Characterisation and Colouring of Planar Graphs

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Abstract

Planar graphs form a well-known class of graphs in graph theory. This essay aims to illustrate and to study some important properties of this family. Among the various ways to characterise planar graphs, a particular attention is given to Kuratowski's Theorem. Details of its proof and some of its applications form the main part of this essay, together with a characterisation of so-called outerplanar graphs using the notion of forbidden subgraphs.

Colouring of planar graphs is also treated: we discuss the proof of the Five Colour Theorem, review the history of the Four Colour Theorem and provide some of its applications. Finally, we consider 3-colourings of outerplanar graphs.

Savaranon'ando

E grafy hay velary ame pla da aṅatin'e kilasy ne grafy tena be pahafatatry ame teôry ne grafy. Ity boky ity da mikendry e handiniky sy haṅasongadiṅy toetoetra vitsivitsy mavesa-danja mikasiky en'io kilasy io. Ame karaja fomba fijaka e maso-siva mamaritry e grafy hay velary ame pla, da nofidinay hoalaligny e Teôremo n'i Kuratowski. E ṅatsipirian'e profony evao da mameno ilany tsakabe am'ity boky ity miaraky ame maso-siva famatara e grafy azo velary ame pla ka e teboki-lovokiny aby da ame sisin'e tare raiky.

Da mbola nodinihy amitôṅao evakoa e fandokoa grafy: e profon'e Teôremo n'e Fandokoa Ame Loko Dimy, tatara sy vokatry e Teôremo n'e Fandokoa Ame Loko Efatry. Ary farany, 3-fandokoa e grafy azo velary ame pla ka e teboki-lovokiny aby da ame sisin'e tare raiky.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Eric Ould Dadah Andriantiana, 22 May 2009

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1. Introduction

In most branches of mathematics, it is a common problem to determine appropriate characterisations of certain classes of structures. In the field of graph theory, the class of planar graphs is one of the most important graph classes. Certain engineering problems require efficient ways to test whether a given network is planar or not. For instance, the electrical model of a printed circuit has to correspond to a planar graph in order to have a possible print.

A considerable step in the characterisation of planar graphs was made in 1930 when K. Kuratowski [Kur30] characterised planar graphs by their forbidden subgraphs K^5 and $K_{3,3}$. Later results give other criteria. For example, S. MacLane [Mac37] found an algebraic criterion for planarity in 1937.

In this essay we focus on the characterisation of the family of planar graphs by presenting some of its most importants characteristics.

Different approaches were used to provide criteria for the planarity of a graph. Euler's formula that relates the number of vertices, edges and faces of a plane graph is discussed in the second chapter among other basic tools.

A detailed proof of the aforementioned theorem by Kuratowski is presented in the third chapter, together with a similar theorem that characterises outerplanar graphs.

The fourth chapter is devoted to the colouring problem; different aspects of colouring problems for planar graphs and maps will be considered, in particular the Four Colour Theorem and its applications.

2. Preliminaries

2.1 Basic Notions in Graph Theory

An undirected graph G is an ordered pair $G = (V(G), E(G))$, where the elements of the set of edges $E(G)$ are two-element subsets of the set of vertices $V(G)$.

The cardinality $|V(G)|$ of $V(G)$ is called the order of G and the number $|E(G)|$ of elements in $E(G)$ is called the size of G.

For simplicity, we will use the notation uv for the edge $\{u, v\}$. Two vertices u and v are adjacent if $uv \in E(G)$, u is called a neighbour of v and vice versa. The vertices u and v are said to be incident to the edge uv.

The set of neighbours of a vertex v is called the neighbourhood of v, its cardinality is called the degree of v.

A graph $H = (V(H), E(H))$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of G; if moreover H is different from G, then it is called a proper subgraph of G.

A walk in G is an alternating sequence, $W(v_0, v_k) = v_0, v_0v_1, v_1, \ldots, v_{k-1}v_k, v_k$ of vertices and edges, beginning and ending with a vertex such that all edges in the sequence are preceded and followed by its ends. A walk without repetition of vertex is called a path; we denote a path that joins u and v by $P(u, v)$ (even if such a path is not unique). A cycle is a walk with no repetition of a vertex except that the first and the last vertices are the same.

G is connected if any two of its vertices are joined by a path in G and it is disconnected otherwise. If $k \in \mathbb{N}$, G is called k-connected if it has at least $k+1$ vertices and the removal of at most $k-1$ vertices from G leaves a remaining connected subgraph of G. A maximal connected subgraph of G is called a connected component of G.

A graph G of order n is complete if any two of its vertices are connected by an edge, it is denoted by K^n. Figure 2.1.a shows K^5.

A graph G is said to be a complete k-partite graph if its set of vertices can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ and for two vertices u and v of G, $uv \in E(G)$ if and only if there is no i in $\{1, 2, \ldots, k\}$ such that V_i contains u and v. If for all i in $\{1, 2, \ldots, k\}$ we have $|V_i| = n_i$, then G is denoted by $K_{n_1,n_2,\ldots,n_k}$. In Figure 2.3.a $K_{3,3}$ is shown, where the two sets that form the vertex partition are $\{v_1, v_2, v_3\}$ and $\{v_a, v_b, v_c\}$.

A plane graph is a graph whose vertices are points in the plane and each of whose edges is represented by a curve joining its endpoints, in such a way that edges intersect only at their endpoints. A planar graph is a graph that can be embedded in the plane in this way.

A plane graph G partitions the rest of the plane into several regions that we call faces of G. We denote the set of faces of G by $F(G)$.

An edge on the boundary of a face f is said to be incident to f. An edge which is incident to a single face is called a cut edge. The total number of edges incident to a face f, where the cut edges are counted twice, is the degree of the face f, it is denoted by $\deg(f)$.

A subset $\{v_1, v_2, \ldots, v_k\}$ of $V(G)$ is called k-vertex cut of G if and only if the deletion of its elements along with edges incident leads to a graph with more connected components than G.

A graph G' that is constructed by adding extra vertices on some edges of G, is called a subdivision of G. In Figure 2.1.b, a subdivision of K^5 is shown. A subdivision of a subgraph of a graph G is called a topological minor of G.

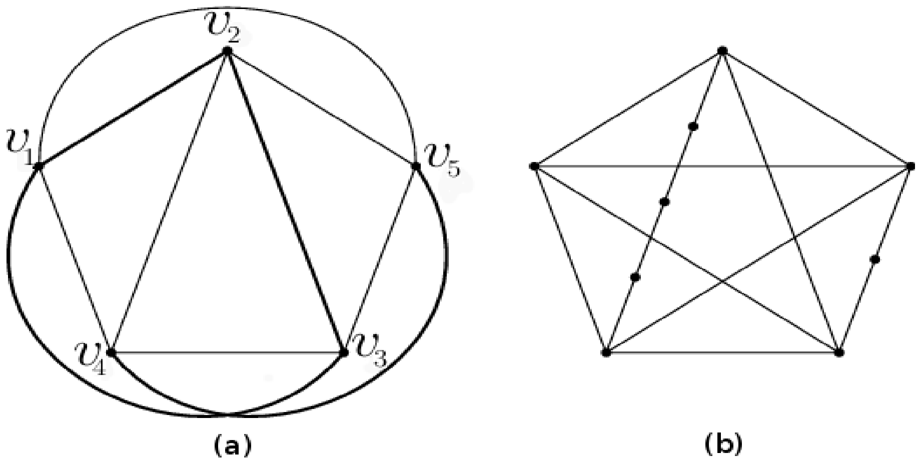

Figure 2.1: K^5 on the left and one of its subdivisions on the right

If a graph G' can be created from G by contracting one or more edges, then G is said to be contractible to G'. If a subgraph of G is contractible to a graph H, then H is called a minor of G.

Remark 2.1.1. *All undirected graphs can be drawn by representing a vertex by a point and an edge by a line*[1].

2.2 Euler's Formula

A formula relating the number of vertices, the number of edges and the number of faces in a plane graph is attributed to Euler.

Theorem 2.2.1. *For a connected plane graph $G = (V(G), E(G))$,*

$$|V(G)| - |E(G)| + |F(G)| = 2. \tag{2.1}$$

A proof of this theorem by induction with respect to the number of faces is given on page 143 of [BM76].

Theorem 2.2.1 cannot provide a direct tool to decide whether a graph is planar or not, because the faces of a graph cannot be determined without finding one of its planar embeddings. However, it has corollaries which are important for the planarity criterion.

