Practice Problems Set 8

Fall 25

1. First-Order Logic

   Consider a neighborhood with different types of houses and residents. Each resident can own zero or more houses. A set of properties are represented as the following predicates and relationships:

   - isResident(x): Person x lives in the neighborhood.
   - isHouse(y): House y is located in the neighborhood.
   - owns(x, y): Person x owns house y.

   Express the following statements in first-order logic using the defined predicates:

   (a) John is a resident.

Solution:

$\text{isResident}(John)$

   (b) Tom owns a house but is not a resident.

Solution:

$\big(\exists y : \text{isHouse}(y) \land \text{owns}(Tom, y)\big) \land \neg\text{isResident}(Tom)$

   (c) Mary owns two houses.

Solution:

$\exists y_1, y_2 : \big(\text{isHouse}(y_1) \land \text{isHouse}(y_2) \land y_1 \neq y_2 \land \text{owns}(Mary, y_1) \land \text{owns}(Mary, y_2)\big)$

   (d) Every resident owns at least one house.

Solution:

$\forall x : \big(\text{isResident}(x) \rightarrow \exists y : (\text{isHouse}(y) \land \text{owns}(x, y))\big)$

   (e) No two residents own the same house.

Solution:

$\neg\exists x_1, x_2, y : \big(\text{isResident}(x_1) \land \text{isResident}(x_2) \land x_1 \neq x_2 \land \text{isHouse}(y) \land \text{owns}(x_1, y) \land \text{owns}(x_2, y)\big)$

2. 3-COLOR Problem

A graph is *3-colorable* if its vertices can be colored with three colors (say Red, Blue, Green) such that no two adjacent vertices have the same color. The 3-COLOR problem tests if a finite undirected graph is 3-colorable. Show that 3-COLOR$\in$ **NP** using Fagin's Theorem.

Solution:

By Fagin's Theorem, it suffices to write a $\exists$SO sentence to describe the 3-colorability of undirected graphs. Let E be the binary predicate indicating edges of the graph, then the sentence can be written as:

$$\exists B \exists G \exists R : \Bigg(\forall x : \big(B(x) \vee G(x) \vee R(x) \big) \wedge$$

$$\forall x \forall y : \Bigg(E(x,y) \rightarrow \Big(\big(\neg B(x) \vee \neg B(y) \big) \wedge \big(\neg G(x) \vee \neg G(y) \big) \wedge \big(\neg R(x) \vee \neg R(y) \big) \Big) \Bigg) \Bigg)$$

The formula is actually a $\exists$MSO formula because the second-order variables B, G, R are all existentially quantified.