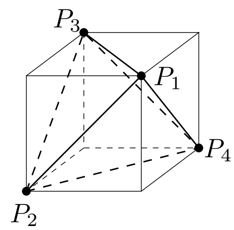
18.02 Fall 2007 – Problem Set 1, Part B Solutions

1. a) $P_1 = (1,1,1)$, $P_2 = (1,-1,-1)$, $P_3 = (-1,-1,1)$, $P_4 = (-1,1,-1)$. Each pair of points differs by two sign changes from the others. All six edges of the tetrahedron are diagonals of faces of the cube and hence have the same length. For instance,

$$|\overrightarrow{P_1P_2}| = |\langle 1-1, -1-1, -1-1\rangle| = |\langle 0,-2,-2\rangle| = 2\sqrt{2}$$



b) $\cos\theta = \boldsymbol{A}\cdot\boldsymbol{B}/|\boldsymbol{A}||\boldsymbol{B}| = \langle 1,1,1\rangle\cdot\langle 1,-1,-1\rangle/(\sqrt{3})^2 = -1/3$. $\theta \approx 1.91$ radians (109.5°).

c) Adjacent edges: $\overrightarrow{P_1P_2} = \langle 0,-2,-2\rangle$, $\overrightarrow{P_1P_3} = \langle -2,-2,0\rangle$.

$$\cos\alpha = \frac{\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}}{|\overrightarrow{P_1P_2}||\overrightarrow{P_1P_3}|} = \frac{4}{(2\sqrt{2})^2} = \frac{1}{2}; \quad \alpha = \pi/3 = 60^\circ$$

The faces are equilateral triangles, so the angles are 60°.

Opposite edges: $\overrightarrow{P_1P_2} = \langle 0,-2,-2\rangle$, $\overrightarrow{P_3P_4} = \langle 0,2,-2\rangle$.

$$\cos\beta = \frac{\overrightarrow{P_1P_2}\cdot\overrightarrow{P_3P_4}}{|\overrightarrow{P_1P_2}||\overrightarrow{P_3P_4}|} = \frac{0}{(2\sqrt{2})^2} = 0; \quad \beta = \pi/2 = 90^\circ$$

By symmetry the perpendicular bisector to an edge contains the opposite edge, so these two edges are perpendicular to each other.

d) $\overrightarrow{P_1P_2} = \langle 0,-2,-2\rangle$, $\overrightarrow{P_1P_3} = \langle -2,-2,0\rangle$.

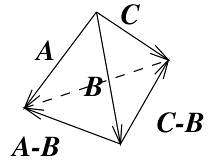
$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 0 & -2 & -2 \\ -2 & -2 & 0 \end{vmatrix} = -4\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}}; \quad \text{area} = \frac{1}{2}\sqrt{(-4)^2+4^2+(-4)^2} = 2\sqrt{3}.$$

2. a) P is on the altitude from $P_1 \iff \overrightarrow{P_1P}\cdot\overrightarrow{P_2P_3} = 0 \iff \boldsymbol{v}_1\cdot(\boldsymbol{v}_2 - \boldsymbol{v}_3) = 0$ (recall that $\boldsymbol{v}_i = \overrightarrow{P_iP}$). Equivalently this condition can be rewritten as: $\boldsymbol{v}_1\cdot\boldsymbol{v}_2 = \boldsymbol{v}_1\cdot\boldsymbol{v}_3$.

b) From (a): P is on the altitude from P_1, so $\boldsymbol{v}_1\cdot\boldsymbol{v}_2 = \boldsymbol{v}_1\cdot\boldsymbol{v}_3$. Similarly P is on the altitude from P_2, so $\boldsymbol{v}_2\cdot\boldsymbol{v}_1 = \boldsymbol{v}_2\cdot\boldsymbol{v}_3$ (same argument as in (a), exchanging the roles of P_1 and P_2). Therefore $\boldsymbol{v}_1\cdot\boldsymbol{v}_2 = \boldsymbol{v}_1\cdot\boldsymbol{v}_3 = \boldsymbol{v}_2\cdot\boldsymbol{v}_3$.

c) From (b), $\boldsymbol{v}_1\cdot\boldsymbol{v}_3 = \boldsymbol{v}_2\cdot\boldsymbol{v}_3$, so $(\boldsymbol{v}_1 - \boldsymbol{v}_2)\cdot\boldsymbol{v}_3 = 0$, i.e. $\overrightarrow{P_1P_2}\cdot\overrightarrow{P_3P} = 0$, which means that $\overrightarrow{P_1P_2}\perp\overrightarrow{P_3P}$, i.e. P is on the altitude from P_3.