Corollary 2.2.2. *If $G = (V(G), E(G))$ is a planar graph such that $|V(G)| \geqslant 3$, then*

$$|E(G)| \leqslant 3|V(G)| - 6. \tag{2.2}$$

[1] Not necessarily without intersections

Proof. Let us begin by considering the case where the graph G is connected. If G is a tree consisting of three vertices and two edges, then we have $|E(G)| = 2 \leqslant 3|V(G)| - 6 = 3$, which satisfies the corollary. Otherwise, the degree of any face in a planar embedding $\tilde{G}$ of G is at least 3 so

$$\sum_{f \in F(G)} \deg(f) = 2|E(G)| \geqslant 3|F(\tilde{G})|, \tag{2.3}$$

and applying Theorem 2.2.1 to G, it follows that $|F(\tilde{G})| = |E(G)| - |V(G)| + 2$. Hence the inequality (2.3) becomes

$$2|E(G)| \geqslant 3(|E(G)| - |V(G)| + 2) \tag{2.4}$$
$$3|V(G)| - 6 \geqslant |E(G)|. \tag{2.5}$$

If the graph G is not connected, then we can add $n \geqslant 1$ more edges to join all connected components and have, as a result of the first part of the proof, $3|V(G)| - 6 \geqslant |E(G)| + n$ and obviously

$$3|V(G)| - 6 \geqslant |E(G)|, \tag{2.6}$$

which completes the proof. □

Corollary 2.2.3. *In a planar graph, the minimum vertex degree is at most* 5.

Proof. Let us assume that G is a graph which has a minimum degree greater or equal to 6. Since

$$2E(G) = \sum_{v \in V(G)} \deg(v), \tag{2.7}$$

it follows that

$$E(G) \geqslant \frac{6|V(G)|}{2} \tag{2.8}$$
$$E(G) > 3|V(G)| - 6. \tag{2.9}$$
$$\tag{2.10}$$

So by Corollary 2.2.2, G is nonplanar. □

This number of the highest value of the minimum degree in a planar graph cannot be reduced to be 4. Figure 2.2 shows an example of a planar graph whose vertices do not have degrees less than five.

Remark 2.2.4. *No criterion that only involves the number of vertices and the number of edges can be sufficient to decide whether a graph is planar or not, because there exist graphs of the same size and the same order such that one of them is planar while the other is nonplanar. This is illustrated in Figure 2.3, the nonplanarity of* $K_{3,3}$*, on the left, will be proven in the next chapter.*

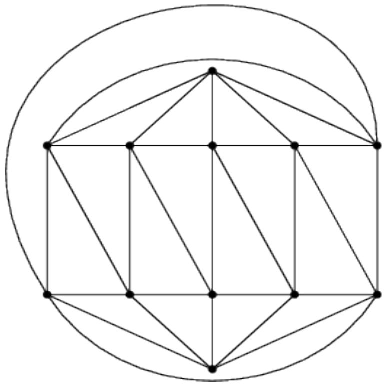

Figure 2.2: Example of a planar graph without a vertex degree less than five

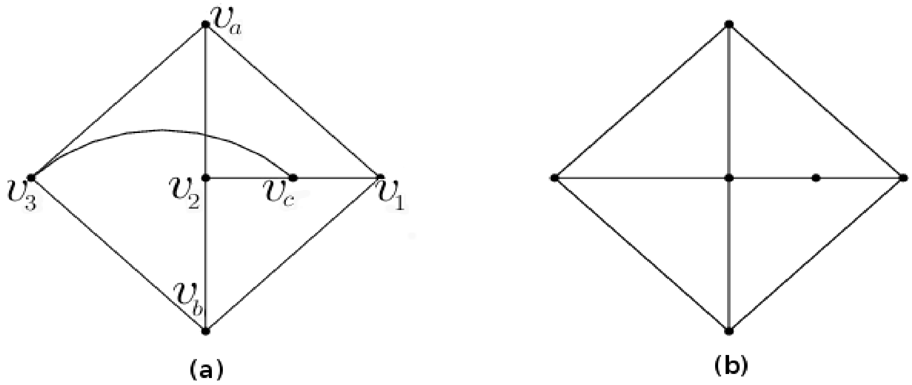

Figure 2.3: $K_{3,3}$ on the left and a planar graph with the same order and the same size as $K_{3,3}$ on the right

2.3 Jordan Curves

Definition 2.3.1. *A closed, continuous and non-self-intersecting curve is called a Jordan curve.*

This family of curves has an obvious, but interesting, property as stated in the following theorem.

Theorem 2.3.2. *The planar embedding of a Jordan curve G divides the rest of the plane — without the curve — into two parts:*

- *a bounded region called interior: int(G), and*
- *an unbounded region which is the exterior: ext(G).*

Any continuous path linking a point of the interior region to a point of the exterior region must cross the Jordan curve.

Even if it seems to be a straightforward theorem, its formal proof is not trivial, the original proof is provided in [Jor87]. In the second chapter we will see an application of this theorem.

Example 2.3.3. *In Figure 2.1, the cycle C formed by v_1, v_2, v_3 and the edges joining them is a Jordan curve, $v_5 \in$ ext(C) and $v_4 \in$ int(C) so the edge, v_4v_5 must cross C for being able to join v_4 and v_5.*

2.4 Bridges of a Cycle

In order to prove Kuratowski's Theorem, we will need to familiarise ourselves with the notion of bridges. In fact bridges can be defined relatively to any kind of subgraph, but in the following section we will restrict ourselves to the study of bridges of a cycle. Given a cycle C of a graph G that is embedded in the plane, once we choose one of the two possible directions, any two vertices of C will correspond to exactly one path in the cycle; we will consider the direction to be anticlockwise and denote by $C[u,v]$ the path in C that joins the vertices u and v in this order . We will indicate that an endpoint of the path is not included by turning the square bracket nearest to it to the exterior. For instance, $C[u,v[$ does not contain v.

Definition 2.4.1. *Let G be a connected graph and C be a cycle in G. We can define an equivalence relation R^C on $E(G)\backslash E(C)$ as follows:*

For all edges uv and $u'v'$ in $E(G)\backslash E(C)$, $uvR^Cu'v'$ if and only if there exists a walk W, where the first and last edges are, respectively, uv and $u'v'$, such that it does not have common vertices with C except for its end vertices. In other words, W is internally disjoint from C.

Proof. Let us verify that R^C is an equivalence relation.

- *Reflexivity*

  Let uv be an element of $E(G)\backslash E(C)$. If u and v are in $E(C)$, then we considere $W = u, uv, v$, otherwise one of u and v is not in $V(C)$, we can assume that $v \notin V(C)$ and consider $W = u, uv, v, uv, u$. In the two cases, the walk W satisfies the condition of Definition 2.4.1 corresponding to uvR^Cuv.

- *Symmetry*

  Let uv and $u'v'$ be elements of $E(G)\backslash E(C)$ such that $uvR^Cu'v'$. By definition there is a walk W in G, internally disjoint from C and such that its first and last edges are, respectively, uv and $u'v'$. The reverse walk will also be internally disjoint with C but its first and last edges are, respectively, $u'v'$ and uv. According to the definition of R^C, this implies that $u'v'R^Cuv$.

- *Transitivity*

  If, for $uv, u'v'$ and $u''v''$ in $E(G)\backslash E(C)$, we have $uvR^Cu'v'$ and $u'v'R^Cu''v''$, then by definition there are two walks

  $$W = v_1, uv, v_2, v_2v_3, \ldots, v_{k-1}, u'v', v_k \text{ and} \tag{2.11}$$
  $$W' = v'_1, u'v', v'_2, v'_2v'_3, \ldots, v_{l-1}, u''v'', v'_k \tag{2.12}$$

  internally disjoint from C, such that the first and last edges of W are, respectively, uv and $u'v'$ while those of W' are, respectively, $u'v'$ and $u''v''$. There are two possibilities:

  – If $v_k = v'_1$, then $v_{k-1} = v'_2$, and we can form a walk

  $$v_1, uv, v_2, \ldots, v_{k-1}, v'_2v'_3, \ldots, v_{l-1}, u''v'', v'_k \tag{2.13}$$

  which is clearly internally disjoint from C. So by Definition 2.4.1 it follows that $uvR^Cu''v''$.

– If $v_k = v'_2$, then $v_k \notin E(C)$, therefore the walk

$$v_1, uv, v_2, v_2v_3, \ldots, v_{k-1}, u'v', v_k, v'_2v'_3, \ldots, v_{l-1}, u''v'', v'_k \tag{2.14}$$

satisfies the condition of Definition 2.4.1 corresponding to $uvR^C u''v''$.

It follows that R^C is an equivalence relation. □

Definition 2.4.2. *If C is a cycle in a graph G and R^C is the equivalence relation defined in Definition 2.4.1, then a subgraph which consists of all the elements in an equivalence class of R^C and their end vertices is called a bridge of C in G.*

From the construction, it follows that two arbitrarily chosen vertices of a bridge B are connected by a path.Therefore, B is connected.

If G is connected, it follows that any bridge B of a cycle C in G has a common vertex with C. Such a vertex, which may not be unique, is called a vertex of attachment of B.

Definition 2.4.3. *A k-bridge B of C is a bridge such that $|V(B) \cap V(C)| = k$.*

Two k-bridges B and B' of C are equivalent if

$$V(B) \cap V(C) = V(B') \cap V(C), \tag{2.15}$$

which means that they have the same vertices of attachment.

In Figure 2.4, B_4 and B_5 are equivalent 4-bridges.

