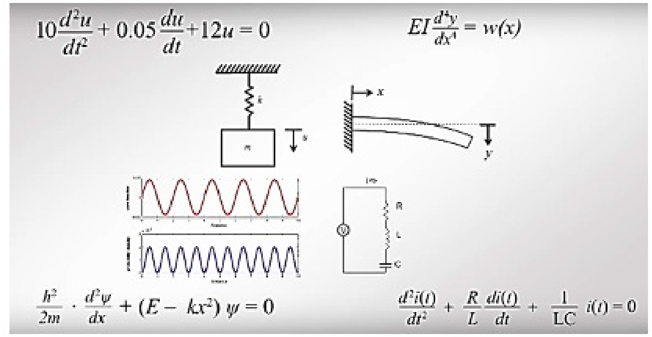
SOLUCION DE SISTEMAS LINEALES

La solución de sistemas de ecuaciones lineales es un tema de gran interés en el mundo de la computación actual. Este tipo de sistemas se encuentran en diversas áreas como optimizaciones lineales y no lineales, análisis de elementos finitos, problemas inversos de valores propios, entre otros.



$$\left.\begin{array}{l}
a_1^1x_1+a_1^2x_2+a_1^3x_3+a_1^4x_4+\cdots a_1^nx_n=b_1\\
a_2^1x_1+a_2^2x_2+a_2^3x_3+a_2^4x_4+\cdots a_2^nx_n=b_2\\
a_3^1x_1+a_3^2x_2+a_3^3x_3+a_3^4x_4+\cdots a_3^nx_n=b_3\\
\vdots\\
a_n^1x_1+a_n^2x_2+a_n^3x_3+a_n^4x_4+\cdots a_n^nx_n=b_n
\end{array}\right\}$$

Sistema de Ecuaciones

$$\rightarrow\begin{pmatrix}
a_1^1 & a_1^2 & a_1^3 & \cdots & a_1^n\\
a_2^1 & a_2^2 & a_2^3 & \cdots & a_2^n\\
a_3^1 & a_3^2 & a_3^3 & \cdots & a_3^n\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
a_n^1 & a_n^2 & a_n^3 & \cdots & a_n^n
\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\\\vdots\\b_n\end{pmatrix}$$

Representación Matricial

$$\rightarrow\left(\begin{array}{ccccc|c}
a_1^1 & a_1^2 & a_1^3 & \cdots & a_1^n & b_1\\
a_2^1 & a_2^2 & a_2^3 & \cdots & a_2^n & b_2\\
a_3^1 & a_3^2 & a_3^3 & \cdots & a_3^n & b_3\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
a_n^1 & a_n^2 & a_n^3 & \cdots & a_n^n & b_n
\end{array}\right)$$

Matriz Ampliada

O de forma más compacta: $AX=b$

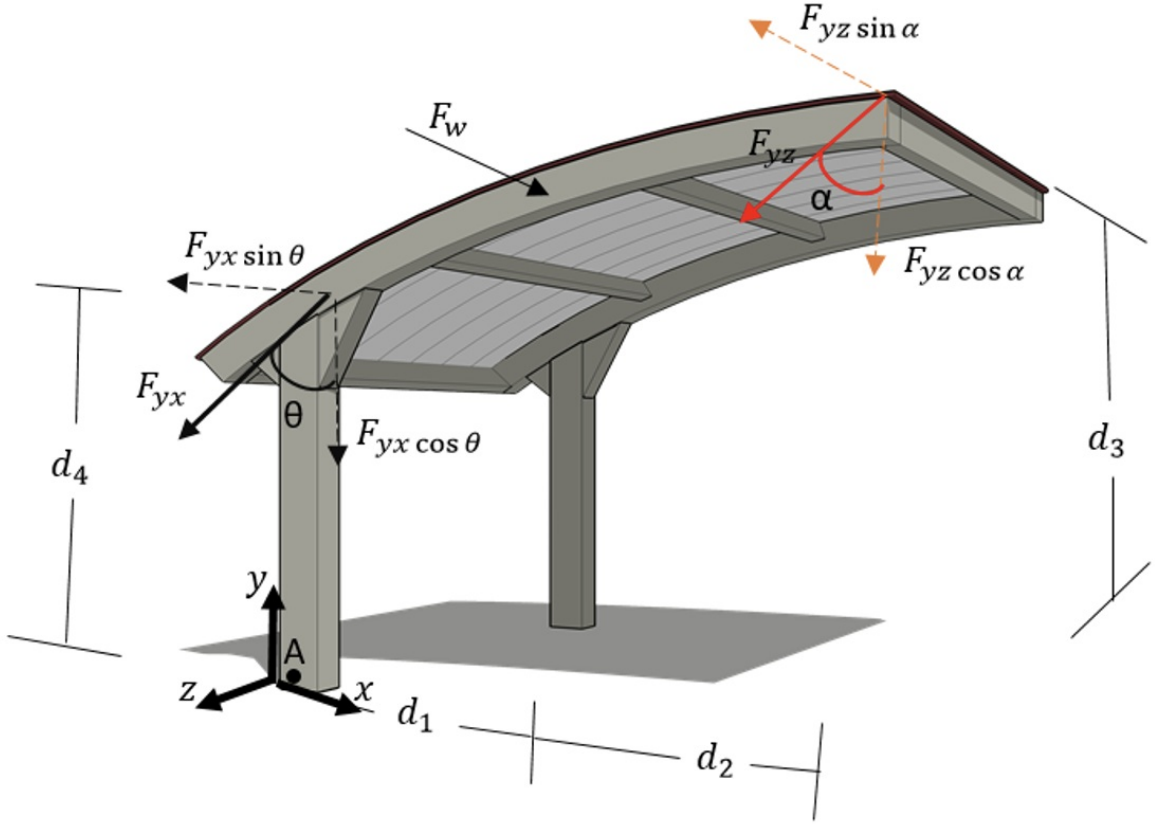
LINEAL:

$$\begin{array}{rcrcrcr}
2x_1 & - & x_2 & + & 2x_3 & = & -1\\
x_1 & & & + & 3x_3 & = & 4\\
-3x_1 & + & 3x_2 & - & 6x_3 & = & 21
\end{array}$$

