Problem 1

a)

Our model is: $y_t = \alpha y_{t-1} + e_t + \theta e_{t-1}$. Using the lag operator:

$$
\begin{aligned}
y_t (1 - \alpha L) &= e_t (1 + \theta L) \Rightarrow \\
y_t &= \frac{(1 + \theta L)}{(1 - \alpha L)} e_t \\
&= e_t (1 + \theta L) \sum_{j=0}^{\infty} (\alpha L)^j \Rightarrow \\
&= e_t \left(\sum_{j=0}^{\infty} (\alpha L)^j + \sum_{j=0}^{\infty} \theta (\alpha L)^{j+1} \right)
\end{aligned}
$$

By rearranging terms and collecting the terms with the same L^j we find:

$$
\psi_j = \begin{cases} 1 & j = 0 \\ \alpha^{j-1} (\alpha + \theta) & j \geq 1 \end{cases}
$$

b)

Recall that we can write y_t as:

$$
y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}
$$

Since e_t are independent from each other:

$$
\operatorname{var}(y_t) = \operatorname{var}(e_t) \sum_{j=0}^{\infty} \psi_j^2
$$

Now,

$$
\begin{aligned}
\sum_{j=0}^{\infty} \psi_j^2 &= 1 + (\alpha + \theta)^2 \sum_{j=1}^{\infty} \alpha^{2(j-1)} \\
&= 1 + \frac{(\alpha + \theta)^2}{1 - \alpha^2}
\end{aligned}
$$

Let, $\operatorname{var}(e_t) = \sigma^2$, then:

$$
\operatorname{var}(y_t) = \left(1 + \frac{(\alpha + \theta)^2}{1 - \alpha^2} \right) \sigma^2
$$

We now calculate the first five autocovariances:

$$
\begin{aligned}
\gamma(1) &= \mathbb{E}\left[y_t y_{t-1}\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{\infty}\psi_j e_{t-j}\sum_{k=0}^{\infty}\psi_k e_{t-1-k}\right] \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\psi_j\psi_k\mathbb{E}\left[e_{t-j}e_{t-1-k}\right] \\
&= \sum_{j=1}^{\infty}\psi_j\psi_{j-1}\sigma^2 \\
&= \sigma^2\left[(\alpha+\theta)+(\alpha+\theta)^2\sum_{j=2}^{\infty}\alpha^{j-1}\alpha^{j-2}\right] \\
&= \sigma^2\left[(\alpha+\theta)+\frac{(\alpha+\theta)^2}{\alpha^3}\sum_{j=2}^{\infty}\alpha^{2j}\right] \\
&= \sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right]
\end{aligned}
$$

Now to find $\gamma(k)$ for $k \geq 2$ note that:

$$
\begin{aligned}
y_t y_{t-k} &= \alpha y_{t-1}y_{t-k} + e_t y_{t-k} + \theta e_{t-1}y_{t-k} \Rightarrow \\
\gamma(k) &= \mathbb{E}\left[y_t y_{t-k}\right] \\
&= \alpha\gamma(k-1)
\end{aligned}
$$

So we have the following:

$$
\gamma(k) = \alpha^{k-1}\sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right]
$$

for $k \geq 1$. Writing this explicitly for the first 5 k:

$$
\begin{aligned}
\gamma(1) &= \sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right] \\
\gamma(2) &= \alpha\sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right] \\
\gamma(3) &= \alpha^2\sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right] \\
\gamma(4) &= \alpha^3\sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right] \\
\gamma(5) &= \alpha^4\sigma^2(\alpha+\theta)\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right]
\end{aligned}
$$