Theorem 2.4.4. *For $k \geqslant 3$, if v_1, v_2 and v_3 are three vertices of attachment of a k-bridge B of a cycle C, then there exists a vertex $v \in V(B)\backslash V(C)$ and three paths $P(v, v_1), P(v, v_2)$ and $P(v, v_3)$ in B joining v to v_1, v_2 and v_3, respectively, such that for all i and j in $\{1, 2, 3\}$, $P(v, v_i)$ and $P(v, v_j)$ are either equal or have only v in common.*

Proof. Let B be a k-bridge of a cycle C in a graph G and let v_1, v_2, v_3 be vertices of attachment of B. By definition of a bridge, there is a path $P(v_1, v_2)$ in B, joining v_1 and v_2, internally disjoint from C. $P(v_1, v_2)$ must contain another vertex v' different from its end points. For if it is just an edge, then there is no possible walk internally disjoint from C which can relate v_1v_2 to an edge in B containing v_3, this is in contradiction with Definition 2.4.2. Since v' and v_3 are in B, there is a path $P(v', v_3)$ internally disjoint from C joining v' to v_3. Let us denote the last vertex of $P(v', v_3)$ in $P(v_1, v_2)$ by v. We can then take P_1 to be the inverse of the first part of $P(v_1, v_2)$ joining v_1 and v, P_2 the second part of $P(v_1, v_2)$ from v to v_2 and P_3 the part of $P(v', v_3)$ from v to v_3. By construction, it follows that if $i \neq j$, P_i and P_j have only v in common. □

For $k \geqslant 2$, any k-bridge B corresponds to a partition of C into edge-disjoint paths; each of them is delimited by two consecutive vertices of attachment of B and called a segment of B.

Definition 2.4.5. *If a bridge B of a cycle C has all of its vertices of attachment lying in a single segment of another bridge B' of the same cycle, then the two bridges are said to avoid one another, otherwise they overlap.*

Two bridges B and B' of C are skew if there exist two vertices of attachment v_1 and v_2 of B and two other vertices of attachment v'_1 and v'_2 of B' such that they appear in C in the cyclic order v_1, v'_1, v_2, v'_2.

Remark 2.4.6. *A bridge with a single vertex of attachment and any other bridge of the same cycle avoid one another.*

If $k \geqslant 4$, then two equivalent k-bridges are skew.

In Figure 2.4, the bridge B_2 avoids B_4 and the bridges B_2 and B_3, as well as B_4 and B_5, are skew.

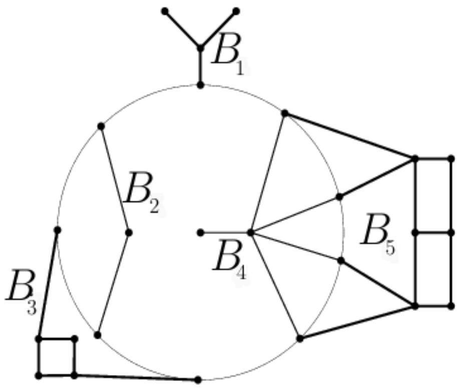

Figure 2.4: A cycle C and its bridges.

Theorem 2.4.7. *If two bridges overlap, then either they are skew or they are equivalent 3-bridges.*

Proof. Let B and B' be two bridges of a cycle C in G that overlap. As a result of Remark 2.4.6, we can assume that each of the two bridges has at least two vertices of attachment.

If one of them has exactly two vertices of attachment, we can assume that it is B, then since the bridges overlap, B' must have vertices of attachment in the interior of each of the two segments of B meaning that they are skew.

So now we can assume that B and B' have at least 3 vertices of attachment. Let us first consider them as not equivalent; then there is a vertex of attachment u of B situated in a segment of C delimited by two consecutive vertices of attachment u' and v' of B' and, as the two bridges overlap, there must be another vertex of attachment v of B outside the segment $C_{[u',v']}$. By construction, u', u, v', v occur on C in this order which implies that they are skew.

Let us now assume that B and B' are equivalent k-bridges where $k \geqslant 3$. As mentioned in Remark 2.4.6, if $k \geqslant 4$, then the bridges are skew. So we are left with the case of $k = 3$, where B and B' are equivalent 3-bridges. □

Let $\tilde{G}$ be a planar embedding of a planar graph G. Then any cycle C of G, as a Jordan curve, partitions the rest of the plane $\mathscr{P}$, where $\tilde{G}$ lies, into int(C) and ext(C).

Definition 2.4.8. *In $\tilde{G}$, a bridge of C is either included in int(C) or it is contained in ext(C). In the former case, it is called an inner bridge, and an outer bridge otherwise.*

A bridge of C cannot have vertices in int(C) and some other vertices in ext(C) otherwise, if $u \in V(B)$ is in int(C) and $v \in V(B)$ is in ext(C), then the path $P(u,v)$ which is internally disjoint from C must cross C, which is a contradiction to the assumption that $\tilde{G}$ is plane.

Definition 2.4.9. *An inner bridge (respectively, an outer bridge) B of C in a planar embedding $\tilde{G}$ of G is transferable if there is a planar embedding $\tilde{G}'$ of G which is equal to $\tilde{G}$ except that B becomes an outer bridge (respectively, inner bridge) of C.*

An inner (respectively, outer) bridge that avoids every outer (respectively, inner) bridge is transferable. For example, in Figure 2.4, B_1 is a transferable outer bridge.

Theorem 2.4.10. *Two inner bridges (respectively, outer bridges) of a cycle in a plane graph avoid one another.*

Proof. We are only going to prove the case of inner bridges, since the case of the outer bridges can be treated in a similar way. Let B and B' be two inner bridges of a cycle C in a plane graph G. Reasoning by contradiction, assume that they overlap; by Theorem 2.4.7, it follows that they are either skew or equivalent 3-bridges.

Let us first consider the case where they are skew. It follows that two vertices of attachment u and v of B and two vertices of attachment u' and v' of B' appear in the cyclic order u, u', v, v' in C. By the definition of a bridge, there are two paths, $P(u, v)$ in B joining u to v and $P(u', v')$ in B' joining u' to v', which are internally disjoint from C. The two paths are disjoint because they belong to different bridges and they do not have common end points. These two paths have to cross each other, which is in contradiction to the fact that G is plane. Figure 2.5.a illustrates an example of this case.

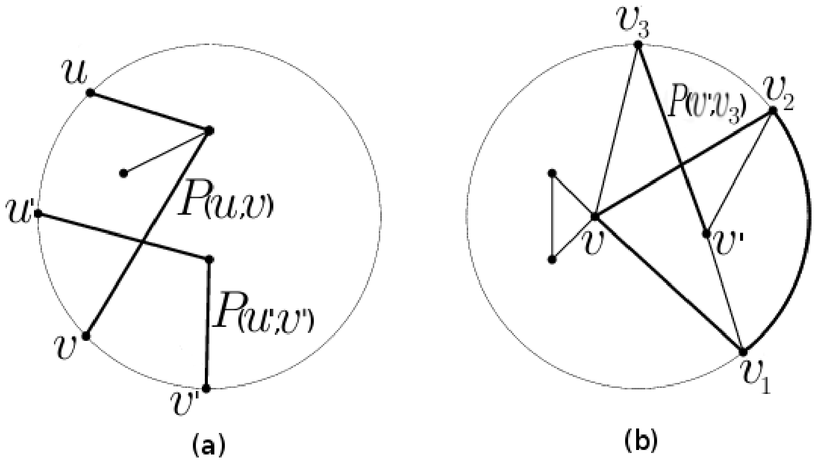

Figure 2.5: Crossing paths in two equivalent or skew inner bridges

Secondly, if B and B' are equivalent 3-bridges and $\{v_1, v_2, v_3\}$ is their set of vertices of attachment, then Theorem 2.4.4 ensures the existence of $v \in V(B)\backslash V(C)$ such that there are three paths $P(v, v_1), P(v, v_2)$ and $P(v, v_3)$ in B and $i \neq j$ implies that $P(v, v_i)$ and $P(v, v_j)$ have only v in common. But $v \in \text{int}(C)$, therefore the subgraph $P(v, v_1) \cup P(v, v_2) \cup P(v, v_3)$ partitions the rest of int(C) into three regions int(C_1), int(C_2) and int(C_3) where

$$C_1 = P(v, v_1) \cup P(v, v_2) \cup C[v_1, v_2], \tag{2.16}$$
$$C_2 = P(v, v_2) \cup P(v, v_3) \cup C[v_2, v_3], \tag{2.17}$$
$$C_3 = P(v, v_3) \cup P(v, v_1) \cup C[v_3, v_1]. \tag{2.18}$$

The same theorem guarantees the existence of a vertex $v' \in V(B')\backslash V(C)$ and three paths $P(v', v_1), P(v', v_2)$ and $P(v', v_3)$ in B' such that any $P(v', v_i)$ and $P(v', v_j)$ are either equal or have only v' in common. Since v' is in int$(C)\backslash(P(v, v_1) \cup P(v, v_2) \cup P(v, v_3))$, it must be contained in one of the three regions; by permuting the indices if necessary, we can assume that it is in int(C_1). But $v_3 \in$ ext(C_1) ; the path $P(v', v_3)$ has to cross C_1 which contradicts the assumption on G. See Figure 2.5.b for an example. □

3. Kuratowski's Theorem

Since equations or inequalities involving size and/or order are not sufficient to characterise planar graphs, we use a different approach. We construct a criterion for planarity in terms of forbidden subgraphs: a graph is planar if and only if it does not contain certain graphs as subgraphs.