NO LINEAL:

$$\begin{aligned}
3x_1-\cos(x_2x_3)-\frac{1}{2}&=0,\\
x_1^2-81(x_2+0.1)^2+\sin x_3+1.06&=0,\\
e^{-x_1x_2}+20x_3+\frac{10\pi-3}{3}&=0
\end{aligned}$$

APLICACION



$$\sum M_X = F_{yz} \sin\alpha \, d_3 - F_w \frac{5}{4} d_4 = 0$$

$$\sum M_y = -F_{yz} \sin\alpha \, (d_1 + d_2) + F_w d_1 = 0$$

$$\sum M_z = -F_{yz} \cos\alpha \, (d_1 + d_2) - F_{yx} \cos\theta \left(\frac{d_1}{3}\right) + F_{yx} \sin\theta \, (d_1) = 0$$

CLASIFICACION DE LOS SISTEMAS LINEALES

- **Su tamaño**
  - Pequeños: $n \leq 300$ donde n representa el número de ecuaciones.
  - Grandes: $n > 300$

En cuanto a los métodos de resolución de sistemas de ecuaciones lineales, podemos clasificarlos en

- **Métodos directos**

  Aquellos que resuelven un sistema de ecuaciones lineales en un número finito de pasos.

  Se utilizan para resolver sistemas pequeños.

- **Métodos iterados** Aquellos que crean una sucesión de vectores que convergen a la solución del sistema.

  Se utilizan para la resolución de sistemas grandes

- **Su estructura**
  - Si la matriz posee pocos elementos nulos diremos que se trata de un sistema *lleno*.
  - Si, por el contrario, la matriz contiene muchos elementos nulos, diremos que la matriz y, por tanto, que el sistema es *disperso* o *sparce*. Matrices de este tipo son las denominadas
    * Tridiagonales: $\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}$
    * Triangulares superiores: $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$
    * Triangulares inferiores: $\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$

Dense Matrix

1	2	31	2	9	7	34	22	11	5
11	92	4	3	2	2	3	3	2	1
3	9	13	8	21	17	4	2	1	4
8	32	1	2	34	18	7	78	10	7
9	22	3	9	8	71	12	22	17	3
13	21	21	9	2	47	1	81	21	9
21	12	53	12	91	24	81	8	91	2
61	8	33	82	19	87	16	3	1	55
54	4	78	24	18	11	4	2	99	5
13	22	32	42	9	15	9	22	1	21

no estructurada

Sparse Matrix

1	.	3	.	9	.	3	.	.	.
11	.	4	.	.	.	.	.	2	1
.	.	1	.	.	.	4	.	1	.
8	.	.	.	3	1	.	.	.	.
.	.	.	9	.	.	1	.	17	.
13	21	.	9	2	47	1	81	21	9
.	.	.	.	.	.	.	.	.	.
.	.	.	.	19	8	16	.	.	55
54	4	.	.	.	11	.	.	.	.
.	.	2	.	.	.	.	22	.	21

SPARSE : estructurada

Graphical method

When solving a system with two linear equations in two variables, we are looking for the point where the two lines cross.

Teorema de Rouché–Frobenius

Assume $\mathbf{A}$ is $n \times m$ and consider the linear system

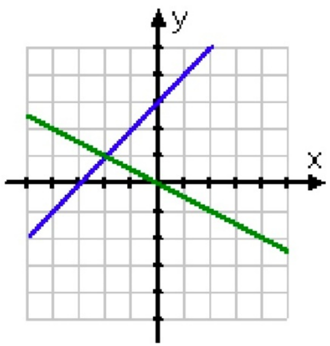
$$\mathbf{Ax} = \mathbf{b}$$

Construct the augmented matrix $\mathbf{B} = [\mathbf{A}\ \mathbf{b}]$ then

Condition	*Solution*
$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = m$	Unique solution $\mathbf{x}$
$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) < m$	Many solutions $\mathbf{x}$
$\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{B})$	No solutions $\mathbf{x}$

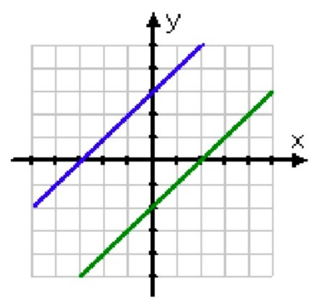
Case 1

Independent system: one solution point



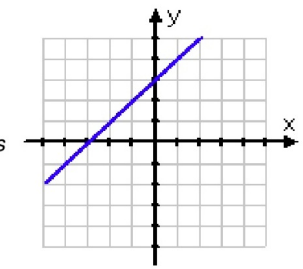
Case 2

Inconsistent system: no solution and no intersection point



Case 3

Dependent system: the solution is the whole line

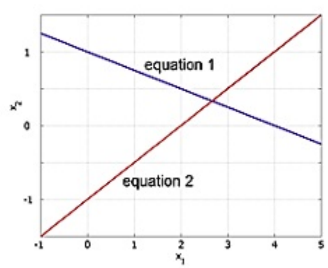


Ejemplo:

A B

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 4 \ \checkmark \\ x_1 - x_2 = 2 \ \checkmark \end{cases}$$



Para recordar:

Matriz singular

The *inverse* $\mathbf{A}^{-1}$ of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is defined such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. If $\mathbf{A}^{-1}$ exists, $\mathbf{A}$ is said to be *nonsingular*. Otherwise, $\mathbf{A}$ is said to be *singular*

$$\det(A) = 0$$

Matriz de cofactores y adjunto

Ejemplo :
Si tenemos el sistema 3x3: B

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 18 \\ 4x_1 + 5x_2 + 6x_3 &= 24 \\ 3x_1 + 1x_2 - 2x_3 &= 4 \end{aligned}$$

Se puede plantear la expresión:

$$A * \vec{x} = \vec{b}$$

Donde **A** es la matriz de coeficientes "**x**" el vector de variables y "**b**" el vector de resultados. Para solucionar el sistema resolvemos para "**x**" :

$$\vec{x} = A^{-1} * \vec{b}$$

$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{pmatrix}$$

Calculamos en primera instancia la matriz de cofactores.
Para el calculo de cada cofactor tenemos:

Para el cofactor C_{11} tenemos:

$$C_{11} = (-1)^{1+1} * \begin{vmatrix} 5 & 6 \\ 1 & -2 \end{vmatrix} = -16$$

$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{pmatrix}$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = +10$$

$-1 \quad -10$

C_{11}

$$matriz\ de\ cofactores = \begin{pmatrix} -16 & 26 & -11 \\ 14 & -22 & 10 \\ -6 & 12 & -6 \end{pmatrix}$$

Calculamos la transpuesta de la matriz de cofactores (cambiar filas por columnas) para hallar la matriz adjunta:

$$Matriz\ Adjunta = \begin{pmatrix} -16 & 14 & -6 \\ 26 & -22 & 12 \\ -11 & 10 & -6 \end{pmatrix}$$

Reemplazamos el Determinante y la Matriz Adjunta en la formula: $A^{-1} = \dfrac{adjA}{|A|}$

$$AX = B \qquad \underbrace{A^{-1}A}_{I}X = A^{-1}B$$

Y la solución del sistema : $\vec{x} = A^{-1} * \vec{b}$ $\quad \boxed{X = A^{-1}B}$

$$\vec{x} = \begin{pmatrix} \frac{-16}{6} & \frac{14}{6} & \frac{-6}{6} \\ \frac{26}{6} & \frac{-22}{6} & \frac{12}{6} \\ \frac{-11}{6} & \frac{10}{6} & \frac{-6}{6} \end{pmatrix} * \begin{pmatrix} 18 \\ 24 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \begin{matrix} \to x_1 \\ \to x_2 \\ \to x_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} =$$

ALTERNATIVAS DE SOLUCION NUMERICA

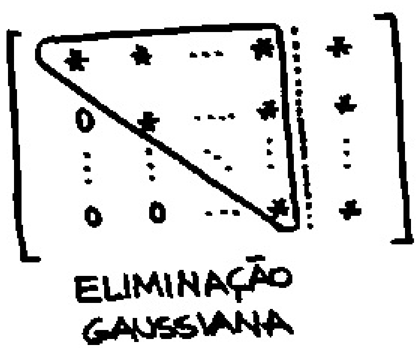
- **Direct Methods for solving (linear) algebraic equations**
  - Gauss Elimination
  - LU decomposition/factorization
  - Error Analysis for Linear Systems and Condition Numbers
  - Special Matrices (Tri-diagonal, banded, sparse, positive-definite, etc)

- **Iterative Methods:** $\boxed{\mathbf{x}^{k+1} = \mathbf{B}\,\mathbf{x}^{k} + \mathbf{c} \qquad k = 0,1,2,\ldots}$

  "Stationary" methods:
  - **Jacobi's method** $\boxed{\mathbf{x}^{k+1} = -\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})\,\mathbf{x}^{k} + \mathbf{D}^{-1}\mathbf{b}}$
  - **Gauss-Seidel iteration** $\boxed{\mathbf{x}^{k+1} = -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U}\,\mathbf{x}^{k} + (\mathbf{D}+\mathbf{L})^{-1}\mathbf{b}}$

ELIMINACION GAUSSIANA

Karl Friedrich Gauss



Los sistemas escalonados son más rápidos de resolver mediante sustitución hacia arriba

Example:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 2x_3 & = & -1 \\ x_1 & & & + & 3x_3 & = & 4 \\ -3x_1 & + & 3x_2 & - & 6x_3 & = & 21 \end{array}$$

Solution:

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{rrr|r} 2 & -1 & 2 & -1 \\ 1 & 0 & 3 & 4 \\ -3 & 3 & 6 & 21 \end{array}\right)$$

augmented matrix

- Forward elimination

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{rrr|r} 2 & -1 & 2 & -1 \\ 1 & 0 & 3 & 4 \\ -3 & 3 & 6 & 21 \end{array}\right) \begin{array}{l} \longrightarrow \\ r_2 - \frac{1}{2}r_1 \\ r_3 + \frac{3}{2}r_1 \end{array}$$

$$\left(\begin{array}{rrr|r} 2 & -1 & 2 & -1 \\ 0 & 1/2 & 2 & 9/2 \\ 0 & 3/2 & 9 & 39/2 \end{array}\right) \begin{array}{l} \longrightarrow \\ \\ r_3 - 3r_2 \end{array}$$

$$\left(\begin{array}{rrr|r} 2 & -1 & 2 & -1 \\ 0 & 1/2 & 2 & 9/2 \\ 0 & 0 & 3 & 6 \end{array}\right) = (\mathbf{U}|\mathbf{y})$$

- Back substitution

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 2x_3 & = & -1 \\ & & \frac{1}{2}x_2 & + & 2x_3 & = & \frac{9}{2} \\ & & & & 3x_3 & = & 6 \end{array}$$

$$\begin{array}{rcl} x_1 & = & -2 \\ x_2 & = & 1 \\ x_3 & = & 2 \end{array}$$

$$\mathbf{x} = (-2, 1, 2)^T$$

Transformaciones elementales de sistemas

Transformaciones elementales	Ejemplo
Permutar dos ecuaciones	$\left.\begin{array}{ll}(E_1) & x-y=3 \\ (E_2) & x+4y=8\end{array}\right\}$ equivale a $\left.\begin{array}{ll}(E_2) & x+4y=8 \\ (E_1) & x-y=3\end{array}\right\}$
Multiplicar una ecuación del sistema por un número distinto de cero	$\left.\begin{array}{ll}(E_1) & x-y=3 \\ (E_2) & x+4y=8\end{array}\right\}$ equivale a $\left.\begin{array}{ll}(2E_1) & 2x-2y=6 \\ (E_2) & x+4y=8\end{array}\right\}$
Sumar a una ecuación del sistema otra multiplicada por un número	$\left.\begin{array}{ll}(E_1) & x-y=3 \\ (E_2) & x+4y=8\end{array}\right\}$ equivale a $\left.\begin{array}{ll}(E_1) & x-y=3 \\ (E_2+3E_1) & 4x+y=17\end{array}\right\}$

Son modificaciones de un sistema lineal que lo transforman en otro.
Ejemplo para un sistema de 2x2:

$$\left.\begin{array}{r}x+2y=7 \\ 3x-4y=1\end{array}\right\} \xrightarrow{E_2 \to E_2-3E_1} \left.\begin{array}{r}x+2y=7 \\ -10y=-20\end{array}\right\} \xrightarrow{E_2/2} \left.\begin{array}{r}x+2y=7 \\ y=2\end{array}\right\} \left.\begin{array}{r}x+2\cdot 2=7 \\ y=2\end{array}\right\}$$

$$\begin{array}{rl}(-3E_1): & -3x-6y=-21 \\ E_2: & 3x-4y=1 \\ \hline E_2-3E_1: & -10y=-20\end{array}$$

Sustituye E_2 por E_2-3E_1

$$\left.\begin{array}{r}x=3 \\ y=2\end{array}\right\}$$

Teorema Todo sistema de m ecuaciones con n incógnitas, puede reducirse a un sistema equivalente del tipo:

$$\left.\begin{array}{r}c_{11}x_1+c_{12}x_2+c_{13}x_3+\ldots\ldots\ldots+c_{1n}x_n=d'_1 \\ c_{22}x_2+c_{23}x_3+\ldots\ldots\ldots+c_{2n}x_n=d'_2 \\ c_{33}x_3+\ldots\ldots\ldots+c_{3n}x_n=d'_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{kk}x_k+\ldots+c_{kn}x_n=d'_k \\ 0=d'_{k+1} \\ \ldots\ldots\ldots\ldots \\ 0=d'_m\end{array}\right\}$$

Ejemplo: eliminacion gausiana

$$AX = B$$

$$A = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & 0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{bmatrix}$$

$$[A \mid B]$$

$$AB \quad \begin{array}{c} \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & 0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right] \end{array} \begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array}$$

$$f_{21} = \frac{A_{21}}{A_{11}}$$ pivote

$$F_2 = F_2 - f_{21} F_1 \quad (1)$$

$$AB = \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.003 & -2.93 & -14.56 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right] \; F_2$$

$$A_{21} = A_{21} - f_{21} A_{11}$$

$$A_{22} = A_{22} - f_{21} A_{12}$$

$$A_{23} = A_{23} - f_{21} A_{13}$$

$$B_2 = B_2 - f_{21} B_1$$

$$f_{31} = \frac{A_{31}}{A_{11}}$$ pivote

$$F_3 = F_3 - f_{31} F_1 \quad (2)$$

$$A_{31} = A_{31} - f_{31} A_{11}$$

$$A_{32} = A_{32} - f_{31} A_{12}$$

$$A_{33} = A_{33} - f_{31} A_{13}$$

$$B_3 = B_3 - f_{31} B_1$$

ELIMINACION GAUSIANA

$$AB = \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.003 & -2.93 & -14.56 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right] \quad F_3$$

$$F_{31} = \frac{A_{31}}{A_{11}}$$

pivote

$$\boxed{F_3 = F_3 - F_{31} F_1} \quad (2)$$

$$A_{31} = A_{31} - F_{31} A_{11}$$
$$A_{32} = A_{32} - F_{31} A_{12}$$
$$A_{33} = A_{33} - F_{31} A_{13}$$
$$B_3 = B_3 - F_{31} B_1$$

