singularity avoiding reaching $\alpha = 0$ before the singularity. This is modified in the CCZ4 scheme to,

$$\partial_t \alpha = -2\alpha \left[\mathcal{K} - 2\Theta\right] + \beta^i \partial_i \alpha, \tag{2.1.105}$$

which is very similar to before; it should be noted that the advection term is important for black hole simulations.

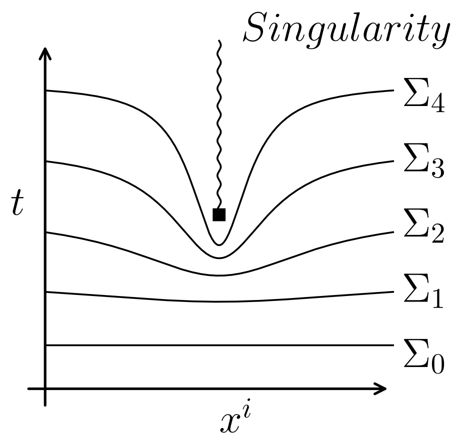


Figure 2.2: Diagram showing the time evolution of a hypersurface using a singularity avoiding lapse gauge condition. The vertical squiggled line represents a physical singularty that is formed at some point in spacetime, potentially from the collapse of matter to a black hole.

Shift Gauge Conditions

So far the shift vector β^i has been ignored, the simplest choice for it would be $\beta^i = 0$ as in 1+log slicing. Using 1+log slicing causes great stretching and shearing of Σ_t in the neghbourhood of a singularity as in Fig. (2.2); the effect of this is that neighbouring gridpoints have large differences in field values leading to inaccurate and unstable evolutions. Another negative side effect is that the computational domain can fall inside an event horizon in black hole simulations. To counteract this we want to pick a shift vector that minimises hypersurface shear σ_{ij} which can be defined as [REF] [ref 1977 smarr and york],

$$\sigma_{ij} := \perp_i^\mu \perp_j^\nu \left[\nabla_{(\mu} n_{\nu)} - \frac{1}{3} \gamma^{ab} \nabla_{(a} n_{b)} \gamma_{\mu\nu} \right], \tag{2.1.106}$$

where σ_{ij} is tracefree corresponding to shearing rather than inflation or expansion. Minimising the total shear Σ,

$$\Sigma = \int \sigma_{ij} \sigma^{ij} \sqrt{\gamma}\, \mathrm{d}x^3, \tag{2.1.107}$$

with respect to β^i leads to an elliptic PDE to be solved for each β^i at each time step that minimises shear,

$$\delta\Sigma = 0 \to \mathcal{D}_i \sigma^{ij} = 0. \tag{2.1.108}$$

This is known as the minimal shift condition. As before, promoting β^i to be evolution variables is computationally cheaper that solving a set of PDE's at each time step. A very common choice is to promote the elliptic PDE for β^i into a hyperbolic equation via introducing a $\partial_t^2 \beta^i$ term and an artificial damping term parameterised by η. This becomes a damped wave equation and is supposed to transport away any part of β^i which does not satisfy $\mathcal{D}_i \sigma^{ij} = 0$. This requires Sommerfeld (outgoing wave) boundary conditions as in [REF]. In GRChombo the standard gamma driver shift condition is used,

$$\partial_t \beta^i = F B^i, \tag{2.1.109}$$

$$\partial_t B^i = \partial_t \tilde{\Gamma}^i - \eta B^i, \tag{2.1.110}$$

where $F = 3/4$ and $\eta = 1$ are used.

TALK ABOUT MOVING PUNCTURE GAUGE AND CHI CLOOR OF CHI TEN MINUS 6 (OR IS IT 5?)

2.1.11 stuff

FIX THE FUCKING HAND REFERNECES!

2.2 Mathematical Modelling of Boson Stars

2.2.1 Action

The Boson Stars considered are a complex Klein Gordon Scalar field, φ, minimally coupled to gravity. The action is the Einstein-Hilbert vacuum action plus the matter action for curved space,

$$S = \int_{\mathcal{M}} \left[\mathcal{L}_{EH} + \mathcal{L}_M\right] \sqrt{-g}\,\mathrm{d}x^4, \tag{2.2.1}$$

$$\mathcal{L}_{EH} = \frac{1}{16\pi G} R, \tag{2.2.2}$$

$$\mathcal{L}_M = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi^* \nabla_\nu \varphi - \frac{1}{2} V(|\varphi|^2), \tag{2.2.3}$$

Here V is the Klein-Gordon potential and it's effect on boson stars is discussed in []. Some common choices of poteantials are,

$$V = \frac{m^2 c^2}{\hbar^2} |\varphi|^2, \tag{2.2.4}$$

$$V = \frac{m^2 c^2}{\hbar^2} |\varphi|^2 + \frac{1}{2}\Lambda_4 |\varphi|^4, \tag{2.2.5}$$

$$V = \frac{m^2 c^2}{\hbar^2} |\varphi|^2 \left(1 - \frac{|\varphi|^2}{2\sigma^2}\right)^2, \tag{2.2.6}$$

where $\hbar$ and c are given for completeness but will be set to unity from now on. Considering only the m^2 term, which corresponds to the squared mass of the particle in the quantum theory, we get a massive wave equation linear in φ, leading to so called mini Boson stars. Having $\Lambda_4 \neq 0$ gives self-interacting stars which have a nonlinear wave equation corresponding to particle creation and annihilation at the quantum level; self interacting potentials can include higher order terms in φ such as $|\varphi|^6$, $|\varphi|^8$ and more. These self interacting potentials tend to have star solutions with a higher density. Finally, Eq. (2.2.6) describes the solitonic potential, giving rise to boson stars with compactnesses comparable to neutron stars.

Varying the action with respect to the metric and scalar field return the Einstein Field equations and the Klein Gordon equation of curved space respectively,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.2.7}$$

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi = \frac{\partial V}{\partial |\varphi|^2} \varphi. \tag{2.2.8}$$

Collectively these are known as the Einstein-Klein-Gordon (EKG) equations. From Eq. (1.4.73), the boson-specific stress energy tensors are,

$$T_{\mu\nu} := -2 \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M, \tag{2.2.9}$$

$$T_{\mu\nu} = \frac{1}{2} \nabla_\mu \varphi^* \nabla_\nu \varphi + \frac{1}{2} \nabla_\nu \varphi^* \nabla_\mu \varphi - \frac{1}{2} g_{\mu\nu} \left[g^{\alpha\beta} \nabla_\alpha \varphi^* \nabla_\beta \varphi + V\right]. \tag{2.2.10}$$

Studying neutron stars requires the fermionic, or ordinary fluid, stress tensor $\boldsymbol{T}_F$;

$$T_F^{\mu\nu} = \left[\rho c^2 + P\right] \frac{u^\mu u^\nu}{c^2} + P g^{\mu\nu} + 2u^{(\mu} q^{\nu)} + \pi^{\mu\nu} + \dots\,. \tag{2.2.11}$$

The continuity equation from Eq. (1.4.26), $\nabla_\nu T^{\mu\nu} = 0$, returns the highly nonlinear relativistic Navier-Stokes equations of curved space. The viscosity term $\pi^{\mu\nu}$ and heat flux q^μ are often omitted for simplicity. The remaining variables ρ, P and u^μ are the fluid density, pressure and worldline tangent. Just as in flat space, the Navier-Stokes equations can develop shockwaves and the use of sophisticated shock capturing schemes is required, unlike the linear Klein-Gordon equation which is linear in the principal part.

2.2.2 Solitons

A soliton is a wave that exhibits particle-like behaviour. More precisely, in classical field theory, a soliton is a field or set of fields in a localised configuration that can travel at constant speed but not disperse. For our purposes, we look for solitons in the Einstein-Klein-Gordon (EKG) system which are self gravitating localised scalar field and metric configurations. In the case of the real scalar field it was shown by [REF] that there are no long lived stars; however promoting the field to a complex scalar we can find a spherically symmetric stationary soliton with the following scalar field,

$$\varphi = \Phi(r)e^{i\omega t}, \tag{2.2.12}$$

in spherical polar coordinates $x^\mu \in \{t, r, \theta, \phi\}$. Traditionally, the polar areal gauge has been used for the metric's ansatz,

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -a^2(r)\mathrm{d}t^2 + b^2(r)\mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right], \tag{2.2.13}$$

where the boundary condition $b^2(0) = 1$ is demanded to avoid a conical singularity at the origin. However an isotropic gauge is more useful for simulations due to easier conversion to cartesian space-coordinates, for more information on isotropic coordinates see section 1.4.6. The polar areal solution must then be transformed into an isotropic solution. Alternatively, the approach taken in this report, is to start with an isotropic ansatz,

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\Omega^2(r)\mathrm{d}t^2 + \Psi^2(r)\mathrm{d}\boldsymbol{x}^2, \tag{2.2.14}$$

where $\mathrm{d}\boldsymbol{x}^2$ denotes the euclidean 3D line element; this changes between spherical polar or cartesian coordinates trivially. This ends up being slightly harder to solve for numerically, but no conversion to isotropic coordinates is needed afterwards.

To get a set of ODE's to solve for the functions $\{\Omega(r), \Psi(r), \Phi(r)\}$ we must turn to the Einstein Equation and Klein Gordon Equation. The Einstein Equations for $\{\mu, \nu\} = \{0, 0\}, \{1, 1\}, \{2, 2\}$ are the only components that give unique non-zero equations in spherical symmetry; they are,

$$\frac{\Omega^2\left[r\Psi'^2 - 2\Psi\left[r\Psi'' + 2\Psi'\right]\right]}{r\Psi^4} = 4\pi G\left[\Omega^2\left[\frac{P'^2}{\Psi^2} + V\right] + \omega^2 P^2\right], \tag{2.2.15}$$

$$\frac{2\Psi\Psi'\left[r\Omega' + \Omega\right] + r\Omega\Psi'^2 + 2\Psi^2\Omega'}{r\Psi^2\Omega} = 4\pi G\left[P'^2 - \Psi^2 V + \frac{\omega^2 P^2\Psi^2}{\Omega^2}\right], \tag{2.2.16}$$

$$r\left[-\frac{r\Psi'^2}{\Psi^2} + \frac{r\Psi'' + \Psi'}{\Psi} + \frac{r\Omega'' + \Omega'}{\Omega}\right] = -4\pi G r^2\Psi^2\left[\frac{P'^2}{\Psi^2} + V - \frac{\omega^2 P^2}{\Omega^2}\right]. \tag{2.2.17}$$

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ (left above) and the stress tensor $T_{\mu\nu}$ (right above) were obtained with a private, self written Mathematica notebook. Substituting the metric ansatz Eq. (2.2.14) and the field ansatz Eq. (2.2.12) into Eq. (2.2.8), the Klein Gordon equation becomes,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi = \frac{\partial V}{\partial|\varphi|^2}\varphi, \tag{2.2.18}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi(r)e^{i\omega t}\right] = \frac{\partial V}{\partial|\varphi|^2}\Phi(r)e^{i\omega t}, \tag{2.2.19}$$

$$\Phi'' = \Phi\Psi^2\left[V' - \frac{\omega^2}{\Omega^2}\right] - \Phi'\left[\frac{\Omega'}{\Omega} + \frac{\Psi'}{\Psi} + \frac{2}{r}\right]. \tag{2.2.20}$$

Simplifying the Einstein Equations and combinig with the Klein Gordon equation we get three ODE's to solve; the EKG system has been reduced to two second order ODE's and a first order ODE,

$$\Omega' = \frac{\Omega}{r\Psi' + \Psi}\left[2\pi G r\Psi\left[\Phi'^2 - \Psi^2 V + \frac{\omega^2\Phi^2\Psi^2}{\Omega^2}\right] - \Psi' - \frac{r\Psi'^2}{2\Psi}\right], \tag{2.2.21}$$

$$\Psi'' = \frac{\Psi'^2}{2\Psi} - \frac{2\Psi'}{r} - 2\pi G\left[V\Psi^3 + \Phi'^2\Psi + \frac{\omega^2\Phi^2\Psi^3}{\Omega^2}\right], \tag{2.2.22}$$

$$\Phi'' = \Phi\Psi^2\left[V' - \frac{\omega^2}{\Omega^2}\right] - \Phi'\left[\frac{\Omega'}{\Omega} + \frac{\Psi'}{\Psi} + \frac{2}{r}\right]. \tag{2.2.23}$$

This is turned into a set of five first order ODE's to numerically integrate from $r = 0$ out to large radius. Note that if we had used the polar areal ansatz in Eq. (2.2.13) the equation for Φ would also be first order; reducing the EKG system to four first order ODE's.

2.2.3 3+1 Klein Gordon System

Now let's project the Klein Gordon equation in a 3+1 split to get the evolution equations. The first step is to turn the second order Klein-Gordon equation into two first order ones

$$\mathcal{L}_m\varphi = ..., \tag{2.2.24}$$

$$\mathcal{L}_m\Pi = ..., \tag{2.2.25}$$

where Π is the foliation dependant definition of conjugate momentum to the complex scalar field,

$$\Pi := -\mathcal{L}_n\varphi. \tag{2.2.26}$$

Decomposing the Klein Gordon Equation,

$$\nabla^\mu\nabla_\mu\varphi = V'\varphi = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\left[\gamma^{\mu\nu} - n^\mu n^\nu\right]\partial_\nu\varphi\right] = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\left[\mathcal{D}^\mu\varphi - n^\mu\mathcal{L}_n\varphi\right]\right]. \tag{2.2.27}$$

The term with $\mathcal{D}^\mu$ simplifies like,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\mathcal{D}^\mu\varphi\right] = \frac{1}{\alpha\sqrt{\gamma}}\partial_\mu\left[\alpha\sqrt{\gamma}\mathcal{D}^\mu\varphi\right] = \mathcal{D}_\mu\mathcal{D}^\mu\varphi + \mathcal{D}^\mu\varphi\,\partial_\mu\ln\alpha, \tag{2.2.28}$$

and the remainder becomes,

$$-\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}n^\mu\mathcal{L}_n\varphi\right] = -\left[\nabla\cdot n + n\cdot\partial\right]\mathcal{L}_n\varphi = -\mathcal{K}\Pi + \mathcal{L}_n\Pi. \tag{2.2.29}$$

Combining these results, the full Klein Gordon system is constructed,

$$\mathcal{L}_m\Pi = -\mathcal{D}^\mu\varphi\,\partial_\mu\alpha + \alpha\left[\mathcal{K}\Pi - \mathcal{D}_\mu\mathcal{D}^\mu\varphi + V'\varphi\right], \tag{2.2.30}$$

$$\mathcal{L}_m\varphi = -\alpha\Pi. \tag{2.2.31}$$

The final matter term we must decompose is the Klein-Gordon stress tensor in Eq. (2.2.10) with Eqs. (2.1.45), (2.1.46) and (2.1.47).

$$\rho = n^\mu n^\nu T_{\mu\nu} = \frac{1}{2}|\Pi|^2 + \frac{1}{2}\gamma^{ij}\mathcal{D}_i\varphi^*\mathcal{D}_j\varphi + \frac{1}{2}V(|\varphi|^2), \tag{2.2.32}$$

$$S_i = -\perp_i^\mu n^\nu T_{\mu\nu} = \frac{1}{2}\left[\Pi^*\mathcal{D}_i\varphi + \Pi\mathcal{D}_i\varphi^*\right], \tag{2.2.33}$$

$$S_{ij} = \perp_i^\mu\perp_j^\nu T_{\mu\nu} = \mathcal{D}_{(i}\varphi\mathcal{D}_{j)}\varphi^* - \frac{1}{2}\left[\gamma^{ij}\mathcal{D}_i\varphi\mathcal{D}_j\varphi^* - |\Pi|^2 + V(|\varphi|^2)\right]. \tag{2.2.34}$$

2.2.4 Klein Gordon's Noether Charge

For the complex scalar field, we have the U(1) symmetry,

$$\varphi \to \varphi e^{i\epsilon} \approx \varphi + i\epsilon\varphi, \tag{2.2.35}$$
$$\varphi^* \to \varphi^* e^{-i\epsilon} \approx \varphi^* - i\epsilon\varphi^*, \tag{2.2.36}$$

which leaves the Lagrangian unchanged and therefore the total action. The associated conserved current j and current density $\mathcal{J}$ are then,

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\nabla_\mu\varphi}\delta\varphi + \frac{\delta\mathcal{L}}{\delta\nabla_\mu\varphi^*}\delta\varphi^*, \tag{2.2.37}$$
$$j^\mu = ig^{\mu\nu}\left[\varphi\nabla_\nu\varphi^* - \varphi^*\nabla_\nu\varphi\right], \tag{2.2.38}$$

where the current satisfies the continuity equation,

$$\nabla_\mu j^\mu = 0. \tag{2.2.39}$$

Using Eq. (1.3.142), the continuity equation can be re-written as,

$$\nabla_\mu j^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}j^\mu\right) = \frac{1}{\sqrt{-g}}\partial_\mu\mathcal{J}^\mu = 0, \tag{2.2.40}$$

where $\mathcal{J}^\mu = \sqrt{-g}j^\mu$ is the current expressed as a tensor density. Therefore,

$$\partial_\mu\mathcal{J}^\mu = 0, \tag{2.2.41}$$

is also true and even in curved space the current $\boldsymbol{\mathcal{J}}$ obeys a conservation equation; this means there must be some conserved charge $\mathcal{Q}$ associated with the current. Integrating Eq. (2.2.41) over a manifold $\mathcal{M}$ gives,

$$\int_\mathcal{M}\nabla_\mu j^\mu\sqrt{-g}\,\mathrm{d}x^4 = 0, \tag{2.2.42}$$
$$\int_\mathcal{M}\partial_\mu\left[\sqrt{-g}j^\mu\right]\mathrm{d}x^4 = 0, \tag{2.2.43}$$
$$\int_\mathcal{M}\partial_\mu\mathcal{J}^\mu\mathrm{d}x^4 = 0, \tag{2.2.44}$$
$$\int_{t_0}^{t_1}\left[\int_{\Sigma_t}\partial_0\mathcal{J}^0\mathrm{d}x^3\right]\mathrm{d}t = -\int_{t_0}^{t_1}\left[\int_{\Sigma_t}\partial_i\mathcal{J}^i\mathrm{d}x^3\right]\mathrm{d}t, \tag{2.2.45}$$
$$\int_{t_0}^{t_1}\left[\int_{\Sigma_t}\partial_0\mathcal{J}^0\mathrm{d}x^3\right]\mathrm{d}t = -\int_{t_0}^{t_1}\left[\int_{\partial\Sigma_t}\hat{s}_i\mathcal{J}^i\mathrm{d}x^2\right]\mathrm{d}t, \tag{2.2.46}$$
$$\int_{t_0}^{t_1}\left[\int_{\Sigma_t}\partial_0\mathcal{J}^0\mathrm{d}x^3\right]\mathrm{d}t = 0, \tag{2.2.47}$$

where the flat space divergence theorem was used and $\hat{\boldsymbol{s}}$ is the flat space normal to $\partial\Sigma_t$, the boundary of Σ_t. The term containing $\hat{s}_i\mathcal{J}^i$ integrates to zero over $\partial\Sigma_t$ due to $\mathcal{J}$ vanishing on $\partial\Sigma_t$. The $\mathcal{J}^0$ term can be simplified by permuting the time derivative using,

$$\partial_0\int_{\Sigma_t}\mathcal{J}^0\mathrm{d}x^3 = \int_{\Sigma_t}\partial_0\mathcal{J}^0\mathrm{d}x^3 + \lim_{\Delta x^0\to 0}\left[\frac{1}{\Delta x^0}\int_{\Delta\Sigma_t}\left[\mathcal{J}^0 + \Delta x^0\partial_0\mathcal{J}^0\right]\mathrm{d}x^3\right], \tag{2.2.48}$$

where the last term vanishes as $\mathcal{J}$ vanishes near $\partial\Sigma$, and Eq. (2.2.47) becomes,

$$\int_{t_0}^{t_1}\left[\partial_0\int_{\Sigma_t}\mathcal{J}^0\mathrm{d}x^3\right]\mathrm{d}t = 0, \tag{2.2.49}$$

and the formula for the conserved charge Q can be read off as,

$$\partial_0 Q = 0, \tag{2.2.50}$$

$$Q = \int_{\Sigma_t} \mathcal{J}^0 \mathrm{d}x^3. \tag{2.2.51}$$

The charge density $\mathcal{Q}$ is defined as,

$$Q := \int_{\Sigma_t} \mathcal{Q}\sqrt{\gamma}\,\mathrm{d}x^3, \tag{2.2.52}$$

$$\mathcal{Q} = \frac{\mathcal{J}^0}{\sqrt{\gamma}} = \frac{\sqrt{-g}j^0}{\sqrt{\gamma}} = \alpha j^0, \tag{2.2.53}$$

where Eq. (2.1.68), saying $\sqrt{-g} = \alpha\sqrt{\gamma}$, was used. Finally we get an expression for the total Noether charge $N = Q$,

$$N = i \int_{\Sigma_t} \sqrt{-g}\left[\varphi\nabla^0\varphi^* - \varphi^*\nabla^0\varphi\right]\mathrm{d}x^3. \tag{2.2.54}$$

Using $\sqrt{-g} = \alpha\sqrt{\gamma}$ again and $\alpha\nabla^0\varphi = -n_\mu\nabla^\mu\varphi = \Pi$ from Eq. (2.2.26) we get the following neat formula,

$$N = i \int_{\Sigma_t} \left[\varphi\Pi^* - \varphi^*\Pi\right]\sqrt{\gamma}\,\mathrm{d}x^3. \tag{2.2.55}$$

Equivalently, the Noether charge density $\mathcal{N}$ is ,

$$\mathcal{N} = i\left(\varphi\Pi^* - \varphi^*\Pi\right). \tag{2.2.56}$$

MAYBE POINT THIS SECTION AT THE CONTINUITY PAPER, OR JSUT SAY WE GENERALISE IT IN THERE?

THE IDEAS OF THIS SECTION ARE FURTHAR EXPLORED FOR ENERGY, MOMENTUM, ANGULAR MOMENTUM AND PROCA FIELDS IN SECTION REFFF

2.2.5 Boosted Boson Stars and Black Holes

Let us now consider a moving star, this corresponds to boosting a stationary soliton solution. There is no unique way of doing this as any coordinate transformation that reduces to a Minkowski spacetime boost at large radius will do the job. All the degrees of freedom we have can be absorbed into a coordinate gauge choice, so it makes sense to choose the trivial, constant valued boost, with rapidity $\chi = \operatorname{arctanh}(v)$ for a velocity v, from Special Relativity. Using Cartesian coordinates, boost matrix $\mathbf{\Lambda}$ for a boost in the x direction is,

$$\Lambda^\mu_\nu = \exp\begin{pmatrix} 0 & -\chi & 0 & 0 \\ -\chi & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh(\chi) & -\sinh(\chi) & 0 & 0 \\ -\sinh(\chi) & \cosh(\chi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2.57}$$

Declaring the boosted and rest frame to have coordinates x^μ and $\tilde{x}^\mu$,

$$\tilde{x}^\mu = [\Lambda^{-1}]^\mu{}_\nu x^\nu, \tag{2.2.58}$$

and the metric transforms via the tensor transformation law like,

$$\tilde{g}_{\mu\nu}(\tilde{x}^\sigma) = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}g_{\alpha\beta}(\tilde{x}^\sigma) = \Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu g_{\alpha\beta}(\tilde{x}^\sigma), \tag{2.2.59}$$

where the inverse boost matrix $\mathbf{\Lambda}^{-1}$ can be found simply by $\mathbf{\Lambda}^{-1}(\chi) = \mathbf{\Lambda}(-\chi)$. We choose the boosted soliton's initial Cauchy surface to be the level set of $\tilde{t} = 0$; this corresponds to the star begin stationary in the boosted frame. The coordinates and metric transform as,

$$x^\mu = \{t, x, y, z\} = \{\tilde{t}\cosh(\chi) + \tilde{x}\sinh(\chi), \tilde{x}\cosh(\chi) + \tilde{t}\sinh(\chi), \tilde{y}, \tilde{z}\}, \tag{2.2.60}$$

$$g_{\mu\nu} = \mathrm{diag}\{-\Omega^2, \Psi^2, \Psi^2, \Psi^2\}, \tag{2.2.61}$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -\Omega^2\cosh^2(\chi) + \Psi^2\sinh^2(\chi) & \sinh(\chi)\cosh(\chi)\left[\Omega^2 - \Psi^2\right] & 0 & 0 \\ \sinh(\chi)\cosh(\chi)\left[\Omega^2 - \Psi^2\right] & \Psi^2\cosh^2(\chi) - \Omega^2\sinh^2(\chi) & 0 & 0 \\ 0 & 0 & \Psi^2 & 0 \\ 0 & 0 & 0 & \Psi^2 \end{pmatrix}, \tag{2.2.62}$$

as the star is at rest using coordinates x^μ rather than $\tilde{x}^\mu$. Comparing this boosted metric to the $3+1$ decomposed metric in Eq. (2.1.65) we can read off the shift vector $\tilde{\beta}_i$, the 3 metric $\tilde{\gamma}_{ij}$ and obtain the lapse and metric determinant,

$$\tilde{\alpha}^2 = \frac{\Psi^2\Omega^2}{\Psi^2\cosh^2(\chi) - \Omega^2\sinh^2(\chi)}, \tag{2.2.63}$$

$$\tilde{\gamma} = \det\tilde{\gamma}_{ij} = \Psi^4\left[\Psi^2\cosh^2(\chi) - \Omega^2\sinh^2(\chi)\right]. \tag{2.2.64}$$

Finally, the conformal 3-metric with unit determinant is,

$$\bar{\gamma}_{ij} = \tilde{\gamma}^{-\frac{1}{3}}\begin{pmatrix} \Psi^2\cosh^2(\chi) - \Omega^2\sinh^2(\chi) & 0 & 0 \\ 0 & \Psi^2 & 0 \\ 0 & 0 & \Psi^2 \end{pmatrix}, \tag{2.2.65}$$

where normally $\tilde{\gamma}_{ij}$ is the conformal 3-metric, but to avoid confusion it is denoted $\bar{\gamma}_{ij}$ in this section. Turning our attention to the matter fields now we only need to change the coordinate dependance, like $\varphi(x) \to \varphi(\tilde{x})$, given that φ and Π are (complex) scalar fields. Given that $\tilde{t} = 0$ describes a time slice in the rest frame (where the star has non-zero velocity), $t = \tilde{x}\sinh(\chi)$ in the rest frame and we get the following boosted complex scalar field,

$$\varphi = \Phi(r)e^{i\omega\tilde{x}\sinh(\chi)}, \tag{2.2.66}$$

where r is the radius in the boosted frame; $r = \sqrt{\tilde{x}^2\cosh^2(\chi) + \tilde{y}^2 + \tilde{z}^2}$. Note the field is modulated by an oscillatory phase now with wavenumber $k = \omega\tilde{x}\sinh(\chi)$; nodal planes in $\mathrm{Re}(\varphi)$ appear perpendicular to velocity. The conjugate momentum $\tilde{\Pi}$, defined in Eq. (2.2.26), in the rest frame it becomes,

$$\tilde{\Pi}(\tilde{x}^\mu) = -\mathcal{L}_{\tilde{n}}\varphi(\tilde{x}^\mu) = -\frac{1}{\tilde{\alpha}}\tilde{m}\cdot\tilde{\partial}\varphi = -\frac{1}{\tilde{\alpha}}\left[\tilde{\partial}_0 - \tilde{\beta}^i\tilde{\partial}_i\right]\Phi(r)e^{i\omega t}. \tag{2.2.67}$$

Inconveniently we cannot simply evaluate $\tilde{\Pi}$ in the boosted frame as this has a different spacetime foliation and the normal vector $\boldsymbol{n} \neq \tilde{\boldsymbol{n}}$ is genuinely changed; not just transforming components under coordinate transformation. Explicitly writing the contravariant components of the shift vector,

$$\tilde{\beta}^i = \left(\frac{\sinh(\chi)\cosh(\chi)\left[\Omega^2 - \Psi^2\right]}{\Psi^2\cosh^2(\chi) - \Omega^2\sinh^2(\chi)}, 0, 0\right), \tag{2.2.68}$$

and using the following derivative formulae,

$$\partial_{\tilde{t}} = \cosh(\chi)\partial_t + \sinh(\chi)\partial_x, \tag{2.2.69}$$

$$\partial_{\tilde{x}} = \cosh(\chi)\partial_x + \sinh(\chi)\partial_t, \tag{2.2.70}$$

$$\partial_t\varphi = \Phi\partial_t e^{i\omega t} = i\omega\Phi e^{i\omega t}, \tag{2.2.71}$$

$$\partial_x\varphi = \frac{\partial r}{\partial x}\Phi' e^{i\omega t} = \frac{x}{r}\Phi' e^{i\omega t}, \tag{2.2.72}$$

we get an expression for the conjugate momentum of a boosted star. Again, setting $\tilde{t} = 0$ gives the conjugate momentum on the surface $\tilde{t} = 0$ to be used as initial conditions,

$$\widetilde{\Pi} = -\frac{1}{\tilde{\alpha}} \left[\left[\sinh(\chi) - \tilde{\beta}^1 \cosh(\chi) \right] \frac{\tilde{x} \cosh(\chi)}{r} \Phi' + i\omega \left[\cosh(\chi) - \tilde{\beta}^1 \sinh(\chi) \right] \Phi \right] e^{i\omega \tilde{x} \sinh(\chi)}. \tag{2.2.73}$$

The penultimate ingredient is the intrinsic curvature $\tilde{\boldsymbol{K}}$, defined in Eq. (2.1.10). Similarly to the conjugate momentum, the definition of $\tilde{\boldsymbol{K}}$ depends on the spacetime foliation so using $K_{ij} = 0$ in the stars rest frame and using the tensor transformation to conclude that $\tilde{K}_{ij} = 0$ in the rest frame (where the star moves) is incorrect. Instead the components $\tilde{K}_{ij}$ must be calculated from scratch with the correct normal vector $\boldsymbol{n}$ like,

$$\widetilde{\mathcal{K}}_{\mu\nu} := -\frac{1}{2} \mathcal{L}_{\tilde{n}} \tilde{\gamma}_{\mu\nu} = -\frac{1}{2\tilde{\alpha}} \mathcal{L}_{\tilde{m}} \tilde{\gamma}_{\mu\nu} = -\frac{1}{2\tilde{\alpha}} \left[\tilde{m} \cdot \tilde{\partial} \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik} \tilde{\partial}_j \tilde{m}^k + \tilde{\gamma}_{jk} \tilde{\partial}_i \tilde{m}^k \right]. \tag{2.2.74}$$

A private, self written mathematica script gives the following explicit form for the components of $\tilde{K}_{ij}$,

$$\begin{aligned}
\alpha^{-1} \widetilde{\mathcal{K}}_{xx} &= \cosh^2(\chi) \sinh(\chi) \frac{x}{r} \frac{\left[v^2 \Omega^2 \Omega' + \Psi \Omega \Psi' - 2\Psi^2 \Omega' \right]}{\Psi^2 \Omega}, & (2.2.75) \\
\alpha^{-1} \widetilde{\mathcal{K}}_{xy} &= \cosh(\chi) \sinh(\chi) \frac{y}{r} \frac{\left[\Omega \Psi' - \Psi \Omega' \right]}{\Psi \Omega}, & (2.2.76) \\
\alpha^{-1} \widetilde{\mathcal{K}}_{xz} &= \cosh(\chi) \sinh(\chi) \frac{z}{r} \frac{\left[\Omega \Psi' - \Psi \Omega' \right]}{\Psi \Omega}, & (2.2.77) \\
\alpha^{-1} \widetilde{\mathcal{K}}_{yy} &= -\sinh(\chi) \frac{x}{r} \frac{\Psi'}{\Psi}, & (2.2.78) \\
\widetilde{\mathcal{K}}_{zz} &= \widetilde{\mathcal{K}}_{yy}, & (2.2.79)
\end{aligned}$$

where the $\{x, y, z\}$ need to be expanded in terms of $\{\tilde{x}, \tilde{y}, \tilde{z}\}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

The final object needed is the three-dimensional connection symbols $\Upsilon^i{}_{jk}$, these are calculated numerically after the initial data is loaded in using the definition from Eq. (1.3.77),

$$\Upsilon^i_{jk} = \frac{1}{2} \gamma^{im} \left(\partial_k g_{jm} + \partial_j g_{mk} - \partial_m g_{jk} \right). \tag{2.2.80}$$

The boost formalism described here can apply this to the Black Hole spacetime by an identical procedure setting

$$\begin{aligned}
\varphi &= 0, & (2.2.81) \\
\Pi &= 0, & (2.2.82) \\
\Omega &= \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}, & (2.2.83) \\
\Psi &= \left[1 + \frac{M}{2r} \right]^2, & (2.2.84)
\end{aligned}$$

corresponding to the isotropic Schwarzschild black hole given in section 1.4.6.

2.2.6 Spherical Harmonics in Curved Space DO I KEEP THIS SECTION? MAYBE JUST FOR INTERPITING SOME SIMS

Spherical harmonics are an orthonormal function basis for the surface of a sphere. They arise when looking for solutions to the 3D spherical polar laplacian

$$\nabla^2 \varphi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right), \quad \mu, \nu \in \{1, 2, 3\}. \tag{2.2.85}$$

On the hypersurface $r = 1$ we get the following metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \tag{2.2.86}$$

and on this surface the spherical harmonics $Y_{lm}(\theta,\phi)$ satisfy the following condition.

$$\mathcal{D}_\mu \mathcal{D}^\mu Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi), \quad x^\mu \in \{\theta,\phi\} \tag{2.2.87}$$

This means we can take any spherically symmetric and static metric with $g_{\phi\phi} = \sin^2\theta g_{\theta\theta}$ and replace the angular part of the wave equation with $l(l+1)$. For a spherically symmetric spacetime this gives the Klein Gordon equation for scalar hair.

$$\nabla_\mu \nabla^\mu \varphi = V'\varphi, \quad \varphi = T(t)R(r)Y_{lm}(\theta,\phi) \tag{2.2.88}$$

$$\varphi^{-1}\nabla_\mu \nabla^\mu \varphi = g^{tt}\frac{\ddot{T}}{T} + \frac{1}{R\sqrt{|g|}}\partial_r\left(\sqrt{|g|}g^{rr}\partial_r R\right) - l(l+1)g^{\theta\theta} \tag{2.2.89}$$

This is a second order ODE for the radial profile R and is an eigenvalue problem for ω if we assume $T = e^{i\omega t}$. Assuming $T = e^{-kt}$ can be done on-top of the black hole metric () and requires the assumption of no back reaction of the scalar field on the metric; this gives an eigenvalue problem in k instead. This leads to the following ODE for the radial profile.

$$\frac{1}{R\sqrt{|g|}}\partial_r\left(\sqrt{|g|}\Psi^{-4}\partial_r R\right) = k^2\frac{\Omega^2}{\Psi^2} + \frac{l(l+1)}{r^2\Psi^4} + V' \tag{2.2.90}$$

Simulations shown later involve boson stars of mass M and black holes of mass $M \to 10M$; these simulations often produce scalar hair about these black holes that is orders of magnitude less massive than the boson stars. In this regime the above equation is assumed relevant.

MAYBE JUST USE THIS SECTION TO TALK ABOUT THE COLLISIONS THAT MAYBE SEEM TO FOLLOW THIS RULE ... SAY HOW EVEN THOUGHT ITS NOT REALLY SPHERICALLY SYMMETRIC MAYBE IT CAN BE APPROXIMATES BY THIS? AND WE CAN'T USE SUPERPOSISION OF SOLUTIONS REALLY

Chapter 3

Numerical Methods and GRChombo

3.1 Numerical Methods

3.1.1 Numerical Discretisation of Spacetime

The are many ways to numerically time evolve a field theory on some spatial surface. Some popular methods include spectral methods, fourrier methods, finite element models and finite volume methods [MAYBE ADD REFS?]. The method used throughout this work is the finite difference framework. This entails modelling a continuous spacetime with a discrete lattice of points; usually this lattice is cuboidal or cubic. A field $\phi(x^\mu)$ over a manifold $\mathcal{M}$ is expressed as a set of discrete values ϕ^n_{ijk}, for integer $\{n, i, j, k\}$, where the coordinate $(x^n_{ijk})^\mu = \{n\Delta t, i\Delta x, j\Delta y, k\Delta z\}$ where Δt, Δx, Δy and Δz represent the grid spacing for example in cartesian coordinates. In the limit that the grid spacing tends to zero, the lattice and discretized fields perfectly model the continuum. For a detailed introduction to numerical methods the reader is directed to [REF] [NUMERICAL RECIPES IN C].

To calculate gradients on a lattice, we can consider a two dimensional manifold spanned by coordinates $\{t, x\}$. We can no longer lean on the traditional definition of $\mathrm{d}f/\mathrm{d}x$,

$$\frac{\mathrm{d}f}{\mathrm{d}x} := \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right), \tag{3.1.1}$$

as there no longer exists two points x and $x+\Delta x$ that are infinitessimally close to each other. Derivatives are now calculated by comparing gridpoints a finite distance from each other; this requires the well known formula for the taylor expansion of a function about a point x_0,

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \dots . \tag{3.1.2}$$

To elucidate this idea, a formula for the second derivative we will calculate an expression for the second derivative. Picking five lattice points, equally spaced by Δx, with coordinates $\{x_{-2}, x_{-1}, x_0, x_1, x_2\}$, and writing $f(x_i) = f_i$, we can use Eq. (3.1.2) to write,

$$f_2 \approx f_0 + 2\Delta x f_0' + 2(\Delta x)^2 f_0'' + \frac{4}{3}(\Delta x)^3 f_0''' + \frac{2}{3}(\Delta x)^4 f_0'''', \tag{3.1.3}$$

$$f_1 \approx f_0 + \Delta x f_0' + \frac{1}{2}(\Delta x)^2 f_0'' + \frac{1}{6}(\Delta x)^3 f_0''' + \frac{1}{24}(\Delta x)^4 f_0'''', \tag{3.1.4}$$

$$f_0 \approx f_0, \tag{3.1.5}$$

$$f_{-1} \approx f_0 - \Delta x f_0' + \frac{1}{2}(\Delta x)^2 f_0'' - \frac{1}{6}(\Delta x)^3 f_0''' + \frac{1}{24}(\Delta x)^4 f_0'''', \tag{3.1.6}$$

$$f_{-2} \approx f_0 - 2\Delta x f_0' + 2(\Delta x)^2 f_0'' - \frac{4}{3}(\Delta x)^3 f_0''' + \frac{2}{3}(\Delta x)^4 f_0'''', \tag{3.1.7}$$

where the taylor expansion is stopped at terms of order $(\Delta x)^4$. The equations above can be inverted to give,

$$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x_0} := f_0'' = \frac{-f_2 + 16 f_1 - 30 f_0 + 16 f_{-1} - f_{-2}}{12(\Delta x)^2}, \tag{3.1.8}$$

and an approximation for the second derivative using a discrete sum of neighbouring points, also called a stencil, has been obtained. Adding terms of order $(\Delta x)^5$ would not affect the result as the pairs of terms $\{f_2, f_{-2}\}$ and $\{f_1, f_{-1}\}$ appear in equal amounts; therefore Eq. (3.1.8) is correct upto a taylor expansion of order $(\Delta x)^6$. Given that f_0'' appears with a $(\Delta x)^2$ and Eq. (3.1.8) is correct untill terms of order $(\Delta x)^6$, then the expression is accurate up to fourth order. [CHECK THIS?] Any other derivate to any order accuracy (in any dimension) can be calculated in a similar fashion. In the limit that the grid spacing $\Delta x \to 0$ the approximations for the derivatives approach the exact contiuum limit; higher order accurate stencils approach the continuum limit more quickly.

3.1.2 Boundary Conditions

Another artefact of evolving field equations over a volume on a computer is that the volume must have finite volume; a computer does not have infinite memory to store an infinite amount of gridpoints. The usual way to deal with this problem is to enforce an algebraic condition on the fields on a surface surrounding the region of interest. Alternatively, an infinite volume can be modeled if coordinates are used which compactify the volume to some finite region, this corresponds to the grid space diverging or resolution becoming infintely low quality towards the boundary.

Common boundary conditions include Dirichlet (fixed value), Von-Neumann (fixed derivative) or some mix of these conditions. When discretising a field theory over a lattice of points, it is more common to re-categorize boundary conditions into types such as reflective, periodic or symmetric.

As an example in one spatial dimension, symmetric boundary conditions for a field ϕ about a point x_n could be implemented by creating extra *ghost cells* beyond the desired simulation domain with coordinates $\{x_{n+1}, x_{n+2}, x_{n+3}, ...\}$ separated by Δx and setting,

$$\phi_{n+1} = \phi_{n-1}, \quad \phi_{n+2} = \phi_{n-2}, \quad \phi_{n+3} = \phi_{n-3}, \quad \dots, \tag{3.1.9}$$

where $\phi_i = \phi(x_i)$. These extra ghost cells allow the calculation of derivatives at x_n using stencils, as shown in section 3.1.1; without these ghost cells the stencil would not be able to access points x_m where $m > n$. The number of ghost cells should be chosen to be the minimum required to allow the calculation of derivcative stencils at each point in the simulation domain. The desired location of the boundary condition does not have to coincide with a gridpoint. As an example, modifying the symmetric boundary condition to be centered on $x_n + \Delta x/2$ instead results in,

$$\phi_{n+1} = \phi_n, \quad \phi_{n+2} = \phi_{n-1}, \quad \phi_{n+3} = \phi_{n-2}, \quad \dots. \tag{3.1.10}$$

Generic boundary conditions can be imposed by assigning values to ghost cells similarly to above. Although the examples given in this section are in one dimension, the ideas generalise to arbitrary dimensions.

Sommerfeld Boundary Conditions

A very useful type of boundary condition is the Sommerfeld boundary condition [REF], used to approximate an outgoing wave being transmitted through the boundary condition. Sommerfeld boundary condtions can be derived from studying the solution to the wave equation in spherical symmetry in flat space,

$$-\frac{1}{v^2}\partial_t^2\Psi + \gamma^{ij}\mathcal{D}_i\mathcal{D}_j\Psi = -\frac{1}{v^2}\partial_t^2\Psi + \frac{1}{\sqrt{-\gamma}}\partial_i\left(\sqrt{\gamma}\gamma^{ij}\partial_j\Psi\right) = -\partial_t^2\Psi + \frac{1}{r^2}\partial_r\left(r^2\partial_r\Psi\right) = 0, \tag{3.1.11}$$

for some field Ψ with wavespeed v. In spherical polar coordinates, $\gamma^{ij} = \text{diag}\{1, r^{-2}, r^{-2}\text{cosec}^2(\theta)\}$. It can easily be shown that this has an outgoing wave solution,

$$\Psi(r,t) = \Psi_\infty + \frac{A}{r}\psi(r - vt), \tag{3.1.12}$$

for an asymptotic value Ψ_∞ and constant A. In differential form this can be written as,

$$\frac{1}{v}\partial_t\Psi + \partial_r\Psi + \frac{1}{r}(\Psi - \Psi_\infty) = 0. \tag{3.1.13}$$

The equation of motion for Ψ can be used to write ∂_t in terms of spatial derivatives and field values giving the new boundary condition which can be applied numerically as a regular mixed type boundary condition.

In general relativity, Sommerfeld boundary conditions are commonly used with $v = c = 1$ (the speed of light) to avoid reflections of matter and gravitational waves from the boundary of a simulation. It should be noted that Sommerfeld boundary conditions are approximate in general relativity for a number of reason. Matter often doesn't propogate at the speed of light, gravitational waves only obey a wave equation in flat space and the Sommerfeld boundary conditions were derived in spherical symmetry and flat space. These conditions are satisfied in the limit the simulation boundary is infinately far away from the centre of an asymptotically flat vacuum spacetime. It has been found experimentally in my work that ensuring the boundary conditions are sufficiently far away from any compact objects is very important in maintaining accuracy of the simulation when using Sommerfeld boundary conditions.

3.1.3 The Method of Lines

Assuming adequate boundary conditions are in place, the time evolution of initial data ϕ^n_{ijk} covering a spacelike computational grid can be done by applying a time integration scheme to the PDE governing the field $\phi(x^\mu)$. There are many ways to do this and the reader is directed to [REF] [NUMERICAL RECIPES IN C] for a comprehensive introduction. A common and simple method is the method of lines (MoL). The MoL reduces the dimensionality of the PDE problem to set of ODE's in time at each fixed coordinate and treats spatial derivatives as a function on this line through spacetime. For example, consider the partial differential equation,

$$\partial_t \phi(\boldsymbol{x}, t) = \hat{O}\phi(\boldsymbol{x}, t) + f(\phi, \boldsymbol{x}, t), \tag{3.1.14}$$

where $\hat{O}$ is some spatial derivative operator, f is a function and $\boldsymbol{x}$ are spacelike coordinates. In the MoL the operator $\hat{O}$ can be discretised on a grid like,

$$\hat{O}\phi(\boldsymbol{x}, t) \to (O\phi)_{ijk}(t), \tag{3.1.15}$$

where $(O\phi)_{ijk}(t) \approx \hat{O}\phi(x_{ijk}, t)$ is a sum of field values at neighbouring gridpoints generated by the method of finite differences. Discretising the function f on the grid gives the following ODE for each gridpoint with spatial indeces $\{i, j, k\}$,

$$\partial_t \phi_{ijk}(t) = (\hat{O}\phi)_{ijk}(t) + f_{ijk}(\phi, t) = F_{ijk}(\phi, t), \tag{3.1.16}$$

where $F_{ijk}(\phi, t)$ can be treated as the discretisation of a continuum function $F(\phi, \boldsymbol{x}, t)$.

3.1.4 Integration of ODE's

To perform the time evolution of the ODE Eq. (3.1.16), one now needs to pick a time integration method for an ODE (rather than a PDE if MoL was not used). The obvious choice might be to discretise the time integral like,

$$\partial_t \phi^n_{ijk} = \frac{\phi^{n+1}_{ijk} - \phi^n_{ijk}}{\Delta t}, \tag{3.1.17}$$

where Δt is the time-step of evolution. This can be substituted into Eq. (3.1.16) to give,

$$\phi^{n+1}_{ijk} = \phi^n_{ijk} + F^n_{ijk}\Delta t, \tag{3.1.18}$$

which is an explicit scheme; the desired object ϕ^n_{ijk} is given by an explicit formula. This is known as the Eular method and is often unstable; it can be shown to be completely unstable no matter how small Δt is taken to be for $F^n_{ijk} = A\phi^n_{ijk}$ for positive constant A. There is nothing stopping us instead writing,

$$\phi^{n+1}_{ijk} - F^{n+1}_{ijk}\Delta t = \phi^n_{ijk}, \tag{3.1.19}$$

which is known as an implicit scheme as the desired object ϕ^{n+1}_{ijk} is given by an implicit equation. This is called the backwards Eular method and is often stable, but only first order accurate in time. This can be improved to the second order accurate Crank-Nicolson method,

$$\phi^{n+1}_{ijk} - \frac{1}{2}F^{n+1}_{ijk}\Delta t = \phi^n_{ijk} + \frac{1}{2}F^n_{ijk}\Delta t, \tag{3.1.20}$$

but is still implicit in ϕ^n_{ijk}. These problem with implicit methods is that the F^{n+1}_{ijk} are some combination of ϕ^{n+1}_{ijk} and multiple other neighbouring gridpoints to $\{i, j, k\}$; the number of gridpoints increases with higher order accurate spatial derivatives. Solving the set of simultanious equations in Eq. (3.1.20), for example, requires the inverting of a very big (albeit sparse) matrix whose size scales with the number of gridpoints. This can be done in a single step if the F^n_{ijk} are linear in ϕ^n. For non-linear ODE's the F^n_{ijk} are non-linear in ϕ^n_{ijk}; in the best case one can linearise if and the ϕ^{n+1}_{ijk} can be solved with an iterative method, in the worst the implicit scheme is impossible to solve.

A stable and explicit method can be obtained by seeking a higher order accurate time derivative. In a similar fashion to the derivation of Eq. (3.1.8), one can write,

$$\phi^n_{ijk} = \phi^n_{ijk}, \tag{3.1.21}$$

$$\phi^{n-1}_{ijk} = \phi^n_{ijk} - \Delta t \dot{\phi}^n_{ijk} + \frac{1}{2}(\Delta t)^2 \ddot{\phi}^n_{ijk}, \tag{3.1.22}$$

$$\phi^{n-2}_{ijk} = \phi^n_{ijk} - 2\Delta t \dot{\phi}^n_{ijk} + 2(\Delta t)^2 \ddot{\phi}^n_{ijk}, \tag{3.1.23}$$

where the dot represents a time derivative and the taylor expansion has been given up to $(\Delta t)^2$ terms. Rearranging, these give

$$\partial_t \phi^n_{ijk} = \frac{-3\phi^n_{ijk} + 4\phi^{n-1}_{ijk} - \phi^{n-2}_{ijk}}{6\Delta t}. \tag{3.1.24}$$

Substituting this into Eq. (3.1.16) gives an explicit equation for ϕ^n_{ijk},

$$\phi^n_{ijk} = \frac{4}{3}\phi^{n-1}_{ijk} - \frac{1}{3}\phi^{n-2}_{ijk} - 2F_{ijk}\Delta t, \tag{3.1.25}$$

where the index n has been omitted from F_{ijk} as is can be replaced with any combination of F^n_{ijk}, F^n_{ijk} and F^n_{ijk} such as $F^n_{ijk} = \frac{4}{3}F^{n-1}_{ijk} - \frac{1}{3}F^{n-2}_{ijk}$. Even thought this method is explicit it requires two sets of initial data, ϕ^{n-1}_{ijk} and ϕ^{n-2}_{ijk}.

MAYBE CUT THIS BIT SHORT AT EQ 3.1.24

The Runge Kutta Method

The Runge-Kutta method is an explicit ODE integration scheme that can be made accurate to arbitrary order in Δt. For an ODE like

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} + F(\phi, t), \tag{3.1.26}$$

the widely used fourth order accurate Runge-Kutta (RK4) method first calculates four intermediate gradients $\{k_1, k_2, k_3, k_4\}$,

$$k_1 = F(\phi, t), \tag{3.1.27}$$

$$k_2 = F(\phi + \frac{1}{2}k_1\Delta t, t + \frac{1}{2}\Delta t), \tag{3.1.28}$$

$$k_3 = F(\phi + \frac{1}{2}k_2\Delta t, t + \frac{1}{2}\Delta t), \tag{3.1.29}$$

$$k_4 = F(\phi + k_3\Delta t, t + \Delta t), \tag{3.1.30}$$

that are summed in a way that calculates $\phi(t + \Delta t)$ from $\phi(t)$,

$$\phi(t + \Delta t) = \phi(t) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)\Delta t + \mathcal{O}(\Delta t^5), \tag{3.1.31}$$

which removes errors upto and including $(\Delta t)^4$ terms. A similar procedure can be done for any desired accuracy with higher order methods becoming quite involved. The simplicity of this RK4 scheme along with it's robustness in use has no doubt led to it being the go-to method for integrating ODEs. Lower order Runge-Kutta methods exist such as RK1 which is the Eular method in Eq. (3.1.18).

3.2 GRChombo

3.2.1 Overview of GRChombo

GRChombo [REF] [JOSS AND OLD PAPER] is an open-source fully non-linear Numerical Relativity (NR) code built on top of Chombo [REF], a PDE solver with adaptive mesh refinement (AMR). GRChombo is written in C++ making extensive use of templating, classes and object oriented programming. GRChombo also supports vectorisation and parallelisation with OpenMP and MPI making it scale efficiently to large problems suitable for use on supercomputer clusters. Current public examples of GRChombo include a black hole binary with seperate spins, a single kerr black hole and a compact real scalar configuration.

The AMR in Chombo relies on the Berger-Oliger style AMR [REF] with block-structured Berger-Rigoutsos grid generation [REF]. The labelling of which regions to regrid, called tagging, is specifiable by the user; a common example to use a variable ψ to make a new, gradient sensitive variable $\aleph$,

$$\aleph = \Delta x \sqrt{\frac{\left|\sum_{ij}(\partial_i\partial_j\psi)(\partial_i\partial_j\psi)\right|}{\left|\sum_k(\partial_k\psi)(\partial_k\psi)\right| + \epsilon}}, \tag{3.2.1}$$

for grid spacing Δx and small positive ϵ to avoid division by zero. Some threshold ψ_0 is prespecified and any gridpoint where $\aleph > \psi_0$ is flagged for regridding. The reason for premultiplying by Δx is so that as the grid spacing gets smaller on more deeper levels the tagging criterion is not flagged unless gradients become more extreme.

In order to evolve a spatial hypersurface with the Einstein equation, GRChombo uses either the BSSN formalism [REF] described in section 2.1.7 or the CCZ4 formalism [REF]. A summary of the CCZ4 formulation is given in section 2.1.9 along with the equations of motion used in this work. The conformal factor used is $\chi = \gamma^{-\frac{1}{3}}$ where γ is the metric determinant from Eq. (2.1.9) on the three dimensional hypersurface Σ_t. The time evolution scheme is the method of lines 3.1.3 using 4th order spatial derivatives 3.1.1 and runge kutta 4th order time integration 3.1.4. While 6th order spatial derivatives have been implemented, they are not used in this work due to their increasing the number of ghost cells needed. Kreiss-Oliger dissipation [REF] is used to dampen high frequency noise occuring from interpolation, regridding of AMR and high field gradients inside black hole regions. As described in section 2.1.10, the moving puncture gauge with conformal factor χ is used to evolve moving black hole singularities.