First proven in 1930 by Kazimierz Kuratowski, Kuratowski's Theorem is the first theorem in graph theory that states necessary and sufficient conditions for a graph to be planar. The aim of this chapter is to present a proof of the theorem and some applications. The proof that is presented in this chapter follows Chapter 9 of [BM76].

3.1 Statement and Proof of Kuratowski's Theorem

Theorem 3.1.1. *A graph is planar if and only if it does not contain a subgraph that is a subdivision of* K^5 *or* $K_{3,3}$.

3.1.1 Preparation for the Proof

Let us first begin by showing some lemmas.

Lemma 3.1.2. K^5 *is not a planar graph.*

Proof. Since $|V(K^5)| = 5$ and $|E(K^5)| = 10$, it follows that $|E(K^5)| > 3V(K^5) - 6 = 9$. So by Corollary 2.2.2, K^5 cannot be planar. □

Lemma 3.1.3. $K_{3,3}$ *is not planar.*

Proof. By contradiction, let us assume that $K_{3,3}$ is planar and let $\tilde{K}_{3,3}$ be its planar embedding. Since it is 2-connected and it does not contain a triangle, each face of $\tilde{K}_{3,3}$ is incident to at least four edges. Therefore

$$2E(K_{3,3}) = \sum_{f \in F(\tilde{K}_{3,3})} \deg(f) \tag{3.1}$$

$$\geqslant 4|F(\tilde{K}_{3,3})| \tag{3.2}$$

$$E(K_{3,3}) \geqslant 2|F(\tilde{K}_{3,3})|, \tag{3.3}$$

and by Theorem 2.2.1 we have

$$E(K_{3,3}) \geqslant 2(2 + E(K_{3,3}) - V(K_{3,3})) \tag{3.4}$$

$$2V(K_{3,3}) - 4 \geqslant E(K_{3,3}). \tag{3.5}$$

But as $E(K_{3,3}) = 9$ and $V(K_{3,3}) = 6$, the inequality (3.5) leads to $8 \geqslant 9$ which is absurd. So we conclude that $K_{3,3}$ is not planar. □

Lemma 3.1.4. *If* G *is planar, then all subgraphs of* G *are planar.*

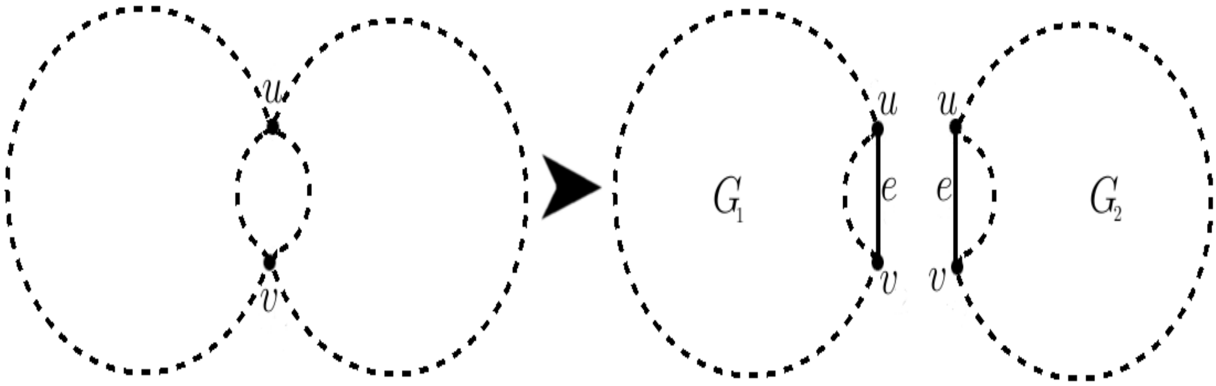

Figure 3.1: A 2-vertex cut $\{u, v\}$

Proof. Let H be a subgraph of a planar graph G. By definition G is isomorphic to a plane graph $\tilde{G}$. If f is an isomorphism from G to $\tilde{G}$, then H has a planar embedding $f(H)$ that can be obtained by deleting some edges or vertices from the plane graph $\tilde{G}$. □

Lemma 3.1.5. *If G is nonplanar, then all subdivisions of G are nonplanar.*

Proof. Assume that we have a planar embedding $\tilde{G}_s$ of a subdivision G_s of a nonplanar graph G. In $\tilde{G}_s$, edges intersect only at their end points, so smoothing vertices of degree two in $\tilde{G}_s$ leads us to a new plane graph. A planar embedding of G can be obtained by repeating this operation, so G would be planar which contradicts the hypothesis. □

Lemma 3.1.6. *A nonplanar graph that contains no subdivision of K^5 or $K_{3,3}$, and has as few edges as possible, is 3-connected.*

Proof. Let G be a graph as defined in the statement of the lemma. It is clear that G has to be 2-connected, otherwise it would have two edge-disjoint proper subgraphs G_1 and G_2 which have only a cut vertex in common. G_1 and G_2 are planar because of the minimality of G. So by merging two planar embeddings of G_1 and G_2 we would have a planar embedding of G and this contradicts the nonplanarity of G.

By contradiction, assume that G is not 3-connected, then it has a 2-vertex cut $\{u, v\}$; therefore it has two edge-disjoint proper subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{u, v\}$. As shown in Figure 3.1 let us join, in each of G_1 and G_2, the two vertices u and v by an edge e, if such an edge does not already exist. Then at least one of the two subgraphs G_1 and G_2 is nonplanar, for otherwise we can take two planar embeddings of G_1 and G_2, which have u and v incident to the exterior face, and merge the two. After deleting e if necessary, we end up with a planar embedding of G which is impossible.

By exchanging the names, if required, we can assume that it is G_1 which is not planar. Since G_1 contains fewer edges than G, if it does not contain any subdivision of K^5 or $K_{3,3}$, then it contradicts the minimality of G. So G_1 contains a subdivision K of K^5 or $K_{3,3}$. If it does not contain the edge e, then the subgraph K is also contained in G, which is impossible. So G_1 contains the edge e which must be included in K. In G_2, v_1 and v_2 must belong to a connected component, otherwise each of them would be a cut vertex of G which is not possible. Therefore G_2 contains a path $P(v_1, v_2)$ joining v_1 to v_2. Replacing e in K by the path $P(v_1, v_2)$ we find a subdivision of K^5 or $K_{3,3}$ contained in G, but this contradicts an assumption on G. So we can conclude that G must be 3-connected. □

3.1.2 Main Proof

Let us now prove Theorem 3.1.1.

Proof. **Necessary Condition**

Let G be a graph containing a subdivision K of either K^5 or $K_{3,3}$. By Lemma 3.1.5, K is nonplanar and by Lemma 3.1.4, we can conclude that G itself is nonplanar.

Sufficient Condition

By contradiction, assume that the set of nonplanar graphs, which do not contain either a subdivision of K^5 or a subdivision of $K_{3,3}$, is not empty. Let G be an element of the set chosen in such a way that it has as few edges as possible. Lemma 3.1.6 ensures that it is 3-connected. By deleting an edge uv, with endpoints u and v, we obtain a 2-connected graph H. By the minimality of G, H is planar. Let us denote by $\tilde{H}$ a planar embedding of H. There are two internally disjoint paths joining u to v, otherwise H would not be 2-connected. Therefore u and v are contained in at least one cycle of $\tilde{H}$. Let C be a cycle in $\tilde{H}$ containing u and v such that there are as many edges as possible in the interior region of C. Each bridge of C in $\tilde{H}$ must have at least two vertices of attachment since H is 2-connected.

Next we show that each outer bridge of C is a single edge that overlaps uv: if an outer bridge avoids uv, then it has two vertices of attachment v_1 and v_2 joined by a path $P(v_1, v_2)$ internally disjoint from C. So the cycle $C' = (C \backslash C[v_1, v_2]) \cup P(v_1, v_2)$ will contain u and v and have more edges in the interior than in that of C –the path $C[v_1, v_2]$ is in the interior of C'. But this contradicts the choice of C. If an outer 2-bridge, overlapping uv or not, has a third vertex, then its vertices of attachment would form a 2-vertex cut of G which is not possible as G is 3-connected. And finally, if an outer bridge B of C has more than two vertices of attachment, then one of the segments of C whose endpoints are u and v will contain two of its vertices of attachment say v_1' and v_2'. By Theorem 2.4.4, B has an interior vertex w and paths $P(w, v_1')$ and $P(w, v_2')$, joining w to v_1' and to v_2', internally disjoint with C, that have only w in common. So the cycle $(C \backslash C[v_1', v_2']) \cup P(w, v_1') \cup P(w, v_2')$ will contradict the choice of C. So the only possibility of an outer bridge is to be a single edge and skew to uv.

There is an inner bridge that is skew to uv, because otherwise the edge uv could be drawn in int(C) and this would lead us to a planar embedding of G, which is impossible. Among the inner bridges that are skew to uv, there is one which overlaps an outer bridge and is thus skew to the outer bridge by Theorem 2.4.7, since there are only outer 2-bridges. If this was not the case, we could transfer all inner bridges out and draw uv to obtain a planar embedding of G. Now we can conclude that there is an inner bridge B_i that is skew to uv and skew to an outer bridge $u'v'$. Notice that $u'v'$ and uv are also skew. Now we discuss the possibilities for the vertices of attachment of B_i.