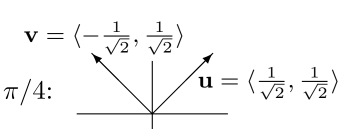
3.



$(\boldsymbol{A}-\boldsymbol{B})\times(\boldsymbol{C}-\boldsymbol{B}) = \boldsymbol{A}\times\boldsymbol{C} - \boldsymbol{A}\times\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{C} + \boldsymbol{B}\times\boldsymbol{B}$;
using $\boldsymbol{A}\times\boldsymbol{C} = -\boldsymbol{C}\times\boldsymbol{A}$ and $\boldsymbol{B}\times\boldsymbol{B} = 0$, one gets

$$\tfrac{1}{2}\boldsymbol{A}\times\boldsymbol{B} + \tfrac{1}{2}\boldsymbol{B}\times\boldsymbol{C} + \tfrac{1}{2}\boldsymbol{C}\times\boldsymbol{A} + \tfrac{1}{2}(\boldsymbol{A}-\boldsymbol{B})\times(\boldsymbol{C}-\boldsymbol{B}) = 0$$

4. a) $\mathbf{u} = A_\theta\hat{\boldsymbol{\imath}} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$; $\mathbf{v} = A_\theta\hat{\boldsymbol{\jmath}} = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$. For $\theta = \pi/4$:



b) $$A_{\theta_1}A_{\theta_2} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = A_{\theta_1+\theta_2}.$$

Rotating a vector $\boldsymbol{v}$ by θ_2 and then by θ_1 is the same operation as rotating it by $\theta_1+\theta_2$.

c) $\det A_\theta = cos^2\theta + \sin^2\theta = 1$, so using cofactors,

$$A_\theta^{-1} = \frac{1}{\det A_\theta}\begin{pmatrix} \cos\theta & -\sin\theta \\ -(-\sin\theta) & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A_\theta^T.$$

Since $A_\theta A_\theta^{-1} = I$, we have $A_\theta A_\theta^T = I$. Moreover,

$$A_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = A_\theta^{-1},$$

using that cosine is even and sine is odd. The reason why $A_\theta^{-1} = A_{-\theta}$ is that a rotation by angle θ is reversed by a rotation by the same angle in the opposite direction.

d) e) With $\det(A) = +1$: $\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$: F $\longrightarrow$ 

$\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$: F $\longrightarrow$ 

With $\det(A) = -1$: $\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$: F $\longrightarrow$ 

$\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$: F $\longrightarrow$ 

When det $= +1$, we get an ordinary F, rotated. When det $= -1$, we get a rotated ꟻ (backwards F).

5. a) $M\boldsymbol{X}$ is a column vector, whose 4 entries correspond to the amounts of flour, sugar, egg, and butter (in grams) contained in the assortment of pastries.

b) Consider $\boldsymbol{Y} = NM\boldsymbol{X}$: then $\boldsymbol{Y}$ is a column vector, whose 3 entries correspond to the amounts of protein, carbohydrates, and fat (in grams) in the assortment of pastries.

c) The matrix NM is a square (3×3) matrix, and it is invertible (one checks that $\det(NM) \neq 0$). Therefore $\boldsymbol{X} = (NM)^{-1}\boldsymbol{Y}$. Numerically,

$$A = (NM)^{-1} \approx \begin{pmatrix} -0.469 & 0.070 & -0.013 \\ 0.384 & -0.005 & -0.103 \\ 0.018 & -0.030 & 0.113 \end{pmatrix}$$

(in Matlab: `M=[22 40 50; 18 10 3; 5 14 5; 10 10 22]; N=[0.10 0 0.13 0; 0.76 1 0.01 0; 0.01 0 0.10 0.82]; A=inv(N*M)`)

d) Using (c),

$$\boldsymbol{X} = A\begin{pmatrix} 50 \\ 300 \\ 65 \end{pmatrix} \approx \begin{pmatrix} -3.14 \\ 10.99 \\ -0.89 \end{pmatrix}.$$

One would need to eat about -3 cookies, 11 doughnuts, and -1 croissant a day. Of course the negative entries do not make any physical sense; this unfortunately means that a diet consisting exclusively of cookies, doughnuts and croissants cannot be balanced.