Numerical solution of restricted n-body problems, using the central differences approximation.

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October 14, 2013

Abstract

In order to simulate the trajectories of celestial bodies, we want a numerical solution to the n-body problem created by the modeling of the gravitational interactions between each body.

We want to be able to simulate these trajectories for a space scale with the same order of magnitude as the solar system, and a time scale varying from a few days to a few years.

At first we use a simple finite difference method on a discrete time scale to approximate the second order derivative component in the formulation of Newton's second law. Then we use Heun's scheme to get a higher order approximation that allows us to evaluate the local error and therefore the stiffness of the solution, which can be used as a hint for collision detection.

Instead of focusing on precision or accuracy, the main goal of the study is the ease of implementation. Therefore we only study here the stability and convergence of the solution.

Contents

1 Introduction

The gravitational interaction between two bodies, not accounting for relativistic effects, can be be expressed using Newton's law of universal gravitation. In a system consisting of only those two bodies, their relative trajectories can then be easily computed using Newton's second law of motion.

However, in a system where an arbitrary number of bodies interact with each other, solving the ODE system describing the trajectories becomes non-trivial. It has been shown that no analytical solution could be found for the general three body problem[1]. Several ways have been designed to numerically solve this problem for an arbitrary number of bodies (the n-body problem). Here we use the central difference approximation to express the acceleration of a body at a certain time t as a function of its position at the times $t - \delta t$, t and $t + \delta t$.

2 Problem formulation

Let there be n bodies $B_i; \quad i \in [[1, n]]$ in a Euclidian space and m_i their respective mass. Let $\vec{R_i}(t); \quad i \in [[1, n]]$ be their respective positions in the associated Cartesian coordinate system at the time $t \geq 0$.
For two bodies B_i and $B_j; \quad i, j \in [[1, n]]^2$ the force applied by B_i upon B_j is given by Newton's law of universal gravitation :

$$\vec{F_{i/j}} = G\frac{m_i m_j}{||\vec{R_i} - \vec{R_j}||^3}(\vec{R_i} - \vec{R_j}) \tag{1}$$

Moreover, following Newtown's law of motion, the acceleration of B_j is :

$$\forall j \in [[1, n]] : \quad m_j \frac{d^2\vec{R_j}}{dt^2} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \vec{F_{i/j}} \tag{2}$$

Therefore, the acceleration of every body in the system is described by the following system :

$$\forall j \in [[1, n]] : \quad \frac{d^2\vec{R_j}}{dt^2} = G * \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{m_i}{||\vec{R_i} - \vec{R_j}||^3}(\vec{R_i} - \vec{R_j}) \tag{3}$$

We want to solve this system for all $\vec{R_j}; \quad i \in [[1, n]]$.

[1]Bruns H.E., 1887

3 Discretization of the time dimension and Central differences approximation

Let δt be an arbitrarily small, positive, time interval and t_0 an initial time. Furthermore let's define $\forall k \in \mathbf{N}: \quad \vec{R_{j,k}} = \vec{R_j}(t_0 + k\delta t)$.
Thus, using the 2^{nd} order central difference approximation we obtain :

$$\forall k \in \mathbf{N}*, \quad \forall j \in [[1,n]]: \quad \frac{d^2\vec{R_{j,k}}}{dt^2} = \frac{\vec{R_{j,k+1}} - 2\vec{R_{j,k}} + \vec{R_{j,k-1}}}{\delta t^2} + O(\delta t^2) \quad (4)$$

Therefore in the discretized time dimension, (3) becomes :

$$\forall k \in \mathbf{N}*, \quad \forall j \in [[1,n]]:$$
$$\frac{\vec{R_{j,k+1}} - 2\vec{R_{j,k}} + \vec{R_{j,k-1}}}{\delta t^2} = G * \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{m_i}{||\vec{R_{i,k}} - \vec{R_{j,k}}||^3}(\vec{R_{i,k}} - \vec{R_{j,k}}) \quad (5)$$

