Randomized Complete Block Design

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The Blocking Principle

- A **nuisance** factor is a factor that probably has some effect on the response, but it's of no interest to the experimenter. We hope to minimize the effect of a nuisance factor.
- Typical nuisance factors include batches of raw material, operators, pieces of test equipment, time (shifts, days, etc.), gender, age, …
- **Randomization** is the design technique used to guard against such nuisance factors, especially when they are **unknown and uncontrolled**
- **Blocking** is a technique for dealing with **nuisance factors when they are known and controllable**
- Blocking can be used to systematically eliminate the effect of nuisance factors
- **Many** experiments involve blocking

Examples

1. Covid vaccine trials

   In Covid vaccine trials, **gender and age** are *known and controllable* nuisance factors, so blocking subjects into gender and age groups helps reduce variability. In contrast, **diet and exercise habits** are *less controllable or difficult to measure reliably*, so randomization helps balance these unknown factors across treatment arms.

2. Tire durability study

   In a study comparing four tire types for durability (measured by miles until wear-out), nuisance factors include the **brand of car, road surface, and driver skill.** Since these are known and controllable, the experimenter can block on them (e.g., test all four tire types on the same car, same track, and with the same driver). Randomization is still needed to guard against uncontrollable factors like **weather changes or slight variations** in traffic.

Examples

3. Fertilizer effectiveness in agriculture

   When testing different fertilizers on crop yield, **field location and soil type** are known nuisance factors (they can be blocked by assigning each fertilizer to plots within the same field/soil type). Meanwhile, **daily weather variation** (sunlight, rainfall) is an uncontrollable nuisance factor, better addressed by randomization.

Blocking in Practice

- Step 1. Identify nuisance factors that are known and controllable
- Step 2. Form homogeneous blocks. Divide experimental units into groups that have the same values of nuisance factors in Step 1
  - For example, all males in one block, all females in another
  - Each driver assigned their own block
- Step 3. Randomize treatments within each block
  - Within each block, randomly assign the treatments (e.g., vaccine vs placebo, tire type A–D)
- Step 4. Analyze results accounting for blocks
  - ANOVA / regression

Illustration (Covid Vaccine Example)

Block (Nuisance Factor)	Randomized Treatment Assignment
Male, Age < 50	Vaccine / Placebo randomly assigned within block
Male, Age ≥ 50	Vaccine / Placebo randomly assigned within block
Female, Age < 50	Vaccine / Placebo randomly assigned within block
Female, Age ≥ 50	Vaccine / Placebo randomly assigned within block

Randomized Complete Block Design (RCBD)

- Units are grouped into **blocks** that are same on known nuisance factors
- **All treatments appear the same number of times (e.g., once) in every block** (complete)
- **Randomize within each block** to assign treatments

Why use it

- Removes variation due to blocks, improves precision
- Fair within-block comparisons of treatments

When to use

- There are known and controllable nuisance factors
- It is feasible to apply all treatments in every block

RCBD

Important points:

- Variability of the nuisance factor **between** blocks can be large, variability **within** a block should be relatively small
- In general, a **block** is a specific level of the nuisance factor
- A **complete replicate** of the basic experiment is conducted in each block
- All runs **within** a block are **randomized**
- A block represents a **restriction on randomization**
- Notice the **two-way structure** of the experiment
- Once again, we are interested in testing the equality of treatment means.

Paired Comparison Design

Example: consider a hardness testing machine that presses a rod with a pointed tip into a metal specimen with a known force. By measuring the depth of the depression caused by the tip, the hardness of the specimen is determined. Two different tips are available for this machine. Determine whether one tip produces different mean hardness readings than the other.

■ TABLE 2.6
Data for the Hardness Testing Experiment

Specimen	Tip 1	Tip 2
1	7	6
2	3	3
3	3	5
4	4	3
5	8	8
6	3	2
7	2	4
8	9	9
9	5	4
10	4	5

Issue with two-sample t-test for this example:
Metal specimens are not homogeneous, which will inflate the variance.

Blocking: each pair forms a block

Paired t-test

Note that if we compute the jth paired difference

$$d_j = y_{1j} - y_{2j} \qquad j = 1, 2, \ldots, 10$$

the expected value of this difference is $\mu_d = E(d_j)$

We want to test

$$h_0 : \mu_d = 0$$
$$h_1 : \mu_d \neq 0$$

This is a single-sample t-test. The test statistic for this hypothesis is

$$t_0 = \frac{\bar{d}}{S_d/\sqrt{n}}$$

t_0 follows a t-distribution with n-1 degrees of freedom

For the example,

$$\begin{array}{llll} d_1 = 7-6=1 & & d_6 = 3-2=1 \\ d_2 = 3-3=0 & & d_7 = 2-4=-2 \\ d_3 = 3-5=-2 & & d_8 = 9-9=0 \\ d_4 = 4-3=1 & & d_9 = 5-4=1 \\ d_5 = 8-8=0 & & d_{10} = 4-5=-1 \end{array}$$

$$\bar{d} = \frac{1}{n}\sum_{j=1}^{n} d_j = \frac{1}{10}(-1) = -0.10$$

$$S_d = \sqrt{\frac{\sum_{j=1}^{10}(d_j - \bar{d})^2}{n-1}} = 1.20$$

$$t_0 = \frac{\bar{d}}{S_d/\sqrt{n}} = \frac{-0.10}{1.20/\sqrt{10}} = -0.26$$

$|t_0| = 0.26 \not> t_{0.025,9} = 2.262$, we cannot reject the hypothesis H_0

If two-sample t-test were used, $S_p = 2.32$

Paired

Margin of error

$$\bar{d} \pm t_{0.025,9} s_d / \sqrt{n}$$

$$-0.10 \pm (2.262)(1.20)/\sqrt{10}$$

$$-0.10 \pm 0.86$$

Two-sample

$$\bar{y}_1 - \bar{y}_2 \pm t_{0.025,18} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$4.80 - 4.90 \pm (2.101)(2.32)\sqrt{\tfrac{1}{10} + \tfrac{1}{10}}$$

$$-0.10 \pm 2.18$$

The Hardness Testing Example

- Now we wish to determine whether 4 different tips produce different (mean) hardness reading.
- We do trials for each tip on 5 metal specimen. To conduct this experiment, assign all 4 tips to each specimen
- Can we still have 10 metal specimen?

Block (Specimen)				
1	Tip 3	Tip 2	Tip 4	Tip 1
2	Tip 4	Tip 3	Tip 1	Tip 2
3	Tip 2	Tip 4	Tip 3	Tip 1
4	Tip 2	Tip 3	Tip 1	Tip 4
5	Tip 2	Tip 3	Tip 4	Tip 1

- Each specimen is called a "**block**", providing more homogenous experimental unit to test the tips
- Within a block, the order in which the four tips are tested is **randomly** determined.

Randomized Complete Block Design

- A generalization of the paired design.
- Generally, suppose that there are *a* treatments (factor levels) and *b* blocks, each treatment tested once in each block

