EXTENDABLE CODIMENSION TWO SUBVARIETIES IN A GENERAL HYPERSURFACE

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ABSTRACT. We exhibit a class of *extendable* codimension 2 subvarieties in a general hypersurface of dimension at least 4 in projective space. As a consequence, we prove that a general hypersurface of degree d and dimension at least 4 does not support globally generated indecomposable ACM bundles of any rank if their first Chern class $e \ll d$.

1. INTRODUCTION

Let Y be a smooth projective variety and $X \subset Y$ be a smooth subvariety. Relating the geometry of X and Y has been a long standing theme in algebraic geometry. Results in this context are usually referred to as Lefschetz theorems. The best known results are the *Grothendieck-Lefschetz* and *Noether-Lefschetz* theorems. A special case of the Noether-Lefschetz theorem says that for a very general hypersurface $X \subset \mathbb{P}^3$ of degree $d \geqslant 4$, any curve $C \subset X$ is a complete intersection in $\mathbb{P}^3$. In particular, $C = X \cap S$ for a surface $S \subset \mathbb{P}^3$ and thus *extendable* (there is a related notion of extendability in the literature, a very nice survey on which can be found in [Lop23]. See the references therein, especially [Wah87] and [BM87]).

More generally, throughout this article, we will say a codimension k subscheme $Z \subset X$ of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ is *extendable* if $Z = X \cap \Sigma$ scheme-theoretically, where $\Sigma \subset \mathbb{P}^{n+1}$ is a pure codimension k subscheme.

With a view to finding a generalisation of the Noether-Lefschetz theorem, Griffiths and Harris in [GH85], asked whether any curve in a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geqslant 6$ is extendable. The main idea is that codimension 2 subvarieties in projective spaces are already more complicated (for instance, not all of them are defined by 2 homogeneous polynomials) and the expectation was that perhaps the codimension 2 geometry of general hypersurfaces is no more complicated, thus establishing a Lefschetz type result.

C. Voisin in [Voi88] showed the existence of curves in smooth hypersurfaces $X \subset \mathbb{P}^4$ which were not cut out by surfaces in $\mathbb{P}^4$. One of the fundamental differences in these two cases is the following. Consider the normal bundle sequence for the inclusions $C \subset X \subset \mathbb{P}^{n+1}$:

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_C(d) \longrightarrow 0.$$

For smooth hypersurfaces in $\mathbb{P}^3$, this sequence splits if and only if C is extendable and hence a complete intersection (see [GH83]). However, this is no longer true once C is a curve in a smooth hypersurface $X \subset \mathbb{P}^4$. In this case, the splitting of the above sequence only implies that C is *infinitesimally extendable*, i.e., there exists a curve $D \subset X_{(1)}$ where $X_{(1)}$ is the first order thickening of X in $\mathbb{P}^4$ such that $C = D \cap X$. If $C \subset X$ (or more generally a codimension 2 subvariety Z in a smooth hypersurface of dimension $n \geqslant 4$) is, in addition, *arithmetically Cohen Macaulay* (henceforth, we abbreviate this as *ACM*), then it was shown in [MKRR09] that if C extends infinitesimally, then it is in fact extendable in the above sense

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(see Corollary 2 for the precise statement). This fact was used to show the existence of a large class of counterexamples generalising Voisin's examples in [Voi88]. There are also examples of non-extendable subvarieties in higher codimension (see for instance [IN02]).

Coming back to the case of curves in hypersurfaces in $\mathbb{P}^4$, and their extendability, a conjecture in [RT19] proposes that any ACM curve C in a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geqslant 6$ is extendable if the number of generators of the canonical module of the curve C is less than or equal to 2. When the canonical module has a single generator, the curve C is *subcanonical* and the main result of [Rav09] (see also [MKRR07]) states that C is in fact a complete intersection. When the number of generators of the canonical module is 2, barring a few exceptions, this conjecture was settled in [RT22].

Extendability of codimension 2 ACM subvarieties in smooth hypersurfaces is related to a conjecture of Buchweitz-Greuel-Schreyer ([BGS87]) on the non-existence of low rank indecomposable ACM vector bundles and a generalisation of this conjecture (see [Fae13] and [RT19]), results on which are proven, for example, in [Tri16, Tri17, RT19, RT22]. It is also related to the *Ulrich complexity* of hypersurfaces ([Bea00, ES03]); we refer to [Cos17, Bea18, CMR+21] for an overview of this topic, see also [RT22, LR24a, LR24b].

In this article, we exhibit a bigger class of extendable surfaces Z in a general hypersurface $X \subset \mathbb{P}^5$ of degree d. As a consequence, we prove a splitting result for ACM bundles E on X. The expert will immediately see that the results in this article are far from being sharp. Indeed our aim here has been to showcase how well-known and beautiful results available in the literature can be brought together to answer some rather long standing questions of interest.

Conventions. We work over the field of complex numbers $\mathbb{C}$. A *variety* is an integral separated scheme of finite type over $\mathbb{C}$. A *curve* (resp. *surface*) is a variety of dimension one (resp. two).

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2. Statements of the main results

In this section, we provide the statements of our main results. We start by recalling that a subvariety $Z \subset W$ is said *ACM* if $H^i_*(W, I_{Z/W}) = 0$ for $1 \leqslant i \leqslant \dim Z$ where $I_{Z/W}$ is the ideal sheaf of Z in W.

Given any coherent sheaf $\mathcal{F}$ on an ACM variety $Y \subset \mathbb{P}^N$, note that the module of global sections $\Gamma(\mathcal{F}) := \bigoplus_{a\in\mathbb{Z}} H^0(X, \mathcal{F}(a))$ is a finitely generated module over the polynomial ring in $N+1$-variables. Any choice of a set of *generators* $\{s_i\}$ with $\operatorname{degree}(s_i) = a_i$, yields a surjection

$$\bigoplus_i \mathcal{O}_Y(-a_i) \twoheadrightarrow \mathcal{F}$$

which induces a surjection at the level of the Γ modules, i.e., a surjection

$$\bigoplus_i H^0(Y, \mathcal{O}_Y(a - a_i)) \longrightarrow H^0(Y, \mathcal{F}(a)) \text{ for all } a \in \mathbb{Z}.$$

Definition 1. We say that the cohomology of a coherent sheaf $\mathcal{F}$ on an ACM variety $Y \subset \mathbb{P}^N$ is *generated in degree* k (by m sections) if there exists a set of generators $\{s_i\}$ with $\operatorname{degree}(s_i) = k$ (consisting of m elements).

Remark 1. Note that if the cohomology of $\mathcal{F}$ is generated in degree k, then $\mathcal{F}(k)$ is globally generated.

The aim of this article is to prove the following:

Theorem 1. *Let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of dimension $n \geqslant 4$ and degree d. A local complete intersection, ACM codimension 2 subvariety $Z \subset X$ is extendable if there exists a positive integer e such that*

(i) $\binom{e+5}{4} \leqslant 2d-4$,
(ii) $I_{Z/X}(e)$ *is globally generated, and*
(iii) the cohomology of the line bundle $\omega_Z \otimes \omega_X^{-1}$ *is generated in degree* $-e$.

