Graph Isomorphism

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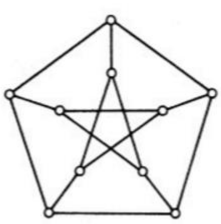
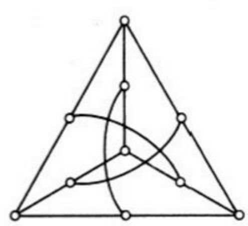
When do we consider two graphs "the same"?

- **G_1 and G_2 represent the same graph?**



G_1



G_2

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Graph Isomorphism

Definition:

- The **simple graphs** $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are **isomorphic** if there exists a bijective function f, f: $V_1 \to V_2$, with the property that:
  - u and v are adjacent in G_1, $(u,v) \in E_1$, if and only if f(u) and f(v) are adjacent in G_2, $(f(u),f(v)) \in E_2$, for all u and v in V_1.
- Such a function f is called an **isomorphism**.

- **Note that:**
  - **Bijective function** f: $A \to B$:
    - **one-to-one**: $a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
    - **onto**: for all $b \in B$, there exists $a \in A$ and $f(a) = b$
  - Isomorphism of simple graphs is an equivalence relation.

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Graph Isomorphism (cont…)

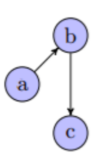
Remember that:

- The **simple graphs** $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are **isomorphic** if there exists a **bijection** f: $V_1 \to V_2$ such that $(u,v) \in E_1$ if and only if $(f(u),f(v)) \in E_2$. The bijection f is called the isomorphism from G_1 to G_2, and we use the notation $G_2 = f(G_1)$.

Example:

- The **simple graphs** G_1 and G_2 are clearly **isomorphic**.

G_1

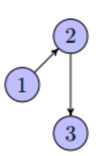


f(a) = 1; f(b) = 2; f(c) = 3

$(a,b) \in G_1 \to (f(a),f(b)) = (1,2) \in G_2$

$(b,c) \in G_1 \to (f(b),f(c)) = (2,3) \in G_2$

G_2



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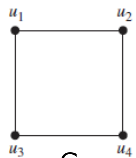
Graph Isomorphism (cont…)

Remember that:

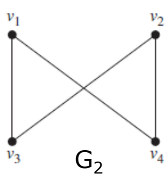
- The **simple graphs** $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are **isomorphic** if there exists a bijection $f : V_1 \rightarrow V_2$ such that $(u,v) \in E_1$ if and only if $(f(u),f(v)) \in E_2$. The bijection f is called the isomorphism from G_1 to G_2, and we use the notation $G_2 = f(G_1)$.

Example:

- Show that the **graphs** G_1 and G_2 are **isomorphic**.



G_1



G_2

- The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one function between V_1 and V_2.
- To see that this function preserves adjacency, note that adjacent vertices in G_1 are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and
- each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in G_2.

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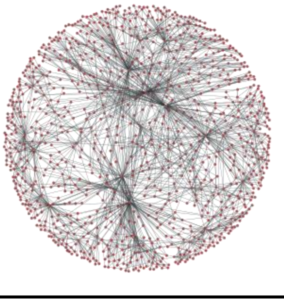
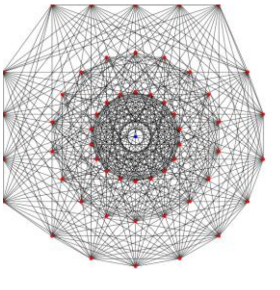
Graph Isomorphism (cont…)

Determining whether two simple graphs are isomorphic

- **Facts:**
  - It is **<u>often difficult</u>** to determine whether two simple graphs are **isomorphic**.
  - There are **n! possible one-to-one functions** between the vertex sets of two simple graphs with n vertices.
  - Testing each such correspondence to see whether it preserves adjacency and non-adjacency is impractical if n is a large number.



