Chapter 2C

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July 29, 2022

1 Notes

Definition 1 (Dimension)**.** The dimension of a finite-dimensional vector space is the length of all bases of the space. It is denoted as dim V.

Note that all bases must have the same length as they are both linearly independent in the space and span it. Bases of different length will lead to a conflict with the Length of Independent/Spanning Lists Theorem.

Fact: spaces like $\mathbb{F}^n$ have dimension n, and $\mathcal{P}_m(\mathbb{F})$ has dimension $m+1$.

The dimension of a subspace is not greater than the dimension of the parent space. (Note that any basis of the subspace is a linearly independent list in the parent space.) Additionally, the dimension of a 'proper subspace' must be less than the dimension of the parent space. If it was equal, then we could append a vector missing from the proper subspace to one of its bases to create a linearly independent list one longer than the dimension of the parent space.

Theorem 1 (Linearly independent or spanning list of the right length is a basis)**.** *Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length dim V is a basis of V. And every spanning list of vectors in V with length dim V is also a basis of V.*

Both cases can be proven by a trivial expansion or reduction to a basis which must have length dim V, which leaves the list unchanged.

Theorem 2 (Dimension of a sum)**.** *If U_1, U_2 are subspaces of a finite-dimensional vector space, then dim $U_1 + U_2 =$ dim $U_1 +$ dim $U_2 -$ dim $U_1 \cap U_2$. (This is analogous to counting elements in the union of sets.)*

Take a basis of $U_1 \cup U_2$ and extend it to a basis of U_1 and a basis of U_2. Combine all the vectors into a single list, which should span $U_1 + U_2$ and be of the correct length. Show that it is linearly independent in $U_1 + U_2$ by some algebraic manipulation, the dual role of the basis vectors of $U_1 \cup U_2$ as being in both U_1 and U_2, and closedness under addition and scalar multiplication.

2 Problems

Problem 1. Suppose V is finite-dimensional and U is a subspace of V such that dim $U =$ dim V. Prove that $U = V$.

If $U \neq V$, i.e. U is a proper subspace of V, then the dimension of U must be less than V (see notes for reason). So we must have $U = V$.

Problem 2. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.

We can consider the nature of all subspaces with dimensions 0, 1, and 2. The dimension 0 subspace is $\{0\}$ and by Problem 1 the dimension 2 subspaces is $\mathbb{R}^2$. For a subspace with dimension 1, its basis is a single (non-zero) vector in $\mathbb{R}^2$, and its span will be a line formed by the scalar multiplies of that single vector.

Problem 3. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

Again, we may consider the general form of subspaces generated as the span of between 0 and $\dim \mathbb{R}^3 = 3$ linearly independent vectors in $\mathbb{R}^3$. Dimension 0 corresponds to $\{0\}$ and dimension 3 to $\mathbb{R}^3$. For dimension 1, taking any non-zero vector v the span is av, which is a line through the origin. For dimension 2, the span is $a_1v_1 + a_2v_2$, which we recognize as the form of a plane passing through $(0, 0, 0)$.

Problem 4. Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0\}$. Find a basis in U, then extend to a basis of $\mathcal{P}_4(\mathbb{F})$, and find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

To find a basis, remember the work done in proving Problem 2A.17. For a space of polynomials $\mathcal{P}_{x-c,m}(\mathbb{F}) = \{p(x) \in \mathcal{P}_4(\mathbb{F}) : p(x - c) = 0\}$, the space is spanned by $\{(x - c), (x - c)^2, \dots, (x - c)^m\}$, so is m-dimensional (a reduction of 1 dimension), which we may verify by linear substitution. So for $U = \mathcal{P}_{x-6,m}(\mathbb{F})$, a valid basis is $(x - 6), (x - 6)^2, (x - 6)^3, (x - 6)^4$. We can extend this to a basis of $\mathcal{P}_4(\mathbb{F})$ by the addition of 1, which is not a member of U so must be outside the span of our basis for U and therefore linearly independent from the other vectors. So by the Every Subspace Is Part Of A Direct Sum Theorem, the direct sum of U and the subspace that is the span of 1, the constant functions, is $\mathcal{P}_4(\mathbb{F})$. (This basically allows shifting the polynomial so that $p(6)$ can take on any value.)

