MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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Problem Set 6 Solutions

Problem 1: Short Questions

(a) If charges flow very slowly through a metal, why does it not require several hours for light to come on when you throw a switch?

Even though the average velocity of charges is very slow, when you connect a potential difference across a circuit (wire & light bulb), an electric field is established almost instantly (propagating near the speed of light) everywhere in the circuit, and the electrons begin to drift everywhere all at once.

(b) What advantage does 120-V operation offer over 240 V? What are the disadvantages?

Both 120-V and 240-V lines can deliver injurious or lethal shocks, but there is a somewhat better safety factor with the lower voltage. On the other hand, a 240-V device requires less current to operate at the same power, and hence generates less heating and loss (I^2R) in wiring.

(c) Why is it possible for a bird to stand on a high-voltage wire without getting electrocuted?

The reason is because the potential on the entire wire is nearly uniform, and the potential difference between the bird's feet is approximately zero. Thus, the amount of current flowing through the bird is negligible, since the resistance through the bird's body between its feet is much greater than the resistance through the wire between the same two points.

(d) If your car's headlights are on when you start the ignition, why do they dim while the car is starting?

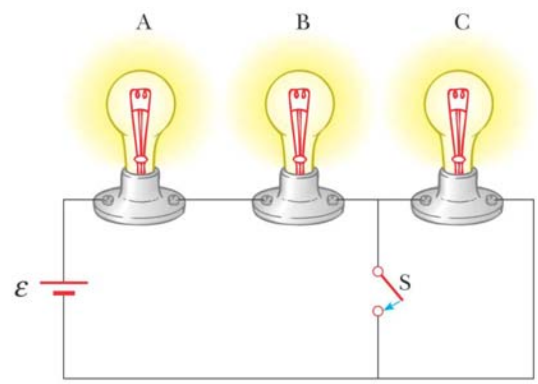
The starter motor draws a significant amount of current from the battery while it is starting the car. This, coupled with the internal resistance of the battery, decreases the output voltage of the battery below its the nominal 12 V. This decrease in voltage decreases the current through (and brightness of) the headlights.

(e) Suppose a person falling from a building on the way down grabs a high-voltage wire. If the wire supports him as he hangs from it, will he be electrocuted? If the wire then breaks, should he continue to hold onto the end of the wire as he falls?

Problem 1: Short Questions *continued…*

As long as he only grabs one wire and does not touch anything that is grounded, he will be safe. If the wire breaks, *let go!* If he continues to hold on to the wire, there will be a large—and rather lethal—potential difference between the wire and his feet when he hits the ground.

(f) A series circuit consists of three identical lamps connected to a battery as shown in the figure below. When the switch S is closed, what happens to the brightness of the light bulbs? Explain your answer.



Closing the switch makes the switch and the wires connected to it a zero-resistance branch. All of the current through A and B will go through the switch and lamp C goes out, with zero voltage across it. With less total resistance, the current in the battery becomes larger than before and lamps A and B get brighter.

Problem 2: Battery Life

AAA, AA, … D batteries have an open circuit voltage (EMF) of 1.5 V. The difference between different sizes is in their lifetime (total energy storage). A AAA battery has a life of about 0.5 A-hr while a D battery has a life of about 10 A-hr. Of course these numbers depend on how quickly you discharge them and on the manufacturer, but these numbers are roughly correct. One important difference between batteries is their internal resistance – alkaline (now the standard) D cells are about 0.1Ω.

Suppose that you have a multi-speed winch that is 50% efficient (50% of energy used does useful work) run off a D cell, and that you are trying to lift a mass of 60 kg (hmmm, I wonder what mass that would be). The winch acts as load with a variable resistance R_L that is speed dependent.

a) Suppose the winch is set to super-slow speed. Then the load (winch motor) resistance is much greater than the battery's internal resistance and you can assume that there is no loss of energy to internal resistance. How high can the winch lift the mass before discharging the battery?

Problem 2: Battery Life *continued…*

This is just a question of energy. The battery has an energy storage of (1.5 V)(10 A-hr) = 15 W-hr or 54 kJ. So it can lift the mass:

$$U = mgh \Rightarrow \quad h = \frac{U}{mg} = \frac{54\,\text{kJ}\cdot\frac{1}{2}}{(60\text{ kg})(9.8\text{ m/s})} = \boxed{46\text{ m}}$$

The factor of a half is there because the winch is only 50% efficient.

b) To what resistance R_L should the winch be set in order to have the battery lift the mass at the fastest rate? What is this fastest rate (m/sec)? HINT: You want to maximize the power delivery to the winch (power dissipated by R_L).