$$\begin{array}{cc} & \overbrace{\hphantom{3 \quad -0.1 \quad -0.2}}^{A} \quad B \\ AB = & \left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.003 & -2.93 & -19.56 \\ 0 & -0.19 & 10.02 & 70.6150 \end{array}\right] \end{array} \quad F_3$$

$$F_{32} = \frac{A_{32}}{A_{22}}$$

$$F_3 = F_3 - F_{32} F_2$$

$$A_{31} = A_{31} - F_{32} A_{21}$$
$$A_{32} = A_{32} - F_{32} A_{22}$$
$$A_{33} = A_{33} - F_{32} A_{23}$$
$$B_3 = B_3 - F_{32} B_2$$

$$AB = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{A^*} & B^* \\ 3 & -0.1 & -0.2 & 7.85 \\ 0 & 7.003 & -0.293 & -19.56 \\ 0 & 0 & 10.02 & 70.084 \end{array}\right]$$

$$
\begin{array}{l}
\\
AB
\end{array}
\left[\begin{array}{ccc:c}
\multicolumn{3}{c:}{A^*} & B^* \\
3 & -0.1 & -0.2 & 7.85 \\
0 & 7.003 & -0.293 & -19.56 \\
0 & 0 & 10.02 & 70.084
\end{array}\right]
$$

$$0x_1 + 0x_2 + 10.02x_3 = 7.084$$

$$x_3 = \frac{b_3}{A_{33}} \quad //$$

$$0x_1 + 7.003x_2 - 0.293x_3 = -19.56$$

$$x_2 = \frac{b_2 + A_{23}x_3}{A_{22}} \quad //$$

$$3x_1 - 0.1x_2 - 0.2x_3 = b_1$$

$$x_1 = \frac{b_1 - \overset{-0.1}{\cancel{A_{12}}}x_2 - \overset{-0.2}{\cancel{A_{13}}}x_3}{A_{11}} \quad //$$

FORWARD ELIMINATION

```
Para k=1 ... n-1
    Para i=k+1 ... n
        fik=Aik/Akk
        Para j=k ...  n
            Aij=Aij-fik*Akj
        Fin para
        Bi=Bi-fik*Bk
    Fin para
Fin para
```

Prueba de escritorio

n	k	i	j	f_{ik}	A_{ij}	B_i
3	1	2	$\cancel{1}$	$f_{21} = A_{21}/A_{11}$	$A_{21} = A_{21} - f_{21} A_{11}$	$B_2 = B_2 - f_{21} * B_1$
			$\cancel{2}$		$A_{22} = A_{22} - f_{21} A_{12}$	
			3		$A_{23} = A_{23} - f_{21} A_{13}$	

Backward substitution

```
Xn=Bn/Ann

Para i=(n-1) ... 1
    Suma=Bi
    Para j=i+1  ... n
        Suma=Suma-Aij*Xj
    Fin para
    Xi=Suma/Aii
Fin para
```

Prueba de escritorio:

n	i	j	Suma	x_i
3	2	3	$B_2 - A_{23} x_3$	$x_3 = B_3 / A_{33}$
				$x_2 = \frac{B_2 - A_{23} x_3}{A_{22}}$

Algoritmo para matrices tridiagonales o de Thomas

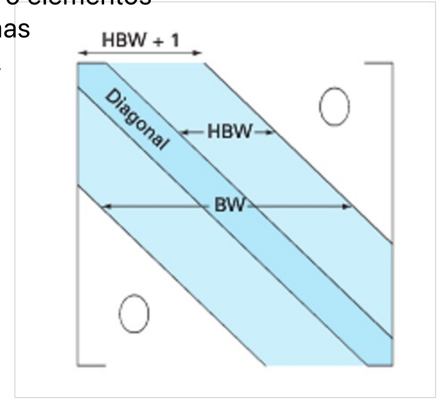
Una matriz tridiagonal se corresponde a un sistema de ecuaciones de la forma

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i,$$

donde $a_1 = 0$ y $c_n = 0$. lo que se puede representar matricialmente como

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}.$$

Este tipo de matrices suelen salir al plantear discretizaciones por métodos de diferencias finitas, volúmenes finitos o elementos finitos de problemas unidimensionales.



Fuente: https://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_(Thomas_algorithm)

El primer paso del método es modificar los coeficientes como sigue:

$$c'_i = \begin{cases} \dfrac{c_i}{b_i} & ; \quad i = 1 \\ \dfrac{c_i}{b_i - c'_{i-1} a_i} & ; \quad i = 2, 3, \dots, n-1 \end{cases}$$

donde se marcan con superíndice ' los nuevos coeficientes.