Problem 5. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$. Find a basis in U, then extend to a basis of $\mathcal{P}_4(\mathbb{R})$, and find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Consider a general polynomial $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, so $p''(x) = 12a_4x^2 + 6a_3x + 2a_2$. We want $p''(6) = 432a_4 + 36a_3 + 2a_2 = 0$. This imposes one constraint, so the dimension of the solution space for coefficients, and therefore polynomial determined by them, will be 4, so a list of four linearly independent vectors is a basis for U. One such list is $1, x, x^3 - 3x^2, x^4 - 12x^3$. We may extend this to a basis of $\mathcal{P}_4(\mathbb{R})$ with the addition of x^2, and we can set $W = \{f(x) = ax^2; a, x \in \mathbb{R}\}$.

Problem 6. Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5)\}$. Find a basis in U, then extend to a basis of $\mathcal{P}_4(\mathbb{F})$, and find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

U can also be expressed as all polynomials that are shifted by a constant from $\mathbb{F}_{x-2,x-5,4}$. A basis of U is $1, (x - 2)(x - 5), x(x - 2)(x - 5), x^2(x - 2)(x - 5)$. As U is a proper subset of $\mathcal{P}_4(\mathbb{F})$ which has dimension 5, U must be finite-dimensional and have dimension not greater than 4 (dimension 5 is impossible by Problem 1). We may extend this to a basis of $\mathcal{P}_4(\mathbb{F})$ with the addition of x, and we may set $W = \{f(x) = ax : x \in \mathbb{F}, a \in \mathbb{F}\}$.

Problem 7. Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5) = p(6)\}$. Find a basis in U, then extend to a basis of $\mathcal{P}_4(\mathbb{F})$, and find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.

U can also be expressed as all polynomials that are shifted by a constant from $\mathbb{F}_{x-2,x-5,x-6,4}$. A basis of U is $1, (x - 2)(x - 5)(x - 6), x(x - 2)(x - 5)(x - 6)$. As U is a proper subset of Problem 6's set U, which has dimension 4, U must be finite-dimensional and have dimension not greater than 3. We may extend this to a basis of $\mathcal{P}_4(\mathbb{F})$ with the addition of x and x^2, and we may set $W = \{f(x) = a_2x^2 + a_1x : x \in \mathbb{F}; a_1, a_2 \in \mathbb{F}\}$.

Problem 8. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$. Find a basis in U, then extend to a basis of $\mathcal{P}_4(\mathbb{R})$, and find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

We may express U as all polynomials that are derivatives of some function $P(x)$ in $\mathcal{P}_5(\mathbb{R})$ where $P(-1) = P(1)$, which includes $\mathcal{P}_{x+1,x-1,5}(\mathbb{R})$ since shifting by a constant doesn't matter for differentiation. Four such functions are $x^2, x^3 - x, x^4, x^5 - x^3$. Their derivatives are $2x, 3x^2 - 1, 4x^3, 5x^4 - 3x^2$, which forms a basis of U. We may expand this to a basis of $\mathcal{P}_4(\mathbb{R})$ by the addition of 1, and we may set $W = \{f : f(x) = c; c \in \mathbb{R}\}$, i.e. the constant functions.

Problem 9. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that $\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

A good approach is to find a linearly independent list in $\operatorname{span}(v_1 + w, \dots, v_m + w)$ or a subspace that is of length at least $m - 1$, which implies that any basis must be at least as long in this finite-dimensional space. (We have turned this problem to prove into a problem to find via related theorems.)

