First-Order Logic (continued)

(or predicate logic)

Natural numbers

Language

- $L_{arith} = (<,\ +,\ \cdot,\ 0,1,2,\dots)$
- Alternatively, also $(<,\ +,\ \cdot,\ 0, S)$ --- 1 can be $S(0)$, 2 is can be $S(S(0))$

Structure

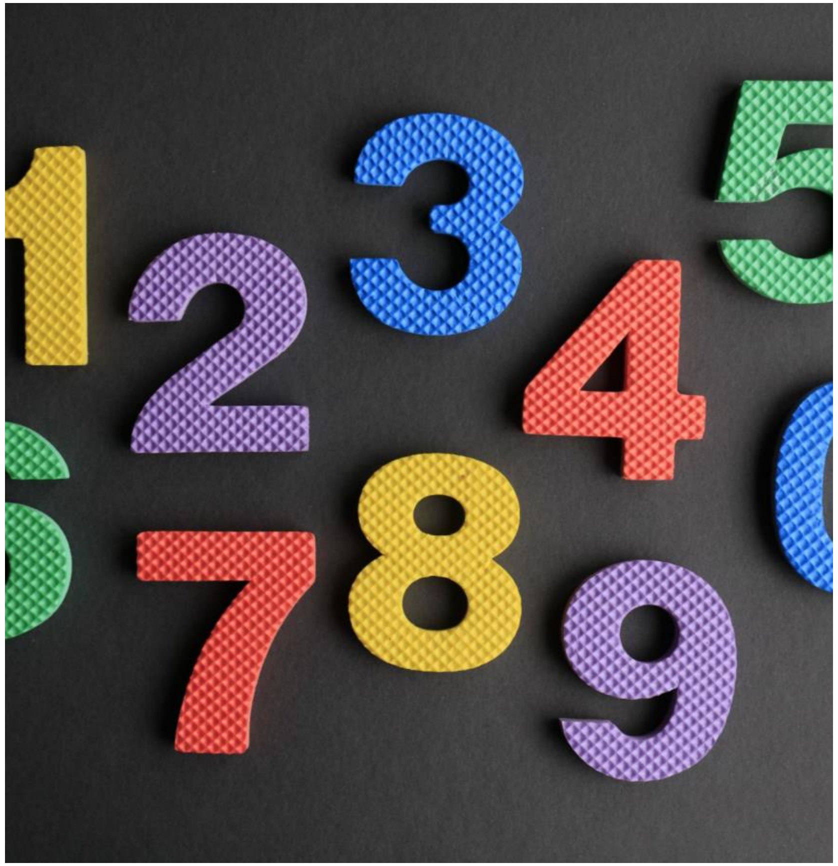
- $\mathbb{N}$ as the universe with standard interpretation of $<,\ +,\ \cdot$
- Nonstandard structures exist!

Axioms

- $\forall x: \neg(x < 0)$
- $\forall x, y: x < y \lor x = y \lor y < x$
- …

Propositions

- Addition is associative:
  - $\forall x, y, z: (x + y) + z = x + (y + z)$
- Prime number:
  - $\mathrm{Prime}(x) \equiv \neg\exists y, z\ (1 < y \land 1 < z \land x = y \cdot z)$
- Twin prime conjecture:
  - $\forall x \exists y: x < y \land \mathrm{Prime}(y) \land \mathrm{Prime}(y + 2)$



(Finite) Graphs

Language

- $L_{graph} = (E, <)$

Structure

- Any graph (V, E) with $<$ interpreted (arbitrarily)

Axioms ($<$ is the transitive closure of E)

