MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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Problem Set 1 Solutions

Problem 1: Two Vectors

Given two vectors, $\vec{\mathbf{A}} = (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ and $\vec{\mathbf{B}} = (7\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$, evaluate the following:

(a) $2\vec{\mathbf{A}} + \vec{\mathbf{B}}$;

$$2\vec{\mathbf{A}} + \vec{\mathbf{B}} = 2(4\hat{\mathrm{i}} - 3\hat{\mathrm{j}} + 5\hat{\mathrm{k}}) + (7\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 15\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 14\hat{\mathbf{k}}$$

(b) $\vec{\mathbf{A}} - 3\vec{\mathbf{B}}$;

$$\vec{\mathbf{A}} - 3\vec{\mathbf{B}} = (4\hat{\mathrm{i}} - 3\hat{\mathrm{j}} + 5\hat{\mathrm{k}}) - 3(7\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = -17\,\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$$

(c) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$;

Since $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$, the dot product is

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (4)(7) + (-3)(4) + (5)(4) = 36$$

(d) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$;

With $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$, the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is given by

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & -3 & 5 \\ 7 & 4 & 4 \end{vmatrix} = -32\,\hat{\mathbf{i}} + 19\,\hat{\mathbf{j}} + 37\,\hat{\mathbf{k}}$$

(e) What is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$?

The dot product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\cos\theta$ where θ is the angle between the two vectors. With:

$$A = |\vec{\mathbf{A}}| = \sqrt{(4)^2 + (-3)^2 + (5)^2} = \sqrt{50} = 5\sqrt{2}$$

$$B = |\vec{\mathbf{B}}| = \sqrt{(7)^2 + (4)^2 + (4)^2} = \sqrt{81} = 9,$$

and using the result from part (c), we obtain

$$\cos\theta = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|} = \frac{30}{5\sqrt{2} \cdot 9} = 0.6 \quad \Rightarrow \quad \theta = 52.5°.$$

Problem 1: Two Vectors *continued*

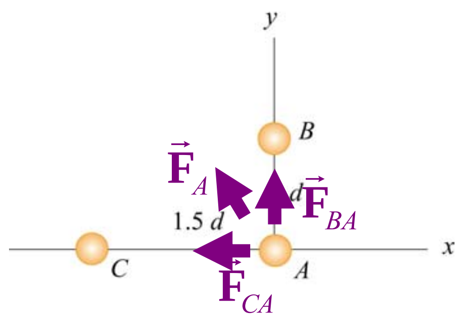
(f) Find two unit vectors that are perpendicular to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

The cross product $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ (or $\vec{\mathbf{B}}\times\vec{\mathbf{A}}$) is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore, from the result of part (d), the unit vectors may be obtained as

$$\hat{\mathbf{n}} = \pm\frac{\vec{\mathbf{A}}\times\vec{\mathbf{B}}}{\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right|} = \pm\frac{\left(-32\,\hat{\mathbf{i}}+19\,\hat{\mathbf{j}}+37\,\hat{\mathbf{k}}\right)}{\sqrt{(-32)^2+(19)^2+(37)^2}} = \pm\frac{1}{\sqrt{2754}}\left(-32\,\hat{\mathbf{i}}+19\,\hat{\mathbf{j}}+37\,\hat{\mathbf{k}}\right)$$

Problem 2: Electrostatic Force

Consider three point charges (A, B, C) located as shown in figure at right (where d is 9.0 cm). A has positive charge 3.0 μC while B & C both have negative charge -1.0 μC. Calculate the resultant electric force on A. Be sure to specify both the magnitude and direction.



Let d_{ca} =1.5d be the distance between A & C. The displacement from C to A is $\vec{\mathbf{r}}_{CA} = d_{CA}\hat{\mathbf{i}}$. The displacement from B to A is $\vec{\mathbf{r}}_{BA} = -d\hat{\mathbf{j}}$. Using Coulomb's law, the force exerted on A is

$$\vec{\mathbf{F}}_A = \vec{\mathbf{F}}_{CA}+\vec{\mathbf{F}}_{BA} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_C q_A \vec{\mathbf{r}}_{CA}}{r_{CA}^3}+\frac{q_B q_A \vec{\mathbf{r}}_{BA}}{r_{BA}^3}\right)$$

where q_A, q_B, q_C are the charges on A, B and C respectively. Since $q_B = q_C \equiv q_{BC} = -1.0\,\mu C$ this becomes:

$$\vec{\mathbf{F}}_A = \frac{q_{BC}q_A}{4\pi\varepsilon_0}\left(\frac{\vec{\mathbf{r}}_{CA}}{r_{CA}^3}+\frac{\vec{\mathbf{r}}_{BA}}{r_{BA}^3}\right) = \frac{q_{BC}q_A}{4\pi\varepsilon_0 d^2}\left(\frac{1.5}{(1.5)^3}\hat{\mathbf{i}}+\hat{\mathbf{j}}\right)$$

Note that this is easier than the more typical method of finding the magnitudes first and then multiplying by the sine and cosine of angles to get the components.