The same conclusion holds when $\dim X = 3$ if condition (ii) above is replaced by the stronger condition that $I_{Z/\mathbb{P}^4}(e)$ is globally generated.

Remark 2. Note that if $N_{Z/X}$ is the normal bundle of $Z \subset X$ in the above, then

$$\omega_Z \otimes \omega_X^{-1} = \det N_{Z/X}.$$

Here's an example of a situation in which such codimension two subvarieties arise. Let E be a globally generated ACM bundle of rank r on a smooth, degree d hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geqslant 4$. Any choice of $r-1$ general sections yields an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0.$$

Here $Z \subset X$ is either empty, or a codimension 2 subvariety defined by the vanishing of these $r-1$ sections, $I_{Z/X}$ is its ideal sheaf and e is the first Chern class of E. If e satisfies the inequality (i) in Theorem 2, then Z, if non-empty, is extendable; i.e., $Z = X \cap \Sigma$ for some pure codimension 2 subscheme $\Sigma \subset \mathbb{P}^{n+1}$. This is because in this case (ii) is obvious as E is assumed to be globally generated, and (iii) can be seen by dualizing the above exact sequence and passing to cohomology (the proof is similar to that of Proposition 1).

The proof of Theorem 2 is based on an induction argument, the main step of which is proving the assertion when $n = 4$. The main ingredient of the proof in this case is the following:

Theorem 2. *Let X be a general hypersurface in $\mathbb{P}^5$ of degree d and let $Z \subset X$ be an ACM local complete intersection surface. Suppose that Z satisfies the following conditions for some integer $e > 0$:*

(i) $\binom{e+5}{4} \leqslant 2d-4$,
(ii) *there exists a smooth member* $\Theta \in |I_{Z/X}(e)|$,
(iii) $I_{Z/X}(e+1)$ *is globally generated, and*
(iv) *the cohomology of* $\omega_Z \otimes \omega_X^{-1}$ *is generated in degree* $-e$.

For a general hyperplane $H \cong \mathbb{P}^4 \subset \mathbb{P}^5$, let $C := Z \cap H$, $Y := X \cap H$ be the hyperplane sections. Then the normal bundle sequence

$$0 \longrightarrow N_{C/Y} \longrightarrow N_{C/\mathbb{P}^4} \longrightarrow \mathcal{O}_C(d) \longrightarrow 0. \tag{1}$$

associated to the inclusions $C \subset Y \subset \mathbb{P}^4$ splits.

When X is a general hypersurface in $\mathbb{P}^4$ of degree d and $C \subset X$ is an ACM curve, we show that the splitting of (1) implies extendability of C. Similar ideas were used by Voisin in [Voi92] in the context of the extendability of curves in K3 surfaces (in the sense discussed in [Lop23]).

As mentioned in the Introduction, extendability of pure codimension 2 subvarieties is intimately related with the splitting of ACM vector bundles (cf. Lemma 1 and Corollary 2). In this direction, we deduce the following by-product of our results:

Theorem 3. *Fix a positive integer* e*. Then a general hypersurface of dimension* $n \geqslant 4$ *and degree* d *satisfying the inequality* $\binom{e+5}{4} \leqslant 2d-4$ *does not support any globally generated ACM bundle* E *with first Chern class* $c_1(E) = \mathcal{O}_X(e)$ *that is not a direct sum of line bundles.*

Our proof of Theorem 1 makes use of the Beauville-Mérindol criterion (see [BM87]) for splitting of short exact sequences, combining it with Green's exactness criterion for Koszul complexes (see [Gre88]).

3. Preliminaries on Hartshorne-Serre correspondence

We recall the Hartshorne-Serre correspondence for codimension 2 subschemes in a smooth variety that will be crucial for us the sequel:

Theorem 4 ([Arr07, Theorem 1])**.** *Let* X *be a smooth, projective variety and* $Z \subset X$ *be a locally complete intersection subvariety of codimension* 2*. Let* L *be a line bundle such that*

(i) $H^2(X, L^{-1}) = 0$*, and*
(ii) $\omega_Z \otimes (\omega_X \otimes L)^{-1}$ *is globally generated by* $(r-1)$ *sections.*

Then there exists a rank r *vector bundle* E *and an exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X} \otimes L \longrightarrow 0. \tag{2}$$

Furthermore, if $H^1(X, L^{-1}) = 0$*, then* E *is unique up to an unique isomorphism.*

Remark 3. If $Z \subset X$ satisfies the assumptions of Theorem 4, then by definition there is a surjection

$$\mathcal{O}_X^{\oplus r-1} \longrightarrow \omega_Z \otimes (\omega_X \otimes L)^{-1}.$$

This is the same map that we obtain as connecting map by dualizing the exact sequence (2).

Remark 4. We note the following that will be used without any further reference:

- $Y \subset \mathbb{P}^N$ is ACM if and only if it is projectively normal and $H^i_*(Y, \mathcal{O}_Y) = 0$ for all $1 \leqslant i \leqslant \dim Y - 1$.
- If $Y \subset \mathbb{P}^N$ is ACM and $X \in |\mathcal{O}_Y(d)|$ then $X \subset \mathbb{P}^N$ is ACM.
- Let $Y \subset \mathbb{P}^N$ be an ACM variety and $X \in |\mathcal{O}_Y(d)|$. Let $Z \subset X$ be a subvariety of codimension 2. By the exact sequence
$$0 \longrightarrow \mathcal{O}_Y(-d) \longrightarrow I_{Z/Y} \longrightarrow I_{Z/X} \longrightarrow 0,$$
we see that $Z \subset X$ is ACM if and only if $Z \subset Y$ is ACM.

Recall that we say $Y \subset \mathbb{P}^N$ is *AG* (i.e. *arithmetically Gorenstein*) if it is ACM and subcanonical (i.e. $\omega_Y = \mathcal{O}_Y(s)$ for some $s \in \mathbb{Z}$). Let us now record the following useful

Proposition 1. *Let* $Y \subset \mathbb{P}^N$ *be a smooth AG variety and let* $X \in |\mathcal{O}_Y(d)|$ *be a smooth hypersurface and assume that* $\dim X = n \geqslant 3$*. Let* $e \in \mathbb{Z}$ *and let* $Z \subset X$ *be an ACM local complete intersection subvariety of codimension* 2 *for which the cohomology of* $\omega_Z \otimes \omega_X^{-1}$ *is generated in degree* $-e$ *by* $(r-1)$ *sections. Then the associated vector bundle* E *(coming from Theorem 4) sitting in the exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0. \tag{3}$$

is ACM. Moreover, E *is globally generated if and only if* $I_{Z/X}(e)$ *is globally generated.*

Proof. Taking dual of (3) gives rise to the 4-term exact sequence

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow E^\vee \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow \mathcal{E}xt^1_X(I_{Z/X}(e), \mathcal{O}_X) \longrightarrow 0. \tag{4}$$

One has the identification $\mathcal{E}xt^1_X(I_{Z/X}, \omega_X) \cong \omega_Z$ using which (4) may be rewritten as

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow E^\vee \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow \ell \longrightarrow 0 \tag{5}$$

where $\ell := \omega_Z \otimes \omega_X^{-1}(-e)$. Also, by assumption, we have

$$H^0(X, \mathcal{O}_X(a)^{\oplus r-1}) \longrightarrow H^0(Z, \ell(a)) \text{ surjects for all } a \in \mathbb{Z} \tag{6}$$

where the map above is induced by the map $\mathcal{O}_X^{\oplus r-1} \longrightarrow \ell$ in (5) (see Remark 3).

