

# Free-Electron Lasers with Variable Parameter Wigglers

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**Abstract**—A general discussion of free-electron lasers (FEL's) with variable parameter wigglers is presented with a view towards their potential for the production of high power optical radiation at reasonable efficiency. The theoretical analysis is based upon a one-dimensional relativistic Hamiltonian formulation and is developed in a manner to take advantage of the analogy between the FEL process and radio frequency accelerators. Three promising operational modes are identified and analyzed. The first may be thought of as an electron decelerator and is thought to have the most promise for single-pass devices. Both oscillator and amplifier configurations are studied. The second is based upon adiabatic trapping and detrapping, intended to reduce the spread in electron energy typically induced by the FEL process. The third is based upon the method of phase area displacement. It has the advantage of wide gain bandwidth and small induced energy spread, and is thought to have the most promise for storage ring applications. Generally speaking, it is found that high peak power is intrinsic to these modes of operation. Potential problems from parasitic oscillations analogous to the stimulated Raman effect are analyzed, and some others arising from transverse inhomogeneity are identified.

## I. INTRODUCTION

THE recent successful operation of the “free-electron laser” (FEL) by the group at Stanford and the availability of high power electron beams has stimulated a great deal of interest in the use of the FEL to produce a high power tunable laser beam [1]. When an electron travels through a periodic transverse magnetic field, the electron is given a transverse velocity which allows it to either receive or give energy to the transverse electric field of a plane electromagnetic wave. If the longitudinal velocity of the electron is such that the electron slips behind the radiation wave by one radiation wavelength while traveling a distance of one magnetic field period, the transverse velocity of the electron remains in-phase with the electric field of the radiation. Such electrons will continue to have their energy increased or decreased, depending on their phase relative to the radiation field. Phillips investigated such a device, called the

Ubitron, and used it to produce microwaves [2]. The FEL experiment by the Stanford group [3] used a helical wiggler magnet with a constant magnetic field and periodicity and produced optical radiation in a similar manner as the Ubitron. Most of the analysis of the FEL, as well as the methods envisioned for FEL operation, started with the assumption that the magnetic field and period of the wiggler are constant [4], [5]. A brief review of the properties and limitations of such a device is presented in Section III.

The maximum fractional energy that can be extracted from a constant parameter FEL is equal to  $1/2N$  where  $N$  is the number of wiggler periods, while the resulting energy spread of the beam of electrons emerging from the FEL is comparable to or greater than the mean energy loss. This places severe limitations on the overall gain, efficiency, and modes of operation for the constant parameter FEL. While there are many schemes to overcome some of the limitations of the constant parameter FEL, such as the use of transverse gradients in the magnetic field of the wiggler [6] and the coupling of the FEL with an isochronous storage ring [7], the use of a variable parameter wiggler may be an even more attractive method to remove these limitations.

It is the purpose of this paper to discuss the operation of a variable parameter FEL and to illustrate many of the possible operational modes that may be used to overcome these limitations of the constant parameter FEL. For the case where the wiggler parameters, such as magnetic field and wavelength, are variable, the possible operating modes envisioned for an FEL are considerably different than those for a constant parameter wiggler; for example, it becomes possible to extract a reasonable fraction of the electron's energy with an emerging energy spread a great deal less than the mean energy loss. The method of describing the electrons' motion in this paper is similar to the treatment used by the accelerator physicist in describing the motion of charged particles in a radio frequency accelerating system [8], [9]. We assume from the beginning that the magnetic field and period vary along the wiggler and derive the equations of motion for an electron passing through the FEL, from which we can obtain the energy lost or gained by the electron. The change in the radiation field is taken to be given by the sum of the energies lost by all of the electrons which pass through the FEL.

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In Section II-A, we present the equations that define the energy and relative phase of a resonant or synchronous electron in terms of the wiggler magnetic field, the wiggler period, the optical field, and the optical wavenumbers. In Section II-B, we discuss the motion of the electrons about the synchronous energy, using the Hamiltonian formulation similar to that used in the treatment of charged particle acceleration with radio frequency fields. The summary of the "bucket" or maximum stable phase trajectories for the electrons is presented in Section II-C. The results of Section II are used in Section III to briefly discuss the well-known properties of the constant parameter wiggler, both as an illustration of the use of the Hamiltonian formulation and to obtain the results in terms of notation of this paper.

In Section IV, the use of a variable parameter wiggler to trap a significant fraction of the electron beam in a decelerating bucket is discussed [10]. The wiggler parameters are varied so as to gradually reduce the resonant energy of the trapped electrons, which results in a value for the transfer of energy from the trapped electrons to the radiation approximately equal to the reduction in the resonant energy. This reduction in the resonant energy can be much larger than the energy that can be transferred in the constant parameter wiggler. The operation of a laser oscillator which utilizes this scheme is discussed in Section IV-A, where the amplification of the radiation field in passing through the wiggler just compensates for the reflection loss. Amplifier designs for this trapping and deceleration mode of FEL operation are discussed in Section IV-B. While the trapped electrons emerge with an energy spread comparable to that of the constant parameter FEL, the mean energy loss of electrons is an order of magnitude larger than that from the constant parameter FEL.

In Section V, a method of adiabatically capturing, deceleration, and decapture of the electrons is discussed. The purpose of this scheme is to achieve a small energy spread for the emerging beam, while at the same time to obtain a significant reduction of the average electron energy. Such a scheme would be useful if the same electron beam was reaccelerated with an external source and then passed through the FEL again.

The last method of extracting energy from the electrons, discussed in Section VI, is called phase area displacement, and is one in which the initial resonant energy is below the energy of the injected electrons [11]. The wiggler parameters are varied such that the resonant energy is increased to a final value above the energy of the electrons, a process which can result in a decrease in the mean energy of the electrons while producing a rather modest increase in the energy spread of the beam. While the average energy lost by this process, for monoenergetic electrons, is on the same order as the energy lost from a constant parameter FEL, it has the advantage of being less sensitive to the initial energy spread of the electrons and of producing a lower energy spread in the emerging beam.

There is a possibility of an instability that can arise from the interaction between a radiation signal at a frequency shifted from the main radiation frequency by an amount equal to the oscillation frequency of the electrons in the ponderomotive well;

this instability is investigated in Section VII. The results show that the instability that must build up from noise is probably not important for the amplifier, but in the design of an oscillator some type of damping will probably need to be included.

Some related considerations are presented in the Appendixes. Appendix I gives a more general derivation of the Madey theorem [12] relating small-signal energy gain and spread with application to the variable wiggler. Appendix II presents some mathematical details relating to Section VI. Appendix III is concerned with the requirements on *e*-beam emittance. Appendix IV discusses the justification at high  $\gamma$  for neglecting electrostatic effects. Appendix V discusses a different variable wiggler concept, the Klystron [13], which is related to  $\pi$ -pulse designs [14], in which one wiggler is used for prebunching and a second for energy extraction.

## II. EQUATIONS OF MOTION

The equations of motion of an individual electron are determined by a Hamiltonian,  $\mathcal{H}_0(x, P_x, y, P_y, z, P_z, t)$ , which satisfies a modified Hamiltonian variational principle

$$\delta \int_{t_1}^{t_2} (P_x \dot{x} + P_y \dot{y} + P_z \dot{z} - \mathcal{H}_0) dt = 0. \quad (2.1)$$

Throughout this derivation we shall assume that the electrostatic interaction between the electrons can be neglected, deferring until later the determination of the circumstances in which this assumption is justified. With this assumption we can write

$$\mathcal{H}_0(r, P, t) = c \sqrt{m^2 c^2 + [P - (e/c) A(r, t)]^2} \quad (2.2)$$

where the total energy  $E$  of the electron is to be identified with the Hamiltonian, i.e.,  $E = \mathcal{H}_0$ . We expect many of the various parameters to vary with the longitudinal coordinate  $z$ ; for this reason it is useful to identify  $z$  as the independent variable rather than the time  $t$ . We rewrite (2.1) as

$$\delta \int_{z_1}^{z_2} [P_x x' + P_y y' + (-E)t' - (-P_z)] dz = 0 \quad (2.3)$$

where we define the derivatives with respect to  $z$  with a prime, i.e.,  $x' = dx/dz$ , etc.

It is convenient to designate the quantity  $(-P_z)$  as a new Hamiltonian  $\mathcal{H}_1$  such that

$$\mathcal{H}_1(x, P_x, y, P_y, (-E), t, z) = -P_z \quad (2.4)$$

where  $(-E)$  plays the role of the momentum conjugate to  $t$ , and  $z$  is the independent variable. The solution of (2.2) for  $P_z$  yields

$$\begin{aligned} \mathcal{H}_1 = & -\{(E^2/c^2 - m^2 c^2) - (P_x - e/c A_x)^2 \\ & - (P_y - e/c A_y)^2\}^{1/2} - e/c A_z. \end{aligned} \quad (2.5)$$

For the vector potential, we shall write

$$A = A_w + A_s \quad (2.6)$$

where  $A_w$  refers to the vector potential of the wiggler field and  $A_s$  refers to the signal (or optical) field. We assume these potentials to have the special forms

$$\begin{aligned} A_w &= -A_w(z) \\ &\cdot \left[ \hat{x} \cos \left( \int_0^z k_w(z_1) dz_1 \right) + \hat{y} \sin \left( \int_0^z k_w(z_1) dz_1 \right) \right] \\ A_s &= A_s(z) \left[ \hat{x} \cos \left( \int_0^z k_s(z_1) dz_1 - \omega_s t \right) \right. \\ &\quad \left. - \hat{y} \sin \left( \int_0^z k_s(z_1) dz_1 - \omega_s t \right) \right]. \end{aligned} \quad (2.7)$$

$$(2.8)$$

Equation (2.7) corresponds to a circularly polarized static magnetic field of the sort used in the original Stanford FEL experiments. The assumption of no transverse variation can only be an approximation in practice, but we assume that the electrons are confined to a region in which transverse variation can be neglected. The use of a nonstatic or linearly polarized wiggler field makes no essential difference in the theory, but the assumption of the form (2.7) allows some simplification of the exposition.  $A_w$  and  $k_w$  are both taken to be  $z$  dependent to allow for the use of specialized magnet designs. We assume  $A_w$ ,  $k_w$ ,  $A_s$ , and  $k_s$  to be slowly varying in  $z$  in the sense that it will generally be permissible to neglect  $A'_w = dA_w/dz$  as compared to  $k_w A_w$ , etc. The signal field has also been chosen to be circularly polarized (in the direction that is driven by the FEL amplification process).

The assumption that  $A$  is independent of the transverse coordinate implies that  $\mathbf{P}_\perp = P_x \hat{x} + P_y \hat{y}$  is a constant of the motion. We shall, throughout most of this paper, assume this constant to be zero, an assumption which corresponds to the neglect of transverse velocities in the incoming beam before it enters the wiggler. Some aspects of the effect of nonzero  $P_\perp$  on operation are discussed in Appendix III and applied to a discussion of the relation of the characteristics of realizable electron beams to other design parameters.

Taking account of the above assumptions, we find that the equations of motion for the energy and time can be obtained from the following Hamiltonian:

$$\begin{aligned} \mathcal{H}_1((-E), t, z) &= -mc \\ &\cdot \{-\mu^2(z) + \gamma^2 + 2a_w a_s \cos \psi(z, t)\}^{1/2} \end{aligned} \quad (2.9)$$

where we have introduced the dimensionless vector potentials  $a = eA/mc^2 (mc^2/e = 1706 \text{ G} \cdot \text{cm} = 0.511 \times 10^6 \text{ V})$ ,

$$\mu^2(z) = 1 + a_w^2(z) + a_s^2(z) \quad (2.10)$$

$$\psi = \int_0^z (k_w + k_s) dz_1 - \omega_s t \quad (2.11)$$

and

$$\gamma = E/mc^2. \quad (2.12)$$

With the special assumptions that we have made, the problem has been reduced to a straightforward one of one-dimensional relativistic motion with specified external forces. The quantity,  $m\mu$  may be thought of as an effective electron mass, increased above the rest mass by the transverse kinetic energy.

From Hamilton's equations [see (2.9)],  $\partial \mathcal{H}_1/\partial(-E) = t'$ ,  $\partial \mathcal{H}_1/\partial t = -(-E')$ , and the definitions of  $\psi$  and  $\gamma$  in (2.11) and (2.12), we obtain

$$\psi' = (k_w + k_s) + \frac{\omega_s}{mc^2} \frac{\partial \mathcal{H}_1}{\partial \gamma} \quad (2.13)$$

and

$$\gamma' = -\frac{\omega_s}{mc^2} \frac{\partial \mathcal{H}_1}{\partial \psi}. \quad (2.14)$$

Strong beam-wave interaction occurs when the beam velocity is approximately matched to that of the bunching wave, in which case  $\psi$  is approximately constant. While we expect to allow the various parameters to vary with  $z$ , it will still be the case that the orbits of principal interest will be those which remain in approximate velocity resonance and for which  $\psi$  is slowly varying. For this reason it proves to be more convenient to use  $\psi$  and  $\gamma$  as (canonical) coordinates rather than  $t$  and  $(-E)$  with  $z$  still the independent coordinate. This leads us to use the following Hamiltonian to describe the motion in the  $\psi, \gamma$  coordinates:

$$\mathcal{H}_2 = \gamma(k_w + k_s) - \frac{\omega_s}{c} \{\gamma^2 - \mu^2 + 2a_w a_s \cos \psi\}^{1/2}. \quad (2.15)$$

The equations of motion for  $\psi$  and  $\gamma$  which follow from this new Hamiltonian are

$$\psi' = \frac{\partial \mathcal{H}_2}{\partial \gamma} = (k_w + k_s) - \frac{\omega_s}{\sqrt{\gamma^2 - \mu^2 + 2a_w a_s \cos \psi}} \gamma \quad (2.16)$$

$$\gamma' = -\frac{\partial \mathcal{H}_2}{\partial \psi} = -\frac{\omega_s a_w a_s \sin \psi}{\sqrt{\gamma^2 - \mu^2 + 2a_w a_s \cos \psi}} \quad (2.17)$$

which of course are the same equations that are obtained by the use of (2.13) and (2.14) with the old Hamiltonian  $\mathcal{H}_1(-E, t, z)$ .

For the applications to be discussed in this paper,  $\gamma^2 \gg \mu^2$ ,  $\gamma^2 \gg a_w a_s$ , and  $a_w \gg a_s$ . With these approximations the Hamiltonian may be approximated by

$$\mathcal{H}_3 = \gamma(k_w + \delta k_s) + \frac{\omega_s}{2\gamma c} (\mu^2 - 2a_w a_s \cos \psi) \quad (2.18)$$

with

$$\mu^2 \approx 1 + a_w^2 \text{ and } \delta k_s = k_s - \frac{\omega_s}{c}. \quad (2.19)$$

Evidently the term

$$U_p = \frac{a_w a_s \omega_s}{\gamma c} \cos \left[ \int_0^z (k_s + k_w) dz_1 - \omega_s t \right] \quad (2.20)$$

plays the role of a potential (the "ponderomotive potential"). It may be thought of as providing a bunching force which moves at the resonant velocity:

$$v_r = \frac{\omega_s}{k_w + k_s}. \quad (2.21)$$

The equations of motion that correspond to the approximate Hamiltonian  $\mathcal{H}_3$  are

$$\psi' = (k_w + \delta k_s) - \frac{\omega_s}{2\gamma^2 c} (\mu^2 - 2a_w a_s \cos \psi) \quad (2.22)$$

and

$$\gamma' = -\frac{\omega_s a_s a_w}{\gamma c} \sin \psi. \quad (2.23)$$

The corresponding equations for the particle energy and phase may be written as

$$\begin{aligned} \gamma^2 &= \frac{\omega_s}{2c} \frac{\mu^2 - 2a_w a_s \cos \psi}{(k_w + \delta k_s - \psi')} \\ &\approx \frac{\omega_s \mu^2}{2c(k_w + \delta k_s - \psi')} \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} \psi'' &= (k'_w + \delta k'_s) - \frac{2(k_w + \delta k_s - \psi') a_w}{\mu^2} \\ &\quad \cdot [a'_w + (2k_w + 2\delta k_s - \psi') a_s \sin \psi] \end{aligned} \quad (2.25)$$

where we have used the approximation that  $a_w \gg a_s$ .

The  $z$  dependence of  $a_w$  and  $k_w$  are determined by the physical design of the FEL hardware, and in subsequent sections we will discuss the design considerations upon which these specifications may be based. The optical field amplitude, and, to some extent, the phase, depend explicitly upon the FEL amplification process and will be affected by such additional properties as the design of the optical cavity and the design and magnitude of the electron beam. In keeping with our assumption of the absence of variation in the transverse direction, we shall assume that the electron beam and optical beam overlap perfectly and that the electron beam is also uniform in electron density. Under these conditions the variation of  $a_s$  is determined by energy conservation. Thus, we write

$$\begin{aligned} \frac{c}{4\pi} (|E_s \times B_s|_z - |E_s \times B_s|_0) &= \frac{c}{4\pi} k_s^2 \left( \frac{mc^2}{e} \right)^2 \\ &\quad \cdot (a_s^2(z) - a_s^2(0)) \\ &= n_e mc^2 (\langle \gamma(0) \rangle - \langle \gamma(z) \rangle) \end{aligned} \quad (2.26)$$

or

$$a_s^2(z) = a_s^2(0) + \frac{\omega_p^2}{\omega_s^2} (\langle \gamma(0) \rangle - \langle \gamma(z) \rangle) \quad (2.26)$$

where  $\langle \rangle$  implies averages over initial distributions of energy and phase and  $\omega_p^2 = 4\pi n_e e^2/m$  is the plasma frequency.  $n_e$  is

the electron density measured in the laboratory frame, and we have assumed  $v \approx c$ .

An alternative derivation of (2.26), along with a determination of  $\delta k_s$ , can be determined from the Maxwell equation

$$\frac{\partial_2 A_s}{\partial z^2} - \frac{1}{c^2} \frac{\partial_2 A_s}{\partial t^2} = -\frac{4\pi}{c} J_{\perp}. \quad (2.27)$$

Substitution of the form (2.8) yields

$$\frac{2\omega_s}{c} \delta k_s A_s \hat{e}_1 - 2 \frac{\omega_s}{c} A_s' \hat{e}_2 = \frac{4\pi}{c} J_{\perp} \quad (2.28)$$

where  $\hat{e}_1$  and  $\hat{e}_2$  are defined by

$$\begin{aligned} \hat{e}_1 &= \left[ \hat{x} \cos \left( \int_0^z k_s(z_1) dz_1 - \omega_s t \right) \right. \\ &\quad \left. - \hat{y} \sin \left( \int_0^z k_s(z_1) dz_1 - \omega_s t \right) \right] \\ \hat{e}'_1 &= k_s \hat{e}_2 \end{aligned} \quad (2.29)$$

and we have neglected derivatives of  $A_s'$  and  $\delta k_s$  compared to  $\omega_s/c$ . Because  $\hat{e}_1$  is a unit vector and  $\hat{e}_1 \cdot \hat{e}_2$  vanishes, we find

$$\frac{2\omega_s}{c} \delta k_s A_s = \frac{4\pi}{c} J_{\perp} \cdot \hat{e}_1. \quad (2.30)$$

In order that the form originally chosen for  $A_s$  be strictly correct, it is necessary that the right-hand side of (2.30) be time independent. In actual fact, the  $J_{\perp} \cdot \hat{e}_1$  generated by the assumed form for  $A_s$  will be a periodic function of time with period  $2\pi/\omega_s$ . We shall eliminate the oscillating terms by carrying out a long time average of  $J_{\perp} \cdot \hat{e}_1$ . Such a procedure is valid [15] provided  $\delta k_s \ll k_w$ .

For a single electron, we have

$$J_{\perp} \cdot \hat{e}_1 = ev_{\perp} \cdot \hat{e}_1 \delta(x - x_0(t)) \delta(y - y_0(t)) \delta(z - z_0(t)) \quad (2.31)$$

where

$$v_{\perp} = \nabla P_{\perp} \mathcal{H}_0 \Big|_{P_{\perp}=0} = \frac{-eA}{\gamma mc} \approx \frac{-eA_w}{\gamma mc} \quad (2.32)$$

for  $A_s \ll A_w$ . Hence,

$$\begin{aligned} (J_{\perp} \cdot \hat{e}_1)_{\text{single particle}} &= \frac{e^2 A_w(z)}{\gamma mc} \cos \psi \delta(x - x_0(t)) \\ &\quad \cdot \delta(y - y_0(t)) \delta(z - z_0(t)). \end{aligned} \quad (2.33)$$

We time average by integrating over time from  $-T/2$  to  $T/2$  to obtain

$$\begin{aligned} (J_{\perp} \cdot \hat{e}_1)_{\text{single particle}} &= \frac{e^2 A_w(z)}{v_z T \gamma mc} \cos \psi(z) \\ &\quad \cdot \delta(x - x_0(z)) \delta(y - y_0(z)) \\ &= 0 \quad \text{for } -T/2 < t(z) < T/2 \\ &\quad \text{otherwise.} \end{aligned} \quad (2.34)$$

Summing over all electrons and averaging over the beam cross section to eliminate the transverse  $\delta$  functions, we obtain

$$(J_{\perp} \cdot \hat{e}_1)_{\text{time average}} = \frac{n_e e^2 A_w}{mc} \langle \cos \psi / \gamma \rangle \quad (2.35)$$

where again  $\langle \rangle$  means average over initial  $\psi$  and energy. Substitution of (2.35) with (2.30) yields

$$\delta k_s = \frac{\omega_p^2}{2c\omega_s} \frac{a_w}{a_s} \langle \cos \psi / \gamma \rangle. \quad (2.36)$$

Again from (2.28), we find

$$2 \frac{\omega_s}{c} A'_s = - \frac{4\pi}{c} J_{\perp} \cdot \hat{e}_2$$

and proceeding in a similar manner we find

$$a'_s = \frac{\omega_p^2}{2\omega_s c} a_w \langle \sin \psi / \gamma \rangle. \quad (2.37)$$

To compare (2.37) with (2.26), we note that

$$\frac{d\gamma}{dz} \approx \frac{1}{mc^2} \frac{\partial H_1}{\partial t} = - \frac{a_s a_w \omega_s}{c} (\sin \psi / \gamma)$$

so that

$$\frac{d\langle \gamma \rangle}{dz} = - \frac{a_s a_w \omega_s}{c} \langle \sin \psi / \gamma \rangle. \quad (2.38)$$

The equivalence of (2.37) and (2.38) to (2.26) is apparent. From (2.36) and (2.37), we have

$$\delta k_s = \frac{a'_s \langle \cos \psi / \gamma \rangle}{a_s \langle \sin \psi / \gamma \rangle} \quad (2.39)$$

so that the assumption  $a'_s/a_s \ll k_w$  made at the outset would appear to typically imply  $\delta k_s \ll k_w$  as well. Since we expect the particles to bunch in the  $0 < \psi < \pi/2$  range,  $\langle \cos \psi / \gamma \rangle$  will be positive. This implies a tendency of the electron beam to trap the optical beam and hence to counter to some extent the effects of diffraction.

The equations of motion derived above [(2.24) and (2.25)] are valid (under the special assumptions that were made) for every electron, and given the initial conditions  $\psi(0)$  and  $\gamma(0)$  for every electron along with the expressions for  $k_w$ ,  $a_w$ ,  $a_s$ , and  $\delta k_s$ , these equations may be integrated to yield the values of  $\psi$  and  $\gamma$  as functions of the longitudinal position  $z$ . Of course, for a high gain FEL it is necessary to make sure that the signal field  $a_s$  used in the equations of motion is also a self-consistent solution to (2.26) and (2.39); by using a computer one can solve the equations of motion for a large number of particles and obtain the final coordinates in phase space for every particle as well as the final signal field and phase. In principle, this procedure could be used to choose the optimum functional form for  $k_w(z)$  and  $a_w(z)$ . It is possible to gain physical insight into the content of these equations without the need of integrating the equations of motion for a large number of initial conditions by referring the electrons energy variable  $\gamma$  to the synchronous or resonant value  $\gamma_r$ . This approach is extremely useful in determining the functions  $k_w(z)$  and  $a_w(z)$  to be used in different modes of operation for the FEL.

