Thm (Master Theorem)

$T(n) = aT\left(\frac{n}{b}\right) + cn$ where a, b and c are $\begin{cases} \text{positive integers} \\ \text{independent of } n \end{cases}$

with $T(1) = 1$ and n is a power of b

Case 1 $a = b$ $T(n) = O(n \log n)$

Case 2 $a < b$ $T(n) = O(n)$

Case 3 $a > b$ $T(n) = O(n^{\log_b a})$

INTUITION

Case 1: $T(n) = 2T\left(\frac{n}{2}\right) + n$

total n — n — $T(n)$

total n — $\frac{n}{2}$ $\frac{n}{2}$ — $2T\left(\frac{n}{2}\right)$

total n — $\frac{n}{2^2}$ $\frac{n}{2^2}$ $\frac{n}{2^2}$ $\frac{n}{2^2}$ — $2^2T\left(\frac{n}{2^2}\right)$

$\vdots$

end when $\frac{n}{2^k} = 1 \Leftrightarrow k = \log_2 n$

$\log_2 n$ lines

total $n \log_2 n$

Case 2: $T(n) = T\left(\frac{n}{2}\right) + n$

total n — n — $T(n)$

total $\frac{n}{2}$ — $\frac{n}{2}$ — $T\left(\frac{n}{2}\right)$

total $\frac{n}{2^2}$ — $\frac{n}{2^2}$ — $T\left(\frac{n}{2^2}\right)$

end when $\frac{n}{2^k} = 1 \Leftrightarrow k = \log_2 n$

total $n + \frac{n}{2} + \frac{n}{2^2} + \dots + \frac{n}{2^{\log_2 n}}$

$= n\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{\log_2 n}}\right)$

total $< 2n$

STEP 1 — STEP 2 — STEP 3

Proof by substitution — guessing general form — using base case

$$T(n) = aT\left(\frac{n}{b}\right) + cn$$

substitute $T\left(\frac{n}{b}\right)$

$$= a\left(aT\left(\frac{n}{b^2}\right) + c\frac{n}{b}\right) + cn$$

calculations

$$= a^2T\left(\frac{n}{b^2}\right) + cn\frac{a}{b} + cn$$

substitute $T\left(\frac{n}{b^2}\right)$

$$= a^2\left(aT\left(\frac{n}{b^3}\right) + c\frac{n}{b^2}\right) + cn\frac{a}{b} + cn$$

calculation

$$= a^3T\left(\frac{n}{b^3}\right) + cn\left(\frac{a}{b}\right)^2 + cn\left(\frac{a}{b}\right) + cn$$

$$\vdots$$

guessing general form

$$= a^kT\left(\frac{n}{b^k}\right) + cn\left(\left(\frac{a}{b}\right)^{k-1} + \left(\frac{a}{b}\right)^{k-2} + \dots + \left(\frac{a}{b}\right) + 1\right)$$

In summary: $(*)$

$$\boxed{\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) + cn \\ &= a^kT\left(\frac{n}{b^k}\right) + cn\left(\left(\frac{a}{b}\right)^{k-1} + \left(\frac{a}{b}\right)^{k-2} + \dots + \left(\frac{a}{b}\right) + 1\right) \end{aligned}}$$

Case 1 $a = b$

(*) becomes

$$T(n) = aT\left(\frac{n}{a}\right) + cn$$

$$= a^K T\left(\frac{n}{a^K}\right) + cn(\underbrace{1 + 1 + \dots + 1 + 1}_{K \text{ times}})$$

$$= a^K T\left(\frac{n}{a^K}\right) + cnK$$

What is a suitable value of K to stop?

Know $T(1) = 1$, so want $\frac{n}{a^K} = 1 \Leftrightarrow K = \log_a n$

$$= a^{\log_a n} T\left(\frac{n}{a^{\log_a n}}\right) + cn \log_a n$$

$$= n\,T\left(\frac{n}{n}\right) + cn \log_a n$$

$$= n\,T(1) + cn \log_a n$$

$$1$$

$$= n + cn \log_a n = O(n \log n)$$

Case 2 $\quad a < b$

$(*)$ becomes

$$T(n) = aT\left(\frac{n}{b}\right) + cn$$

$$= a^k T\left(\frac{n}{b^k}\right) + cn\left(\left(\frac{a}{b}\right)^{k-1} + \left(\frac{a}{b}\right)^{k-2} + \dots + \left(\frac{a}{b}\right) + 1\right)$$

$$= a^k T\left(\frac{n}{b^k}\right) + cn\left(\frac{1 - \left(\frac{a}{b}\right)^k}{1 - \frac{a}{b}}\right)$$

using power series for $x < 1$ see appendix (A) number (1)

what is a suitable value for k to stop?