Now we can plug in numerical values:

$$\vec{\mathbf{F}}_A = \left(8.99\times10^9\ \mathrm{N\,m^2\,C^{-2}}\right)\frac{\left(-1.0\times10^{-6}\mathrm{C}\right)\left(3.0\times10^{-6}\mathrm{C}\right)}{\left(9.0\times10^{-2}\mathrm{m}\right)^2}\left(\frac{1.5}{(1.5)^3}\hat{\mathbf{i}}-\hat{\mathbf{j}}\right) = \boxed{\left(-1.5\hat{\mathbf{i}}+3.3\hat{\mathbf{j}}\right)\mathrm{N}}$$

It is sufficient to leave the answer as this (both the magnitude and direction are indicated) but if you wanted to be explicit, you can calculate. The magnitude of $\vec{\mathbf{F}}_{\mathbf{A}}$ is

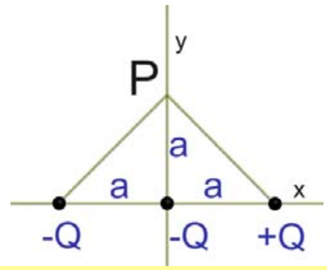
$$F = \sqrt{F_x^2+F_y^2} = \sqrt{(-1.5\,\mathrm{N})^2+(3.3\,\mathrm{N})^2} = 3.6\,\mathrm{N}$$

and the angle with respect to the $+x$ axis is

$$\varphi = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{3.3\,\mathrm{N}}{-1.5\,\mathrm{N}}\right) = 114^{\circ}$$

Problem 3: Charges

Three charges equal to $-Q$, $-Q$ and $+Q$ are located a distance a apart along the x axis (see sketch). The point P is located on the positive y-axis a distance a from the origin.



(a) What is the electric field $\vec{\mathbf{E}}$ at point P?

If we number the charges from left to right, then at point P, the E field due to the charges are

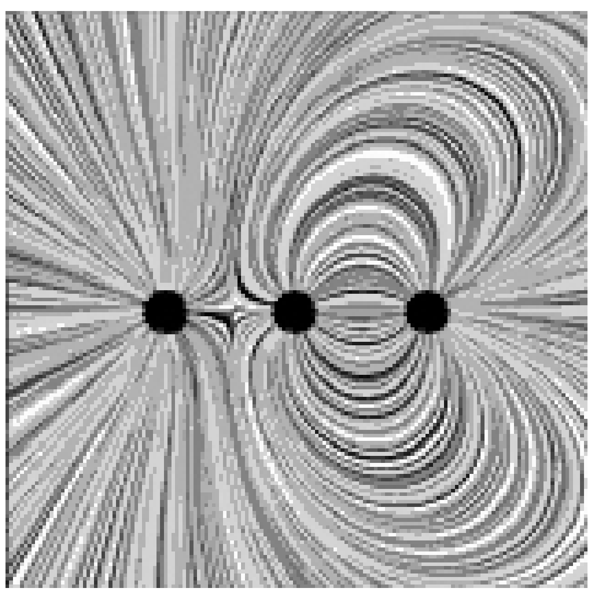
$$\vec{\mathbf{E}}_1 = k_e \frac{-Q}{\left(\sqrt{2}a\right)^3}\left(a\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}}\right) = -k_e \frac{Q}{2\sqrt{2}a^2}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$

$$\vec{\mathbf{E}}_2 = k_e \frac{-Q}{a^2}\left(\hat{\mathbf{j}}\right)$$

$$\vec{\mathbf{E}}_3 = k_e \frac{Q}{\left(\sqrt{2}a\right)^3}\left(-a\hat{\mathbf{i}} + a\,\hat{\mathbf{j}}\right) = -k_e \frac{Q}{2\sqrt{2}a^2}\left(\hat{\mathbf{i}} - \hat{\mathbf{j}}\right)$$

Thus, the total electric field is $\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 = -k_e \frac{Q}{\sqrt{2}a^2}\hat{\mathbf{i}} - k_e \frac{Q}{a^2}\hat{\mathbf{j}}$

(b) _(a)_ (enter one letter) is the correct field line representation for this problem



You can tell this because the center and rightmost charges are of opposite signs (field lines start on one and end on the other), while the center and leftmost charges are of the same sign (their field lines repel each other).