#### A. Definition of Synchronous Energy and Phase

The synchronous energy  $\gamma_r$  and phase  $\psi_r$  are defined by

$$\gamma_r^2 = \frac{\omega_s \mu^2}{2(k_w + \delta k_s)c} \quad (2.40)$$

and

$$\frac{d\gamma_r}{dz} = - \frac{a_w a_s \omega_s}{\gamma_r c} \sin \psi_r. \quad (2.41)$$

If (2.41) yields a  $z$  independent value for  $\psi_r$ , then substitution in (2.24) and (2.25) shows that a particle with values of  $\gamma$  and  $\psi$  equal to  $\gamma_r$  and  $\psi_r$ , respectively, will retain these values throughout its motion through the wiggler. Such a particle may be thought of as a synchronous particle. More generally, as will become apparent later, if  $\gamma - \gamma_r$ ,  $\psi - \psi_r$  are initially small, they tend to remain small.

It is possible to look at (2.40) and (2.41) as definitions of  $\gamma_r$  and  $\psi_r$  assuming that  $k_w$ ,  $a_w$ ,  $a_s$ , and  $\delta k_s$  are known functions of  $z$ . However, it is also possible to consider these as design equations where the wiggler functions  $k_w$  and  $a_w$  are to be determined to achieve the desired functions  $\gamma_r$ ,  $\psi_r$ , and  $a_s$ .

Much of the remainder of the paper will deal with different ways to choose  $\psi_r$  and  $\gamma_r$  and hence the wiggler parameters for various operating modes of the FEL. We first must study the motion of electrons with phase coordinates different from the synchronous values and demonstrate that for sufficiently small deviations the electrons will perform stable oscillations about the synchronous values. It must be noted that one is restricted in the choice for the functions  $\gamma_r$  and  $\psi_r$ ; for example,  $\psi_r$  is not defined if

$$\left| \frac{d\gamma_r}{dz} \right| > \left| \frac{\omega_s a_w a_s}{c \gamma_r} \right|.$$

#### B. Motion About the Synchronous Energy

In this section we study the motion of the electrons about the synchronous energy by writing

$$\gamma = \delta \gamma + \gamma_r. \quad (2.42)$$

We regard  $\delta \gamma$  as the new canonical momentum with a new Hamiltonian  $H_4$  given by

$$H_4(\psi, \delta \gamma, z) = H_3(\psi, \gamma_r + \delta \gamma, z) + \psi \frac{d\gamma_r}{dz}. \quad (2.43)$$

With the definitions for  $\gamma_r$  and  $\psi_r$  as chosen in (2.40) and (2.41) and with the assumption of  $\delta \gamma \ll \gamma_r$ , the new Hamiltonian  $H_4$ , which is derived from the approximate Hamiltonian  $H_3$  [see (2.18)], is

$$H_4 = \frac{k_w + \delta k_s}{\gamma_r} (\delta \gamma)^2 - \frac{\omega_s a_w a_s}{c \gamma_r} (\cos \psi + \psi \sin \psi_r). \quad (2.44)$$

This form for the Hamiltonian and the resulting equations of motion are very familiar to accelerator physicists. The following analysis is quite similar to their treatment of RF acceleration in standing wave linacs. In writing (2.44), we have omitted terms which are functions only of the independent variable  $z$ ,

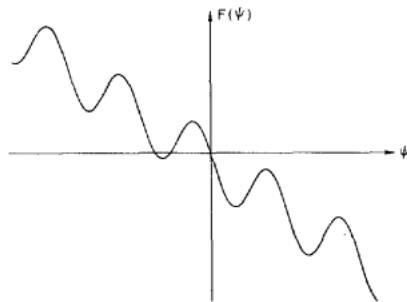


Fig. 1. The ponderomotive potential  $F(\psi)$ . The case shown is for positive  $\psi_r$ , corresponding to the case in which energy is extracted from the electrons.

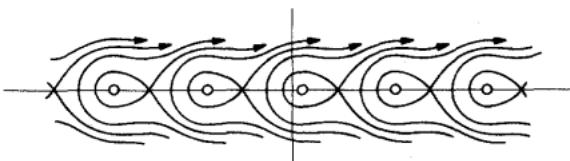


Fig. 2. Trajectories in the  $\psi, \delta\gamma$  phase plane for  $\psi_r > 0$ .

since they have no effect upon the equations of motion, and it has been assumed that a value of  $\psi_r$  exists, i.e., that

$$\left| \frac{d\gamma_r}{dz} \right| < \left| \frac{\omega_s a_w a_s}{\gamma_r} \right|. \quad (2.45)$$

$\mathcal{H}_4$  has the form of a nonrelative single particle Hamiltonian with  $z$  dependent "mass" and a  $z$  dependent potential function  $F$  given by

$$F(\psi) = -C(\cos \psi + \psi \sin \psi_r) \quad (2.46)$$

with

$$C(z) = \frac{\omega_s a_s a_w}{\gamma_r c}. \quad (2.47)$$

We assume  $C > 0$  and  $-\pi/2 < \psi_r < \pi/2$  so that  $\psi = \psi_r$  corresponds to a minimum of the potential and  $(\pi \operatorname{sgn} \psi_r - \psi_r)$  to a maximum. There are, of course, a succession of minima at  $\psi = \psi_r + 2\pi n$  and maxima at  $\psi = \pi - \psi_r + 2\pi n$ , and one may conveniently think of the successive troughs as buckets in which particles may become trapped.  $F(\psi)$  is illustrated in Fig. 1.

Of particular interest are design parameters chosen so that the behavior changes adiabatically with  $z$ . In that case, electrons trace out trajectories in the  $\psi, \delta\gamma$  phase plane given by

$$\delta\gamma(\bar{H}, \psi, z) = \mp \sqrt{\frac{\gamma_r}{k_w + \delta k_s} [\bar{H} - F(\psi)]} \quad (2.48)$$

and illustrated for  $\psi_r > 0$  in Fig. 2 at  $z \approx 0$ . The case  $\psi_r > 0$  corresponds to  $d\gamma_r/dz < 0$  and therefore represents decelerating buckets.  $\bar{H}(0)$  is determined by the initial values of  $\psi$  and  $\delta\gamma$ . The closed orbits correspond to the particles trapped in the buckets. If the change of parameters with  $z$  is adiabatic, then  $H(z)$  is determined by the requirement that the area of the closed phase curve given by

$$J = \oint \delta\gamma d\psi \quad (2.49)$$

remains constant as  $z$  changes. For the unbound orbits, the representation is valid only over a range of  $z$  for which the

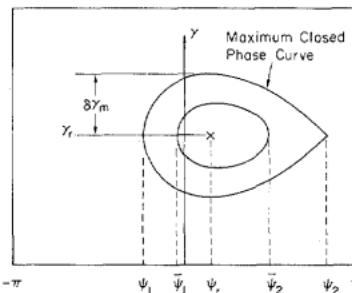


Fig. 3. Stable phase plane trajectories.

change in parameters is small. For small oscillations about  $\psi_r$ , one can expand  $F(\psi)$  about  $\psi_r$ . The motion for these orbits is harmonic with period of oscillation

$$Z = \frac{\pi \mu}{(k_w + \delta k_s) \sqrt{a_s a_w \cos \psi_r}} \approx \frac{\mu \lambda_w}{2 \sqrt{a_s a_w \cos \psi_r}} \quad (\lambda_w \equiv 2\pi/k_w). \quad (2.50)$$

The period for weakly trapped particles, with large excursion in  $\psi$ , is, of course, larger. The parameter variation may be considered to be adiabatic if it is small over a distance of the order of the period [16]. In order that a particle be trapped in a bucket, it is necessary that  $|\delta\gamma| < \delta\gamma_{\max}$  with

$$\frac{\delta\gamma_{\max}}{\gamma_r} = \frac{2}{\mu} \sqrt{a_s a_w \left[ \cos \psi_r - \left( \frac{\pi}{2} \operatorname{sgn} \psi_r - \psi_r \right) \sin \psi_r \right]}. \quad (2.51)$$

If one wishes to avoid the approximations associated with the use of (2.44), it is straightforward to base the definition of  $\gamma_r$ ,  $\psi_r$ , and  $J$  upon (2.15) and (2.43).

#### C. Summary of Bucket Parameters

Because the pattern of the stable trajectories traced out by the electrons in the  $\psi, \delta\gamma$  phase plane repeat in  $\psi$  at intervals of  $2\pi$ , we will only discuss the phase motion in the interval for  $-\pi < \psi < \pi$ . The maximum stable phase curve or bucket is shown in Fig. 3 and corresponds to the trajectory given by (2.48) for

$$\bar{H} = \bar{H}_m = C [\cos \psi_r - (\pi \operatorname{sgn} \psi_r - \psi_r) \sin \psi_r] \quad (2.52)$$

with

$$C = \frac{\omega_s a_s a_w}{\gamma_r c}. \quad (2.53)$$

The bucket intercepts of the  $\psi$  axis, designated by  $\psi_1$  and  $\psi_2$  in Fig. 3, are given by

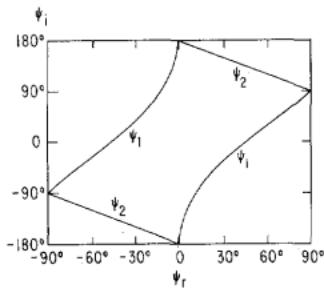
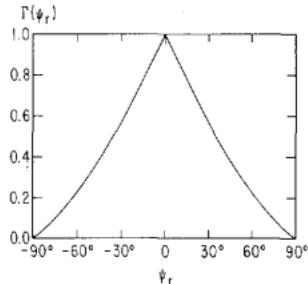
$$\psi_2 \equiv \pi \operatorname{sgn} \psi_r - \psi_r \quad (2.54)$$

and

$$\cos \psi_1 + \psi_1 \sin \psi_r = \cos \psi_2 + \psi_2 \sin \psi_r. \quad (2.55)$$

The values of  $\psi_1$  and  $\psi_2$  are shown as a function of  $\psi_r$  in Fig. 4. The maximum bucket height  $\delta\gamma_{\max}$  given by (2.51) can be written as

$$\frac{\delta\gamma_{\max}}{\gamma} = \frac{2\sqrt{a_s a_w}}{\mu} \Gamma(\psi_r) \quad (2.56)$$

Fig. 4.  $\psi_1 \psi_2$  as functions of  $\psi_r$ .Fig. 5. The bucket height function  $\Gamma(\psi_r)$ .

with

$$\Gamma(\psi_r) = \sqrt{\cos \psi_r - \left(\frac{\pi}{2} \operatorname{sgn} \psi_r - \psi_r\right) \sin \psi_r}. \quad (2.57)$$

$\Gamma(\psi_r)$  is shown in Fig. 5.

The area of the bucket shown in Fig. 3 may be obtained from (2.49) and is given by

$$J = 2 \frac{\gamma_r}{\mu} \sqrt{2a_s a_w} \int_{\psi_1}^{\psi_2} [\cos \psi_r + \cos \psi - (\pi - \psi_r - \psi) \sin \psi_r]^{1/2} d\psi \quad (2.58)$$

$$= \frac{16 \gamma_r}{\mu} \sqrt{a_s a_w} \alpha(\psi_r) \quad (2.59)$$

where

$$\alpha(\psi_r) = \frac{\sqrt{2}}{8} \int_{\psi_1}^{\psi_2} [\cos \psi_r + \cos \psi - (\pi - \psi_r - \psi) \sin \psi_r]^{1/2} d\psi \quad (2.60)$$

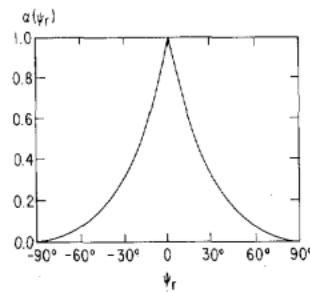
is the moving bucket area and is plotted in Fig. 6. Note that  $\alpha = 1$  for  $\psi_r = 0$  and decreases to zero for  $\psi_r = \mp \pi/2$ . [Equations (2.58) and (2.60) are for  $\psi_r > 0$ . For  $\psi_r < 0$ , use  $\alpha(\psi_r) = \alpha(-\psi_r)$ .]

### III. CONSTANT PARAMETER WIGGLER

The original mode of FEL operation demonstrated [3] by Madley and co-workers utilized a wiggler with a fixed wave-number  $k_w$  and field amplitude [4]  $a_w$ . For such a case, the operational mode is one in which

$$\psi_r = \text{constant} = 0 \quad (3.1)$$

and

Fig. 6. The phase space area function  $\alpha(\psi_r)$ .

$$\gamma_r^2 = \text{constant} = \frac{\omega_s \mu^2}{2 k_w c} \quad (3.2)$$

where the fact that  $\delta k_s \ll k_w$  has been used.

The Hamiltonian [5], which describes the motion of the electrons, is given by

$$H_4 = \frac{k_w}{\gamma_r} (\delta \gamma)^2 - C \cos \psi \quad (3.3)$$

where

$$C = \frac{\omega_s a_s a_w}{2 \gamma_r c}. \quad (3.4)$$

For this mode of operation, the buckets are stationary or non-accelerating with a half-height

$$\frac{\delta \gamma_m}{\gamma_r} = \frac{2}{\mu} \sqrt{a_w a_s} \quad (3.5)$$

and the length of one synchrotron period

$$Z = \frac{\mu \lambda_w}{2 \sqrt{a_w a_s}} = \frac{\lambda_w \gamma_r}{(\delta \gamma)_m}. \quad (3.6)$$

The motion of the electrons in such a bucket is one in which some of the electrons gain energy while others lose energy in a manner that depends upon the total length of the FEL as well as the initial phase and energy of the electron. If the average injection energy equals the resonance energy, the average change in the electron energy will be zero, provided that the initial phase distribution is uniform. While there are modes of operation for the constant parameter wiggler which assume a nonuniform phase distribution for the injected electrons, we will consider only the case where the initial phase distribution is uniform.

Electrons that have an initial energy different from the resonant energy by a large amount compared to the maximum bucket height will not be appreciably affected by the FEL, and hence will have only a small change in their energy (they will not be resonant with the optical wave), while electrons that are too near the resonant energy will also, on the average, have only a small energy change. The electrons with an initial energy difference from resonance  $(\delta \gamma)_i$  will have the largest energy change in traveling through the FEL when

$$(\delta \gamma)_i \sim (\delta \gamma_r)_m.$$

Clearly, if a significant fraction of the electrons are to lose energy, it is necessary for the initial energy spread in the beam

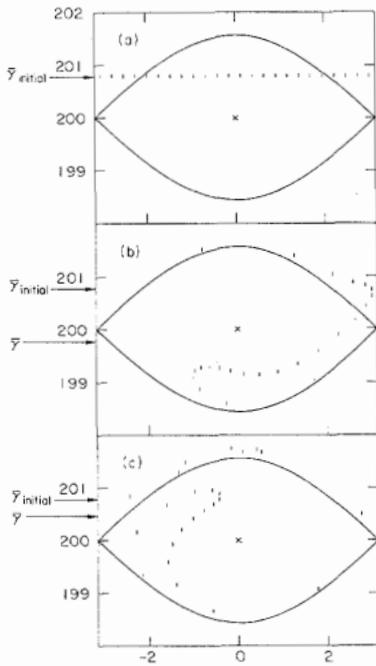


Fig. 7. Phase space plots for monoenergetic electrons in a constant parameter wiggler: (a) at  $z = 0$ ; (b)  $L$  corresponds to  $\frac{1}{2}$  synchrotron oscillation; and (c)  $L$  is  $\frac{2}{3}$  the value shown in (b).

to be less than the maximum bucket height. Consider a monoenergetic beam of electrons with an initial energy difference  $(\delta\gamma)_i > 0$  as shown in Fig. 7(a). After a distance  $L$  which corresponds to approximately one-half of a synchrotron oscillation, many particles will have their energy shifted below the resonant energy as shown in Fig. 7(b). For longer distances, these electrons will continue to oscillate about the resonant energy and the average energy will begin to increase back to the initial energy as shown in Fig. 7(c).

The performance of an FEL operating with an electron beam of an initial uniform phase distribution and a constant parameter wiggler will have the following properties. First, for reasonable performance, the energy spread of the beam must not be significantly larger than the bucket height. Secondly, the maximum energy loss by the beam will be on the order of the bucket height. Thirdly, the optimum length of the FEL is on the order of one-half of a synchrotron period. From (3.5) and (3.6), we see that the maximum allowable energy spread, maximum energy loss, and optimum FEL length are related by

$$\left(\frac{\Delta\gamma}{\gamma}\right)_{\text{max spread}} \sim \left(\frac{\Delta\gamma}{\gamma}\right)_{\text{max loss}} \sim \left(\frac{\delta\gamma_m}{\gamma_r}\right)_{\text{bucket height}} \sim \frac{\lambda_w}{2L} \sim \frac{1}{2N} \quad (3.7)$$

where  $N$  is the number of wiggler periods

Another important characteristic of the constant parameter wiggler is the fact that, due to the dependence of final energy upon initial phase, electrons which enter at a particular energy emerge with an energy spread that is on the same order (typically larger) as the mean energy loss. This property is qualitatively apparent in Fig. 7(b). As a result, the energy spread in the emerging electrons tends to be increased, over that of the entering electrons, by an amount comparable to the mean energy loss. The above refers to the situation in which the sig-

nal amplitude is such that the synchrotron period is on the order of twice the length. When the signal amplitude is small compared to this characteristic amplitude, one is in the linear regime. Then the relation between energy loss and induced energy spread is governed by the Madey theorem [12] (Appendix I)

$$\langle\gamma_f - \gamma_i\rangle = \frac{1}{2} \frac{\partial}{\partial\gamma_i} \langle(\gamma_f - \gamma_i)^2\rangle$$

where  $\langle \cdot \rangle$  denotes average over entry phase, and implies that the induced energy spread becomes much larger than the mean energy loss as the optical amplitude becomes small. These results have important consequences for the operation of an FEL as an oscillator in a storage ring. Renieri [17], Deacon, *et al.* [18], and Elias *et al.* [19] have shown that if one depends upon the synchrotron radiation to damp the growth of velocity spread, then the maximum signal power which can be extracted from the electrons is on the order of the product of the synchrotron radiation rate with the maximum value of the fractional energy spread that will circulate in the ring; hence, on the order of 1 percent of the synchrotron radiation power.

#### IV. IMPROVED ENERGY EXTRACTION BY MEANS OF ADIABATIC DECREASE OF THE RESONANT ENERGY

We have seen in Section III that the electron energy which can be extracted in passing through a fixed parameter wiggler designed to operate at some specified  $z$  independent optical power level is limited to some fraction ( $\sim 0.5$ ) of the bucket height given by (2.56), so that  $\langle(\delta\gamma)/\gamma_r\rangle_{\text{max}} \approx \sqrt{a_w a_s}/\mu$ . On the other hand, if the wiggler parameters are varied so as to gradually reduce  $\gamma_r$  as the bucket moves through the wiggler, electrons which remain trapped in the bucket give up an energy approximately equal to the reduction in  $\gamma_r$ . In this section we discuss, in an exploratory way, some characteristic features of this mode of operation.

##### A. Operation at Constant Signal Amplitude

A laser oscillator, fitted with an optical cavity formed of highly reflecting mirrors, operates at steady state so that the amplification of the signal field  $a_s$  in passing through the wiggler just compensates for the reflection loss. Accordingly, it is a reasonable approximation to assume that the electrons see a constant  $a_s$ . On the other hand, as we shall find that very large  $a_s$  is desirable, we may suppose that only a small portion of the resonator is occupied with coincident electron beam and laser "micropulses." Due to (2.39), it is also a good approximation to neglect  $\delta k_s$ .

It is convenient for discussion and probably desirable as a design characteristic to choose  $\gamma_r$  so that  $\psi_r$  is constant. Then (2.41) immediately provides the relation

$$\gamma_i^2 - \gamma_f^2 = 2 \frac{\omega_s}{c} L a_s \bar{a}_w \sin \psi_r \quad (4.1)$$

where

$$\gamma_i \equiv \gamma_r(z = 0), \quad \gamma_f \equiv \gamma_r(L)$$

and

$$\bar{a}_w L \equiv \int_0^L a_w(z) dz.$$

The assumption of constant  $\psi_r$  together with (2.40) and (2.41) provides a useful constraint on the wiggler design. One sees, for example, that if one chooses to keep  $k_w$  constant, then (2.40) and (2.41) are satisfied by decreasing  $a_w$  linearly. For this choice, the bucket (more accurately, the bucket minimum) moves with constant velocity; the electron energy loss comes entirely from the reduction in  $\mu^2$ , the effective electron mass. Alternatively, if one keeps  $a_w$  constant, then the wiggler wavelength decreases linearly with  $z$ . The effective mass remains constant, but the bucket slows down. An intermediate case is provided by keeping the wiggler field,  $b_w = k_w a_w$ , constant. Equation (2.41) is again readily integrable.

Equation (4.1) implies the inequality

$$(\gamma_i^2 - \gamma_f^2) \leq 2 \frac{\omega_s}{c} L a_s \bar{a}_w$$

which is equivalent to

$$2La_s \geq \frac{1}{2\bar{a}_w} \left( \frac{\mu^2(0)}{k_w(0)} - \frac{\mu^2(L)}{k_w(L)} \right). \quad (4.2)$$

Equation (4.2) represents a constraint on the length and/or optical power density which is required to obtain a desired change in  $\gamma_r$ . It is probably desirable, therefore, to design the wiggler magnet so as to minimize the right-hand side subject to a specified value [20] of  $k_w(L)$  and bucket efficiency, which we define by  $\eta_b = (\gamma_i - \gamma_f)/\gamma_i$ . The required relations are easily established for the constant  $k_w$  and constant  $a_w$  cases. One obtains the following simple results.

*Constant  $k_w$  Case:*

$$k_w La_s \geq \frac{1}{2} \eta_b \frac{2 - \eta_b}{1 - \eta_b} \quad (4.3)$$

$$a_w(0) = \frac{\gamma_i}{\gamma_f}, \quad a_w(L) = \frac{\gamma_f}{\gamma_i} \quad (4.4)$$

$$a_w(z) = a_w(0)(1 - z/L) + a_w(L)z/L. \quad (4.5)$$

*Constant  $a_w$  Case:*

$$k_w(L) La_s \geq \frac{1}{2} \eta_b \frac{2 - \eta_b}{(1 - \eta_b)^2} \quad (4.6)$$

$$a_w = 1$$

$$\lambda_w(z) = \lambda_w(0)(1 - z/L) + \lambda_w(L)z/L. \quad (4.7)$$

It is apparent from these results that the constraint becomes severe as  $\eta_b$  approaches unity, and it is less severe in the constant  $k_w$  case. The constant  $b_w$  case turns out to be intermediate between the two (see Fig. 14). Much of the discussion for the remaining part of this section will, for definiteness, be with reference to the constant  $k_w$  case. The above discussion suggests that it is likely to be one of the better methods of varying  $\gamma_r$ . It also may offer hardware advantages.

For the system described by (4.3) and (4.4), we have from (4.1)

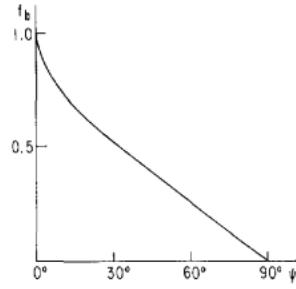


Fig. 8. Trapping function  $f_b$  as a function of  $\psi_r$ .

$$k_w La_s = \frac{1}{2} \eta_b \frac{2 - \eta_b}{1 - \eta_b} \csc \psi_r. \quad (4.8)$$

In the absence of prebunching, electrons will enter the wiggler at arbitrary initial phase  $\psi(0)$ . For electrons of energy  $\gamma_r$ , only those for which

$$\psi_1 < \psi(0) < \pi - \psi_r \equiv \psi_2 \quad (4.9)$$

will be trapped in the bucket, where  $\psi_1$  is defined by (2.55). The fraction trapped in the bucket is therefore  $f_b = (\psi_2 - \psi_1)/2\pi$ . For electrons with initial energy different from  $\gamma_r$ , the trapped fraction will be smaller and no electrons are trapped if the deviation exceeds  $\delta\gamma_{\max}$  [see (2.56)]. The product  $f_b \eta_b$  can be thought of as a sort of idealized electronic efficiency. Fig. 8 provides a plot of  $f_b$  as a function of  $\psi_r$ . It is apparent that increases of idealized efficiency either by increasing  $\eta_b$  or increasing  $f_b$  drives the needed magnitude of  $La_s$  to larger values.