First we need to determine how to maximize power delivery. If a battery V is connected to two resistances, r_i (the internal resistance) and R, the load resistance, the power dissipated in the load is:

$$P = I^2 R = \left(\frac{V_0}{R+r_i}\right)^2 R = V_0^2 \frac{R}{(R+r_i)^2}$$

We want to maximize this by varying R:

$$\frac{dP}{dR} = \frac{d}{dR}\left(V_0^2 R (R+r_i)^{-2}\right) = V_0^2\left[(R+r_i)^{-2} - 2R(R+r_i)^{-3}\right] = 0$$

Multiply both sides by $V_0^{-2}(R+r_i)^3 : \left[(R+r_i) - 2R\right] = r_i - R = 0 \qquad \Rightarrow \qquad \boxed{R = r_i}$

So, to get the fastest rate of lift (most power dissipation in the winch) we need the winch resistance to equal the battery internal resistance, $R_L = r_i = 0.1\ \Omega$.

Using this we can get the lift rate from the power:

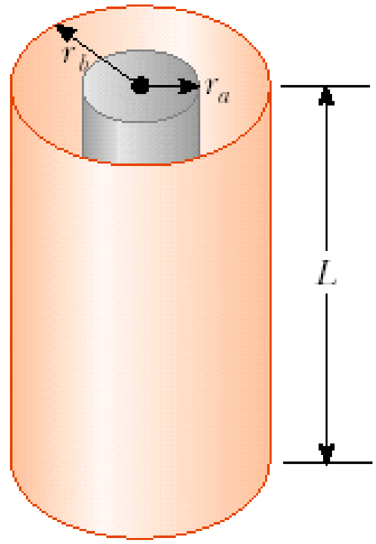
$$P = I^2 R_L = \left(\frac{V_0}{R_L + r_i}\right)^2 R_L = \frac{V_0^2}{4r_i} = \overbrace{2}^{50\%\,eff}\frac{d}{dt}(mgh) \Rightarrow v = \frac{dh}{dt} = \frac{V_0^2}{8r_i mg}$$

Thus we find a list rate of $\boxed{v = 4.8\text{ mm/s}}$

c) At this fastest lift rate how high can the winch lift the mass before discharging the battery?

This is just part a over again, except now we waste half the energy in the internal resistor, so the winch will only rise half as high, to $\boxed{23\text{ m}}$

Problem 3: Sea Water



An oceanographer is studying how the ion concentration in sea water depends on depth. She does this by lowering into the water (until completely submerged) a pair of concentric metallic cylinders (see figure) at the end of a cable and taking data to determine the resistance between these electrodes as a function of depth. The water between the two cylinders forms a cylindrical shell of inner radius r_a , outer radius r_b, and length L much larger than r_b. The scientist applies a potential difference ΔV between the inner and outer surfaces, producing an outward radial current I. Let ρ represent the resistivity of the water.

Find the resistance of the water between the cylinders in terms of L, ρ, r_a, and r_b.

We will build up the total resistance of the water by considering a number of cylindrical shells of water in series. Consider a thin cylindrical shell of radius r, thickness dr, and length L. Its contribution to the overall resistance of the water is

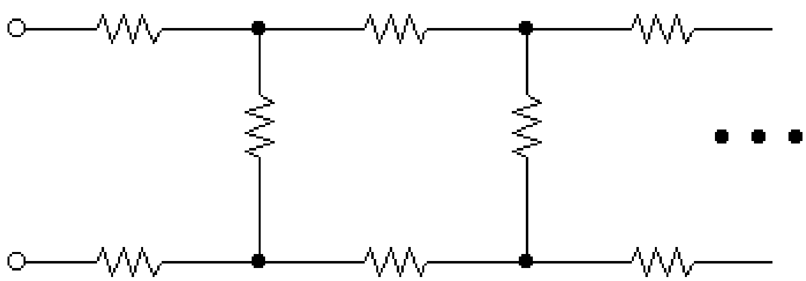
$$dR = \frac{\rho d\ell}{A} = \frac{\rho dr}{(2\pi r)L} = \left(\frac{\rho}{2\pi L}\right)\frac{dr}{r}$$

The resistance of the whole annulus is the series summation of the contributions of the thin shells:

$$R = \frac{\rho}{2\pi L}\int_{r_a}^{r_b}\frac{dr}{r} = \boxed{\frac{\rho}{2\pi L}\ln\left(\frac{r_b}{r_a}\right) = R}$$

Problem 4: Resistor Ladder

Consider the infinite ladder of resistors below formed by using identical 1 Ω resistors.



