

Hoare Logic

Axiomatic Semantics (AKA program logics)

A system for proving properties about programs

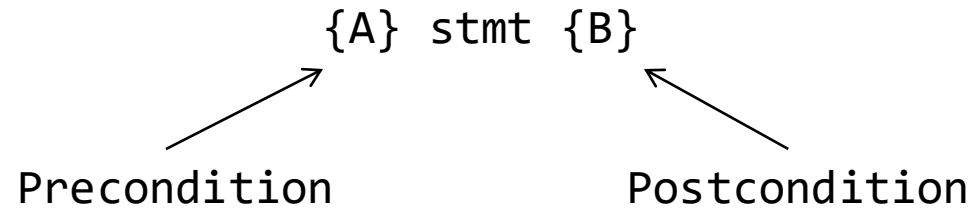
Key idea:

- We can define the semantics of a construct by describing its effect on **assertions** about the program state.

Two components

- A language for stating assertions (“the assertion logic”)
 - Can be First-Order Logic (FOL), a specialized logic such as separation logic, or Higher-Order Logic (HOL), which can encode the others.
 - Many specialized languages developed over the years: Z, Larch, JML, Spec#
- Deductive rules (“the program logic”) for establishing the truth of such assertions

Hoare Triples



Hoare triple

- If the program state *before* execution satisfies A , and the execution of stmt *terminates*, the program state *after* execution satisfies B
- This is a partial correctness assertion.
- We sometimes use the notation

$[A] \text{ stmt } [B]$

to denote a total correctness assertion

which means you also have to prove termination.

What do assertions mean?

The language of assertions:

- $A ::= \text{true} \mid \text{false} \mid e_1 = e_2 \mid e_1 \leq e_2 \mid A_1 \wedge A_2 \mid \neg A \mid \forall x. A$
- $e ::= 0 \mid 1 \mid \dots \mid x \mid y \mid \dots \mid e_1 + e_2 \mid e_1 \cdot e_2$

Notation $\sigma \models A$ means that the assertion holds on state σ .

- A is interpreted inductively over state σ as a FO structure.
- Ex. $\sigma \models x = 1$ iff. $\sigma[x] = 1$
- Ex. $\sigma \models A \wedge B$ iff. $\sigma \models A$ and $\sigma \models B$

Derivation Rules

Derivation rules for each language construct

$$\frac{}{\vdash \{A[x \rightarrow e]\}x := e \{A\}} \quad \frac{\vdash \{A \wedge b\}c_1 \{B\} \quad \vdash \{A \wedge \text{not } b\}c_2 \{B\}}{\vdash \{A\}\text{if } b \text{ then } c_1 \text{ else } c_2 \{B\}}$$

$$\frac{\vdash \{A\}c_1 \{C\} \quad \vdash \{C\}c_2 \{B\}}{\vdash \{A\}c_1; c_2 \{B\}} \quad \frac{\vdash \{A \wedge b\}c \{A\}}{\vdash \{A\}\text{while } b \text{ do } c \{A \wedge \text{not } b\}}$$

Can be combined with the rule of consequence

$$\frac{\vdash A' \Rightarrow A \quad \vdash \{A\}c \{B\} \quad \vdash B \Rightarrow B'}{\vdash \{A'\}c \{B'\}}$$

Example

The following program purports to compute the square of a given integer n (not necessarily positive).

```
int i, j;  
i := 1;  
j := 1;  
while (j != n) {  
    i := i + 2*j + 1;  
    j := j+1;  
}  
return i;
```

Example

{true}

int i, j;

i := 1;

j := 1;

while (j != n) {

 i := i + 2*j + 1;

 j := j+1;

}

return i;

{i = n*n}

Example

{true}

int i, j;

{??}

i := 1;

{??}

j := 1;

{??}

while (j != n) {

 i := i + 2*j + 1;

 j := j+1;

}

{??}

return i;

{i = n*n}

Example

```
{true}
int i, j;
{true} //strongest postcondition
i := 1;
{i=1} //strongest postcondition
j := 1;
{i=1 ∧ j=1} //strongest postcondition
{??} //loop invariant
while (j != n) {
    i := i + 2*j + 1;
    j := j+1;
}
{i = n*n} //weakest precondition
return i;
{i = n*n}
```

Example

```
{true}
int i, j;
{true} //strongest postcondition
i := 1;
{i=1} //strongest postcondition
j := 1;
{i=1 ∧ j=1} //strongest postcondition
{??} //loop invariant
while (j != n) {
    i := i + 2*j + 1;
    j := j+1;
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{i = n*n} //weakest precondition
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{i=1 ∧ j=1} //strongest postcondition
{i = j*j} //loop invariant
while (j != n) {
    i := i + 2*j + 1;
    j := j+1;
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{i = n*n} //weakest postcondition
return i;
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Example

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{true}
int i, j;
{true} //strongest postcondition
i := 1;
{i=1} //strongest postcondition
j := 1;
{i=1 ∧ j=1} //strongest postcondition
{i = j*j} //loop invariant
while (j != n) {
    {i = j*j ∧ j != n}
    {i + 2*j + 1 = (j+1)*(j+1)}
    i := i + 2*j + 1;
    {i = (j+1)*(j+1)}
    j := j+1;
    {i = j*j}
}
{i = n*n} //weakest postcondition
return i;
{i = n*n}
```

Soundness and Completeness

What does it mean for our derivation rules to be sound?

What does it mean for them to be complete?

So, are they complete?

$\{\text{true}\} x:=x \{p\}$

$\{\text{true}\} c \{\text{false}\}$

Relative Completeness in the sense of Cook (1974)

Expressible enough to express intermediate assertions, e.g., loop invariants