Maximum Flow

Adapted from the CLRS book slides

OVERVIEW

Use a graph to model material that flows through conduits.

Each edge represents one conduit, and has a ***capacity***, which is an upper bound on the ***flow rate*** = units/time.

Can think of edges as pipes of different sizes. But flows don't have to be of liquids. The textbook has an example where a flow is how many trucks per day can ship hockey pucks between cities.

Want to compute the maximum rate that material can be shipped from a designated ***source*** to a designated ***sink***.

FLOW NETWORKS

$G = (V, E)$ directed.

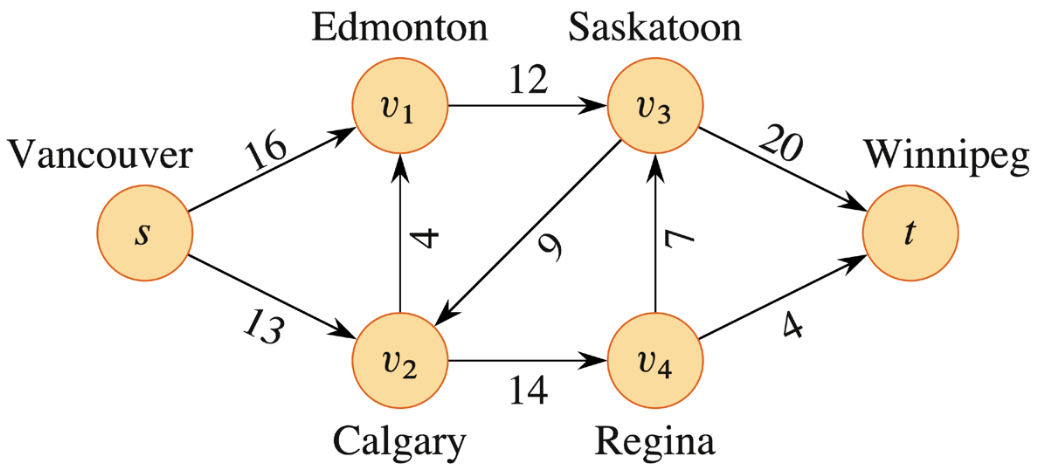
Each edge (u, v) has a ***capacity*** $c(u, v) \geq 0$.

If $(u, v) \notin E$, then $c(u, v) = 0$.

If $(u, v) \in E$, then reverse edge $(v, u) \notin E$. (Can work around this restriction.)

Source vertex s, ***sink*** vertex t, assume $s \leadsto v \leadsto t$ for all $v \in V$, so that each vertex lies on a path from source to sink.

FLOW NETWORKS



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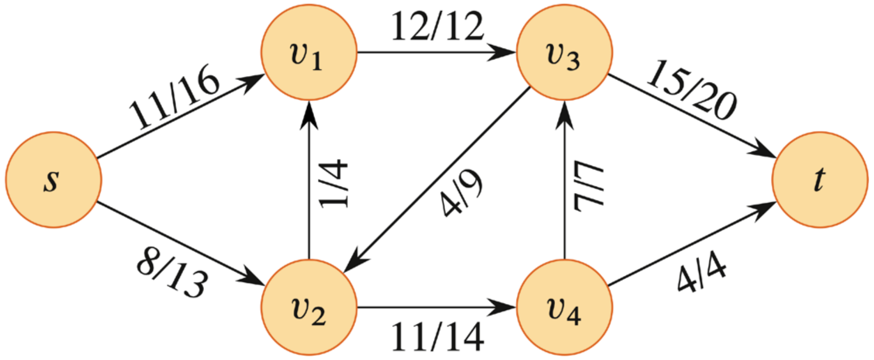


