Introduction to Matrix Algebra

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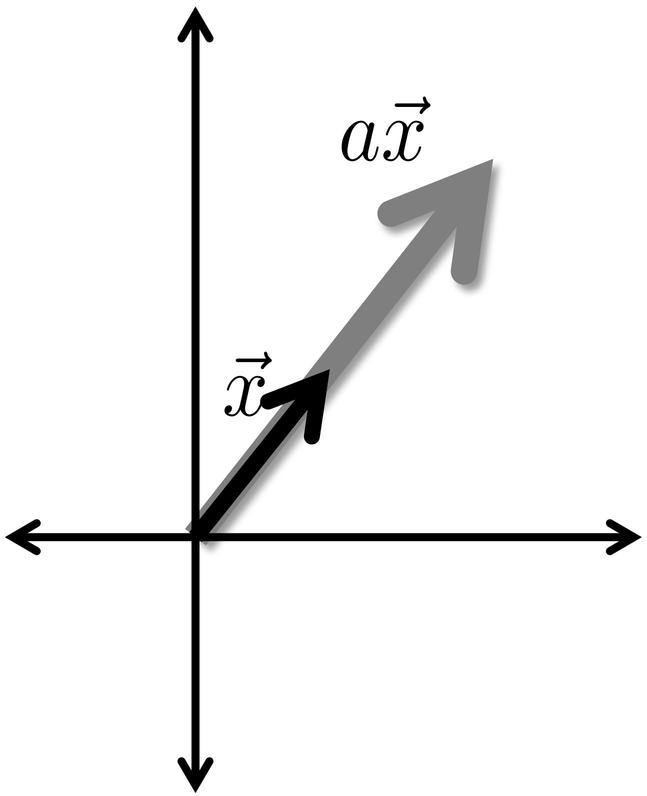
Vectors

A vector has **magnitude** and **direction**.

Scalar times vector:



$$a\vec{x} = a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}$$

Vectors Addition

- Vectors with the same size (*N*) are addible.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_N + y_N \end{pmatrix}$$

Multiplication between vectors
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

$$= \sum_{i=1}^{N} x_i y_i$$

Matrix

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

n observations of p independent variables

form a matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ & \cdots & & \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix}$$

A matrix contains column vectors and row vectors.

In X, each row is an observation; each column accommodates a factor (variable)

A column vector with *n* entries is a $n \times 1$ matrix
A row vector with *n* entries is a $1 \times n$ matrix

Matrix addition

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Matrix Transpose

- $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

- The transpose of a $n \times m$ matrix is a $m \times n$ matrix

Matrix times a vector

$$y = W x$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

M X 1 M X N N X 1

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**
- Inner matrix dimensions must agree

Matrix times a vector: **inner product interpretation**

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the ith element of **y** is the dot product of the ith row of W with **x**

Matrix times a vector: **inner product interpretation**

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the ith element of **y** is the dot product of the ith row of W with **x**

Product of Two Matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

N X P PX M N X M

- Inner matrix dimensions must agree
- **Note:** Matrix multiplication doesn't (generally) commute, **AB** $\neq$ **BA**
- C_{ij} is the inner product of the i^{th} row with the j^{th} column

Matrix times Matrix:
by inner products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{iP} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & C_{ij} & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$C_{ij} = \sum_{k=1}^{P} A_{ik} B_{kj}$$

- C_{ij} is the inner product of the i^{th} row of **A** with the j^{th} column of **B**

Special matrices: **diagonal matrix**

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \qquad D\,x = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{pmatrix}$$

- This acts like scalar multiplication

Special matrices: **identity matrix**

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

For any matrix $A, AI = A, IA = A$
Like 1 in the scalar world.

Inverse matrix

An *n*-by-*n* square matrix A is called invertible if there exists an *n*-by-*n* square matrix **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

- B is denoted as A^{-1}
- The inverse of a matrix does not always exist.
- To find the inverse of a matrix is not simple.
- Fortunately, we have R to help us. In R, we can use “solve(A)” to find the inverse of A.

Matrix Equations

- The linear system of equations

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \\ 3x_1 + 2x_2 - x_3 = 3 \end{cases}$$

is equivalent to

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$$

$$Ax = b$$

$$x = A^{-1}b = \begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \\ 3 & 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$$

Matrix Equations

Generally,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$x = A^{-1}b$$