Practice Problems Set 11

Fall 25

1. Cycle Property (attributed to CLRS Exercise 21.1-5)

   Let e be a maximum-weight edge on some cycle of connected graph $G = (V, E)$. Prove that there is a minimum spanning tree of $G' = (V, E - \{e\})$ that is also a minimum spanning tree of G. That is, there is a minimum spanning tree of G that does not include e.

Solution:

Let T be a minimum spanning tree for G. If T does not contain e, then we are done.

So now suppose that T contains e. We will construct another minimum spanning tree that does not contain e. Let $e = (u, v)$ and let $T' = T - \{(u, v)\}$ be T with edge (u, v) removed. Define the cut $(S, V - S)$ such that

- $S = \{x \in V : \text{there is a path } u \rightsquigarrow x \text{ in } T'\}$
- $V - S = \{x \in V : \text{there is a path } v \rightsquigarrow x \text{ in } T'\}$

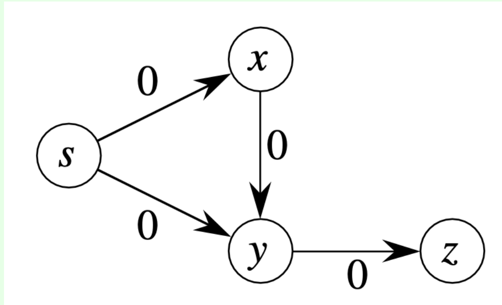
Because e is on a cycle, some other edge e' in the cycle crosses the cut $(S, V - S)$, and by the definition of e, we have $w(e') \leq w(e)$. Construct the tree $T'' = T' \cup \{e'\}$. Tree T'' is a spanning tree for G with weight $w(T'') = w(T') + w(e') = w(T) - w(e) + w(e') \leq w(T)$. Since we assume that T is a minimum spanning tree for G, T'' must be one as well, and it does not include e.

2. Dijkstra Misunderstood (attributed to CLRS Exercise 22.3-6)

   Professor Newman thinks that he has worked out a simpler proof of correctness for Dijkstra's algorithm. He claims that Dijkstra's algorithm relaxes the edges of every shortest path in the graph in the order in which they appear on the path, and therefore the path-relaxation property applies to every vertex reachable from the source. Show that the professor is mistaken by constructing a directed graph for which Dijkstra's algorithm relaxes the edges of a shortest path out of order.

Solution:

Consider this graph:



Dijkstra's algorithm could relax edges in the order (s, y), (s, x), (y, z), (x, y). The graph has two shortest paths from s to z: (s, x, y, z) and (s, y, z), both with weight 0. The edges on the shortest path (s, x, y, z) are relaxed out of order, because (x, y) is relaxed after (y, z).