De igual manera se opera:

$$d'_i = \begin{cases} \dfrac{d_i}{b_i} & ; \quad i = 1 \\ \dfrac{d_i - d'_{i-1} a_i}{b_i - c'_{i-1} a_i} & ; \quad i = 2, 3, \dots, n. \end{cases}$$

a lo que se llama barrido hacia adelante. A continuación se obtiene la solución por sustitución hacia atrás:

$$x_n = d'_n$$

$$x_i = d'_i - c'_i x_{i+1} \qquad ; \; i = n-1, n-2, \dots, 1.$$

```python
def TDMASolve(a, b, c, d):
    n = len(d)  # número de filas

    # Modifica los coeficientes de la primera fila
    c[0] /= b[0]  # Posible división por cero
    d[0] /= b[0]

    for i in range(1, n):
        ptemp = b[i] - (a[i] * c[i-1])
        c[i] /= ptemp
        d[i] = (d[i] - a[i] * d[i-1])/ptemp

    # Sustitución hacia atrás
    x = [0 for i in range(n)]
    x[-1] = d[-1]

    for i in range(-2, -n-1, -1):
        x[i] = d[i] - c[i] * x[i+1]

    return x
```

Aplicable a matrices diagonalmente dominante

Descomposición LU

Definimos la factorización LU de una matriz cuadrada $A \in M_n(\mathbb{R})$, como encontrar dos matrices $L, U \in M_n(\mathbb{R})$, tales que

1. Son triangulares
2. $\mathrm{A} = LU$

- Forma de Doolittle

Obtenida por
Eliminación Gaussiana

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Forma de Crout

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- Forma de Cholesky

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Doolittle Algorithm : LU

Lower Triangular

$$\begin{bmatrix} A00 & A01 & A02 \\ A10 & A11 & A12 \\ A20 & A21 & A22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L10 & 1 & 0 \\ L20 & L21 & 1 \end{bmatrix} \begin{bmatrix} U00 & U01 & U02 \\ 0 & U11 & U12 \\ 0 & 0 & U22 \end{bmatrix}$$

Upper Triangular

Ejemplo

$$x_1 + x_2 + x_3 = 1$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$3x_1 + 5x_2 + 3x_3 = 4$$

Solución:

A = $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ such that A X = C.

Por eliminación gaussiana:

1) $R_2 \to R_2 - 4R_1$

2) $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \Rightarrow \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

3) $R_3 \to R_3 - (-2)R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

L = $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$

Resolver sistema escalonado 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$z_1 = 1$, $z_2 = 2$

$z_3 = 5$

Resolver sistema escalonado 2:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

R/ $x_1 = 1$, $x_2 = 0.5$, $x_3 = -0.5$

Método de Cholesky

descomponer la matriz A en 2 matrices triangulares

$[A] = [L][U]$

A

$$\begin{array}{c|cccccc} & & & j & & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & a_{11} & & & & & \\ 2 & a_{21} & a_{22} & & \text{Sim} & & \\ 3 & a_{31} & a_{32} & a_{33} & & & \\ 4 & a_{41} & a_{42} & a_{43} & a_{44} & & \\ 5 & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \\ 6 & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array}$$

L

$$\begin{array}{c|cccccc} & & & & j & & \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline & l_{11} & 0 & 0 & 0 & 0 & 0 \\ & l_{21} & l_{22} & 0 & 0 & 0 & 0 \\ & l_{31} & l_{32} & l_{33} & 0 & 0 & 0 \\ i & l_{41} & l_{42} & l_{43} & \boxed{l_{44}} & 0 & 0 \\ & l_{51} & l_{52} & l_{53} & l_{54} & l_{55} & 0 \\ k & l_{61} & l_{62} & l_{63} & l_{64} & l_{65} & l_{66} \end{array}$$

terminos fuera de la diagonal

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} l_{kj}}{l_{ii}}$$

terminos en la diagonal

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

$[A][X] = [B]$

$\downarrow$

$[L][U]$

① $[L][Y] = [B]$ $\qquad AX = B$

② $[L^T][X] = [Y]$ $\qquad AX = B$

$l_{64} = ??$

$K = 4 \qquad J = 1, 2, 3$

$$l_{44} = \sqrt{a_{44} - [\,l_{41}^2 + l_{42}^2 + l_{43}^2\,]}$$

$K = 4 \quad i = 1 \quad J = 1$

$$l_{41} = \frac{a_{41}}{l_{11}}$$

$K = 4 \quad i = 2 \quad J = 1 \,..\, 1$

$$l_{42} = \frac{a_{42} - l_{21} l_{41}}{l_{22}}$$

$K = 4 \quad i = 3 \quad J = 1, 2$

$$l_{43} = \frac{a_{43} - [\,l_{31} l_{41} + l_{32} l_{42}\,]}{l_{33}}$$

$K = 2 \quad J = 1 \,..\, 1$

$$l_{22} = \sqrt{a_{22} - l_{21}^2}$$

Solve by Cholesky's method:

$$4x_1 + 2x_2 + 14x_3 = 14$$
$$2x_1 + 17x_2 - 5x_3 = -101$$
$$14x_1 - 5x_2 + 83x_3 = 155.$$

¿positiva definida? SI

¿simetrica? SI

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

$$\det([4]) = \underline{4} > 0$$

$$\det\left(\begin{bmatrix} 4 & 2 \\ 2 & 17 \end{bmatrix}\right) = \underline{64} > 0$$

$$\det\left(\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}\right) = \underline{1600} > 0$$

Métodos Iterativos

- **Direct methods**
  - small systems (n = 100, 1000)
  - full matrix (almost all elements are nonzero)
  - well-conditioned matrices
- **Iterative methods**
  - large systems (n>100, 1000)
  - sparse matrix (containing a lot of zero elements)
  - Ill-conditioned matrices (each iteration can be understood as initial)
  - iterative refinement solution computed by the direct method
  - iterative methods can be generalized for complex problems (with constraints)

Forma General de un Método Iterativo

- The system Ax = b can be written in iterative form

$$\mathbf{x} = \mathbf{C}\mathbf{x} + \mathbf{d}$$

- Choice the initial approximation: $\mathbf{x}^{(0)}$
- Computing of approximations $\mathbf{x}^{(k+1)}$:

Punto Estacionario $$\mathbf{x}^{(k+1)} = \mathbf{C}\mathbf{x}^{(k)} + \mathbf{d}, \qquad \mathbf{k} = 0, 1, 2, \ldots$$

- Stopping criterion: $\| \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} \| \le \varepsilon$,
  where ε > 0 is a prescribed tolerance

Examples of iterative methods includes the Jacobi, Gauss-Seidel methods, and more recent Conjugate Gradient, GMRES Generalized minimal Residual Methods and many others.