Although we are not carrying out any serious studies of the effects of transverse variation in this paper, we do wish to take into account some obvious constraints which are thereby imposed upon operating parameters. To control the effects of transverse variation in  $a_w$ , we write

$$k_w r_e = \frac{1}{3} \sigma_w. \quad (4.10a)$$

The quantity  $\sigma_w$  is a parameter, which we shall typically set equal to 1, which represents our ignorance of what it should really be. It may be dependent upon other operating parameters. In addition, we assume the oscillator to be provided with an optical resonator formed by a pair of mirrors and producing a Gaussian beam centered in the wiggler. This causes an amplitude variation both transverse to and along the optical beam of the form

$$|a_s(r, z)| = \left| a_s \left( r, \frac{L}{2} \right) \right| \frac{\exp [-r^2 f^2 / r_s^2 (f^2 + d^2)]}{\left[ 1 + \frac{d^2}{f^2} \right]^{1/2}}$$

where  $f = \frac{1}{2} (\omega_s/c) r_s^2$  and  $d = z - L/2$ . To control transverse and longitudinal variation in  $a_s$ , we write

$$r_e = \frac{2}{3} r_s \sigma_e \quad (4.10b)$$

and

$$L = \frac{1}{2} \frac{\omega_s}{c} r_s^2 \sigma_L \quad (4.10c)$$

where  $\sigma_e$  and  $\sigma_L$  play a role analogous to  $\sigma_w$ . Combining these relations, we obtain

$$k_w L a_s \approx \frac{1}{4} a_s r_s \frac{\omega_s}{c} \frac{\sigma_w \sigma_L}{\sigma_e} \quad (4.10d)$$

which tells us that, apart from the  $\sigma$  factors, the left-hand side of (4.8) is proportional to the square root of the peak power circulating in the optical resonator. Returning to dimensional variables, we obtain for this power

$$P = \frac{1}{2} \left( \frac{m^2 c^5}{e^2} \right) \eta_b^2 \left( \frac{2 - \eta_b}{1 - \eta_b} \right)^2 \csc^2 \psi_r \sigma_P \quad (4.11)$$

where

$$\frac{m^2 c^5}{e^2} = 8.7 \text{ gW}$$

and

$$\sigma_P = \left( \frac{\sigma_e}{\sigma_w \sigma_L} \right)^2.$$

Equation (4.11) provides us with some working relations which we find to be useful for further discussion. It suggests that there is a minimum circulating power required to reduce the  $\gamma$  of any electron by  $\delta\gamma = \gamma_i \eta_b$ . We call this the threshold power  $P_T$  with the working definition

$$P_T = \frac{1}{2} \frac{m^2 c^5}{e^2} \eta_b^2 \left( \frac{2 - \eta_b}{1 - \eta_b} \right)^2. \quad (4.12)$$

We also note that for fixed  $\sigma_P$  and idealized efficiency  $\eta_i = \eta_b f_b$ , it is possible to minimize the right-hand side of (4.8) with respect to  $\eta_b$ . We define  $P_m$  as the result of this minimization, with  $\sigma_P$  set equal to 1. This procedure provides us with a useful estimate of circulating power required for a given idealized efficiency and a reasonable basis for selection of  $\eta_b$  and  $\psi_r$ . Fig. 9(a) provides a plot of  $P_T$  and  $P_m$  as functions of  $\eta_b$  and  $\eta_i$ , respectively. Fig. 9(b) provides a plot of the  $\psi_r$  and  $\eta_b$  functions of  $\eta_i$  which emerge from the minimization.

Computer simulation, to be described later, yields electronic efficiencies which are of the order of 80–90 percent of  $\eta_i$ . There are two reasons for departure from the ideal. First, electrons which are initially trapped do not necessarily remain trapped for the entire transit through the wiggler. An electron which escapes from a bucket at some  $z = z_e$  experiences a fractional energy loss of

$$\delta\gamma/\gamma = \frac{\gamma(0) - \gamma(z_e)}{\gamma(0)} < \eta_b.$$

Second, the electrons which are not initially trapped, i.e., those for which the initial phase is outside the  $\psi_1 < \psi < \psi_2$  range, all tend to gain energy. The average gain of these electrons is on the order of  $\frac{1}{2}(\delta\gamma)_{\max}$ ; both the fact that there is an increase and its order of magnitude can be inferred from Fig. 2.

Detrapping appears to be the more important of the two effects. Its occurrence can be readily related to the adiabatic

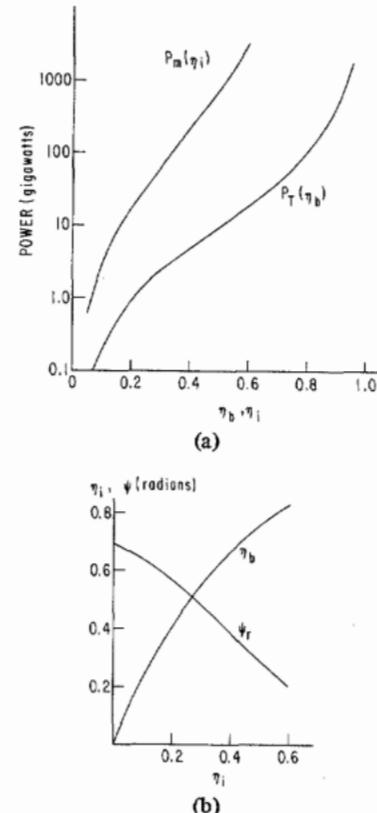


Fig. 9. (a) Threshold power for trapping  $P_T$  as a function of bucket efficiency. Minimum required power  $P_m$  as a function of ideal efficiency  $\eta_i$ . (b) Values of bucket efficiencies  $\eta_b$  and resonant phase  $\psi_r$  for minimum power design as functions of ideal efficiency  $\eta_i$ .

invariant defined in (2.49). For an electron which enters at  $\delta\gamma = 0$ ,  $\psi = \psi_0$ , we have from (2.44) and (2.46)–(2.49)

$$J(\psi_r, \psi_0) = 8 \sqrt{\frac{2\omega_s a_w(0) a_s(0)}{c[k_w(0) + \delta k_w(0)]}} \bar{\alpha}(\psi_r(0), \psi_0) \quad (4.13)$$

where

$$\begin{aligned} \bar{\alpha}(\psi_r, \psi_0) = & \frac{\sqrt{2}}{8} \int_{\psi_1}^{\psi_2} [\cos \psi + \psi \sin \psi_r - \cos \psi_0 \\ & + \psi_0 \sin \psi_r]^{1/2} d\psi. \end{aligned} \quad (4.14)$$

One of the limits is given by  $\psi_0$  and the other is related to it by (see Fig. 3)

$$\begin{aligned} \cos \bar{\psi}_1 + \bar{\psi}_1 \sin \psi_r &= \cos \bar{\psi}_2 + \psi_2 \sin \psi_r, \\ &= \cos \psi_0 + \psi_0 \sin \psi_r. \end{aligned} \quad (4.15)$$

In general, one has

$$\psi_1 < \bar{\psi}_1 < \psi_r, \quad \psi_r < \bar{\psi}_2 < \psi_2 = \pi - \psi_r,$$

so that which of the two is equal to  $\psi_0$  depends upon the range in which  $\psi_0$  is located. In addition, at any  $z$  there is a maximum possible value for  $J$  which is given by

$$J_M(\psi_r, z) = 8 \sqrt{\frac{2\omega_s a_w(z) a_s(z)}{c(k_w(z) + \delta k_w(z))}} \bar{\alpha}(\psi_r(z), \pi - \psi_r(z)). \quad (4.16)$$

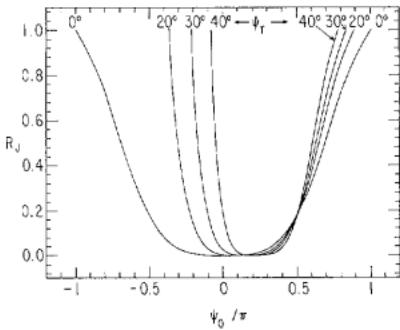


Fig. 10. The detrapping function  $R_j(\psi_0 \psi_r) \equiv \bar{\alpha}^2(\psi_r \psi_0)/\alpha^2(\psi_r)$  plotted as a function of initial phase  $\psi_0$  for various values of resonant phase  $\psi_r$ .

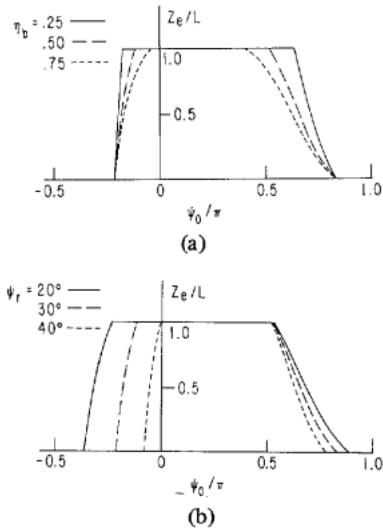


Fig. 11. (a) Detrapping distance  $Z_e$  shown as a function of initial phase  $\psi_0$  with (a) bucket efficiency as a parameter, and (b) resonant phase  $\psi_r$  as a parameter.

The function  $\bar{\alpha}(\psi_r, \pi - \psi_r) = \alpha(\psi_r)$  is plotted as a function of  $\psi_r$  in Fig. 6. The value ( $z_e$ ) of  $z$ , at which a particle escapes from the bucket, is determined by

$$J_M(\psi_r(z_e), z_e) = J(\psi_r(0), \psi_0).$$

For the oscillator problem which we have been studying, with constant  $a_s$ ,  $\psi_r$ ,  $k_w$  and  $\delta k_s = 0$ , we have

$$a_w(z_e) = a_w(0) \frac{\bar{\alpha}^2(\psi_r, \psi_0)}{\alpha^2(\psi_r)} \quad (4.17)$$

and hence, from (4.4) and (4.5)

$$z_e = \frac{L}{\eta_b(2 - \eta_b)} \left( 1 - \frac{\bar{\alpha}^2(\psi_r, \psi_0)}{\alpha^2(\psi_r)} \right). \quad (4.18)$$

One easily sees that for a given  $\eta_b$  there is a range of  $\psi_0$  about  $\psi_r$  such that (4.18) yields values of  $z_e > L$ . These correspond, of course, to particles which do not detrap. Fig. 10 provides a family of plots of  $[\bar{\alpha}^2(\psi_r, \psi_0)/\alpha^2(\psi_r)]$  as a function of  $\psi_0$  for various values of  $\psi_r$ , and Fig. 11(a) and (b) present  $z_e$  as a function of  $\psi_0$  with  $\eta_b$  and  $\psi_r$  as parameters. Incidentally, the adiabatic theory of detrapping typically implies an upper bound of electronic efficiency for some designs. For the constant  $k_w$ ,  $\psi_r$ ,  $a_s$  wiggler, the bound is found to be 0.672. If

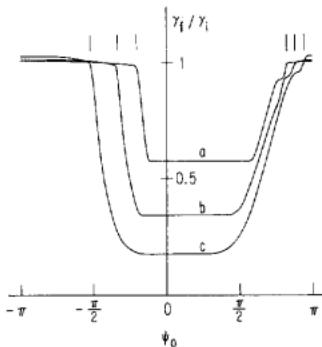


Fig. 12. Energy spectrum of emerging electrons as a function of entry phase. All electrons enter with energy equal to  $\gamma_r(0)$ . The three curves refer to minimum power configurations of the sort illustrated in Fig. 9. The parameters of the three curves are:

	$\eta_b$	$\psi_r$	$\eta_i$	$\eta_e$
a	0.425	31.9°	0.215	0.181
b	0.66	21.9°	0.400	0.314
c	0.823	11.4°	0.600	0.448

The vertical lines above the curves denote positions of the trapping boundaries  $\psi_1$  and  $\psi_2$ .

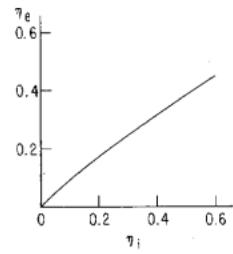


Fig. 13. Electronic efficiency  $\eta_e$  as a function of ideal efficiency  $\eta_i$  for the maximum power configuration of Fig. 9.

$a_w$  rather than  $k_w$  is held constant, the bound is reduced to 0.544.

A numerical simulation of the constant  $k_w$ ,  $\psi_r$ ,  $a_s$  case has been carried out, based upon (2.25). The solution is used to compute  $\gamma(L)$  from (2.24) as a function of  $\psi_0$  for various values of  $\eta_b$  and  $\psi_r$ . The results of some of these computations are shown in Fig. 12 for some of the minimum power configurations of Fig. 9. The flat portions at  $\gamma(L)/\gamma_i = 1 - \eta_b$  corresponds to the range of  $\psi$  over which there is no detrapping.

The fact that the rapidly changing portions are not vertical is an indication that detrapping is taking place. The peculiar jog at the right-hand end just before  $\psi_0$  reaches  $\psi_2$  is due to particles which detrap after the first synchrotron oscillation. Note that untrapped particles gain energy. All of the curves shown are for  $\gamma_i = \gamma_r$ , with no energy spread included. They should be representative for a spread small compared to  $\delta\gamma_{\max}$ . The numerical simulation also provides a value for the electronic efficiency  $\eta_e$ . Fig. 13 shows the extent to which  $\eta_e$  differs from  $\eta_i$  for the configurations of Fig. 9.

It is clear from (4.16) that detrapping can be avoided for the constant  $a_s$  case by designing the wiggler so that  $a_w/k_w$  is constant. As in constant  $k_w$ , constant  $a_w$ , and constant  $b_w$  cases discussed previously, one can use the condition of constant  $\psi_r$  and specified  $\eta_b$  to determine the  $z$  dependence of  $a_w$  and  $k_w$ , and the required power. The required threshold powers in the constant  $k_w$ ,  $a_w$ ,  $b_w$ , and  $a_w/k_w$  cases are com-

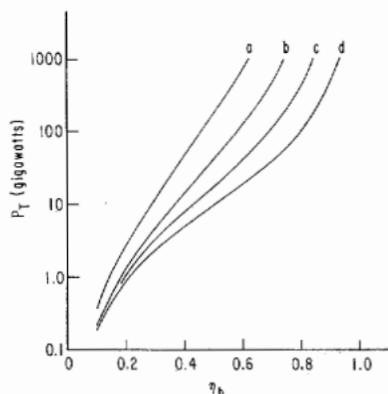


Fig. 14. Comparison of threshold powers as a function of  $\eta_b$  for various constants  $\psi_r$ , constant  $a_s$  wiggler designs. The four cases are the following. Curve *a*: constant  $a_w/k_w$ . Curve *b*: constant  $a_w$ . Curve *c*: constant  $b_w$ . Curve *d*: constant  $k_w$ .

pared in Fig. 14 as functions of  $\eta_b$ . Since the penalty due to detrapping is rather small, one actually obtains a better net efficiency from a constant  $k_w$  wiggler than from one which avoids detrapping. Some small improvement can probably be obtained over the constant  $k_w$  case by taking detrapping into account, but we have not pursued this question further. This assessment applies to the case in which the incident electron energy spread is small compared to  $\delta\gamma_{\max}$ . When they are comparable, as is likely to be the case in practice, the penalty imposed by detrapping increases, and its avoidance acquires more significance as a design consideration. It should be noted that in the case of an amplifier or heavily loaded oscillator, where  $a_s$  increases with  $z$ , detrapping tends to be avoided automatically, as discussed later in this section.

In all of the previous discussion we have imposed the requirement (largely to simplify the discussion) that  $\psi_r$  be  $z$  independent. One may ask whether this choice leads to an approximately optimized wiggler design. From the point of view of detrapping, it would be useful to have  $\psi_r$  decrease with  $z$ , to compensate for the decrease in  $a_w/k_w$  which seems to occur for the more optimum constant  $\psi_r$  designs. This is apparent from the behavior of  $\alpha(\psi_r)$  [see (4.13), (4.14), and (4.18)] shown in Fig. 6. On the other hand, from the point of view of initial capture, it would be best to have  $\psi_r$  have a very small, or even vanishing initial value. This can be seen from the plot of the trapping fraction  $f_b$  shown in Fig. 8. Thus, to maximize trapping,  $\psi_r$  should increase with  $z$ , while to avoid detrapping,  $\psi_r$  should decrease with  $z$ . One can, in principle, increase  $a_w/k_w$  to overcome the effect of decreasing  $\psi_r$  and thus avoid detrapping, but experience with the constant  $\psi_r$  case suggests that this will not improve matters either. Hence, the constant  $\psi_r$ , constant  $k_w$  design is probably not very far from an optimum one. Again, in anticipation of the amplifier case, we note that an increase of  $a_s$  with  $z$  can eliminate the detrapping associated with an increase in  $\psi_r$  so that an increase of  $\psi_r$  with  $z$  should have some advantages for an amplifier.

We conclude this discussion by listing the properties of a "standard" wiggler design, based upon the constant  $k_w$  case. It is certainly not unique and probably not optimal, but it does provide a convenient point of reference for future discussion. We begin by specifying an idealized efficiency  $\eta_i$ , a

signal frequency  $\omega_s$ , and an electron beam radius  $r_e$ . The associated bucket efficiency  $\eta_b$  and resonant phase  $\psi_r$  are determined from  $\eta_i$  by Fig. 9. For wiggler design parameters, we then propose

$$k_w \cong \frac{1}{3r_e} \quad (4.19)$$

$$L \cong \frac{9}{8} \frac{\omega_s}{c} r_e^2 \quad (4.20)$$

$$a_w(z) = \frac{1 - (2\eta_b - \eta_b^2)(z/L)}{1 - \eta_b} \quad (4.21)$$

$$B_w(z) = 1.7 k_w a_w(z) \cong 1.7 \frac{a_w(z)}{3r_e(\text{cm})} \text{ kG.} \quad (4.22)$$

The electron beam energy  $\epsilon_e$  is given by

$$\epsilon_e = mc^2 \gamma_i \cong .51 \left[ \frac{3}{2} \frac{\omega_s r_e}{c} \frac{2 - 2\eta_b + \eta_b^2}{(1 - \eta_b)^2} \right]^{1/2} \text{ MeV.} \quad (4.23)$$

The required peak beam current is given by

$$I_e = \frac{eP_m}{mc^2 \gamma_i \eta_e Q} \quad (4.24)$$

where  $Q$  is the "Q" of the optical resonator, and  $\eta_e \leq \eta_i$  is the electronic efficiency. Rewriting (4.24) yields

$$I_e = \frac{1}{Q(6\omega_s r_e/c)^{1/2}} \left( \frac{\eta_b^2 \csc^2 \psi_r}{\eta_e} \right) \cdot \frac{(2 - \eta_b)^2}{(1 - \eta_b)(2 - 2\eta_b + \eta_b^2)^{1/2}} \left( \frac{mc^3}{e} \right) \quad (4.25)$$

where

$$\left( \frac{mc^3}{e} \right) = 17.0 \text{ kA.}$$

The degradations associated with the variation of the resonator mode over the electrons, as well as the effect of energy spread of the incident electrons, have been neglected in the previous discussion. For assessing the effect of the latter, we note the relation [using (4.8) and (2.56) together with the above]

$$\frac{\delta\gamma_{\max}}{\gamma} = \frac{4\sqrt{3}}{3} \left( \frac{cn_b(2 - \eta_b) \csc \psi_r}{\omega_s r_e (2 - 2\eta_b + \eta_b^2)} \right)^{1/2} \Gamma(\psi_r). \quad (4.26)$$

The electron beam radius which can be used is primarily determined by the electron beam emittance. It is shown in Appendix III that if one assumes ideal emittance scaling, then the beam radius must exceed a lower limit proportional to the emittance constant. Finally, we repeat that we have neglected the possibility that some modification in the parametric dependencies may emerge if the omitted "σ" factors are taken into account.

It is instructive to illustrate the above expressions with a reasonable numerical example. Let us suppose we wish to construct a 1 μm oscillator. A preliminary perusal suggests that  $\eta_i \approx 0.215$  might yield parameters achievable with current technology. Considerations bearing upon the choice of electron radius will be discussed in Appendix III, but for the moment

we simply specify  $r_e = 1.25$  mm. With these choices the following numbers emerge:

$$\begin{aligned} \eta_e &= 0.18 & a_w &= 1.75 - 1.18 z/L \\ \eta_b &= 0.43 & B_w &= 4.56 - 3.07 z/L \text{ kG} \\ \psi_r &= 32^\circ & E_e &= 111 \text{ MeV} \\ P_m &= 21 \text{ GW} & QI_e &= 1100 \text{ A} \\ \lambda_w &= 2.3 \text{ cm} & \delta\gamma_{\max}/\gamma &= 0.014 \\ L &= 10.9 \text{ m} \end{aligned}$$

As indicated, the beam current and, accordingly, the optical power generated ( $P_m/Q$ , of course) will depend upon the resonator  $Q$  which is chosen, and, due to mirror losses, the optical power which emerges will be reduced from that generated. Furthermore, we again emphasize that these are peak values. If the electron beam is provided by an RF linac, "peak" refers to values which obtain during a micropulse. Average values during a macropulse are reduced by a factor which typically lies between 25 and 100.

### B. Operation with Amplifying Signal Amplitude

In discussing the behavior of the standard design, the increase in the signal amplitude  $a_s$  was neglected. This neglect is justified for large values of  $Q$ . As one decreases  $Q$ , we expect  $\eta_e$  to approach  $\eta_i$  more closely, but otherwise the basic formulas for the operation of the device are still expected to hold approximately despite the fact that  $a_s$  is increasing. The decrease in  $Q$  is likely to be associated with increased output coupling. This fact, together with the increase in  $\eta_e$  means that the extra beam power input associated with the increase in  $I_e$  is being more efficiently converted to signal power output.

These expectations are easily understood by making reference to (2.41), which we rewrite as

$$a_s \sin \psi_r \equiv - \frac{c\gamma_r}{\omega_s a_w} \frac{d\gamma_r}{dz}. \quad (4.27)$$

The right-hand side of this equation has already been determined by the standard design to be a constant. Associated with this design was an input value of  $a_s$  and an input value of  $\psi_r$ . However, if  $a_s$  increases,  $\psi_r$  simply decreases. Reference to (2.59) and Fig. 6 tells us that the phase area of the bucket increases. Hence, particles initially trapped remain trapped and apart from the small increase in energy of the untrapped particles  $\eta_e = \eta_i = \eta_b f_b$ . Due to the fact that the bucket is actually widening, some initially untrapped particles may even become trapped, and  $\eta_e$  can exceed  $\eta_i$ . The important lesson to be learned is the fact that a more rapid than "planned" increase in  $a_s$  does no harm to the trapping and deceleration process. On the other hand, if  $a_s$  increases less rapidly than planned, a reduction in phase area occurs which can lead to some detrappling. Indeed, if  $a_s$  drops sufficiently below its planned value to require  $\sin \psi_r > 1$ , total detrappling takes place.

It should now be apparent that a simple way to design an amplifier is simply to remove the mirrors from the standard design and supply a beam current given by (4.25) with  $1/Q$  replaced by  $(G - 1)$ , where  $G$  is the power gain. The required input signal power is given by  $P_m$ .

In our discussion of high  $Q$  oscillators, we established a

quasi-optimized connection between the minimum circulating power  $P_m$  and  $\eta_i$ . Clearly, similar optimization could be desirable for an amplifier. The simple amplifier designed above makes no attempt to take a programmed increase of  $a_s$  into account in specifying its  $\gamma_r$  profile and hence is unlikely to be optimum in this respect. What we would wish to have is a procedure for designing the wiggler so as to maximize the efficiency for a specified power output and specified gain.

