

FEniCS-arclength: A numerical continuation package in FEniCS for nonlinear problems in solid mechanics

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Summary

FEniCS-arclength is a package that implements the numerical arclength continuation method, following the pioneering work in developing the method ([Crisfield, 1981](#); [Riks, 1979](#)), as a nonlinear solver in the open source finite element software framework FEniCS ([Alnæs et al., 2015](#); [Logg et al., 2012](#)). The arclength method is a robust nonlinear solver that is capable of capturing complex equilibrium paths including unstable states in solid mechanics problems with structural instabilities. Such structural instabilities are induced by geometric nonlinearities, whereas material and contact nonlinearities can also be accommodated in the framework.

FEniCS ([Alnæs et al., 2015](#); [Logg et al., 2012](#)) is an open-source software that solves partial differential equations (PDEs) through finite element analysis (FEA). Building on the versatile FEniCS platform, researchers have created both detailed tutorials ([Bleyer, 2018](#)) and software packages ([Finsberg, 2019](#); [Rodriguez et al., 2019](#)) to facilitate research in specific application domains. For example, there are multiple open-source software libraries designed to increase the existing capabilities of FEniCS ([Farrell et al., 2016](#); [Kamensky & Bazilevs, 2019](#); [Mitsch et al., 2019](#)). Building on this established precedent, we have implemented our arclength solver as an add-on library on top of FEniCS, and also demonstrated usage of our solver via multiple documented examples involving nonlinear continuum and beam finite elements at large deformations. Notably, our intent is to keep the solver usage similar to already established FEniCS solvers to enable off-the-shelf implementation within FEniCS workflows. To this end, our arclength solver can be integrated into existing FEniCS code that is based on a Newton solver just by changing the choice of solver. Furthermore, our code and documentation are organized such that more advanced users can make custom modifications as needed, for example pinpointing and branch switching algorithms ([Wriggers et al., 1988](#)).

In brief, our implementation follows a recently published method ([Kadapa, 2021](#)) adopting an extrapolated predictor scheme using the previously converged two solution steps, and solves the augmented arclength matrix equation as the corrector scheme. The main advantage of the extrapolation prediction scheme is in its ability to determine the forward direction of the equilibrium path without doing extra computations such as evaluating the determinant of the stiffness matrix ([Crisfield, 1981](#)), although calculating the sign of the determinant can be simple depending on the type of linear solver (i.e., the sign of the determinant can be extracted during numerical factorization). For displacement-controlled arclength problems, similar to a multifreedom constraint problem, we introduced a constraint matrix to augment the prescribed non-zero Dirichlet boundary conditions and the free degrees of freedoms to solve the arclength equations ([Batoz & Dhatt, 1979](#); [Verhoosel et al., 2009](#)). More

45 details about the theory, implementation, and usage of our arclength solver and general
46 mathematical preliminaries can be found in our extensive ReadTheDocs documentation (<https://fenics-arclength.readthedocs.io/en/latest/>).
47

48 Statement of Need

49 Materials and structures that undergo large deformation often exhibit fascinating mechanical
50 instabilities due to their geometrically nonlinear behavior. Recently, there has been a renewed
51 interest in studying these mechanical instabilities in engineered architected materials (Medina
52 et al., 2020), and in leveraging our understanding of mechanics to interpret the non-linear
53 behavior of natural materials (J.-H. Lee et al., 2021) and the elastocapillary behavior of soft
54 solids (Dortdivanlioglu & Javili, 2022). In these applied mechanics problems, the finite element
55 method is often used to computationally simulate these complex structures, particularly when
56 geometry and/or material behavior are complex and consequently previously derived and
57 validated analytical solutions are not applicable. In the solutions of geometrically nonlinear
58 problems with finite element analysis, the equilibrium path of the structure is traced either
59 in a force-controlled scheme (i.e., prescribed incremental Neumann boundary conditions)
60 or displacement-controlled scheme (i.e., prescribed incremental non-zero Dirichlet boundary
61 conditions). However, in extreme cases, such as in the case of snap-back or snap-through
62 instabilities, standard Newton solver based force/displacement-controlled schemes fail. To
63 address this shortcoming of standard Newton solvers, various nonlinear solvers have been
64 proposed based on the common approach of adding an extra constraint equation on top of the
65 Newton solver (Leon et al., 2012). In particular, the arclength method addresses this challenge
66 through an arclength constraint equation where an incremental load or displacement factor at
67 each load iteration, an additional unknown, is modified. Various arclength-type solvers are
68 available in commercial FEA software (e.g., see Riks method in ABAQUS (Smith, 2009)) to
69 capture the full equilibrium path. The goal of this work is to provide the research community
70 with an open-source and robust arclength solver that can readily be applied to a myriad of
71 nonlinear solid mechanics problems.

