Chapter 1

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1 Notes

Notation (F, V, W). Let $\mathbb{F}$ denote the fields $\mathbb{R}$ or $\mathbb{C}$, and let V, W denote vector spaces over $\mathbb{F}$.

Definition 1 (Linear map)**.** A linear map (also known as a LINEAR TRANSFORMATION) from V to W is a function $T : V \to W$ that satisfies additivity and homogeneity, i.e.

- $T(u + v) = T(u) + T(v)$ for all $u, v \in V$
- $T(\lambda u) = \lambda T(u)$ for all $u \in V \lambda \in \mathbb{F}$

We may use Tv in place of $T(v)$.

Notation. The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$ (using the mathcal L symbol).

Some linear transformations include differentiation, integration, scaling, rotation in $\mathbb{R}^n$, etc. Linear transformations may be between infinite-dimensional vector spaces or finite-dimensional vector spaces, with the same or changing dimension.

Theorem 1 (Linear maps and basis of domain)**.** *Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that $T(v_j) = w_j$ for each $j = 1 \ldots n$.*

In particular, this map is defined by $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$. Note that every vector in V can be written as a unique linear combination of the basis vectors so the domain of this map is indeed V. We find that a map in $\mathcal{L}(V, W)$ that satisfies the condition $T(v_j) = w_j$ gets us to this definition by linearity, so it is unique.

We can perform algebraic operations on maps in $\mathcal{L}(V, W)$, analogous to operations on functions. We define the sum and (scalar) product operations:

- $(S + T)(v) = Sv + Tv$
- $(\lambda T)(v) = \lambda(Tv)$

The sum and product of two linear maps in $\mathcal{L}(V, W)$ remain linear maps in $\mathcal{L}(V, W)$, which we can verify by plugging in $v + w$ and λv. So $\mathcal{L}(V, W)$ is a vector space–the zero identity is just the zero map, and the additive inverse of a map is what you get from scalar multiplication by -1.

Definition 2 (Product of linear maps)**.** If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by $(ST)(u) = S(Tu)$ for $u \in U$.

This is effectively a composition of one linear map with another, just like any other composition of two functions. Remember that we perform these products, like function compositions, from right to left. Three important properties are:

- Associativity: $(T_1T_2)T_3 = T_1T_2T_3 = T_1(T_2T_3)$
- Identity: $TI = IT = T$
- Distributive properties: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$

Always keep the products in order! Products of linear maps are not commutative!

Theorem 2 (Linear maps take 0 to 0)**.** *Suppose T is a linear map from V to W. Then $T(0) = 0$.*

Proof. By additivity, we have $T(0) = T(0 + 0) = T(0) + T(0)$. Adding $-T(0)$ to both sides, we get $0 = T(0)$. ∎

2 Problems

Problem 1. For the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$ where $b, c \in \mathbb{R}$, show that T is linear if and only if $b, c = 0$.

If $b, c = 0$, then $T(x, y, z) = (2x-4y+3z, 6x)$. Show that T now satisfies additivity and homogeneity.

If $b \neq 0$, then the first component is neither additive (try adding 0) nor homogenous. If $c \neq 0$, then the second component is not homogenous (when scaling by a factor a the $cxyz$ term gets scaled by a^3) (and probably not additive). So it is not linear. This proves the contrapositive.

Problem 2.