# Numerical solution of restricted n-body problems, using the central differences approximation.

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**Abstract**

In order to simulate the trajectories of celestial bodies, we want a numerical solution to the n-body problem created by the modeling of the gravitational interactions between each body.

We want to be able to simulate these trajectories for a space scale with the same order of magnitude as the solar system, and a time scale varying from a few days to a few years.

We use a simple finite difference method on a discrete time scale to approximate the second order derivative component in the formulation of Newton's second law. No consideration has been given to the collision case.

Instead of focusing on precision or accuracy, the main goal of the study is the ease of implementation. Therefore we only study here the stability and convergence of the solution.

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# 1 Introduction

The gravitational interaction between two bodies, not accounting for relativistic effects, can be be expressed using Newton's law of universal gravitation. In a system consisting of only those two bodies, their relative trajectories can then be easily computed using Newton's second law of motion.

However, in a system where an arbitrary number of bodies interact with each other, solving the ODE system describing the trajectories becomes non-trivial. It has been shown that no analytical solution could be found for the general three body problem[1]. Several ways have been designed to numerically solve this problem for an arbitrary number of bodies (the n-body problem). Here we use the central difference approximation to express the acceleration of a body at a certain time $t$ as a function of its postilion at the times $t - \delta t$, $t$ and $t + \delta t$.

# 2 Problem formulation

Let there be $n$ bodies $B_i; \quad i \in [[1, n]]$ in a Euclidian space and $m_i$ their respective mass. Let $\vec{R_i}(t); \quad i \in [[1, n]]$ be their respective positions in the associated Cartesian coordinate system at the time $t \geq 0$.
For two bodies $B_i$ and $B_j; \quad i, j \in [[1, n]]^2$ the force applied by $B_i$ upon $B_j$ is given by Newton's law of universal gravitation :

$$\vec{F_{i/j}} = G \frac{m_i m_j}{|\vec{R_i} - \vec{R_j}|^3} (\vec{R_i} - \vec{R_j}) \tag{1}$$

Moreover, following Newtown's law of motion, the acceleration of $B_j$ is :

$$\forall j \in [[1, n]] : \quad m_j \frac{d^2 \vec{R_j}}{dt^2} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \vec{F_{i/j}} \tag{2}$$

Therefore, the acceleration of every body in the system is described by the following system :

$$\forall j \in [[1, n]] : \quad \frac{d^2 \vec{R_j}}{dt^2} = G * \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{m_i}{|\vec{R_i} - \vec{R_j}|^3} (\vec{R_i} - \vec{R_j}) \tag{3}$$

We want to solve this system for all $\vec{R_j}; \quad i \in [[1, n]]$.

[1] Bruns H.E., 1887

## 3 Discretization of the time dimension and Central differences approximation

Let $\delta t$ be an arbitrarily small, positive, time interval and $t_0$ an initial time. Furthermore let's define $\forall k \in \mathbf{N}: \quad \vec{R_{j,k}} = \vec{R_j}(t_0 + k\delta t)$.
Thus, using the $2^{nd}$ order central difference approximation we obtain :

$$\forall k \in \mathbf{N}*, \quad \forall j \in [[1,n]]: \quad \frac{d^2\vec{R_{j,k}}}{dt^2} = \frac{\vec{R_{j,k+1}} - 2\vec{R_{j,k}} + \vec{R_{j,k-1}}}{\delta t^2} + O(\delta t^2) \quad (4)$$

Therefore in the discretized time dimension, (3) becomes :

$$\forall k \in \mathbf{N}*, \quad \forall j \in [[1,n]]: \\ \frac{\vec{R_{j,k+1}} - 2\vec{R_{j,k}} + \vec{R_{j,k-1}}}{\delta t^2} = G * \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{m_i}{|\vec{R_{i,k}} - \vec{R_{j,k}}|^3}(\vec{R_{i,k}} - \vec{R_{j,k}}) \quad (5)$$

Thus we can express each the position of each body at a certain time step $k \geq 2$ as a function of the position of all the bodies at the time steps $k-1$ and $k-2$ :

$$\forall k \in \mathbf{N} - \{\mathbf{0}, \mathbf{1}\}, \quad \forall j \in [[1,n]]: \\ \vec{R_{j,k}} = 2\vec{R_{j,k-1}} - \vec{R_{j,k-2}} + \delta t^2 G * \sum_{\substack{i=1 \\ i \neq j}}^{n} m_i \frac{\vec{R_{i,k-1}} - \vec{R_{j,k-1}}}{|\vec{R_{i,k-1}} - \vec{R_{j,k-1}}|^3} \quad (6)$$

## 4 Initial conditions

In order to use (6) to compute the positions of the bodies, we need their initial positions for the first two time steps, for $k = 0$ and $k = 1$. We will use the initial position and speed of the bodies and derive their position for the time step $k = 1$ from it.
Let's note $\vec{R_{j0}}$ and $\vec{V_{j0}}$ the position and speed vectors for $k = 0$ and $j \in [[1,n]]$. Substituting the speed by it's forward difference approximation we obtain :

$$\forall j \in [[1,n]]: \quad \vec{V_{j0}} = \frac{R_{j,1} - Rj,0}{\delta t} + O(\delta t) \Leftrightarrow \vec{R_{j,1}} = \vec{R_{j0}} + \delta t \vec{V_{j0}} + O(\delta t) \quad (7)$$

It's worth noting that using this approximation causes the leading error to go down from $O(\delta t^2)$ to $O(\delta t)$.

## 5 Stability and convergence study

## 6 Conclusion and results