The Latin Square Design

- These designs are used to simultaneously control (or eliminate) **two sources of nuisance variability**
- The treatments (factorial levels) are denoted by the Latin letters A, B, C, …
- The rows and columns represent **two blocks,** that is, two restrictions on randomization.

4×4

A	*B*	*D*	*C*
B	*C*	*A*	*D*
C	*D*	*B*	*A*
D	*A*	*C*	*B*

5×5

A	*D*	*B*	*E*	*C*
D	*A*	*C*	*B*	*E*
C	*B*	*E*	*D*	*A*
B	*E*	*A*	*C*	*D*
E	*C*	*D*	*A*	*B*

6×6

A	*D*	*C*	*E*	*B*	*F*
B	*A*	*E*	*C*	*F*	*D*
C	*E*	*D*	*F*	*A*	*B*
D	*C*	*F*	*B*	*E*	*A*
F	*B*	*A*	*D*	*C*	*E*
E	*F*	*B*	*A*	*D*	*C*

The Latin Square Design

- In general, a Latin square for p treatments, is a square containing p rows and p columns. Each of the resulting p^2 cells contains one of the p letters that corresponds to the treatments, and each letter occurs once and only once in each row and column.
- closely related to a popular puzzle called a sudoku puzzle that originated in Japan (sudoku means "single number" in Japanese).
- Easy to check, but hard to construct.

Sudoku

The puzzle typically consists of a 9×9 grid, with nine additional 3×3 blocks contained within. A few of the squares contain numbers and the others are blank. The goal is to fill the blanks with the integers from 1 to 9 so that each row, each column, and each of the nine 3×3 blocks making up the grid contains just one of each of the nine integers.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

The Rocket Propellant Problem

- An experimenter is studying the effects of five different formulations of a rocket propellant used in aircrew escape systems on the observed burning rate.
- Each formulation is mixed from a batch of raw material, and the formulations are prepared by several operators.
- There are two nuisance factors to be "averaged out".

Latin Square Design for the Rocket Propellant Problem

	Operators				
Batches of Raw Material	**1**	**2**	**3**	**4**	**5**
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

Statistical Analysis of the Latin Square Design

- The statistical (effects) model is

$$y_{ijk} = \mu + \tau_j + \alpha_i + \beta_k + \varepsilon_{ijk} \begin{cases} i = 1,2,\ldots,p \\ j = 1,2,\ldots,p \\ k = 1,2,\ldots,p \end{cases}$$

where y_{ijk} is the observation in the ith row and kth column for the jth treatment, μ is the overall mean, α_i is the ith row effect, τ_j is the jth treatment effect, β_k is the kth column effect

ANOVA Partitioning

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Rows}} + SS_{\text{Columns}} + SS_E$$

with respective degrees of freedom

$$p^2 - 1 = p - 1 + p - 1 + p - 1 + (p-2)(p-1)$$

$$SS_T = \sum_{i,j,k} (y_{ijk} - \bar{y}_{...})^2$$

Here the $\sum_{i,j,k}$ takes sum over all observed responses

$$SS_{Treatments} = p \sum_{j=1}^{p} (\bar{y}_{.j.} - \bar{y}_{...})^2$$

$$SS_{Rows} = p \sum_{i=1}^{p} (\bar{y}_{i..} - \bar{y}_{...})^2$$

$$SS_{Columns} = p \sum_{k=1}^{p} (\bar{y}_{..k} - \bar{y}_{...})^2$$

ANOVA Table

Analysis of Variance for the Latin Square Design				
Source of Variation	**Sum of Squares**	**Degrees of Freedom**	**Mean Square**	F_0
Treatments	$SS_{\text{Treatments}}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	SS_{Rows}	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	SS_{Columns}	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	SS_T	p^2-1		

$$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$$

which is distributed as $F_{p-1,(p-2)(p-1)}$ under the null hypothesis.

The Rocket Propellant Problem

TABLE 4.12

Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

Prediction and Residual

As in any design problem, the experimenter should investigate the adequacy of the model by inspecting and plotting the residuals. For a Latin square, the residuals are given by

$$\begin{aligned} e_{ijk} &= y_{ijk} - \hat{y}_{ijk} \\ &= y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}_{...} \end{aligned}$$