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# A

**Algorithm 1** create_geometric_map

Given the mesh elements $[\Omega_1, ..., \Omega_m]$;
Given the map coefficients $X^1_{kl}$, $X^2_{kl}$;
Given the solution space $\mathcal{P}_{\mathbf{p}}$ with $\mathbf{p} = (p_1, p_2)$;
Given the quadrature rule $\{(\boldsymbol{\xi}_r, w_r) : r = 1, ..., q\}$, the basis functions $\tilde{N}_k(\boldsymbol{\xi}_r)$ and their derivatives $\frac{\partial \tilde{N}_k}{\partial \tilde{\xi}^k_r}(\boldsymbol{\xi}_r)$ on the reference domain;
Initialize $\phi = \mathbf{0}$, $\frac{\partial \phi}{\partial \xi} = \mathbf{0}$, $\frac{\partial \phi^{-1}}{\partial \mathbf{x}} = \mathbf{0}$;

**for** $l = 1$ to $m$ **do**
  **for** $r = 1$ to $q$ **do**
    **Compute the geometric map:**
    $\phi^1_l(\boldsymbol{\xi}_r) = \sum_{k=1}^{(p_1+1)(p_2+1)} X^1_{kl} \tilde{N}_k(\boldsymbol{\xi}_r)$;
    $\phi^2_l(\boldsymbol{\xi}_r) = \sum_{k=1}^{(p_1+1)(p_2+1)} X^2_{kl} \tilde{N}_k(\boldsymbol{\xi}_r)$;

    **Compute map derivatives:**
    $\frac{\partial \phi^1_l}{\partial \xi^1}(\boldsymbol{\xi}_r) = \sum_{k=1}^{(p_1+1)(p_2+1)} X^1_{kl} \frac{\partial \tilde{N}_k}{\partial \xi^1}(\boldsymbol{\xi}_r)$;
    $\frac{\partial \phi^1_l}{\partial \xi^2}(\boldsymbol{\xi}_r) = \sum_{k=1}^{(p_1+1)(p_2+1)} X^1_{kl} \frac{\partial \tilde{N}_k}{\partial \xi^2}(\boldsymbol{\xi}_r)$;
    $\frac{\partial \phi^2_l}{\partial \xi^1}(\boldsymbol{\xi}_r) = \sum_{k=1}^{(p_1+1)(p_2+1)} X^2_{kl} \frac{\partial \tilde{N}_k}{\partial \xi^1}(\boldsymbol{\xi}_r)$;
    $\frac{\partial \phi^2_l}{\partial \xi^2}(\boldsymbol{\xi}_r) = \sum_{k=1}^{(p_1+1)(p_2+1)} X^2_{kl} \frac{\partial \tilde{N}_k}{\partial \xi^2}(\boldsymbol{\xi}_r)$;

    **Compute inverse map derivatives:**
    $\det = \det(\frac{\partial \phi_l}{\partial \xi}(\boldsymbol{\xi}_r)) = \frac{\partial \phi^1_l}{\partial \xi^1}(\boldsymbol{\xi}_r) \frac{\partial \phi^2_l}{\partial \xi^2}(\boldsymbol{\xi}_r) - \frac{\partial \phi^1_l}{\partial \xi^2}(\boldsymbol{\xi}_r) \frac{\partial \phi^2_l}{\partial \xi^1}(\boldsymbol{\xi}_r)$;
    $\frac{\partial (\phi^{-1}_l)^1}{\partial x^1}(\phi_l(\boldsymbol{\xi}_r)) = \frac{1}{\det} \frac{\partial \phi^2_l}{\partial \xi^2}(\boldsymbol{\xi}_r)$;
    $\frac{\partial (\phi^{-1}_l)^1}{\partial x^2}(\phi_l(\boldsymbol{\xi}_r)) = \frac{-1}{\det} \frac{\partial \phi^1_l}{\partial \xi^2}(\boldsymbol{\xi}_r)$;
    $\frac{\partial (\phi^{-1}_l)^2}{\partial x^1}(\phi_l(\boldsymbol{\xi}_r)) = \frac{-1}{\det} \frac{\partial \phi^2_l}{\partial \xi^1}(\boldsymbol{\xi}_r)$;
    $\frac{\partial (\phi^{-1}_l)^2}{\partial x^2}(\phi_l(\boldsymbol{\xi}_r)) = \frac{1}{\det} \frac{\partial \phi^1_l}{\partial \xi^1}(\boldsymbol{\xi}_r)$;
  **end for**
**end for**

