Technical Companion Note for: “Local Projections vs. VARs: Lessons From Thousands of DGPs”

Dake Li Mikkel Plagborg-Møller Christian K. Wolf

September 26, 2023

This companion note provides technical details for some of the computations in the replication code for the simulation study of Li et al. (2023).[1] Our calculations rely on the ABCD representation (Fernández-Villaverde et al., 2007) of the encompassing dynamic factor model, discussed in Section 1. We then in Section 2 describe how this representations can be used to derive various population parameters of interest.

1 ABCD representation

We begin with the baseline dynamic factor model of Stock & Watson (2016):

$$
\begin{aligned}
f_t &= \Phi(L) f_{t-1} + H\varepsilon_t && (1)\\
X_t &= \Lambda f_t + v_t && (2)\\
v_{i,t} &= \Delta_i(L) v_{i,t-1} + \Xi_i \xi_{i,t}, && (3)
\end{aligned}
$$

where $v_t = (v_{1,t}, \dots, v_{n_X,t})'$, $\Phi(L) = \sum_{\ell=1}^{p_f} \Phi_\ell L^\ell$, $\Delta_i(L) = \sum_{\ell=1}^{p_v} \delta_{i,\ell} L^\ell$, and the shocks ε_t and $\xi_t = (\xi_{1,t}, \dots, \xi_{n_X,t})'$ are i.i.d. standard normal and mutually independent.

We map the DFM (1) - (3) into the “ABCD” form of Fernández-Villaverde et al. (2007) as follows. The general ABCD representation takes the form

$$
s_t = As_{t-1} + Be_t \qquad (4)
$$

[1] https://github.com/dake-li/lp_var_simul

$$y_t = Cs_{t-1} + De_t \tag{5}$$

where $e_t \sim \mathcal{N}(0, I)$. Define the notation $\Phi_{1:p} = (\Phi_1, \dots, \Phi_p)$, $\Delta_\ell = \text{diag}(\delta_{1,\ell}, \dots, \delta_{n_X,\ell})$, $\Delta_{1:p} = (\Delta_1, \dots, \Delta_p)$, and $\Xi = \text{diag}(\Xi_1, \dots, \Xi_{n_X})$. Then the mapping from (1) - (3) to (4) - (5) given a selected set of observables $\bar{w}_t = \bar{S}X_t$ is

$$\underbrace{\begin{pmatrix} f_t \\ \vdots \\ f_{t-p_f+1} \\ v_t \\ \vdots \\ v_{t-p_v+1} \end{pmatrix}}_{s_t} = \underbrace{\begin{pmatrix} \Phi_{1:p_f-1} & \Phi_{p_f} & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & \Delta_{1:p_v-1} & \Delta_{p_v} \\ 0 & 0 & I & 0 \end{pmatrix}}_{A} \begin{pmatrix} f_{t-1} \\ \vdots \\ f_{t-p_f} \\ v_{t-1} \\ \vdots \\ v_{t-p_v} \end{pmatrix} + \underbrace{\begin{pmatrix} H & 0 \\ 0 & 0 \\ 0 & \Xi \\ 0 & 0 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix}}_{e_t} \tag{6}$$

$$\bar{w}_t = \underbrace{\bar{S}\begin{pmatrix} \Lambda\Phi_{1:p_f} & \Delta_{1:p_v} \end{pmatrix}}_{C} \begin{pmatrix} f_{t-1} \\ \vdots \\ f_{t-p_f} \\ v_{t-1} \\ \vdots \\ v_{t-p_v} \end{pmatrix} + \underbrace{\bar{S}\begin{pmatrix} \Lambda H & \Xi \end{pmatrix}}_{D} \begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \tag{7}$$

External instrument. When we additionally consider an IV satisfying the equation

$$z_t = \rho_z z_{t-1} + \alpha\varepsilon_{1,t} + \sigma_\nu \nu_t, \tag{8}$$

we can augment the above ABCD representation with the IV as follows. Consider the expanded observation vector $w_t = (\bar{w}_t', \tilde{z}_t)'$, where $\tilde{z}_t = z_t - \rho z_{t-1}$ is the residualized IV (we can treat this as observed for the purposes of our population calculations). Then w_t satisfies an ABCD representation (4) - (5) with the same A matrix and state variables s_t as in (6). We augment the shock vector with the IV measurement error: $e_t = (\varepsilon_t', \xi_t', \nu_t)'$. The corresponding B matrix simply appends a column of zeros on the right of the corresponding matrix in (6). The C matrix appends a row of zeros at the bottom of the corresponding matrix in (7), and the D matrix now equals

$$\begin{pmatrix} \bar{S}\Lambda H & \bar{S}\Xi & 0 \\ \alpha \mathbf{e}_1' & 0 & \sigma_\nu \end{pmatrix},$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0)'$.