The code can implement Sommerfeld, periodic, reflective and extrapolating boundary conditions. Simulations in this work all use some combination of Sommerfeld boundary conditions for boundaries far away and reflective boundary conditions in a plane of symmetry. Headon collision of the same object can use *octant* symmetry, three planes of symmetry on the planes $x = 0$, $y = 0$ and $z = 0$. This means one only needs to simulate the region $x > 0$, $y > 0$ and $z > 0$ with reflective boundary conditions on the symmetry planes; the overall problem size (or number of gridpoints) reduces by a factor of eight. Similarly, headon collisions of non-similar objects can use *quadrant* symmetry, reducing the problem size by four. A general inspiral of two dissimilar compact obejcts has one plane of symmetry, the plane of inspiral; *bitant* symmetry can be used to half the problem size. For all the pre-mentioned symmetries, a suitable rest frame must be used aligning with the symmetries of the initial data. For detail on numerical implementation of boundary conditions see section 3.1.2.

A selection of diagnostic tools have been included into GRChombo. These include, but are not limited to, black hole horizon finders, gravitational wave extraction and the calculation for ADM mass and momentum, Noether charges and energy-momentum densities and fluxes. The diagnostic for the angular momentum density and flux are the result of section **??**.

While GRChombo can be used to simulate traditional spacetimes, such as binary black hole inspirals [REF] [REF MIREN PAPER], it excells at simulating novel and theoretical physics due to its adaptable code and AMR. The advantage of AMR is that regions needing higher resolution are assigned (and de-assigned) dynamically during run time; this requires no a-priori knowledge or pre-determined grid structure unlike other NR codes. [DO I LIST CODES HERE?] AMR is especially useful for matter simulations that can develop features requiring higher resolution in places a human may not expect making pre-specified mesh refinement hard to use. GRChombo has also sucessfully simulated ring-like configurations [REF] [Thomas,Josu] and inhomogeneous spacetimes [REF] [ref pau katy ...] which would be tricky with a conventional pre-specified grid structure. GRChombo has been designed to be straight forward to modify making it an excellent choice for exotic matter, modified gravity theories and higher dimensional spacetimes.

Simulation Units

MAKE THIS WORK WITH THE CONVENTIONS SECTION ASND THEN DESCRIBE WHAT IS NEW FOR ONLY THE SIMULATIONS

GRChombo defaults to geometric units, as described in section (1.2). The scalar field φ appears in the action as

$$S = \int_{\mathcal{M}} \left[g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi^* + ... \right] \mathrm{d}x^4 \tag{3.2.2}$$

and given S and the metric are dimensionless, dimensional analysis tells us φ has units of inverse length in natural units, or units of energy. The Klein-Gordon mass m

$$\Box \varphi = m^2 \varphi \tag{3.2.3}$$

can be absorbed into new dimensionless spatial coordinates $\tilde{x}^i = x^i m$ changing the KG equation to the scale invariant form.

$$\Box \varphi = \varphi \tag{3.2.4}$$

3.2.2 Boson Star Initial Data

Following on from the EKG ODE's in Eqs. (2.2.21), (2.2.22) and (2.2.23) we now seek to solve them numerically to obtain initial data for a single static boson star. The system can be reduced to a set of five first order ODE's with five boundary conditions. For a physical star we would like to impose $\Phi(0) = \Phi_c$, $\Phi'(0) = 0$, $\Phi(r \to \infty) \to 0$, $\Omega'(0) = 0$, $\Omega(r \to \infty) \to 1$, $\Psi'(0) = 0$ and $\Psi(r \to \infty) \to 1$ to be regular at the origin and match the Schwarzschild vacuum solution at large radius; however this is seven boundary conditions and we can only impose five. The condition $\Omega(0)' = 0$ cannot be specified as Eq. (2.2.21) is first order in derivatives of Ω but given that r and Ψ' both vanish at the origin then Ω' must also vanish at the origin automatically. One more boundary condition can be removed by asking for the boson star solution to match the isotropic Schwarzschild solution in Eq. (1.4.43) and therefore,

$$\Omega = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right) \quad \& \quad \Psi = \left(1 + \frac{m}{2r} \right)^2 \tag{3.2.5}$$

where m can be interpreted as the mass of the boson star; this mass will not enter the boundary condition so can be safely ignored. Combining the two equations above gives

$$\sqrt{\Psi} \, (1 + \Omega) = 2 \tag{3.2.6}$$

for a vacuum spacetime. Imposing the single condition $\sqrt{\Psi(\infty)} \, (1 + \Omega(\infty)) = 2$, rather than both $\Omega(\infty) = 1$ and $= \Psi(\infty) = 1$, then gives asymptotic flatness in just one boundary condition. One final point of importance is the frequency ω turns the Klein-Gordon ODE into an eigenvalue problem, admitting only discrete values of ω.

The problem has now been reduced to five ODE's with the following five boundary conditions,

$$\{\Phi(0), \Phi'(0), \Psi'(0), \Omega(0), \sqrt{\Psi(\infty)} \, (1 + \Omega(\infty)) \, ; \omega\} = \{\Phi_c, 0, 0, \omega_0, 2; \omega_0\}, \tag{3.2.7}$$

subjected to the condition of an eigenvalue $\omega = \omega_0$. The first attempt to find the radial profile $\{\Phi(r), \Omega(r), \Psi(r)\}$ of the boson star was to use a relaxation method [REF] as it trivially incorporates the above two-point boundary conditions. In practice this method did not work well with the eigenvalue problem in ω. Unlike with a shooting method, there was no obvious way of telling whether the guess ω was larger or smaller than the correct value. Even if this problem were overcome, a numerical solution with relaxation is computationally slow, even with Successive Over-Relaxation [REF] [20]; perhaps a multigrid method could work here but a simpler method was used.

Shooting Method

To find the initial data for a single Boson star, a private c++ script was written using RK4 [REF] to integrate the EKG system taking five initial conditions, and eigenvalue guess ω_0,

$$\{\Phi(0), \Phi'(0), \Psi(0), \Psi'(0), \Omega(0); \omega\} = \{\Phi_c, 0, \Psi_c, 0, \omega_0; \omega_0\}. \tag{3.2.8}$$

Unfortunately ω_0 and Ψ_c are unknown apriori, but guessing any values reasonably close to untity, such as $\omega_0 = 0.5$ and $\Psi_c = 2$, still give a boson star. This will generally result in the following asymptotic metric,

$$g_{\mu\nu}(r \to \infty) \to \mathrm{diag}(-A^2, B^2, B^2, B^2), \tag{3.2.9}$$

for constant A and B.

Before we discuss how to find the correct value of ω, there is a subtle numerical problem to adress. Using spherical polar coordinates in flat space, the Klein-Gordon equation (2.2.8) with $V = m^2|\varphi|^2$ and ansatz $\varphi = \Phi_{flat}(r)e^{\mathrm{i}\omega t}$ reduces to,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\varphi = \frac{\partial V}{\partial|\varphi|^2}\varphi, \tag{3.2.10}$$

$$\partial_t\left(g^{tt}\partial_t\right)\Phi_{flat}(r)e^{\mathrm{i}\omega t} + \frac{1}{r^2}\partial_r\left(r^2 g^{rr}\partial_r\right)\Phi_{flat}(r)e^{\mathrm{i}\omega t} = m^2\Phi_{flat}(r)e^{\mathrm{i}\omega t}, \tag{3.2.11}$$

$$\omega^2\Phi_{flat}(r) + \frac{1}{r^2}\partial_r\left(r^2\partial_r\right)\Phi_{flat}(r) = m^2\Phi_{flat}(r), \tag{3.2.12}$$

$$\tag{3.2.13}$$

where $\sqrt{-g} = r^2\sin(\theta)$, $g^{tt} = -1$ and $g^{rr} = 1$. This has general solution,

$$\Phi_{flat}(r) = \frac{1}{r}\left(C_1 e^{-r\sqrt{m^2-\omega^2}} + C_2 e^{r\sqrt{m^2-\omega^2}}\right), \tag{3.2.14}$$

for $\Phi_{flat}(r)$ and two constants C_1 and C_2. Due to finite resolution during numerical integration, at large radius C_2 will never be exactly zero and will eventually grow (along increasing radius) and spoil the numerical integration; even though this behavour was derived in flat space it is still present in curved space with spherical symmetry - especially at such large radius that space is approximately flat. In practice, the scalar field Φ will decay to some value roughly twenty orders of magnitude smaller than the central density $\Phi(0)$ and is effectively zero within numerical noise. At this point the coefficient C_2 is excited by noise and starts to grow exponentially. At a radius r_* when the growing mode is deemed to be dominating, usually detected by an axis crossing ($\Phi(r_*) = 0$ or a turning point $\Phi'(r_*) = 0$) the conditions $\Phi(r > r_*) = \Phi'(r > r_*) = 0$ are enforced during integration. This creates a vacuum for $r > r_*$ and the spacetime is pure Schwarzschild. After this point, an exponentially growing stepsize was used to reach radii of order 10^8 to 10^{10} times larger than desired for evolutions and the values $A = \Omega_\infty = \sqrt{-g_{00}}$ and $B = \Psi_\infty = \sqrt{g_{ii}}$ can be read off.

Interval bisection was used to find the best value of ω, ω_0, to machine precision; for the ground state we can tell that $\omega > \omega_0$ if $\Phi(r)$ develops a turning point before an axis crossing and $\omega < \omega_0$ if $\Phi(r)$ develops an axis crossing before a turning point. To find the n'th excited state, which has n axis crossing for

$\Phi(r)$ and $\Phi(r \to \infty) \to 0$ a similar scheme is followed to find the eigenvalue ω_n. If $\Phi(r)$ has $n+1$ axis crossings then $\omega > \omega_n$ and if $\Phi(r)$ has n axis crossings followed by a turning point then $\omega < \omega_n$. This method of doing a numerical integration and iteratively restarting to get closer to the target solution is known as a shooting method.

Putting everything together, a boson star solution with eigen value ω_0 (or ω_n for excited stars) and asymptotic metric Eq. (3.2.9) can be obtained. To find a star with asymptotic metric $\eta_{\mu\nu}$ of flat space, the initial conditions are iteratively improved like $\Omega_c \to \Omega_c/\Omega_\infty$ and $\Psi_c \to \Psi_c/\Psi_\infty$; the interval bisection for ω is then restarted. This is iterated three to five times which leaves $A = \Omega_\infty = 1$ and $B = \Psi_\infty = 1$ to extreme precision and the isotropic boson star is created. This whole process requires a few seconds runtime for a high resolution 200,000 grid-point simulation on a regular laptop.

