Q1

a) $\frac{1+i}{i-7} = \frac{1+i}{i-7} \cdot \frac{i+7}{i+7} = \frac{i+7+i^2+7i}{i^2-49}$

$= \frac{8i+7-1}{-1-49} = \frac{8i+6}{-50} = \frac{4i+3}{-25} = \frac{-3}{25} - \frac{4i}{25}$

Therefore $\frac{1+i}{i-7} = \frac{-3}{25} - \frac{4i}{25}$ //

b) $(\sqrt{3}-i)^{20} = \left[2 \cdot \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)\right]^{20} = \left[2 \cdot e^{-\frac{i\pi}{6}}\right]^{20}$

$= 2^{20} \cdot e^{-\frac{10i\pi}{3}} = 2^{20} e^{-\frac{4i\pi}{3}} = 2^{20}(-1/2 + i\sqrt{3}/2)$

$= -2^{19} + i\sqrt{3} \cdot 2^{19} = -524{,}288 + i\,524{,}288\sqrt{3}$

therefore $(\sqrt{3}-i)^{20} = -524{,}288 + i \cdot 524{,}288\sqrt{3}$ //

Q2. $A = (-1,1,2)$, $B = (2,1,0)$

$C = (0,-1,5)$

$$A - B = (-1,1,2) - (2,1,0) = (-3,0,2)$$

$$A - C = (1,1,2) - (0,-1,5) = (1,2,-3)$$

Let r and $s \in \mathbb{R}$.
The equation of the plane passing through A, B, C is, therefore, given by:

$$(x,y,z) = (-1,1,2) + r\cdot(-3,0,2) + s\,(1,2,-3)$$

Q3.

$$z = a + i \cdot b$$

In order to know whether F is a subfield of $\mathbb{C}$, it must be verified whether F is a field. Thus, all 5 field axioms must be verified and for each pair of elements in F, multiplication and addition must also result in an element of F.

(F1) commutativity of addition and multiplication

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2 \in F$

a) Regarding addition: $a+b = b+a$

$$z_1 + z_2 = a_1 + ib_1 + a_2 + ib_2$$

$$z_2 + z_1 = a_2 + ib_2 + a_1 + ib_1$$

$$a_1 + ib_1 + a_2 + ib_2 = a_2 + ib_2 + a_1 + ib_1$$

therefore $z_1 + z_2 = z_2 + z_1$ //

b) Regarding multiplication: $a \cdot b = b \cdot a$

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= a_1 \cdot a_2 + ia_1b_2 + ia_2b_1 - b_1 \cdot b_2$$

$$z_2 \cdot z_1 = (a_2 + ib_2)(a_1 + ib_1)$$
$$= a_1 \cdot a_2 + ia_2b_1 + ia_1b_2 - b_1 \cdot b_2$$

therefore, $z_1 \cdot z_2 = z_2 \cdot z_1$ //

(F2) Associativity of addition and multiplication

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $z_3 = a_3 + ib_3$
where $z_1, z_2, z_3 \in F$

a) Regarding addition: $(a+b)+c = a+(b+c)$

by F1, (LHS): $(z_1 + z_2) + z_3 = (a_1 + a_2 + ib_1 + ib_2) + a_3 + ib_3$
$$= a_1 + a_2 + a_3 + ib_1 + ib_2 + ib_3$$

by F1, (RHS): $z_1 + (z_2 + z_3) = a_1 + ib_1 + (a_2 + a_3 + ib_2 + ib_3)$
$$= a_1 + a_2 + a_3 + ib_1 + ib_2 + ib_3$$

therefore, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ //

b) Regarding multiplication: $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

by F1, (LHS): $(z_1\cdot z_2)\cdot z_3 = (a_1a_2 + ia_1b_2 + ia_2b_1 - b_1b_2)\cdot(a_3 + ib_3)$

by F5, $= a_1a_2a_3 + ia_1a_2b_3 + ia_1a_3b_2 - a_1b_2b_3 + ia_2a_3b_1 - a_2b_1b_3 - a_3b_1b_2 - ib_1b_2b_3$

by F1, (RHS): $z_1(z_2\cdot z_3) = (a_1 + ib_1)(a_2a_3 + ia_2b_3 + ia_3b_2 - b_2b_3)$

$= a_1a_2a_3 + ia_1a_2b_3 + ia_1a_3b_2 - a_1b_2b_3 + ia_2a_3b_1 - a_2b_1b_3 - a_3b_1b_2 - ib_1b_2b_3$

therefore, $(z_1\cdot z_2)\cdot z_3 = z_1\cdot(z_2\cdot z_3)$ //

(F3) Existence of identity elements for addition and multiplication.

Let $z_1 = a + ib$, $z_2 = c + id \in F$

a) Regarding addition: $a + 0 = a$

$z_1 = a + ib + 0 + i\cdot 0 = a + ib$

identity element is $0 + i0$

$z_2 = c + id + 0 + i0 = c + id$

b) Regarding multiplication : $a \cdot 1 = a$

$$a + ib \cdot (1 + i0) = a + ib = z_1$$ //

(F4) Existence of inverses for addition and multiplication

Let $z_1 = a + ib$, $z_2 = c + id \in F$

a) Regarding addition : $a' + c' = 0$

$$z_1 + z_2 = a + ib + c + id = 0$$

$$a = -c \quad , \quad b = -d$$

therefore z_1 and z_2 are additive inverses. //

b) Regarding multiplication: $b' \cdot d' = 0$

Let $z_3 = e + if = \dfrac{a - ib}{a^2 + b^2}$, $(a^2 + b^2) \neq 0$

$$z_1 \cdot z_3 = 1 = (a + ib) \cdot (e + if)$$

$$= (a + ib) \cdot \left(\frac{a - ib}{a^2 + b^2} \right) = \frac{a^2 - iab + iab + b^2}{a^2 + b^2}$$

$$= 1$$

therefore z_1 and z_3 are mult. inverses. //

(F5) Distributivity of multiplication over addition : $a \cdot (b+c) = a \cdot b + a \cdot c$

let $z_1 = a+ib$, $z_2 = c+id$, $z_3 = e+if \in F$

$a+ib(c+id+e+if)$

$= ac + iad + a \cdot e + i \cdot a \cdot f + ib \cdot c - bd$

$+ ieb - bf = z_1(z_2+z_3)$ //

all field axioms are satisfied and it is also possible to check closure under addition and multiplication in F.

a) Closure under addition:

$z_1 + z_2 = z_3$, where $z_1, z_2, z_3 \in F$

let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $z_3 = a_3 + ib_3$

$z_1 + z_2 = (a_1 + a_2) + i \cdot (b_1 + b_2) = z_3$

therefore, $a_1 + a_2 = a_3$ and

$b_1 + b_2 = b_3$

a_3 and $b_3 \in \mathbb{Q}$, since $a_1, a_2 \in \mathbb{Q}$ and due to closure under addition in $\mathbb{Q}$.

b) Closure under multiplication: $z_1 \cdot z_2 = z_3$

let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $z_3 = a_3 + ib_3$
and $z_1, z_2, z_3 \in F$

$z_1 \cdot z_2 = z_3 \Longleftrightarrow (a_1 + ib_1)(a_2 + ib_2)$

$= a_1 \cdot a_2 + i a_1 b_2 + i a_2 b_1 - b_1 b_2$

$= (a_1 \cdot a_2 - b_1 \cdot b_2) + i \cdot (a_1 \cdot b_2 - b_1 \cdot b_2)$

$= a_3 + ib_3$

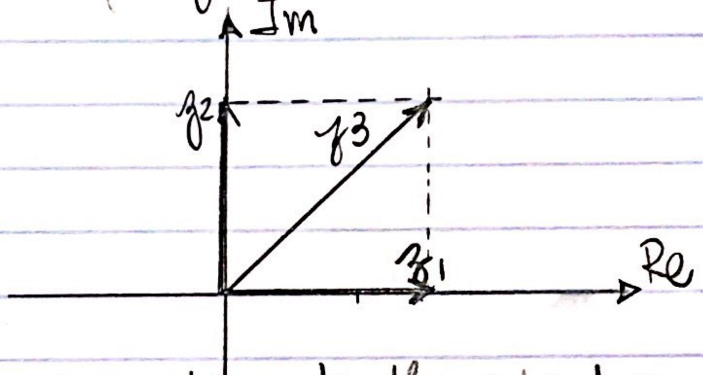
therefore, $a_1 \cdot a_2 - b_1 \cdot b_2 = a_3$
and $a_1 \cdot b_2 - b_1 \cdot b_2 = b_3$

$a_1, a_2, b_1, b_2 \in \mathbb{Q}$ because
$z_1, z_2 \in F$.

Due to closure under addition
(and subtraction) in $\mathbb{Q}$,
a_3 and $b_3 \in \mathbb{Q}$ and
therefore, $z_3 \in F$

Q4.

Let $z_1 = 1 \cdot e^{i2\pi}$, $z_2 = 1 \cdot e^{i\pi/2}$,
where $z_1, z_2 \in F$.
Let $z_3 = z_1 + z_2$
Thus, in order to F be a field,
$z_3 \in F$, which is not true.



according to the graphic, $z_3 = \sqrt{2} \cdot e^{i\frac{\pi}{4}} = r_{z_3} \cdot e^{i\frac{\pi}{4}}$

Since $r_{z_3} = \sqrt{2}$ is not a rational number, z_3 is not in F, violating a field property in F. Thus F is not a field. Closure under addition is not satisfied.

Q5

Using F5,

$(a+jb)(c+jd) = ac + jad + jbc + j^2 bd = 1$

Since $j^2 + j + 1 = 0$,

$ac + jad + jbc + j^2 bd = (ac - bd) + j(ad + bc - bd)$

because $j^2 bd = -bd - jbd$.

Using $j^2 + j + 1 = 0$ and $ac + jad + jbc + j^2 bd = 1$,

$$\begin{cases} ac - 1 = 1 \\ ad + bc = 1 \\ bd = 1 \end{cases}$$, therefore $c = 2/a$ and $d = 1/b$

Therefore, the multiplicative inverse of $(a + jb)$ is $(c + jd)$, where:

$$c + jd = \frac{2}{a} + j\frac{1}{b}$$, where $a, b \neq 0$

and $c, d \in F$