The Graeco-Latin Square Design

- To average out the effects of three nuisance factors

Consider a $p \times p$ Latin square, and superimpose on it a second $p \times p$ Latin square in which the treatments are denoted by Greek letters. If the two squares when superimposed have the property that each Greek letter appears once and only once with each Latin letter, the two Latin squares are said to be **orthogonal**, and the design obtained is called a **Graeco-Latin square**. An example of a 4×4 Graeco-Latin square is shown in Table 4.18.

TABLE 4.18

4 × 4 Graeco-Latin Square Design

	Column			
Row	**1**	**2**	**3**	**4**
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

Statistical Analysis of the Graeco-Latin Square Design

$$y_{ijkl} = \mu + \theta_i + \tau_j + \omega_k + \Psi_l + \varepsilon_{ijkl} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, p \end{cases}$$

ANOVA Partitioning

$$SS_T = SS_L + SS_G + SS_{Rows} + SS_{Columns} + SS_E$$

$$SS_T = \sum_{i,j,k,l} (y_{ijkl} - \bar{y}_{....})^2$$

$$SS_L = p \sum_{j=1}^{p} (\bar{y}_{.j..} - \bar{y}_{....})^2$$

$$SS_G = p \sum_{k=1}^{p} (\bar{y}_{..k.} - \bar{y}_{....})^2$$

$$SS_{Rows} = p \sum_{i=1}^{p} (\bar{y}_{i...} - \bar{y}_{....})^2$$

$$SS_{Columns} = p \sum_{l=1}^{p} (\bar{y}_{...l} - \bar{y}_{....})^2$$

Here $\sum_{i,j,k,l}$ takes sum over all observed responses

TABLE 4.19

Analysis of Variance for a Graeco-Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom
Latin letter treatments	$SS_L = \frac{1}{p} \sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$
Greek letter treatments	$SS_G = \frac{1}{p} \sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$
Rows	$SS_{\text{Rows}} = \frac{1}{p} \sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p-1$
Columns	$SS_{\text{Columns}} = \frac{1}{p} \sum_{l=1}^{p} y_{...l}^2 - \frac{y_{....}^2}{N}$	$p-1$
Error	SS_E (by subtraction)	$(p-3)(p-1)$
Total	$SS_T = \sum_i \sum_j \sum_k \sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	p^2-1

The Rocket Propellant Problem

- An experimenter is studying the effects of five different formulations of a rocket propellant used in aircrew escape systems on the observed burning rate.
- Each formulation is mixed from a batch of raw material, and the formulations are prepared by several operators.
- There are two nuisance factors to be "averaged out".

Latin Square Design for the Rocket Propellant Problem

	Operators				
Batches of Raw Material	**1**	**2**	**3**	**4**	**5**
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

- Suppose that in the rocket propellant experiment an additional factor, test assemblies, could be of importance.

Graeco-Latin Square Design for the Rocket Propellant Problem

	Operators					
Batches of Raw Material	**1**	**2**	**3**	**4**	**5**	$y_{i\ldots}$
1	$A\alpha = -1$	$B\gamma = -5$	$C\varepsilon = -6$	$D\beta = -1$	$E\delta = -1$	-14
2	$B\beta = -8$	$C\delta = -1$	$D\alpha = 5$	$E\gamma = 2$	$A\varepsilon = 11$	9
3	$C\gamma = -7$	$D\varepsilon = 13$	$E\beta = 1$	$A\delta = 2$	$B\alpha = -4$	5
4	$D\delta = 1$	$E\alpha = 6$	$A\gamma = 1$	$B\varepsilon = -2$	$C\beta = -3$	3
5	$E\varepsilon = -3$	$A\beta = 5$	$B\delta = -5$	$C\alpha = 4$	$D\gamma = 6$	7
$y_{\ldots l}$	-18	18	-4	5	9	$10 = y_{\ldots}$

The ANOVA Table for Graeco-Latin Square

TABLE 4.21

Analysis of Variance for the Rocket Propellant Problem

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Formulations	330.00	4	82.50	10.00	0.0033
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Test assemblies	62.00	4	15.50		
Error	66.00	8	8.25		
Total	676.00	24			

Recall The ANOVA Table for Latin Square

TABLE 4.12

Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

So far, we have introduced blocking techniques:

- RCBD
- LSD
- GLSD

In some experiments, we may not be able to run all treatment combinations in each block.

We need incomplete blocking techniques.

Balanced Incomplete Block Design (BIBD)

- In certain experiments, we may not be able to run all the treatment combinations in each block.
- For example, suppose that each batch of material is only large enough to accommodate testing three treatments.
- A BIBD is a design in which any treatment appears the same number of times, and any two treatments appear together an equal number of times.

Recall the Vascular Graft Experiment

Randomized Complete Block Design for the Vascular Graft Experiment

	Batch of Resin (Block)			
Extrusion Pressure (PSI)	**1**	**2**	**3**	**4**
8500	90.3	89.2	98.2	93.9
8700	92.5	89.5	90.6	94.7
8900	85.5	90.8	89.6	86.2
9100	82.5	89.5	85.6	87.4

BIBD

	Batch of Resin (Block)			
Extrusion Pressure (PSI)	**1**	**2**	**3**	**4**
8500	90.3	89.2	—	93.9
8700	—	89.5	90.6	94.7
8900	85.5	90.8	89.6	—
9100	82.5	—	85.6	87.4

Analysis of BIBD

- As usual, we assume that there are a treatments and b blocks. In addition, we assume that each block **contains k treatments**, that **each treatment occurs r times** in the design, and that there are

$$N = ar = bk$$

total observations.

- The number of times each pair of treatments appears in the same block is

$$\lambda = \frac{r(k-1)}{a-1}$$ (must be an integer)

Effects Model and ANOVA for BIBD

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$$

$$SS_T = SS_{\text{Treatments(adjusted)}} + SS_{\text{Blocks}} + SS_E$$

Recall the ANOVA for RCBD

ANOVA partitioning of total variability:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}[(\bar{y}_{i.}-\bar{y}_{..})+(\bar{y}_{.j}-\bar{y}_{..})$$

$$+(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})]^2$$

$$= b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$$

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

ANOVA

TABLE 4.23

Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\frac{k\sum Q_i^2}{\lambda a}$	$a-1$	$\frac{SS_{\text{Treatments(adjusted)}}}{a-1}$	$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\frac{1}{k}\sum y_{.j}^2 - \frac{y_{..}^2}{N}$	$b-1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$N-a-b+1$	$\frac{SS_E}{N-a-b+1}$	
Total	$\sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$	$N-1$		

$$Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{.j} \qquad i = 1, 2, \ldots, a$$

with $n_{ij} = 1$ if treatment i appears in block j and $n_{ij} = 0$ otherwise. '