Chapter 1

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July 2022

1 Notes

Notation. Let $\mathbb{F}$ stand for either $\mathbb{R}$ or $\mathbb{C}$, as many properties apply to both fields.

Notation. Let $\mathbb{F}^n$ represent all lists of length n consisting of elements in $\mathbb{F}$.

Notation. Let $\mathbb{F}^S$ represent the set of functions from set S to F.

(a) For $f, g \in S$, define the sum $(f+g)(x) = f(x) + g(x)$.

(b) For $\lambda \in \mathbb{F}$ and $f \in S$, define the scalar product $(\lambda f)(x) = \lambda \cdot f(x)$.

Definition 1 (Vector Space)**.** A vector space is a set V along with addition and scalar multiplication on V that satisfies the following properties:

- commutativity of addition
- associativity of addition and scalar multiplication
- additive identity
- additive inverse for every element
- multiplicative identity
- distributive properties

The elements of V are called vectors or points. From now on, we will also refer to the vector space as V, where we have defined addition and multiplication (or assumed it).

We can prove some basic properties of vector spaces in general that we see in $\mathbb{Z}$:

- Unique additive identity (0)
- Unique additive inverse for each element $(-v)$
- $0v = 0$ and $a0 = 0$
- $(-1)v = -v$

Definition 2 (Subspace)**.** A subset U of V is a subspace of V if it remains a vector space using the same rules of addition and multiplication as V.

Condition. U is a subset of V if and only if $0 \in U$ and U is closed under addition and scalar multiplication. Since the same rules of addition and multiplication apply as in V, we know the rest of the vector space properties are satisfied.

Definition 3 (Sum of subsets)**.** Suppose $U_1, \ldots, U_m$ are subsets of V. The sum of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums taking one element from U_1, one from U_2, and so on until U_m. Mathematically:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

Example. Suppose that $U = \{(x_1, x_1, y_1, y_1) \in \mathbb{F}^4 : x_1, y_1 \in \mathbb{F}\}$ and $W = \{(x_2, x_2, x_2, y_2) \in \mathbb{F}^4 : x_2, y_2 \in \mathbb{F}\}$. Then:

$$U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}$$

Proof. How can we get 3 independent variables in four components from this sum of subsets? It's obvious that the first and second components of $U + W$ are the same, $x = x_1 + x_2$, so there are at most three independent components. Let's try to find a solution for arbitrary x, y, z by solving the system of linear equations. We get:

$$\begin{cases} x_1 + x_2 = x \\ y_1 + x_2 = y \\ y_1 + y_2 = z \end{cases}$$

When written in row-echelon form, it is obvious that there exists a solution set with one parameter. If we let $y_2 = t, t \in \mathbb{F}$, then we get $y_1 = z - t$, $x_2 = y - y_1 = y - z + t$, and $x_1 = x - x_2 = x - y + z - t$. So the sum does have 3 independent components. □

Theorem 1 (Sum of subspaces is the smallest containing subspace)**.** *Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$.*

Proof. It's easy to see that $U_1 + \cdots + U_m$ contains 0 and is closed under addition and multiplication, hence it is a subspace of V. (See addendum)

When we claim that something is the smallest possible set satisfying some property, a common proof strategy is to show that all sets (including the smallest) satisfying the property must also contain this smallest set, i.e. that the smallest set is a subset of all sets satisfying the property. How might we go about this here?

Suppose that W is a subspace of V that contains the subspaces $U_1, \dots, U_m$. Pick arbitrary elements $u_1 \in U_1, \dots, u_m \in U_m$. Note that these also belong to W, which is a vector space in its own right. Since addition is closed in vector spaces, $u_1 + \cdots + u_m \in W$. Because our selection of elements was arbitrary, this works for all possible sums of elements, i.e. all members of the sum $U_1 + \cdots + U_m$. Therefore, W contains $U_1 + \cdots + U_m$.

Since all subspaces of V that contain $U_1, \dots, U_m$ also contain $U_1 + \cdots + U_m$, $U_1 + \cdots + U_m$ is the smallest subspace of V that contains $U_1, \dots, U_m$. □

Definition 4 (Direct sum)**.** Suppose $U_1, \dots, U_m$ are subspaces of V. Then the sum $U_1 + \cdots + U_m$ is called a direct sum if each element of the sum $u_1 + \cdots + u_m$, $u_i \in U_i$, can only be written in one way. We will denote known direct sums with the $\oplus$, e.g. $U_1 \oplus \cdots \oplus U_m$.

