Practice Problems Set 3

Fall 25

1. Solving Recurrences (adapted from CLRS Exercise 4.5-1)

   Use the master method to give tight asymptotic bounds for the following recurrences.

   (a) $T(n) = 2T(n/4) + 2^{2024}$

   Solution:
   $T(n) = \Theta(\sqrt{n})$. Here, $f(n) = O(n^{\frac{1}{2}-\epsilon})$ for $\epsilon = 1/2$, as $f(n)$ is a constant. Case 1 applies, and $T(n) = \Theta(n^{\frac{1}{2}}) = \Theta(\sqrt{n})$.

   (b) $T(n) = 2T(n/4) + \sqrt{n} + \lg n$

   Solution:
   $T(n) = \Theta(\sqrt{n} \lg n)$. Now $f(n) = \sqrt{n} + \lg n = \Theta(n^{\log_b a})$. Case 2 applies, with $k = 0$.

   (c) $T(n) = 2T(n/4) + \Theta(\sqrt{n} \lg^2 n)$

   Solution:
   $T(n) = \Theta(\sqrt{n} \lg^3 n)$. Now $f(n) = \Theta(n^{\log_b a} \lg^2 n)$. Case 2 applies, with $k = 2$.

   (d) $T(n) = 2T(n/4) + n$

   Solution:
   $T(n) = \Theta(n)$. This time, $f(n) = \Omega(n)$, so $f(n) = \Omega(n^{\log_b a + \epsilon})$ for $\epsilon = 1/2$. In order for Case 3 to apply, we have to check the regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$. Here, $af(n/b) = n/2$, and so the regularity condition holds for $c = 1/2$. Therefore, Case 3 applies.

   (e) $T(n) = T(\frac{n}{4}) + T(\frac{n}{5}) + T(\frac{n}{6}) + \Theta(n)$

   Solution:
   $T(n) = \Theta(n)$. By monotonicity, $T(n) \leq 3T(\frac{n}{4}) + \Theta(n)$. Then $T(n) = O(n)$ by Case 3 of the Master Theorem, since $\Theta(n) \subset \Omega(n^{\log_4 3 + \epsilon})$ for some $\epsilon > 0$. On the other hand, if we ignore the recursive component, $T(n) = \Omega(n)$. Combining the two gives $T(n) = \Theta(n)$.

2. Asymptotic Notations (Attributed to CLRS Exercise 4.5-2)

Professor Caesar wants to develop a matrix-multiplication algorithm that is asymptotically faster than Strassen's algorithm. His algorithm will use the divide-and-conquer method, dividing each matrix into $n/4 \times n/4$ submatrices, and the divide and combine steps together will take $\Theta(n^2)$ time. Suppose that the professor's algorithm creates a recursive subproblems of size $n/4$. What is the largest integer value of a for which his algorithm could possibly run asymptotically faster than Strassen's?

Solution:

We need to find the largest integer a such that $\log_4 a < \lg 7$. The answer is $a = 48$.