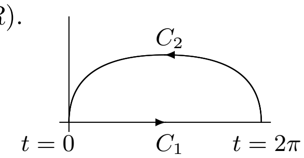
18.02 Fall 2007 – Problem Set 9, Part B Solutions

1. a) Green's theorem: $\mathrm{curl}(x\,\hat{\boldsymbol{\jmath}}) = 1$, so $\oint_C x\,dy = \iint_R dA = \mathrm{Area}(R)$.

Similarly, $\mathrm{curl}(-y\,\hat{\boldsymbol{\imath}}) = 1$, so $\oint_C -y\,dx = \iint_R dA = \mathrm{Area}(R)$.



b) $\mathrm{Area}(R) = -\oint_C y\,dx$. $\int_{C_1} y\,dx = 0$ since $y = 0$. So

$$\text{Area} = -\int_{C_2} y\,dx = -\int_{2\pi}^{0} a(1-\cos t)\,\underbrace{a(1-\cos t)\,dt}_{dx} = \int_0^{2\pi} a^2(1-2\cos t+\cos^2 t)\,dt$$

$$= a^2\Big[t - 2\sin t + (\frac{t}{2} + \frac{1}{4}\sin 2t)\Big]_0^{2\pi} = 3\pi a^2.$$

2. a)

$$\oint_C (x^2y+y^3-y)\,dx + (3x+2y^2x+e^y)\,dy = \iint_R (3+2y^2)-(x^2+3y^2-1)\,dA = \iint_R (4-x^2-y^2)\,dA,$$

where R is the region enclosed by C (by Green's theorem). The integrand is > 0 inside the circle $x^2+y^2 = 4$ and < 0 outside, so the integral is biggest when C is the circle $x^2+y^2 = 4$.

b) $$\iint_{x^2+y^2<4} (4-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta = 2\pi\left[2r^2 - \tfrac{1}{4}r^4\right]_0^2 = 8\pi.$$

3. a) The normal vector to C at (x,y) is $\hat{\boldsymbol{n}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$. Therefore $\vec{F}\cdot\hat{\boldsymbol{n}} = \langle xy, y^2\rangle \cdot \langle x, y\rangle = x^2y + y^3 = (x^2+y^2)y = y$.

The contribution to flux is positive when $\vec{F}\cdot\hat{\boldsymbol{n}} = y > 0$, which corresponds to the upper half-circle, and negative for the lower half-circle.

b) When $\vec{F} = \langle P, Q\rangle$, $\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C -Q\,dx + P\,dy$. Parametrizing the circle by $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, we calculate

$$\begin{aligned}
\int_C -y^2\,dx + xy\,dy &= \int_C -\cos^2\theta\,(-\cos\theta\,d\theta) + \cos\theta\sin\theta\,(\cos\theta\,d\theta) \\
&= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\sin\theta\,d\theta \\
&= \int_0^{2\pi} \sin\theta\,d\theta = 0.
\end{aligned}$$

(or, using (a) and observing that $ds = d\theta$, $\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C y\,ds = \int_0^{2\pi}\sin\theta\,d\theta = 0$)

We get zero because, as seen in (a), $\vec{F}\cdot\hat{\boldsymbol{n}} = y$, so the (negative) flux through the lower half-circle exactly compensates the (positive) flux through the upper half-circle.

c) $\mathrm{div}\,\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2) = 3y$. So by Green's theorem, we have

$$\oint_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \iint_R 3y\,dA = 3\iint_R y\,dA = 0$$

where R is the unit disk, and $\iint_R y\,dA$ is seen to be zero either by symmetry or by computation in polar coordinates

$$\iint_R y\,dA = \int_0^{2\pi}\int_0^1 r\sin\theta\,r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}\sin\theta\,d\theta = 0.$$