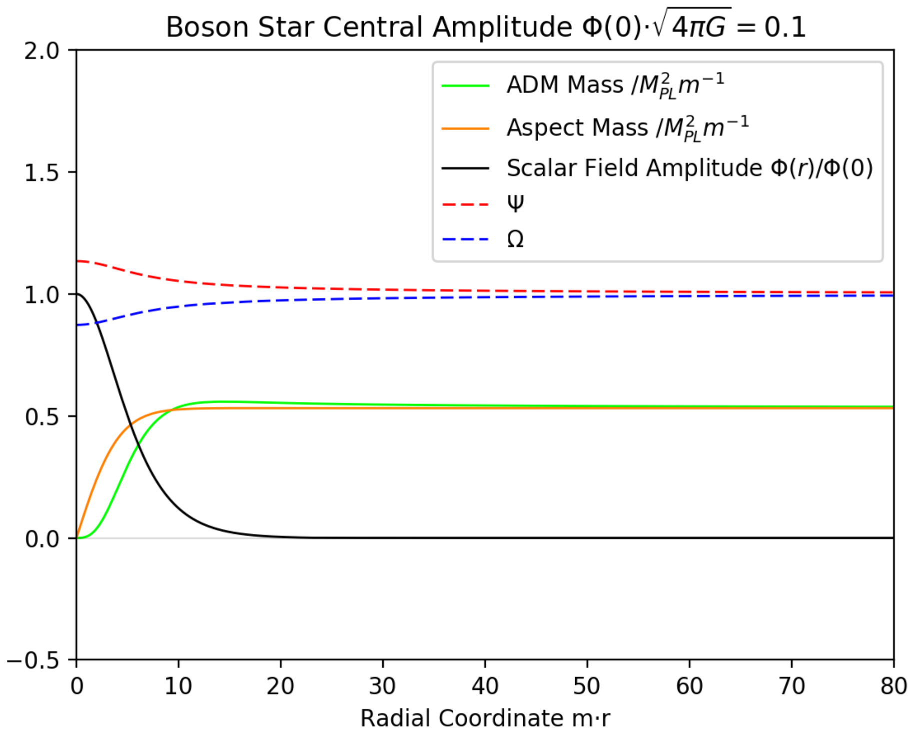


Figure 3.1: Boson star radial profile for the ground state

Figures 3.3 and 3.4 show the numerical result for the radial profile of a mini boson star ($\Lambda = 0$) and an excited mini boson star. Note two mass definitions are plotted; the ADM mass (calculated as a function of finite r) and the aspect mass $M_A(r)$ which corresponds to assuming the metric's solution is Schwarzschild with $M_A(r)$ rather than M. Polytropic fluid stars were also simulated as a preliminary test of the code; they are much easier to create not needing to solve an eigenvalue problem and don't have an asymptotically growing mode. Figures () show how the ADM mass of boson stars varies with central amplitude $\Phi(0)$ and r_{99}, the radius which $\Phi(r_{99}) = \Phi(0)/100$. It should be noted that the $\Lambda = 0$ case agrees with the known maximum mass, the Kaup limit [] $M_{max} \approx 0.633 M_{PL}^2 m^{-1}$ with the highest measured mass being $M_{max} = 0.63299(3) M_{PL}^2 m^{-1}$ corresponding to a central amplitude of $\sqrt{4\pi G}\Phi(0)_{max} = 0.271(0)$.

While many different boson stars have been made to test the initial data code, all the following evolutions use the same boson star with parameters $\Lambda = 0$, $\sqrt{4\pi G}\Phi(0) = 0.1 \to \Phi(0) \approx 0.0282$ and ADM mass $M = 0.532(7)$. This is as the stars are heavy enough to form black holes under collisions and large deformations, but stable enough to not collapse to a black hole for moderate perturbations.

EXPLAIN EIGENVALUES AND GROUND STATES BETTER.

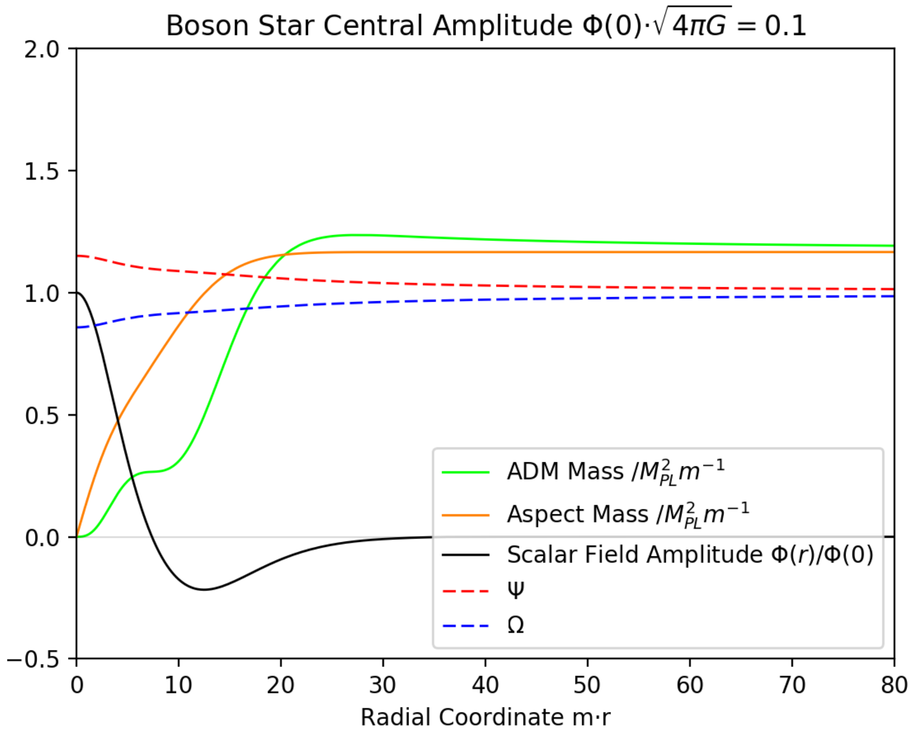


Figure 3.2: Boson Star radial profile : 1st Excited state

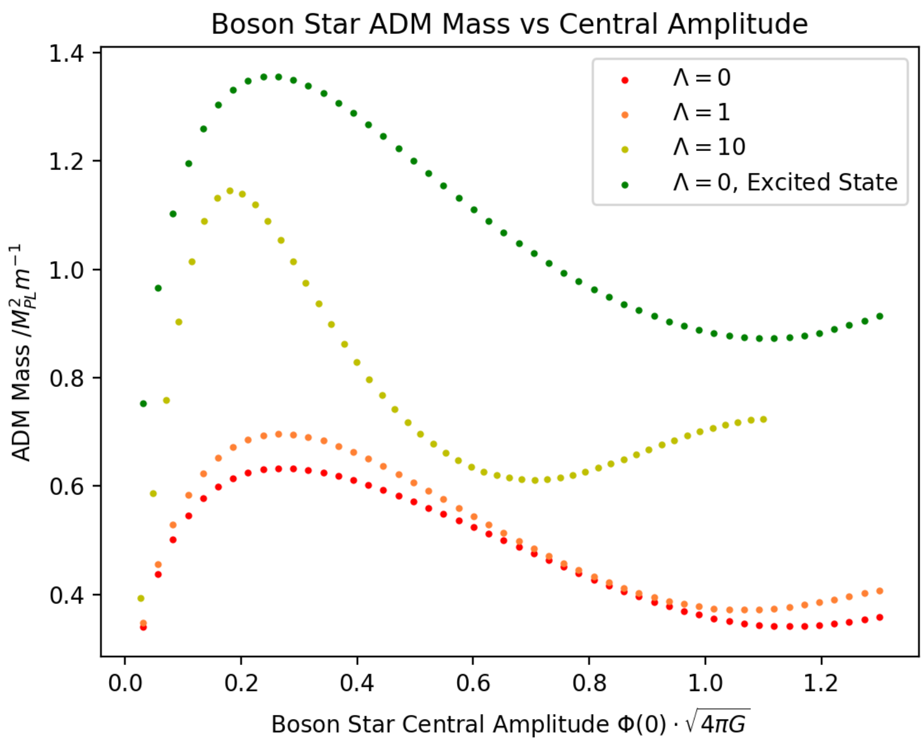


Figure 3.3: ADM mass vs $\Phi(0)$

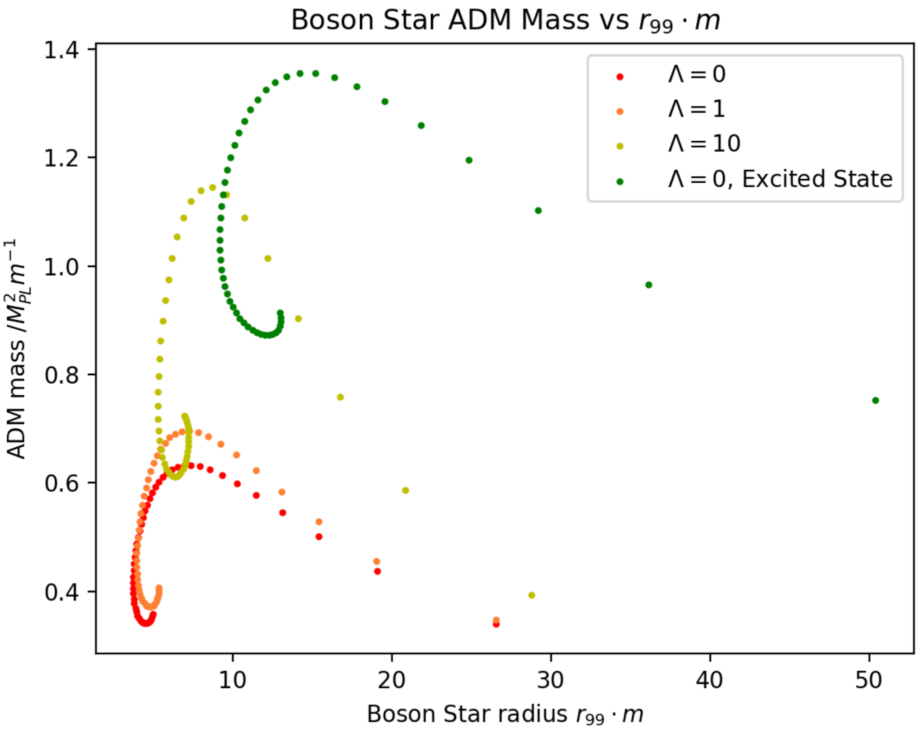


Figure 3.4: ADM mass vs r_{99}

mod_phi_nice0000.png	modphimax.png

Figure 3.5: Left: 2D slice of initial $|\varphi|$, Right: Maximum of $|\varphi|$ during evolution.

3.2.3 Single Star Evolution

The first simulation done was of the $\Phi(0) = 0.02820$ mini boson star; as mentioned before all simulations are done with this star. The star is supposed to remain in the centre of the grid and not change as it is a rest frame soliton; this is observed through evolution with GRChombo. Figure () shows a rough initial phase in $|\varphi|$ which changes significantly upon changing the AMR regridding, hence it is likely only a side effect of the interpolation errors at the boundary of AMR regions. Figure () shows that the star conserves $\mathcal{N}$ upto 4 figures and the constraint $\mathcal{H}$ is driven towards zero as desired.

