Question 1

$v_1, v_2, v_3 \in \mathbb{R}^3$

$v_1 = (3,2,0)$
$v_2 = (1,1,1)$
$v_3 = (1,0,2)$

$v = (-1,2,3) = a_1 v_1 + a_2 v_2 + a_3 v_3$

Therefore,

$-1 = a_1 3 + a_2 \cdot 1 + a_3 \cdot 1$
$2 = a_1 \cdot 2 + a_2 \cdot 1 + a_3 \cdot 0$
$3 = a_1 \cdot 0 + a_2 \cdot 1 + a_3 \cdot 2$

Gaussian Elimination:

3	1	1	-1
2	1	0	2
0	1	2	3
	1	-2	8
	3	6	9
		12	-15

$\therefore a_3 = -1.25$
$\therefore 3 + 2.5 = a_2$
$\therefore a_2 = 5.5$
$\therefore \frac{2-5.5}{2} = a_1 = -1.75$

$a_1 = -1.75$
$a_2 = 5.5$
$a_3 = -1.25$ //

Question 2

$$\begin{cases} x_1 - x_2 + x_3 - x_4 = 0 & (\mathrm{I}) \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0 & (\mathrm{II}) \end{cases}$$

$(\mathrm{II}) - (\mathrm{I})$: $3x_2 + 2x_3 + 5x_4 = 0$ (III)

The set $\beta = \{(1,-1,-1,1), (1,1,1,1)\}$ is a basis for the system of equations.

Thus it is possible to check β in equation (III):

$$1 \cdot (1,-1,-1,1) + 0 \cdot (1,1,1,1) = (1,-1,-1,1)$$

therefore, $x_1 = 1$, $x_2 = -1$, $x_3 = -1$, $x_4 = 1$

(III); $3x_2 + 2x_3 + 5x_4 = 0$

$$\iff 3 \cdot (-1) + 2(-1) + 5(1) = 0$$

$$\iff -3 - 2 + 5 = 0$$

Thus the solution of the system is a linear combination of β, and β is a basis.

Question 4

Let $k \in \mathbb{N}^0$, $0 \leq k \leq n$
Let $p_k = x^k$, $x \in \mathbb{R}$
Let $Pol_n(\mathbb{R})$ be a space of polynomials of degree $\leq n$.

$Pol_n(\mathbb{R})$ may be represented as:

$$\sum_{k=0}^{k=n} a_k \cdot x^k = 0 \quad , \text{ where } a_k \in \mathbb{R}$$

The set $\beta = \{ p_k : k \in \mathbb{N}^0, 0 \leq k \leq n \}$ may be a basis for $Pol_n(\mathbb{R})$, since:
- Any polynomial in $Pol_n(\mathbb{R})$ can be a linear combination of the elements in β;
- the elements in β are linearly independent from each other.

$$\begin{cases} a_k \cdot x^k + 0x^{k-1} + \ldots + 0x^0 = 0 \\ 0 \cdot x^k + a_{k-1} \cdot x^{k-1} + \ldots + 0x^0 = 0 \\ \ldots \quad = 0 \\ \ldots \quad = 0 \\ \ldots \end{cases}$$

Since the only solution for the system above is the trivial one, where $a_k = a_{k-1} = \ldots = a_0 = 0$, β is linearly independent.