Gauge Field Regularization as a categorical generalization of weak equivalence: Applications to cosmological string detection, jazz harmony, and the structure of human language.

Abstract:

This paper aims at giving a category theoretical description of gauge field regularization as a combined technique to implement Bayesian Learning under SU(2) invariance. As a result, we obtain a formulation of gauge field regularization as a generalized notion of weak equivalence in the sense of Brown (1977), Quillen (1967), and Baues (1993). This allows us to formulate efficient algorithms for type inference in an intuitionistic Martin-Löf style type theory that are equivalent to sequentially improving approximations of exact Bayesian inference in sequence modeling.

We study gauge field regularization as a new instrument for three learning tasks:

1. Using numerical solutions of cosmological equations as a sequential model for cosmic string detection;
2. Creating a generative model for modal jazz harmony from a database of songs and a series of known principles used in jazz analysis;
3. deriving phonotactic rules for the Finno-Ugric family of languages from a database of IPA transcribed conversations.

The main motivation is to show that Brouwerian techniques, i.e., tools from the field of Intuitionistic Mathematics, are able to solve problems in other fields of science in a natural and effective way.

Introduction

Gauge fields are differential equations with a symmetry group G that have been extensively used in quantum field theory, quantum field gravity, and quantum gravity (Mendelsohn et al., 2016). It is characterized by a differential equation of the form $D_k\Phi(x) = (\partial_k + igA_k(x))\,\Phi(x)$, where A is the gauge potential, g is the gauge coupling constant, and Φ is the dynamical field. The invariance arises from the fact that D_k can be decomposed into a commutative and an anticommutative part. The commutative part is given by the gauge potential, while the anticommutative part involves the field $\Phi(x)$ and the symmetry group G (Mendelsohn et al., 2016). The integration of gauge fields and the Lie group G leads to the so-called Brouwer's Apery-Weyl fixed point theorem, which states that every point in a bounded convex subset of a Banach space is a fixed point of a continuous function from the normed space into itself (Brouwer, 1997). In other words, we obtain a fixed point of the path integral for a given measure on the space of field and potential configurations. This can be seen as a corollary of the more general theorem that, for a continuous map $f : X \to X$ between compact Hausdorff spaces, there is a fixed point of f in the closure of the image of f (Gleason and Dugundji, 1974). The important relationship between gauge fields and Brouwer's Apery-Weyl theorem allows us to describe gauge field regularization as a categorical structure that is a generalization of weak equivalence. Indeed, the path integral of gauge fields can be considered as a regularization of the classical field theory, which is obtained in the limit of infinite gauge coupling constant. In this way, it can be seen as a homotopy equivalence between the classical theory and the gauge fixed limit.

In probabilistic models, Bayesian inference can be written as a series of field operators applied to the exact posterior density, which is not a density function (Gleason and Dugundji, 1974; Brouwer, 1997). In this way, Bayesian inference is a construction that, starting from the a priori (i.e., the initial data that we have), successively integrates the field of the a posteriori (i.e., the data that we have at each timestep) and estimates the exact posterior. Such estimation is exact in the limit of infinite coupling constants (Gleason and Dugundji, 1974; Brouwer, 1997). This is due to the fact that the application of the field operator is a homeomorphism, and the inverse of such operator is the integration over the field configuration space with respect to the Haar measure of the underlying symmetry group (Mendelsohn et al., 2021). In practice, this corresponds to taking the limit of an infinite number of iterations of the field operator. Here we show that if the underlying symmetry group G is SU(2), then the underlying complex structure is the one of a unitary representation, and the infinite number of iterations of a field operator is achieved by the spectral projection of a self-adjoint operator on a generalized gauge theory with a corresponding Apery-Weyl fixed point the anticommutative part of which corresponds to the exact posterior.

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