N.png	H.png

Figure 3.6: Left: Total Noether charge $\mathcal{N}$ during evolution, Right: $||\mathcal{H}||_2$ during evolution.

3.2.4 Superposition of Initial Data

In order to simulate a spacetime consisting of two stars (or a star and a black hole) we must choose a way of superposing the initial data of two objects, centred at $x^{(1)}$ and $x^{(2)}$. For some field $\psi^{(j)}$ associated with the compact object at $x^{(j)}$,

$$\psi^{(j)} = \psi(x - x^{(j)}), \tag{3.2.15}$$

where ψ refers to the object centred about the origin. Taking two compact objects with fields φ, Π, γ_{ij}, $\mathcal{K}_{ij}$, α and β^i, a naive superposition scheme was chosen;

$$\varphi = \varphi^{(1)} + \varphi^{(2)}, \tag{3.2.16}$$

$$\Pi = \Pi^{(1)} + \Pi^{(2)}, \tag{3.2.17}$$

$$\mathcal{K}^i_j = \mathcal{K}^{(1)}{}^i_j + \mathcal{K}^{(2)}{}^i_j, \tag{3.2.18}$$

$$\gamma_{\mu\nu} = \gamma^{(1)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu}, \tag{3.2.19}$$

$$\beta_i = \beta^{(1)}_i + \beta^{(2)}_i, \tag{3.2.20}$$

$$\alpha = \sqrt{\alpha^2_{(1)} + \alpha^2_{(2)} - 1}, \tag{3.2.21}$$

$$\chi = \det\left\{\gamma^{(1)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu}\right\}^{-1/3}, \tag{3.2.22}$$

where the super-scripts $^{(1)}$ and $^{(2)}$ refer to the separate compact objects. The extrinsic curvature is chosen to be superposed with mixed indeces so that it implies the trace $\mathcal{K}$ is also superposed. If one of the compact objects is a black hole the lapse $\alpha \to 0$ on the horizon; this is circumvented by setting,

$$\alpha = \sqrt{\chi}, \tag{3.2.23}$$

ensuring that the lapse is real and non-negative for a all of Σ_t. Superposing two solutions in general relativity usually no longer satisfies the Einstein equation, the Hamiltonian constraint Eq. (2.1.50) and momentum contraints Eq. (2.1.52) are violated. For asymptotically flat compact objects, the constraint violation reduces to zero as the object separation tends to inifinity. In the case of finite separations, the CCZ4 scheme in section 2.1.9 aims to drive the constraint violation towards zero and hence a true solution of Einstein's equation. The preliminary test of collisions of compact objects in sections [REF] use this naive superposition scheme. Section [REF] explores a technique to improve the naive superposition of compact objects.

3.2.5 Head-on Collisions

All the cases studied here are for stationary initial data $\tilde{\mathcal{A}}_{ij} = 0, \mathcal{K} = 0$ in-falling from an initial separation of $d \cdot m = 32$ due to gravitational attraction. Firstly we consider the equal mass Boson star binary, initial data in figure (). At first they slowly infall creating a short lived object with three maxima, shown in figure (,left), then collapse to a black hole with a decaying spherical harmonic cloud (figures) outside. As with all the simulations from now on we assume a black hole forms if $\chi \ll 16^{-1}$ where $\chi = 16^{-}1$ is the value taken on the horizon for the isotropic Schwarzschild metric.

`headon_bs/chi0000.png`	`headon_bs/modphi0000.png`

Figure 3.7: Initial Data, Left: χ, Right: $|\varphi|$.

The second case considered is the same Boson star outside a black hole parameterised by $M = 10M_{BS}$ where M_{BS} is the ADM mass of the Boson star. The scalar field tidally deforms into an ellipsoid with high central density, well beyond Kaup limit of $\varphi(0) \sim 0.0764$, and spontaneous collapse to a smaller external black hole is observed. After collapse, figure (), there is an elongated cloud about the new small black hole; there are many nodal lines in $\text{Re}(\varphi)$ which focus on the large black hole showing the cloud is in-falling.

Final case consists of an equal mass Black Hole and Boson Star, initial configuration in Figure 11. As can be seen from the plot of $\text{Re}(\phi)$ and $|\phi|$, in Figure 12, most of the star falls into the black hole, however some scalar field manages to excite an intricate spherical harmonic cloud pattern.

In all three cases, the Noether charge drops rapidly upon the formation of a black hole; this will be explained in the next section. However some scalar field lingers after collapse, in each case the hair takes headon_bs/modphi0003.png | headon_bs/modphi0004.png

Figure 3.8: Scalar field amplitude before and after black hole formation, Left: Time $t \cdot m = 150$, Right: Time $t \cdot m = 200$.

headon_bs/phi_re0004.png | headon_bs/phi_re_perp0004.png

Figure 3.9: Real part of scalar field after black hole formation, Left: xy plane, Right: yz plane, perpendicular to initial star separation.

the form of spherical harmonics discussed in (). Also observed is the decay of the spherical harmonics to zero amplitude in these simulations with no angular momentum.

LMR/chi0000.png LMR/modphi0000.png

Figure 3.10: Initial Data, Left: χ, Right: $|\varphi|$.

3.2.6 Binary Inspiral

The only considered case here is the Quasi-circular orbit and inspiral of two equal mass boson stars. The initial boosts were determined by a newtonian calculation yielding

$$v^2 = \frac{M}{2d} \tag{3.2.24}$$

where $M = 0.53(29)$ is the ADM mass of the Boson Star and d is the initial separation. For a relatively low separation of $d \cdot m = 32$ code units, shown in Figures 13,14 (Left), we get $v \sim 0.0915$. The boson stars are observed to complete roughly half an orbit before merging and collapse, forming a black hole. Here a Kerr black hole is assumed to have formed as the spacetime has a significant angular momentum which should partially infall with the scalar field.

Figure 15 shows the total Noether charge, which is no longer conserved. When the black hole forms, the in-falling scalar field moves towards zero radius and gets hugely compressed. The huge compression causes extreme field gradients which induces continuous regridding in the AMR, but this is capped at 7 layers in this simulation to make runtime feasible. When the 8th AMR level is needed it is simply not added and resolution becomes low enough that any Noether charge near the centre is so under-resolved that it seems to fall between the gridpoints. Interestingly, Figures 13,14 (Right) show a scalar field configuration lingering around the black hole, mostly outside the contour $\chi = 0.7$, looking like scalar hair. Also Figure 15 (Left) of the Noether charge appears to take significantly longer to decay than in the linear collision simulations. It can be seen in Figure 14 in the plot of $\mathrm{Re}(\varphi)$ that the cloud has angular nodes corresponding to angular momentum similarly to boosted stars picking up nodal planes perpendicular to momentum and Figure 10 (Bottom) of the infalling cloud.

The gravitational wave (GW) signal can be seen in Figure 15 (Right), extracted at a radius $r \cdot m = 90$. At the time $t \cdot m \approx 700$ the gravitational wave signal due to the merger appears; in order to record a longer inspiral simulations with larger initial separations need to be simulated. Currently there appears to be some small problem with the initial data, also observed in the collisions with no angular momentum, that manifests itself in noisy GW extraction and the Hamiltonian constraint initially sharply rising.

LMR/chi0005.png

LMR/modphi0005.png

LMR/phi_re0000.png

Figure 3.11: Final Data at time $t \cdot m = 125$, Left: Conformal factor χ, Right: Scalar field modulus $|\varphi|$, Bottom: Real part of scalarfield $\mathrm{Re}(\varphi)$.

3.2.7 STUFF

I think I want a stationary star headon collision maybe collison with angular momentum collision of bs with bh headon/chi0000.png

headon/mod_phi0000.png

Figure 3.12: Initial data for equal mass Boson Star and Black Hole, Left: χ, Right: $|\varphi|$

headon/phi_re0000.png

headon/mod_phi0007.png

Figure 3.13: Time $t \cdot m = 905$ for equal mass Boson Star and Black Hole, Left: $\mathrm{Re}(\varphi)$, Right: $|\varphi|$

inspiral/mod_phi_inspiral_nice0000.png

inspiral/mod_phi_inspiral_nice0002.png

Figure 3.14: Boson Star $|\varphi|$. Left: initial data, Right: later time $t \cdot m = 700$

inspiral/phi_inspiral_nice0000.png

inspiral/phi_inspiral_nice0002.png

Figure 3.15: Boson Star $\mathrm{Re}(\varphi)$. Left: initial data, Right: later time $t \cdot m = 700$

inspiral/N.png

inspiral/GW.png

Figure 3.16: Left: Total Noether charge $\mathcal{N}$ during evolution, Right: Ψ_4 over 22 harmonic during evolution.

Chapter 4

STUFF TO DO

PUT A NICE FIGURE ON EACH CHAPTER PAGE?

CODE DEV? LIKE INITIAL DATA AND RANDOM SIMS IVE DONE ESPECIALLY WITH BLACK HOLES, MAYBE AN INITIAL DATA SECTION THEN FOLOWED BY MALAISE?

rename the sections maybe, or maybe not. is NUmerical relativity the best name for section that has no numerics and just does a 3+1 spacetime split?

ADD TO CONVENTIONS THAT LATE LATIN INDECES (I,J,K,...) SYMBOLISE 3D OBJECTS IN 4D SPACE ASWELL AS 3D OBJECST IN 3D SPACE

BOX IMPORTANT EQS? MAYBE NOT.

CHECK THE 3+1 SPLIT STRESS TENSOR AGREES WITH PAPERS ...

CHECK CURLY VS SMOOTH BRACKETS FOR SETS AND VECTORS ALL OVER

REF THE ADM METRIC?

make a note of the dx form notation but don't go into depth, maybe hint at intergating over a mfold?

maybe put BSSN and onwards in the GRCHOMBO SECTION? OR A CODE SECTION? and keep the first half as the adm decomposiiton section?

MAYBE ADD Z4 LAGRANGEAN OR UNDERSTAND IT BETTER? LOOK AT MARKUS THESIS? OR KATY?

CHECK CONVENTIONS ARE OBEYED IN INTRO NR BOSON Q MALAISE OTHER SECTRIONS...

MAKE SOME UNIFORM CONVENTION FOR STRESS TENSOR, MAYBE RHO FOR ENERGY DENSITY MAYBE EPSILON? SIMILAR FOR THE MOMENTA AND THE PROJECTED TENSOR? its in the ccz4 equations too ...

fig or Fig in figure referneceing?

DECIDE ON A FACTOR OF 16PI IN TEH MATTER LAGRANGEAN OF GR BETWEEN THE INTRO SECTION AND THE LATER SECTION. PROBABLY USE R/16PI RATHER THAN 16PI*MATTER ...

standardise complex phi, bar or star for conjugate? or dagger??

MAKE SURE I CREDIT MIREN FOR THE TIME EVOLUTION OF THE SCALAR FIELD

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