vs



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Graph invariants

- Sometimes it is not hard to show that **two graphs are not isomorphic**.
- Two graphs <u>are not isomorphic</u> if we can find a property only one of the two graphs has. A property preserved by isomorphism of graphs is called a **graph invariant**.
- For instance,
  - Isomorphic simple graphs must have **<u>the same number of vertices</u>**, because there is a one-to-one correspondence between the sets of vertices of the graphs.
  - Isomorphic simple graphs also must have **<u>the same number of edges</u>**, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges.
  - The **<u>degrees of the vertices in isomorphic simple graphs must be the same</u>**. That is, a vertex v of degree d in G_1 must correspond to a vertex f(v) of degree d in G_2, because a vertex w in G_1 is adjacent to v if and only if f(v) and f(w) are adjacent in G_2.

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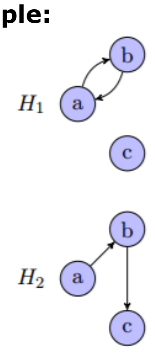
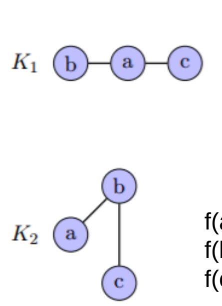
Graph invariant (cont…)

- Isomorphic simple graphs must have **<u>the same number of vertices</u>** …
- Isomorphic simple graphs also must have **<u>the same number of edges</u>** …
- The **<u>degrees of the vertices in isomorphic simple graphs must be the same</u>** …

Example:



Non-isomorphic!



f(a) = b
f(b) = a
f(c) = c

Isomorphic!

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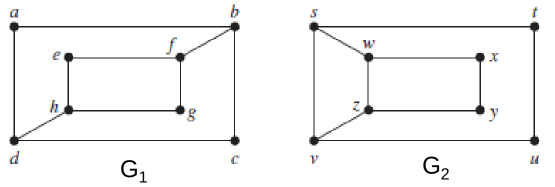
Graph invariant (cont…)



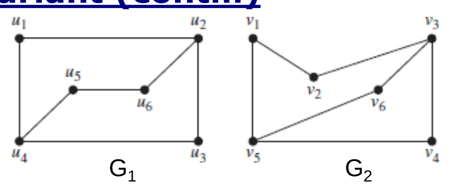
Example: Determine whether the graphs G_1 and G_2 are isomorphic.

- **Facts:**
  - The graphs G_1 and G_2 both have 8 vertices and 10 edges.
  - They also both have 4 vertices of degree 2 and 4 of degree 3.
  - Because these invariants all agree, it is still conceivable that these graphs are isomorphic.

However:

- **G_1 and G_2 are not isomorphic**. To see this, note that because deg(a) = 2 in G_1, a must correspond to either t, u, x, or y in G_2, because these are the vertices of degree 2 in G_2.
- However, each of these 4 vertices in G_2 is adjacent to another vertex of degree 2 in G_2, which is not true for a in G_1.

Graph invariant (cont…)

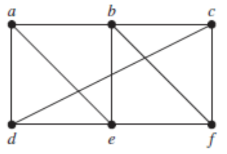


Example: Determine whether the graphs G_1 and G_2 are isomorphic.

- **Facts:**
  - Both G_1 and G_2 have six vertices and seven edges.
  - Both have four vertices of degree two and two vertices of degree three.
  - The subgraphs of G_1 and G_2 consisting of all vertices of degree two and the edges connecting them are isomorphic (Verify!).
  - Because G_1 and G_2 agree with respect to these invariants, it is **acceptable** to try to find an isomorphism f.

Paths

- **Definition:**
  - A **path** or a **walk** in a graph G = (V,E) is a sequence of vertices $(v_0,v_1, \ldots ,v_k)$ such that there exists an edge between any two consecutive vertices, i.e. $(v_i,v_{i+1}) \in E$ for all i, $0 \leq i < k$.
  - The **length of the path** is the number of edges in the path.
  - It is easy to see that the length of the path with n vertices is equal to n-1.
  - A **path** is called **simple** if it does not contain the same edge more than once.
- **Example:**



  - (a, d, c, f , e) is a simple path of length 4, because (a,d), (d,c), (c,f), and (f,e) are all edges.
  - However, (d, e, c, a) is not a path, because (e,c) is not an edge of the graph.
  - (a, b, e, a, b, c) – not a simple path

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Paths (cont…)

- **Definition:**
  - A **path** or a **walk** in a graph G = (V,E) is a sequence of vertices $(v_0,v_1, \ldots ,v_k)$ such that there exists an edge between any two consecutive vertices, i.e. $(v_i,v_{i+1}) \in E$ for all i, $0 \leq i < k$.
  - The **length of the path** is the number of edges in the path.
  - It is easy to see that the length of the path with n vertices is equal to n-1.
- **Examples:**
  - The **Bacon number** of an actor or actress is the shortest path from the actor or actress to Kevin Bacon on the following Hollywood graph: the nodes are actors and actresses, and edges connect people who star together in a movie.
  - The **Erdos number** is similarly defined to be the distance of a mathematician to Paul Erdos on the co-authorship graph.

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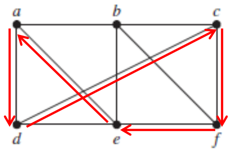
Cycles

- **Definition:**
  - A **cycle/circuit** is a **path/walk** where k ≥ 1 and $v_0 = v_k$ (i.e., starts and ends at the same vertex).

- **Example:**

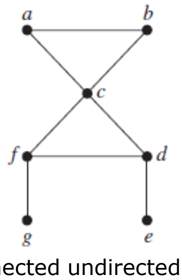


  - (a, d, c, f , e, a) is a cycle of length 5, because (a,d), (d,c), (c,f), (f,e), and (e,a) are all edges.

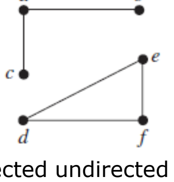
Connectivity

- **Definition:**
  - An **undirected graph** is **connected** if there exists a path between any two nodes u, v ∈ V.
  - By definition a graph containing a single node v is considered connected via the length 0 path (v).
  - An undirected graph that is not connected is called **disconnected**.

- **Examples:**



Connected undirected graph

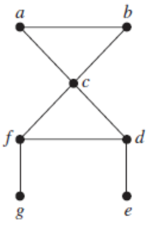


Disconnected undirected graph

Connectivity (cont…)

- **Definition:**
  - An **undirected graph** is **connected** if there exists a path between any two nodes u, v ∈ V.
  - By definition a graph containing a single node v is considered connected via the length 0 path (v).
  - An undirected graph that is not connected is called **disconnected**.

- **Theorem:** There is a simple path between every pair of distinct vertices of a **connected undirected graph**.
- Note that a simple path between every pair of distinct vertices can be not unique

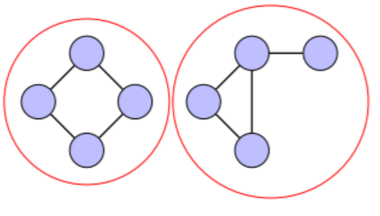


Between a and b: path (a,b)
Between b and a: path (b,a)
Between a and c: path (a,c)
. . .
Between a and e: path (a,c,d,e)
. . .

Connectivity (cont…)

- **Definition:**
  - A **connected component** of a graph G is a connected subgraph of G that is a maximal connected subgraph of G.
  - A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

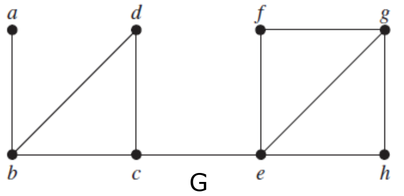
- **Example:** A graph with two connected components.



Connectivity (cont…)

- Sometimes the removal from a **graph** of a vertex and all incident edges produces a new graph with more **connected components**. Such vertices are called **cut vertices**.
- The removal of a **cut vertex** from a connected graph produces a subgraph that is not connected.
- Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**.

Example: Find the cut vertices and cut edges in the graph G.



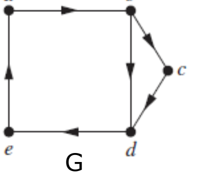
G

- The **cut vertices** of G are b, c, and e. The removal of one of these vertices (and its adjacent edges) disconnects the graph.
- The **cut edges** are (a,b) and (c,e). Removing either one of these edges disconnects G.

Connectivity (cont…)

- The notion of **connectivity** on a **directed graph** is more complicated, because paths are not reversible.
- **Definition:**
  - A directed graph G = (V,E) is **strongly connected** if there exists a path from any node u to any node v

- **Example:** Is G strongly connected graph?

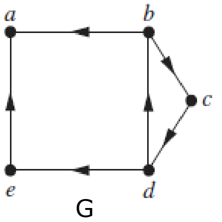


G

Yes!

Connectivity (cont…)

- **Definition:**
  - A directed graph G = (V,E) is **weakly connected** if there exists a path from an node u to any node v in the underlying undirected graph:
- **Example:** Is G **strongly connected** graph?

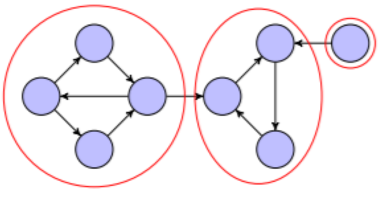


G

  - The graph G is not strongly connected. There is no directed path from a to b in this graph.
  - However, G is weakly connected, because there is a path between any two vertices in the underlying undirected graph.

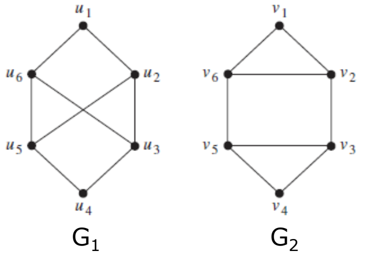
Connectivity (cont…)

- **Example:**
  - **Strongly connected components** of the graph are circled in red.
  - Note that there can still be edges between strongly connected components.



Paths and Isomorphism

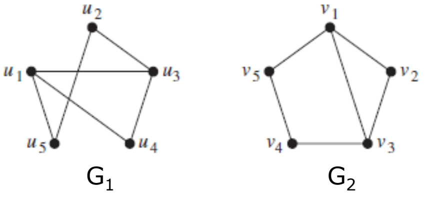
- **Example:** Determine whether the graphs G_1 and G_2 are **isomorphic**.



- **Facts:**
  - Both G_1 and G_2 have six vertices and eight edges.
  - Each has four vertices of degree three, and two vertices of degree two. So, the three invariants — number of vertices, number of edges, and degrees of vertices — all agree for the two graphs.
- **However:**
  - G_2 has a simple circuit of length three, namely, v_1, v_2, v_6, v_1, whereas G_1 has no simple cycle of length three.
  - All simple cycles in G_1 have length at least four.
  - Because the existence of a simple cycle of length three is an isomorphic invariant, **G_1 and G_2 are not isomorphic**.

Paths and Isomorphism (cont…)

- **Example:** Determine whether the graphs G_1 and G_2 are **isomorphic**.



- **Facts:**
  - Both G_1 and G_2 have five vertices and six edges, both have two vertices of degree three and three vertices of degree two.
  - Both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five.
  - Because all these isomorphic invariants agree, **G_1 and G_2 may be isomorphic**.

Graph Isomorphism

- **Complementary Material:**

https://math.unm.edu/~loring/links/graph_s05/hw2.pdf

http://www.ms.uky.edu/~csima/ma111/GraphsLecture2.pdf

https://www.geeksforgeeks.org/mathematics-graph-isomorphisms-connectivity/