Chapter 2A

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1 Notes

Definition 1 (Span)**.** The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the span of $v_1, \dots, v_m$, denoted $\text{span}(v_1, \dots, v_m)$. The span of () is defined as $\{0\}$. Symbolically:

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbb{F}\}$$

Furthermore, we say that if $\text{span}(v_1, \dots, v_m) = V$, then $v_1, \dots, v_m$ spans V.

Theorem 1 (Span is the smallest containing subspace)**.** *The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.*

Proof. The span of the vectors can be shown to be a subspace of V, since 0 is a member and the span is closed under addition and scalar multiplication. And if a subspace contains $v_1, \dots, v_m$, then every member of the span, which can be written as a linear combination of only those vectors, must be in the subspace by closedness under addition and scalar multiplication, so all such subspaces must contain the span. $\square$

Definition 2 (Finite-dimensional vector space)**.** A vector space is finite dimensional if there exists a list of vectors in the space that span it. Otherwise, it is infinite-dimensional.

Definition 3 (Misc. notes on polynomials)**.** We call the set of polynomials in $\mathbb{F}$ as $\mathcal{P}(\mathbb{F})$ (which happens to be a subspace of $\mathbb{F}^{\mathbb{F}}$, and the set of polynomials with degree AT MOST $m, m \geq 0$ as $\mathcal{P}_m(\mathbb{F})$. We consider the polynomial $f(x) = 0$ to have degree $-\infty$, and say that $-\infty < m$, so that it is in all $\mathcal{P}_m(\mathbb{F})$. This makes it also a subspace of $\mathbb{F}^{\mathbb{F}}$. Note that polynomials in $\mathbb{C}$ may have complex coefficients and take complex inputs.

Note that $\mathcal{P}_m(\mathbb{F})$ is spanned by $m+1$ vectors $(1, x, \dots, x^m)$ so is a finite-dimensional vector space, but $\mathcal{P}(\mathbb{F})$ (with no limit on m) is infinite-dimensional because polynomials of degree m cannot be represented without adding x^m to the span for all m.

Definition 4 (Linearly independent)**.** A list $v_1, \dots, v_m$ of vectors in V is called linearly independent if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1, \dots, a_m = 0$. In other words, $\text{span}(v_1, \dots, v_m)$ is a direct sum of $v_1, \dots, v_m$.

The empty list () is defined to be linearly independent.

Otherwise, the list of vectors is linearly dependent. We see this happen if we can find a linear combination of the vectors (with not all coefficients being 0) that sums to 0, i.e. we can write one vector as a linear combination of the others.

Theorem 2 (Linear Dependence Lemma)**.** *Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \dots, m\}$ such that $v_j \in \text{span}(v_1, \dots, v_{j-1}$, and if the jth term is removed from $v_1, \dots, v_m$ then the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.*

Proof. Because the list is linearly independent, we may write $a_1 v_1 + \cdots + a_m v_m = 0$ for some list of coefficients $(a_1, \dots, a_m)$. So letting v_j be the highest-index non-zero vector in the list, we may express it as $\frac{a_1 v_1}{a_j} + \cdots + \frac{a_{j-1} v_{j-1}}{a_j}$, so v_j is in the span of the other vectors. If we removed v_j from the list, for any member of the span we could always replace it with the expression we derived, so the span remains unaffected.

For the special case of $j = 1$ and thus $v_1 = 0$, after removing v_1 the list becomes $\{\}$, which is defined to have span $\{0\}$. $\square$

Theorem 3 (Length of linearly independent and spanning lists). *In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors (for that space).*

Proof. One proof relies on two facts: appending a vector to a spanning list of the space obviously produces a linearly dependent list as the vector is a member of the span, and we can then remove a vector from that linearly dependent list using the Linear Dependence Lemma and the span remains unaffected. Starting with a linearly independent list and a spanning list, we may therefore substitute one element at a time members of the spanning list with members of the linearly independent list, until we have used up all the elements of the latter. There's nothing in the Linear Dependence Lemma that prevents this from working at every step, so we must have at least as many elements in the former as in the latter to avoid a contradiction. □

This fact is useful for identifying linearly independent or non-spanning lists of $\mathbb{F}^n$, which are spanned by the n vectors $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$.

Theorem 4 (Finite-dimensional subspaces). *Every subspace of a finite-dimensional vector space is also finite-dimensional.*

Proof. We can continue adding vectors one at a time to a list of linearly independent vectors in U until we eventually get a spanning list of U. This linearly independent list cannot grow longer than a spanning list of V by the Length of linearly independent and spanning lists theorem. □

2 Problems

Problem 1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ also spans V.

We have $v_3 = (v_3 - v_4) + v_4$, $v_2 = (v_2 - v_3) + (v_3 - v_4) + (v_4)$, and $v_1 = (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + (v_4)$.

Problem 2. Verify the assertions in 2.18 (linearly independent lists).

(a) If $v = 0$, $av = 0$ for any a, hence it's not linearly independent. Otherwise, $a = 0$.