Let E_1 be the torsion-free sheaf defined as the cokernel of the injection $\mathcal{O}_X(-e) \longrightarrow E^\vee$ in (5). Breaking up the sequence (5), we obtain the two short exact sequences

$$0 \longrightarrow \mathcal{O}_X(-e) \longrightarrow E^\vee \longrightarrow E_1 \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow E_1 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow \ell \longrightarrow 0. \tag{8}$$

Recall that $H^i_*(X, \mathcal{O}_X) = 0$ for $1 \leqslant i \leqslant n-1$ as $X \subset \mathbb{P}^N$ is ACM (whence AG by adjunction). Passing to the cohomology of (8), we conclude that $H^1_*(X, E_1) = 0$ by (6). Consequently $H^1_*(X, E^\vee) = 0$ by (7) which by duality implies $H^{n-1}_*(X, E) = 0$. It follows that E is ACM since $H^i_*(X, E) = 0$ for $1 \leqslant i \leqslant n-2$ by (3). To see the second assertion, consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{O}_X^{\oplus r-1}) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & H^0(X, E) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & H^0(X, I_{Z/X}(e)) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus r-1} \arrow[r] & E \arrow[r] & I_{Z/X}(e) \arrow[r] & 0.
\end{tikzcd}$$

Since the left vertical map is surjective, it follows that the middle one is surjective if and only if the right one is so, whence the conclusion follows. □

As an useful consequence, we deduce the following:

Corollary 1. *Let the hypotheses be as in Proposition 1. Then the multiplication map*

$$H^0(Z, \ell(a)) \otimes H^0(Z, \mathcal{O}_Z(b)) \longrightarrow H^0(Z, \ell(a+b))$$

is surjective whenever $a, b \geqslant 0$.

Proof. Thanks to (6) (and the fact that $H^0(X, \mathcal{O}_X(m)) \longrightarrow H^0(Z, \mathcal{O}_Z(m))$ is surjective for all m), it is enough to check that

$$H^0(Z, \mathcal{O}_Z(a)) \otimes H^0(Z, \mathcal{O}_Z(b)) \longrightarrow H^0(Z, \mathcal{O}_Z(a+b))$$

is surjective whenever $a, b \geqslant 0$. For this, we note that we have a commutative diagram:

$$\begin{tikzcd}
H^0(Y, \mathcal{O}_Y(a)) \otimes H^0(Y, \mathcal{O}_Y(b)) \arrow[r] \arrow[d] & H^0(Y, \mathcal{O}_Y(a+b)) \arrow[d] \\
H^0(Z, \mathcal{O}_Z(a)) \otimes H^0(Z, \mathcal{O}_Z(b)) \arrow[r] & H^0(Z, \mathcal{O}_Z(a+b))
\end{tikzcd} \tag{9}$$

The horizontal map on the top row is a surjection as $Y \subset \mathbb{P}^N$ is ACM (hence projectively normal), and the vertical maps are surjective since $Z \subset Y$ is ACM. It follows that the bottom horizontal map is also a surjection. □

In what follows, we use the results of this section when $Y = \mathbb{P}^{n+1}$, $n \geqslant 3$, $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface and $L = \mathcal{O}_X(e)$. Note that in this case, we have

$$\ell := \omega_Z \otimes (\omega_X \otimes L)^{-1} = \omega_Z(n+2-d-e)$$

by adjunction.

4. Equivalent characterizations of extendability

In this section, we prove one of the central results that we use to prove our three main theorems. This result and the following corollary are probably well-known to the experts; they had been implicitly used in various articles of the first author, but had not been stated in this explicit form before.

Lemma 1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d, and let $Z \subset X$ be a codimension 2 local complete intersection subvariety defined by the exact sequence*

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0$$

where E is a bundle of rank r.

(1) *If E is a direct sum of line bundles, then Z is extendable.*
(2) *If Z is extendable then the normal bundle sequence*

$$0 \longrightarrow N_{Z/X} \longrightarrow N_{Z/\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_Z(d) \longrightarrow 0 \tag{10}$$

splits.

(3) *If the normal bundle sequence (10) for the inclusions $Z \subset X \subset \mathbb{P}^{n+1}$ splits, then there exists a subscheme $Z_{(1)} \supset Z$ in the first-order thickening $X_{(1)}$ of the hypersurface X in $\mathbb{P}^{n+1}$ such that the following sequence is exact*

$$I_{Z_{(1)}/X_{(1)}}(-d) \xrightarrow{\times f} I_{Z_{(1)}/X_{(1)}} \longrightarrow I_{Z/X} \longrightarrow 0.$$

Furthermore $fI_{Z_{(1)}/X_{(1)}}(-d) = I_{Z/X}(-d)$.

Proof. (1) Since E splits into a sum of line bundles, the map

$$\mathcal{O}_X^{\oplus r-1} \longrightarrow E \cong \bigoplus_{i=1}^{r} \mathcal{O}_X(a_i)$$

lifts to a map

$$\mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus r-1} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n+1}}(a_i).$$

The cokernel of this map is (a twist of) the ideal sheaf of a codimension 2 subscheme $\Sigma \subset \mathbb{P}^{n+1}$ which satisfies the condition that $Z = X \cap \Sigma$ scheme-theoretically. Indeed, arguing locally, suppose $X = \mathrm{Spec}(A/(f))$ and let the defining ideals of $\Sigma \subset \mathbb{P}^{n+1}$ and $Z \subset \mathbb{P}^{n+1}$ be $J \subset A$ and $I \subset A$ respectively. Now, $(f) \subset I$ as $Z \subset X$. Moreover, $I_{\Sigma/\mathbb{P}^{n+1}}$ restricts to $I_{Z/X}$ which implies $\pi(I) = \pi(J)$ where $\pi : A \longrightarrow A/(f)$ is the natural map. This implies $I = J + (f)$, or equivalently, $Z = X \cap \Sigma$. This shows in particular that Σ doesn't have a divisorial component. Since $\mathrm{codim}_{\mathbb{P}^{n+1}}(\Sigma) \leqslant 2$ (see for e.g. [Ott95, Lemma 2.7]), we conclude that Σ is of pure codimension 2, whence Z is extendable.