Remember from Problem 2A.10 that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, \ldots, v_m)$. So if $w \notin \text{span}(v_1, \ldots, v_m)$, then $v_1 + w, \ldots, v_m + w$ is linearly independent, in which case it is a basis of its span which must have dimension m. Now show that if w is in the span that only one dimension is lost. We may write w as a unique linear combination of $v_1, \ldots, v_m$. Let v_j be the last vector featured in the linear combination (so that it has a non-zero coefficient), and consider the subspace $U = \text{span}(v_1 + w, \ldots, v_{j-1} + w, v_{j+1} + w, \ldots, v_m + w)$ with $v_j + m$ removed. The list $v_1 + w, \ldots, v_{j-1} + w, v_{j+1} + w, \ldots, v_m + w$ cannot be linearly dependent, otherwise we could rearrange to write w as a linear combination of a different linear combination excluding v_j, which is a contradiction. So the list is linearly independent, hence the dimension of U (the length of any basis and spanning list) is at least $m - 1$. Being its superspace, $\text{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$ as well.

Alternately, consider that for $2 \leq j \leq m$, that $v_j - v_1 = (v_j + w) - (v_1 + w)$, hence all $v_j - v_1 \in \text{span}(v_1 + w, \ldots, v_m + w)$. Given that the list $v_1, \ldots, v_j$ is linearly independent, it's obvious that $v_2 - v_1, \ldots, v_m - v_1$ of length $m - 1$ is also linearly independent. We can rewrite $a_1(v_2 - v_1) + \cdots + a_{m-1}(v_m - v_1) = 0$ as $-(a_1 + \cdots + a_{m-1})v_1 + a_1 v_2 + \cdots + a_{m-1} v_m$, which by linear independence must have all coefficients be 0. So any spanning list, and therefore any basis, of the space must have length at least $m - 1$.

Problem 10. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Since their degree is less than or equal to m, $p_0, p_1, \ldots, p_m \in \mathcal{P}_m(\mathbb{F})$. Note that $\mathcal{P}_m(\mathbb{F})$ is spanned by $1, x, \ldots, x_m$ so has dimension $m + 1$. Also, each p_j is not in the span of the previous vectors (as we introduce a higher degree term from the previous), so by the Linear Dependence Lemma the list is linearly independent in $\mathcal{P}_m(\mathbb{F})$. By the Right Length Lemma, $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

Problem 11. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim\, U = 3, \dim\, W = 5, U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Note that $U + W = \mathbb{R}^8$, which has dimension 8. So by the Dimension of a Sum Theorem, we must have $\dim\, (U \cap W) = 0$, which can only be the case if $U \cap W = \{0\}$. By the Condition for Direct Sum of Two Subspaces, $U + W$ is indeed a direct sum, so $\mathbb{R}^8 = U \oplus W$.

Problem 12. Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap V \neq \{0\}$.

Suppose that $U \cap W = \{0\}$. Then $\dim\, (U \cap W) = 0$, so by the Dimension of a Sum Theorem, $\dim\, (U + W) = 5 + 5 - 0 = 10$. But this cannot be the case because $U + W$ must remain a subspace of $\mathbb{R}^9$ so must have at most dimension 9. Therefore, $U \cap W$ must include elements other than 0.

Problem 13. Suppose U and W are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

By the Dimension of a Sum Theorem, $\dim\, (U \cap W) = 2$. This implies the existence of a pair of 2 linearly independent vectors in $U \cap W$ that forms a basis, and if the pair are linearly independent they cannot be scalar multiples of each other.

Problem 14. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1, \ldots, U_m$ is finite-dimensional and $\dim(U_1 + \cdots + U_m) \leq \dim\, U_1 + \cdots + \dim\, U_m$.

This follows by induction from the Dimension of a Sum theorem, since the sum of two finite-dimensional subspaces remains a finite-dimensional subspace so sums can be chained indefinitely.

Problem 15. Suppose V is finite-dimensional, with $\dim\, V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that $V = U_1 + \oplus \cdots \oplus U_n$.

There exists a basis $v_1, \ldots, v_n$ of V with length n. This list is also linearly independent. Let $U_j = \text{span} v_j = \{a_j v_j : a_j \in \mathbb{F}\}$. Then $U_1 + \cdots + U_m = \{a_1 v_1 + \ldots a_n v_n : a_1, \ldots, a_n \in \mathbb{F}\} = \text{span}(v_1, \ldots, v_j) = V$. By linear independence, the only way to make this linear combination equal to 0 is to set all coefficients to 0, which in the sum corresponds to choosing the zero vector from each subspace. This satisfies the Condition for Direct Sum.

