18.02 Fall 2007 – Problem Set 5, Part B Solutions

1. In general, $\left.\dfrac{dg}{ds}\right|_{\hat{\boldsymbol{u}}} = \nabla g \cdot \hat{\boldsymbol{u}}$ is largest when $\hat{\boldsymbol{u}} = \operatorname{dir} \nabla g$ and

$$\nabla g \cdot (\operatorname{dir} \nabla g) = \nabla g \cdot \frac{\nabla g}{|\nabla g|} = \frac{|\nabla g|^2}{|\nabla g|} = |\nabla g|.$$

Similarly, the directional derivative is smallest when $\hat{\boldsymbol{u}} = -\operatorname{dir} \nabla g$ and its value is $-|\nabla g|$. Therefore the answers to (a) and (b) are as follows.

a) Since $\nabla f = \langle 3x^2 - y^2 - 8x + 3 + 2xy, -2xy + x^2 \rangle = \langle -1/4, -3/4 \rangle$, the maximum directional derivative is $|\langle -1/4, -3/4 \rangle| = \sqrt{10}/4$; the minimum is $-\sqrt{10}/4$.

b) The maximum is achieved in the direction $\hat{\boldsymbol{u}} = \operatorname{dir} \langle -1/4, -3/4 \rangle = \langle -1, -3 \rangle/\sqrt{10}$. The minimum is achieved in the opposite direction, $\hat{\boldsymbol{u}} = -\operatorname{dir} \langle -1/4, -3/4 \rangle = \langle 1, 3 \rangle/\sqrt{10}$.

c) We need $\hat{\boldsymbol{u}} \cdot \langle -1/4, -3/4 \rangle = 0$. So $\hat{\boldsymbol{u}} = \pm\operatorname{dir} \langle 3, -1 \rangle = \pm\dfrac{\langle 3, -1 \rangle}{\sqrt{10}}$.

d) The maximum of df/ds is approximately 0.8 ($\pm.05$) achieved when $\hat{\boldsymbol{u}} = \langle \cos\theta_1, \sin\theta_1 \rangle$, $\theta_1 \approx 252°$ ($\pm 8°$). $\hat{\boldsymbol{u}}$ points parallel to (and in the same direction as) the gradient vector and perpendicular to the contour line.

The minimum of df/ds is approximately -0.8 ($\pm.05$) achieved when $\hat{\boldsymbol{u}} = \langle \cos\theta_2, \sin\theta_2 \rangle$, $\theta_2 = -180° + \theta_1 \approx 72°$ ($\pm 8°$). $\hat{\boldsymbol{u}}$ points parallel to (and opposite) the gradient vector and perpendicular to the contour line.

$df/ds = 0$ when $\hat{\boldsymbol{u}} = \langle \cos\theta, \sin\theta \rangle$, $\theta = \theta_1 \pm 90° \approx 162°$ and $342°$ ($\pm 5°$). $\hat{\boldsymbol{u}}$ points parallel to the contour line and perpendicular to the gradient vector.

Remark: because $\operatorname{dir} \nabla f = -\operatorname{dir} \langle 1, 3 \rangle$, the exact value of the angle θ_2 is $\tan^{-1}(3) \approx 71.6°$.

2. a) $\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, z, y \rangle = \langle 4, 1, -1 \rangle$ at $(2, -1, 1)$. The direction of greatest decrease is that of $-\nabla g$, i.e. the unit vector $-\dfrac{\nabla g}{|\nabla g|} = \dfrac{\langle -4, -1, 1 \rangle}{\sqrt{18}} = \dfrac{\langle -4, -1, 1 \rangle}{3\sqrt{2}}$.

b) Let $\Delta x = x - 2$, $\Delta y = y + 1$, $\Delta z = z - 1$; then the line in the direction of $\langle -4, -1, 1 \rangle$ can be parametrized by $\Delta x = -4t$, $\Delta y = -t$, $\Delta z = t$. (Dividing by $3\sqrt{2}$ is unnecessary and just makes calculations more complicated.) At $P_0 = (2, -1, 1)$, we have $g = 3$, and $\nabla g = \langle 4, 1, -1 \rangle$ (from part (a)), so linear approximation gives

$$\begin{aligned} g(x, y, z) \approx g(P_0) + \nabla g(P_0) \cdot \langle \Delta x, \Delta y, \Delta z \rangle &= 3 + 4\Delta x + \Delta y - \Delta z \\ &= 3 + 4(-4t) + (-t) - t = 3 - 18t. \end{aligned}$$

Therefore, $g = 2$ when $t \approx 1/18$. At $t = 1/18$, we obtain $(x, y, z) = (2 - 4t, -1 - t, 1 + t) = (2 - \frac{2}{9}, -1 - \frac{1}{18}, 1 + \frac{1}{18})$. Evaluating g at this point, we find ≈ 2.046, which is close to 2.

3. a) Minimizing $f(x, y, z) = (x - 2)^2 + (y + 1)^2 + (z - 1)^2$ subject to the constraint $g(x, y, z) = x^2 + yz = 2$ gives the Lagrange multiplier equations $\nabla f = \lambda \nabla g$, or

$$\begin{aligned} 2(x - 2) &= \lambda\,(2x) &\quad (1) \\ 2(y + 1) &= \lambda\, z &\quad (2) \\ 2(z - 1) &= \lambda\, y &\quad (3) \end{aligned}$$

We also need to remember a fourth equation, the constraint equation: $x^2 + yz = 2$.

b) Summing (2) and (3) gives $2(y+z) = \lambda(y+z)$, so either $\lambda = 2$ or $y+z=0$.

We first assume that $z = -y$. Then (1) $\Leftrightarrow x - 2 = \lambda x \Leftrightarrow x = 2/(1-\lambda)$, while plugging $y = -z$ into (3) gives $2(z-1) = -\lambda z \Leftrightarrow z = 2/(2+\lambda)$, and $y = -2/(2+\lambda)$. The constraint equation $x^2 + yz = 2$ then becomes

$$\left(\frac{2}{1-\lambda}\right)^2 - \left(\frac{2}{2+\lambda}\right)^2 = 2.$$

Clearing denominators we get $4(2+\lambda)^2 - 4(1-\lambda)^2 = 2(2+\lambda)^2(1-\lambda)^2$, or

$$\lambda^4 + 2\lambda^3 - 3\lambda^2 - 16\lambda - 2 = 0.$$

The two roots of this equation are $\lambda_1 \approx -0.1283$ and $\lambda_2 \approx 2.3448$. Using the above expressions of x, y, z in terms of λ, the first possibility gives , while the second one gives $(x, y, z) \approx (-1.4872, -0.4603, 0.4603)$ which is much further from $(2, -1, 1)$.

Finally we look at the other possibility $\lambda = 2$: then (1) gives $x = -2$, so any solution with $\lambda = 2$ lies far away from $(2, -1, 1)$. So the point closest to $(2, -1, 1)$ at which $x^2 + yz = 2$ is

$$(x, y, z) \approx (1.7725, -1.0686, 1.0686).$$

The approximate answer obtained in Problem 2, namely $(x, y, z) = (2 - \frac{2}{9}, -1 - \frac{1}{18}, 1 + \frac{1}{18}) \approx (1.7778, -1.0556, 1.0556)$, is close to the exact solution but not quite within $1/100$.