(2) Since Z is extendable, $Z = \Sigma \cap X$ for some pure codimension 2 subscheme Σ in $\mathbb{P}^{n+1}$. Hence, $Z \in |\mathcal{O}_\Sigma(d)|$, and so we have an inclusion

$$\mathcal{O}_Z(d) \cong N_{Z/\Sigma} \hookrightarrow N_{Z/\mathbb{P}^{n+1}}.$$

We claim that this inclusion composed with the surjection

$$N_{Z/\mathbb{P}^{n+1}} \twoheadrightarrow N_{X/\mathbb{P}^{n+1}|Z} \cong \mathcal{O}_Z(d)$$

yields a splitting of the normal bundle $N_{Z/\mathbb{P}^{n+1}}$ and hence the normal bundle sequence (10) splits. To verify the claim, we start with the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & I_{X/\mathbb{P}^{n+1}} \cong \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \arrow[d] & & \\
0 \arrow[r] & I_{\Sigma/\mathbb{P}^{n+1}} \arrow[r] & I_{Z/\mathbb{P}^{n+1}} \arrow[r] \arrow[d] & I_{Z/\Sigma} \cong \mathcal{O}_\Sigma(-d) \arrow[r] & 0 \\
 & & I_{Z/X} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Here the vertical sequence is the sequence of ideal sheaves for the inclusions $Z \subset X \subset \mathbb{P}^{n+1}$, while the horizontal sequence is for the inclusions $Z \subset \Sigma \subset \mathbb{P}^{n+1}$. We also have from the local description in the proof of (1) above that the composite map

$$I_{\Sigma/\mathbb{P}^{n+1}} \longrightarrow I_{Z/\mathbb{P}^{n+1}} \longrightarrow I_{Z/X}$$

is surjective.

We will now show that the map $I_{X/\mathbb{P}^{n+1}} \longrightarrow I_{Z/\Sigma}$ is surjective and that it is indeed the natural restriction map $\mathcal{O}_{\mathbb{P}^{n+1}}(-d) \longrightarrow \mathcal{O}_\Sigma(-d)$. For this, we complete the diagram above:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & I_{\Sigma/\mathbb{P}^{n+1}}(-d) \arrow[r] \arrow[d] & I_{X/\mathbb{P}^{n+1}} \arrow[d] & & \\
0 \arrow[r] & I_{\Sigma/\mathbb{P}^{n+1}} \arrow[r] \arrow[d] & I_{Z/\mathbb{P}^{n+1}} \arrow[r] \arrow[d] & I_{Z/\Sigma} \arrow[r] & 0 \\
 & I_{Z/X} \arrow[r, equal] \arrow[d] & I_{Z/X} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

It follows from the snake lemma applied to the two vertical short exact sequences that the map $I_{X/\mathbb{P}^{n+1}} \longrightarrow I_{Z/\Sigma}$ is surjective.

To complete the proof of the claim, we consider the diagram obtained by various quotient maps $\mathcal{I} \longrightarrow \mathcal{I}/\mathcal{I}^2$ from an ideal of a subvariety to its conormal sheaf:

$$\begin{tikzcd}
I_{X/\mathbb{P}^{n+1}} \arrow[r, hook] \arrow[d, two heads] & I_{Z/\mathbb{P}^{n+1}} \arrow[r, two heads] \arrow[d, two heads] & I_{Z/\Sigma} \arrow[d, two heads] \\
N^{\vee}_{X/\mathbb{P}^{n+1}} \arrow[r] & N^{\vee}_{Z/\mathbb{P}^{n+1}} \arrow[r] & N^{\vee}_{Z/\Sigma}
\end{tikzcd}$$

The composition of the top horizontal arrows is a surjection as noted in the above paragraph. Since the right most vertical arrow is a surjection, this means that the composition of the bottom horizontal arrows is also a surjection. Consequently, the composition

$$N^{\vee}_{X/\mathbb{P}^{n+1}} \otimes \mathcal{O}_Z \longrightarrow N^{\vee}_{Z/\mathbb{P}^{n+1}} \longrightarrow N^{\vee}_{Z/\Sigma}$$

is also a surjection, and hence an isomorphism since they are both isomorphic to $\mathcal{O}_Z(-d)$.

(3) This is [MKRR09, Lemma 2.2]. $\square$

The key to our analysis is the following (although we will not use (4) in the sequel):

Corollary 2. *Assume in the situation of Lemma 1 that $Z \subset X$ (equivalently E when the conditions of Proposition 1 are satisfied) is ACM. Then the following are equivalent:*

(1) E is a direct sum of line bundles.
(2) Z is extendable.
(3) The normal bundle sequence (10) for the inclusions $Z \subset X \subset \mathbb{P}^{n+1}$ splits.
(4) There exists a subscheme $Z_{(1)} \supset Z$ in the first-order thickening $X_{(1)}$ of the hypersurface X in $\mathbb{P}^{n+1}$ such that the following sequence is exact

$$I_{Z_{(1)}/X_{(1)}}(-d) \xrightarrow{f} I_{Z_{(1)}/X_{(1)}} \longrightarrow I_{Z/X} \longrightarrow 0.$$

Furthermore, $fI_{Z_{(1)}/X_{(1)}}(-d) = I_{Z/X}(-d)$.

Proof. Indeed, by Lemma 1, all that remains to be proved is that (4) implies (1) under the ACM hypothesis. We recall from [MKRR09, Section 2] that there exists a short exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow I_{Z/X} \longrightarrow 0 \tag{11}$$

such that

- F is a direct sum of line bundles,
- $H^0_*(X, F) \longrightarrow H^0_*(X, I_{Z/X})$ is surjective, and
- G is ACM.

By [MKRR09, Lemma 2.3], G extends to a bundle $\mathcal{G}$ on $X_{(1)}$. Applying [MKRR09, Proposition 2.4], we see that G is a direct sum of line bundles. Since F is a direct sum of line bundles, we conclude that the map

$$H^0(X, F^{\vee} \otimes E(-e)) \longrightarrow H^0(X, F^{\vee} \otimes I_{Z/X})$$

induced by the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-e)^{\oplus r-1} \longrightarrow E(-e) \longrightarrow I_{Z/X} \longrightarrow 0 \tag{12}$$

is surjective as $H^1(X, F^{\vee} \otimes \mathcal{O}_X(-e)) = 0$. Thus the map $F \longrightarrow I_{Z/X}$ in (11) lifts to a map $F \longrightarrow E(-e)$. Consequently, defining

$$\widetilde{F} := F \oplus \mathcal{O}_X(-e)^{\oplus r-1}$$

and using snake lemma, we obtain the following diagram with exact rows and columns:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_X(-e)^{\oplus r-1} \arrow[r, equal] \arrow[d] & \mathcal{O}_X(-e)^{\oplus r-1} \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & \widetilde{F} \arrow[r] \arrow[d] & E(-e) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & F \arrow[r] \arrow[d] & I_{Z/X} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{13}
$$

Since G is a direct sum of line bundles, we obtain $\mathrm{Ext}^1(E(-e), G) = 0$ (as E is ACM). Consequently the middle row of (13) is split. Since $\widetilde{F}$ is a direct sum of line bundles, so is E. $\square$

5. Surjectivity via Green's theorem

We now proceed to prove the main technical result that is needed in the proof of Theorem 1. Throughout this section unless stated otherwise, $X \subset \mathbb{P}^5$ is a general hypersurface of degree d, and $Z \subset X$ is an ACM local complete intersection surface. We also assume that

(A) $I_{Z/X}(e+1)$ is globally generated, and
(B) there is a smooth member $\Theta \in |I_{Z/X}(e)|$ (in particular $e \geqslant 1$).