I might verify or disprove direct sums by checking the independence of linear systems of equations. It turns out we only have to check whether or not 0 can only be written in one way, which is obviously taking all elements of the sum to be 0, which corresponds to checking the homogeneous system.

Condition (Condition for direct sum)**.** Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$ is by taking each u_i equal to 0.

Where the proof approach is to show that if some member v of $U+W$ could be written in two ways, i.e. $v = u_1 + \cdots + u_m$ and $v = v_1 + \cdots + v_m$, then by subtraction we get $0 = (u_1 - v_1) + \cdots + (u_m - v_m)$. Since this is also a member of the sum, we get $u_i - v_i = 0$ and thus $u_i = v_i$, showing that all solutions must be the same one.

This is probably related to the fact that if we change the constants in our linear system of equations to solve for a different solution that if any solutions exist (i.e. our constant vector is a member of the sum), that the solution set is only shifted from the homogeneous system (by any particular solution of the non-homogeneous equation), hence the size of that solution set is the same for any vector.

Theorem 2 (Direct sum of two subspaces)**.** *Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$, i.e. they only share the additive identity.*

Proof. Firstly, if there were some non-zero shared element, say $\exists u_i, w_i : u_i = w_i$, then we could write that element as both $u_i + 0$ and $0 + w_i$, so $U + W$ would not be a direct sum. In the other direction, what if $U \cap W = \{0\}$? Suppose some member of $U + W$ could be written in two ways, i.e.

$\exists u_1, u_2 \in U, w_1, w_2 \in W : u_1 + w_1 = u_2 + w_2$ (which makes sense in V–I'm shortcutting the declaration of an element v of $U + W$ being equal to two things). Then $(u_1 - u_2) = (w_1 - w_2)$. Since $u_1 - u_2 \in U$ and $w_1 - w_2 \in W$ by closedness of addition and subtraction on the vector spaces U and W, this implies that $u_1 - u_2$ and $v_1 - v_2$ are an element common to U and V, and thus by the hypothesis must be equal to 0. This leads to the conclusion that $u_1 = u_2$ and $v_1 = v_2$, and therefore the two sums must be the same.

More concisely, we could use the Condition for a Direct Sum if we knew that the only way to write 0 as a sum was $0 + 0$. Suppose that for $u \in U$ and $w \in W$ that $0 = u + w$. Then $u = -w \in W$ (by closedness of scalar multiplication), but then by hypothesis $u = 0$, and subsequently $w = 0$. □

Note that this theorem only works for the sum of two subspaces.

2 Problems (1.C)

Problem 1. For each of the following subsets of $\mathbb{F}^3$, determine whether is is a subspace of $\mathbb{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$

(a) $x_1, x_2, x_3 = 0$ is a valid solution, so 0 is in the set. For addition and scalar multiplication:

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3)$$

And testing whether these still belong to the vector space:

$$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3) = x_1+y_1+2x_2+2y_2+3x_3+3y_3 = (x_1+x_2+x_3)+(y_1+y_2+y_3) = 0+0 = 0$$

$$ax_1 + 2(ax_2) + 3(ax_3) = a(x_1 + 2x_2 + 3x_3) = a(0) = 0$$

using the distributive properties, and commutativity and associativity of addition and multiplication on the vector space $\mathbb{F}^3$. Hence the set is closed under addition and scalar multiplication. By the Condition for Subspaces, the set is a subspace of $\mathbb{F}^3$.

(b) In this case, $(0, 0, 0)$ is not a valid solution, so 0 is not in the set. This is not a subspace of $\mathbb{F}^3$.

(c) The set is not closed under addition. A counterexample is $(1, 0, 1) + (1, 1, 0) = (2, 1, 1)$.

(d) We can rewrite this set as $\{(5x_1, x_2, x_1)\}$. $(0, 0, 0)$ is clearly a member of this set, and this is also obviously closed under addition and multiplication. (Just factor out the coefficient of 5 when you need to using the distributive properties.)

Problem 2. Verify the assertions in Example 1.35, having to do with verifying subspaces

(a) If $b \neq 0$ then x_3 and x_4 cannot both be 0, so 0 would not be in the set. Otherwise, we get $(x_1, x_2, 5x_3, x_3)$, which is fine.

(b) $f(x) = 0$ is continuous on $[0, 1]$ and the sum and scalar multiples of continuous functions are also continuous. (This comes from the limit laws.)