Flow

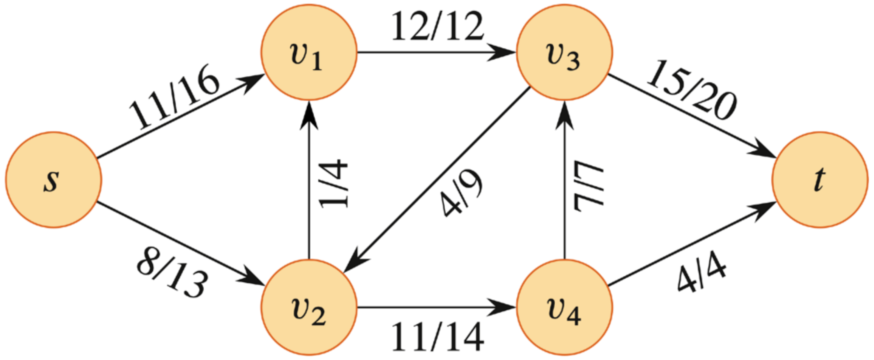
A function $f : V \times V \to \mathbb{R}$ satisfying

- ***Capacity constraint:*** For all $u, v \in V, 0 \leq f(u, v) \leq c(u, v)$,
- ***Flow conservation:*** For all $u \in V - \{s, t\}$, $\underbrace{\sum_{v \in V} f(v, u)}_{\text{flow into } u} = \underbrace{\sum_{v \in V} f(u, v)}_{\text{flow out of } u}$.

Equivalently, $\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$.

WLOG, assume one source, one sink, and edges in at most one direction between any vertex pair.

FLOW



$$
\begin{aligned}
\textit{\textbf{Value of flow}}\ f &= |f| \\
&= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) \\
&= \text{flow out of source} - \text{flow into source}\ .
\end{aligned}
$$

In the example above, value of flow $f = |f| =$ 19

- Note that all flows are $\leq$ capacities.
- Verify flow conservation by adding up flows at a couple of vertices.
- Note that all flows $= 0$ is legitimate.

CUTS

A ***cut*** (S, T) of flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.

- Similar to cut used in minimum spanning trees, except that here the graph is directed, and require $s \in S$ and $t \in T$.

For flow f, the ***net flow*** across cut (S, T) is

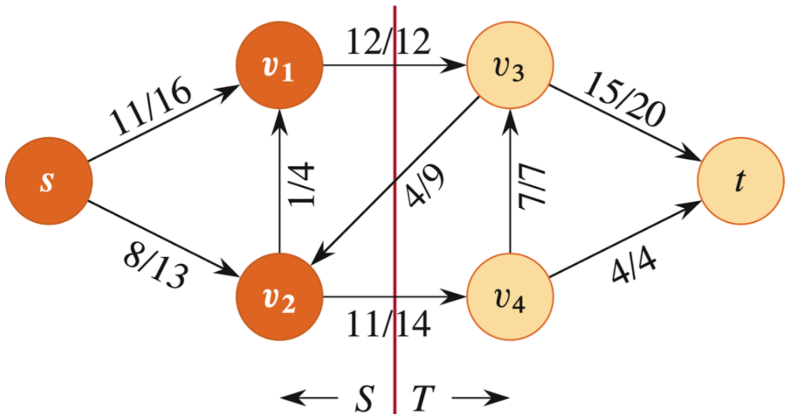
$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) .$$

Capacity of cut (S, T) is

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v) .$$

A ***minimum cut*** of G is a cut whose capacity is minimum over all cuts of G.

CUTS (continued)



Consider the cut $S = \{w, v_1, v_2\}, T = \{v_3, v_4, t\}$.

$$
\begin{aligned}
f(S,T) &= f(v_1, v_3) + f(v_2, v_4) \quad - \quad f(v_3, v_2) \\
&= 12 + 11 - 4 \\
&= 19
\end{aligned}
$$

$$
\begin{aligned}
c(S,T) &= c(v_1, v_3) + c(v_2, v_4) \\
&= 12 + 14 \\
&= 26
\end{aligned}
$$

Lemma

For any cut (S, T), $f(S, T) = |f|$.

Corollary

The value of any flow $\leq$ capacity of any cut.

Therefore, maximum flow $\leq$ capacity of minimum cut.

Will see a little later that this is in fact an equality.