By assumption (B), we have inclusions

$$Z \subset \Theta \subset X \subset \mathbb{P}^5$$

and the corresponding normal bundle sequence

$$0 \longrightarrow N_{Z/\Theta} \longrightarrow N_{Z/X} \longrightarrow N_{\Theta/X|Z} \longrightarrow 0. \tag{14}$$

Since $N_{\Theta/X} \cong \mathcal{O}_\Theta(e)$, taking determinants, we have the identification

$$N_{Z/\Theta} \cong \det N_{Z/X} \otimes \mathcal{O}_Z(-e) \cong \omega_Z \otimes {\omega_\Theta}^{-1} = \ell$$

whence the normal bundle sequence in (14) may be rewritten as

$$0 \longrightarrow \ell \longrightarrow N_{Z/X} \longrightarrow \mathcal{O}_Z(e) \longrightarrow 0. \tag{15}$$

Taking cohomology, we get the sequence

$$0 \longrightarrow H^0(Z, \ell) \longrightarrow H^0(Z, N_{Z/X}) \xrightarrow{\alpha} H^0(Z, \mathcal{O}_Z(e)) \longrightarrow \cdots .$$

Setting $W := \mathrm{Image}(\alpha)$, we have an exact sequence

$$0 \longrightarrow H^0(Z, \ell) \longrightarrow H^0(Z, N_{Z/X}) \longrightarrow W \longrightarrow 0.$$

More generally, twisting (15) with $\mathcal{O}_Z(b)$ for any $b \in \mathbb{Z}$, we also have exact sequences

$$0 \longrightarrow H^0(Z, \ell(b)) \longrightarrow H^0(Z, N_{Z/X}(b)) \longrightarrow W_{b+e} \longrightarrow 0, \tag{16}$$

where

$$W_{b+e} := \mathrm{Image}\left[\, H^0(Z, N_{Z/X}(b)) \longrightarrow H^0(Z, \mathcal{O}_Z(b+e)) \,\right].$$

Evidently $W = W_e$ in the above notation.

Lemma 2. *The vector spaces* W_{b+e} *for* $b > 0$ *are base point free linear subsystems of the space of global sections* $H^0(Z, \mathcal{O}_Z(b+e))$.

Proof. We have commutative diagrams

$$\begin{tikzcd} H^0(Z, N_{Z/X}(b)) \otimes \mathcal{O}_Z \arrow[r, two heads] \arrow[d] & W_{b+e} \otimes \mathcal{O}_Z \arrow[d] \\ N_{Z/X}(b) \arrow[r, two heads] & \mathcal{O}_Z(b+e) \end{tikzcd}$$

with surjective horizontal maps. Arguing as in the proof of [BR22, Proposition 2.1] (which expands on results implicit in [Pac04, Voi96]), we see that $N_{Z/X}(b)$ is globally generated for $b > 0$ and hence the left vertical arrow is surjective for $b > 0$. This implies that the right vertical map is also surjective, i.e., W_{b+e} is a base point free linear subsystem of $H^0(Z, \mathcal{O}_Z(b+e))$ for $b > 0$. □

We have the following preliminary result which will be used in the proof of Theorem 2.

Proposition 2. *Let* $X \subset \mathbb{P}^5$ *be a smooth hypersurface of degree* d *and* $Z \subset X$ *be a local complete intersection surface. Under the assumptions* (A) *and* (B) *above, there is a linear subsystem* $\widetilde{W} \subset H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(e+1))$ *whose base locus is supported on a finite set in the complement* $\mathbb{P}^5 \setminus X$.

Proof. We have the sequence

$$0 \longrightarrow I_{Z/X}(e+1) \longrightarrow \mathcal{O}_X(e+1) \longrightarrow \mathcal{O}_Z(e+1) \longrightarrow 0.$$

Taking global sections, we get an exact sequence

$$0 \longrightarrow H^0(X, I_{Z/X}(e+1)) \longrightarrow H^0(X, \mathcal{O}_X(e+1)) \longrightarrow H^0(Z, \mathcal{O}_Z(e+1) \longrightarrow 0.$$

Note that right exactness follows from the fact that Z is ACM. Let $\widetilde{W}_X$ denote the lift of W_{e+1} under the surjective map

$$H^0(X, \mathcal{O}_X(e+1)) \twoheadrightarrow H^0(Z, \mathcal{O}_Z(e+1)),$$

so that we have an exact sequence

$$0 \longrightarrow H^0(X, I_{Z/X}(e+1)) \longrightarrow \widetilde{W}_X \longrightarrow W_{e+1} \longrightarrow 0.$$

The linear system $\widetilde{W}_X$ is base point free. Indeed, it has no base point on Z by Lemma 2. Since $\widetilde{W}_X$ contains the linear subsystem $H^0(X, I_{Z/X}(e+1))$, using (A) we conclude that $\widetilde{W}_X$ has no base points in the complement $X \setminus Z$ either.

Now let $\widetilde{W}$ denote the lift of W_{e+1} under the surjective map

$$H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(e+1)) \twoheadrightarrow H^0(Z, \mathcal{O}_Z(e+1)),$$

so that we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathbb{P}^5, I_{Z/\mathbb{P}^5}(e+1)) \arrow[r] \arrow[d, two heads] & \widetilde{W} \arrow[r, two heads] \arrow[d] & W_{e+1} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H^0(X, I_{Z/X}(e+1)) \arrow[r] & \widetilde{W}_X \arrow[r, two heads] & W_{e+1} \arrow[r] & 0. \end{tikzcd}$$

In the above, the surjection of the leftmost vertical map follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(e+1-d) \longrightarrow I_{Z/\mathbb{P}^5}(e+1) \longrightarrow I_{Z/X}(e+1) \longrightarrow 0.$$

By the snake lemma, it follows that the middle arrow

$$\widetilde{W} \longrightarrow \widetilde{W}_X$$

is also a surjection (in fact, for $e+1<d$, this map is an isomorphism).

Let T denote the base locus of the linear system $\widetilde{W}$. From the surjection above, it follows that T does not meet the hypersurface X. Since X is ample, we conclude that T is a finite set in the complement $\mathbb{P}^5 \setminus X$. This finishes the proof. $\square$

With hypotheses as in Proposition 2, let $C \subset Y \subset \mathbb{P}^4$ be general hyperlane sections of the inclusions $Z \subset X \subset \mathbb{P}^5$ above. Define the subspaces $W_{b+e,C} \subset H^0(C, \mathcal{O}_C(b+e))$ as the image under the composite map

$$W_{b+e} \hookrightarrow H^0(Z, \mathcal{O}_Z(b+e)) \twoheadrightarrow H^0(C, \mathcal{O}_C(b+e)).$$

Claim 1. *$W_{e+1,C}$ is base point free.*

Proof of Claim 1: For $b=1$, we have a commutative square

$$\begin{array}{ccc} W_{e+1}\otimes \mathcal{O}_Z & \twoheadrightarrow & W_{e+1,C}\otimes \mathcal{O}_C \\ \downarrow\!\!\downarrow & & \downarrow \\ \mathcal{O}_Z(e+1) & \twoheadrightarrow & \mathcal{O}_C(e+1) \end{array} \tag{17}$$

where the left vertical map is surjective by Lemma 2, and the bottom horizontal map, being the restriction map, is surjective. Consequently it follows that the right vertical arrow is a surjection as well. ♠