- Case 1: B_i has a vertex of attachment $w_1 \notin \{u, v, u', v'\}$. By exchanging the names if necessary, we can assume that $w_1 \in C[u', u]$. There are then two possibilities for B_i to be skew to $u'v'$ and uv.
  - Case 1.a: B_i has another vertex of attachment in $C]v', v[$, then as in Figure 3.2, we obtain a subdivision of $K_{3,3}$.

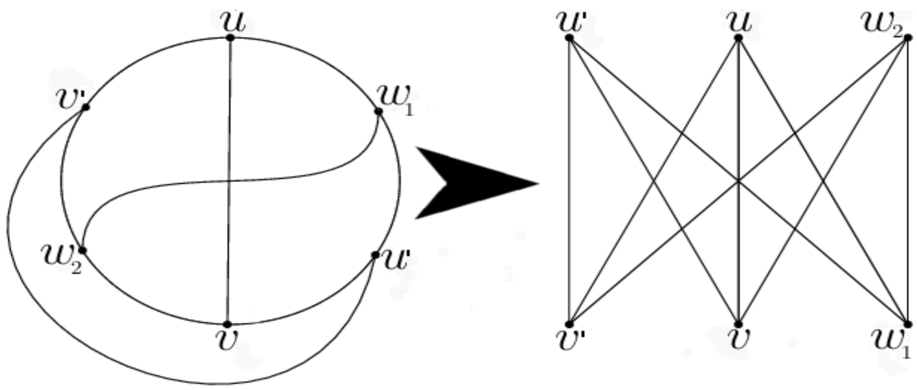

Figure 3.2: Example for case 1.a

- Case 1.b: B_i has no vertices of attachment in $C]v', v[$, then it has to have two other vertices of attachment $w_2 \in C]v, u'[$ and $w_3 \in C]u, v'[$. So it has an internal vertex w_0 and three paths $P(w_0, w_1), P(w_0, w_2)$ and $P(w_0, w_3)$, internally disjoint from C, joining w_0 to w_1, w_2 and w_3 respectively, such that any two of the three paths have only w_0 in common. As illustrated in Figure 3.3 we have again a subdivision of $K_{3,3}$.

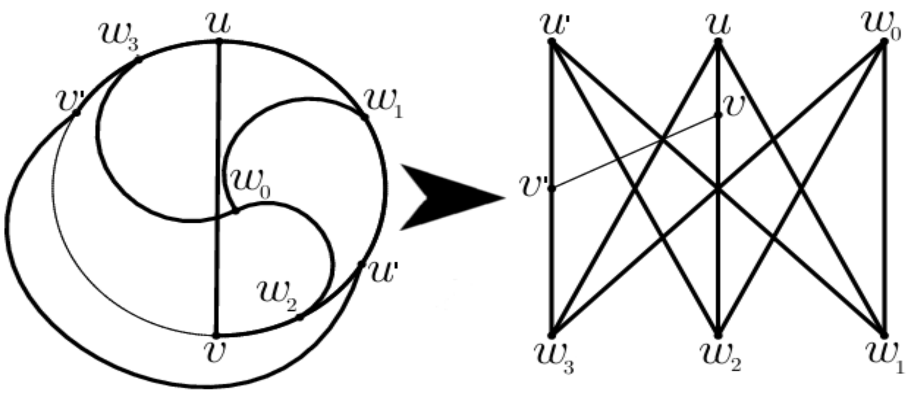

Figure 3.3: Example for case 1.b

- Case 2: B_i has no vertex of attachment that is not in $\{u, v, u', v'\}$; then all of u, v, u' and v' are vertices of attachment of B_i, otherwise it avoids uv or $u'v'$. So the case that we are going to discuss will depend on the number of common vertices of the paths $P(u, v)$ and $P(u', v')$, in B_i, joining u to v and u' to v' respectively.
  - Case 2.a: $P(u, v)$ and $P(u', v')$ have only one vertex in common. In this case we obtain a subdivision of K^5, see Figure 3.4.
  - Case 2.b: $P(u, v)$ and $P(u', v')$ have two or more vertices in common. We can denote by w_1 and w_2, respectively, the first and last vertices that $P(u, v)$ and $P(u', v')$ have in common and find a subdivision of $K_{3,3}$ as illustrated in Figure 3.5.

□

Another theorem giving an alternative planarity criterion was published by K. Wagner in 1937. It states that a graph is nonplanar if and only if it contains a subgraph contractible to K_5 or to $K_{3,3}$. This theorem

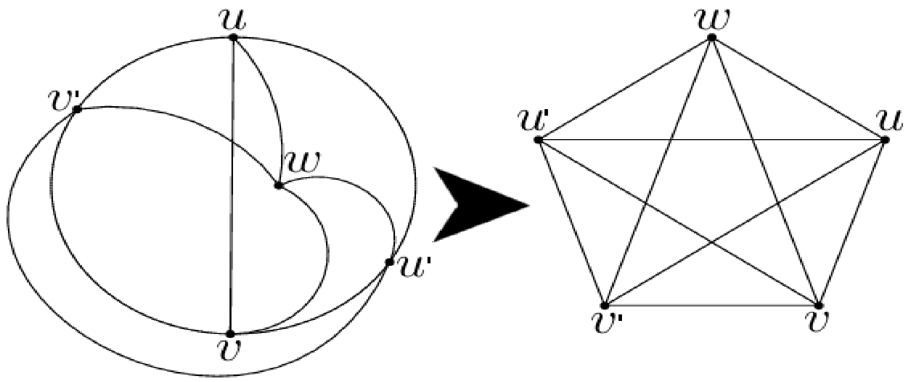

Figure 3.4: Example for the case 2.a

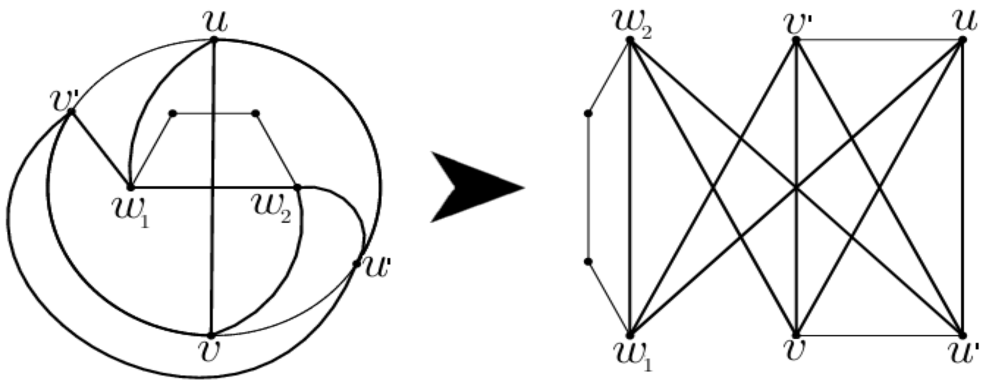

Figure 3.5: Example for the case 2.b

of Wagner is related to Kuratowski's Theorem in the sense that all subdivisions of K_5 (respectively, of $K_{3,3}$) are contractible to K_5 (respectively, to $K_{3,3}$); but in order to understand their difference, let us analyse a situation where we can find a forbidden minor, by the theorem of Wagner, whose type is different from the forbidden subgraph found by Kuratowski's Theorem.

3.2 Applications of Kuratowski's Theorem

A linear time algorithm, developed in detail in Section 8 of [BM04], called Kuratowski Subgraph Isolator, makes Kuratowski's Theorem usable in practice, even for a graph which have big order or size. For instance, the open source software SAGE [sag] uses this work of J. M. Boyer and W. J. Myrvold and gives, inside its function called `is_planar()`, the possibility to check planarity by using Kuratowski's Theorem.

3.2.1 Nonplanarity of the Petersen Graph

The Petersen graph is shown in Figure 3.6, it has 10 vertices and 15 edges. So it satisfies the inequality (2.2) given in Corollary 2.2.2 but this does not allow us to conclude that it is planar. As the graph is in fact nonplanar, any attempt to draw a planar embedding will not lead to a solution. But it is not

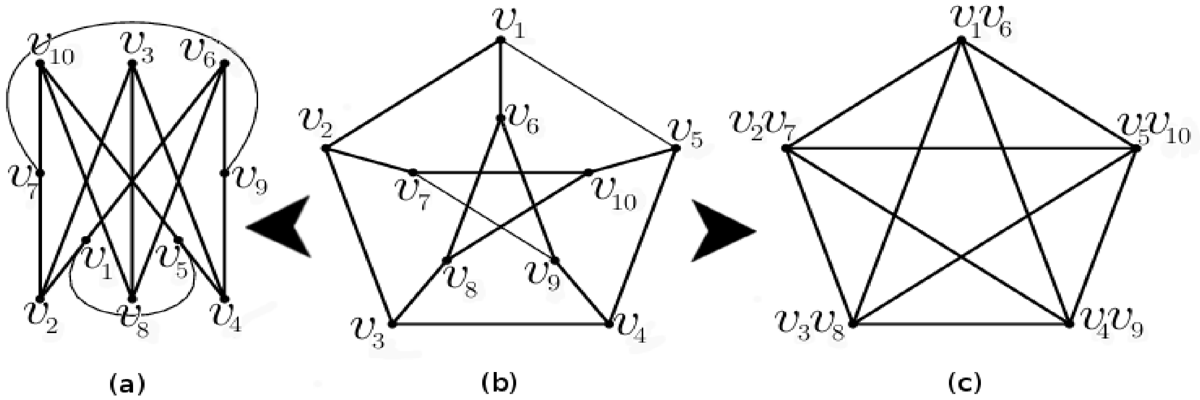

Figure 3.6: Subdivision of K_5 and minor $K_{3,3}$ in the Petersen graph

difficult to find a subdivision of $K_{3,3}$, as it is shown in Figure 3.6.a or a K_5 minor as in Figure 3.6.c. In this later, a vertex with double labels v_iv_j correspond to two merged vertices v_i and v_j.

