The 3CNF-SAT Problem

SAT is NP-Complete

What's next?

- SAT $\leq_p$ [Any_NP_Problem]
- Encoding arbitrary SAT formulae is onerous
- 3CNF-SAT is a middleman:

  SAT $\leq_p$ 3CNF-SAT $\leq_p$ [Any_NP_Problem]

The 3-CNF-SAT problem

Recall Conjunctive Normal Form (CNF):

- $\bigwedge_{i=1}^{m}(\bigvee_{j=1}^{n} l_{i,j})$
- E.g., $(p_1 \vee \neg p_3) \wedge (\neg p_1 \vee \mathrm{p}_2 \vee p_3) \wedge (p_1 \vee \neg \mathrm{p}_2 \vee p_4 \vee p_5)$
- Every $\bigvee_{j=1}^{n} l_{i,j}$ is called a clause/conjunct

3-Conjunctive Normal Form (3-CNF):

- Each clause contains exactly three distinct literals
- E.g., $(p_1 \vee \neg p_1 \vee \neg p_2) \wedge (p_3 \vee \mathrm{p}_2 \vee p_4) \wedge (\neg p_1 \vee \neg \mathrm{p}_3 \vee \neg p_4)$
- Why 3-CNF? Easier to reduce to other problems!

Theorem: The 3-CNF-SAT problem is NP-complete.

Proof:

- **Membership in NP:** The proof for SAT still applies!
- **NP-hardness:** Only need to show SAT $\leq_p$ 3-CNF-SAT

SAT to 3-CNF-SAT reduction

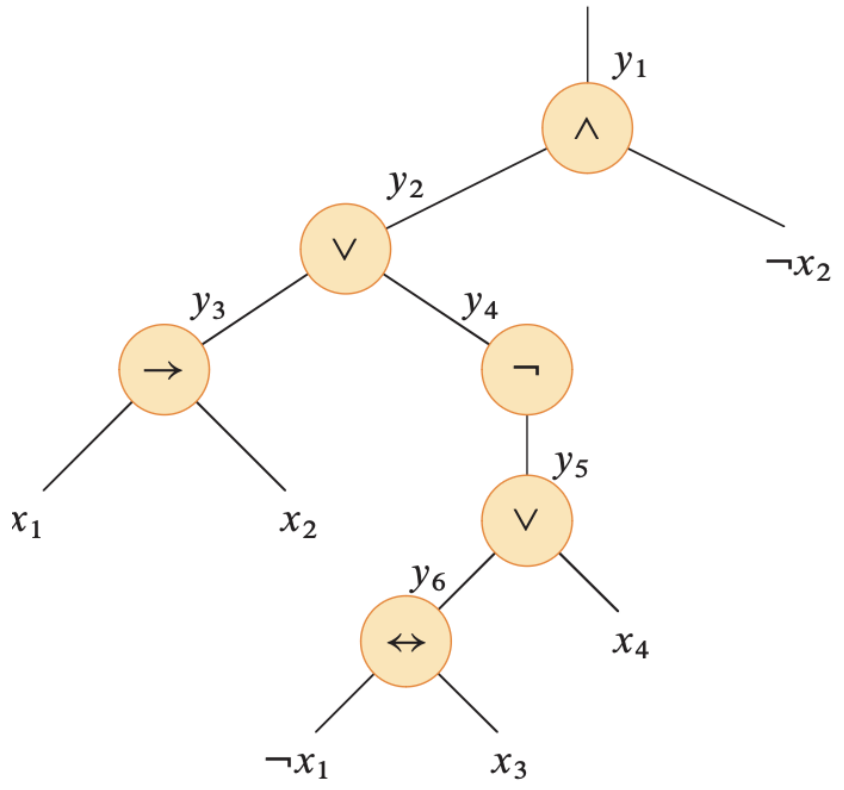
Step 1:

- Build a parse tree and introduce a new variable for each internal node!

Example:

$$\phi = ((x_1 \to x_2) \lor \neg((\neg x_1 \leftrightarrow x_3) \lor x_4)) \land \neg x_2$$

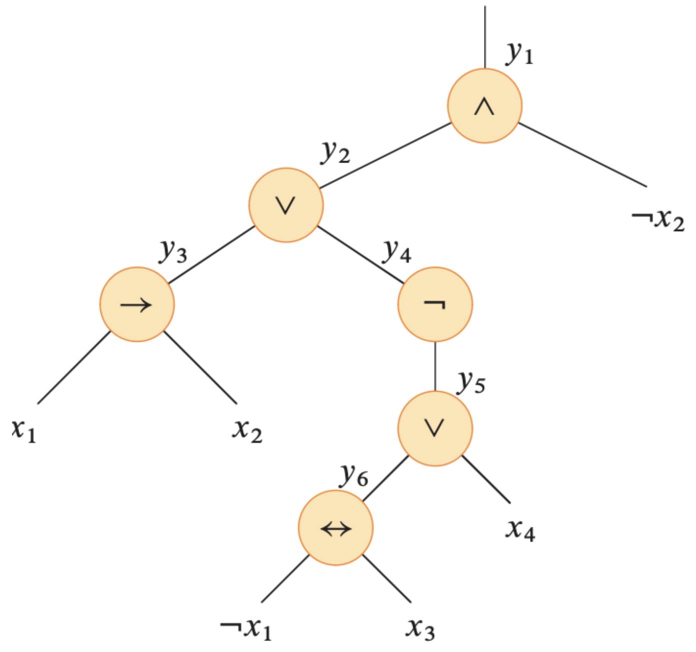
$$\begin{aligned} \phi' \;=\; & y_1 \land (y_1 \leftrightarrow (y_2 \land \neg x_2)) \\ & \land (y_2 \leftrightarrow (y_3 \lor y_4)) \\ & \land (y_3 \leftrightarrow (x_1 \to x_2)) \\ & \land (y_4 \leftrightarrow \neg y_5) \\ & \land (y_5 \leftrightarrow (y_6 \lor x_4)) \\ & \land (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3)) \end{aligned}$$



SAT to 3-CNF-SAT reduction

Step 2:

Convert each clause into CNF



$$\begin{aligned}\phi' = y_1 &\wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2))\\ &\wedge (y_2 \leftrightarrow (y_3 \vee y_4))\\ &\wedge (y_3 \leftrightarrow (x_1 \rightarrow x_2))\\ &\wedge (y_4 \leftrightarrow \neg y_5)\\ &\wedge (y_5 \leftrightarrow (y_6 \vee x_4))\\ &\wedge (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3))\end{aligned}$$

$$\begin{aligned}\phi''_1 = {} & (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2)\\ & \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2)\\ & \ldots\end{aligned}$$