# Difference Constraints

Adapted from the CLRS book slides

# DIFFERENCE CONSTRAINTS

Special case of linear programming.

Given a set of inequalities of the form $x_j - x_i \leq b_k$.

- $x$'s are variables, $1 \leq i, j \leq n$,
- $b$'s are constants, $1 \leq k \leq m$.

Want to find a set of values for the $x$'s that satisfy all $m$ inequalities, or determine that no such values exist. Call such a set of values a ***feasible solution***.

# EXAMPLE

$$
\begin{aligned}
x_1 - x_2 &\le 5 \\
x_1 - x_3 &\le 6 \\
x_2 - x_4 &\le -1 \\
x_3 - x_4 &\le -2 \\
x_4 - x_1 &\le -3
\end{aligned}
$$

Solution: $x = (0, -4, -5, -3)$

Also: $x = (5, 1, 0, 2) = [\text{above solution}] + 5$

# EXAMPLE APPLICATION

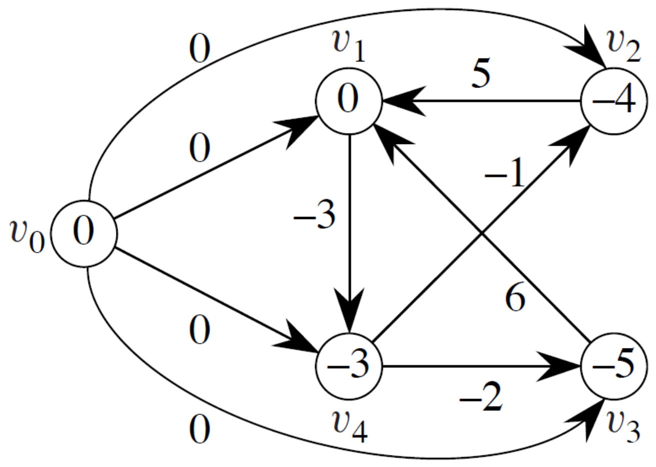
$x_i$ are times when events are to occur.

- Suppose that event $x_j$ must occur after event $x_i$ occurs but no more than $b_k$ time units after event $x_i$ occurs. Get constraints $x_j - x_i \geq 0$, which is equivalent to $x_i - x_j \leq 0$, and $x_j - x_i \leq b_k$.
- What if $x_j$ must occur at least $b_k$ time units after $x_i$? Get $x_j - x_i \geq b_k$, which is equivalent to $x_i - x_j \leq -b_k$.

# CONSTRAINT GRAPH

$G = (V, E)$, weighted, directed.

- $V = \{v_0, v_1, v_2, \ldots, v_n\}$: one vertex per variable $+ v_0$
- $E = \{(v_i, v_j) : x_j - x_i \leq b_k$ is a constraint$\} \cup \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_n)\}$
- $w(v_0, v_j) = 0$ for all $j = 1, 2, \ldots, n$
- $w(v_i, v_j) = b_k$ if $x_j - x_i \leq b_k$



# THEOREM

Given a system of difference constraints, let $G = (V, E)$ be the corresponding constraint graph.

1. If $G$ has no negative-weight cycles, then

   $$x = (\delta(v_0, v_1), \delta(v_0, v_2), \ldots, \delta(v_0, v_n))$$

   is a feasible solution.
2. If $G$ has a negative-weight cycle, then there is no feasible solution.

# HOW TO FIND A FEASIBLE SOLUTION

1. Form constraint graph.
   - $n + 1$ vertices.
   - $m + n$ edges.
   - $\Theta(m + n)$ time.
2. Run BELLMAN-FORD from $v_0$.
   - $O((n + 1)(m + n)) = O(n^2 + nm)$ time.
3. BELLMAN-FORD returns FALSE $\Rightarrow$ no feasible solution.

   BELLMAN-FORD returns TRUE $\Rightarrow$ set $x_i = \delta(v_0, v_i)$ for all $i = 1, 2, \ldots, n$.