As before, let $\widetilde{W}_{\mathbb{P}^4}$ be the lift of $W_{e+1,C}$ under the map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(e+1)) \twoheadrightarrow H^0(C, \mathcal{O}_C(e+1)).$$

We then have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{Z/\mathbb{P}^5}(e+1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(e+1) & \longrightarrow & \mathcal{O}_Z(e+1) & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\ 0 & \longrightarrow & I_{C/\mathbb{P}^4}(e+1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(e+1) & \twoheadrightarrow & \mathcal{O}_C(e+1) & \longrightarrow & 0. \end{array}$$

which gives the exact sequence

$$0 \longrightarrow I_{Z/\mathbb{P}^5}(e) \longrightarrow I_{Z/\mathbb{P}^5}(e+1) \longrightarrow I_{C/\mathbb{P}^4}(e+1) \longrightarrow 0.$$

Consequently, we obtain the following surjection as $Z \subset \mathbb{P}^5$ is ACM:

$$H^0(\mathbb{P}^5, I_{Z/\mathbb{P}^5}(e+1)) \twoheadrightarrow H^0(\mathbb{P}^4, I_{C/\mathbb{P}^4}(e+1)).$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbb{P}^5, I_{Z/\mathbb{P}^5}(e+1)) & \longrightarrow & \widetilde{W} & \twoheadrightarrow & W_{e+1} & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow & & \downarrow\!\!\downarrow & & \\ 0 & \longrightarrow & H^0(\mathbb{P}^4, I_{C/\mathbb{P}^4}(e+1)) & \longrightarrow & \widetilde{W}_{\mathbb{P}^4} & \twoheadrightarrow & W_{e+1,C} & \longrightarrow & 0. \end{array}$$

The middle vertical arrow is surjective since the left and right verticals are surjective.

Lemma 3. *With notation as above, $\widetilde{W}_{\mathbb{P}^4} \subset H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(e+1))$ is a base point free linear subsystem.*

Proof. Recall from Proposition 2 that $\widetilde{W}$ is base point free away from a finite set $T \subset \mathbb{P}^5 \setminus X$. Since a general hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$ will avoid the finite set T, and $\widetilde{W} \longrightarrow \widetilde{W}_{\mathbb{P}^4}$ is a surjection by the previous discussion, it follows that $\widetilde{W}_{\mathbb{P}^4}$ is base point free. $\square$

Proposition 3. *With notation as above, if $\binom{e+5}{4} \leqslant 2d-4$, then the multiplication map*

$$W_{d+e-5,C} \otimes H^0(C, \mathcal{O}_C(d)) \longrightarrow H^0(C, \mathcal{O}_C(2d+e-5))$$

is surjective.

Before we work out the proof of Proposition 3, we recall the following result of Green which will play a key role for us:

Theorem 5 ([Gre88, Theorem 2]). *Let $\widetilde{W} \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(a))$ be a base point free linear system. Then the Koszul complex*

$$\bigwedge^{p+1} \widetilde{W} \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k-a)) \longrightarrow \bigwedge^{p} \widetilde{W} \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \longrightarrow \bigwedge^{p-1} \widetilde{W} \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k+a))$$

is exact in the middle provided that $\operatorname{codim}(\widetilde{W}) \leqslant k-p-a$.

Proof of Proposition 3. We have a commutative square where the vertical maps are the multiplication maps, and the horizontal maps are the restriction maps:

$$\begin{tikzcd}
\widetilde{W}_{\mathbb{P}^4} \otimes H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-6)) \arrow[r] \arrow[d, "\widetilde{\mu}"'] & W_{e+1,C} \otimes H^0(C, \mathcal{O}_C(2d-6)) \arrow[d, "\mu"] \\
H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d+e-5)) \arrow[r, two heads] & H^0(C, \mathcal{O}_C(2d+e-5))
\end{tikzcd}$$

Claim 2. *Under the assumptions of Proposition 3, the map*

$$\mu : W_{e+1,C} \otimes H^0(C, \mathcal{O}_C(2d-6)) \longrightarrow H^0(C, \mathcal{O}_C(2d+e-5))$$

is surjective.

Proof of Claim 2: First, recall that $\widetilde{W}_{\mathbb{P}^4}$ is base point free. In Green's result (Theorem 5 above), letting $p=0$, $k=2d+e-5$, and $a=e+1$, we see that if

$$\operatorname{codim}(W_{e+1,C}) \ = \ \operatorname{codim}(\widetilde{W}_{\mathbb{P}^4}) \leqslant \ 2d-6, \tag{18}$$

then the left vertical map $\widetilde{\mu}$ is surjective which implies that the right vertical map μ is surjective as well. To prove (18), note that $W_{e+1,C}$ is a base point free subspace of $H^0(C, \mathcal{O}_C(e+1))$ by Claim 1. As the restriction map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(e+1)) \longrightarrow H^0(C, \mathcal{O}_C(e+1))$$

is a surjection, it follows that

$$\operatorname{codim}(W_{e+1,C}) \leqslant h^0(\mathcal{O}_{\mathbb{P}^4}(e+1)) - 2 = \binom{e+5}{4} - 2.$$

Thus, (18) holds since $\binom{e+5}{4} \leqslant 2d-4$, whence the multiplication map μ is surjective. ♠

We continue with the proof of Proposition 3. Recall that $d > 6$ by hypothesis as $e \geqslant 1$. We have the commutative diagram

$$\begin{tikzcd} W_{e+1,C} \otimes H^0(C, \mathcal{O}_C(1))^{\otimes(2d-6)} \arrow[r, two heads] \arrow[d, "\mu_1"'] & W_{e+1,C} \otimes H^0(C, \mathcal{O}_C(2d-6)) \arrow[d, "\mu"] \\ W_{d+e-5,C} \otimes H^0(C, \mathcal{O}_C(d)) \arrow[r, "\mu_d"] & H^0(C, \mathcal{O}_C(2d+e-5)). \end{tikzcd}$$

That the top horizontal map is surjective follows by a diagram similar to (9). The surjectivity of μ_d now follows by the surjectivity of μ proven in Claim 2. $\square$

6. Proof of Theorem 2 via the Beauville-Mérindol criterion

We recall a very elegant splitting criterion, due to Beauville and Mérindol (see [BM87, Lemme 1]) for a sequence of vector bundles on a curve to be split. Since the proof is very short, we include it to enhance the ease of reading.

Lemma 4 (The Beauville-Mérindol criterion)**.** *Let* C *be a local complete intersection projective curve and*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0 \tag{19}$$

be a short exact sequence of bundles. This sequence splits if

(i) $H^0(C, F) \longrightarrow H^0(C, G)$ *is surjective, and*

(ii) the cup product map

$$\cup : H^0(C, G) \otimes H^0(C, E^\vee \otimes \omega_C) \longrightarrow H^0(C, E^\vee \otimes G \otimes \omega_C)$$

is surjective.

Proof. We first note that the boundary map $H^0(C, G) \xrightarrow{\partial} H^1(C, E)$ yields the map

$$\partial : H^0(C, G) \otimes H^0(C, E^\vee \otimes \omega_C) \longrightarrow \mathbb{C}.$$

The short exact sequence (19) corresponds to an element $\eta \in \mathrm{Ext}^1(G, E) \cong H^1(C, G^\vee \otimes E)$, and via Serre duality, we treat the element η as a map

$$\eta : H^0(C, G \otimes E^\vee \otimes \omega_C) \longrightarrow \mathbb{C}.$$

To this end, we note the following commutative diagram

$$\begin{tikzcd} H^0(C, G) \otimes H^0(C, E^\vee \otimes \omega_C) \arrow[r, "\partial"] \arrow[d, "\cup"'] & \mathbb{C} \\ H^0(C, G \otimes E^\vee \otimes \omega_C) \arrow[ur, "\eta"'] & \end{tikzcd} \tag{20}$$

Consequently, we have that $\partial = \eta \circ \cup$. Since the cup product map $\cup$ is surjective, we have $\eta = 0 \iff \partial = 0$, and the latter is zero by assumption. $\square$

Proof of Theorem 2. By taking a general hyperplane section of the inclusions

$$Z \subset \Theta \subset X \subset \mathbb{P}^5$$

we obtain inclusions

$$C \subset S \subset Y \subset \mathbb{P}^4.$$

Here S is a smooth complete intersection surface.

Recall that the normal bundle $N_{C/Y}$ is a rank 2 bundle and as such we have

$$N^{\vee}_{C/Y} \cong N_{C/Y} \otimes \left(\det N_{C/Y}\right)^{-1} \cong N_{C/Y} \otimes \omega_Y \otimes \omega_C^{-1}.$$

Consequently,

$$N^{\vee}_{C/Y} \otimes \omega_C \cong N_{C/Y} \otimes \omega_Y. \tag{21}$$

By the Beauville-Mérindol criterion, we need to check that

(a) the map $\alpha : H^0(C, N_{C/\mathbb{P}^4}) \longrightarrow H^0(C, \mathcal{O}_C(d))$ is surjective, and
(b) the cup product map

$$H^0(C, \mathcal{O}_C(d)) \otimes H^0(C, N^{\vee}_{C/Y} \otimes \omega_C) \longrightarrow H^0(C, N^{\vee}_{C/Y} \otimes \omega_C(d))$$

is surjective.

Since Y is a *general* hypersurface of degree d in $\mathbb{P}^4$, we have (see, for example, [BMK13, Proposition 3.2])

$$\text{Image}\left[H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \longrightarrow H^0(C, \mathcal{O}_C(d))\right] \subset \text{Image}\left[H^0(\mathbb{P}^4, N_{C/\mathbb{P}^4}(d)) \longrightarrow H^0(C, \mathcal{O}_C(d))\right].$$

Recall that C is ACM, whence the map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \longrightarrow H^0(C, \mathcal{O}_C(d))$$

is surjective, which verifies condition (a).

For (b), using the identification in (21), we are reduced to proving that the cup product map

$$H^0(C, \mathcal{O}_C(d)) \otimes H^0(C, N_{C/Y}(d-5)) \longrightarrow H^0(C, N_{C/Y}(2d-5))$$

is surjective. Let us define

$$V_d := H^0(C, \mathcal{O}_C(d)).$$

The normal bundle sequence for the inclusions $C \subset S \subset Y$

$$0 \longrightarrow \ell_C := \ell \otimes \mathcal{O}_C \longrightarrow N_{C/Y} \longrightarrow \mathcal{O}_C(e) \longrightarrow 0 \tag{22}$$

is obtained by restricting the normal sequence (15)

$$0 \longrightarrow \ell \longrightarrow N_{Z/X} \longrightarrow \mathcal{O}_Z(e) \longrightarrow 0.$$

Taking cohomology, we get the sequence

$$0 \longrightarrow H^0(C, \ell_C) \longrightarrow H^0(C, N_{C/Y}) \xrightarrow{\bar{\alpha}} H^0(C, \mathcal{O}_C(e)) \longrightarrow \cdots.$$

Setting $\overline{W} := \text{Image}(\bar{\alpha})$, we have an exact sequence

$$0 \longrightarrow H^0(C, \ell_C) \longrightarrow H^0(C, N_{C/Y}) \longrightarrow \overline{W} \longrightarrow 0.$$

More generally, twisting (22) with $\mathcal{O}_C(b)$ for any $b \in \mathbb{Z}$, we also have exact sequences

$$0 \longrightarrow H^0(C, \ell_C(b)) \longrightarrow H^0(C, N_{C/Y}(b)) \longrightarrow \overline{W}_{b+e} \longrightarrow 0, \tag{23}$$

where

$$\overline{W}_{b+e} := \text{Image}\left[\, H^0(C, N_{C/Y}(b)) \longrightarrow H^0(C, \mathcal{O}_C(b+e)) \,\right].$$

As before, it is evident that $\overline{W} = \overline{W}_e$ in the above notation.

We have the following commutative diagram, where the vertical maps are the multiplication maps
(24)

$$\begin{tikzcd}
0 \arrow[r] & H^0(C, \ell_C(d-5)) \otimes V_d \arrow[r] \arrow[d] & H^0(C, N_{C/Y}(d-5)) \otimes V_d \arrow[r] \arrow[d] & \overline{W}_{d+e-5} \otimes V_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(C, \ell_C(2d-5)) \arrow[r] & H^0(C, N_{C/Y}(2d-5)) \arrow[r] & \overline{W}_{2d+e-5} \arrow[r] & 0.
\end{tikzcd}$$

To prove (b) above, which is to say that the vertical map in the middle in the above diagram is surjective, it suffices to prove that the left and the right vertical arrows are surjective which we now proceed to prove.

The left vertical map: We first note that $d > 5$ by assumption. Next, we have the exact sequence defining the ACM bundle E associated to Z

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0.$$

Restricting this sequence to the hyperplane section $Y \subset X$, we get the sequence

$$0 \longrightarrow \mathcal{O}_Y^{\oplus r-1} \longrightarrow E \otimes \mathcal{O}_Y \longrightarrow I_{C/Y}(e) \longrightarrow 0.$$

Note that $E \otimes \mathcal{O}_Y$ is also ACM. Taking duals and arguing as in the proof of Proposition 1, we see that ℓ_C is also generated in degree 0. The surjectivity of the left vertical map now follows from Corollary 1.

The right vertical map: Note that by definition $W_{d+e-5,C} \subset \overline{W}_{d+e-5}$, whence the surjectivity of the right vertical map now follows from Proposition 3. □

We end this section with the following observation in the $n = 3$ case.

Remark 5. Let X be a general hypersurface of dimension 3, and $C \subset X$ an ACM local complete intersection curve. Assume that condition (A) above is replaced by the condition

(A') $I_{C/\mathbb{P}^4}(e+1)$ is globally generated, and as before that
(B) there is a smooth member $S \in |I_{Z/X}(e)|$ (in particular $e \geqslant 1$).