As a first example of a step in the desired direction, we describe a procedure for choosing  $a_s$  (and hence,  $\gamma_r$ ) so as to maintain an approximately constant  $\psi_r$ . In order to be specific, we consider the constant  $k_w$  case. We then find from (2.40) and (2.41), neglecting  $\delta k_s$ ,

$$a'_w = -(2k_w a_s(0) \sin \psi_r) \frac{a_s(z)}{a_s(0)}. \quad (4.28)$$

Note that the coefficient in brackets is a constant. To relate  $a_s(z)/a_s(0)$  to  $a_w$ , we make use of (2.26), assuming that

$$\langle \gamma(0) \rangle - \langle \gamma(z) \rangle = \text{constant} \times (\gamma_r(0) - \gamma_r(z)). \quad (4.29a)$$

This assumption is based upon the idea that the trapped electrons represent a constant fraction of the electrons, and that they give up energy at the rate  $d\gamma_r/dz$ . It yields the equation

$$\frac{a_s(z)}{a_s(0)} = \left[ 1 + (G - 1) \frac{\mu(0) - \mu(z)}{\mu(0) - \mu(L)} \right]^{1/2} \quad (4.29b)$$

where  $G$  is again the power gain. Combining (4.28) and (4.29) yields

$$\int_{a_w(L)}^{a_w(0)} \frac{da_w}{\left[ 1 + (G - 1) \frac{\mu(0) - \sqrt{1 + a_w^2}}{\mu(0) - \mu(L)} \right]^{1/2}} = 2k_w L a_s(0) \sin \psi_r. \quad (4.30)$$

The left-hand side of (4.30) can be regarded as the analog of the right-hand side of (4.2), that is, as a quantity which in this instance should be minimized with respect to  $a_w(L)$  and  $a_w(0)$  subject to the constraint  $1 - (\mu(L)/\mu(0)) = \eta_b$ . This condition then determines the appropriate values of  $a_w(0)$  and  $a_w(L)$  and hence  $\mu(0)$  and  $\mu(L)$ . Subsequently,  $a_w(z)$  is determined from

$$\begin{aligned} &\int_{a_w(z)}^{a_w(0)} \frac{da_w}{\left[ 1 + (G - 1) \frac{\mu(0) - \sqrt{1 + a_w^2}}{\mu(0) - \mu(L)} \right]^{1/2}} \\ &= \frac{z}{L} (2k_w L a_s(0) \sin \psi_r). \end{aligned} \quad (4.31)$$

As an example, we consider the case  $\eta_b = 0.43$ , and for simplicity neglect the final optimization. Instead, we obtain  $a_w(0)$  and  $a_w(L)$  from (4.4), which holds for the  $G = 1$  case. For  $G = 100$ , the value of the left-hand side of (4.30) is found to be 0.203. This number for the right-hand side of (4.30) should be compared to  $a_w(0) - a_w(L) = 1.18$ , which would be required by the simple amplifier design discussed previously in which  $a_w$  decreases linearly. Thus, the improved wiggler profile permits a factor of thirty reduction in the operating power level.

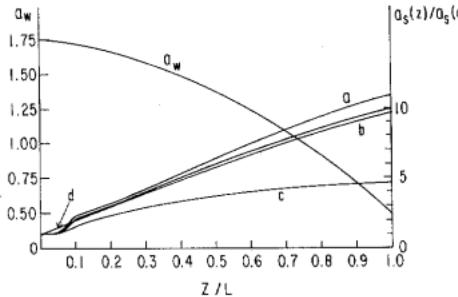


Fig. 15. Simulated performance of a matched profile amplifier.

	peak electron beam power	electronic efficiency
a	350 GW	0.214
b	295 GW	0.203
c	145 GW	0.091

Curve d shows the  $a_s(z)/a_s(0)$  profile obtained using (4.29a).

The minimum of the left-hand side of (4.30) is found to be very broad for  $G = 100$ , so that the value obtained by choosing  $a_w(0)$ ,  $a_w(L)$ , as noted above, is within  $\frac{1}{2}$  percent of the minimum.

The predicted behavior of  $a_s$  is obtained from (4.29) by making use of  $a_w(z)$  as given by (4.31). To complete this example we apply the standard procedure to the rest of the design. Hence, we choose  $\sin \psi_r$  from Fig. 9 to obtain the value 0.53 and an  $\eta_i$  of 0.215. The required optical input power is then 640 MW. Assuming an electronic efficiency equal to  $\eta_i$  and a gain of 100, we obtain a required electron beam power of 295 gW. Fig. 15 shows the design behavior of  $a_w$  as given by (4.31) and  $a_s$  as given by (4.29) (curve d). It also shows the computed values of  $a_s$  and electronic efficiency as obtained from numerical simulation. The numerical simulation was based upon (2.25) and (2.26), neglecting the  $\delta k_s$  terms, but explicitly carrying out the averages required for (2.26). Three cases are shown corresponding to the design beam power, a somewhat higher power and a lower power. The design beam power does not quite yield the design performance. First, we note that the expectation of the ideal electronic efficiency failed to take into account the energy gain from the untrapped electrons. Second, (4.29a), the assumption of a constant ratio between  $\gamma_r(0) - \gamma_r(z)$  and  $\langle \gamma(0) \rangle - \langle \gamma(z) \rangle$ , is clearly not satisfied at the input of the amplifier. The flat behavior of  $a_s$  at the input reflects the fact that bunching must take place before amplification can begin. We see that increasing the beam power improves the efficiency as well as providing more than the design gain. The reduced beam power example illustrates the effect of a smaller increase in  $a_s$  than was anticipated in the wiggler design. Equation (4.27) causes  $\psi_r$  to constantly increase continually detrapping particles, until at  $z/L \sim 0.8$   $\psi_r$  has reached  $\pi/2$ . All particles are detrapped and amplification has ceased.

Because  $a_s$  is an increasing function of  $z$ , it should be possible to allow  $\psi_r$  to increase with  $z$  without detrapping particles. One simply requires that the phase area  $J$ , given by (2.59), not decrease. Indeed, by choosing  $d\gamma/dz$  to vanish at the input, one can begin by trapping all particles since  $\psi_r$  will initially be zero. Since  $a_s$  will, at each  $z$ , increase monotonically with increasing beam power, there will always be a critical

beam power above which all particles remain trapped so that  $f_b = 1$  and  $\eta_e = \eta_b$ . Whether it is better to use the beam power to increase  $f_b$  to unity rather than to increase  $\eta_b$  is a question which we have not explored.

The amplifier length provides another quantity which can be varied as a part of an optimization procedure. In our discussion up to this point, we have assumed the length to be given by (4.20), but there is, however, no compelling reason to limit the length in this way [21]. It can be argued that as long as the rate of amplification is sufficiently great to maintain  $a_s$  constant on the electron beam (i.e., to counter the effect of diffraction spreading), the device can, in principle, be arbitrarily long. It appears likely that  $\delta k_s$  can also play an important role in this situation. We have not, however, carried out a complete analysis of this super-long amplifier and will not discuss it further here.

In our discussion of the amplifier, we have consistently neglected  $\delta k_s$ . It would, of course, be a straightforward matter to include it in carrying out simulations. It appears as an order  $\delta k_s/k_w$  correction to (2.24) and (2.25). Just as a local value of  $a_s^2(z)$  is obtained from (2.26), a local value of  $\delta k_s$  can be obtained from (2.36). For the case illustrated by Fig. 15 we consider that  $\langle \cos \psi/\gamma \rangle \approx 0$  for the first 5 percent of the amplifier, so that  $\delta k_s$  is in fact zero initially. For the region in which  $a_s$  increases approximately linearly, it is likely that  $\langle \cos \psi/\gamma \rangle / \langle \sin \psi/\gamma \rangle \approx 1$  so that  $\delta k_s = (a'_s/a_s) \approx 9/(L+9z)$  and we estimate  $\delta k_s < 10/L$  everywhere. Thus,  $\delta k_s/k_w < 0.003$  everywhere, so that its neglect appears to be justified as long as transverse effects are ignored.

We conclude this section by pointing out the fact that the neglect of transverse effects appears to be especially suspect for the design and analysis of amplifiers. In the amplifier illustrated in Fig. 15, and in the qualitative discussion of other designs that followed, we have depended upon the predicted growth of  $a_s$  to maintain the electron traps. We have assumed in carrying out the design and in the simulations that there is no transverse variation in the signal amplitude. This certainly will not be the case in actuality. Suppose, for example, that the electron beam has a Gaussian profile. Then we may write for (2.37)

$$a'_s = \frac{\omega_p^2(0)}{2\omega_{sc}c} a_w \langle \sin \psi/\gamma \rangle \exp(-r^2/r_e^2) \quad (4.32)$$

where  $\langle \sin \psi/\gamma \rangle$  should also be assumed to be  $r$  dependent. We also have (4.27), which implies that  $a_s \sin \psi_r$  is  $r$  independent. Suppose the amplifier has been designed so that  $\psi_r$  remains constant in  $z$  at some value of  $r = r_d$ . For smaller values of  $r$ , we expect  $a_s$  to increase more rapidly,  $\psi_r$  decreases, and trapping is unaffected. For particles outside  $r_d$ ,  $a_s$  increases less rapidly than programmed. Detrapping can be expected to develop at larger  $r$  and progressively work its way into  $r_d$ . This could lead to a decrease of amplitude at  $r_d$  itself, and hence a progressive erosion of the edge. A similar erosion may be expected to occur at the trailing edge of the pulse. Thus, while electrons at the trailing edge can initially be trapped by the input signal, the failure of the signal to amplify at the rear edge will lead to a progressive detrapping that moves up the pulse. These considerations suggest that some of the advan-

tages of sophisticated designs over the "simple" design (in which one depends upon the initial amplitude to maintain trapping throughout) may be substantially reduced.

The situation is further complicated by the fact that

$$\delta k_s = \frac{\omega_p^2(0)}{2\omega_s c} \frac{a'_s}{a_s(r, z)} \langle \cos \psi/\gamma \rangle \exp(-r^2/r_e^2). \quad (4.33)$$

The presence of this variation tends to focus light towards the center, reinforcing the tendencies discussed in the preceding paragraph. In the amplifier illustrated in Fig. 15, we estimated  $\delta k_s \sim 9/L + 9z$  for  $z > 0.1 L$ . According to (4.33), this value should be construed as referring to the center of the beam as long as the variation of  $a_s$  and  $\langle \cos \psi/\gamma \rangle$  with  $r$  is neglected. Because  $\delta k_s$  decreases with  $z$ , the focusing tendency is largest near the beginning of the amplifier. To obtain a crude quantitative estimate, we consider the problem to be one of propagation of rays in a lens-like medium, with index of refraction given by  $\delta n = c\delta k_s/\omega_s$ . According to the paraxial ray equation, we have

$$\frac{d^2 r(z)}{dz^2} = \frac{d\delta n}{dr} \quad (4.34)$$

where  $r(z)$  is the ray coordinate. We confine our attention to the central portion and write

$$\frac{d\delta n}{dr} = -\frac{2r}{r_e^2} (\delta n)_{r=0} \equiv -\frac{2r}{r_e^2} \delta n_0. \quad (4.35)$$

Taking advantage of the fact that  $r_s dn_0/dz$  is small, we obtain, for a ray emitted at  $z = z_1$  with zero slope

$$r(z) = r(z_1) \left( \frac{\delta n_0(z_1)}{\delta n_0(z)} \right)^{1/4} \left[ \cos \int_{z_1}^z \sqrt{2\delta n_0} dz / r_e \right. \\ \left. + \frac{1}{2} \left( \frac{\delta n'_0 r_e}{(2\delta n_0)^{3/2}} \right)_{z_1} \sin \int_{z_1}^z \sqrt{2\delta n_0} dz / r_e \right]. \quad (4.36)$$

For the amplifier of Fig. 15, using the above mentioned estimate of  $\delta k_s$ , we find, for  $z_1 = 0.1 L$ , that focus takes place at  $z = 0.65 L$ . This presumably overestimates the effect because we have neglected the fact that the focusing effect further increases  $a_s$ , thus reducing  $\delta k_s$ . (The factor  $\langle \cos \psi/\gamma \rangle$  also increases, but the size of the increases is limited by the fact that  $|\cos \psi| < 1$ , and the increase in  $a_s$  is the dominant effect.) Nevertheless, this estimate suffices to indicate that the effect can be significant. We remark that (4.33) does not necessarily imply that  $\delta k_s$  is a maximum in the center as the increase in  $a_s$  can dominate other effects. The presence of this factor in the denominator also indicates that a true self-focusing instability does not exist. The effect clearly decreases when the gain is smaller, so that the net outcome could simply be a limit on useable gain. On the positive side, the focusing effect can also lead to some optical beam trapping, which could be helpful for devices with  $L > kr_e^2$ . Clearly, the complex of issues raised here constitutes an important subject for further research.

As a final comment, we note that for an oscillator

$$\int_0^L \delta k dz \sim \frac{1}{2Q}. \quad (4.37)$$

This should be compared to the total phase shift,  $2 \tan^{-1} 0.5$ , which occurs in the traverse of the wiggler due to diffraction. As a consequence, low  $Q$  oscillators may require some small optical correction to take into account both the effect and the fact that it may vary during buildup of oscillation. Apart from this possibility, the  $\delta k_s$  effect does not appear to affect oscillator operation.

## V. ADIABATIC CAPTURE, DECELERATION, AND DECAPTURE

In this section we will treat the problem of capturing the electrons into a stationary bucket with a resonant energy equal to the mean electron energy, while at the same time minimizing the increase in the phase area occupied by the electrons. After capture, the average phase angle of the electrons must be changed to a positive value, so that the electrons can be placed in the center of a moving bucket and subsequently decelerated. For a single-pass device, it will only be necessary to make sure that the phase area of the captured electrons is less than the area of the decelerating bucket. However, for a multiple pass system, we want to minimize the increase in phase area occupied by the beam. After the beam has been decelerated, the average phase angle of the electron is returned to zero and placed in a new stationary bucket with a resonant energy equal to the final mean electron energy. Finally, the decapture is performed in such a manner as to achieve the small desired energy spread. The total purpose of all of these processes is to obtain a minimum increase in the energy spread while at the same time to reduce significantly the average electron energy.

To illustrate these processes, we divide the wiggler into five regions: region 1, with  $0 < z < z_1$ , is where the adiabatic capture occurs; region 2, with  $z_1 < z < z_2$ , is where the average phase angle is increased; region 3, with  $z_2 < z < z_3$ , is where the deceleration occurs; region 4, with  $z_3 < z < z_4$ , is where the average phase angle is decreased to zero; and region 5, with  $z_4 < z < z_5$ , is where the decapture occurs. The phase space occupied by the electrons during various stages of these processes is shown as the shaded area in Fig. 16(a)-(h).

### A. Region 1

In order to capture all of the electrons without increasing the phase area occupied by the electrons, it is necessary to use a stationary bucket, i.e.,  $\psi_r = 0$ , and to adiabatically increase the height of the bucket. From (2.51), we see that, for  $\psi_r = 0$ , the bucket height  $\delta\gamma_{\max}$  is given by

$$\delta\gamma_{\max} = \frac{2\gamma_r \sqrt{a_w a_s}}{\sqrt{1 + a_w^2}}. \quad (5.1)$$

We want to start with  $\delta\gamma_{\max} = 0$  at  $z = 0$  and increase  $\delta\gamma_{\max}$  with  $z$  while keeping  $\psi_r = 0$ . This may be accomplished by having  $a_w$  increase with  $z$  while demanding that  $k_w$  also increase with  $z$  to maintain the condition

$$k_w(z) = \frac{k_s}{2\gamma_r^2} [1 + a_w^2]. \quad (5.2)$$

Since  $\delta\gamma_{\max}(0) = 0$  and  $\gamma_r$  is a constant equal to  $\gamma_r(0)$ , we must have

$$a_w(0) = 0 \quad \text{and} \quad k_w(0) = \frac{k_s}{2\gamma_r^2}. \quad (5.3)$$

We combine (5.1) and (5.2) and obtain

$$[\delta\gamma_{\max}(z)]^4 = \frac{4k_s^2 a_s^2 [k_w(z) - k_w(0)]}{k_w^2(z) k_w(0)}. \quad (5.4)$$

From (5.4), it follows that the maximum value of the bucket height occurs for a value  $k_w(z) = 2k_w(0)$  corresponding to a value of  $a_w(z)$  equal to 1. It is desirable to have a large value for the final bucket height so we assume that region one of the wiggler is designed to increase both the magnetic field  $a_w$  and the wavenumber  $k_w$  to the final values:

$$k_w(z_1) = 2k_w(0) \quad (5.5)$$

$$a_w(z_1) = 1 \quad (5.6)$$

which yields a final bucket height

$$\delta\gamma_m(z_1) = \gamma_r \sqrt{2a_s}. \quad (5.7)$$

Note that the final value of the fractional bucket height,  $\delta\gamma_m(z_1)/\gamma_r$ , is dependent only upon the field strength and the wavelength of the radiation. In order to capture all of the electrons in the bucket, it is necessary for the initial energy spread in the beam before capture to be less than the final bucket area, i.e.,

$$2\pi \left( \frac{\delta\gamma}{\gamma} \right)_{\text{initial}} < 8 \frac{\delta\gamma_m}{\gamma}. \quad (5.8)$$

While (5.5) and (5.6) give the final values for the wiggler wavelength and magnetic field necessary to achieve the maximum bucket area for capture, the exact speed and method of achieving these final values have yet to be determined. In order to satisfy the adiabatic condition, the change in  $k_w$  and  $a_w$  must be sufficiently slow. The exact number of phase oscillations that are needed for the adiabatic condition depends upon with what accuracy one needs the phase area to remain constant, and will require computer studies for the motion of a large number of particles. However, as a rule of thumb, the capture length must be long compared to the final phase oscillation period, i.e.,

$$z_1 \gg \frac{\pi\gamma_r^2}{k_s} \sqrt{\frac{2}{a_s}}. \quad (5.9)$$

From (5.9) we see that it is desirable to have a low resonant energy and a high optical field in order to have a short capture length. The exact form of the change in  $k_w$  and  $a_w$  has only one constraint given by (5.2); thus we are free to choose the simple form

$$k_w(z) = k_w(0) \left[ 1 + \left( \frac{z}{z_1} \right)^{2\alpha} \right] \quad (5.10)$$

and

$$a_w(z) = \left( \frac{z}{z_1} \right)^\alpha \quad (5.11)$$

with  $\alpha$  a constant.

### B. Region 2

In order to decelerate the electrons after the adiabatic capture has been completed, it is first necessary to change the average phase of the electrons from zero to a positive value

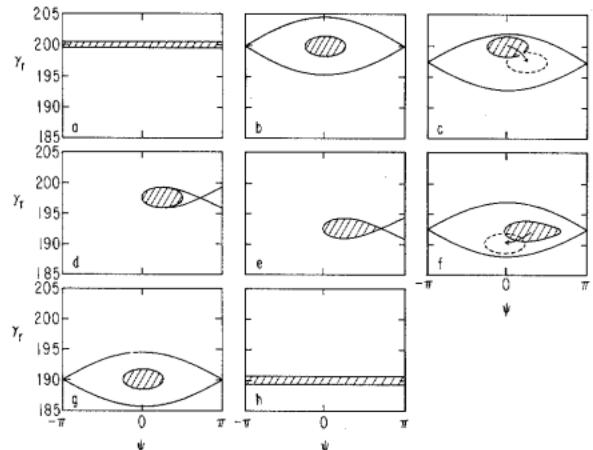


Fig. 16. Schematic behavior of the  $\gamma$ ,  $\psi$  phase space distribution as a function of  $z$  during adiabatic capture, deceleration, and decapture processes: (a) initial distribution; (b) during capture; (c) during increase of average  $\psi$ ; (d) and (e) deceleration process; (f) during the decrease of the average  $\psi$ ; (g) during decapture; and (h) after decapture.

$\bar{\psi}(z_2)$ . There are many possible programs for the variation of  $k_w$  and  $a_w$  that can be used to accomplish this change; as one example, we will describe one simple program below. From the preceding section we have shown that at the end of the adiabatic capture region ( $z = z_1$ ), the values for the wiggler period, magnetic field, and bucket height are given by (5.5)-(5.7), and the average particle phase and energy are equal to  $\bar{\psi}(z_1) = \psi_r(0) = 0$  and  $\bar{\gamma}(z_1) = \gamma_r(0)$ . Consider the case where the wiggler period and magnetic field are changed discontinuously at  $z = z_1$  such that, for  $z > z_1$ ,

$$k_w = k_w(z_1) [1 + f] \quad (5.12)$$

and

$$a_w = a_w(z_1) \quad (5.13)$$

with  $f \ll 1$ .

We can regard this as the discontinuous change in the definition of  $\gamma_r$  such that, for  $z > z_1$ ,

$$\gamma_r = \gamma_r(0) [1 - f/2] \quad (5.14)$$

with  $\psi_r$  still equal to zero. For the case where the change in  $\gamma_r$  is less than the bucket height  $\delta\gamma_m$ , the center of the electron bunch will proceed to perform synchrotron oscillations about the new value of  $\gamma_r$  [see Fig. 16(c)], with the maximum change in  $\bar{\psi}$  occurring after one-quarter of a synchrotron oscillation period. If, on the other hand, the change in the value of  $\gamma_r$  is greater than the bucket height, the magnitude of the increase of  $\bar{\psi}$  will not be limited (i.e., the electrons are de-trapped). Because we want to prevent an increase in the phase area occupied by the bunch, we need to change the phase and energy of all of the electrons in the bunch in a coherent manner. This can be achieved if all of the electrons in the bunch are near enough to the new bucket center to be in the linear region. In order to be able to accomplish this, it is necessary that the phase area occupied by the bunch must be small compared to the bucket area, and the change in the bucket center must be smaller than its height. This second condition yields

$$(f/2)\gamma_r < \delta\gamma_m. \quad (5.15)$$

If the restriction of (5.15) is satisfied, the average values of  $\psi$  and  $\gamma$  are equal to the central values, and we find that after one-quarter of an oscillation  $\bar{\psi}$  reaches its maximum value

$$\bar{\psi}_m = f \frac{\gamma_r}{\delta \gamma_m} \quad (5.16)$$

while the average energy of the bunch has changed by

$$\Delta \bar{\gamma} = -f/2\gamma_r(0). \quad (5.17)$$

### C. Region 3

With the new values of  $\bar{\psi}$  and  $\bar{\gamma}$ , it is now possible to use a tapered wiggler, as was discussed in detail in Section IV, to produce a decelerating bucket and to decrease the energy further. The energy of the bunch has already been decreased during the process of increasing  $\bar{\psi}$ , and if the desired energy extraction is sufficiently small it may be possible to skip the decelerating region entirely.

### D. Region 4

In this region, with  $z_3 < z < z_4$ , we must reduce the average phase angle to zero in preparation for the decapture; as an example, we choose a program similar to that used in region 2. That is, we choose  $k_w$  and  $a_w$  to produce a stationary bucket with a resonant energy equal to  $\bar{\gamma}(z_3)$ , and allow the bunch to perform one-quarter of an oscillation so that  $\bar{\psi}$  returns to zero [see Fig. 16(f)]. The new average phase angle and energy at  $z = z_4$  is given by

$$\bar{\psi}(z_4) = 0 \quad (5.18)$$

and

$$\bar{\gamma}(z_4) = (1 - f/2) \bar{\gamma}(z_3). \quad (5.19)$$

Note that if major decelerating region 3 is eliminated, the total energy change is

$$\Delta \gamma = \bar{\gamma}(z_4) - \bar{\gamma}(0) = -f\gamma_r \quad (5.20)$$

which corresponds to jumping the stationary bucket down in energy by  $-f/2\gamma_r$  and allowing the bunch to perform one-half of a phase oscillation.