72 Examples of FEniCS Arclength

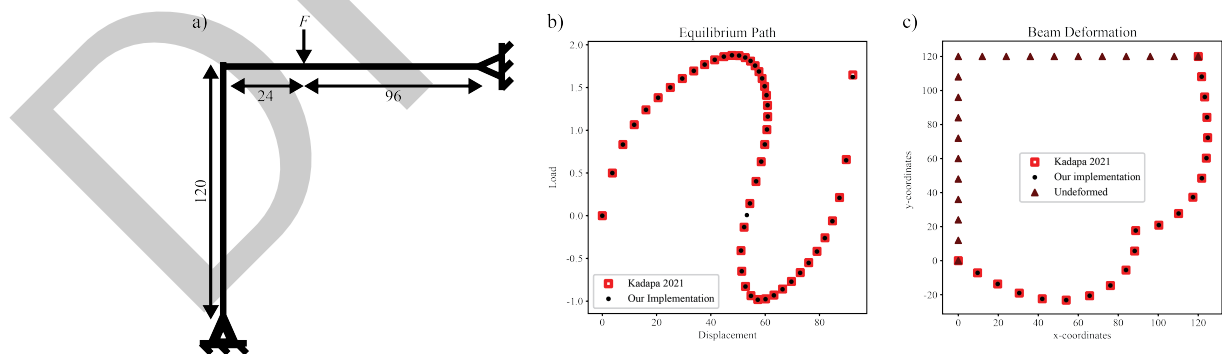


Figure 1: A demonstration of our arclength solver via the snap-back instability behavior of Lee's frame, a popular benchmark problem in nonlinear structural mechanics: a) The problem definition for Lee's frame; b) The equilibrium path from our solver compared to the literature (Kadapa, 2021); c) The final deformed configuration from our solver compared to the literature (Kadapa, 2021).

73 In Figure 1, we show an example of our arclength implementation for solving the Lee's frame
74 problem (S.-L. Lee et al., 1968), a popular benchmark problem in nonlinear structural mechanics.
75 In this problem, the frame is discretized using 2D Simo-Reissner beams (Simo & Vu-Quoc,
76 1986) with mixed elements. The elements have linear Lagrange interpolation functions for

77 both displacement and rotation, and use reduced integration to avoid locking. Figure 1a gives
 78 an overview of the Lee's frame problem and illustrates the equilibrium path (Figure 1b) and
 79 the deformed configuration (Figure 1c) in comparison with the literature (Kadapa, 2021).
 80 The Lee's frame example demonstrates that our solver is able to trace unstable states, where
 81 traditional Newton solvers would fail. Additional validation examples on problems in nonlinear
 82 structural mechanics with beam elements (compared to solutions from Kadapa et al. (Kadapa,
 83 2021)), and bilayer wrinkling with continuum elements (compared to analytical solutions
 84 (Allen, 1969; Budday et al., 2017; Cao & Hutchinson, 2012)) can be found in the examples
 85 folder of the Github Repository ([https://github.com/pprachas/fenics_arclength/tree/master/](https://github.com/pprachas/fenics_arclength/tree/master/examples)
 86 [examples](https://github.com/pprachas/fenics_arclength/tree/master/examples)), and in the ReadTheDocs ([https://fenics-arclength.readthedocs.io/en/latest/index.](https://fenics-arclength.readthedocs.io/en/latest/index.html#notebook-examples)
 87 [html#notebook-examples](https://fenics-arclength.readthedocs.io/en/latest/index.html#notebook-examples)) documentation. Critically, we demonstrated that our solver is able
 88 to trace the complex equilibrium paths of popular benchmark problems such as 215° and
 89 180° arcs (see Table 1 for all the provided examples in our documentation). In addition to
 90 validation problems, our documentation also contains the solver's API, example usage of our
 91 solver for continuum problems, and example usage of our displacement controlled arclength
 92 scheme.

Table 1: A list of all available examples in our GitHub Repository and ReadTheDocs documentation. Examples marked with (§) are compared with solutions from the literature obtained from (Kadapa, 2021). Examples with (★) are compared with solutions from (Allen, 1969; Budday et al., 2017; Cao & Hutchinson, 2012)

Force-controlled Scheme
<i>Continuum elements</i>
▪ Lee's Frame (Continuum Elements)
<i>Beam elements</i>
▪ Lee's Frame §
▪ 215° arc §
▪ 180° arc §
▪ 3D Helical Beam
Displacement-controlled Scheme
<i>Continuum Elements</i>
▪ Bilayer Wrinkling ★
<i>Beam Elements</i>
▪ Random Fiber Network

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