First-Order Logic

(or predicate logic)

Predicate

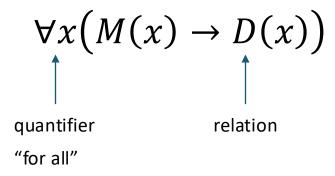
Syllogism

- Major premise: Every man is mortal
- Minor premise: Socrates is a man
- Conclusion: Socrates is mortal

Predicates:

- M for "is a man": M(Socrates), M(Aristotle), M(Purdue), ...
- D for "is mortal": for D(Socrates), D(Aristotle), D(Purdue), ...

First-Order Logic



Number theory:

$$prime(x) \equiv \neg \exists y, z(y > 1 \land z > 1 \land y \cdot z = x)$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$quantifier relation \qquad function$$

$$"exists" \qquad (GT(y, 1))$$

FOL: Syntax

A first-order language is a tuple (R, F, C, arity)

- R is a countable set of relations
- F is a countable set of functions
- C is a countable set of constants
- $arity: R \cup F \rightarrow \mathbb{N}^+$ is an arity function

We also assume a countably infinite set of variables Var

E.g.,
$$L_{arith} = (<, +, \cdot, 0,1,2,...)$$

 $L_{group} = (_^{-1}, \cdot, 1)$

FOL: Syntax

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Term t :: -c \mid f(t_1, ... t_m) \mid x (c \in C, f \in F, x \in Var, Arity(f) = m)

Formula \varphi, \psi :: - \bot \mid \top \mid t_1 = t_2 \mid r(t_1, ... t_m)

\neg \varphi \mid \varphi \lor \psi \mid \exists x \psi \quad (t_1, t_2 \in C, r \in R, x \in Var, Arity(r) = m)
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Bounded variables: $\exists x (...x...) \equiv \exists y (...y...)$

 $FV(\varphi)$: set of free variables in φ (not bounded by any quantifier)

Sentence : $FV(\varphi) = \emptyset$

Defined symbols : \land , \forall , \rightarrow , \leftrightarrow

FOL: Semantics

Let L = (R, F, C, arity) be a first-order language. An L-structure is a tuple (A, τ) :

- *A* is a universe
- τ is a function over $R \cup F \cup C$ s.t.
 - $\tau(r) \subseteq A^m$ for every $r \in R$
 - $\tau(f): A^m \to A$ for every $f \in F$
 - $\tau(c) \in A$ for every $c \in C$

FOL: Semantics

Let L = (R, F, C, arity) be a first-order language, $\mathcal{A} = (A, \tau)$ be an L-structure, then τ can be extended inductively:

- $\tau(a) = a$ for any $a \in A$
- $\tau(f(t_1, ..., t_m)) = \tau(f)(\tau(t_1), ..., \tau(t_m))$ for any m-ary function f

$\mathcal{A} \vDash \varphi$ is defined inductively:

- $\mathcal{A} \vDash \mathsf{T} \text{ and } \mathcal{A} \not\vDash \bot$
- $\mathcal{A} \models t_1 = t_2$ if and only if $\tau(t_1) = \tau(t_2)$
- $\mathcal{A} \models r(t_1, ..., t_m)$ if and only if $\tau(r)(\tau(t_1), ..., \tau(t_m))$
- $\mathcal{A} \vDash \neg \varphi$ if and only if $\mathcal{A} \not\vDash \varphi$
- $\mathcal{A} \vDash \varphi \lor \psi$ if and only if $\mathcal{A} \vDash \varphi$ or $\mathcal{A} \vDash \psi$
- $\mathcal{A} \models \exists x \psi$ if and only if $\mathcal{A} \models \psi(a)$ for some $a \in A$