### E. Region 5

In the fifth and final region, with  $z_4 < z < z_5$ , the electrons are debunched by allowing the magnetic field to decrease to zero. This process is the reverse of the capture process of region 1 and also must be done adiabatically in order to conserve the electron phase space density. It follows that the wiggler period must increase such that  $\bar{\psi}$  remains zero and  $\gamma_r = \text{constant}$ . First, the center of the bucket must be placed at a value of  $\gamma_r = \bar{\gamma}(z_4)$  by a discontinuous change in the wiggler period and field such that, for  $z > z_4$ ,

$$k_w(z) = (1 - f) k_w(z_4) \quad (5.21)$$

$$a_w(z) = a_w(z_4). \quad (5.22)$$

Next, we must decrease the bucket area to zero to decapture the electrons; for simplicity, we choose the reverse of the form chosen for the adiabatic capture in region 1, i.e., for  $z_4 < z < z_5$ ,

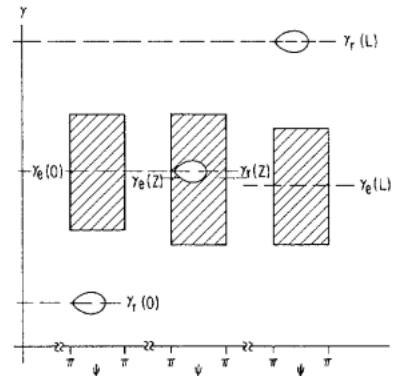


Fig. 17. Position of empty bucket and phase area of electrons at various positions in the FEL.

$$k_w(z) = \frac{k_w(z_4)}{2} \left[ 1 + \left( \frac{z_5 - z}{z_5 - z_4} \right)^{2\alpha} \right] \quad (5.23)$$

and

$$a_w(z) = \left( \frac{z_5 - z}{z_5 - z_4} \right)^\alpha. \quad (5.24)$$

It is clear that an arbitrarily long wiggler magnet could, in this way, decelerate electrons while producing little energy spread. However, in order to determine whether or not the scheme is a promising one, further analysis is required. Especially, a more quantitative estimate is needed of the number of oscillations required in the adiabatic trapping and de-trapping steps of the process. Furthermore, because of the additional length associated with these steps, a more severe sideband instability problem (see Section VII) may be anticipated.

## VI. PHASE AREA DISPLACEMENT

In order that a large fraction of the electrons be decelerated and hence transfer energy into the optical field for the operational modes of the FEL described in the previous sections, it was necessary that the energy spread of the incoming electron beam be less than the maximum bucket height. In this section we will discuss the method of phase area displacement [11] which can allow all of the electrons to be decelerated even when the initial energy spread (or effective energy spread when transverse emittance and magnetic field variation with beam size are included) is considerably larger than the bucket height. Phase area displacement refers to an operational mode in which an empty bucket is accelerated through the phase area of the beam with the result that the phase area occupied by the electrons is displaced downward in energy. Consider the case where the accelerating bucket starts with a resonant energy far below the energy of the electrons in the beam and is adiabatically moved through the beam until the final resonant energy is far above the electron's energy, as illustrated in Fig. 17. Note that for an accelerating bucket  $\psi_r < 0$ . The final mean energy of the electrons is lowered by the phase area of the empty accelerating bucket divided by  $2\pi$ , while the final energy spread of the beam is nearly equal to the initial energy spread, i.e.,

$$\langle \gamma(0) \rangle - \langle \gamma(L) \rangle = \frac{1}{2\pi} J \quad (6.1)$$

$$\Delta\gamma_f \sim \Delta\gamma_i \quad (6.2)$$

where  $J$  is the area enclosed by the accelerating bucket and is given by (2.59). From Fig. 17, we see that the total change in  $\gamma_r$  must be much larger than the sum of the bucket height and the energy spread in the beam, i.e.,

$$[\gamma_r(L) - \gamma_r(0)] \sim \gamma'_r L \gg 2(\delta\gamma)_{\text{max}} + \Delta\gamma_{\text{spread}}. \quad (6.3)$$

As long as both the adiabatic condition, which is discussed below, and the above condition are met, the energy spread of the beam is not greatly increased and the average energy loss of the electrons is independent of the initial energy spread. We note that the Hamiltonian for the particle motion, as given by (2.44), is

$$\mathcal{H}_4 = \frac{k_w}{\gamma_r} (\delta\gamma)^2 - \frac{\omega_s a_w a_s}{c\gamma_r} (\cos\psi + \psi \sin\psi_r). \quad (6.4)$$

It is convenient to use a transformation of the independent variable

$$du = \frac{2k_w}{\gamma_r} dz \quad (6.5)$$

to define a new Hamiltonian

$$\mathcal{H}_5(\psi, \delta\gamma, u) = \frac{(\delta\gamma)^2}{2} - \Lambda(\cos\psi + \psi \sin\psi_r) \quad (6.6)$$

with

$$\Lambda = \left[ \frac{\omega_s a_w a_s}{2k_w c} \right]. \quad (6.7)$$

If  $\psi_r$  is a constant less than zero for an accelerating bucket, then, for the wiggler design with  $a_w/k_w = \text{constant}$ , the Hamiltonian is a constant of the motion; for simplicity, we will specialize to this case to derive an analytical expression for the rms energy spread for the beam after passing through the FEL.

The equation for the rate of change in the synchronous energy in terms of  $u$  can be obtained from (2.41) and (6.5):

$$\frac{d}{du} \gamma_r = -\Lambda \sin\psi_r. \quad (6.8)$$

The rate of change for the total energy,  $\gamma = (\delta\gamma + \gamma_r)$ , follows directly from Hamilton's equation  $d(\delta\gamma)/du = -\partial\mathcal{H}_5/\partial\psi$  which, when combined with (6.8), yields

$$\frac{d}{du} \gamma = -\Lambda \sin\psi. \quad (6.9)$$

We can use the fact that  $\mathcal{H}_5$  is a constant of the motion along with Hamilton's equation  $d\psi/du = \partial\mathcal{H}_5/\partial(\delta\gamma)$  to obtain

$$\frac{d\psi}{du} = \delta\gamma \quad (6.10)$$

and

$$\delta\gamma = \mp\sqrt{2} \{ \bar{H} + \Lambda(\cos\psi + \psi \sin\psi_r) \}^{1/2}. \quad (6.11)$$

Thus, the change in energy is given by

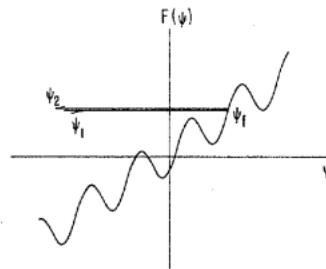


Fig. 18. Motion of the electron in a potential well of an accelerating bucket.

$$\Delta\gamma = -\frac{\Lambda}{\sqrt{2}} \int \frac{\mp(\sin\psi) d\psi}{\sqrt{\bar{H} + \Lambda(\cos\psi + \psi \sin\psi_r)}}. \quad (6.12)$$

We are considering an electron which passes completely around the bucket, i.e.,  $\psi$  starts at a large negative value  $\psi_1$ , reflects at  $\psi_f$ , and returns to a large negative value  $\psi_2$  as shown in Fig. 18. We neglect the quantities which oscillate rapidly as  $\psi_1$  and  $\psi_2 \rightarrow -\infty$ , neglect  $\psi_f$  compared to  $\psi_1$  and  $\psi_2$ , and note that  $\bar{H} = -\Lambda(\cos\psi_f + \psi_f \sin\psi_r)$  to find

$$\Delta\gamma = -\Lambda\sqrt{2} \int_{-\infty}^{\psi_f} \sin\psi_r d\psi + \left[ \frac{1}{\sqrt{\bar{H} + \Lambda(\cos\psi + \psi \sin\psi_r)}} - \frac{1}{\sqrt{\bar{H} + \Lambda(\cos\psi_f + \psi_f \sin\psi_r)}} \right]. \quad (6.13)$$

If we know the probability of an electron having a certain value of  $\psi_f$ , then we can use (6.13) to calculate both the average energy change of electrons  $\bar{\Delta}\gamma$ , and the final rms spread.

The evaluation of (6.13) is rather complicated and details are given in Appendix II. A further energy spread is expected from the fact that the bucket is, of course, not moved up from  $-\infty$  to  $+\infty$  but has a finite energy traverse

$$2\bar{\Delta}\tilde{\gamma} = \gamma_r(L) - \gamma_r(0). \quad (6.14)$$

The additional energy spread due to the end points is uncorrelated with the spread due to the spread calculated above and the two spreads should be added in quadrature. This effect also is considered in Appendix II, where it is shown that, for  $\psi_r \ll 1$ , the average energy gain is given by

$$\bar{\Delta}\gamma = \frac{8}{\pi} \sqrt{\Lambda} \quad (6.15)$$

and the ratio of rms energy spread to energy gain is approximately given by

$$\frac{(\Delta\gamma)_{\text{rms}}}{\bar{\Delta}\gamma} = \sqrt{\sin^2\psi_r + \frac{\pi^4}{(64)^2} \left( \frac{\Delta\gamma}{\bar{\Delta}\tilde{\gamma}} \right)^2}. \quad (6.16)$$

These equations are in good agreement with numerical calculations for small values of  $\psi_r$ .

For a reasonable wiggler it is not feasible to allow a very large value of  $\bar{\Delta}\tilde{\gamma}$  or a very small value of  $\psi_r$ , and because of the approximations made in deriving the Hamiltonian  $\mathcal{H}_4$ , it is

necessary to numerically integrate the equations of motion (2.22) and (2.23) for a large number of electrons with various initial conditions. For the case where  $\psi_r$  is held constant, a small fraction of the particles are often captured by the moving bucket and will have their energy increased, thereby significantly contributing to the increase in the energy spread of the final electron beam. Whether this trapping, which would not occur for a static Hamiltonian, is due to numerical errors or the small nonstatic corrections to the Hamiltonian is now not completely clear. However, it would probably be present due to small errors in any physical realization of the wiggler. The particles that may become captured are those which are near the unstable fixed point of the bucket when the bucket passes through the phase space of the beam. One method that may be used to prevent this capture is to increase the magnitude of the resonant phase angle  $\psi_r$  as the bucket is accelerated; this moves the unstable fixed point slightly, preventing trapping of the electron. This results in electrons with larger energies being decelerated less than those at the lower energies producing an extra source of energy spread in the emerging beam. A balance between all of these effects, as well as a desire to minimize the length of the wiggler, requires computer simulation to arrive at a reasonable design.

While this method will allow a much larger entrance bandwidth and produce a relatively small additional energy spread as compared to a constant wiggler, one must pay the price with a longer wiggler magnet. It also appears that the small-signal gain of such a device behaves differently from that of the constant parameter FEL [12]. At this writing, it does not appear to be out of the question that the phase area displacement wiggler has adequate characteristics to be used in a CW storage ring, although much more simulation is required and the start-up problem must be addressed.

## VII. INSTABILITIES

### A. Introduction

We have been concerned in this paper primarily with FEL's operating at very high  $\gamma$ , and our principal interest has been the mode of operation in which the electrons are trapped in decelerating potential wells for a single pass through the wiggler. In this mode of operation, collective effects should be small. In particular, the relativistic longitudinal plasma frequency  $\omega_{PL} = (4\pi ne^2/m\gamma^3)^{1/2}$  is small enough so that  $\int \omega_{PL} dt \ll 1$  for a pass through the wiggler. Thus, we will concentrate our attention on instabilities which arise essentially from interactions of individual electrons with the electromagnetic field.

These are of two kinds. First, there are problems arising from the transverse structure of the beam which occur because the effective refractive index depends on the density of trapped electrons. A qualitative discussion has been given in Section IV; a more complete quantitative treatment is in progress and will be published later.

A second type of interaction which can occur is the unstable generation of parasitic electromagnetic waves, i.e., waves with a different frequency from that of the signal which we are trying to grow. Such generation can be expected in any situation other than the simple linear regime operated at the maximum

of the signal gain curve. Thus, in the "descending bucket" scheme, a highly nonlinear mode of operation, the electrons are all effectively trapped and moving on the average at a common velocity. We might therefore worry that a wave of somewhat lower frequency than the signal wave, corresponding to a ponderomotive phase velocity  $v = \omega_R/k_R + k_w$  which is slightly smaller than the bucket velocity, could be unstably amplified. In the remainder of this section we will show that such amplification does occur.

### B. Linear Theory of Raman Instability

The problem which we will treat is that of the stability of an infinitesimal electromagnetic wave as it perturbs the equilibrium of electrons trapped in the ponderomotive buckets of the signal wave. In a single-pass wiggler, this equilibrium is not quite a static one. The potential wells may change slowly in shape and depth as the wiggler is traversed, the electron oscillation phase may not be quite random due to the trapping process at the entrance to the wiggler, and the wiggler is, of course, of finite length. In order to make the stability problem tractable, we will idealize to a true static equilibrium. This would correspond physically to a constant parameter wiggler in which the electrons are uniformly accelerated by a dc (or resonant RF) longitudinal electric field of such a magnitude as to exactly balance the radiative loss to the signal wave. This equilibrium may be described by the Hamiltonian  $\mathcal{H}_4$  given in (2.44):

$$\mathcal{H}_4 = \frac{k_w + \delta k_s}{\gamma_r} (\delta\gamma)^2 - \frac{\omega_s a_w a_s}{c\gamma_r} (\cos\psi + \psi \sin\psi_r) \quad (7.1)$$

where we recall that  $\gamma_r$  is the resonant  $\gamma$  for particles in the well,  $k_w$  is the wiggler wavenumber,  $\delta k_s$  is the modification of signal wavenumber which may be lumped into  $k_w$  and will be subsequently ignored,  $a_w$  and  $a_s$  are the nondimensional vector potentials of the wiggler and signal fields, and the phase  $\psi \equiv \int_0^z (k_w + k_s) dz - \omega_s t$ . We recall that the canonical variables in (7.1) are  $\delta\gamma$  and  $\psi$ . The independent variable is  $z$ .

For the static situation which we are considering, all quantities except  $\delta\gamma$  and  $\psi$  are constants, and from (2.41) we see that the dc electric field necessary to maintain the static equilibrium is given by

$$\frac{eE}{mc^2} = \frac{a_w a_s \omega_s}{\gamma_r c} \sin\psi_r. \quad (7.2)$$

We note further, from (2.41), that the rate at which a given electron is radiating into the signal wave depends on its phase and is given by (in units of  $mc^2$ )

$$\dot{E}_s = \frac{a_s a_w \omega_s}{\gamma_r} \sin\psi. \quad (7.3)$$

In order to eliminate unnecessary constants, it is convenient to define  $U^2 = (2k_w \omega_s a_s a_w / c\gamma_r^2) z^2$ , and  $p = \sqrt{2k_w c / \omega_s a_s a_w} \delta\gamma$ . In these units, the Hamiltonian becomes simply

$$\mathcal{H}_5 = \frac{p^2}{2} - (\cos\psi + \psi \sin\psi_r) \quad (7.4)$$

where  $p$  and  $\psi$  are the new canonical momentum and coordinate, effectively, and  $U$  is the independent variable.

This is, of course, the Hamiltonian for the pendulum equation which we have discussed in Sections II and III with a succession of potential wells as shown in Fig. 1. Electrons trapped in the potential well oscillate with frequency  $\Omega(\mathcal{H}_5)$ . This frequency ranges from  $\Omega = 0$  for particles at the top of the potential well to  $\Omega = \sqrt{\cos \psi_r}$  for particles at the bottom. Thus,  $\Delta U \approx 2\pi$  corresponds to a synchrotron period, while a typical wiggler length would be on the order of  $U \sim (a_s)^{-1/2}$  for an efficient single-pass design. To specify the equilibrium, we must give the distribution of electrons  $f(p, \psi)$  in the potential well. In steady state, we expect  $f$  to be independent of the phase of oscillation in the potential. It is therefore convenient to introduce action angle variables:

$$J(\mathcal{H}_5) = \oint p d\psi = \oint d\psi' \sqrt{2(\mathcal{H}_5 + \cos \psi' + \psi' \sin \psi_r)} \quad (7.5)$$

and

$$\phi(\mathcal{H}_5, \psi) = \Omega \int_0^\psi \frac{d\psi'}{\sqrt{2(\mathcal{H}_5 + \cos \psi' + \psi' \sin \psi_r)}} \quad (7.6)$$

with

$$\Omega(\mathcal{H}_5) = 2\pi \left[ \oint \frac{d\psi}{p} \right]^{-1}.$$

The symbol  $\oint$  means a complete cycle of oscillation in the potential well. From (7.6), we see that  $\phi$  varies between 0 and  $2\pi$  in an oscillation period. The area in-phase space is given by  $(1/2\pi) dJ d\phi$  and the equilibrium distribution function is independent of  $\phi$ . Thus, we specify the equilibrium by giving

$$f_0 = f_0(J). \quad (7.7)$$

The equations of motion of a particle are

$$\frac{dJ}{dU} = 0 \quad (7.8)$$

and

$$\frac{d\phi}{dU} = \Omega(J). \quad (7.9)$$

Note, that because of the well-known adiabatic properties of  $J$ , we might expect our results to be applicable also to slowly varying wigglers. For convenience, we normalize

$$\int_0^{J_{\max}} f_0(J) dJ = 1.$$

The rate at which energy is given to the signal wave may be seen from (7.3) to be

$$\frac{dE_s}{dz} = \frac{a_s a_w \omega_s}{c \gamma_r} \frac{1}{2\pi} \iint f \sin \psi dJ d\phi. \quad (7.10)$$

For the equilibrium distribution, this is

$$\frac{dE_s}{dz} = \frac{a_s a_w \omega_s}{c \gamma_r} \sin \psi_r \quad (7.11)$$

since  $\langle \sin \psi \rangle = \sin \psi_r$  (the bracket indicating a phase average, i.e.,  $\langle A \rangle = (1/2\pi) \int Ad\phi$ ). Finally, we note that the phase space above the potential well must be empty, i.e., the phase space density of untrapped electrons must be much smaller than that for trapped electrons which have been held together by the ponderomotive wave. Moreover, we expect the distribution to be continuous, since various diffusive processes which we will discuss later are operative. Hence, it is reasonable to impose the condition

$$f_0(J_{\max}) = 0 \quad (7.12)$$

where  $J_{\max}$  corresponds to the top of the well.

Next we consider the introduction of a small perturbing electromagnetic wave defined by a helical vector potential analogous to that given for the signal wave in (2.8):

$$\vec{A}_R = \epsilon A_s [\hat{x} \cos(k_R z - \omega_R t) - y \sin(k_R z - \omega_R t)]. \quad (7.13)$$

Here,  $\epsilon$  specifies the relative strength of the perturbing wave. The Hamiltonian  $\mathcal{H}_1$  of (2.9) then becomes

$$\mathcal{H}'_1 = mc^2 \{ \mu^2 + \gamma^2 + 2a_w a_s \cos \psi + 2\epsilon a_w a_s \cos \psi_R \}^{1/2}$$

where the phase of the perturbing ponderomotive wave is given by

$$\psi_R \equiv (k_w + k_R) z - \omega_R t = k_w \left( 1 - \frac{\omega_R}{\omega_s} \right) z + \frac{\omega_R}{\omega_s} \psi. \quad (7.14)$$

The development of Section II is easily modified to include both radiation fields,  $A_r$  and  $A_s$ , to yield the new Hamiltonian:

$$\mathcal{H}'_5 = \frac{p^2}{2} - (\cos \psi + \psi \sin \psi_r) - \epsilon \cos \psi_R \quad (7.15)$$

with, in these units,  $\psi_R = \psi + \lambda U$ . The canonical variables, of course, remain  $p$  and  $\psi$ . We will retain the angle action variables defined by (7.5) and (7.6), although the evolution of  $p$  and  $\psi$  are now given by Hamilton's equations for (7.15).

In this expression, we have neglected  $[(\omega_R/\omega_s) - 1] \psi$  since  $|(\omega_R/\omega_s) - 1| \ll 1$  and  $\psi$  remains bounded for trapped particles. Furthermore,

$$\lambda \equiv \frac{1}{2} \sqrt{\frac{1 + a_w^2}{a_s a_w}} \left[ 1 - \frac{\omega_R}{\omega_s} \right]. \quad (7.16)$$

(We note parenthetically that for  $a_w \approx 1$ ,  $\lambda$  would be nearly constant even for a variable wiggler oscillator. For an amplifier, in which  $a_s$  is a strong function of  $z$ ,  $\lambda$  may vary appreciably.)

Similarly following the developments in Section II, we find that the energy given to the perturbing wave is analogous to that described by (7.10), i.e.,

$$\frac{dE_R}{dz} = \epsilon \frac{a_s a_w \omega_s}{\gamma_r c} \frac{1}{2\pi} \iint f \sin \psi_R dJ d\phi. \quad (7.17)$$

Since  $f_0$  is independent of  $U$ , the energy transfer given by (7.17) will be oscillatory in  $U$ , i.e., vanish, on the average, for the equilibrium distribution  $f=f_0$ . Hence, it is necessary to proceed to the next order and determine  $f_1$ , the distribution function to first order in  $\epsilon$ , the amplitude of the sideband.

Noting that the phase space element  $2\pi dpd\psi \equiv dJd\phi$ , we can write the Liouville equation for the distribution function:

$$\frac{\partial f}{\partial U} + \frac{d\phi}{dU} \frac{\partial f}{\partial \phi} + \frac{dJ}{dU} \frac{\partial f}{\partial J} = 0.$$

Recalling that  $\partial f_0/\partial U = \partial f_0/\partial \phi = 0$ , the linearized Liouville equation is

$$\frac{\partial f_1}{\partial U} + \Omega \frac{\partial f_1}{\partial \phi} + \left. \frac{dJ}{dU} \right|_1 \frac{\partial f_0}{\partial J} = 0. \quad (7.18)$$

It is thus necessary to find the rate of change of  $J$  caused by the perturbation.