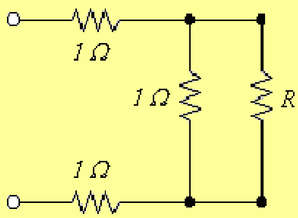
What is the resistance that you will measure between the two nodes at left? HINT: This is conceptually similar to calculating the value of repeating fractions, e.g.

Problem 4: Resistor Ladder *continued…*

$$f = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}$$

It pays to remember that ∞-1≅∞.

The important thing to realize here is that an infinite ladder can have one set of rungs added or removed and not have its resistance change. So the total resistance R must equal that resistance of the below circuit:



$$R = 1\Omega + 1\Omega + \frac{1}{\frac{1}{1\Omega} + \frac{1}{R}} = 2\Omega + \frac{R \cdot 1\Omega}{R + 1\Omega} \qquad \Rightarrow \qquad (R - 2\Omega)(R + 1\Omega) = R \cdot 1\Omega$$

$$R^2 - 2\Omega \cdot R + R \cdot 1\Omega - 2\Omega^2 - R \cdot 1\Omega = R^2 - 2\Omega \cdot R - 2\Omega^2 = 0$$

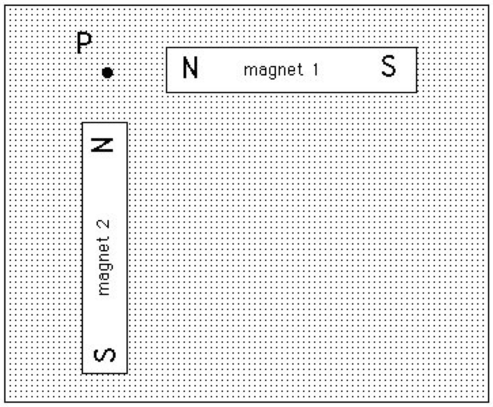
$$R = \frac{2\Omega \pm \sqrt{(2\Omega)^2 - 4 \cdot (-2\Omega^2)}}{2} = \boxed{\left(1 + \sqrt{3}\right)\Omega}$$

Problem 5: Energy Cost

Compare the cost of powering a desk light with D cells as opposed to plugging it into the wall. Does it make sense to use rechargeable batteries? HINT: You can find some useful numbers in problem 2.

A D cell (according to problem 2) has a battery life of 10 A-hr, meaning a total energy storage of (1.5 V)(10 A-hr) = 15 Watt-hrs. We could convert that to about 50 kJ but Watt-hours are a useful unit to use because electricity is typically charged by the kW-hour so this will make comparison easier. A D battery costs about \$1 (you can pay more, but why?) So D batteries cost about \$1/0.015 kwh or \$70/kwh.
Residential electricity costs about \$0.1/kwh. So the battery is nearly three orders of magnitude more expensive. It definitely makes sense to use rechargeable batteries – even though the upfront cost is slightly more expensive you will get it back in a couple recharges. As for your desk light, or anything that can run on batteries or wall power, plug it in. If it is 60 Watts, for every hour you pay only 0.6¢ with wall power but run through \$4 in D batteries.

Problem 6: Superposition

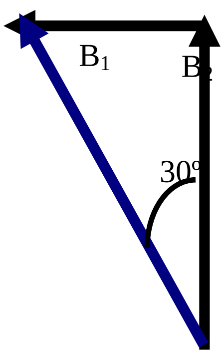


Consider two bar magnets placed at right angles to each other, as pictured at left.

(a) If a small compass is placed at point P, what direction does the painted end of the compass needle point?

It points away from each magnetic North, which means toward the upper left corner (45º if they are the same magnitude).

(b) If the compass needle instead pointed 15 degrees clockwise of where you predicted in (a), what could you qualitatively conclude about the relative strengths of the two magnets?



In order for it to point 15 degrees clockwise the second magnet must be stronger than the first. Since the total field is just a vector sum of the two we can see how much stronger.

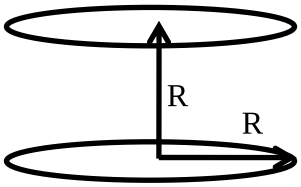
$$\tan 30^\circ = \frac{B_1}{B_2} = \frac{1}{\sqrt{3}} \Rightarrow \boxed{B_2 = \sqrt{3} B_1}$$

Problem 7: Helmholtz Coil

In class you calculated the magnetic field along the axis of a coil to be given by:

$$B_{axial} = \frac{N \mu_0 I R^2}{2} \frac{1}{(z^2 + R^2)^{3/2}}$$

where z is measured from the center of the coil.