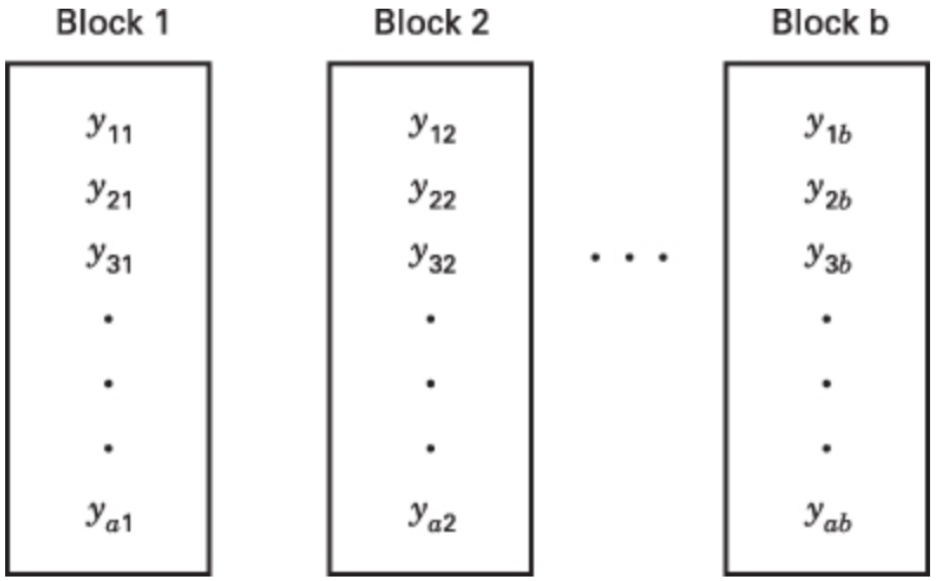


FIGURE 4.1 The randomized complete block design

Extension of the ANOVA to the RCBD

A **statistical model** (effects model) for the RCBD is

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

where μ is the overall mean, τ_i is the effect as a deviation from the overall mean for the ith treatment, and β_j is the effect as a deviation from the overall mean for the jth block.

$$\sum_{i=1}^{a} \tau_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \beta_j = 0$$

Extension of the ANOVA to the RCBD

The relevant hypotheses:

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_a = 0$$
$$H_1 : \tau_i \neq 0 \text{ at least one } i$$

equivalent to

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a$$
$$H_1 : \text{at least one } \mu_i \neq \mu_j$$

where $\mu_i = \mu + \tau_i$.

Extension of the ANOVA to the RCBD

ANOVA partitioning of total variability:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}[(\bar{y}_{i.}-\bar{y}_{..})+(\bar{y}_{.j}-\bar{y}_{..})$$

$$+(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})]^2$$

$$= b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$$

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

Extension of the ANOVA to the RCBD

The degrees of freedom for the sums of squares in

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

are as follows:

$$ab - 1 = a - 1 + b - 1 + (a - 1)(b - 1)$$

$$MS_{Treatments} = \frac{SS_{Treatment}}{a-1}, MS_{Blocks} = \frac{SS_{Blocks}}{b-1},$$

$$MS_E = \frac{SS_E}{(a-1)(b-1)}$$

$$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$$

ANOVA Display for the RCBD

TABLE 4.2

Analysis of Variance for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$\frac{SS_{\text{Treatments}}}{a-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Blocks	SS_{Blocks}	$b-1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E	$(a-1)(b-1)$	$\frac{SS_E}{(a-1)(b-1)}$	
Total	SS_T	$N-1$		

The table does not contain testing for difference between blocks. Sometimes we may also be interested in comparing block means because, if these means do not differ greatly, blocking may not be necessary in future experiments. $MS_{\text{Blocks}} / MS_E$ can be used to compare block means if we treat blocks as fixed effects. However, in practice we usually regard block effects as random, then the F test for blocks is invalid.

Example

- A medical device manufacturer produces vascular grafts. These grafts are produced by extruding resin combined with a lubricant into tubes.
- The product developer suspects that the extrusion pressure affects the defective rates.
- The resin is manufactured by an external supplier and is delivered to the medical device manufacturer in batches. There may be significant batch-to-batch variation.
- Investigate the effect of four different levels of **extrusion pressure** using a **RCBD** for **six batches of resin (blocks)**.

Vascular Graft Example (page 120)

- To conduct this experiment as a RCBD, assign all 4 pressures to each of the 6 batches of resin
- Each batch of resin is called a "**block**"; that is, it's a more homogenous experimental unit on which to test the extrusion pressures

RCBD - Vascular Graft Data

	Extrusion Pressure (I	Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Treatment Total
1	8500	90.3	89.2	98.2	93.9	87.4	97.9	556.9
2	8700	92.5	89.5	90.6	94.7	87.0	95.8	550.1
3	8900	85.5	90.8	89.6	86.2	88.0	93.4	533.5
4	9100	82.5	89.5	85.6	87.4	78.9	90.7	514.6
5	Block Totals	350.8	359.0	364.0	362.2	341.3	377.8	2155.1

Two-way ANOVA Table

TABLE 4.4

Analysis of Variance for the Vascular Graft Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Treatments (extrusion pressure)	178.17	3	59.39	8.11	0.0019
Blocks (batches)	192.25	5	38.45		
Error	109.89	15	7.33		
Total	480.31	23			

TABLE 4.5

Incorrect Analysis of the Vascular Graft Experiment as a Completely Randomized Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Extrusion pressure	178.17	3	59.39	3.95	0.0235
Error	302.14	20	15.11		
Total	480.31	23			

Estimation, Prediction, and Residual

- Parameter estimation: (satisfy the zero-sum constraint)

$$\hat{\mu} = \bar{y}_{..}$$
$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$$
$$\hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{..}$$

- Prediction:

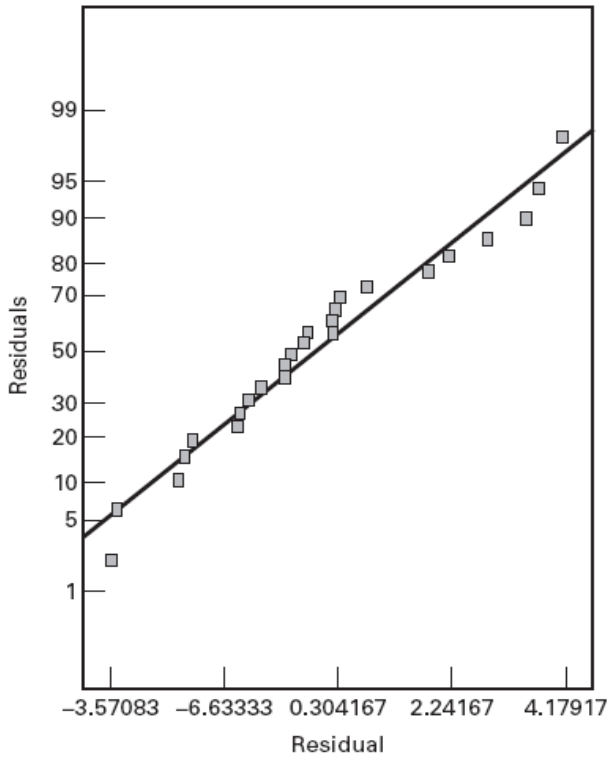
$$\hat{y}_{ij} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j = \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}$$

- Residual:

$$\hat{e}_{ij} = y_{ij} - \hat{y}_{ij}$$

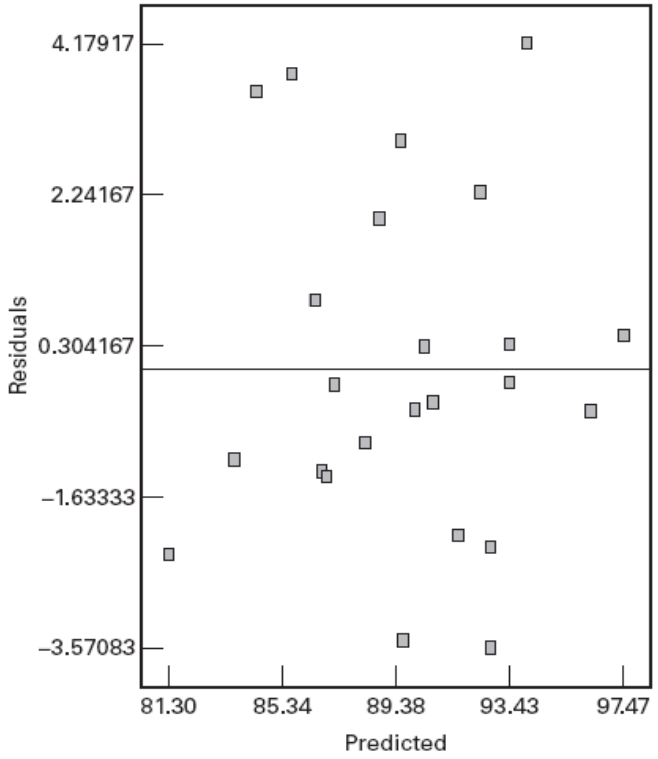
Residual Analysis for the Vascular Graft Example

Test normality



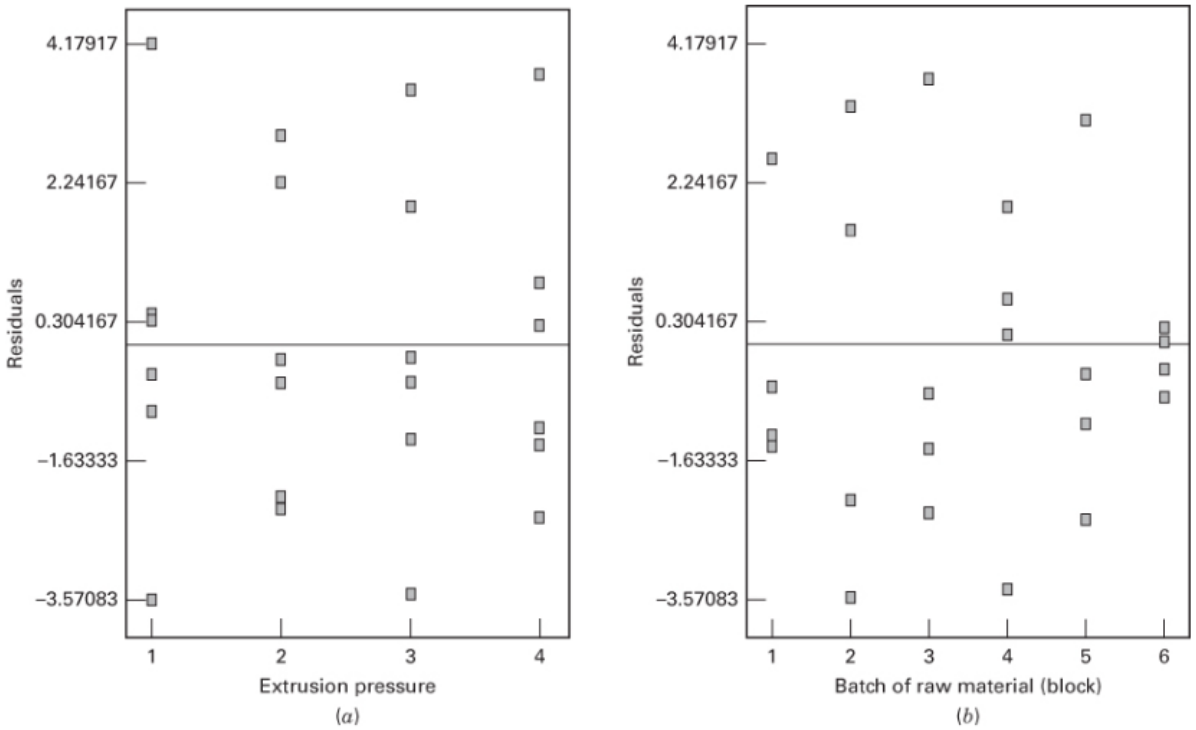
■ FIGURE 4.4 Normal probability plot of residuals for Example 4.1

Test constant variance



■ FIGURE 4.5 Plot of residuals versus $\hat{y}_{ij}$ for Example 4.1

Test constant variance by treatment and block



- If there is more scatter in the residuals for a particular treatment, it could indicate that this treatment produces more erratic response readings than the others.
- More scatter in the residuals for a particular block could indicate that the block is not homogeneous.