First-Order Logic

(or predicate logic)

Predicate

Syllogism

- Major premise: Every man is mortal
- Minor premise: Socrates is a man
- Conclusion: Socrates is mortal

Predicates:

- M for “is a man”: M(Socrates), M(Aristotle), M(Purdue), …
- D for “is mortal”: for D(Socrates), D(Aristotle), D(Purdue), …

First-Order Logic

$$\forall x \big(M(x) \rightarrow D(x)\big)$$

quantifier
"for all"

relation

Number theory:

$$prime(x) \equiv \neg\exists y, z(y > 1 \wedge z > 1 \wedge y \cdot z = x)$$

quantifier
"exists"

relation
$(GT(y, 1))$

function

FOL: Syntax

A first-order language is a tuple $(R, F, C, arity)$

- R is a countable set of relations
- F is a countable set of functions
- C is a countable set of constants
- $arity: R \cup F \to \mathbb{N}^+$ is an arity function

We also assume a countably infinite set of variables Var

E.g., $L_{arith} = (<,\ +,\ \cdot,\ 0,1,2,\dots)$

$L_{group} = (\_^{-1}, \cdot, 1)$

FOL: Syntax

Term $t ::- \ c \mid f(t_1, \ldots t_m) \mid x \qquad (c \in C, f \in F, x \in Var, Arity(f) = m)$

Formula $\varphi, \psi ::- \ \bot \mid \top \mid t_1 = t_2 \mid r(t_1, \ldots t_m)$

$\neg\varphi \mid \varphi \vee \psi \mid \exists x \psi \quad (t_1, t_2 \in C, r \in R, x \in Var, Arity(r) = m)$

Bounded variables: $\exists x(\ldots x \ldots) \equiv \exists y(\ldots y \ldots)$

$FV(\varphi)$: set of free variables in φ (not bounded by any quantifier)

Sentence : $FV(\varphi) = \emptyset$

Defined symbols : $\wedge, \forall, \rightarrow, \leftrightarrow$

FOL: Semantics

Let $L = (R, F, C, arity)$ be a first-order language. An L-structure is a tuple $(A,\ \tau)$:

- A is a universe
- τ is a function over $R \cup F \cup C$ s.t.
  - $\tau(r) \subseteq A^m$ for every $r \in R$
  - $\tau(f) : A^m \to A$ for every $f \in F$
  - $\tau(c) \in A$ for every $c \in C$

FOL: Semantics

Let $L = (R, F, C, arity)$ be a first-order language, $\mathcal{A} = (A,\ \tau)$ be an L-structure, then τ can be extended inductively:

- $\tau(a) = a$ for any $a \in A$
- $\tau\big(f(t_1, \dots, t_m)\big) = \tau(f)(\tau(t_1), \dots, \tau(t_m))$ for any m-ary function f

$\mathcal{A} \vDash \varphi$ is defined inductively:

- $\mathcal{A} \vDash \top$ and $\mathcal{A} \nvDash \perp$
- $\mathcal{A} \vDash t_1 = t_2$ if and only if $\tau(t_1) = \tau(t_2)$
- $\mathcal{A} \vDash r(t_1, \dots, t_m)$ if and only if $\tau(r)(\tau(t_1), \dots, \tau(t_m))$
- $\mathcal{A} \vDash \neg\varphi$ if and only if $\mathcal{A} \nvDash \varphi$
- $\mathcal{A} \vDash \varphi \vee \psi$ if and only if $\mathcal{A} \vDash \varphi$ or $\mathcal{A} \vDash \psi$
- $\mathcal{A} \vDash \exists x \psi$ if and only if $\mathcal{A} \vDash \psi(a)$ for some $a \in A$