Chapter 2B

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1 Notes

Definition 1 (basis)**.** A basis of V is a list of vectors in V that is linearly independent and spans V.

Condition (Criterion for basis)**.** A list $v_1, \ldots, v_n$ of vectors in V is a basis if and only if every $v \in V$ can be **uniquely** written in the form $v = a_1 v_1 + \cdots + a_n v_n$, where $a_1, \ldots, a_n \in \mathbb{F}$.

Forwards: consider the case $v = 0$, since setting all coefficients to 0 is a valid solution which is unique by hypothesis, the list is linearly independent. Backwards: consider what should happen if there existed another set of coefficients that summed to v and subtract to get 0 on one side. Then use linear independence to show the coefficients must be the same.

Note that all bases of a vector spaces must be the same length, so as to not violate the Length of Linearly Independent and Spanning Lists theorem.

Theorem 1 (Spanning list contains a basis)**.** *Every spanning list in a vector space can be reduced to a basis of the vector space. (i.e. either the list is a basis or we can remove some vectors to get a basis.)*

Proof. Suppose $v_1, \ldots, v_m$ spans V. If the list of vectors is linearly independent, we are done. Otherwise, the Linearly Dependence Lemma implies there exists a vector (in the span of the others) that we can remove without affecting the span of the list. We may continue removing vectors until our list is linearly independent, and we have a basis.

In fact, we can make this procedural by going through each vector v_j, $1 \leq j \leq n$, removing it only if it is in $\operatorname{span}(v_1, \ldots, v_{j-1})$. Then since no vector remains that is in the span of the previous ones, the Linear Dependence Lemma implies by modus tollens that the list cannot be linearly dependent. □

Corollary 1 (Basis of finite-dimensional vector spaces)**.** *Every finite-dimensional vector space has a basis.*

Proof. If a vector space is finite-dimensional, it has a spanning list. And that spanning list can be reduced to a basis by the previous theorem. □

Theorem 2 (Linearly independent list extends to a basis)**.** *Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

Proof. A direct proof is easy, but we can also use the Spanning List Contains a Basis Theorem. Suppose $u_1, \ldots, u_m$ is a linearly independent list in V and $w_1, \ldots, w_n$ is a basis of V. Then doing the same manual reduction as in the proof of that theorem, we will get as a basis a list containing u_1 through u_m, and some of the ws, which are the vectors we can add. □

Theorem 3 (Every subspace of V is part of a direct sum equal to V)**.** *Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$. In particular, given a basis of U, W can be constructed as the span of the vectors appended to extend the basis of U to a basis of V.*

Proof. Recall the Direct Sum of Two Subspaces Theorem, which states that in order for $V = U \oplus W$, we require $V = U + W$ and $U \cap W = \{0\}$.

We may take a basis $u_1, \ldots, u_m$ of U, which must be a list of linearly independent vectors in V, and extend it into a basis of V. We may take whatever vectors we added, $w_1, \ldots, w_n$ and let them be the basis of a new subspace $W = \operatorname{span}(w_1, \ldots, w_n)$. This construction ensures that $U + W = V$.

Now we need to verify that U and W share no elements apart from 0. Suppose that there exists some vector $v \in V$ so that $v = a_1 u_1, \ldots, a_m u_m$ and $v = c_1 w_1, \ldots, c_n w_n$ for some $a_1, \ldots, a_m, c_1, \ldots, c_n \in \mathbb{F}$. Then subtracting, we have

$$0 = a_1 u_1 + \cdots + a_m u_m + (-c_1) w_1 + \cdots + (-c_m) w_m$$

And since these vectors are a basis of V and thus linearly independent in V, we must have all coefficients being 0, so $v = 0$. □

2 Problems

Problem 1. Find all vector spaces that have exactly one basis.

Normally, you can multiply one basis vector by a non-zero scalar constant and get a different basis (easily verifiable). The only vector space on which you can't do that is $\{0\}$.

Problem 2. Verify the assertions in Example 2.28 (bases).

Yeah, nah mate.

Problem 3. (a) Let U be the subspace of $\mathbb{R}^5$ defined by $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \wedge x_3 = 7x^4\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

(a) This is like $(3x_1, x_1, 7x_2, x_2, x_3)$. A basis is $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$.

(b) The basis can be extended to $\mathbb{R}^5$ with the addition of $(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$.

(c) $W = \operatorname{span}((1, 0, 0, 0, 0), (0, 0, 1, 0, 0))$, or $\{(x_1, 0, x_2, 0, 0) : x_1, x_2 \in \mathbb{R}\}$.

Problem 4. (a) Let U be the subspace of $\mathbb{C}^5$ defined by $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2 \wedge z_3 + 2z_4 + 3z_5 = 0\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

(c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

(a) This is like $(6z_1, z_1, -2z_2 - 3z_3, z_2, z_3)$. A basis is $(6, 1, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$.