In the following section, we provide a forbidden subgraph characterisation for a specific subclass of planar graphs called outerplanar graph.

3.2.2 Characterisation of the Class of Outerplanar Graphs

Definition 3.2.1. *A planar graph which has a planar embedding where all of its vertices lie on the boundary of one face is called an outerplanar graph.*

Theorem 3.2.2. *A graph is outerplanar if and only if it does not contain a subgraph that is a subdivision of K^4 or $K_{2,3}$.*

This theorem is a consequence of Kuratowski's Theorem and the following lemmas:

Lemma 3.2.3. *K^4 is nonouterplanar.*

Proof. Let the set of vertices $V(K^4)$ of K^4 be $\{v_1, v_2, v_3, v_4\}$. Assume that the lemma was wrong, then K^4 has a planar embedding $\tilde{K}^4$ which contains a face f that is incident to all the vertices; we can choose f to be the outer face. By permuting the names of the vertices if necessary, we can assume that f is bounded by the cycle $C = v_1, v_1v_2, v_2, v_2v_3, v_3, v_3v_4, v_4, v_4v_1, v_1$. But then the two edges v_1v_3 and v_2v_4, as two overlapping inner bridges, will cross each other and that contradicts the planarity of $\tilde{K}^4$. This is illustrated in the left of Figure 3.7. □

Lemma 3.2.4. *$K_{2,3}$ is nonouterplanar.*

Proof. The nonouterplanarity of $K_{2,3}$ is due to the fact that it does not contain any cycle containing all the vertices, which is a necessary condition for a 2-connected graph to be outerplanar. $K_{2,3}$ is shown in the right of Figure 3.7. □

Now we are going to prove Theorem 3.2.2.

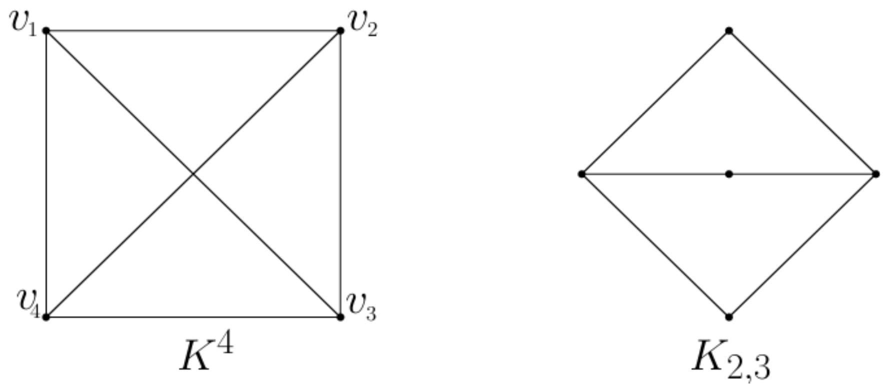

Figure 3.7: K^4 and $K_{2,3}$

Proof. We are going to establish the theorem from Kuratowski's Theorem; a proof using a different approach is presented in [CH67].

Necessary Condition

If one of K^4 and $K_{2,3}$ has a nonouterplanar subdivision K, then by smoothing the vertices of degree 2 in an outerplanar embedding of K we will find an outerplanar embedding of K^4 or $K_{2,3}$ which is in contradiction to Lemma 3.2.3 or to Lemma 3.2.4. So all subdivisions of K^4 or $K_{2,3}$ are nonouterplanar and it follows that a graph which contains one of them cannot be outerplanar.

Sufficient Condition

Let G be a nonouterplanar graph. If it is not planar, then by Kuratowski's Theorem it has a subgraph which is a subdivision of K^5 or $K_{3,3}$ and thus it contains a subgraph subdivision of K^4 or $K_{2,3}$. Otherwise it is planar such that in any of its planar embedding there is no face which has all the vertices in its boundary. Therefore if we introduce a new vertex v and join it to all the vertices of G, as in Figure 3.8, then we have a nonplanar graph H. By Kuratowski's Theorem H contains a subgraph K that is a subdivision of K^5 or $K_{3,3}$. v is necessarily contained in K. So if we delete v and all edges incident to

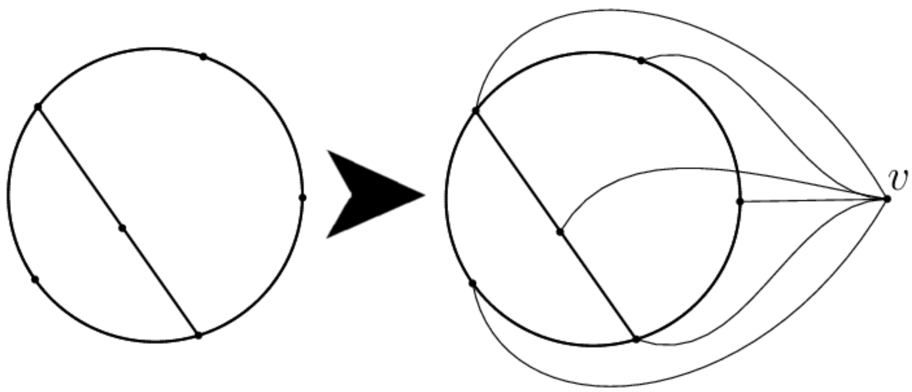

Figure 3.8: Construct a nonplanar graph from a planar and nonouterplanar graph

it from K, we will find a subgraph of G which contains a subdivision of K^4 or $K_{2,3}$. This completes the proof. $\square$

In 1937, Saunders MacLane proved an algebraic criterion stating that a graph is planar if and only if its "cycle space" has a basis such that each of its edges belongs to at most two cycles of this basis. This theorem is established using Kuratowski's Theorem on pages 101 and 102 of [Die06]. A theorem due to Whitney (1933) characterises planar graphs as those graphs that have an "abstract dual"; it is proven on page 106 of [Die06].

Apart from these characterisations of planarity, the Four-Colour Theorem is certainly the most famous result in the context of planar graphs. The following section deals with the problem of colouring planar graphs.

4. Colouring of Planar Graphs

Some of the applications of graph theory, for instance in cartography, require to colour the vertices, the edges or the faces of a given graph. If we impose the condition that only colourings where the ends of each edge have different colours are allowed, then no finite number of colours is sufficient to colour all graphs; this is because for every integer n, the complete graph K^n cannot be coloured with less than n colours satisfying the above condition. The problem of finding the smallest number of colours sufficient to colour a given graph or class of graphs belongs to the oldest and most famous problems in graph theory. In this chapter, we are going to discuss this problem for the class of planar graphs.

Definition 4.0.1. *Let G be a graph and let $\varphi : A \longrightarrow \{c_1, c_2, \ldots, c_k\}$.*

- *Such a map is called k-vertex colouring of G if $A = V(G)$. Moreover if it satisfies the condition*

$$\varphi(u) \neq \varphi(v) \text{ whenever } uv \in E(G), \tag{4.1}$$

  then it is called a proper k-vertex colouring of G.

- *φ is called a k-edge colouring of G if $A = E(G)$. It is a proper k-edge colouring of G if any two edges sharing an endpoint are associated to different elements of $\{c_1, c_2, \ldots, c_k\}$.*

- *φ is called a k-face colouring of G if $A = F(G)$. It is a proper k-face colouring of G if any two faces sharing a common edge are coloured differently.*

We will say that G is k-vertex (respectively k-edge, k-face) colourable, if and only if it has a proper k-vertex (respectively k-edge, k-face) colouring.

4.1 Five Colour Theorem

Let us first discuss the case of 5-colourings of planar graphs. It has a simple proof, so it is a very good start to understand the whole colouring problem.

Theorem 4.1.1. *All planar graphs are 5-vertex colourable.*

Before the main proof of this theorem, let us begin by proving a lemma.