Problem 4: Got Pizza?
How many square inches of pizza are eaten by the MIT student body per semester?

To get an answer to this we need to make several estimates:
Number of students (including undergrad & grad): 10000
For average student (don't forget, this includes grad students!)

Number of weeks per semester:	15 weeks
Number of times pizza eaten per week:	3 times/week
Number of slices eaten per eating:	3 slices/time
Average slice size (16" pizza with 10 slices):	20 in^2/slice

Amount of Pizza Eaten: $15*3*3*20\ in^2*10^4$ $3 \times 10^7\ in^2$

Don't forget to round – you should never need to use a calculator for these problems. You won't necessarily arrive at the same result, but you should be close. Maybe you think that the average student eats 5 slices of pizza daily. Note that this only triples our answer, so we are still in the same order of magnitude.

Problem 5: Charge Imbalance

We know that within the limits of measurement, the magnitudes of the negative charge on the electron and the positive charge on the proton are equal. Suppose, however, that these magnitudes differed from each other by as little as 0.0000001% (1 part in a billion). What would the force be between you and your neighbor in the TEAL classroom? Are you attracted or repelled by them? Using just this line of thought, what limit would you place on equality of electron and proton charge (e.g. 1 part in $10^?$)?

The charge of an electron/proton is 1.6×10^{-19} Coulomb (burn that number into your brain, it's useful). You are basically made out of water which has a molecular mass of about 18 amu (that is 18 g/mole, where a mole contains 6.022×10^{23} atoms – another number you should memorize if you haven't already) and contains 10 protons/electrons. Everything else is approximation:

Mass of person:	100 kg (it's a nice number)
Number of atoms in person:	100 kg/(18 g/mole) ~ 5000 mole ~ 3×10^{27} atoms
Number of electrons in person:	3×10^{27} atoms * 10 e^-/atom ~ 3×10^{28} electrons
Excess charge in person:	3×10^{28} elec. * 1.6×10^{-19} C/elec. * 10^{-9} excess C/C
	6 excess Coulombs

Now we just need to find out what the force on 6 C students do when placed next to each other (about a meter apart):

$$F = \frac{k_e Q^2}{r^2} = \frac{\left(9\times10^9\,\mathrm{Nm^2C^{-2}}\right)\left(6\,\mathrm{C}\right)^2}{\left(1\,\mathrm{m}\right)^2} \sim 3\times10^{11}\,\mathrm{N}$$

The force must be repulsive because, whether the electrons or protons have more charge, the sign of the charge on you and your neighbor will be the same.

This force is ENORMOUS. Let's compare to gravity, which is about 10^3 N using a mass of 100 kg. This means that we can put much more stringent limits on the equality of charge. I would say that two people 10 cm apart feel no noticeable repulsive force (due to electrostatics anyway), with "noticeable" meaning about the weight of a paper clip, $1\text{ g} \times 10\text{ m/s}^2 = 10^{-2}\text{ N}$. Although it's unclear whether you'd really feel the weight of a paper clip if it was spread all over your body, this is around the right magnitude. So by decreasing the distance I've increased the force by a couple of orders of magnitude so I need to drop it by 15 orders of magnitude (from 10^{13} N to 10^{-2} N) which means that I need to reduce the charge by 7 orders of magnitude meaning that the balance is at least 1 part in 10^{16}.

Don't forget to round – you should never need to use a calculator for these problems.

Problem 6: Polygon

Consider a 2N-sided regular polygon, centered on the origin, with each vertex a distance a away from the origin, one of which is located at (x, y) = (a, 0). A charge $+Q$ is placed at each of the 2N vertices. A charge q is placed at the origin.

(a) What is the force on the charge q at the origin?

In a polygon with an even number of vertices, every charge Q has a partner across the origin. So by symmetry the force on the central charge q is 0.

(b) Now remove the $+Q$ charge from the vertex located at (a, 0). Now what is the force on the charge q at the origin? (Don't forget: Force is a vector!)

Removing a $+Q$ charge at a point is the same as adding a $-Q$ charge at the same location (superposition). Since the original force was zero, the force now is just the force from a single $-Q$ charge at (a, 0):

$$\vec{\mathbf{F}} = \frac{qQ}{4\pi\varepsilon_0 a^2}\hat{\mathbf{i}}$$

If q is positive the force is up, towards the hole (the effective $-Q$ charge).

(c) Now instead of a 2N-sided polygon, a 2N+1-sided polygon is used. Now what is the force on the charge q at the origin when $+Q$ charges are placed at all the vertices?

