Practice Problems Set 7

Fall 25

1. 2-CNF-SAT (attributed to CLRS Exercise 34.4-7)

   Let 2-CNF-SAT be the set of satisfiable propositional formulas in CNF with exactly two literals per clause. Show that 2-CNF-SAT$\in$ **P**.

   (Hint: Observe that $x \vee y$ is equivalent to $\neg x \to y$. Reduce 2-CNF-SAT to an efficiently solvable problem on a directed graph.)

Solution:

We show that 2-CNF-SAT can be reduced to the connectivity problem, which can be solved in polynomial time. Given a 2-CNF formula φ with variables X, we construct a graph with $2|X|$ vertices, one representing x and one representing $\neg x$ for each $x \in X$. For each clause of the form $l_1 \vee l_2$, we add two edges to the graph, one from $\neg l_2$ to l_1, one from $\neg l_1$ to l_2. The construction can be done in polynomial time. Note that every satisfying assignment corresponds to set of vertices which contains x or $\neg x$ only for each x. Also note that every path from l to l', if l is in the set, so is l'. Hence φ is *unsatisfiable* if and only if the constructed graph has a path from x to $\neg x$ or from $\neg x$ to x for some $x \in X$, which can be checked in polynomial time.

2. Resolution

   Five mages—Alice, Bob, Carol, David, and Emily—are bound by a complex magical pact. On the night of the solstice, each mage can attempt to channel the pact's energy. The success of each attempt is governed by a set of unbreakable magical laws.

   - For Alice to succeed, it must be true that at that moment, Bob and Carol both fail, or David succeeds.
   - For Bob to succeed, it must be true that at that moment, Alice succeeds and Emily fails.
   - For Carol to succeed, it must be true that at that moment, David succeeds, or Alice and Bob both fail.
   - For David to succeed, it must be true that at that moment, only one of Bob and Carol succeeds.
   - For Emily to succeed, it must be true that at that moment, David and Alice both succeed, or Bob fails.

   As the pact's overseer, you want to determine if Bob may succeed to channel the energy. Solve the problem by encoding the statements to propositional formulae and solve the satisfiability via Resolution.

Solution:

We use propositional variables to represent the statements and the truth values. Let $A/B/C/D/E$ represent "Alice/Bob/Carol/David/Emily succeeds." Then the laws can be encoded as follows:

- $A \to ((\neg B \wedge \neg C) \vee D)$
- $B \to (A \wedge \neg E)$
- $C \to (D \vee (\neg A \wedge \neg B))$
- $D \to (\neg B \leftrightarrow C)$
- $E \to ((D \wedge A) \vee \neg B)$

Now we convert the conjunction of the above five statements and the assumption B to CNF:

$$\begin{aligned}
&(\neg A \vee \neg B \vee D) \wedge (\neg A \vee \neg C \vee D) \\
\wedge\ &(A \vee \neg B) \wedge (\neg B \vee \neg E) \\
\wedge\ &(\neg A \vee \neg C \vee D) \wedge (\neg B \vee \neg C \vee D) \\
\wedge\ &(B \vee C \vee \neg D) \wedge (\neg B \vee \neg C \vee \neg D) \\
\wedge\ &(\neg B \vee D \vee \neg E) \wedge (A \vee \neg B \vee \neg E) \\
\wedge\ &B
\end{aligned}$$

After removing redundant clauses, we can apply resolution as follows:

$$\begin{aligned}
\{\neg A, \neg B, D\} & && (1) \\
\{\neg A, \neg C, D\} & && (2) \\
\{A, \neg B\} & && (3) \\
\{\neg B, \neg E\} & && (4) \\
\{\neg B, \neg C, D\} & && (5) \\
\{\neg B, \neg C, \neg D\} & && (6) \\
\{B\} & && (7) \\
\{A\} & \quad \text{(resolvent of 3 and 7)} && (8) \\
\{\neg E\} & \quad \text{(resolvent of 4 and 7)} && (9) \\
\{\neg B, D\} & \quad \text{(resolvent of 1 and 8)} && (10) \\
\{D\} & \quad \text{(resolvent of 7 and 10)} && (11) \\
\{\neg C, \neg D\} & \quad \text{(resolvent of 6 and 7)} && (12) \\
\{\neg C\} & \quad \text{(resolvent of 11 and 12)} && (13)
\end{aligned}$$

No more resolution can be done and the resulting CNF is not closed. Therefore Bob can succeed and there is only one possibility: Alice, Bob and David succeed, and Carol and Emily fail.