- Skipped

Undirected graphs

- $\forall u, v: E(u, v) \rightarrow E(v, u)$

Linked lists

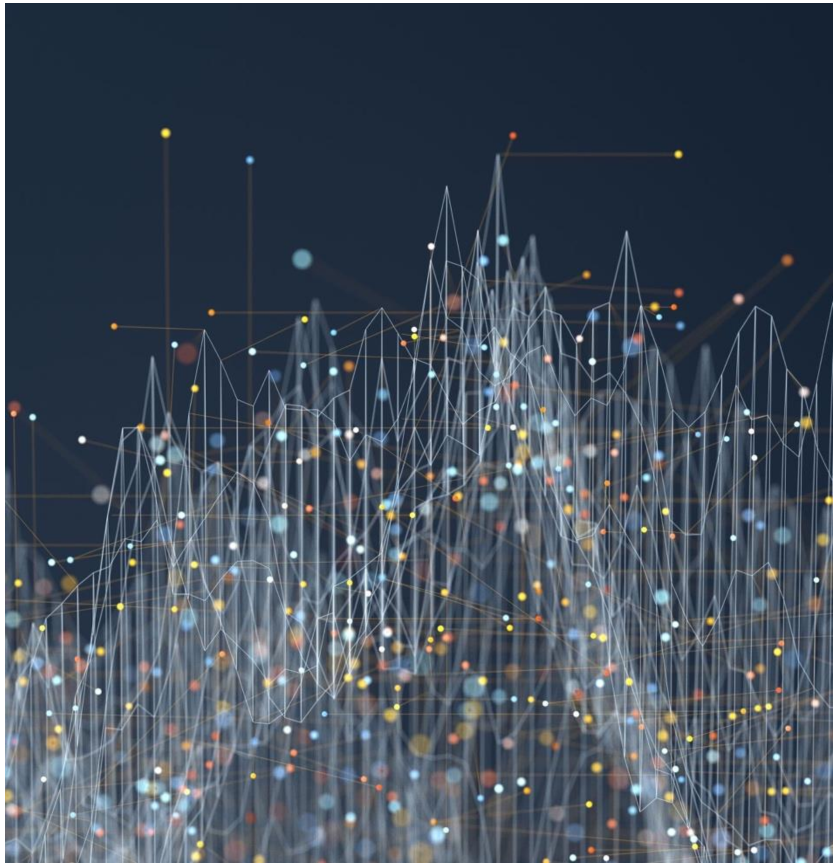
- $\forall u: \neg u < u$
- $\forall u, v: u < v \vee v < u$

n-clique

- $\exists u_1 \dots u_n: \bigwedge_{i,j} u_i < u_j$

Hamiltonicity

- Inexpressible in FOL!



Second-Order Logic

Second-Order (SO) logic extends FOL with second-order variables

- E.g., R^1 (a set variable), R^2 (a relation variable), F^1 (a unary-function variable), F^2 (a binary-function variable) …
- $\exists F^2 \forall x, y(F^2(x,y) = F^2(y,x) \land \ \dots)$
- Sound and complete proof system does not exist

Infinity:

- $\exists z \exists F^1\big((\forall x, y{:}\, F^1(x) = F^1(y) \rightarrow x = y) \land \forall x(F^1(x) \neq z)\big)$

Hamiltonicity:

- $\exists L \exists S \begin{pmatrix} \text{linearOrder}(L) \\ \land \text{"}S \text{ is the successor relation of } L\text{"} \\ \land \forall x \exists y\big(L(x,y) \lor L(y,x)\big) \\ \land \forall x \forall y\big(S(x,y) \rightarrow E(x,y)\big) \end{pmatrix}$

Subclasses of second-order logic

Monadic Second-Order (MSO) logic allows set variables only

- E.g., $\exists S\big(\ldots \exists x(x \in S \land \ \ldots)\big)$

Existential SO (∃SO) consists of formulae of the form $\exists X_1 \ldots \exists X_n\colon \varphi$

How about ∀SO? ∀MSO?

FOL: Models

Let Σ be a set of L-sentences, and $\mathcal{A}$ be an L-structure.

$\mathcal{A}$ is a model of Σ if $\mathcal{A} \vDash \sigma$ for each $\sigma \in \Sigma$, denoted as $\mathcal{A} \vDash \Sigma$

Σ is satisfiable if it has a model

φ (with free variables $x_1, \dots, x_m$) is a logical consequence of Σ if $\mathcal{A} \vDash \forall x_1, \dots x_m \varphi(x_1, \dots, x_m)$ for every model $\mathcal{A}$ of Σ, denoted $\Sigma \vDash \varphi$

- Special case $\Sigma = \emptyset$: $\vDash \varphi$ means φ is satisfied by all structures, i.e., φ is valid

How to check the satisfiability/validity of FOL?

Undecidability of FOL

Church's Theorem (1935): The validity of FOL is undecidable.

- Turing proved independently in 1936

Proof: Reduce from the halting problem of 2-Counter Machines.

Undecidability of FOL

For each 2CM M, construct a formula φ_M such that M halts iff. φ_M is valid

$$\varphi_M \equiv (\varphi_{init} \wedge \varphi_{trans}) \rightarrow \varphi_{final}$$

- φ_{init}: "the initial configuration is reachable"
- φ_{trans}: "if conf A is reachable and A can transition to B, B is also reachable"
- φ_{final}: "A final state is reachable"

Undecidability of FOL

Language

- $C = Q \cup \{0\}$
- $F = \{s\}$ with $arity(s) = 1$
- $R = reach$ with $arity(reach) = 3$
- Intuitively, $reach(q, s(0), 0)$ means the configuration $(q, 1,0)$ is reachable

$\varphi_{init} \equiv reach(q_0, 0,0)$

$\varphi_{final} \equiv \exists x, y\colon \bigvee_{f \in FinalState} reach(f, x, y)$

$\varphi_{trans} \equiv \bigwedge_{t \in \Delta} \varphi_t$

- E.g., if $t = (q, z, nz, q', 1, -1)$, then
- $\varphi_t \equiv \forall x, y\big((reach(q, 0, x) \land x \neq 0 \land s(y) = x) \to reach(q', s(0), y)\big)$