From (7.4)–(7.7), we have

$$\begin{aligned} \frac{dJ}{dU} &= \frac{2\pi}{\Omega} \frac{d\mathcal{H}_5}{dU} = \frac{2\pi}{\Omega} \left[ -\frac{\partial \mathcal{H}_5}{\partial p} \frac{\partial \mathcal{H}'_5}{\partial \psi} + \frac{\partial \mathcal{H}_5}{\partial \psi} \frac{\partial \mathcal{H}'_5}{\partial p} \right] \\ &= -\frac{2\pi}{\Omega} p\epsilon \sin \psi_R = -2\pi\epsilon \frac{\partial \psi}{\partial \phi} \sin \psi_R \\ &= 2\pi\epsilon \frac{\partial}{\partial \phi} \cos \psi_R. \end{aligned}$$

We may now substitute, into (7.18),

$$f_1 = -\frac{2\pi\epsilon}{\Omega} \cos \psi_R \frac{\partial f_0}{\partial J} + \hat{f}_1$$

to see that  $\hat{f}_1$  satisfies

$$\frac{\partial \hat{f}_1}{\partial U} + \Omega \frac{\partial \hat{f}_1}{\partial \phi} + \epsilon \frac{2\pi\lambda}{\Omega} \sin [\psi(J, \phi) + \lambda U] \frac{\partial f_0}{\partial J} = 0. \quad (7.19)$$

The first term in the above expression for  $f_1$  will give no nonoscillatory contribution to  $dE_R/dz$  as may be seen from (7.17) since  $\cos \psi_R \sin \psi_R$  oscillates with frequency  $2\lambda U$ . Hence, only  $\hat{f}_1$  contributes to energy transfer. We solve (7.19) by first substituting

$$\hat{f}_1 = f_+(J, \phi) e^{-i\lambda U} + \text{c.c.}$$

Furthermore, writing  $f_+ = e^{i(\lambda/\Omega)\phi}$  it is easy to solve for

$$f_+ = -i\epsilon \frac{\pi\lambda}{\Omega^2} \left( \exp i \frac{\lambda}{\Omega} \phi \right) \frac{\partial f_0}{\partial J} \int_{-\infty}^{\phi} d\phi' \exp -i \left( \psi' + \frac{\lambda}{\Omega} \phi' \right). \quad (7.20)$$

The lower limit has been specified by the following argument familiar in plasma physics. We must suppose the perturbation to be a growing wave, i.e.,  $\lambda$  has a small positive imaginary part. This makes the integral convergent. Moreover, we see that  $f_+(\phi + 2\pi) = f_+(\phi)$  as is required for a single-valued function. Hence, (7.20) is the unique solution of (7.19). We may now find the energy transfer rate from (7.17):

$$\frac{dE_R}{dz} = \epsilon \frac{a_s a_w \omega_s}{\gamma_r c} \frac{1}{2\pi} \iint dJ d\phi \sin \psi_R [f_+ e^{-i\lambda U} + \text{c.c.}].$$

We are, in fact, concerned with that part of the radiated energy which is nonoscillatory in  $U$ , i.e.,

$$\begin{aligned} \frac{dE_R}{dz} &= -\epsilon^2 \frac{a_s a_w \omega_s}{\gamma_r c} \frac{\pi}{2} \lambda \int_0^{J_{\max}} dJ \frac{1}{\Omega^2} \frac{\partial f_0}{\partial J} \\ &\cdot \left[ \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp i \left( \frac{\lambda}{\Omega} \phi + \psi \right) \right. \\ &\left. \cdot \int_{-\infty}^{\phi} d\phi' \exp -i \left( \psi' + \frac{\lambda}{\Omega} \phi' \right) + \text{c.c.} \right]. \end{aligned} \quad (7.21)$$

We may simplify the  $\phi'$  integral in the following way. Note that  $\psi$  is a periodic function of  $\phi$  so that the integrals for successive periods differ by factors of  $e^{-i\lambda 2\pi/\Omega}$ . Thus,

$$\begin{aligned} I &= \int_0^{2\pi} d\phi \exp i \left( \frac{\lambda}{\Omega} \phi + \psi \right) \int_{-\infty}^{\phi} d\phi' \exp -i \left( \psi' + \frac{\lambda}{\Omega} \phi' \right) \\ &= \int_0^{2\pi} d\phi \exp i \left( \frac{\lambda}{\Omega} \phi + \psi \right) \left[ \sum_{n=0}^{\infty} \exp 2\pi i n \frac{\lambda}{\Omega} \right. \\ &\cdot \int_0^{\phi} \exp -i \left( \psi' + \frac{\lambda}{\Omega} \phi' \right) d\phi' \\ &\left. + \sum_{n=0}^{\infty} \exp 2\pi i(n+1) \frac{\lambda}{\Omega} \int_{\phi}^{2\pi} \exp -i \left( \psi' + \frac{\lambda}{\Omega} \phi' \right) d\phi' \right]. \end{aligned}$$

We may exchange labels  $\phi$  and  $\phi'$  in the second term to obtain

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \int_0^{2\pi} d\phi \int_0^{\phi} d\phi' \left[ \exp 2\pi i n \frac{\lambda}{\Omega} \exp -i \left( \psi' + \frac{\lambda}{\Omega} \phi' \right) \right. \\ &\cdot \exp i \left( \frac{\lambda}{\Omega} \phi + \psi \right) + \exp 2\pi i(n+1) \frac{\lambda}{\Omega} \\ &\left. \cdot \exp -i \left( \psi + \frac{\lambda}{\Omega} \phi \right) \exp i \left( \frac{\lambda}{\Omega} \phi' + \psi' \right) \right]. \end{aligned}$$

Finally,  $I$  must be added to its complex conjugate to yield

$$\begin{aligned} I &= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} d\phi \int_0^{\phi} d\phi' \exp 2\pi i n \frac{\lambda}{\Omega} \\ &\cdot \left\{ \exp -i \left[ \psi' - \psi + \frac{\lambda}{\Omega} (\phi' - \phi) \right] \right. \\ &\left. + \exp -i \left[ \psi - \psi' + \frac{\lambda}{\Omega} (\phi - \phi') \right] \right\}. \end{aligned}$$

However,

$$\sum_{n=-\infty}^{\infty} \exp 2\pi i n \frac{\lambda}{\Omega} = \sum_{m=-\infty}^{\infty} \delta\left(\frac{\lambda}{\Omega} - m\right) = \sum_{m=-\infty}^{\infty} \Omega \delta(\lambda - m\Omega)$$

where  $m$  is any integer. We may again exchange  $\phi$  and  $\phi'$  in the second term to obtain, finally,

$$I = \sum_{m=-\infty}^{\infty} \delta\left(\frac{\lambda}{\Omega} - m\right) \left| \int_0^{2\pi} d\phi \exp i\left(\psi + \frac{\lambda}{\Omega}\phi\right) \right|^2. \quad (7.22)$$

This result may now be substituted into (7.21). However, it is useful first to define the relative gain of the unstable wave to the signal gain. Thus,

$$G(\lambda) = \frac{1}{e^2 a_s^2} \frac{dE_R}{dz} / \frac{1}{a_s^2} \frac{dE_s}{dz}. \quad (7.23)$$

From (7.11), (7.22), and (7.23), we have

$$G(\lambda) = - \sum_{m=-\infty}^{\infty} m \int_0^{J_{\max}} dJ \left\{ \frac{\partial f_0}{\partial J} \delta(\lambda - m\Omega(J)) \cdot \left| \int_0^{2\pi} \exp i(\psi + m\phi) d\phi \right|^2 \right\} / 4 \sin \psi_r. \quad (7.24)$$

Recall that we have normalized  $\int f_0 dJ = 1$ .

Equation (7.24) is our principal result. We note that in an amplifier,  $G$  directly gives us the number of  $e$ -foldings of an unstable mode relative to the signal gain. In general, it will turn out that  $G$  is only slightly larger than 1 and therefore the buildup from noise is not significant. Furthermore, we see from (7.16) that since in an amplifier  $a_s$  varies strongly with  $U$ , the resonant condition  $\lambda = m\Omega$  is only met for a short time. We conclude that the instability is not important for an amplifier.

For an oscillator, however, where the optical waves are reflected many times, we must require  $G(\lambda) < 1$  if the external circuit is nondispersive, i.e., if the external fractional attenuation of the signal and the sideband are equal. For  $G > 1$ , the sideband would grow when the signal is in steady state, and as we will discuss in the following section, would presumably destroy the particle trapping. Hence, an effective requirement for oscillator stability is  $G(\lambda) < 1$  for all  $\lambda$ .

The form of (7.24) immediately suggests the physical interpretation of our results. Recall that the action  $J$  is analogous to the quantum level of the electron in the potential well. Thus, the instability consists of a stimulated Raman scattering in which the signal wave  $\omega_s$  decays into the sideband  $\omega_R$  plus  $m$  "quanta" of particle oscillation. Since both this process and its inverse are possible, the net energy transfer depends on the derivative of the distribution function  $\partial f_0 / \partial J$ . If  $\partial f_0 / \partial J < 0$ , the usual case, it is waves with  $\lambda > 0$ , i.e., the lower sideband, which grow. When many waves of different frequencies  $\lambda$  are excited, we may expect the particles to diffuse in the potential well leading eventually to detrapping as will be discussed later.

Equation (7.24) is, of course, a rather complicated one, with the gain depending in detail on the distribution function. We have not quite been able to prove that it is impossible to make

$G(\lambda) < 1$  for all  $\lambda$ . However, the following sum rule indicates strongly that this is the case. Consider

$$\Gamma = \int_0^{\infty} G\lambda d\lambda = - \sum_{m=0}^{\infty} \int dJ \frac{\partial f_0}{\partial J} m^2 \Omega \cdot \left| \int_0^{2\pi} d\phi \exp i(\psi + m\phi) \right|^2 (4 \sin \psi_r)^{-1}. \quad (7.25)$$

Now,

$$\begin{aligned} & \sum_{m=0}^{\infty} m^2 \left| \int_0^{2\pi} \exp i(\psi + m\phi) d\phi \right|^2 \\ &= \sum_{m=0}^{\infty} \int_0^{2\pi} e^{i\psi} \frac{\partial}{\partial \phi} e^{im\phi} d\phi \int_0^{2\pi} e^{-i\psi'} \frac{\partial}{\partial \phi'} e^{-im\phi'} d\phi' \\ &= \frac{1}{2} \sum_{m=-\infty}^{\infty} \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{\partial \psi}{\partial \phi} \frac{\partial \psi'}{\partial \phi'} \\ &\quad \cdot \exp i(\psi - \psi') \exp im(\phi - \phi') = \pi \int_0^{2\pi} d\phi \left( \frac{\partial \psi}{\partial \phi} \right)^2 \\ &= \pi \oint \frac{\partial \psi}{\partial \phi} d\psi = \pi \frac{J}{\Omega}. \end{aligned}$$

In the last line we have used definitions (7.5) and (7.6). Substituting this result into (7.25) and integrating by parts on  $J$  while using the fact that  $f_0(J_{\max}) = 0$ , we find that

$$\Gamma = \int_0^{\infty} G\lambda d\lambda = \frac{\pi}{4 \sin \psi_r}. \quad (7.26)$$

We may compare this to  $\Gamma_0$ , the value which we would have for a marginal case where  $G \equiv 1$  from  $\lambda = 0$  to  $\lambda = \Omega_{\max} = (\cos \psi_r)^{1/2}$ , and zero for  $\lambda > \Omega_{\max}$ . We see that

$$\frac{\Gamma}{\Gamma_0} = \frac{\pi}{\sin 2\psi_r} > 1. \quad (7.27)$$

This result is not quite conclusive proof of instability since  $G$  is nonzero even for  $\lambda > \Omega_{\max}$  due to harmonic emission. However, in practice, these growth rates appear to be small, and are not enough to offset the numerical factor in (7.27). We would expect therefore that values of  $G$  somewhat greater than unity would typically be found. This is indeed true for all distributions we have looked at.

There is a particular distribution function  $f_0$ , of some physical interest, for which (7.24) simplifies somewhat. This distribution is given by

$$f_0 = \frac{1}{J_{\max}} \quad \text{for } J < J_{\max}$$

and

$$f_0 = 0 \quad \text{for } J \geq J_{\max}. \quad (7.28)$$

If electrons are introduced into the wiggler from a distribution

which has some energy spread and is random in optical phase, we would expect the phase space to be uniformly occupied as in (7.28). Such a distribution would also tend to arise from a slow diffusive process in which electrons were boiled off from the top of the well. One might also expect this distribution to be relatively favorable since the derivative of  $f_0$  is large only near the top of the well where the coupling to the radiation is weak.

Since the integrand of (7.24) is nonvanishing, only for  $J \sim J_{\max}$  we may take the orbit integrals in (7.24) to be those for the particle at the top of the well, i.e., in (7.5) we should put

$$H = H_M = \cos \psi_r - (\pi - \psi_r) \sin \psi_r.$$

It is useful to take advantage of periodicity to improve convergence of the orbit integral for particles near the top of the well. We may thus put

$$\begin{aligned} \int_0^{2\pi} e^{i(\psi - m\phi)} d\phi &= \Omega \oint [e^{i\psi} + e^{-i\psi_r}] \\ &\cdot \frac{\exp \left[ im\Omega \int_0^\psi d\psi' [2(H_M + \cos \psi' + \psi' \sin \psi_r)]^{-1/2} \right]}{\sqrt{2(H_M + \cos \psi + \psi \sin \psi_r)}} d\psi \\ &= \Omega K(\lambda) \end{aligned}$$

where

$$\begin{aligned} K(\lambda) &= \oint [e^{i\psi} + e^{-i\psi_r}] \\ &\cdot \frac{\exp \left[ i\lambda \int_0^\psi d\psi' [2(H_M + \cos \psi' + \psi' \sin \psi_r)]^{-1/2} \right]}{\sqrt{2(H_M + \cos \psi + \psi \sin \psi_r)}} d\psi \end{aligned} \quad (7.29)$$

is independent of  $J$ . Note that the subtraction we have performed makes the integrals smoothly convergent. Equation (7.24) then becomes

$$\begin{aligned} G(\lambda) &= + \sum m \int_0^{\Omega_{\max}} d\Omega \frac{\partial f}{\partial \Omega} \\ &\cdot \delta(\lambda - m\Omega) \Omega^2 |K(\lambda)|^2 / 4 \sin \psi_r \\ &= + \sum_{m=0}^{\infty} \Omega^2 \frac{\partial f}{\partial \Omega} \Big|_{\Omega=\lambda/m} \frac{|K(\lambda)|^2}{4 \sin \psi_r}. \end{aligned}$$

But  $\Omega^2 (\partial f / \partial \Omega) = -[\partial f / \partial(1/\Omega)]$  and the sum over  $m$  may be converted to an integral  $\lambda d(1/\Omega)$ . Hence, we obtain the growth rate

$$G(\lambda) = \frac{\lambda |K(\lambda)|^2}{4 \sin \psi_r J_{\max}}. \quad (7.30)$$

We have evaluated (7.30) numerically and find that the growth rates indeed have the properties we have described. The results are illustrated in Fig. 19 for various  $\psi_r$ . Thus, peak gains of about  $G = 2$  occur typically for values of  $\lambda \sim 0.4-0.5$ . The optimum value of  $\psi_r$  is seen to be around  $30^\circ$ . A signifi-

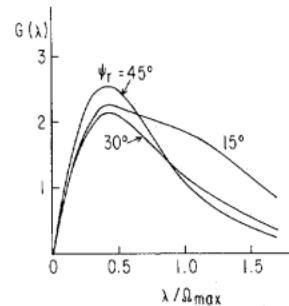


Fig. 19. Relative gain of sideband compared to the signal.  $G(\lambda)$  is defined by (7.30) and plotted as a function of the frequency shift of the test wave for different values of  $\psi_r$ .

cant feature of the results is that there is a range  $\lambda < 0.1$  for which  $G < 1$  even for a distribution in which the wall is filled to the top. This implies that the growing sideband is separated by a finite frequency interval from the signal so that dispersive external optics may be applied to correct the modest parasitic excess growth.

From (7.16), we note that the unstable region is given by

$$0.2 \sqrt{\frac{a_s a_w}{1 + a_w^2}} < \left[ 1 - \frac{\omega_R}{w_s} \right] < 2 \sqrt{\frac{a_s a_w}{1 + a_w^2}}. \quad (7.31)$$

Here,  $a_w \sim 1$  and as a typical value is  $a_s = 10^{-4}$ , the unstable sideband extends in the range  $10^{-2} > (\Delta\omega/\omega) > 10^{-3}$ .

### C. Discussion and Conclusions

In Section VII-B, we have shown that an idealized constant wiggler oscillator is linearly unstable against sideband generation. In this section we will describe briefly the results of some particle simulations and some nonlinear estimates.

In principle it appears attractive to do a one-dimensional particle simulation, calculating particle orbits and interactions with the electromagnetic waves. We have, in fact, done some simulations of the pendulum equations (2.25) including the linearized equation for  $\delta\psi$  resulting from the presence of the sideband. By integrating (7.24) by parts, we see that the rate of energy transfer by a particle to the sideband field should be proportional to

$$\frac{dE}{dU} \sim \epsilon^2 \frac{\partial}{\partial J} \left\{ \delta(\lambda - m\Omega(J)) \left| \int_0^{2\pi} \exp i(\psi + m\phi) d\phi \right|^2 \right\}. \quad (7.38)$$

Here,  $E$  is in units of the bucket height  $\gamma \sqrt{a_s}$  and all other quantities are  $\Theta(1)$ .

This result would evidently be a bit difficult to simulate. In fact, the wigglers which were simulated were of finite length and variable properties. If we try to take the finite length into account by solving (7.19) over a finite interval in  $U$ , the infinite sums leading to (7.22) are replaced by finite sums and we find that (7.38) is approximately modified (averaging over the initial phases of the particles or the sideband waves) by replacing

$$\delta(\lambda - m\Omega) \rightarrow \frac{1}{\pi} \frac{\sin(\lambda - m\Omega) U}{\lambda - m\Omega}. \quad (7.39)$$

Variable wiggler profiles lead to  $\lambda$  depending on  $U$  as is seen from (7.16), and it should be so interpreted in the argument of the  $\delta$  function in (7.38). We have not found a simple way to represent both effects simultaneously.

The results of the simulations so interpreted were in agreement with (7.38), and unstable over-all relative gains  $G = 3\text{--}4$ , consistent with the distributions studied, were observed. There was, however, one striking result which has not been quantitatively analyzed. Particles near the top of the well sometimes gained or lost large amounts of energy. This came about because such particles, as they are about to detrap, are in an unstable equilibrium perched at the top of the well, and strongly perturbed by the sideband. For such particles, the assumption of many oscillation periods is clearly incorrect. The fluctuations of energy were large and of either sign and we have not determined whether such particles produce any net effect, either stabilizing or destabilizing. Very good statistics would be required to determine this, but on the basis of our limited simulations it seems unlikely that this effect is strong enough to modify our overall picture.

We pass now to a brief discussion of the nonlinear behavior of the excited wave. We first consider the situation where only a single unstable mode is excited in addition to the signal. In this case, we expect the wave to be stabilized by eventual detuning of resonant particles from the wave as their energy is changed. We will not present here the lengthy calculation of this effect but will give a qualitative estimate based on (7.38) and (7.39).

We imagine a sideband wave with amplitude  $\epsilon$  interacting with an electron beam weak enough so that the amplitude  $\epsilon$  is unchanged during the interaction. After a certain distance  $U$ , which we wish to determine, the resonant electrons will become detuned and further transfer of energy to the wave will not occur. Integrating (7.38) with respect to  $U$  from 0 to  $U$ , we find for a nearly resonant particle, i.e., a particle for which  $(\lambda - m\Omega)U \sim \mathcal{O}(1)$ , an energy change at position  $U$

$$\Delta E \sim \epsilon^2 \frac{\partial}{\partial J} \frac{\sin^2(\lambda - m\Omega) U}{2} \left| \int_0^{2\pi} \exp i(\psi - m\phi) d\phi \right|^2. \quad (7.40)$$

This may be estimated by expanding the sine and putting  $(\lambda - m\Omega)U \sim 1$ , while recalling that in our units all parameters are of order unity. Thus,

$$\Delta E \sim \epsilon^2 U^3.$$

However, detuning will occur for  $(\partial\Omega/\partial E)\Delta EU \sim 1$ , i.e., the resonant electrons will be detuned for  $U \gtrsim \epsilon^{-1/2}$ . The width of the resonance, i.e., the number of electrons involved at this point, will be of order  $U^{-1} \sim \epsilon^{1/2}$ . Integrating (7.38) over  $J$ , we see that the peak energy transfer rate  $\Delta E' \sim \epsilon^2$  and the maximum energy transferred to the wave in an infinite length wiggler is then

$$\Delta E_{\text{sat}} \sim \epsilon^{3/2}. \quad (7.41)$$

This contrasts with the signal wave growth which continues for the length of the wiggler. For good extraction,  $\Delta\gamma/\gamma \sim \mathcal{O}(1)$ , this length is  $\Delta U \sim a_s^{-1/2}$  as noted earlier. Hence, the relative gain may be estimated by  $G = (1/\epsilon^2) \Delta E_{\text{sat}} a_s^{1/2} = (a_s/\epsilon)^{1/2}$ . Another way to see this is to note that over the distance in which it is amplified,  $\epsilon^{-1/2}$ , the sideband growth rate is only slightly larger than that of the signal, while the latter continues to grow for the full length  $a_s^{-1/2}$ . Thus, a single wave will saturate at a very low level,  $\epsilon \sim a_s \sim 10^{-4}$ , since above this level its growth is less than that of the signal for each pass of the wiggler.

Unfortunately, there may be of order  $10^9$  optical wavelengths between mirrors, hence,  $10^7$  possible unstable modes (even neglecting off-angle propagation). Under these circumstances even the relatively narrow resonances for the saturated single modes overlap and we can expect that the resonant detuning will not occur, since particles will simply move from one wave to the next as their energy changes. In these circumstances, it seems appropriate to assume random phases for the waves, leading to a quasi-linear diffusion of the particles in the well. The quasi-linear diffusion equations may be obtained by carrying (7.19) to second order. Omitting the details of the derivation, we find

$$\frac{\partial f_0}{\partial U} = \sum_m m^2 \frac{\pi}{2} \frac{\partial}{\partial J} \epsilon^2 (m\Omega) \left| \int_0^{2\pi} \exp i(\psi + m\phi) d\phi \right|^2 \cdot \frac{\partial f_0}{\partial J}. \quad (7.42)$$

Here we have normalized  $\epsilon^2(\lambda)d\lambda$  as the energy in sideband waves between  $\lambda$  and  $\lambda + d\lambda$  relative to the signal. Equation (7.42) is consistent with our physical picture of the instability, describing a Brownian motion of the electrons in the potential well as the Raman scattering proceeds. If we multiply (7.42) by  $J$  and integrate over a small interval, we see that the rate of change in the number of action "quanta" in that interval is proportional to  $m$  times the rate of energy gain of the waves in that interval as given by (7.24). (Recall that  $f_0 dJ$  is the number of electrons.) The form of (7.42) can be simplified if we assume a flat sideband spectrum, i.e., put  $\epsilon^2(\lambda) \equiv \epsilon^2$  for all  $\lambda$ . The sum on  $m$  can then be done as in the derivation of (7.26) to give

$$\frac{\partial f_0}{\partial U} = \epsilon^2 \frac{\pi^2}{2} \frac{\partial}{\partial J} \frac{J}{\Omega} \frac{\partial f_0}{\partial J}. \quad (7.43)$$

This is a diffusion equation with diffusion coefficient of order  $\epsilon^2$ . Hence, a characteristic diffusion distance for detrapping to occur is  $U \sim \epsilon^{-2}$ . It follows that when the energy in the sidebands is greater than  $a_s^{1/2}$  times the signal energy, the buckets will be destroyed before the wiggler has been traversed. We have, of course, assumed random phases in the sideband waves, but any kind of phase locking which could restore stable buckets seems improbable.

In summary we find that the scheme of decelerating electrons trapped in potential wells is subject to a Raman sideband instability where the gain of the unstable wave is several times

the signal gain. It seems probable that any mode of FEL operation which tries to suppress energy spread, e.g., adiabatic trapping, gain expanders, etc., would be subject to similar instabilities, which are inherent to operation away from the peak of the linear gain curve.

We note finally that rather modest dispersive optics would be required to stabilize a high  $Q$  trapped electron oscillator. If the cavity  $Q$  were 50, it would have to be reduced to 20 for the sidebands. This could presumably be accomplished by incorporating a grazing incident diffraction grating in the optical system or by providing an etalon grating in the outer part of the mirrors. It would also be possible to introduce a gas or liquid with enough dispersion so that the sideband pulses would arrive back out of phase with the electron micropulses, if breakdown could be avoided at high power.

## VII. CONCLUSION

In the preceding sections we have described and analyzed a number of techniques for making use of the properties of variable parameter wigglers to substantially improve the efficiency of energy extraction from the electrons. These techniques have the common feature of requiring the existence of substantial ponderomotive potential wells or "buckets" which are then manipulated to produce desirable properties. Because of this feature, they all require high optical peak power, either circulating in a resonator or emitted from an amplifier, and are thus intrinsically high peak power devices.

A number of potential problems have been identified for "filled" bucket devices. One of these, especially applicable to oscillators, is the potential instability to parasitic oscillations at frequencies displaced from the main frequency by an amount on the order of the frequency of particle oscillation in the trapping buckets. It appears that some frequency discrimination device, such as a diffraction grating, may be required to suppress these oscillations. The second, especially applicable to amplifiers, has to do with the effects of transverse variation. The amplification process may introduce growing inhomogeneity in the optical field, arising both from inhomogeneity in the growth rate and transverse inhomogeneity in the induced nonlinear index of refraction. The assessment of this problem requires further work and clarification before one can judge its magnitude. More generally, the one-dimensional character of our analysis is a major deficiency. Serious attempts to design practical devices should be based upon an analysis which includes three-dimensional effects more adequately.

Despite the above comments, it is our overall conclusion that the use of a variable parameter wiggler in an FEL laser to produce high power optical radiation at selectable frequency and reasonable efficiency is a promising prospect. The exact scheme that is the best one to use (or combination of various schemes) will depend upon the type of electron beam that is available, as well as the type of use for the radiation. At the present time there are several groups that are pursuing the variable wiggler approach, and we have greatly enjoyed our interaction with them.