Thus we can express each the position of each body at a certain time step $k \geq 2$ as a function of the position of all the bodies at the time steps $k-1$ and $k-2$:

$$\forall k \in \mathbf{N} - \{\mathbf{0}, \mathbf{1}\}, \quad \forall j \in [[1,n]]:$$
$$\vec{R_{j,k}} = 2\vec{R_{j,k-1}} - \vec{R_{j,k-2}} + \delta t^2 G * \sum_{\substack{i=1 \\ i\neq j}}^{n} m_i \frac{\vec{R_{i,k-1}} - \vec{R_{j,k-1}}}{||\vec{R_{i,k-1}} - \vec{R_{j,k-1}}||^3} \quad (6)$$

4 Initial value problem

In order to use (6) to compute the positions of the bodies, we need their initial positions for the first two time steps, for $k = 0$ and $k = 1$. We will use the initial position and speed of the bodies and derive their position for the time step $k = 1$ from it.
Let's note $\vec{R_{j0}}$ and $\vec{V_{j0}}$ the position and speed vectors for $k = 0$ and $j \in [[1,n]]$. Substituting the speed by it's forward difference approximation we obtain :

$$\forall j \in [[1,n]]: \quad \vec{V_{j0}} = \frac{R_{j,1} - Rj,0}{\delta t} + O(\delta t) \Leftrightarrow \vec{R_{j,1}} = \vec{R_{j0}} + \delta t\vec{V_{j0}} + O(\delta t) \quad (7)$$

It's worth noting that using this approximation causes the leading error to go down from $O(\delta t^2)$ to $O(\delta t)$.

5 Stability and convergence study

6 Heun's scheme and local error evaluation

When two bodies enter in a collision trajectory the equation to solve becomes stiff. The explicit scheme used to solve the ODE system is then inadequate to solve it. In case of a collision between two of the bodies, the influence of the rest is likely to become negligible. A simple two-body model is therefore more adapted for the situation. In order to detect these collision courses and switch the model, we use a higher order scheme to approximate the trajectory with a better precision at each step, and compare the two results.

$$Let \quad F_j : \quad \vec{R_1}, .., \vec{R_n} \to G * \sum_{\substack{i=1 \\ i \neq j}}^{n} m_i \frac{\vec{R_i} - \vec{R_j}}{||\vec{R_i} - \vec{R_j}||^3} \tag{F}$$

We can then rewrite (6) :

$$\forall k \in \mathbf{N} - \{\mathbf{0}, \mathbf{1}\}, \quad \forall j \in [[1, n]] : \\ \vec{R_{j,k}} = 2\vec{R_{j,k-1}} - \vec{R_{j,k-2}} + \delta t^2 F_j(\vec{R_{1,k-1}}, .., \vec{R_{n,k-1}}) \tag{8}$$

Therefore using Heun method gives us :

$$\forall k \in \mathbf{N} - \{\mathbf{0}, \mathbf{1}\}, \quad \forall j \in [[1, n]] : \\ \vec{T_{j,k}} = 2\vec{R_{j,k-1}} - \vec{R_{j,k-2}} + \delta t^2 F_j((\vec{R_i})_{i \in [[1,n]]}) \\ \vec{R_{j,k}} = 2\vec{R_{j,k-1}} - \vec{R_{j,k-2}} + \frac{\delta t^2}{2}(F_j((\vec{R_{i,k-1}})_{i \in [[1,n]]}) + F_j((\vec{T_{i,k-1}})_{i \in [[1,n]]}) \tag{9}$$

We can then get an evaluation of the error at each step :

$$\forall k \in \mathbf{N} - \{\mathbf{0}, \mathbf{1}\}, \quad \forall j \in [[1, n]] : E_{j,k} = \frac{||\vec{R_{j,k}} - \vec{T_{j,k}}||}{||\vec{R_{j,k}}||} \tag{10}$$

7 Practical example : Earth-like celestial body sitting at the L4 point of a binary star system

8 Conclusion and results