THE FORD-FULKERSON METHOD

Residual network

Given a flow f in network $G = (V, E)$.

Consider a pair of vertices $u, v \in V$.

How much additional flow can be pushed directly from u to v?
That's the ***residual capacity***,

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \ , \\ f(v, u) & \text{if } (v, u) \in E \ , \\ 0 & \text{otherwise (i.e., } (u, v), (v, u) \notin E) \ . \end{cases}$$

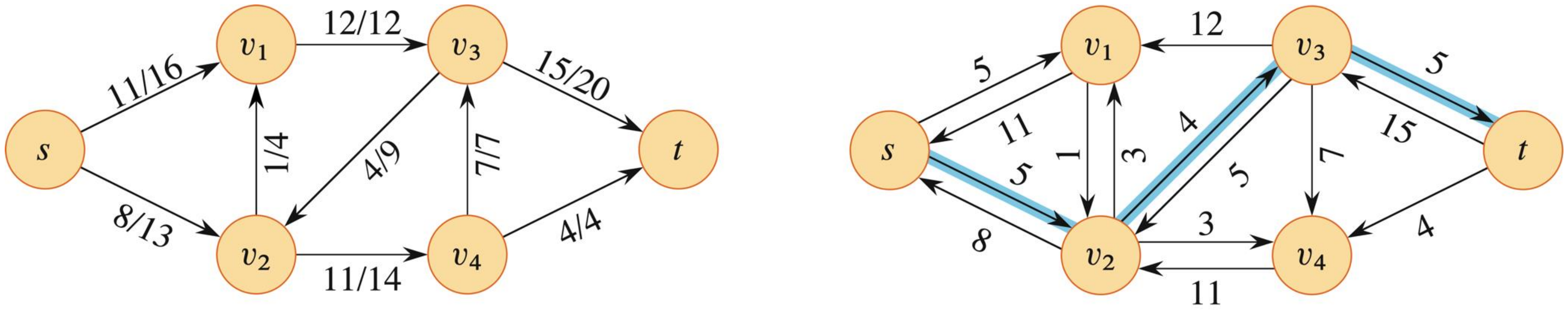
THE FORD-FULKERSON METHOD (continued)

The ***residual network*** is $G_f = (V, E_f)$, where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\} \ .$$

Each edge of the residual network can admit a positive flow.

For our example:



Residual network is similar to a flow network, except that it may contain antiparallel edges ((u, v) and (v, u)).

THE FORD-FULKERSON METHOD (continued)

Given flows f in G and f' in G_f, define $(f \uparrow f')$, the ***augmentation*** of f by f', as a function $V \times V \to \mathbb{R}$:

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E, \\ 0 & \text{otherwise} \end{cases}$$

for all $u, v \in V$.

Intuition: Increase the flow on (u, v) by $f'(u, v)$ but decrease it by $f'(v, u)$ because pushing flow on the reverse edge in the residual network decreases the flow in the original network. Also known as ***cancellation***.

Lemma

Given a flow network G, a flow f in G, and the residual network G_f, let f' be a flow in G_f. Then $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.

AUGMENTING PATH

A simple path $s \leadsto t$ in G_f.

- Admits more flow along each edge.
- Like a sequence of pipes through which can squirt more flow from s to t.

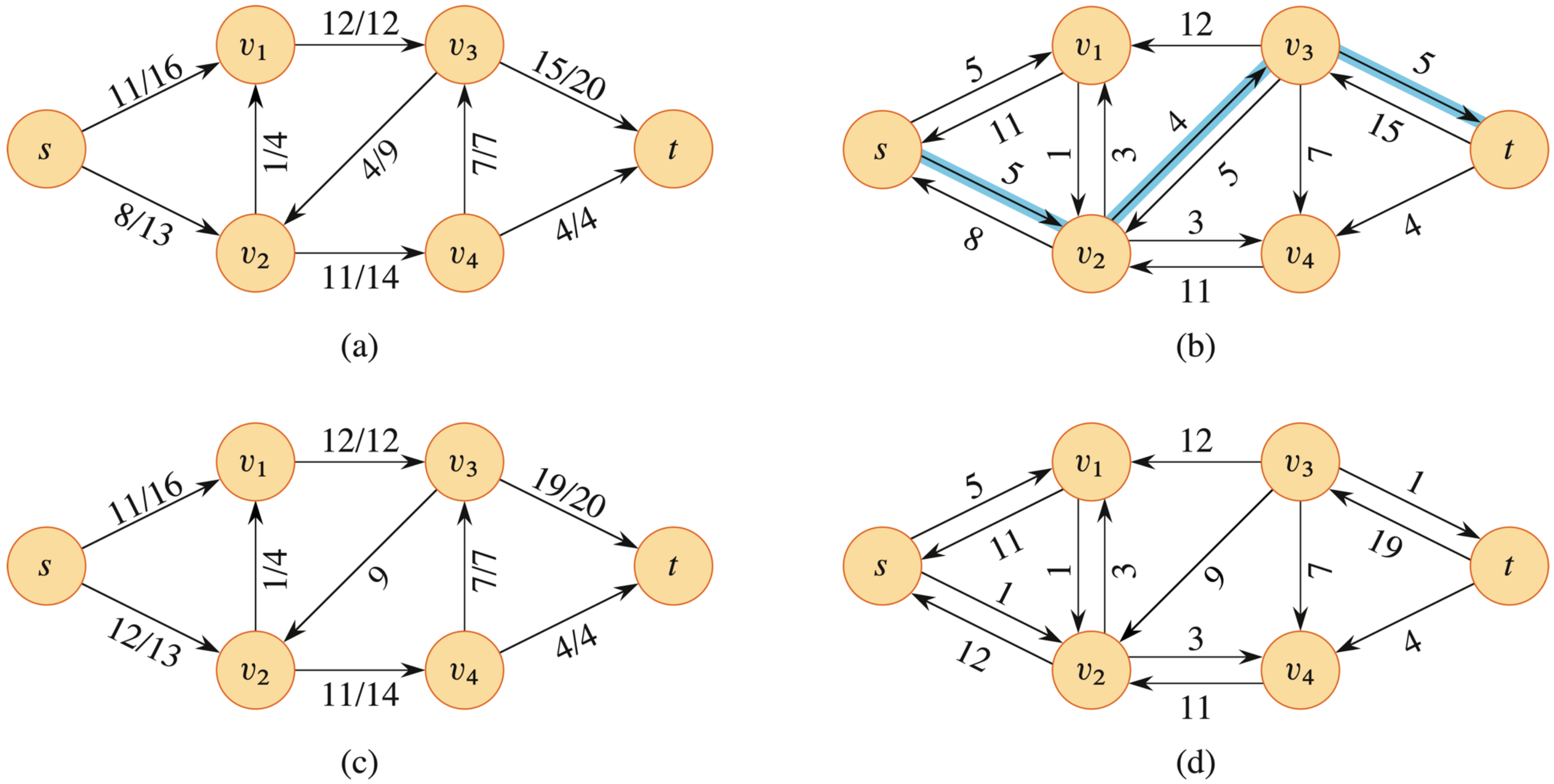
How much more flow can be pushed from s to t along augmenting path p? That is the ***residual capacity*** of p:

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\} .$$

EXAMPLE

For our example, consider the highlighted augmenting path, whose residual capacity is 4.

After pushing 4 additional units along the path.



EXAMPLE (continued)

Lemma

Given flow network G, flow f in G, residual network G_f. Let p be an augmenting path in G_f. Define $f_p : V \times V \rightarrow \mathbb{R}$:

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \text{ ,} \\ 0 & \text{otherwise .} \end{cases}$$

Then f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Corollary

Given flow network G, flow f in G, and an augmenting path p in G_f, define f_p as in lemma. Then $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$.

THEOREM

Theorem (Max-flow min-cut theorem)

The following are equivalent:

1. f is a maximum flow.
2. G_f has no augmenting path.
3. $|f| = c(S, T)$ for some cut (S, T).