In a polygon with an odd number of vertices, with one vertex on the y-axis, the remaining vertices are mirror symmetric across the y-axis. This means that the x-component of the force must be zero (every push to the right is balanced by an equal push to the left). But what about the y-component? If you have the feeling that it will also cancel that is the correct intuition, but how do we prove it? Let's assume that the y-component isn't zero, that is, the net force is either up or down. That means that q is attracted to/repelled from the top charge. By symmetry then it must be attracted to/repelled from every charge. This, however, is impossible. So, again, there is no net force on q.

(d) Finally, the $+Q$ charge at $(a, 0)$ is again removed. What now is the force on q?

Just as in part (b),

$$\vec{\mathbf{F}} = \frac{qQ}{4\pi\varepsilon_0 a^2}\hat{\mathbf{i}}$$

Problem 7: Oscillating Charge

Two point charges $4Q$ and $-Q$ are fixed on the x-axis at x = $-d$ and x = 0 respectively.

(a) There is one point on the x-axis, x = x_0, where the electric field is zero. What is x_0?

The charges are of opposite sign, so the field zero is not between them but to one side or the other. Since the $4Q$ charge at x = $-d$ is bigger, the zero will be to the right of the $-Q$ charge ($x_0 > 0$). Now we can solve:

$$\vec{\mathbf{E}}(x_0) = \frac{1}{4\pi\varepsilon_0}\left(\frac{4Q}{(d+x_0)^2} + \frac{-Q}{x_0^2}\right) = 0 \quad \Rightarrow \quad \frac{4Q}{(d+x_0)^2} = \frac{Q}{x_0^2} \quad \Rightarrow \quad 4x_0^2 = (d+x_0)^2 \quad \Rightarrow \quad x_0 = d$$

(b) A third point charge q of mass m is free to move along the x-axis. What force does it feel if it is placed at x = x_0 (the location you just found)?

Since $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$ and $\vec{\mathbf{E}}(x_0) = 0$, the charge feels no force at x = x_0.

(c) Now q is displaced along the x-axis by a small distance a to the right. What sign of charge should q be so that it feels a force pulling it back to x = x_0?

To the right of x = x_0 = d the $4Q$ charge will dominate the $-Q$ charge. You can see this most clearly if you go a far distance to the right. Looking back at the two charges they will appear as a single $3Q$ charge. In order to be attracted back to the net positive charge, q must be negative.

(d) Show that if a is small compared to d ($a << d$) q will undergo simple harmonic motion. Determine the period of that motion.

To show that the motion is simple harmonic we have to show that the acceleration of (and hence force on) q is proportional to its displacement. The force on it at x = $d + a$ is:

$$F = \frac{q}{4\pi\varepsilon_0}\left(\frac{4Q}{(d+d+a)^2} + \frac{-Q}{(d+a)^2}\right) = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{4}{(2d+a)^2} + \frac{-1}{(d+a)^2}\right)$$

Now we have to make use of the approximation that $a << d$. The usual way to make use of such conditions is to make a small number out of the ratio ($a/d << 1$). So:

$$F=\frac{qQ}{4\pi\varepsilon_0 d^2}\left(\frac{1}{(1+a/2d)^2}+\frac{-1}{(1+a/d)^2}\right)$$

where we have factored out 2d from the first denominator and d from the second. Now we can do a Taylor expansion, keeping just the first order term: $(1+x)^{-2}\approx 1-2x$

$$F\approx\frac{qQ}{4\pi\varepsilon_0 d^2}\big((1-2a/2d)-(1-2a/d)\big)=\frac{qQ}{4\pi\varepsilon_0 d^3}a$$

Since $F=m\frac{d^2x}{dt^2}$, we have (at x = $d + a$)

$$\frac{d^2x}{dt^2}=-\frac{(-q)Q}{4\pi\varepsilon_0 md^3}a$$

Since the acceleration is proportional to the displacement (a) but oppositely directed (recall that q is negative), we conclude that the motion of q is simple harmonic with

$$\omega^2=\frac{|q|Q}{4\pi\varepsilon_0 md^3}$$

The period is defined as $T=\frac{2\pi}{\omega}$, thus

$$T=2\pi\sqrt{\frac{4\pi\varepsilon_0 md^3}{|q|Q}}$$

(e) How fast will the charge q be moving when it is at the midpoint of its periodic motion?

Since ω is the angular frequency of q, we can describe the position of q, $x(t)$, to be

$$x(t)=d+a\cos(\omega t)$$

To find the velocity, $v(t)$, we just differentiate x with respect to t:

$$v(t)=\frac{d}{dt}x(t)=-a\omega\sin\omega t$$

Since $-1<\sin\omega t<1$, the maximum speed is thus

$$v_{\max}=a\omega$$

which is the speed of q at the midpoint of its motion (x = d).