Last Name: .... **First Name:** .... **Email:** ....

Problem 1: Subset Sum, Dynamic Programming (25 points)

You are given n items $\{1, 2, \ldots n\}$, where each item has a given positive weight w_i, $1 \leq i \leq n$. You are also given an upper bound W. You would line to select a subset S of the items so that $\sum_{i \in S} w_i \leq W$, and, subject to this restriction, $\sum_{i \in S} w_i \leq W$ is as large as possible. Give an $O(nW)$ algorithm, justify correctness and running time.

Answer:

Problem 2: Balanced Tree, Dynamic Programming (25 points).
Let $T(V,E)$ be a directed acyclic graph with $V = \{v_1, \ldots v_n\}$.
Suppose that T is given in topologically sorted order, that is, if v_i is an ancestor of v_j then $i < j$.
Suppose further that each vertex $v_i \in V$ has a given positive cost $c(v_i) > 0$. Define the weight of a vertex $v_i \in V$ as the sum of the costs of all vertices that can be reached from v_i (equivalently belong to the subtree rooted at v_i) : $weight(v_i) = \sum_{\substack{v_j \in V: \\ v_j \text{ is reachable from } v_i}} c(v_j)$
Say that T is balanced if and only if, for every vertex $v_i \in V$, if v_i has children $u_1, \ldots, u_k$, then

$$weight(u_1) = weight(u_2) = \ldots = weight(u_k)$$

Give a polynomial time algorithm that decides if a directed tree with costs on its vertices is balanced. Justify your answer and argue running time.

Answer:

Problem 3: Max Independent Set, Dynamic Programming (25 points)
(a) Consider a line graph on vertices $\{1, \ldots, n\}$ and edges $\{1, 2\}$, $\{2, 3\}$, ... , $\{(n-1), n\}$. Each vertex has a positive weight w_i, $1 \leq i \leq n$. Give an $O(n)$ algorithm that outputs the weight of a maximum weight independent set of the line graph. You may give a simple description of the algorithm, and/or pseudocode. You should include a short argument of correctness and running time.
(b) Consider a cycle graph on vertices $\{1, \ldots, n\}$ and edges $\{1, 2\}$, $\{2, 3\}$, ... , $\{(n-1), n\}$, $\{n, 1\}$. Each vertex has a positive weight w_i, $1 \leq i \leq n$. Give an $O(n)$ algorithm that outputs the weight of a maximum weight independent set of the cycle graph. You may give a simple description of the algorithm, and/or pseudocode. You should include a short argument of correctness and running time.

Answer:

Problem 4: Longest Path, Dynamic Programming (25 points)
Let $G(V, E)$ be a directed acyclic graph, where $V = \{v_1, \ldots, v_n\}$. The graph is presented in adjacency list representation, and with the property that $v_i \rightarrow v_j \in E$ only if $i < j$. Give an $O(|V| + |E|)$ algorithm that finds the length of the longest path (maximum number of edges) from v_1 to v_n. If there is no path from v_1 to v_2 then your algorithm should output ∞.
Give a short justification of correctness and running time.

Answer: