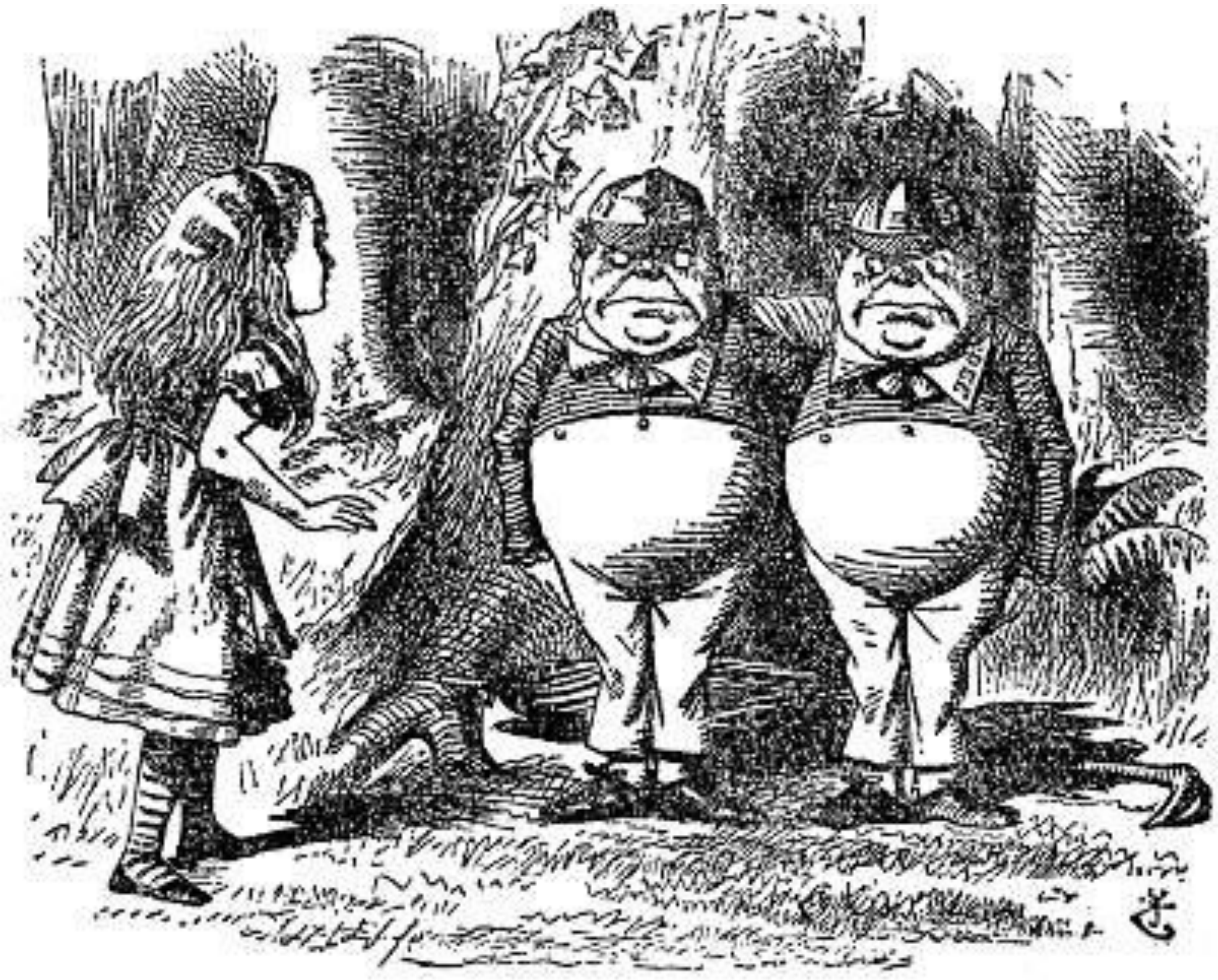
Logic

“Contrariwise,” ... “if it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic.”

- Tweedledum and Tweedledee, Lewis Carroll



Why logic?

- Logic: the Calculus of Computation
  - the study of the principles of valid inference and demonstration.
  - A formal language for describing problems (and their complexity!)
  - Algorithm correctness verification
  - A lot of interesting algorithms!

- Syntax + Semantics + Inference

  (form) (meaning) (reasoning)

$$\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}': (\boldsymbol{x} > \boldsymbol{y} \wedge \boldsymbol{x}' = \boldsymbol{x} + \mathbf{1}) \rightarrow \boldsymbol{x}' > \boldsymbol{y}$$

Propositional Logic

(or zeroth-order logic)

What are propositions?

Propositions are sentences that can be determined as true or false

- p = “Tomorrow is raining.”
- q = “U.S. has 50 states.”

Propositional logic: Syntax

P: a countably infinite set of propositions $\{p_1, p_2, \dots\}$

Well Formed Formulae (WFF) is the smallest set that satisfies:

- $\top, \bot \in WFF$, $p \in WFF$ for any $p \in P$
- If $\alpha \in WFF$, then $(\neg\alpha) \in WFF$
- If $\alpha_1, \alpha_2 \in WFF$, then $(\alpha_1 \sim \alpha_2) \in WFF$
- $\sim$ is a connective, e.g., $\wedge, \vee, \rightarrow$

Propositional Logic: Syntax

Precedence:

- $\neg$ takes precedence over $\wedge$
- $\wedge$ takes precedence over $\vee$
- E.g., $\neg p \wedge q \vee r$

Alternative Syntax

- $\alpha \rightarrow \beta \equiv \neg \alpha \vee \beta$
- $\alpha \wedge \beta \equiv \neg(\neg \alpha \vee \neg \beta)$

Propositional Logic: Semantics

A model/valuation/interpretation is a function

$v: P \rightarrow \{T, F\}$

v can be extended to $\hat{v}$, mapping every formula to $\{T, F\}$:

- $\hat{v}(p) = v(p)$
- $\hat{v}(\neg\alpha) = \begin{cases} T \ if \ \hat{v}(\alpha) = F \\ F \ if \ \hat{v}(\alpha) = T \end{cases}$
- $\hat{v}(\alpha_1 \wedge \alpha_2) = \begin{cases} T \ if \ \hat{v}(\alpha_1) = T \ and \ \hat{v}(\alpha_2) = T \\ F \qquad otherwise \end{cases}$
- $\hat{v}(\alpha_1 \vee \alpha_2) = \begin{cases} T \ if \ \hat{v}(\alpha_1) = T \ or \ \hat{v}(\alpha_2) = T \\ F \qquad otherwise \end{cases}$

Propositional Logic: Semantics

$\hat{v}(\alpha) = T$ is also denoted as $v \vDash \alpha$

A wff α is a tautology if $v \vDash \alpha$ for every model v,

denoted as $\vDash \alpha$

- E.g., $p \to p$

A wff α is satisfiable if $v \vDash \alpha$ for some model v

- E.g., $p \land \neg q$

Theorem: α is a tautology iff. $\neg\alpha$ is unsatisfiable.

Propositional Logic: Inference

The SAT problem: How to check if α is satisfiable (or $\neg\alpha$ is a tautology) ?

- Naïve algorithm: enumerate all possible models (exponentially many)
- The first known NP-complete problem (Cook 1971)
- At least as hard as *all* NP problems

Algorithms for the SAT problem

- Hilbert System
- Tableaux
- Resolution
- DPLL
- Natural Deduction
- Sequent Calculus