**Return the geometric map:**
**return** $\phi$, $\frac{\partial \phi}{\partial \xi}$, $\frac{\partial \phi^{-1}}{\partial \mathbf{x}}$

# B

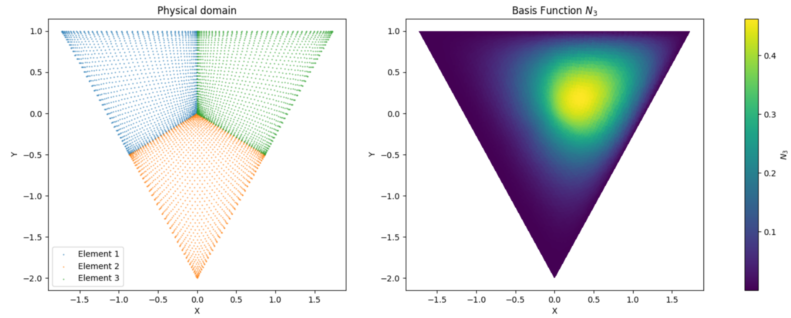


Figure 1:

# C

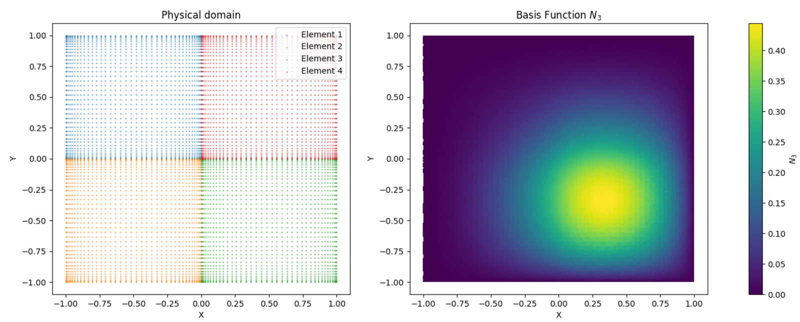


Figure 2:

# D

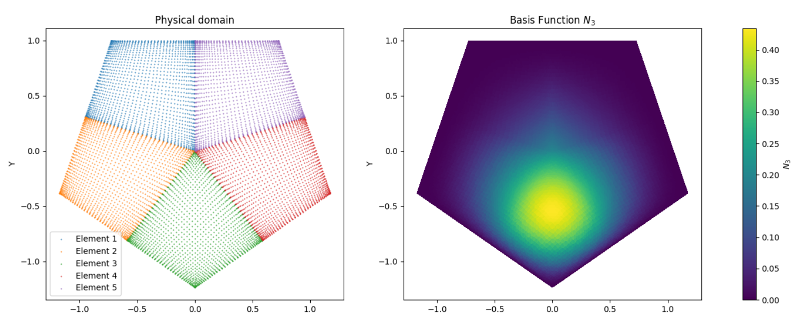


Figure 3:

# E

**Algorithm 2** Assemble 2D FE problem.

Given the mesh elements $[\Omega_1, ..., \Omega_m]$;
Given the solution space $\mathcal{P}_{\mathbf{p}}$ with $\mathbf{p} = (p_1, p_2)$;
Read the extraction matrix $\mathbf{E}$;
Choose quadrature rule $\{(\boldsymbol{\xi}_r, w_r) : r = 1, ..., q\}$ on the reference domain;
Initialize $\mathbf{A} = \mathbf{0}$;
Initialize $\mathbf{b} = \mathbf{0}$;
**for** $l = 1$ to $m$ **do**
  **for** $i \in \mathcal{I}(l)$ **do**
    $N_{i|\Omega_l} = \sum_{k=1}^{(p_1+1)(p_2+1)} e_{ikl} \tilde{N}_{kl}$;
    $N_{i,x_1|\Omega_l} = \sum_{k=1}^{(p_1+1)(p_2+1)} e_{ikl} (\tilde{N}_{kl,\xi_1} \phi^{-1}_{l1,x_1} + \tilde{N}_{kl,\xi_2} \phi^{-1}_{l2,x_1})$;
    $N_{i,x_2|\Omega_l} = \sum_{k=1}^{(p_1+1)(p_2+1)} e_{ikl} (\tilde{N}_{kl,\xi_1} \phi^{-1}_{l1,x_2} + \tilde{N}_{kl,\xi_2} \phi^{-1}_{l2,x_2})$;
    **for** $j \in \mathcal{I}(l)$ **do**
      $N_{j|\Omega_l} = \sum_{k=1}^{(p_1+1)(p_2+1)} e_{jkl} \tilde{N}_{kl}$;
      $N_{j,x_1|\Omega_l} = \sum_{k=1}^{(p_1+1)(p_2+1)} e_{jkl} (\tilde{N}_{kl,\xi_1} \phi^{-1}_{l1,x_1} + \tilde{N}_{kl,\xi_2} \phi^{-1}_{l2,x_1})$;
      $N_{j,x_2|\Omega_l} = \sum_{k=1}^{(p_1+1)(p_2+1)} e_{jkl} (\tilde{N}_{kl,\xi_1} \phi^{-1}_{l1,x_2} + \tilde{N}_{kl,\xi_2} \phi^{-1}_{l2,x_2})$;
      value $= 0$;
      **for** $r = 1$ to $q$ **do**
        value $+=$ `problem_B`$\Big(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r),\, N_{i|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r)),\, N_{i,x_1|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r)),\, N_{i,x_2|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r)),$
          $N_{j|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r)),\, N_{j,x_1|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r))\,, N_{j,x_2|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r))\Big)\, w_r \det(\nabla_{\boldsymbol{\xi}}\, \boldsymbol{\phi}_l(\boldsymbol{\xi}_r))$;
      **end for**
      $A_{ij} +=$ value;
    **end for**
    value $= 0$;
    **for** $r = 1$ to $q$ **do**
      value $+=$ `problem_L`$\Big(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r),\, N_{i|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r)),\, N_{i,x_1|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r)),$
        $N_{i,x_2|\Omega_l}(\boldsymbol{\phi}_l(\boldsymbol{\xi}_r))\Big)\, w_r \det(\nabla_{\boldsymbol{\xi}}\, \boldsymbol{\phi}_l(\boldsymbol{\xi}_r))$;
    **end for**
    $b_{ij} +=$ value;
  **end for**
**end for**

# F

We want to show that $u_h \in \mathcal{S}_h$ implies that $u_{i_1} = ... = u_{i_{n_D}} = 0$. By definition, $u_h \in \mathcal{S}_h$ implies that $u_h \in \mathcal{F}(p, k; \Omega_h)$ and $u_h|_{\Gamma_D} = 0$. By definition we have that

$$u_h|_{\Gamma_D} = \sum_{\ell=1}^{n_D} u_{i_\ell} N_{i_\ell}|_{\Gamma_D}$$
$$\Rightarrow \sum_{\ell=1}^{n_D} u_{i_\ell} N_{i_\ell}|_{\Gamma_D} = 0$$

Since $N_{i_1}, ..., N_{i_{n_D}}$ are the basis functions that are non-zero on $\Gamma_D$, we must have that $u_{i_\ell} = 0$ for $\ell = 1, ..., n_D$.

## G

We know that $Au = b$. From F we also know that $u_{i_1} = ... = u_{i_{n_D}} = 0$. Thus, while computing $Au$, there are $n_D$ columns of $A$ whose entries get multiplied by $u_{i_1} = ... = u_{i_{n_D}} = 0$. Thus, we can eliminate these $n_D$ columns of $A$ without losing any information. Moreover, the rows of $A$ corresponding to the basis functions that are non-zero on $\Gamma_D$ give zero when we take the dot product of those and $u$. Therefore, we can take them out of $A$, and since there are exactly $n_D$ of them, $A$ in the end becomes a $(n - n_D) \times (n - n_D)$ matrix. We can define this $(n - n_D) \times (n - n_D)$ sub-matrix of $A$ as $\tilde{A}$. Furthermore, we can take away the $n_D$ coefficients of $b$ corresponding to the rows of $A$ that we took away, creating a new vector which we call $\tilde{b}$.