Know $T(1) = 1$, so want $\frac{n}{b^k} = 1 \iff k = \log_b n$

$$= a^{\log_b n}\, T(\overset{1}{1}) + \frac{cnb}{b-a}\left(1 - \frac{a^{\log_b n}}{b^{\log_b n}}\right)$$

(A) see appendix number (3)

$$= b^{\log_b a \log_b n} + \frac{cnb}{b-a} - \frac{c\cancel{n}b}{b-a}\,\frac{b^{\log_b a \log_b n}}{\cancel{n}}$$

$$= b^{\log_b a \log_b n}\left(1 - \frac{cb}{b-a}\right) + \frac{cnb}{b-a}$$

$\longrightarrow$ next page

$$= b^{\log_b n \log_b a}\left(1 - \frac{cb}{b-a}\right) + \frac{cb}{b-a} \cdot n$$

see appendix A
number 3

$$= n^{\log_b a}\left(1 - \frac{cb}{b-a}\right) + \frac{cb}{b-a} n$$

see appendix A
number 4

$$= \frac{cb}{b-a} n + n^{1-\epsilon}\left(1 - \frac{cb}{b-a}\right)$$

this is the leading term

$$= O(n)$$

Case 3 $\quad a > b$

(*) becomes

$$T(n) = aT\left(\frac{n}{b}\right) + cn$$

$$= a^K T\left(\frac{n}{b^K}\right) + cn\left(\left(\frac{a}{b}\right)^{K-1} + \left(\frac{a}{b}\right)^{K-2} + \dots + \left(\frac{a}{b}\right) + 1\right)$$

$$= a^K T\left(\frac{n}{b^K}\right) + cn\left(\frac{\left(\frac{a}{b}\right)^K - 1}{\frac{a}{b} - 1}\right)$$

using Power series for $x \neq 1$ see Appendix (A) number (1)

What is a suitable value for K to stop?

Lik all prior cases, Know $T(1) = 1$, so want $\frac{n}{b^K} = 1 \Leftrightarrow K = \log_b n$

$$= a^{\log_b n} \cancel{T(1)} + \frac{cnb}{a-b}\left(\frac{a^{\log_b n}}{b^{\log_b n}} - 1\right)$$

$\cancel{T(1)} \to 1$

appendix (A) number (3)

$$= b^{\log_b a \, \log_b n} + \frac{c\cancel{n}b}{a-b} \, \frac{b^{\log_b a \, \log_b n}}{b^{\cancel{\log_b n}}} - \frac{cnb}{a-b}$$

$b^{\cancel{\log_b n}} \to n$

next page

$$= b^{\log_b n \log_b a}\left(1+\frac{cb}{a-b}\right) - \frac{cnb}{a-b}$$

$$= n^{\log_b a}\left(1+\frac{cb}{a-b}\right) - \frac{cnb}{a-b}$$

this is leading term see Appendix A number 4

$$= O\left(n^{\log_b a}\right).$$

Mergesort

Suppose that we want an $O(n \log n)$ sorting algorithm and we do not Know where to start...,

Know $T(n) = 2T(\frac{n}{2}) + cn = O(n \log n)$

Approach Try to reduce sorting an array of size n to sorting two arrays of size $\frac{n}{2}$ and do some linear work to combine

But that is exactly mergesort:

Mergesort $(a_1 \ldots\ldots a_n)$

recursive call **Mergesort** $(a_1 \ldots a_{\frac{n}{2}})$

" **Mergesort** $(a_{\frac{n}{2}+1} \ldots a_n)$

merge the two sorted parts in n comparisons

$$T(n) = 2T\left(\frac{n}{2}\right) + n = O(n \log n)$$

Appendix (A)

(1) $$x^N + x^{N-1} + \dots + x^2 + x + 1 = \begin{cases} N+1 & \text{for } x = 1 \\ \dfrac{x^{N+1}-1}{x-1} < \dfrac{x^{N+1}}{x-1} & \text{for } x > 1 \\ \dfrac{1-x^{N+1}}{1-x} < \dfrac{1}{1-x} & \text{for } x < 1 \end{cases}$$

(2) for any constant ϵ, $0 < \epsilon < 1$: $$n^{1-\epsilon} < n < n^{1+c}$$

(3) $$x^{\log_x y} = y$$

(4) $a < b$ $\quad \log_b a < 1$

$a > b$ $\quad \log_b a > 1$

__Note__: We often use the following:

$$a^{\log_b n} = b^{\log_b a \, \log_b n} = b^{\log_b n \, \log_b a} = n^{\log_b a}$$

↓ because of (3) above $a = b^{\log_b a}$

↓ because of (3) above $b^{\log_b n} = n$