Ejemplo 1

$$\left.\begin{array}{rcl} 4x_1 - x_2 + 2x_3 & = & -12 \\ 2x_1 + 5x_2 + x_3 & = & 5 \\ x_1 + x_2 - 3x_3 & = & -4 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} x_1 & = & \frac{1}{4}(-12 + x_2 - 2x_3) \\ x_2 & = & \frac{1}{5}(5 - 2x_1 - x_3) \\ x_3 & = & \frac{1}{-3}(-4 - x_1 - x_2) \end{array}\right.$$

Iterative matrix

$$\boldsymbol{C} = \begin{pmatrix} 0 & 1/4 & -2/4 \\ -2/5 & 0 & -1/5 \\ 1/3 & 1/3 & 0 \end{pmatrix}, \qquad \boldsymbol{d} = \begin{pmatrix} -12/4 \\ 1 \\ 4/3 \end{pmatrix}$$

Recurrent formulas

$$\begin{array}{rcl} x_1^{(k+1)} & = & \frac{1}{4}(-12 + x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} & = & \frac{1}{5}(5 - 2x_1^{(k)} - x_3^{(k)}) \\ x_3^{(k+1)} & = & \frac{1}{-3}(-4 - x_1^{(k)} - x_2^{(k)}) \end{array}$$

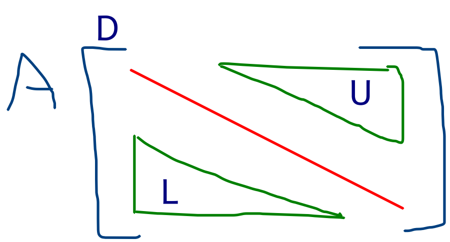
Método Jacobi

- By equations: $\boldsymbol{A} = (a_{ij}),\ a_{ii} \neq 0,\ \boldsymbol{b} = (b_i)$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \qquad i = 1, \ldots, n$$

- Matrix form: $\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{A} = \boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U}$

$$\begin{aligned} (\boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U})\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &= -\boldsymbol{D}^{-1}(\boldsymbol{L} + \boldsymbol{U})\boldsymbol{x} + \boldsymbol{D}^{-1}\boldsymbol{b} \\ \boldsymbol{x}^{(k+1)} &= \underbrace{\boxed{-\boldsymbol{D}^{-1}(\boldsymbol{L} + \boldsymbol{U})}}_{\boldsymbol{C}_J}\boldsymbol{x}^{(k)} + \underbrace{\boxed{\boldsymbol{D}^{-1}\boldsymbol{b}}}_{\boldsymbol{d}_J} \end{aligned}$$



Solución Ejemplo 1

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\| x^{(k+1)} - x^{(k)} \|_R$	
0	0	0	0	-	
1	-3.0000	1.0000	1.3333	**3.0000**	
2	-3.4167	1.9333	0.6667	**0.9333**	
3	-2.8500	2.2333	0.8389	**0.5667**	
4	-2.8611	1.9722	1.1278	**0.2889**	
5	-3.0708	1.9189	1.0370	**0.2097**	
6	-3.0388	2.0209	0.9494	**0.1020**	
7	-2.9694	2.0256	0.9940	**0.0694**	
8	-2.9906	1.9890	1.0187	**0.0367**	
9	-3.0121	1.9925	0.9995	**0.0215**	
10	-3.0016	2.0050	0.9935	**0.0125**	
11	-2.9955	2.0019	1.0011	**0.0077**	$< \varepsilon = 10^{-2}$
	-3	2	1		

Criterios de Parada

- $Ax=b$
- En cualquier iteración k, el término residual

  $r^k=b-Ax^k$
- Verificar la norma del término residual

  $||b-Ax^k||$
- Si esto es menor que la cota del valor de parada

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\| x^{(k+1)} - x^{(k)} \|_R$
0	0	0	0	-
1	-12	5	-4	**12**
2	37	13	-13	**49**
3	-84	-108	-106	**121**
4	344	711	-236	**819**
5	139	-3291	-2003	**4002**
6	286	14894	-4864	**18185**
7	23752	-55279	-34640	**70173**
8	-57267	208257	-107037	**263536**
				divergent

Convergencia

Lemma:

- Jacobi (and also Gauss-Seidel) method **converges** for every initial approximation $x^{(0)}$ if the matrix of the system **Ax = b** is **strictly diagonally dominant**.