## APPENDIX I THE RELATION BETWEEN ENERGY SPREAD AND ENERGY TRANSFER IN THE SMALL-SIGNAL LIMIT

Madey [12] has proved an important theorem which, in the small-signal limit, relates the phase averaged energy spread to the phase averaged energy change experienced by an electron as it passes through the laser interaction region. The theorem states that

$$\langle \gamma_f - \gamma_i \rangle = \frac{1}{2} \frac{\partial}{\partial \gamma_i} \langle (\gamma_f - \gamma_i)^2 \rangle \quad (A1)$$

where  $\langle \cdot \rangle$  denotes an average over entry phase, and  $\gamma_i$  and  $\gamma_f$  denote initial and final values, respectively.  $\gamma_i$  is taken to be fixed so that it is  $\gamma_f$ , which is dependent upon the entry phase before the average is carried out. The left-hand side is obviously the phase averaged energy change,  $\delta E_L$  (measured, of course, in units of  $mc^2$ ). The phase averaged energy spread  $\Delta E_s$  is given by

$$\Delta E_s^2 \equiv \langle \gamma_f^2 \rangle - \langle \gamma_f \rangle^2 = \langle (\gamma_f - \gamma_i)^2 \rangle - \langle (\gamma_f - \gamma_i) \rangle^2.$$

We shall see that  $\langle (\gamma_f - \gamma_i) \rangle^2 / \langle (\gamma_f - \gamma_i)^2 \rangle$  vanishes in the small-signal limit [indeed, it is implied by (A1)] so that (A1) may also be written

$$\delta E_L = \frac{1}{2} \frac{\partial}{\partial E_i} \Delta E_s^2. \quad (A2)$$

Both (A1) and (A2) are valid only in the small-signal limit.

The importance of the theorem in the discussion of the energy spread problem in storage rings has been indicated in Section III. The proof given by Madey contains a number of unnecessary approximations and restrictions which complicate the exposition and limit the generality of the result. It therefore seems worthwhile to present our modified version of his proof here. We shall see that the derivation of (A1) is not significantly more complicated than the evaluation  $\langle \gamma_f - \gamma_i \rangle$  in specific cases, so that one can take advantage of (A1) to obtain  $\langle \gamma_f - \gamma_i \rangle$  from the much simpler evaluation of  $\langle (\gamma_f - \gamma_i)^2 \rangle$ .

For simplicity, we make the (almost certainly) unnecessary assumption that the transverse canonical momentum is both conserved and vanishing. As described in Section II, the motion of an electron through the wiggler magnet is, under these conditions, described by the Hamiltonian

$$H = mc^2 \gamma = mc^2 \left[ \mu^2(z) + \left( \frac{P_z}{mc} \right)^2 - 2a_w(z)a_s(z) \cos \psi(z, t) \right]^{1/2} \quad (A3)$$

where we have introduced the dimensionless vector potentials  $a = eA/mc^2$  and

$$\mu^2(z) = 1 + a_w^2(z) + a_s^2(z) \quad (A4)$$

$$\psi = \int (k_w + k_s) dz' - \omega_s t + \phi \quad (A5)$$

where  $a_w$  and  $a_s$  are the vector potential amplitudes of circu-

larly polarized wiggler and optical fields, respectively. We shall assume that a particle enters at  $z = 0$ ,  $t = 0$  and exits at  $z = L$ . The phase  $\phi$  then represents the random phase of the optical field at which the particle enters. From Hamilton's equations, one easily shows that

$$\frac{d\gamma}{dz} = - \frac{\omega_s a_w a_s \sin \psi}{c \sqrt{\gamma^2 - \mu^2 + 2a_w a_s \cos \psi}} \quad (\text{A6})$$

and

$$\beta \equiv \frac{v_c}{c} = \sqrt{1 - \frac{\mu^2}{\gamma^2} + \frac{2a_w a_s}{\gamma^2} \cos \psi}. \quad (\text{A7})$$

For a more detailed discussion of the equations of motion, the reader is referred to Section II.

We are now in a position to undertake the derivation of (A1). First write

$$\gamma = \gamma_i + \gamma_1 + \gamma_2 + \dots \quad (\text{A8})$$

where  $\gamma_1$  is first order in  $a_s$ ,  $\gamma_2$  second order, etc.

From (A6) we have

$$\gamma(z) - \gamma_i = - \frac{\omega_s}{c} \int_0^z \frac{a_w(z') a_s(z') \sin \psi(z', t(z')) dz'}{\sqrt{\gamma^2(z') - \mu^2(z')} + 2a_w(z') a_s(z') \cos \psi(z', t(z'))}. \quad (\text{A9})$$

where

$$t(z) = \frac{1}{c} \int_0^z \frac{dz'}{\beta(z')} \quad (\text{A10})$$

$$t_0 + t_1 + \dots \quad (\text{A11})$$

and  $t_0$ ,  $t_1$ , etc., also refer to zero, first, etc., order in  $a_s$ . We have, immediately,

$$\gamma_1(z) = - \frac{\omega_s}{c} \int_0^z \frac{a'_w a'_s \sin \psi'_0 dz'}{\sqrt{\gamma_i^2 - \mu_0'^2}} \quad (\text{A12})$$

where  $\psi'_0 \equiv \psi(z', t_0(z'))$ ,  $\mu_0^2 = 1 + a_w^2$ , and, in general, the primes indicate that the argument is  $z'$  (and double primes indicate that the argument is  $z''$ ). Using

$$\langle \sin \psi'_0 \rangle = 0 \quad (\text{A13})$$

and

$$\langle \sin \psi'_0 \sin \psi''_0 \rangle = \frac{1}{2} \cos(\psi'_0 - \psi''_0) \quad (\text{A14})$$

yields

$$\langle \gamma_1 \rangle = 0 \quad (\text{A15})$$

and, to lowest nonvanishing order (second in  $a_s$ ), using  $\langle (\gamma_i - \gamma_f)^2 \rangle = \langle \gamma_i^2 \rangle_{z=L}$  also yields

$$\begin{aligned} \langle (\gamma_i - \gamma_f)^2 \rangle &= \frac{\omega^2}{2c^2} \int_0^L dz \int_0^L dz' \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \\ &\cdot \frac{a''_w a''_s}{\sqrt{\gamma_i^2 - \mu_0''^2}} \cos(\psi'_0 - \psi''_0). \end{aligned} \quad (\text{A16})$$

Expanding (A9), one obtains

$$\begin{aligned} \gamma_2 &= - \frac{\omega_s}{c} \int_0^z \left[ \frac{\partial}{\partial \gamma_i} \left( \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \right) \gamma'_1 \sin \psi'_0 \right. \\ &\quad \left. - \frac{\omega_s a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} t'_1 \cos \psi'_0 dz' \right] + \frac{\omega_s}{c} \int_0^z \\ &\quad \cdot \frac{a''_w a''_s}{(\gamma_i^2 - \mu_0''^2)^{3/2}} \sin \psi'_0 \cos \psi'_0 dz'. \end{aligned} \quad (\text{A17})$$

We write  $\tilde{\gamma}_2$  for the first line of (A17) and note that since

$$\langle \sin \psi'_0 \cos \psi'_0 \rangle = 0, \quad \langle \gamma_2 \rangle = \langle \tilde{\gamma}_2 \rangle.$$

Next, expanding (A10) and using (A7) yields

$$t_0(z') = \frac{1}{c} \int_0^{z'} \frac{dz''}{\beta_0''} \quad (\text{A18})$$

$$\beta_0 = \sqrt{1 - \frac{\mu_0^2}{\gamma_i^2}} \quad (\text{A19})$$

and writing  $t_1 = t_{1a} + t_{1b}$ ,

$$t_{1a} = \frac{1}{c} \int_0^z \gamma''_1 \frac{\partial}{\partial \gamma_i} \frac{1}{\beta_0''} dz'' \quad (\text{A20})$$

$$t_{1b} = \frac{1}{c} \int_0^z \frac{\partial}{\partial \gamma_i} \left( \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \right) \cos \psi'_0 dz'. \quad (\text{A21})$$

In writing (A21), we have, in anticipation of what is to follow, made use of the identity

$$\frac{d}{d\epsilon} \left( 1 - \frac{\mu_0^2}{\gamma_i^2} + \frac{2\epsilon}{\gamma_i^2} \right)^{-1/2} \Big|_{\epsilon=0} = \frac{d}{d\gamma_i} \frac{1}{\sqrt{\gamma_i^2 - \mu_0^2}}. \quad (\text{A22})$$

Substituting (A12) to (A20), and interchanging the order of integration, we obtain

$$\begin{aligned} t_{1a} &= - \frac{\omega_s}{c^2} \int_0^z dz' \frac{a'_w a'_s \sin \psi'_0}{\sqrt{\gamma_i^2 - \mu_0'^2}} \int_{z'}^z dz'' \frac{\partial}{\partial \gamma_i} \frac{1}{\beta_0''} \\ &= - \frac{\omega_s}{c^2} \int_0^z dz' \frac{a'_w a'_s \sin \psi'_0}{\sqrt{\gamma_i^2 - \mu_0'^2}} \frac{\partial}{\partial \gamma_i} (t_0(z) - t_0(z')). \end{aligned} \quad (\text{A23})$$

Now, substitute (A12), (A21), and (A23) into (A17) to obtain

$$\begin{aligned} \tilde{\gamma}_2 &= \frac{\omega_s^2}{c^2} \int_0^z dz' \int_0^{z'} dz'' \left[ \frac{\partial}{\partial \gamma_i} \left( \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \right) \frac{a''_w a''_s}{\sqrt{\gamma_i^2 - \mu_0''^2}} \right. \\ &\quad \cdot \sin \psi'_0 \sin \psi''_0 + \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \frac{\partial}{\partial \gamma_i} \left( \frac{a''_w a''_s}{\sqrt{\gamma_i^2 - \mu_0''^2}} \right) \\ &\quad \cdot \cos \psi'_0 \cos \psi''_0 - \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \frac{a''_w a''_s}{\sqrt{\gamma_i^2 - \mu_0''^2}} \\ &\quad \left. \cdot \cos \psi'_0 \sin \psi''_0 \omega_s \frac{\partial}{\partial \gamma_i} (t'_0 - t''_0) \right]. \end{aligned} \quad (\text{A24})$$

Then phase averaging and supplementing (A14) with

$$\langle \cos \psi'_0 \cos \psi''_0 \rangle = \frac{1}{2} \cos (\psi'_0 - \psi''_0) \quad (A25)$$

$$\langle \cos \psi'_0 \sin \psi''_0 \rangle = -\frac{1}{2} \sin (\psi'_0 - \psi''_0) \quad (A26)$$

yields, to lowest nonvanishing order (second in  $a_s$ )

$$\begin{aligned} \langle \gamma_f - \gamma_i \rangle &= \langle \tilde{\gamma}_2 \rangle_{z=L} \\ &= \frac{\omega_s^2}{2c^2} \frac{\partial}{\partial \gamma_i} \int_0^L dz' \int_0^{z'} dz'' \frac{a'_w a'_s}{\sqrt{\gamma_i^2 - \mu_0'^2}} \\ &\quad \cdot \frac{a''_w a''_s}{\sqrt{\gamma_0^2 - \mu_0''^2}} \cos (\psi'_0 - \psi''_0) \end{aligned} \quad (A27)$$

$$= \frac{1}{2} \frac{\partial}{\partial \gamma_i} \langle (\gamma_f - \gamma_i)^2 \rangle. \quad (A28)$$

In going from (A27) to (A28), we have made use of (A16) and of the symmetry in  $(z', z'')$  of the integrand of (A27). Equation (A28) and (A1) are the same so that the proof is now complete.

It should be added that an identical theorem holds for the Yariv-Shih [22] device. The proof is similar but simpler.

Madey has also derived (A1) by means of a quantum mechanical argument in which stimulated and spontaneous emissions are related to one another and to (A1). In a similar spirit, we offer an even simpler proof based upon the principle of detailed balance. Let  $P(\gamma_1, \gamma_2)$  be the probability density in  $\gamma_2$  that an electron which enters at  $\gamma = \gamma_1$  leaves with  $\gamma = \gamma_2$ . The principle of detailed balance states that

$$P(\gamma_1, \gamma_2) = P(\gamma_2, \gamma_1). \quad (A29)$$

Assuming that the wiggler magnet with optical field satisfies (A29), we have, with  $\Delta\gamma = \gamma_2 - \gamma_1$ ,

$$\begin{aligned} P(\gamma_1, \gamma_1 + \Delta\gamma) &= P(\gamma_1 + \Delta\gamma, \gamma_1 + \Delta\gamma - \Delta\gamma) \\ &= P(\gamma_1, \gamma_1 - \Delta\gamma) + \left( \frac{\partial}{\partial \gamma_1} \right)_{\Delta\gamma} \\ &\quad \cdot P(\gamma_1, \gamma_1 - \Delta\gamma) \Delta\gamma. \end{aligned} \quad (A30)$$

Equation (A30) holds only in the small-signal limit. Now, multiplying (A30) by  $\Delta\gamma$ , integrating over  $\Delta\gamma$ , and using

$$\begin{aligned} \langle (\gamma_f - \gamma_i) \rangle &= \int P(\gamma_1, \gamma_1 + \Delta\gamma) \Delta\gamma d(\Delta\gamma) \\ &= - \int P(\gamma_1, \gamma_1 - \Delta\gamma) \Delta\gamma d(\Delta\gamma) \end{aligned}$$

and

$$\begin{aligned} \int \left( \frac{\partial P}{\partial \gamma_1} \right)_{\Delta\gamma} (\Delta\gamma)^2 d(\Delta\gamma) &= \frac{\partial}{\partial \gamma_1} \int P(\gamma_1, \gamma_1 - \Delta\gamma) \\ &\quad \cdot (\Delta\gamma)^2 d(\Delta\gamma) \\ &= \frac{\partial}{\partial \gamma_1} \langle (\gamma_f - \gamma_i)^2 \rangle, \end{aligned}$$

we obtain (A1) again.

It is perhaps worth emphasizing that (A1) demonstrates that

for finite wigglers in the small-signal limit, zero energy spread implies zero energy transfer. To see this we note that (A16) may be written in the form

$$\langle (\gamma_f - \gamma_i)^2 \rangle = \frac{\omega_s^2}{2c^2} F^* F \quad (A31)$$

with

$$F = \int_0^L dz \frac{a_w a_s}{\sqrt{\gamma_i^2 - \mu_0^2}} \exp i\psi_0. \quad (A32)$$

The vanishing of  $\langle (\gamma_f - \gamma_i)^2 \rangle$  therefore implies that  $F$  vanishes. For finite  $L$ ,  $\partial F / \partial \gamma_i$  cannot be singular, thus  $\partial / \partial \gamma_i \langle (\gamma_f - \gamma_i)^2 \rangle$  must vanish as well.

As a simple example of the application of (A1) we consider the case of a variable parameter wiggler. Equation (A5) specialized to zeroth-order yields

$$\frac{d\psi_0}{dz} = k_w - \frac{\omega_s \mu^2}{2\gamma^2 c} \quad (A33)$$

and for simplicity we consider the case studied by Brau [23]

$$k_w = k_{w0} + k'_w z \quad (A34)$$

with  $a_w, a_s, \mu$  constant and  $\sqrt{\gamma_i^2 - \mu_0^2} \cong \gamma_i$ . Because of (A31), we may choose the integration constant arbitrarily in determining  $\psi_0$  from (A33). Choosing this constant conveniently, we obtain

$$\langle (\gamma_f - \gamma_i)^2 \rangle = \frac{\omega_s^2 a_w^2 a_s^2}{2c^2 \gamma_i^2} F_1 F_1^* \quad (A35)$$

where

$$F_1 = \int_0^L dz \exp i \frac{k'_w}{2} (z - z_0)^2 \quad (A36)$$

with

$$z_0 = \left( \frac{\omega_s \mu^2}{2\gamma_i^2 c} - k_{w0} \right) / k'_w. \quad (A37)$$

The point  $z_0$  corresponds to  $d\psi_0/dz_0 = 0$  as given by (A33). It is, of course, the phase matching point for given  $\omega_s, \gamma_i$  when it occurs within the wiggler. It is apparent from (A36) that the dependence on  $z_0$  is identical to that which occurs for the single slit Fresnel intensity problem, where  $z_0$  is the displacement of the observation point from and parallel to the slit edge and  $k_w/k'_w$  is the distance from the slit plane. The constant wiggler corresponds to the Fraunhofer limit, and hence to the case  $k'_w L^2 \gtrsim 1$ . To express  $F_1$  in terms of standard Fresnel integrals, we define

$$\begin{aligned} F_2(y) &= \frac{y}{|y|} \int_0^{|y|} \exp i \frac{k'_w}{2} z^2 dz = \sqrt{\pi/k'_w} \\ &\quad \cdot [C(y \sqrt{k'_w/\pi}) + iS(y \sqrt{k'_w/\pi})]. \end{aligned} \quad (A38)$$

Then,

$$F_1 = F_2(z_0) + F_2(L - z_0) \quad (A39)$$

and (A1) yields, immediately,

$$\begin{aligned} \langle \gamma_f - \gamma_i \rangle &\approx \frac{\omega_s^2 a_w^2 a_s^2}{2c^2 \gamma_i^2} \frac{dz_0}{d\gamma_i} \operatorname{Re} \left( \frac{dF_1^*}{dz_0} F_1 \right) \\ &= \frac{\omega_s^3 a_w^2 a_s^2 \mu^2}{2c^3 \gamma_i^5 k'_w} \operatorname{Re} \left\{ \left[ \exp -i \frac{k'_w}{2} (L - z_0)^2 \right. \right. \\ &\quad \left. \left. - \exp -i \frac{k'_w}{2} z_0^2 \right] (F_2(z_0) + F_2(L - z_0)) \right\}. \end{aligned} \quad (\text{A41})$$

Equation (A40) reveals the energy transfer to be a rapidly oscillating function of  $z_0$ . While its derivation is simple, the discussion of its behavior is complicated. For  $z_0$  well within the wiggler,  $F_2(z_0) + F_2(L - z_0) \approx ((1 + i)/2) \sqrt{\pi/k'_w}$  and, hence,

$$\begin{aligned} \operatorname{Re} \frac{dF_1^*}{dz_0} F_1 &\cong \sqrt{\frac{\pi}{2k'_w}} \left( \cos \left( \frac{k'_w}{2} (L - z_0)^2 - \frac{\pi}{4} \right) \right. \\ &\quad \left. - \cos \left( \frac{k'_w}{2} z_0^2 - \frac{\pi}{4} \right) \right). \\ &= \sqrt{\frac{2\pi}{k'_w}} \sin k'_w L z_1 \sin \left( \frac{k'_w}{2} \left( \frac{L^2}{4} + z_1^2 \right) - \frac{\pi}{4} \right) \end{aligned}$$

where  $z_1 = z_0 - L/2$  is the distance of  $z_0$  from the wiggler center. The energy transfer is seen to oscillate symmetrically about zero with decreasing period and nondecreasing amplitude as  $z_0$  moves towards the center of the wiggler. An energy spread in the incident beam induces an average in  $z_0$  over a range  $\delta z_0 \approx (2k_w/k'_w)/(\delta\gamma_i/\gamma_i)$ . For typical values of  $\delta\gamma_i/\gamma_i$  and the other parameters, the average extends over several periods ( $\sim 5$ ) near the center. For such a beam the oscillations will decrease strongly in amplitude as one moves towards the center.

The situation near one side of the wiggler is different. For  $z_0$  negative or approximately zero, we write

$$\begin{aligned} \frac{dF_1^*}{dz_0} F_1 &= - \left[ (F_2(z_0) + F_2(L - z_0)) \exp i \frac{k'_w}{2} z_0^2 \right] \\ &\quad \cdot \left[ 1 - \exp i \frac{k'_w}{2} (z_0^2 - (L - z_0)^2) \right]. \end{aligned}$$

The first factor in square brackets is nonoscillatory in the range considered. The second factor is even more rapidly oscillating than before but with average value equal to 1. Again, an energy spread will strongly damp the oscillations. We therefore omit the oscillating term to obtain

$$\operatorname{Re} \frac{dF_1^*}{dz_0} F_1 = - \sqrt{\pi/k'_w} g(-z_0 \sqrt{k'_w/\pi})$$

where

$$\begin{aligned} g(x) &\equiv \left[ \frac{1}{2} - C(x) \right] \cos \frac{\pi}{2} x^2 + \left[ \frac{1}{2} - S(x) \right] \sin \frac{\pi}{2} x^2 \\ &\approx \frac{1}{2 + 4.142x + 3.492x^2 + 6.670x^3} \quad \text{for } x > 0. \end{aligned}$$

For  $x < 0$ ,  $g$  reaches a peak of  $\sim 1.3$  at  $x \sim -0.74$  and then

commences to oscillate at nondecreasing amplitude and decreasing period, being  $\sim \sqrt{2} \sin [(\pi/2)x^2 + (\pi/4)]$  for  $|x|$  large. The situation at  $z_0 > L$  or  $\approx L$  is similar, but with the sign of the effect reversed. A qualitative picture of the behavior as a function of  $\langle \gamma_i \rangle$  that results after energy averaging is shown in Fig. 20.

## APPENDIX II EVALUATION OF PHASE AREA DISPLACEMENT ENERGY TRANSFER AND ENERGY SPREAD

We may obtain a formal result for  $\Delta\gamma$  by rewriting (6.13)

$$\begin{aligned} \Delta\gamma &= - \frac{2\sqrt{2}}{\sqrt{\pi}} \sqrt{\Lambda} \sin \psi_r \int_{-\infty}^{\psi_f} d\psi \int_0^\infty d\lambda \\ &\quad \cdot \{ \exp -\lambda^2 [H + (\cos \psi + \psi \sin \psi_r)] \\ &\quad - \exp -\lambda^2 [H + (\cos \psi_f + \psi \sin \psi_r)] \} \end{aligned} \quad (\text{A41})$$

where  $H = -\cos \psi_f - \psi_f \sin \psi_r$ , and we focus our attention upon the turning point  $\psi_f$  which lies between the minimum at  $\psi_r$  and maximum at  $\pi - \psi_r$  as shown in Fig. 18. The range of  $\psi_f$  within these limits is given by  $\psi_1 < \psi_f < \pi - \psi_r$ , where  $\psi_1$  satisfies

$$\cos \psi_1 + \psi_1 \sin \psi_r = \cos(\pi + \psi_r) - (\pi + \psi_r) \sin \psi_r$$

and is given by  $\psi_1 = (\pi - \sqrt{4\pi|\psi_r|} - \psi_r)$  for small  $|\psi_r|$ . Since  $\psi_r$  is negative here, we find it convenient to write  $\psi_r = -|\psi_r|$  and replace  $\psi_f$  by  $\psi'_f \equiv \psi_f - \pi$  so that  $-\sqrt{4\pi|\psi_r|} < \psi'_f - |\psi_r| < 0$ . Making these changes and also shifting the variable of integration by  $\pi$ , we obtain

$$\begin{aligned} \Delta\gamma &= 2\sqrt{2\Lambda/\pi} \sin |\psi_r| \int_{-\infty}^{\psi'_f} d\psi \int_0^\infty d\lambda \\ &\quad \cdot \{ \exp \lambda^2 [\cos \psi - \cos \psi'_f + (\psi - \psi'_f) \sin |\psi_r|] \\ &\quad - \exp [\lambda^2(\psi - \psi'_f) \sin |\psi_r|] \}. \end{aligned} \quad (\text{A42})$$

Next, we write

$$\begin{aligned} &\int_{-\infty}^{\psi'_f} \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) d\psi \\ &= \sum_{n=0}^{\infty} \int_{\psi'_f - 2\pi(n+1)}^{\psi'_f - 2\pi n} \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) d\psi \\ &= \int_{\psi'_f - 2\pi}^{\psi'_f} d\psi \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) \\ &\quad \cdot \sum_{n=0}^{\infty} [\exp (-2\pi\lambda^2 \sin |\psi_r|)]^n \\ &= \frac{\exp (\pi\lambda^2 \sin |\psi_r|)}{2 \sinh (\pi\lambda^2 \sin |\psi_r|)} \\ &\quad \cdot \int_{\psi'_f - 2\pi}^{\psi'_f} \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) d\psi \end{aligned} \quad (\text{A43})$$

and



Fig. 20. Qualitative picture of the electron energy transfer function in the small-signal limit for a variable parameter wiggler.

$$\begin{aligned} & \int_{\psi_f' - 2\pi}^{\psi_f'} \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) d\psi \\ &= \int_{-\pi}^{\psi_f'} \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) d\psi \\ &+ \exp (-2\pi\lambda^2 \sin |\psi_r|) \\ &\cdot \int_{\psi_f'}^{\pi} \exp \lambda^2 (\cos \psi + \psi \sin |\psi_r|) d\psi. \end{aligned} \quad (\text{A44})$$