As pictured at left, a Helmholtz coil is created by placing two such coils (each of radius *R* and *N* turns) a distance *R* apart.

(a) If the current in the two coils is parallel (Helmholtz configuration), what is the axial field strength at the center of the apparatus (midway between the two coils)? How does this compare to the field strength at the center of the single coil configuration (e.g. what is the ratio)?

Problem 7: Helmholtz Coil *continued…*

We just use superposition to determine the total field. We are the same distance $z = R/2$ from each coil and since the currents are parallel they both create a field in the same direction (for example, if both currents are counter-clockwise they both create an upward magnetic field at the midpoint). The magnitude then is just twice what one would give:

$$B = 2\times\frac{N\,\mu_0\,I\,R^2}{2}\frac{1}{\left(\left(R/2\right)^2+R^2\right)^{3/2}} = \frac{N\,\mu_0\,I}{R}\frac{1}{\left(\left(1/2\right)^2+1\right)^{3/2}} = \frac{8N\,\mu_0\,I}{5^{3/2}R}$$

Comparing this to the field strength at the center of a single coil:

$$B_{\text{sgl coil}} = \frac{N\,\mu_0\,I}{2R}$$

We find that the field of a Helmholtz coil is slightly larger:

$$\frac{B_{\text{Helmholtz}}}{B_{\text{Sgl Coil}}} = \left(\frac{8N\,\mu_0\,I}{5^{3/2}R}\right)\Big/\left(\frac{N\,\mu_0\,I}{2R}\right) = \frac{16}{5^{3/2}} \approx 1.4$$

(b) In the anti-Helmholtz configuration the current in the two coils is anti-parallel. What is field strength at the center of the apparatus in this situation?

In this case the fields from the two coils are in opposite directions so they cancel each other out. That is, $B = 0$.

(c) Our coils have a radius $R = 7$ cm and $N = 168$ turns, and we will run with $I = 0.6$ A in single coil and 0.3 A in Helmholtz and anti-Helmholtz mode. What, approximately, are the largest on-axis fields we should expect in these three configurations? Where (approximately) are the fields the strongest?

For a single coil the maximum is at the center of the coil, for a Helmholtz at the center: .

$$B^{\max}_{\text{sgl coil}} = \frac{(168)\left(4\pi\times10^{-7}\ \text{T}\cdot\text{m/A}\right)(0.6\ \text{A})}{2\left(7\times10^{-2}\ \text{m}\right)} = 9\times10^{-4}\ \text{T} = 9\ \text{Gauss};$$

$$B^{\max}_{\text{Helmholtz}} = \frac{16}{5^{3/2}\cdot 2}\cdot 9\ \text{Gauss} = 6.5\ \text{Gauss}$$

Problem 7: Helmholtz Coil *continued…*

For an Anti-Helmholtz coil the field between the two is given by:

$$B_{\text{Anti-Helmholtz}} = \frac{N\,\mu_0\,I\,R^2}{2}\left[\frac{1}{(z^2+R^2)^{3/2}} - \frac{1}{\left((R-z)^2+R^2\right)^{3/2}}\right]$$

where I have taken the bottom coil to be at $z = 0$ and to have a CCW current, making a positive field. I then had to subtract the field from the top coil, which is now a distance (R-z) away from our observation point at z.
To maximize we could take the derivative and set it equal to zero, but this is a pretty nasty and since we are only asked to approximate, realizing that the max is at about at the centers of each of the coils makes our lives easier:

$$\begin{aligned} B^{\text{max}}_{\text{Anti-Helmholtz}} &\approx \frac{N\,\mu_0\,I}{2R}\left[\frac{1}{(u^2+1)^{3/2}} - \frac{1}{\left((1-u)^2+1\right)^{3/2}}\right] \\ &\approx \frac{(168)\left(4\pi\times10^{-7}\ \text{T}\cdot\text{m/A}\right)(0.3\ \text{A})}{2\left(7\times10^{-2}\ \text{m}\right)}\left[1-\frac{1}{2^{3/2}}\right] = 2.9\ \text{Gauss} \end{aligned}$$

If you were to do the calculation (numerically is the only non-painful way) you find the max is closer to 3.2 Gauss and the maxima are just outside the two coils.