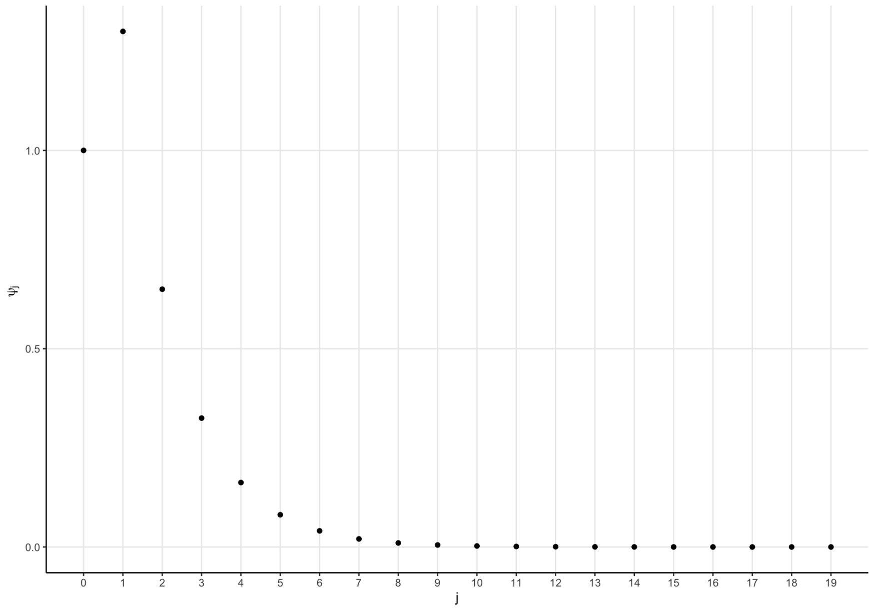
c)

I report the first 5 ψ_j below:

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= 1.3 \\
\psi_2 &= 0.65 \\
\psi_3 &= 0.325 \\
\psi_4 &= 0.1625 \\
\psi_5 &= 0.08125
\end{aligned}
$$

And here is the plot with the first 20 coefficients:

Figure 1: ψ_j



d)

y_t and $\tilde{y}_t$ will be observationally equivalent if they have the same autocovariances function. Recall the expressions we found in part a) for the y_t process:

$$\gamma(0) = \left(1 + \frac{(\alpha+\theta)^2}{1-\alpha^2}\right)\sigma^2$$

$$\gamma(k) = \alpha^{k-1}\sigma^2(\alpha+\theta)\left[1 + \frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right]$$

Rewriting for the $\tilde{y}_t$process:

$$\tilde{\gamma}(0) = \left(1 + \frac{\left(\alpha+\tilde{\theta}\right)^2}{1-\alpha^2}\right)\tilde{\sigma}^2$$

$$\tilde{\gamma}(k) = \alpha^{k-1}\tilde{\sigma}^2\left(\alpha+\tilde{\theta}\right)\left[1 + \frac{\alpha\left(\alpha+\tilde{\theta}\right)}{1-\alpha^2}\right]$$

We first set $\gamma(0) = \tilde{\gamma}(0)$ and isolate $\tilde{\sigma}^2$:

$$\tilde{\sigma}^2 = \left(1 + \frac{(\alpha+\theta)^2}{1-\alpha^2}\right)\frac{\sigma^2}{\left(1 + \frac{\left(\alpha+\tilde{\theta}\right)^2}{1-\alpha^2}\right)} \tag{1}$$

Now, set $\tilde{\gamma}(k) = \gamma(k)$ for $k \geq 1$:

$$\sigma^2(\alpha+\theta)\left[1 + \frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right] = \tilde{\sigma}^2\left(\alpha+\tilde{\theta}\right)\left[1 + \frac{\alpha\left(\alpha+\tilde{\theta}\right)}{1-\alpha^2}\right]$$

Substituting $\tilde{\sigma}^2$ in the expression above and simplifying:

$$(\alpha+\theta)\frac{\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right]}{\left(1+\frac{(\alpha+\theta)^2}{1-\alpha^2}\right)}\left(1+\frac{\left(\alpha+\tilde{\theta}\right)^2}{1-\alpha^2}\right)=\left(\alpha+\tilde{\theta}\right)\left[1+\frac{\alpha\left(\alpha+\tilde{\theta}\right)}{1-\alpha^2}\right]$$

Define the following:

$$\kappa=(\alpha+\theta)\frac{\left[1+\frac{\alpha(\alpha+\theta)}{1-\alpha^2}\right]}{\left(1+\frac{(\alpha+\theta)^2}{1-\alpha^2}\right)}$$
$$x=\left(\alpha+\tilde{\theta}\right)$$