(b) The basis can be extended to $\mathbb{C}^5$ with the addition of $(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$.

(c) $W = \operatorname{span}((1, 0, 0, 0, 0), (0, 0, 1, 0, 0))$, or $\{(z_1, 0, z_2, 0, 0) : z_1, z_2 \in \mathbb{C}\}$.

Problem 5. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbb{F})$ such that none of the polynomials has degree 2.

Obviously true. Consider the basis $1, x, x^3 + x^2, x^3$. Note that the 2nd and 3rd degree coefficients can still be determined uniquely as $x^3 + x^2, x^3$ are linearly independent. Verify this by solving the system of equations for an arbitrary polynomial.

Problem 6. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of V.

Note that $a_1v_1+a_2v_2+a_3v_3+a_4v_4 = a_1(v_1+v_2)+(a_2-a_1)(v_2+v_3)+(a_3-a_2)(v_3+v_4)+(a_4-a_3)v_4$, so every vector in $\operatorname{span}(v_1, v_2, v_3, v_4)$ is also in $\operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$. We can similarly show the other direction as well, so the two spans are the same and $\operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4) = V$.

For the equation $0 = a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4v_4$, we get $0 = a_1v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4$, and so $a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4 = 0$. Since we have $a_1 = 0$, by this chain we get $a_2, a_3, a_4 = 0$. So $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also linearly independent in V. Therefore, it is a basis of V.

Problem 7. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3, v_4 \notin U$, then v_1, v_2 is a basis of U.

A counterexample is $V = \mathbb{F}^4$, $v_1, v_2, v_3, v_4 = (1,1,0,0), (0,1,1,0), (0,0,1,1), (0,0,0,1)$ (which is problem 6 applied to the standard basis), and $U = \{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{F}\}$. Note that a standard basis of U is $(1,0,0,0), (0,1,0,0), (0,0,1,0)$, which is linearly independent. So v_1, v_2 cannot span U, as that would violate the Length of Linearly Independent and Spanning Lists Theorem.

Problem 8. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V.

(EXTERNAL SOLUTION) We can write every element of V as a unique linear combination of two vectors $u \in U$ and $w \in W$ (direct sum), and furthermore write u as a unique linear combination of $u_1, \dots, u_m$ and write v_n as a unique linear combination of $v_1, \dots, v_n$ (basis). Therefore, we can write every element of V as a unique linear combination of $u_1, \dots, u_m, v_1, \dots, v_n$, so that is a basis for V.

(OLD) We have $V = U + W$, $U = a_1u_1 + \cdots + a_mu_m$, and $V = b_1w_1 + \cdots + b_nw_n$. Then:

$$V = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n$$

So $u_1, \dots, u_m, w_1, \dots, w_m$ spans V.

Is this list also linearly independent in V? Suppose for sake of contradiction that $u_1, \dots, u_m, w_1, \dots, w_m$ is linearly dependent in V. Consider the equation:

$$0 = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n$$

There must exist a valid solution with non-zero coefficients. But that solution cannot have all $a_1, \dots, a_m = 0$ as $w_1, \dots, w_m$ are linearly independent so that would force $b_1, \dots, b_n = 0$ too. The same applies for setting $b_1, \dots, b_n = 0$ first. (Linear independence also forces all the vectors to be non-zero.) So we must have some non-zero a_i and b_j, $1 \le i \le m$ and $1 \le j \le n$. But that would allow us to write some linear combination in terms of $u_1, \dots, u_m$, a member of U, as equal to some linear combination in terms of $w_1, \dots, w_m$, a member of W, so U and W must share some non-zero vector. But we know that by the Direct Sum of Two Subspaces Theorem, $U \cap W = \{0\}$, so this is impossible. Therefore, $u_1, \dots, u_m, w_1, \dots, w_m$ must be linearly independent.

(At this point, it's clear enough to do a direct proof saying that since $U \cap W = \{0\}$, there exists no way to express an element of U in terms of an element of V. Since the linear dependence relation always lets us do this in this case, we cannot have linear dependence.)

3 Appendix

Theorem 4 (Basis Theorem)**.** *For any basis $(v_1, \dots, v_m)$ of a vector space V with dimension m, every element of V can be written as a unique linear combination of the basis vectors.*

$\operatorname{span}(v_1, \dots, v_m) = V$, so there exists a linear combination $v = a_1v_1 + \cdots + a_mv_m$. Suppose that there exists some other linear combination $v = b_1v_1 + \cdots + b_mv_m$. Subtract to get $0 = (a_1 - b_1)v_1 + \cdots + (a_m - b_m)v_0$. Since $(v_1, \dots, v_m)$ is linearly independent, we must have $(a_1 - b_1), \dots, (a_m - b_m) = 0$, and thus $a_1 = b_1, \dots, a_m = b_m$. Therefore, the linear combination is unique.