First differences. If A has eigenvalues equal to unity, it is helpful to also derive a state space model for the stationary first differences $\Delta\bar{w}_t$. Define the reduced-rank decomposition $A - I = \alpha\beta'$, where α and β have full column rank. Note that $\Delta s_t = \alpha\beta' s_{t-1} + Be_t$ and $\beta' s_t = \beta'\Delta s_t + \beta' s_{t-1} = (I + \beta'\alpha)\beta' s_{t-1} + \beta' Be_t$. Using $\Delta\bar{w}_t = C\Delta s_{t-1} + D\Delta e_t$, we obtain the stationary representation

$$\begin{aligned}
\underbrace{\begin{pmatrix} \beta' s_{t-1} \\ e_t \end{pmatrix}}_{\tilde{s}_t} &= \underbrace{\begin{pmatrix} I + \beta'\alpha & \beta' B \\ 0 & 0 \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} \beta' s_{t-2} \\ e_{t-1} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ I \end{pmatrix}}_{\tilde{B}} e_t && (9) \\
\Delta\bar{w}_t &= \underbrace{\begin{pmatrix} C\alpha & CB - D \end{pmatrix}}_{\tilde{C}} \tilde{s}_{t-1} + De_t && (10)
\end{aligned}$$

2 Computing population objects

We now discuss how our code maps the ABCD representation derived above into the various structural population objects that are reported in our analysis. These computations are done by the function `dgp_irfs_stats.m`.

Impulse response estimands. We discuss each of the three structural estimands studied in the main analysis of Li et al. (2023).

1. *Observed shock.* Given the matrix H that defines the structural shock of interest $\varepsilon_{1,t}$ as a function of the reduced-form factor forecast errors $\eta_t = H\varepsilon_t$, population impulse response functions for all macro observables X_t are readily computed from the ABCD representation (6) - (7).[2] These computations are done in `compute_irfs.m`.

2. *IV.* For the target IV impulse responses, we scale the population impulse response of the target variable by the impact response of the IV normalization variable.

[2]Note that the matrix H is either exogenously fixed or chosen to maximize the impact impulse response of a certain variable. See the section "Shock Weights" in `run_dfm.m`.

3. *Recursive shock.* We derive the "innovations representation" of the ABCD model as in Fernández-Villaverde et al. (2007):[3]

$$\begin{aligned} \hat{x}_t &= A\hat{x}_{t-1} + K\bar{u}_t \\ \bar{w}_t &= C\hat{x}_{t-1} + \bar{u}_t \end{aligned}$$

where $\hat{x}_t = E[s_t \mid \bar{w}_t, \bar{w}_{t-1}, \dots]$ and $\bar{u}_t = \bar{w}_t - E[\bar{w}_t \mid \bar{w}_{t-1}, \bar{w}_{t-2}, \dots]$. The innovations representation immediately yields the impulse responses of the observables $\bar{w}_t$ with respect to the Wold innovations $\bar{u}_t$. We orthogonalize the Wold innovations using a Cholesky decomposition, given the chosen ordering of the variables described in Li et al. (2023).

Degree of invertibility. For each DGP, we compute the degree of invertibility of the observed shock $\varepsilon_{1,t}$, i.e., the R^2 value in a population regression of $\varepsilon_{1,t}$ onto the Wold innovations $\bar{u}_t$. Since $\mathrm{Var}(\varepsilon_{1,t}) = 1$, this equals

$$\mathrm{Cov}(\varepsilon_{1,t}, \bar{u}_t)\,\mathrm{Var}(\bar{u}_t)^{-1}\,\mathrm{Cov}(\bar{u}_t, \varepsilon_{1,t}) = D_{\bullet 1}'\,\mathrm{Var}(\bar{u}_t)^{-1} D_{\bullet 1},$$

where $D_{\bullet 1}$ is the first column of the D matrix, and $\mathrm{Var}(\bar{u}_t)$ has already been obtained when deriving the innovations representation of the ABCD model.

Persistence. We compute several measures of the persistence of the ABCD model (6) - (7), see Table 1 (Section 3.4) of Li et al. (2023). The following ingredients are needed for these computations:

- Variance-covariance matrix of $\bar{w}_t$: The ABCD model implies

$$\mathrm{Var}(\bar{w}_t) = C\,\mathrm{Var}(s_t)C' + DD', \quad \text{where } \mathrm{vec}(\mathrm{Var}(s_t)) = (I - (A \otimes A))^{-1}\,\mathrm{vec}(BB').$$

- Long-run variance-covariance matrix of $\bar{w}_t$: The ABCD model implies

$$\bar{w}_t = [C(I - AL)^{-1}BL + D]e_t,$$

[3]To ensure that the matrix $\Sigma = \mathrm{Var}(s_t \mid \bar{w}_t, \bar{w}_{t-1}, \dots)$ is positive semidefinite despite numerical rounding errors, we write Equation 9 in Fernández-Villaverde et al. (2007) in the following equivalent way:

$$\Sigma = L\tilde{\Sigma}L', \quad \text{where } L = \tilde{A} - \tilde{A}\tilde{\Sigma}\tilde{C}'(\tilde{C}\tilde{\Sigma}\tilde{C}')^{-1}\tilde{C},\ \tilde{\Sigma} = \begin{pmatrix} \Sigma & 0 \\ 0 & I \end{pmatrix},\ \tilde{A} = (A, B),\ \tilde{C} = (C, D).$$

so the long-run variance equals $[C(I-A)^{-1}B+D][C(I-A)^{-1}B+D]'$.

- If A has at least one eigenvalue equal to unity, we report the variance-covariance matrix and long-run variance-covariance matrix of $\Delta\bar{w}_t$ instead of $\bar{w}_t$. We do this by computing the stationary ABCD representation (9) - (10) for $\Delta\bar{w}_t$.

- Coefficient matrices in VAR(∞) representation: We derive the coefficient matrices in the reduced-form VAR(∞) representation of $\bar{w}_t$ by first computing the innovations representation of the ABCD model, and then applying the formula in Chapter 8.6 of Hansen & Sargent (2013). The formula allows us to compute the coefficients out to an arbitrary lag length.

- Largest VAR root: We form the companion matrix of the VAR(∞) representation (truncated to a large, but finite lag length) and report the largest absolute value of its eigenvalues.

IV STRENGTH. Let $i_t \in \bar{w}_t$ denote the normalization variable that we will use to gauge IV strength. Our IV strength measure is defined as the R^2 value in a population regression of i_t onto z_t (or equivalently the residualized IV $\tilde{z}_t$), after controlling for lagged data. Let $u_t = w_t - E[w_t \mid w_{t-1}, w_{t-2}, \dots]$ denote the Wold innovations for the expanded data vector $w_t = (\bar{w}_t', \tilde{z}_t)$, and let u_t^i and u_t^z denote the elements in u_t corresponding to i_t and $\tilde{z}_t$, respectively. Then the IV strength measure can be expressed as

$$\frac{\operatorname{Cov}(u_t^i, u_t^z)^2}{\operatorname{Var}(u_t^i)\operatorname{Var}(u_t^z)}.$$

The variance-covariance matrix of the Wold innovations u_t is readily obtained by computing the innovations representation of the ABCD model augmented with the IV.

References

Fernández-Villaverde, J., Rubio-Ramírez, J. F., Sargent, T. J., & Watson, M. W. (2007). ABCs (and Ds) of Understanding VARs. *American Economic Review*, *97*(3), 1021–1026.

Hansen, L. & Sargent, T. (2013). *Recursive Models of Dynamic Linear Economies*. The Gorman Lectures in Economics. Princeton University Press.

Li, D., Plagborg-Møller, M., & Wolf, C. K. (2023). Local Projections vs. VARs: Lessons From Thousands of DGPs. Working Paper.

Stock, J. H. & Watson, M. W. (2016). Dynamic factor models, factor-augmented vector autoregressions, and structural vector autoregressions in macroeconomics. In *Handbook of Macroeconomics*, volume 2 (pp. 415–525). Elsevier.