Substituting (A44) into (A43), and (A43) into (A42), we obtain

$$\begin{aligned} \Delta\gamma &= 2\sqrt{2\Lambda/\pi} \sin |\psi_r| \int_0^\infty d\lambda \exp -\lambda^2 \cos \psi_f' \\ &\cdot \left\{ \frac{1}{2 \sinh(\pi\lambda^2 \sin |\psi_r|)} \left[ \exp -\pi\lambda^2 \sin |\psi_r| \right. \right. \\ &\cdot \int_{\psi_f'}^{\pi} \exp -\lambda^2 \sin |\psi_r| (\psi_f' - \psi) \exp \lambda^2 \cos \psi d\psi \\ &+ \exp \pi\lambda^2 \sin |\psi_r| \int_{-\pi}^{\psi_f'} \exp -\lambda^2 \sin |\psi_r| (\psi_f' - \psi) \\ &\cdot \exp \lambda^2 \cos \psi d\psi \left. \right] - \frac{\exp \lambda^2 \cos \psi_f'}{\lambda^2 \sin |\psi_r'|} \left. \right\} \end{aligned} \quad (\text{A45})$$

which we rewrite as

$$\begin{aligned} \Delta\gamma &= 2\sqrt{2\Lambda/\pi} \sin |\psi_r| \int_0^\infty d\lambda \exp -\lambda^2 \cos \psi_f' \\ &\cdot \left\{ \frac{1}{2 \sinh \pi\lambda^2 (\sin |\psi_r|)} \left[ \cosh(\pi\lambda^2 \sin |\psi_r|) \right. \right. \\ &\cdot \int_{-\pi}^{\pi} \exp -\lambda^2 \sin |\psi_r| (\psi_f' - \psi) \exp \lambda^2 \cos \psi d\psi \\ &+ \sinh(\pi\lambda^2 \sin |\psi_r|) \left[ \int_{-\pi}^{\psi_f'} - \int_{\psi_f'}^{\pi} d\psi \right. \\ &\cdot \exp -\lambda^2 \sin |\psi_r| (\psi_f' - \psi) \exp \lambda^2 \cos \psi \left. \right] \\ &- \frac{\exp \lambda^2 \cos \psi_f'}{\lambda^2 \sin |\psi_r|} \left. \right\}. \end{aligned} \quad (\text{A46})$$

This is, of course, a bit complicated. However, we may perform the integral over  $\psi$  in the limit  $\psi_f'^2, |\psi_r| \rightarrow 0$  to yield, after some manipulation,

$$\begin{aligned} \Delta\gamma &= 2\sqrt{2\Lambda/\pi} |\psi_r| \int_0^\infty d\lambda \left[ -\frac{1}{\lambda^2 |\psi_r|} \right. \\ &+ \left\{ I_0(\lambda^2) e^{-\lambda^2} \exp \frac{\lambda^2}{2} (\psi_f' - |\psi_r|)^2 \left[ \pi [\coth \pi\lambda^2 |\psi_r| - 1] \right. \right. \\ &\left. \left. + \sqrt{2\pi} \int_\beta^\infty e^{-x^2/2} dx \right] \right] \end{aligned} \quad (\text{A47})$$

where  $\beta \equiv \lambda |\psi_f' - |\psi_r||$  and  $I_0$  is the usual Bessel function of imaginary argument. This is still too hard to do exactly. Splitting it into ranges, we have, for the integrand, if

$$\begin{aligned} \frac{1}{|\psi_f' - |\psi_r||} &> \lambda \\ &- \frac{1}{\lambda^2 |\psi_r|} + \left[ I_0(\lambda^2) e^{-\lambda^2} \left\{ 1 + \frac{\lambda^2}{2} (\psi_f' - |\psi_r|)^2 \right\} \right. \\ &\left. \cdot (\pi \coth \pi\lambda^2 |\psi_r| - \sqrt{2\pi} \lambda |\psi_f' - |\psi_r||) \right] \end{aligned}$$

$$\text{if } \frac{1}{|\psi_f' - |\psi_r||} > \lambda;$$

and

$$\begin{aligned} I_0(\lambda^2) e^{-\lambda^2} \exp \frac{\lambda^2}{2} (\psi_f' - |\psi_r|)^2 &\left[ 2\pi \exp -2\pi\lambda^2 |\psi_r| \right. \\ &\left. - \frac{\sqrt{2\pi} \exp -(\lambda^2/2)(\psi_f' - |\psi_r|)^2}{\lambda |\psi_f' - |\psi_r||} \right] - \frac{1}{\lambda^2 |\psi_r|}, \end{aligned} \quad (\text{A48})$$

if  $\lambda > 1/|\psi_f' - |\psi_r||$ .

The integrations may be done approximately to yield

$$\begin{aligned} \Delta\gamma &= 2\sqrt{2\Lambda/\pi} \left[ \int_0^\infty d\lambda \frac{[I_0(\lambda^2) e^{-\lambda^2} - 1]}{\lambda^2} \right. \\ &+ \frac{1}{2\sqrt{2\pi}} (\psi_f' - |\psi_r|)^2 \ln \frac{1}{\sqrt{\sin |\psi_r|}} \\ &+ |\psi_r| \sqrt{\frac{\pi}{2}} \ln \frac{\sqrt{4\pi} |\psi_r|}{|\psi_f' - |\psi_r||} \\ &+ \sqrt{\frac{\pi}{2}} |\psi_r| Ei \left( \frac{4\pi |\psi_r|}{(\psi_f' - |\psi_r|)^2} - 1 \right) \left. \right] \end{aligned} \quad (\text{A49})$$

with  $Ei(z)$  the exponential integral. We observe that

$$\int_0^\infty d\lambda \frac{e^{-\lambda^2} I_0(\lambda^2) - 1}{\lambda^2} = -2\sqrt{2/\pi}.$$

Note that the result diverges unless  $-\sqrt{4\pi} |\psi_r| < \psi_f'$ .

$|\psi_r| < 0$  which is the allowable range for turning. The derivation is correct only in the limit of very small  $|\psi_r|$ . The logarithmic terms are exact but errors on the order of  $(\ln |\psi_r|)^{-1}$  are to be expected. We have somewhat arbitrarily chosen them so that the leading term goes to zero for  $\sin |\psi_r| = 1$ .

Noting that the range of  $\psi'_f - |\psi_r|$  is from  $-\sqrt{4\pi}|\psi_r|$  to 0, we put  $(\psi'_f - |\psi_r|) = -\sqrt{4\pi}\psi_r \chi$ . Then, as a rough approximation, we have, with  $0 \leq \chi \leq 1$ ,

$$\Delta\gamma = \frac{8}{\pi} \sqrt{\Lambda} \left\{ -1 + \frac{\pi\psi_r}{4} [\chi^2 \ln \sin |\psi_r|] + [\ln \chi + \ln |1 - \chi|] \right\} \quad (\text{A50})$$

and doing the averages over  $2\chi d\chi$ , we find the rms spread

$$\frac{\Delta\gamma_{\text{rms}}}{\Delta\gamma} = \left[ \frac{(\overline{\Delta\gamma})^2 - (\overline{\Delta\gamma})^2}{\Delta\gamma} \right]^{1/2} = \frac{\pi}{4} \sin |\psi_r| \sqrt{\left( 4 - \frac{\pi^2}{3} \right) + \frac{1}{3} [\ln \sin |\psi_r|] + \frac{1}{12} [\ln \sin |\psi_r|]^2}. \quad (\text{A51})$$

Note that  $\overline{\Delta\gamma} = -(8/\pi)\sqrt{\Lambda}$  is exactly what is obtained in the  $|\psi_r| = 0$  limit by taking the phase space displacement at the bump as expected. The terms proportional to  $[\ln \chi]$  and  $[\ln |1 - \chi|]$  in (A50) arise, of course, from particles which just graze the top of the reflecting wells.

A further energy spread is to be expected from the fact that the bucket is, of course, not moved up from  $-\infty$  to  $+\infty$  but has a finite energy traverse  $2\Delta\tilde{\gamma}$ . Thus, in (A52), if we look at the end point contributions to  $\Delta\gamma$ , we see that we may expect a spread from the two end points

$$\Delta\gamma = \frac{\cos \psi \Lambda \sqrt{2}}{\Delta\tilde{\gamma}}.$$

Eliminating  $\Lambda$  by using (A48), we find for the rms contribution to end loss

$$\frac{\Delta\gamma_{\text{rms}}}{\Delta\tilde{\gamma}} = \frac{\pi^2}{64} \frac{\overline{\Delta\gamma}}{\Delta\tilde{\gamma}}. \quad (\text{A52})$$

The spreads (A51) and (A52) are presumably statistically independent so they should be added on in the rms sense. For small resonant angles, we may approximate the slowly varying terms in (A51) and combine it with (A52) to give a simple formula for the ratio of spread to gain

$$\frac{\Delta\gamma_{\text{rms}}}{\Delta\gamma} = \sqrt{\sin^2 \psi_r + \frac{\pi^4}{(64)^2} \left( \frac{\overline{\Delta\gamma}}{\Delta\tilde{\gamma}} \right)^2}. \quad (\text{A53})$$

### APPENDIX III

#### THE LIMITATION ON ELECTRON BEAM RADIUS IMPOSED BY EMITTANCE

All of the discussion in the main body of this paper has been carried out under the assumption  $P_\perp = 0$ . Any real electron beam will, however, have a distribution of transverse velocities so that, at best, one can only expect  $P_\perp/P_z$  to be small. While we are not prepared to discuss the effects of this spread completely, there is one important effect which can be readily estimated and which provides a basis for estimating the magnitude of spread which can be tolerated. This effect is the spread

in the axial velocity  $v_z$  which a spread in  $P_\perp$  implies for a monoenergetic beam.

We assume an azimuthally symmetric beam. The phase area in  $x$ ,  $dx/dz$ , which contains the beam is equal to that in  $y$ ,  $dy/dz$ , and is denoted by  $\pi\epsilon$ . The quantity  $\epsilon$  is referred to as the beam emittance. We therefore take

$$P_\perp = \epsilon \frac{P_z}{r_e} \quad (\text{A54})$$

as an appropriate measure of the variation in  $P_\perp$  ( $r_e$  is the electron beam radius as in Section IV). We take

$$\gamma = \left[ 1 + \left( \frac{P_z}{mc} \right)^2 + \left( a_w - \frac{P_\perp}{mc} \right)^2 \right]^{1/2} \quad (\text{A55})$$

to be fixed, so that the deviation of  $P_z$  oscillates with the wiggler period. Since this period is short compared to the synchrotron period, it is appropriate to average. Hence, we obtain (assuming  $\beta_z \sim 1$ )

$$\langle \delta P_z \rangle = \frac{\langle \delta \beta_z \rangle}{\beta} = \frac{1}{2} \left( \frac{P_\perp}{\gamma mc} \right)^2. \quad (\text{A56})$$

We require

$$\langle \delta \beta_z \rangle < \frac{d\beta_z}{d\gamma} \cdot \delta\gamma_{\text{max}} = \frac{\mu^2}{\gamma^3} \delta\gamma_{\text{max}} \quad (\text{A57})$$

where  $\delta\gamma_{\text{max}}$  is the bucket height defined by (2.56). Combining these relations yields the inequality

$$r_e > \frac{\gamma}{\mu \sqrt{2(\delta\gamma_{\text{max}}/\gamma)}} \epsilon. \quad (\text{A58})$$

For linear accelerators, the following relation has some theoretical and considerable experimental support [24]:

$$\epsilon = \frac{K}{\gamma\beta} \bar{I}_e^{1/2} \quad (\text{A59})$$

where  $K$  is referred to as the emittance constant.  $\bar{I}_e$  is the peak electron beam current averaged over a macropulse. Hence, we obtain

$$r_e > \frac{\bar{I}_e^{1/2}}{\mu \sqrt{2(\delta\gamma_{\text{max}}/\gamma)}} K. \quad (\text{A60})$$

All of the factors which appear in (A60) can be related to the "standard design" parameters discussed in Section IV. Thus, using (4.25) and (4.26), we obtain

$$r_e > \frac{1}{4} \left[ \frac{\sqrt{2} n_b^{3/2} (2 - n_b)^{5/2} \csc^{3/2} \psi_r}{\eta_e (1 - n_b) (2 - 2n_b + n_b^2) \Gamma} \right]^{1/2} \cdot \left( \frac{\bar{I}_e}{Q\bar{I}_e} \right)^{1/2} \left( \frac{mc^3}{e} \right)^{1/2} \cdot K \quad (\text{A61})$$

which, for the numerical example following (4.26), takes the form

$$r_e > 1.28 \left( \frac{\bar{I}_e}{QI_e} \right)^{1/2} \left( \frac{mc^3}{e} \right)^{1/2} K. \quad (\text{A62})$$

Since  $I_e$  refers to the peak current in a micropulse, the factor  $\bar{I}_e/I_e$  varies from  $\frac{1}{25}$  to  $\frac{1}{100}$ , depending upon design. A typical "good" value for  $K$  is  $0.3 \text{ cm} \cdot (\text{kA})^{-1/2}$ . Assuming this value for  $K$ , we see that the choice  $r_e = 1.24 \text{ mm}$  used in the numerical examples satisfies (A62) for  $\bar{I}_e/QI_e < 1/163$ .

In the case of amplifiers, the peak current required is so large that it is probably necessary to use an induction accelerator instead of an RF linac. Then the factor  $\bar{I}_e/I_e$  is simply unity. In addition,  $Q$  is, of course, also unity. Actual numbers depend upon gain required and wiggler profile design details, but for our example of Fig. 15, we find  $r_e > 5.6 K \text{ cm}$ . Reference to (4.20) suggests that an order of magnitude reduction in  $K$  would greatly enhance the practicality of amplifier applications.

It should be mentioned that emittance also affects the beam transport problem [24]. Furthermore, to the extent that one depends upon the wiggler to provide focusing, there is a connection between emittance and the wiggler factor  $\sigma_w$  [see (4.10a)] [24], [25]. These considerations have not been taken into account in deriving (A61).

#### APPENDIX IV

##### NEGLECT OF SPACE CHARGE FORCE

In this Appendix we estimate the importance of space charge forces in the discussion of longitudinal motion. We do not consider transverse space charge effects.

The ponderamotive force on an electron has a  $z$  derivative given by

$$K_p = \frac{a_w a_s}{\gamma} m \omega_s^2 \cos \psi_r \quad (\text{A63})$$

at the bottom of the bucket.  $K_p$  may be thought of as the bucket spring constant. The negative spring constant arising from space charge forces at the bottom of the well is given (approximately) by

$$K_{sc} = 2\pi f_b e \rho_e = \frac{2f_b I_e e}{r_e^2 c} \quad (\text{A64})$$

where  $\rho_e$  is the electron beam charge density and  $f_b$  is the trapped fraction. Hence,

$$\frac{K_{sc}}{K_p} = \frac{2f_b I_e \gamma e}{cm \omega_s^2 a_w a_s r_e^2 \cos \psi_r}. \quad (\text{A65})$$

We consider space charge forces to be unimportant when this ratio is small.

From (2.40) and (2.41) in the standard model,

$$a_s = \frac{\eta_b (2 - \eta_b)}{2k_w L (1 - \eta_b) \sin \psi_r}. \quad (\text{A66})$$

From (4.24) and (4.11) (taking  $\eta_e \approx f_b \eta_b$ ),

$$I_e = \frac{1}{2} \frac{mc^3}{c} \frac{\eta_b}{f_b} \frac{(2 - \eta_b)^2}{(1 - \eta_b)^2 \sin^2 \psi_r Q \gamma_i} \quad (\text{A67})$$

and from (4.4) at  $z = L$ , where  $(K_{sc}/K_p)$  is largest,

$$\frac{a_w}{\gamma} = \frac{1}{\gamma_i}. \quad (\text{A68})$$

Combining the above relations in the standard model, we obtain

$$\frac{K_{sc}}{K_p} = \frac{9}{8Q\gamma_i^2} \frac{(2 - 2\eta_b + \eta_b^2)(2 - \eta_b)}{(1 - \eta_b)^3 \sin \psi_r \cos \psi_r}. \quad (\text{A69})$$

For the values of  $\gamma_i$  which we have in mind, this is always a very small number. For the numerical example given after (4.26),

$$K_{sc}/K_p = 5.8 \times 10^{-4}/Q.$$

For the case of an amplifier, the  $Q$  factor is, of course, missing. The space charge effect is relatively strongest at the input end because the ponderamotive force is weakest there. The numerical factors depend upon design details. For the amplifier of Fig. 15, we find

$$\frac{K_{sc}}{K_p} \approx \frac{200}{\gamma_i^2} \quad (\text{A70})$$

so that space charge effects can be significant for smaller values of  $\gamma_i$ .

#### APPENDIX V THE OPTICAL KLYSTRON

In order to improve the efficiency of the constant parameter FEL in storage ring operation, it is logical to think of bunching the electrons before sending them through the wiggler. One could speculate that by properly choosing the phase of the modulation, energy spread could be greatly reduced [14]. To study this, we consider a system of two wiggler magnets (a velocity buncher and a radiator) separated by a long drift space (see Fig. 21).

This is a special soluble case of a variable wiggler, where bunching becomes large. The electron motion is given by (2.44)

$$\frac{d\psi}{dz} = -A(z)\delta\gamma \quad (\text{A71})$$

$$\frac{d\delta\gamma}{dz} = C(z) \sin \psi. \quad (\text{A72})$$

Here,  $A$  and  $C$  are defined in the main text. For our case, since  $l \ll L$ , we may take  $A = 0$  for  $z < l$ , and  $z > l + L$ , and  $A = \text{constant}$  in the drift space. We take  $C = C_1$  in the first magnet and  $C = C_2$  in the second. Consider an electron with  $\delta\gamma = \delta\gamma_0$ ,  $\psi = \psi_0$  at  $z = 0$ . Then, at  $z = l$ ,

$$\psi = \psi_0, \delta\gamma = \delta\gamma_0 + C_1 l \sin \psi_0$$

at  $z = l + L$ ,

$$\psi = \psi_0 - A[\delta\gamma_0 + C_1 l \sin \psi_0]L + \chi$$

where we have allowed for a phase shift  $\chi$  in the second magnet and

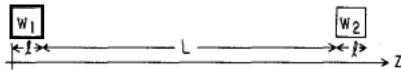


Fig. 21. Schematic representative of optical klystron configuration.

$$\delta\gamma = \delta\gamma_0 + C_1 l \sin \psi_0$$

and finally, at  $z = L + 2l$ ,

$$\begin{aligned} \delta\gamma &= \delta\gamma_0 + C_1 l \sin \psi_0 + C_2 l \sin [\psi_0 + \chi \\ &\quad - AL(\delta\gamma_0 + C_1 l \sin \psi_0)] \\ &= \delta\gamma_0 + C_1 l \sin \psi_0 + C_2 l \sum_{n=-\infty}^{\infty} J_n(AC_1 Li) \\ &\quad \cdot \sin [\psi_0 + \chi - AL\delta\gamma_0 - n\psi_0] \end{aligned} \quad (A73)$$

with  $J_n$  being the  $n$ th order Bessel function.

We may now average over initial phase  $\psi_0$  to obtain

$$\langle \delta\gamma - \delta\gamma_0 \rangle = C_2 J_1(AC_1 Li) \sin (\chi - AL\delta\gamma_0) \quad (A74)$$

and

$$\begin{aligned} \langle (\delta\gamma - \delta\gamma_0)^2 \rangle &= \frac{1}{2} C_1^2 l^2 + \frac{1}{2} C_2^2 l^2 \\ &\quad \cdot [1 - \cos 2(\chi - AL\delta\gamma_0) J_2(2AC_1 Li)] \\ &\quad + C_1 C_2 l^2 \cos (\chi - AL\delta\gamma_0) \\ &\quad \cdot \{J_0(AC_1 Li) - J_2(AC_1 Li)\}. \end{aligned} \quad (A75)$$

If we now minimize the ratio  $\langle (\delta\gamma - \delta\gamma_0)^2 \rangle / \langle \delta\gamma - \delta\gamma_0 \rangle^2$  with respect to  $C_2/C_1$  holding other factors fixed, we have

$$\begin{aligned} \frac{\langle (\delta\gamma - \delta\gamma_0)^2 \rangle}{\langle \delta\gamma - \delta\gamma_0 \rangle^2} &= \frac{1}{2} \left[ \frac{1 - \cos 2(\chi - AL\delta\gamma_0) J_2(2AC_1 Li) - \cos^2 (\chi - AL\delta\gamma_0) [J_0 - J_2]^2}{J_1^2 \sin^2 (\chi - AL\delta\gamma_0)} \right] \\ &= \frac{1}{2} \left[ \frac{1 - J_2(2\beta) - [J_0(\beta) - J_2(\beta)]^2}{J_1^2(\beta) \sin^2 (\chi - AL\delta\gamma_0)} + \frac{2J_2(2\beta) + [J_0(\beta) - J_2(\beta)]^2}{J_1^2(\beta)} \right] \end{aligned}$$

with  $\beta = AC_1 Li$ .

Since  $1 - J_2(2\beta) - [J_0(\beta) - J_2(\beta)]^2 > 0$ , the optimum value for the phase will be such that  $\sin (\chi - AL\delta\gamma_0) = 1$ , i.e.,  $C_1/C_2 = 0$ . Hence, finally, for the optimum energy  $\delta\gamma_0$ , the ratio of spread to gain is given by

$$\frac{\langle (\delta\gamma - \delta\gamma_0)^2 \rangle}{\langle \delta\gamma - \delta\gamma_0 \rangle^2} = \frac{1}{2} \left[ \frac{1 + J_2(2\beta)}{J_1^2(\beta)} \right]. \quad (A76)$$

The ratio is optimal for  $\beta = 2.2$  where  $\langle (\delta\gamma - \delta\gamma_0)^2 \rangle / \langle \delta\gamma - \delta\gamma_0 \rangle^2 = 2$ . This, at first sight, may appear advantageous for finite  $\beta$  operation. However, we may note that for small values of  $\beta$  the moments given by (A74) obey the Madey theorem:

$$\langle \delta\gamma - \delta\gamma_0 \rangle = \frac{1}{2} \frac{\partial}{\partial \delta\gamma_0} \langle (\delta\gamma - \delta\gamma_0)^2 \rangle$$

while as  $\beta$  is increased the bandwidth  $\sim 1/AL \sim C_1 l / \beta$  also becomes very small. For storage ring operation the ratio, bandwidth  $x$  mean energy loss/(energy spread)<sup>2</sup>, optimized with respect to  $C_2/C_1$  for  $\delta\gamma_0$  at the peak gain point ( $\sin (\chi - AL\delta\gamma_0) = -1$ ), is presumably more relevant. The optimized

ratio is found to be  $\approx J_1(\beta)/\beta$  so that low  $\beta$  operation is, in fact, preferable. Similarly, use of the Fokker-Planck equation to determine a steady-state solution from (A4) and (A5) shows that even for large  $\beta$ , the steady state is characterized by an emitted power which is a small fraction of synchrotron radiation (and a decreasing function of  $\beta$ ). However, that is only an indicative result, not a conclusive one, since, for large  $\beta$ , the Fokker-Planck equation is not valid as higher derivatives  $\partial n_f / \partial \gamma^n$  become important as the bandwidth becomes small. Pending a presumably numerical solution of the integral equation for the distribution function in a storage ring at large  $\beta$ , we can only say that the outlook is not promising. In particular, the result (A76) shows that it is not possible, even for a  $\delta$ -function input, to completely eliminate energy spread.

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- [16] More explicitly, one might take for an adiabaticity condition  $|\gamma'_r| \ll (2\pi/Z)\delta\gamma_{\max}$ , or, using (2.50), (2.51) and (2.40), we find
$$|\gamma'_r| \ll 2 \frac{\omega_s a_s a_w}{c\gamma_r} \sqrt{\cos^2 \psi_r - \left(\frac{\pi}{2} \operatorname{sgn} \psi_r - \psi_r\right) \cos \psi_r \sin \psi_r}.$$

This is seen to be consistent with (2.41) for sufficiently small  $\psi_r$ . Numerical simulation to be described later suggests that for smoothly varying wigglers with approximately constant  $\psi_r$  that (2.41) is, in fact, sufficient for adiabatic behavior.

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**Philip L. Morton**, photograph and biography not available at the time of publication.

**Marshall N. Rosenbluth**, photograph and biography not available at the time of publication.