In this way we now have an $(n - n_D) \times (n - n_D)$ sub-matrix $\tilde{A}$ of $A$ and an $(n - n_D) \times 1$ sub-vector $\tilde{b}$ of $b$ such that $\tilde{A}^{-1}\tilde{b}$ will yield the remaining unknown coefficients $u_i$.

## H

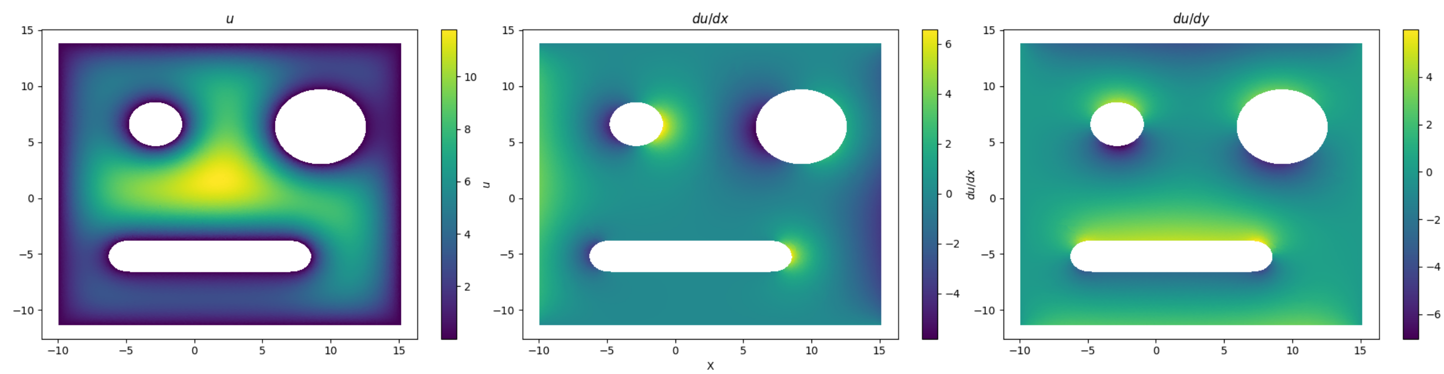


Figure 4: From `distressed_robotD.mat` with neval = 20

# I

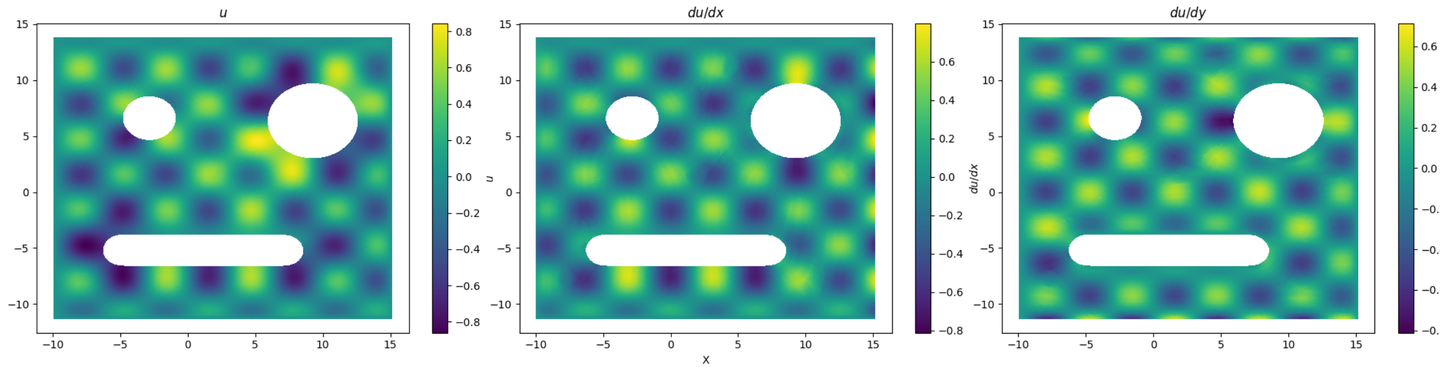


Figure 5: From `distressed_robotDN.mat` with neval = 20