(c) Similarly, the sum and scalar multiple of differentiable everywhere functions is also differentiable everywhere. (Again, from limit laws.)

(d) $f(x) = 0$ is most certainly differentiable on $(0, 3)$, and $f'(x) = 0$. Hence we require $b = 0$ for 0 to be in the set. Because differentiation is linear, we can easily find that the sum or scalar multiple of differentiable functions with $f'(2) = 0$ still has $f'(2) = 0$.

(e) $\{0, 0, \dots\}$ has a limit of 0. Also, our limit laws tell us that the sum or scalar multiple of something with limit 0 also has limit 0.

Problem 3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

$f(x) = 0$ is in the set, as $f'(x) = 0$ also. For two functions $f, g \in R^{(-4,4)}$:

$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f+g)(2)$$

$$cf'(-1) = c \cdot 3f(2) = 3 \cdot cf(2)$$

Hence the set is also closed under addition and scalar multiplication. So it is a subspace of $\mathbb{R}^{(-4,4)}$.

Problem 4.

Since definite integration is linear, we have $\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = b + b = 2b$, and $\int_0^1 cf = c\int_0^1 f = cb$. Both are never equal to b (hence the set is not closed) unless $b = 0$, in which case the result is always equal to 0. In that case 0 is also in the set since $\int_0^1 0 = 0$. Hence the set is a subspace only for $b = 0$.

Problem 5. Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$?

Yes, $(0, 0) \in \mathbb{R}^2$ and it is closed under addition and scalar multiplication. (We can only introduce a complex component by taking the square root of a negative number in one component.)

Problem 6. (a) Is $\{(a, b, c)\} \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

(b) Is $\{(a, b, c)\} \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

In $\mathbb{R}^3$: we can say that $a = b$, so our set just becomes (a, a, b), which is clearly a subspace. In $\mathbb{C}^3$: there can be multiple cube roots for the same real number, so we need to be careful about what happens when we add things together. (Scalar multiplication is obviously fine.) Take the vectors $(1, (-1 + i\sqrt{3})/2, 0)$ and $(1, (-1 - i\sqrt{3})/2, 0))$, where the cubes of 1,$(-1 + i\sqrt{3})/2$, and $(-1 - i\sqrt{3})/2$ are all 1. Their sum is $(2, -1, 0)$, which obviously doesn't satisfy the condition. So the set is not closed under addition, therefore it is not a subspace of $\mathbb{C}^3$.

Problem 7. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses, but U is not a subspace of $\mathbb{R}^2$.

$U = \{(x_1, x_2)\} : x_1, x_2 \in \mathbb{Q}$. The scalar can be irrational, so this is not closed under scalar multiplication. (Actually, even integers work as the scalar can be any real number.)

Problem 8. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

(EXTERNAL SOLUTION) $U = \{(a, a)\} \cup \{(a, 2a)\}$. Consider the vectors $(1, 1), (1, 2) \in U$ where $(1, 1) + (1, 2) = (2, 3) \neq U$, so the set is not closed under addition.

Problem 9. Is the set of periodic functions $\{f(x) : (\exists p \in \mathbb{R} : \forall x \in \mathbb{R}, f(x+p) = f(x))\}$ from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

The sum of two functions whose periods have no common multiple, such as the case where one period is rational and the other is irrational, are no longer periodic. So the set is not closed under addition.

Problem 10. Suppose $U_1, U_2 \subset V$. Prove that $U_1 \cap U_2$ is a subspace of V.

0 must exist in both U_1 and U_2. For arbitrary elements $u, v \in U_1 \cap U_2$, since they are in U_1 and in U_2, their sum $u + v$ must be in U_1 and also U_2, because addition is closed under both subspaces. So $u + v \in U_1 \cap U_2$, and therefore $U_1 \cap U_2$ is closed under addition. A similar fact applies for scalar multiplication.

Problem 11. Prove that the intersection of every (non-empty?) collection of subspaces of V is a subspace of V.

This follows by induction from Problem 10. For the base case, consider a collection of 1 subspace, $\{U\}$. Its intersection of all elements is itself, which is a subspace. Now suppose that the intersection of a collection of n subspaces of V, $\{U_1, \dots, U_n\}$, is a subspace, and consider the addition of another subspace U_{n+1} to the collection (where if there are a finite number of subspaces, $n+1$ is not greater). By Problem 10, the intersection of the two subspaces $(U_1 \cap \dots \cap U_n) \cap U_{n+1}$ is also a subspace. Hence the collection of $n+1$ subspaces is also a subspace. By induction, the intersection of a collection of any number of subspaces of V is also a subspace of V.