Then the above analysis shows that the linear subsystem $W \subset H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(e+1))$ given by the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^4, I_{C/\mathbb{P}^4}(e+1)) \longrightarrow W \longrightarrow \overline{W}_{e+1} \longrightarrow 0$$

is base point free. The same argument as above shows that, by the Beauville-Mérindol criterion, the normal bundle sequence for the inclusions $C \subset X \subset \mathbb{P}^4$ splits.

7. Proofs of Theorem 1 and Theorem 3

Before we provide the proofs of Theorem 1 and Theorem 3, we will need the following elementary result.

Lemma 5. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and $Y \subset \mathbb{P}^n$ be obtained by taking a smooth hyperplane section. Let E be a rank r vector bundle on X. Then*

(1) If E is ACM, then so is its restriction $E \otimes \mathcal{O}_Y$.
(2) If E is ACM and $E \otimes \mathcal{O}_Y$ splits into a sum of line bundles of the form $\mathcal{O}_Y(a)$, then so does E.

Proof. We have the short exact sequence

$$0 \longrightarrow E(-1) \longrightarrow E \longrightarrow E \otimes \mathcal{O}_Y \longrightarrow 0.$$

Taking cohomology, we have the long exact sequence

$$\cdots \longrightarrow H^i(X, E(j)) \longrightarrow H^i(Y, E(j) \otimes \mathcal{O}_Y) \longrightarrow H^{i+1}(X, E(j-1)) \longrightarrow \cdots .$$

Since the extreme terms vanish for $0 < i < n-1$ and $\forall j \in \mathbb{Z}$, (1) follows. For (2), write $E \otimes \mathcal{O}_Y \cong \bigoplus_{j=1}^r \mathcal{O}_Y(a_j)$ and note that the composed map

$$E \longrightarrow E \otimes \mathcal{O}_Y \cong \bigoplus_{j=1}^{r} \mathcal{O}_Y(a_j)$$

lifts to a map

$$E \longrightarrow \bigoplus_{j=1}^{r} \mathcal{O}_X(a_j) \tag{25}$$

via the exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^{r} \mathcal{O}_X(a_j - 1) \longrightarrow \bigoplus_{j=1}^{r} \mathcal{O}_X(a_j) \longrightarrow \bigoplus_{j=1}^{r} \mathcal{O}_Y(a_j) \longrightarrow 0$$

as $H^1_*(X, E^\vee) = 0$. Since (25) is a map between vector bundles of the same rank, we conclude that it is an isomorphism. Indeed, this is a consequence of the fact that the determinant of the map is non-zero as it is so on Y. This implies that E is also a direct sum of line bundles. □

Proof of Theorem 1. The proof is based by induction on the dimension n. Let us first deal with the base case:

Claim 3. *Theorem 1 holds when* $n = 4$.

Proof of Claim 3: Let $Z \subset X$ be a local complete intersection ACM surface satisfying the hypotheses of Theorem 1. By Lemma 1, it is enough to show that E in (3) is a direct sum of line bundles. Note that E is ACM and globally generated by Proposition 1. Pick a general subspace $V_{r-1} \subset H^0(X, E)$ of dimension $r-1$, which by [Ban91] defines a smooth ACM (hence irreducible) surface $Z' \subset X$. Moreover we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z'/X}(e) \longrightarrow 0 \tag{26}$$

Note that the conditions (ii) and (iii) of Theorem 2 continue to hold for $Z' \subset X$.

Now, we note that there is a smooth $\Theta \in |I_{Z'/X}(e)|$ which is obtained by choosing a general r-dimensional subspace $V_r \subset H^0(X, E)$ containing V_{r-1} and taking Θ to be the degeneracy locus $D_{r-1}(\phi)$ where

$$\phi : V_r \otimes \mathcal{O}_X \longrightarrow E.$$

Consequently, the normal bundle sequence for the inclusions $C' \subset Y \subset \mathbb{P}^4$ obtained by taking a general hyperplane section of the inclusions $Z' \subset X \subset \mathbb{P}^5$ splits by Theorem 2, whence $E \otimes \mathcal{O}_Y$ is a direct sum of line bundles by Corollary 2. By Lemma 5, E is a direct sum of line bundles as well. ♠

Let us continue with the proof of Theorem 1. Now we carry out the induction step. Since the assertion is already proven for $n = 4$ in Claim 3, we assume $n \geqslant 5$. Recall the exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r-1} \longrightarrow E \longrightarrow I_{Z/X}(e) \longrightarrow 0,$$

where E is a rank r globally generated ACM bundle on X (see Proposition 1). Setting $X_n := X$, $Z_{n-2} := Z$, and repeatedly restricting this sequence by general hyperplane sections X_i of dimension i, one obtains codimension 2 subvarieties Z_{i-2} of dimension $i-2$, and the exact sequences

$$0 \longrightarrow \mathcal{O}_{X_i}^{\oplus r-1} \longrightarrow E_i \longrightarrow I_{Z_{i-2}/X_i}(e) \longrightarrow 0 \text{ for all } i \geqslant 4, \tag{27}$$

where $E_i := E|_{X_i}$. By Lemma 5, E_i is ACM for $i \geqslant 4$, whence $Z_{i-2} \subset X_i$ is ACM by (27) for i in the same range. As a result, the pair (Z_{i-2}, X_i) satisfies the hypotheses of the Theorem for all $i \geqslant 4$.

Now, E_4 is a direct sum of line bundles by the proof of Claim 3, and by Lemma 5, so is E. Hence Z is extendable by Corollary 2.

When $n = 3$, the proof is as in the proof of the $n = 4$ case, i.e., Claim 3, except that we invoke Remark 5 instead of Theorem 2. □

Proof of Theorem 3. By the arguments in the proof of Theorem 1, it is enough to show that E_4 is a direct sum of line bundles where $E_4 := E|_{X_4}$ and X_4 is a obtained by intersecting $n-4$ general hyperplane sections of X. As E_4 is globally generated with $c_1(E_4) = \mathcal{O}_{X_4}(e)$, a choice of $r-1$ general sections yields an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_4}^{\oplus r-1} \longrightarrow E_4 \longrightarrow I_{Z_2/X_4}(e) \longrightarrow 0,$$

where I_{Z_2/X_4} is the ideal sheaf of a pure codimension 2 smooth ACM subscheme Z_2 in X_4. First assume $Z_2 = \emptyset$ whence $I_{Z_2/X_4} = \mathcal{O}_{X_4}$. In this case, clearly the above exact sequence is split as $H^1_*(X_4, \mathcal{O}_{X_4}) = 0$ whence E_4 is a direct sum of line bundles. So, we may assume $Z_2 \neq \emptyset$, in particular $H^0(X_4, I_{Z_2/X_4}) = 0$. Since $Z_2 \subset X_4$ is ACM, we see that $h^0(Z_2, \mathcal{O}_{Z_2}) = 1$, in particular Z_2 is irreducible. Therefore, applying the proof of Claim 3, we conclude that E_4 is a direct sum of line bundles. □

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