Lemma 4.1.2. *In a proper k-vertex colouring of a graph G, where the colours are $c_1, c_2, \ldots, c_k$, if H is a subgraph of G which consists of all vertices coloured with $c_1, c_2, \ldots, c_l$, where $l \leqslant k$, with all edges joining them and if I is one of the connected components of H, then for all i and j such that $1 \leqslant i, j \leqslant l$, we can exchange the colours c_i and c_j with each other in I and still have a proper k-vertex colouring of G.*

Proof. Let I be as defined in the statement of Lemma 4.1.2. Exchanging c_i with c_j in I does not require any changes in any other components of H, because they are not connected to I. Moreover there is no need to repaint the vertices with colours different from $c_1, c_2, \ldots, c_l$, because, even if some of them might be linked by an edge to vertices of I, they will still have different colours than those in I. □

Now we are ready to prove Theorem 4.1.1.

Proof. We proceed by induction on the number n of vertices.

It is clear that if the number of vertices in a graph is less or equal to 5, then it can be coloured with five colours.

Now, let us assume that all planar graphs with n vertices can be coloured with five colours c_1, c_2, c_3, c_4 and c_5, and let us show that five colours are also enough to colour all graphs with $n+1$ vertices.

Let G be a planar graph with $n+1$ vertices. According to Corollary 2.2.3 the minimum degree d of G is at most 5. Consider a vertex v of G which has degree d. If we delete v and all incident edges, we obtain a smaller graph G' with n vertices. As stated in the induction hypothesis, G' can be coloured with five colours. Let us choose a 5-vertex colouring of G'.

- If $d \leqslant 4$, then there is at least one colour among the five that is not assigned to any of the neighbours of v, say c_5 is one of them. This enables us to add v back to G and assign the colour c_5 to it, which gives us a proper 5-vertex colouring of G.
- If $d = 5$, let v_1, v_2, v_3, v_4 and v_5 be the neighbours of v, numbered clockwise in a planar embedding $\tilde{G}$ of G.
  - If the number n_v of colours involved to colour all neighbours v_1, v_2, v_3, v_4 and v_5 is less than 5, then we will have again at least one free colour for v.
  - If $n_v = 5$, by permuting the names of the colours if necessary, we can assume that the colour of v_i is c_i for all $i \in \{1, 2, 3, 4, 5\}$. We try to reduce n_v by some permutation of colours. First we look at the subgraph $G_{1,3}$ of G' which is made up of all vertices coloured with c_1 and c_3 and the edges connecting them. If v_3 does not belong to the connected component $G'_{1,3}$ of $G_{1,3}$ containing v_1, then we exchange the colours c_1 and c_3 in $G'_{1,3}$, which does not change the validity of the colouring of $G_{1,3}$ as we have seen in Lemma 4.1.2. Consequently v_1 and v_3 will be both coloured by c_3, hence n_v becomes 4. Therefore, v can now be coloured with c_1. Otherwise, v_1 and v_3 are in the same connected component $G'_{1,3}$ of $G_{1,3}$, which proves the existence of a path $P(v_1, v_3)$, coloured with c_1 and c_3 only, connecting them.
    If this last case occurs, we look at the connected component $G_{2,4}$ of G' which consists of all the vertices coloured with colours c_2 and c_4 and all the edges connecting them. If v_4 does not belong to the connected component $G'_{2,4}$ containing v_2, then we permute the colours c_2 and c_4 in $G'_{2,4}$ to obtain a colouring in which v_2 and v_4 receive the same colour c_4. This reduces n_v to 4 and again allows us to colour v with the colour c_2. Otherwise v_2 and v_4 are in $G'_{2,4}$, so there is a path $P(v_2, v_4)$, coloured with c_2 and c_4, connecting them. $P(v_1, v_3)$ and $P(v_2, v_4)$ have to intersect at some point (this is illustrated in Figure 4.1). However, the vertex sets of $P(v_1, v_3)$ and $P(v_2, v_4)$ are disjoint, and so we obtain a contradiction to the assumption that $\tilde{G}$ is plane.

□

Remark 4.1.3. *The 5-vertex colouring of a given planar graph G is never unique, since it is always possible to interchange colours to obtain a different colouring.*

Theorem 4.1.1 has a corollary which is a stronger theorem ensuring the existence of a specific way to colour a planar graph with five colours.

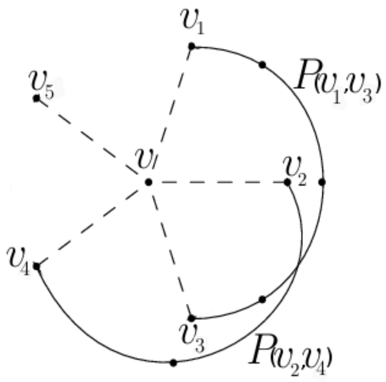

Figure 4.1: Crossing of the paths $P(v_1, v_3)$ and $P(v_2, v_4)$

Corollary 4.1.4. *Every planar graph has a proper* 5*-vertex colouring, where the boundary of each face of its planar embedding is coloured with at most four colours.*

Proof. By contradiction, assume that G is a planar graph that cannot be coloured according to this condition. Then all possible 5-vertex colourings of G leave at least one face whose boundary is coloured with five colours. Let n_f denote the number of faces in the planar embedding of G. If we add n_f more vertices to G such that the interior of each face receives one, and if we join each of these vertices to all the vertices on the boundary of the face containing it, then we end up with a bigger planar graph G'. G' cannot have a 5-colouring, as to any 5-colouring of G corresponds a face f having five different colours in its boundary, so there is no more colour for the vertex of G' placed in the interior of f, and this is in contradiction with Theorem 4.1.1. So we can conclude the corollary. □

The method of this proof fails if we reduce the number of colours to be 4. Therefore, different approach is needed in order to solve the Four Colour Problem.

4.2 Four Colour Theorem

Just like Fermat's famous Last Theorem, the Four Colour Problem is very easy to understand but no formal proof was found for more than hundred years.

4.2.1 Brief History of the Solution of the Four Colour Problem

The list of mathematicians who attempted the Four Colour Problem and published important results is very long, so we cannot list all of them. In the following, we provide a brief overview of the history of the Four Colour Theorem.

The genesis of the Four Colour Theorem was the conjecture made by Francis Guthrie in 1852, when he finished to colour the map of England with four colours. He wondered whether four colours are sufficient to colour all maps. Presented by Arthur Cayley to the London Mathematical Society in 1878, the problem became more public.

Alfred Bray Kempe published a proof for the Four Colour Theorem in 1879. 11 years later Percy Heawood discovered an error in Kempe's proof and modified it to prove the Five Colour Theorem.

Peter Guthrie Tait established in 1880 that the problem could be reduced to the 3-edge colouring of the family of all planar graphs whose vertices all have degree three.

In 1922, Philip Franklin proved that Guthrie's conjecture is satisfied by all maps with at most 25 regions. Further extension of this number of regions in a 4-colourable maps was made later by C. N. Reynolds and P. Franklin himself before C. E. Winn showed in 1938 that all maps containing 35 or less regions obey to the Four Colour Conjecture.

A big step was accomplished with the famous proof given by W. Haken and K. Appel [AH76] in 1976, but the doubts about its computer-based part induced a lot of discussion on its definitive validity. In [RSST97] the proof of Haken and Appel is described as " To check that the members of their "unavoidable set" ... would require us to input by hand into the computer descriptions of some 1400 graphs;".

In 1997 Daniel Sanders, Paul Seymour, Neil Robertson and Robin Thomas published a new computer-based proof for the Four Colour Theorem using the same approach as Haken and Appel. They simplified the non-computer parts of the proof in [RSST97], reduced the number of elements in the unavoidable set of reducible configurations to 633 and improved the computer program involved to have shorter running time.

4.2.2 Consequences of the Four Colour Theorem

The Four Colour Theorem has consequences outside graph theory as well, in particular there is an interesting relation to algebra. Apart from its consequence in cartography, it is also used to prove a property of the cross product.

Colouring in Cartography

Definition 4.2.1. *A map is a finite partition of the plane into connected regions called countries.*

For simplification we consider all regions to be countries, but in fact, in a real map they might be oceans or something else.

To any map, it is always possible to associate a planar graph with a unique vertex in the interior of each country and an edge joining two vertices if and only if the countries containing them have a common boundary which is not reduced to a single point. Figure 4.2 shows an example of this relation by using the map of Africa.

This construction enables us to conclude from the Four Colour Theorem that in fact, what Guthrie conjectured in 1852 is indeed true: "Four colours are enough to colour any map in such a way that distinct colours are used for neighbouring countries

Algebraic Consequence of The Four Colour Theorem

An interesting relation between graph theory and linear algebra was found by L. H. Kauffman when he related the Four Colour Theorem to a property of the cross product in $\mathbb{R}^3$. Let us see how this relation was established.