Problem 12. Prove that the union of two subspaces of V is a subspace if and only if one is contained in the other.

If one subspace is contained in the other, then their union is simply the containing subspace. Suppose that the union of two subspaces U, W of V, $U \cup W$, is also a subspace. We want to show that one of the subsets is contained in the other, i.e. $U \subset W$ or $W \subset U$. Suppose for sake of contradiction that this is not the case. This means that there exist elements of $U \cup W$ that are either not in U or not in W. Consider two such elements $u \in U \setminus W$ and $w \in W \setminus U$, where both elements must be in $U \cup W$. Since $U \cup W$ is closed under addition, $u + w \in U \cup W$ – it may be in U or V (possibly both). If it is in U, then $(u + w) + (-u) = w$, which violates the closedness of U under addition. If it is in V, then $(u + w) + (-w) = u$, which violates the closedness of W under addition. So we get a contradiction. Therefore, we must have one of the subsets containing the other.

Problem 13. Prove that the union of three subspaces of V is a subspace if and only if one contains the other two. (This is not the case if we replace $\mathbb{F}$ by a field with only two elements.)

(EXTERNAL SOLUTION) Suppose some subset contains another: WLOG suppose $U_1 \subseteq U_2$, so that $U_1 \cup U_2 \cup U_3 = U_2 \cup U_3$. Then by Problem 10, This is only a subspace of V if $U_2 \subseteq U_3$, in which case $U_3 \supseteq U_1, U_2$, or if $U_3 \subseteq U_2$, in which case $U_2 \supseteq U_1, U_3$.

Now suppose that no subset contains another (which is the only other case). Then there exists elements $u_1 \in U_1$ but not U_2 or U_3 and $u_2 \in U_2$ but not U_1 or U_3. Then neither $u_1 + u_2$ nor $u_1 - u_2$ can be in U_1 or U_2 without violating the closedness of U_2 or U_1 under addition respectively (by the same argument as in Problem 12). Then either at least one of $u_1 + u_2$ and $u_1 - u_2$ is outside $U_1 \cup U_2 \cup U_3$, or both are in U_3 but not U_1 or U_2. But in the second case, $(u_1 + u_2) + (u_1 - u_2) = 2u_1 \in U_3$. If it is in $U_1 \cap U_3$, then $\frac{1}{2}(2u_1) = u_1 \neq U_3$, which violates closedness of U_3 under scalar multiplication. If it is in U_3 but not U_1, this violates closedness of U_1 under scalar multiplication. We always get a contradiction, so instead at least one of $u_1 + u_2$ and $u_1 - u_2$ is outside $U_1 \cup U_2 \cup U_3$, in which case $U_1 \cup U_2 \cup U_3$ is not closed under addition and is therefore not a subspace of V.

Problem 14. Verify the assertion in Example 1.38, that for the two subsets of $\mathbb{F}^4$, $U = (x, x, y, y)$ and $W = (x, x, x, y)$, that $U + W = (x, x, y, z)$.

Handled in notes.

Problem 15. Suppose U is a subspace of V. What is $U + U$?

Since U is closed under addition, the sum of any two elements in U is also in U. Therefore, $U + U = U$.

Problem 16. Is the operation of addition on the subspaces of V commutative? i.e. if U, W are subspaces of V, is $U + W = W + U$?

Consider two arbitrary vectors $u \in U$ and $w \in W$. Since addition is commutative on the subspace V, $u + w = w + u$, hence whatever that result is must be in both $U + W$ and $W + U$. Therefore $U + W = W + U$.

Problem 17. Is the operation of addition on the subspaces of V associative? i.e. if U_1, U_2, U_3 are subspaces of V, is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?

For any choice of $u_1 \in U_1, u_2 \in U_2, u_3 \in U_3$, $(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$ by commutativity of addition on V. So any sum of vectors is going to appear in both sums of subspaces, so they're the same.

Problem 18. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that $U_1 + W = U_2 + W$, then $U_1 = U_2$.

Not necessarily true: consider the case where both U_1 and U_2 are subspaces of W but distinct, in which case $U_1 + W = U_2 + W = W$. A counterexample is $V = \mathbb{R}^3$, $W = (x_1, x_2, 0)$, $U_1 = (x_1, 0, 0), U_2 = (0, x_1, 0)$.

Problem 19. Suppose $U = (x, x, y, y)$, a subspace of $\mathbb{F}^4$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Consider $W = (x, 0, y, 0)$. It's obvious that U and W share no elements except for 0–try making the two vector templates equal. By the Direct Sum of Two Subspaces Theorem, $U + W$ is a direct sum, (x, y, z, a).

Problem 23. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that $V = U_1 \oplus W$ and $V = U_2 \oplus W$, then $U_1 = U_2$.

Note that by the Direct Sum of Two Subspaces Theorem, $U_1 \cap W, U_2 \cap W = \{0\}$. I think this is true, but I haven't thought up of a complete proof. Maybe the existence of elements in U_1 not in U_2 automatically implies the existence of some sum of an element in U_1 and element in V that doesn't appear in $U_2 \oplus W$, and vice versa, in which case $U_1 + W \neq U_2 + W$, which is a contradiction with the hypothesis. (Note: try finding two solutions in Problem 20 to get a counterexample.)

Problem 24. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions of $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e + U_o$.

I've done this before as MATH 120 HW4 Problem 4a. Let $f(x)$ be a function. I wish to find a pair of an even function $g(x)$ and an odd function $h(x)$ so that $f(x) = g(x) + h(x)$. Note that $f(-x) = g(-x) + h(-x) = g(x) - h(x)$. We can arrive at the two equations $f(x) + f(-x) = 2g(x)$ and $f(x) - f(-x) = 2h(x)$. Therefore, we have $g(x) = \frac{1}{2}(f(x) + f(-x))$ and $h(x) = \frac{1}{2}(f(x) - f(-x))$.

To prove that this is a unique solution, suppose there exists some other pair of functions $g_1(x), h_1(x)$ that satisfies the same property, i.e. $f(x) = g_1(x) - h_1(x)$. Subtracting, we get $0 = g(x) - g_1(x) + h(x) - h_1(x)$, or $g(x) - g_1(x) = h_1(x) - h(x)$. But the difference of two even functions is even, and the difference of two odd functions is odd, so we have that whatever the function on both sides is is both even and odd. The only function that satisfies this is 0, hence $g(x) - g_1(x) = 0$ and $h_1(x) - h(x) = 0$. Then $g(x) = g_1(x)$ and $h(x) = h_1(x)$ – they are the same pair. Therefore, there is only one pair of functions.

3 addendum

A direct proof that the sum of subspaces is also a subspace:

Theorem 3 (Sum of subspaces is the smallest containing subspace)**.** *Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is also a subspace of V.*

Proof. $U_1 + \cdots + U_m$ contains $U_1, \ldots, U_m$ because for any element of a subspace $u_i \in U_i$, we can write it as a valid sum consisting of only u_i with the rest of the elements being 0 (which by condition exists in all subspaces). How about showing that $U_1 + \cdots + U_m$ is itself a subspace? By our previous statement it contains 0. Also, we can show that $U_1 + \cdots + U_m$ is closed under addition and scalar multiplication. For any $v_1, v_2 \in U_1 + \cdots + U_m$ and $c \in \mathbb{F}$:

$$v_1 + v_2 = (u_{1,1} + \cdots + u_{m,1}) + (u_{1,2} + \cdots + u_{m,2}) = (u_{1,1} + u_{1,2}) + \cdots + (u_{m,1} + u_{m,2})$$

$$cv_1 = c(u_1 \ldots u_m) = cu_1 + \cdots + cu_m$$

Where the manipulations in the second step are justified by the existence of the associative, commutative, and distributive properties in the larger subset V. Furthermore, since addition and scalar multiplication are closed on the subspaces $U_1, \ldots, U_m$, we have that $u_{i,1} + u_{i_2} \in U_i$ and $cu_i \in U_i$, hence the results of addition and scalar multiplication of sums remain valid sums. Therefore, by the Condition for Subspaces, $U_1 + \cdots + U_m$ is also a subspace of V. $\square$

Problem 25. Prove that $\mathbb{R}^{\mathbb{R}}$ is the direct sum of the set of even functions $\mathbb{R} \to \mathbb{R}$ and the set of odd functions $\mathbb{R} \to \mathbb{R}$, i.e. every real function defined everywhere can be written as the sum of an even function and an odd function.

This is identical to MATH 120 HW6 Problem 4. The solution is $f(x) = \frac{1}{2}[f(x)+f(-x)]+\frac{1}{2}[f(x)-f(-x)]$, where the first function is even and the second is odd. The only function that is both even and odd is the zero function, which fulfills the Direct Sum of Two Subsets Theorem.