Substituting these in the above expression:

$$\kappa\left(1+\frac{x^2}{1-\alpha^2}\right)=x\left[1+\frac{\alpha x}{1-\alpha^2}\right]\Rightarrow$$
$$x^2(\alpha-\kappa)+x\left(1-\alpha^2\right)-\kappa\left(1-\alpha^2\right)=0$$

This is a quadratic equation in x. The solutions are:

$$x_1=\alpha+\frac{1}{\theta}\Rightarrow\tilde{\theta}_1=\frac{1}{\theta}$$
$$x_2=\alpha+\theta\Rightarrow\tilde{\theta}_2=\theta$$

The second solution is the trivial one. So we focus on the first one to find $\tilde{\sigma}^2$. Substituting $\tilde{\theta}=\frac{1}{\theta}$ in Equation (1) we find:

$$\tilde{\sigma}^2=\sigma^2\frac{1-\alpha^2+(\alpha+\theta)^2}{1-\alpha^2+\left(\alpha+\frac{1}{\theta}\right)^2}$$

To sum up, choosing $\tilde{\theta}=\frac{1}{\theta}$ and $\tilde{\sigma}^2=\sigma^2\frac{1-\alpha^2+(\alpha+\theta)^2}{1-\alpha^2+\left(\alpha+\frac{1}{\theta}\right)^2}$ will make $\tilde{y_t}$ observationally equivalent to y_t.

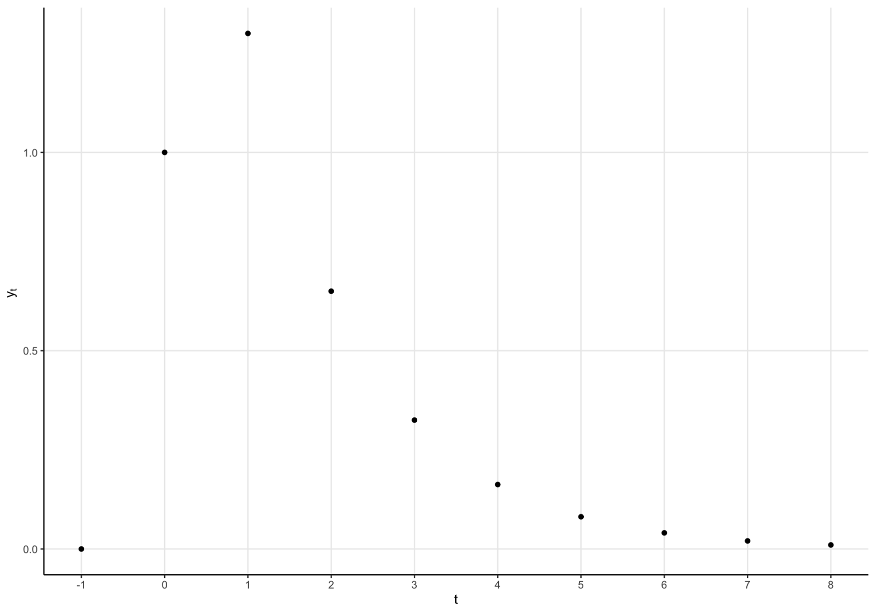
e)

I am assuming I have to compare the results I obtained here with the ψ_j I calculated on part c. As expected they are the same. I report the values of y for the first five instants of time:

$$y_1=1$$
$$y_2=1.3$$
$$y_3=0.65$$
$$y_4=0.325$$
$$y_5=0.1625$$
$$y_6=0.08125$$

And the plot of y_t

Figure 2: y_t



An easy way to see why y_t should match ψ_j is considering the moving average representation of this process:

$$y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$$

Since:

$$e_t = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise} \end{cases}$$

we have:

$$\begin{aligned} y_0 &= \psi_0 e_0 = \psi_0 \\ y_1 &= \psi_0 e_1 + \psi_1 e_0 = \psi_1 \\ y_2 &= \psi_0 e_2 + \psi_1 e_1 + \psi_2 e_0 = \psi_2 \end{aligned}$$

The way we generated y_t is a way of obtaining the impulse response function for this model.

Problem 2

Let $y_{1,t} = \alpha_1 y_{1,t-1} + e_{1,t}$ and $y_{2,t} = \alpha_2 y_{2,t-1} + e_{2,t}$. In what follows, I will assume that $e_{2,t} \perp e_{2,t}$ and $e_{i,t} \sim \left(0, \sigma_i^2\right)$.
Using the lag operator we can write:

$$y_{1,t} \left(1 - \alpha_1 L\right) = e_{1,t}$$

Multiplying both sides by $1 - \alpha_2 L$:

$$y_{1,t} \left(1 - \alpha_1 L\right) \left(1 - \alpha_2 L\right) = e_{1,t} \left(1 - \alpha_2 L\right)$$

Similarly for $y_{2,t}$:

$$y_{2,t} \left(1 - \alpha_2 L\right) \left(1 - \alpha_1 L\right) = e_{2,t} \left(1 - \alpha_1 L\right)$$

Summing both equations:

$$(1-\alpha_2 L)(1-\alpha_1 L)(y_{1,t}+y_{2,t}) = e_{1,t}(1-\alpha_2 L) + e_{2,t}(1-\alpha_1 L)$$

Let $x_t = y_{1,t}+y_{2,t}$, then:

$$x_t - L(\alpha_1+\alpha_2)x_t + \alpha_1\alpha_2 L^2 x_t = e_{1,t} - \alpha_2 e_{1,t-1} + e_{2,t} - \alpha_1 e_{2,t-1} \Rightarrow$$
$$x_t - (\alpha_1+\alpha_2)x_{t-1} + \alpha_1\alpha_2 x_{t-2} = e_{1,t} - \alpha_2 e_{1,t-1} + e_{2,t} - \alpha_1 e_{2,t-1}$$

You can see that the left-hand side of the last equation is an AR(2) process with parameters $\alpha_1+\alpha_2$ and $-\alpha_1\alpha_2$. Now we need to show that the right hand side has the same covariance structure of an MA(1) process. So let $w_t = e_{1,t} - \alpha_2 e_{1,t-1} + e_{2,t} - \alpha_1 e_{2,t-1}$, then:

$$\gamma_w(j) = \begin{cases} \left(1-\alpha_2^2\right)\sigma_1^2 + \left(1-\alpha_1^2\right)\sigma_2^2 & , j=0 \\ -\alpha_2\sigma_1^2 - \alpha_1\sigma_2^2 & , j=1 \\ 0 & , j\geq 2 \end{cases}$$

Which has the same covariance structure as an MA(1). Recall that if $z_t = e_t + \theta e_{t-1}$with $e_t \sim \left(0,\sigma^2\right)$, then:

$$\gamma_z(j) = \begin{cases} \left(1+\theta^2\right)\sigma^2 & , j=0 \\ \theta\sigma^2 & , j=1 \\ 0 & , j\geq 2 \end{cases}$$

In order to express θ, σ as a function of $\alpha_1, \alpha_2, \sigma_1, \sigma_2$ we impose:

$$\gamma_w(0) = \gamma_z(0)$$
$$\gamma_w(1) = \gamma_z(1)$$

Dividing the last two equations and rearranging give us a quadratic equation in θ:

$$\theta^2\left[\alpha_2\sigma_1^2 + \alpha_1\sigma_2^2\right] + \theta\left[\left(1-\alpha_2^2\right)\sigma_1^2 + \left(1-\alpha_1^2\right)\sigma_2^2\right] + \alpha_2\sigma_1^2 + \alpha_1\sigma_2^2 = 0$$

Solving for θ

$$\theta = \frac{-\left(1-\alpha_2^2\right)\sigma_1^2 - \left(1-\alpha_1^2\right)\sigma_2^2 \pm \sqrt{\left[\left(1-\alpha_2^2\right)\sigma_1^2 + \left(1-\alpha_1^2\right)\sigma_2^2\right]^2 - 4\left[\alpha_2\sigma_1^2 + \alpha_1\sigma_2^2\right]^2}}{2\left(\alpha_2\sigma_1^2 + \alpha_1\sigma_2^2\right)}$$

Recall that:

$$\sigma^2 = \frac{-\alpha_2\sigma_1^2 - \alpha_1\sigma_2^2}{\theta}$$

Note that the numerator of the expression we found for both solutions for θ is always negative. So the sign of θ will be determined by the sign of $\alpha_2\sigma_1^2+\alpha_1\sigma_2^2$. In both possible cases $\alpha_2\sigma_1^2+\alpha_1\sigma_2^2 > 0$ and $\alpha_2\sigma_1^2+\alpha_1\sigma_2^2 < 0$, $\sigma^2 > 0$ so both solutions satisfy this condition.
Another condition we should impose is $|\theta| \leq 1$ for invertibility. This will give the criteria to select the appropriate solution for θ.

Problem 3

i)

We estimate the model $y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \alpha_4 y_{t-4} + e_t$ where y_t is the quarterly growth rate for GDP. The model was estimated by OLS and the estimates are shown below:

$$\hat{\alpha_1} = 0.35157$$
$$\hat{\alpha_2} = 0.25944$$
$$\hat{\alpha_3} = 0.03455$$
$$\hat{\alpha_4} = 0.15831$$

We can represent this AR(4) process as an AR(1) 4-vector process: $Y_t = AY_{t-1} + E_t$ where:

$$Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E_t = \begin{bmatrix} e_t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The h step ahead forecast forY_{t+h} is given by:

$$\hat{Y_{t+h}} = A^h Y_t$$

and the forecast errors are given by:

$$\sum_{j=0}^{h-1} A^j E_{t+h-j}$$

and the mean squared error for $\hat{Y_{t+h}}$ is:

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{j=0}^{h-1} A^j E_{t+h-j}\right)\left(\sum_{j=0}^{h-1} A^j E_{t+h-j}\right)'\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{h-1} A^j E_{t+h-j}\right)\left(\sum_{j=0}^{h-1} (A')^j E'_{t+h-j}\right)\right] \\
&= \left(\sum_{j=0}^{h-1} A^j \mathbb{E}\left[E_{t+h-j}E'_{t+h-j}\right](A')^j\right) \\
&= \sigma^2 \left(\sum_{j=0}^{h-1} A^j (A')^j\right)
\end{aligned}$$

where I assumed $e_t \sim \left(0, \sigma^2\right)$.
We estimated σ^2 using the variance of the residuals:

$$\hat{\sigma^2} = 6.247812 \cdot 10^{-5}$$

The forecast and the 95% prediction interval are shown in the following table:

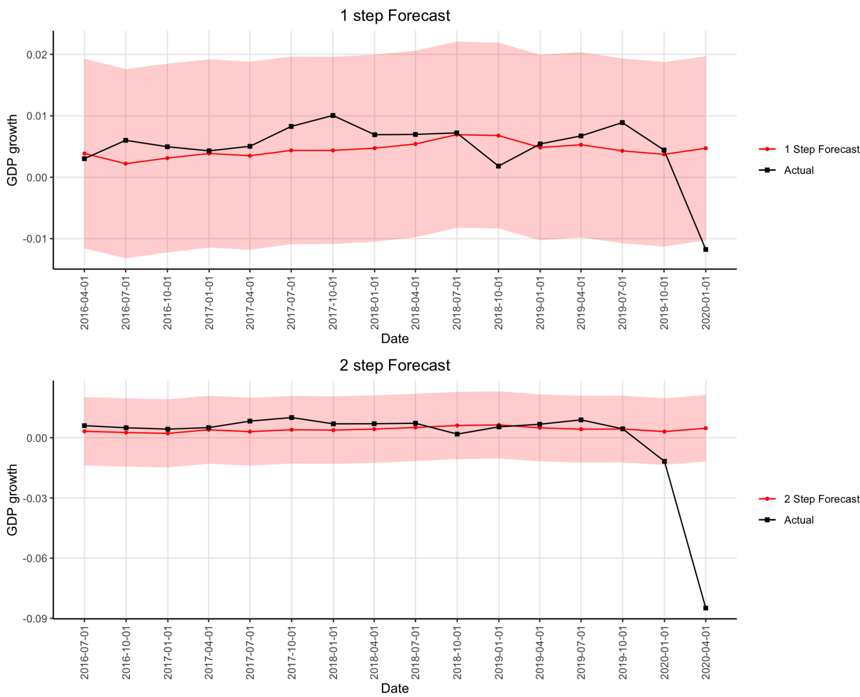
Table 1: Forecasts

h	$\hat{Y}_{t+h}$	IC inf	IC sup
1	0.47%	-1.08%	2.02%
2	0.45%	-1.26%	2.16%

ii)

In order to generate the 1 and 2 step ahead forecast for this model, I fixed the initial date in every estimation sample to 1961-Q1 and increase the end date by 1 quarter in each of the 16 estimation samples.
The following two pictures show the results of our 1 and 2 step ahead forecast together with the 95% confidence interval. As you can see both the 1 and 2 step ahead predictions could not forecast the 2020Q1 and Q2 downturn, since they are outside the 95% confidence interval.

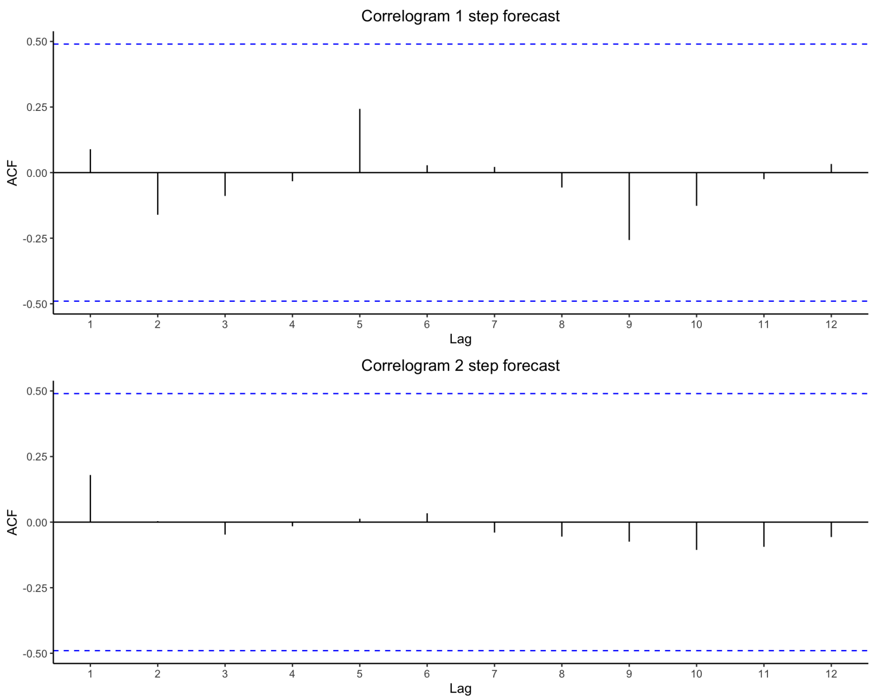
Figure 3: Forecasts



iii)

The correlogram for both errors are shown in the following pictures. The dashed lines are the 2 standard error deviations from 0.
As you can see, both appear to be white noise since we don't have any significant coefficient.

Figure 4: Forecasts



iv)

We compare the predictability of the AR(4) model (Model A) with the predictability of an AR(5) model (Model B).
Let $\hat{e_t}^A$ and $\hat{e_t}^B$ be the forecast errors of model A and B respectively. The two-sided test statistic S is given by:

$$S = \frac{\sqrt{P}\bar{d}}{\text{Avar}(\bar{d})}$$

where $P = 16$ in our case and

$$\bar{d} = \frac{1}{P}\sum_{t=1}^{P}\left(\hat{e_t}^B\right)^2 - \left(\hat{e_t}^A\right)^2$$

Under the null (equal predictability) $S \xrightarrow{d} \mathcal{N}(0,1)$.
I estimate $\text{Avar}(\bar{d})$ using the following:

$$\hat{\text{Avar}}(\bar{d}) = \hat{\gamma_d}(0) + 2\sum_{j=1}^{\tau}\hat{\gamma_d}(j)$$

I used two different estimates: one using only the short run variance $\hat{\gamma_d}(0)$ and another one including one lag : $\hat{\gamma_d}(0) + 2\hat{\gamma_d}(1)$
The results from both were pretty close and gave a p-value of 0.55, so we failed to reject the null with a 5% significance level.