Matriz diagonal dominante o de Hadamard

Una matriz cuadrada de orden n $A = (a_{ij})_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,n}}$ *se dice que es una matriz de diagonal dominante o de Hadamard si*

$$|a_{ii}| > \sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ik}| \quad i = 1, 2, \ldots, n$$

Ejemplo 2.3 La matriz $A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & -1 & 5 \end{pmatrix}$ es de diagonal dominante por verificar que

$$3 > 1 + 1 \qquad 2 > 0 + 1 \quad \text{y} \quad 5 > 2 + |-1| = 3 \qquad \square$$

Matrices fundamentales de una matriz

Se denominan *matrices fundamentales* de una matriz A, y se denotan por A_k, a las submatrices constituidas por los elementos de A situados en las k primeras filas y las k primeras columnas, es decir:

$$A_1 = (a_{11}) \qquad A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \cdots$$

Teorema 2.9 *Las matrices fundamentales A_k de una matriz A de diagonal dominante, son también de diagonal dominante.*

Jacobi y Gauss-Seidel

Jacobi method: (Example 1)

$$
\begin{aligned}
x_1^{(k+1)} &= \tfrac{1}{4}(-12 + x_2^{(k)} - 2x_3^{(k)}) \\
x_2^{(k+1)} &= \tfrac{1}{5}(5 - 2x_1^{(k)} - x_3^{(k)}) \\
x_3^{(k+1)} &= \tfrac{1}{-3}(-4 - x_1^{(k)} - x_2^{(k)})
\end{aligned}
$$

Gauss-Seidel method:

$$
\begin{aligned}
x_1^{(k+1)} &= \tfrac{1}{4}(-12 + x_2^{(k)} - 2x_3^{(k)}) \\
x_2^{(k+1)} &= \tfrac{1}{5}(5 - 2x_1^{(k+1)} - x_3^{(k)}) \\
x_3^{(k+1)} &= \tfrac{1}{-3}(-4 - x_1^{(k+1)} - x_2^{(k+1)})
\end{aligned}
$$

$$\left.\begin{array}{rcl} 4x_1 - x_2 + 2x_3 & = & -12 \\ 2x_1 + 5x_2 + x_3 & = & 5 \\ x_1 + x_2 - 3x_3 & = & -4 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} x_1 & = & \frac{1}{4}(-12 + x_2 - 2x_3) \\ x_2 & = & \frac{1}{5}(5 - 2x_1 - x_3) \\ x_3 & = & \frac{1}{-3}(-4 - x_1 - x_2) \end{array}\right.$$

Ejemplo Jacobi

$$\begin{array}{rcl} x_1^{(k+1)} & = & \frac{1}{4}(-12 + x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} & = & \frac{1}{5}(5 - 2x_1^{(k)} - x_3^{(k)}) \\ x_3^{(k+1)} & = & \frac{1}{-3}(-4 - x_1^{(k)} - x_2^{(k)}) \end{array}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lVert x^{(k+1)} - x^{(k)} \rVert_R$	
0	0	0	0	-	
1	-3.0000	1.0000	1.3333	**3.0000**	
2	-3.4167	1.9333	0.6667	**0.9333**	
3	-2.8500	2.2333	0.8389	**0.5667**	
4	-2.8611	1.9722	1.1278	**0.2889**	
5	-3.0708	1.9189	1.0370	**0.2097**	
6	-3.0388	2.0209	0.9494	**0.1020**	
7	-2.9694	2.0256	0.9940	**0.0694**	
8	-2.9906	1.9890	1.0187	**0.0367**	
9	-3.0121	1.9925	0.9995	**0.0215**	
10	-3.0016	2.0050	0.9935	**0.0125**	
11	-2.9955	2.0019	1.0011	**0.0077**	$< \varepsilon = 10^{-2}$
	-3	2	1		

Ejemplo Gauss-Seidel

$$\begin{array}{rcl} x_1^{(k+1)} & = & \frac{1}{4}(-12 + x_2^{(k)} - 2x_3^{(k)}) \\ x_2^{(k+1)} & = & \frac{1}{5}(5 - 2x_1^{(k+1)} - x_3^{(k)}) \\ x_3^{(k+1)} & = & \frac{1}{-3}(-4 - x_1^{(k+1)} - x_2^{(k+1)}) \end{array}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lVert x^{(k+1)} - x^{(k)} \rVert_R$	
0	0	0	0	-	
1	3.0000	-2.2000	-1.0667	3.0000	
2	2.9833	-1.9800	-0.9989	0.2200	
3	3.0044	-2.0020	-0.9992	0.0220	
4	2.9991	-1.9998	-1.0002	0.0054	$< \varepsilon = 10^{-2}$
	-3	2	1		

Método Gauss-Seidel

- By equations: $\boldsymbol{A} = (a_{ij}),\ a_{ii} \neq 0,\ \boldsymbol{b} = (b_i)$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \qquad i = 1, \ldots, n$$

- Matrix form: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{A} = \boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U}$

$$\begin{aligned}
(\boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U})\boldsymbol{x} &= \boldsymbol{b} \\
(\boldsymbol{L} + \boldsymbol{D})\boldsymbol{x} &= -\boldsymbol{U}\boldsymbol{x} + \boldsymbol{b} \\
(\boldsymbol{L} + \boldsymbol{D})\boldsymbol{x}^{(k+1)} &= -\boldsymbol{U}\boldsymbol{x}^{(k)} + \boldsymbol{b} \\
\boldsymbol{x}^{(k+1)} &= \underbrace{\boxed{-(\boldsymbol{L} + \boldsymbol{D})^{-1}\boldsymbol{U}}}_{\boldsymbol{C}_{GS}} \boldsymbol{x}^{(k)} + \underbrace{\boxed{(\boldsymbol{L} + \boldsymbol{D})^{-1}\boldsymbol{b}}}_{\boldsymbol{d}_{GS}}
\end{aligned}$$