Problem 16. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and $\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m$. (This is analogous to the obvious statement that if a set is written as a disjoint union of finite subsets, then the number of elements in the set equals the sum of the numbers of elements in the disjoint subsets.)

BASE CASE (direct sum of 2 subsets $U_1 \oplus U_2$): The Direct Sum of Two Subsets Theorem implies that $U_1 \cap U_2 = \{0\}$ with dimension 0. So by the Dimension of a Sum Theorem, $\dim U_1 \oplus U_2 = \dim U_1 + \dim U_2$.

INDUCTIVE STEP (direct sum of $m + 1$ subsets given case m): Note that the direct sum of subspaces remains a subspace of V, and that by Problem 1C.17, summation of subspaces is associative. So $\dim U_1 \oplus \cdots \oplus U_{m+1} = \dim (U_1 \oplus \cdots \oplus U_m) \oplus U_{m+1}$. The Direct Sum of Two Subsets Theorem implies that $(U_1 \oplus \cdots \oplus U_m) \cap U_{m+1} = \{0\}$ with dimension 0. So by the Dimension of a Sum Theorem, the dimension is equal to $\dim (U_1 \oplus \cdots \oplus U_m) + \dim U_{m+1}$, where by the inductive hypothesis the first dimension is $\dim U_1 + \cdots + \dim U_m$.

By induction, the statement is proven for all $m \geq 2$. Alternately, we could do this by showing that the direct sum is the same as the merger of spans, and that all the span vectors must be linearly independent.

Problem 17. Prove or provide a counterexample for the statement that $\dim (U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 - \dim (U_1 \cap U_2) - \dim (U_1 \cap U_3) - \dim (U_2 \cap U_3) + \dim (U_1 \cap U_2 \cap U_3)$.

In $V = \mathbb{R}^2$, consider $U_1 = \text{span}(1, 0)$ (the x-axis), $U_2 = \text{span}(0, 1)$ (the y-axis), and $U_3 = \text{span}(1, 1)$ (the line $y = x$), all with dimension 1. The intersection of any pair or all three of these subspaces is $\{0\}$, so the dimension of any intersection is 0. Any two of $(1, 0), (0, 1), (1, 1)$ form a basis for $\mathbb{R}^2$ so $U_1 + U_2 + U_3 = \mathbb{R}^2$, with dimension 2. As a result, our statement says that $2 = 3$, which is a contradiction, so it cannot be true.

Checking for direct proof:

$$
\begin{aligned}
&\dim (U_1 + U_2 + U_3) \\
&= \dim ((U_1 + U_2) + U_3) \\
&= \dim (U_1 + U_2) + \dim U_3 - \dim ((U_1 + U_2) \cap U_3) \\
&= \dim U_1 + \dim U_2 - \dim (U_1 \cap U_2) + \dim U_3 - \dim ((U_1 + U_2) \cap U_3)
\end{aligned}
$$

We want to show that $\dim ((U_1 + U_2) \cap U_3) = \dim U_1 \cap U_3 + \dim U_2 \cap U_3 - \dim (U_1 \cap U_2 \cap U_3)$. By the Dimension of a Sum Theorem, the second part is equal to $\dim ((U_1 \cap U_3) + (U_2 \cap U_3))$. We find that $(U_1 + U_2) \cap U_3 \supseteq U_1 \cap U_3 + U_2 \cap U_3$ as every element of the second set belongs in the first: each is a vector in U_1 add a vector in U_2 which is in $U_1 + U_2$, and since both are in U_3, by closedness under addition the element remains in U_3. However, the reverse is not necessarily true: an element in U_3 may be the sum of two elements not in U_3, and this doesn't break closedness under addition. As a result, the dimension of the two sets may not be the same, which is the case for the counterexample given.