(b) If $w = cv$, $c \in \mathbb{F}$, then $w - cv = 0$. Otherwise, no such relationship exists.

(c) $a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) = (a, b, c, 0)$, where each component is uniquely determined by one coefficient. Getting a sum of 0 requires $a, b, c = 0$.

(d) Powers of z cannot be expressed in terms of linear combinations of other powers of z.

Problem 3. Find a number t such that $(3, 1, 4), (2, -3, 5), (5, 9, t)$ is not linearly independent in $\mathbb{R}^3$.

We have $a_1(3, 1, 4) + a_2(2, -3, 5) + a_3(5, 9, t) = 0$. Solving for a_1, a_2, a_3 in terms of t, we find that $(2 - t)a_3 = 0$. This produces infinitely many solutions when $2 - t = 0$, or $t = 2$, in which case the list will be linearly dependent. Specifically, $(a_1, a_2, a_3) = (-3, 2, 1)$ is a valid solution.

Problem 4. Verify the assertion in the second bullet point in Example 2.20.

Solving $a_1(2, 3, 1) + a_2(1, -1, 2) + a_3(7, 3, c) = 0$, we get that $(8 - c)a_3 = 0$, which produces infinitely many solutions for $c = 8$, otherwise the only solution is $a_1, a_2, a_3 = 0$.

Problem 5. Show that the list $(1 + i, 1 - i)$ is linearly independent if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, but linearly dependent if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$. (The main difference here is whether scalar multiplication allows only scalars in $\mathbb{R}$ or scalars in $\mathbb{C}$.)

Over $\mathbb{R}$: we attempt to solve $a_1(1 + i) + a_2(1 - i) = 0$, where $a_1, a_2 \in \mathbb{R}$ by equating both the real and imaginary components. We get $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$ respectively. In $\mathbb{R}$, we get $a_1, a_2 = 0$ as our only solution, hence the list is linearly independent.

Over $\mathbb{C}$: we attempt to solve $(a_1 + b_1 i)(1 + i) + (a_2 + b_2 i)(1 - i) = 0$. Expanding and separating the real and imaginary components, we get $a_1 - b_1 + a_2 + b_2 = 0$ and $a_1 + b_1 - a_2 + b_2 = 0$ respectively. This is satisfied so long as $a_1 + b_2 = 0$ and $b_1 - a_2 = 0$, or $a_1 = -b_2$ and $b_1 = a_2$, for example $(a_1, b_1, a_2, b_2) = (1, 0, 0, -1)$. Hence the list is linearly dependent.

Problem 6. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is also linearly independent.

We solve $a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4(v_4) = 0$. We get $a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$. Since v_1, v_2, v_3, v_4 is linearly independent, we must have $a_1, a_2 - a_1, a_3 - a_2, a_4 - a_3 = 0$, and since we explicitly have $a_1 = 0$, we get $a_2, a_3, a_4 = 0$ as well. Therefore, the new list is linearly independent.

Problem 7. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then $5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent.

By the same logic as the previous problem, this is linearly independent.

Problem 8. If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, for $\lambda \in \mathbb{F} : \lambda \neq 0$, show whether or not $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

We get $a_1 \lambda v_1 + a_2 \lambda v_2 + \cdots + a_m \lambda v_m = 0$, where by linear independence $a_1\lambda, a_2\lambda, \ldots, a_m\lambda = 0$. Since $\lambda \neq 0$, all the as must be 0. Hence the scalar multiple of the list is also linearly independent.

Problem 9. Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

False: we could for example have the linearly independent lists in $\mathbb{R}$: (1) and (-1), where their sum (0) is linearly dependent. In general, if we have $v_j + w_j = 0$ for some index j, then we get 0 in the new list, which is then linearly dependent.

Problem 10. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

We have $a_1(v_1 + w), \ldots, a_m(v_m + w) = 0$ with some non-zero coefficients. Obviously, $w \neq 0$, otherwise we get our original list which is linearly independent. Rearranging, we get $(a_1 + \cdots + a_m)w = a_1 v_1 + \ldots a_m v_m$. Since $v_1, \ldots, v_m$ is linearly independent and we do not have $a_1, \ldots, a_m = 0$, the right side cannot be equal to 0. Then neither can the left side be equal to 0, and thus $a_1 + \cdots + a_m$ too. So we can safely divide both sides by $a_1 + \cdots + a_m$ to get $w = \frac{a_1 v_1 + \cdots + a_m v_m}{a_1 + \cdots + a_m}$ (which is distributable), and so $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Problem 11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

We know that if $w \in \operatorname{span}(v_1, \ldots, v_m)$ (i.e. it can be expressed as a linear combination of the other vectors), then $v_1, \ldots, v_m, w$ is linearly dependent. Then by modus tollens, if $v_1, \ldots, v_m, w$ is linearly independent, then $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

If $v_1, \ldots, v_m, w$ is linearly dependent, then there exists a linear combination $a_1 v_1 + \cdots + a_m v_m + bw = 0$ with some non-zero coefficients. In particular, this linear combination must have $b \neq 0$, otherwise by the linear independence of $v_1, \ldots, v_m$, we would require $a_1, \ldots, a_m = 0$ which gives all zero coefficients. Then we can write $w = \frac{a_1 v_1 + \cdots + a_m v_m}{b}$ (which is distributable), so $w \in \operatorname{span}(v_1, \ldots, v_m)$. By modus tollens, if $w \notin \operatorname{span}(v_1, \ldots, v_m)$, then $v_1, \ldots, v_m, w$ is linearly independent.

Problem 12. Explain where there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbb{F})$, which is the set of all polynomials with degree less than 4.

A polynomial with degree at most 4 is in the form $f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$, so $\mathcal{P}_4(\mathbb{F})$ is spanned by $(1, x, x^2, x^3, x^4)$, which is a list of length 5. No list of linearly independent vectors is longer than any spanning list, so in this case cannot be of length 6 or greater.

Problem 13. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$.

The list $(1, x, x^2, x^3, x^4)$ of length 5 is linearly independent, so no spanning list can be of length less than 5, lest the linearly independent list be longer.

Problem 14. Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

If such a sequence exists, then for every m, a spanning list cannot have length m without violating the Length of Linearly Independent and Spanning Lists Theorem, as there exists a list of linearly independent vectors of length $m+1$ (the first $m+1$ terms of the sequence) whose length is greater. As there exists no finite spanning list, V is infinite dimensional.

If V is infinite dimensional: the list () of length 0 is linearly independent. For a linearly independent list $v_1, \dots, v_m$ of length m, there must exist some vector $v_{m+1} \in V$ that is not in $\text{span}(v_1, \dots, v_m)$. (If every other vector in V were a member of the span, then $\text{span}(v_1, \dots, v_m) = V$ and V would be finite dimensional.) Then by problem 11, the list $v_1, \dots, v_m, v_{m+1}$ is linearly independent. Therefore, by induction, there exists a list of linearly independent vectors of any length, and hence an infinite sequence of linearly independent vectors. (For generating that sequence, we also know that we can always find a next vector to append to the list, no matter its current contents. This prevents any possibility of finding 'dead ends'.)

Problem 15. Prove that $\mathbb{F}^\infty$ is infinite-dimensional.

The infinite sequence of lists $(1, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, 0, \dots), \dots$ is linearly independent, so by Problem 14 $\mathbb{F}^\infty$ is infinite-dimensional.

Problem 16. Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

This vector space contains the set of all polynomials with restricted domain $[0, 1]$. Since this polynomial set is also a vector space in its own right and thus a subspace of the continuous space, and is infinite-dimensional due to having no upper limit on degree, we can immediately say that by modus tollens on the Finite-Dimensional Subspaces Theorem, the continuous space must also be infinite-dimensional.)

Problem 17. Suppose $p_0, p_1, \dots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \dots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$

Note that all polynomials $p_j(x)$ must either be identically 0 or contain a factor of $x-2$. Furthermore, no degree 0 polynomial (non-zero constant) satisfies the condition. We have $m+1$ polynomials and $m+1$ possible degrees ($-\infty$ and 1 through m), so either we must have the zero polynomial (the only polynomial of degree $-\infty$) in the list, in which case the list is automatically linearly dependent, or else there are $m+1$ polynomials across m degrees from 1 through m, all with a factor $x-2$.

Note that the set of polynomials with degree at most m and being a polynomial multiple of $x-2$, or indeed any other linear factor, is a vector space in its own right (as it contains 0 and is closed under addition and scalar multiplication) and thus a subspace of $\mathcal{P}_m(\mathbb{F})$. Let us call it $\mathcal{P}_{x-2,m}(\mathbb{F})$. The list $x-2, (x-2)^2, \dots, (x-2)^m$ of length m spans it, even though it doesn't contain a constant.

To verify that we can write any polynomial of the subspace as a linear combination of the spanning list, make the substitution $y = x-2$, so the factor of $x-2$ becomes a factor of y. When the polynomial is multiplied out, there will be no constant term, so we get a linear combination in terms of $y, y^2, \dots, y^m$. (This is the same as saying that $\mathcal{P}_{x-2,m}(\mathbb{F})$ can be spanned by $x, x^2, \dots, x^m$.) Undo the substitution and we are done.

Finally, by the Length of Linearly Independent and Spanning Lists Theorem, our list $p_0, p_1, \dots, p_m$ of length $m+1$ must be linearly dependent in $\mathcal{P}_{x-2,m}(\mathbb{F})$, and obviously in $\mathcal{P}_m(\mathbb{F})$ too. (The same logic can be applied for any linear factor $(x-r)$, $r \in \mathbb{R}$.)

(EXTERNAL SOLUTION) Alternately, we can consider that the list $\frac{p_0}{x-2}, \dots, \frac{p_m}{x-2}$ of polynomials in $\mathcal{P}_{m-1}(\mathbb{F})$ is linearly dependent due to being one longer than the spanning list of length m, hence there exists a non-trivial linear combination of that list that sums to 0. Then we can just multiply that linear combination by $(x-2)$.