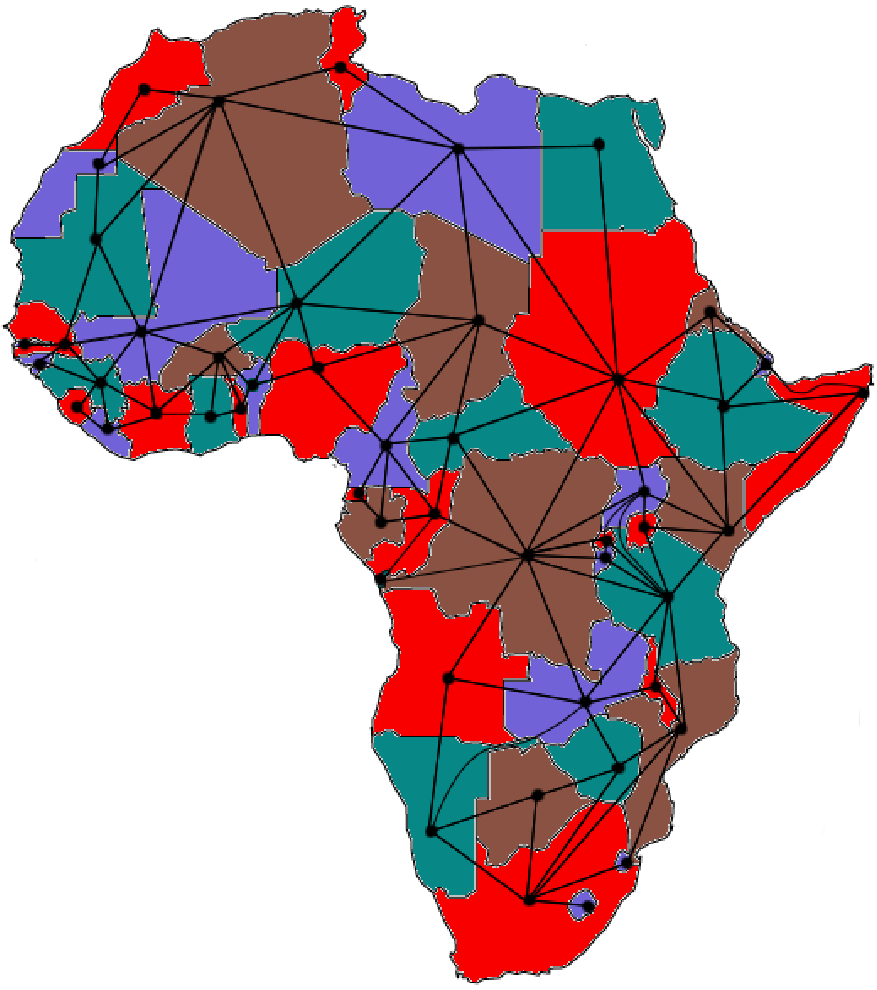

Figure 4.2: 4-colouring of the map of Africa and the corresponding planar graph.

Denote the unit vectors of $\mathbb{R}^3$ by $\vec{i}, \vec{j}$ and $\vec{k}$ and the cross product by $\times$. The cross product is not associative, hence for an integer $i > 2$, if $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_i}$ is a sequence of vectors in $\mathbb{R}^3$, then $\vec{v_1} \times \vec{v_2} \times \cdots \times \vec{v_i}$ is not well defined without describing, by inserting parenthesis, the order in which the operations must be processed. A change made to the position of the parentheses may leads to different results. For example

$$-\vec{j} = \vec{i} \times (\vec{i} \times \vec{j}) \neq (\vec{i} \times \vec{i}) \times \vec{j} = 0. \tag{4.2}$$

Given two associations of the sequence $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_i}$, it was posed as a problem whether it is always possible to find nonzero values for all v_k, $k \in \{1, 2, \ldots, i\}$, such that the two expressions have equal nonzero values. A solution to the problem was given by L. Kauffman [Kau90] by using the Four Colour Theorem.

Theorem 4.2.2. *Given two orders of processing the operations in the expression $\vec{v_1} \times \vec{v_2} \times \cdots \times \vec{v_i}$, there exists an assignment of $\vec{i}, \vec{j}, \vec{k}$ to $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_i}$ such that the two results are equal and nonzero.*

The proof of this theorem needs a lemma which is itself a consequence of the Four Colour Theorem.

Definition 4.2.3. *A graph is cubic if all of its vertices have degree 3.*

Lemma 4.2.4. *Every cubic planar graph without cut edge has a proper* 3*-edge colouring.*

Proof. Let G be a cubic planar graph without cut edge. In its planar embedding $\tilde{G}$, each edge is incident to exactly two faces. According to the Four Colour Theorem, $\tilde{G}$ has a proper 4-face colouring. Let $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$ be the set of the colours that we are using for the faces. Let us colour each edge by the sum, taken in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, of the different colours of the two faces to which it is incident. Because there are no two different elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that sum up to $(0,0)$, the possible colours for the edges are $\{(0,1),(1,0),(1,1)\}$.

If we consider an arbitrary vertex v, and denote by a, b, c the colours of the three faces incident to it, then the colours of the edges containing v are $a+b, a+c, b+c$. If by contradiction we assume that two of the three colours are equal, let us say that $a+b = a+c$, then the addition of the inverse of a to each term leads to $b = c$ which is in contradiction with the hypothesis. So we conclude that the 3-edge colouring of G defined in the previous paragraph is a proper 3-edge colouring. $\square$

The reverse of the implication that we have seen in the proof was proven by Tait [Tai80]. Now let us continue with the proof of Theorem 4.2.2.

Proof. Any two orders O_1 and O_2 of executing the operations in the expression $\vec{v_1} \times \vec{v_2} \times \cdots \times \vec{v_i}$ can be represented, respectively, by two trees T_1 and T_2. For instance, in Figure 4.3.a we have the representations of $((v_1 \times v_2) \times v_3) \times (v_4 \times v_5)$ and $v_1 \times (v_2 \times ((v_3 \times v_4) \times v_5))$. Merging the vertices representing the same vectors in the leaves of the two trees we obtain a graph like the one shown in Figure 4.3.b. By smoothing the vertices of degree 2 and joining the two roots of the trees by an edge e we end up with a planar cubic graph G without cut edge. Denote by $e_k, k \in \{1, 2 \ldots, i\}$, the edge corresponding to v_k.

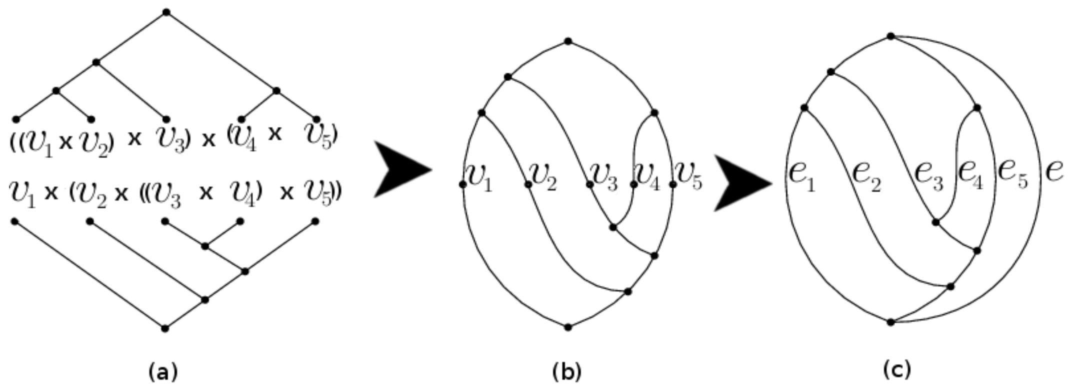

Figure 4.3: Relation between cross product and cubic graphs

According to Lemma 4.2.4, G has a proper 3-edge colouring. We can choose our colours to be $\vec{i}, \vec{j}$ and $\vec{k}$. So if, for $k \in \{1, 2, \ldots, i\}$, we assign the colour of e_k to the vector v_k, then it follows that the results from the two procedures will be plus or minus the colour of the edge e. In [Kau90], Kauffman shows a way to deal with the sign problem and proves that one can find an appropriate colouring which leads to the equality of the two expressions. $\square$

Colouring of Outerplanar Graphs

Because a triangle is an outerplanar graph, it is clear that at least three colours are needed for a proper vertex colouring of all the outerplanar graphs. Let us show that these three colours are in fact sufficient.

Theorem 4.2.5. *All outerplanar graphs have a proper 3-vertex colouring.*

It is a consequence of the Four Colour Theorem.

Proof. Let G be an outerplanar graph and let $\tilde{G}$ be its planar embedding where all the vertices are incident to the exterior face. If we add a vertex v in the exterior face, we can join it to all the vertices of G and still have a planar graph H which may not be outerplanar. By the Four Colour Theorem, we can find a 4-vertex proper colouring for H where colours in all the vertices of G are different from that of v. This is impossible except if G has a 3-vertex colouring. $\square$

A self-contained proof for Theorem 4.2.5 can be done by induction with respect to the number of vertices.

5. Conclusion

We have seen that Kuratowski's Theorem is fundamental in the characterisation of planar graphs. Not only it leads to a linear time algorithm to check planarity, but also its various interpretations give a wide range of graph planarity criteria, including an algebraic criterion.

The latest proof of the Four Colour Theorem, due to N. Robertson, D. Sanders, P. Seymour and R. Thomas, is still a computer-based proof, but substantial progress has been made in recent years. It is interesting to see, however, that the proof of the Five Colour Theorem is very short and simple in comparison. The Four Colour Problem is not an isolated problem: its solution implies solutions to other related problems, even in different areas of mathematics.

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"But by God's grace I am what I am" *Cor. I* 15:10

"Fa ny fahasoavan'Andriamanitra no naha-toy izao ahy" *Kor. I* 15:10

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