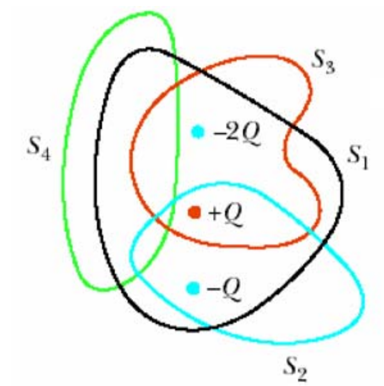
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02 **Fall 2007**

Problem Set 2 Solutions

Problem 1: Closed Surfaces

Four closed surfaces, S_1 through S_4, together with the charges $-2Q$, Q, and $-Q$ are sketched in the figure at right. The colored lines are the intersections of the surfaces with the page. Find the electric flux through each surface.

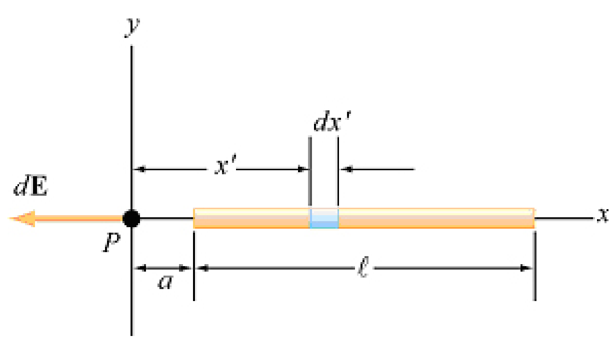


By Gauss's Law, the flux through the closed surfaces is equal to the charge enclosed over ε_0. So,

$$\Phi_{S_1} = -2Q/\varepsilon_0\,;\ \Phi_{S_2} = 0;\ \Phi_{S_3} = -Q/\varepsilon_0\,;\ \Phi_{S_4} = 0$$

Problem 2: Field on Axis of a Line Charge

A wire of length l has a uniform positive linear charge density and a total charge Q. Calculate the electric field at a point P located along the axis of the wire and a distance a from one end:



a. Give an integral expression for the electric field at point P in terms of the variables used in the above figure

$$\mathbf{E} = -k_e \hat{\mathbf{i}} \int_a^{a+l} \frac{\lambda dx'}{(x')^2}$$

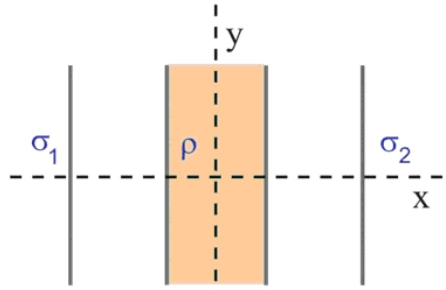
b. Evaluate this integral.

$$\mathbf{E} = k_e \lambda \hat{\mathbf{i}} \left. \frac{1}{x'} \right|_a^{a+l} = k_e \lambda \hat{\mathbf{i}} \left[\frac{1}{a+l} - \frac{1}{a} \right] = k_e \lambda \hat{\mathbf{i}} \left[\frac{-l}{a(a+l)} \right] = -k_e \hat{\mathbf{i}} \frac{Q}{a(a+l)}$$

c. In the limit that the length of the rod goes to zero, does your answer reduce to the right expression?

In the limit that l goes to zero, our expression becomes $\vec{\mathbf{E}} = \left(-k_e \frac{Q}{a^2} \right) \hat{\mathbf{i}}$, as we desire.

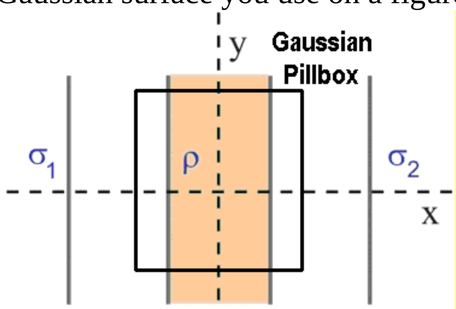
Problem 3: Charged Slab & Sheets



An infinite slab of charge carrying a charge per unit volume ρ has its boundaries located at x = –2 meters and x = +2 meters. It is infinite in the y direction and in the z direction (out of the page). Two similarly infinite charge sheets (zero thickness) are located at x = –6 meters and x = +6 meters, with area charge densities σ_1 and σ_2 respectively. In the accessible regions you've measured the electric field to be:

$$\vec{\mathbf{E}} = \begin{cases} 0\,\hat{\mathbf{i}} & x < -6 \text{ m} \\ -10\,\text{V/m}\,\hat{\mathbf{i}} & \text{-6 m} < \text{x} < \text{-2 m} \\ 10\,\text{V/m}\,\hat{\mathbf{i}} & \text{6 m} > \text{x} > \text{2 m} \\ 0\,\hat{\mathbf{i}} & x > +6 \text{ m} \end{cases}$$

(a) Use Gauss's Law to find the charge density ρ of the slab. You must indicate the Gaussian surface you use on a figure. You can use the symbol ε_o in your answer.

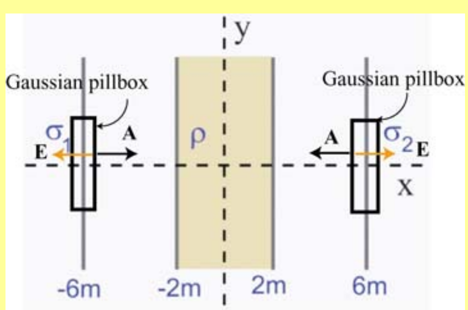


$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \vec{\mathbf{E}}_L \cdot \vec{\mathbf{A}}_L + \vec{\mathbf{E}}_R \cdot \vec{\mathbf{A}}_R = 2\left|E\right|A = \frac{\rho A d}{\varepsilon_0}$$

$$\Rightarrow \rho = \frac{2\left|E\right|\cancel{A}}{\cancel{A}d}\varepsilon_0 = \frac{2\left(10 \text{ V/m}\right)}{\left(4 \text{ m}\right)}\varepsilon_0$$

$$= \left(5 \text{ V/m}^2\right)\varepsilon_0 \quad \left[\text{C/m}^3\right]$$

(b) Use Gauss's Law to find the two surface charge densities of the left and right charged sheets. You must indicate the Gaussian surface you use on a figure. You can use the symbol ε_o in your answer.



Using Gauss's law with the Gaussian pillboxes indicated in the figure, we have

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = -(10 \text{ V/m})A = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma_1 A}{\varepsilon_0} \quad \Rightarrow \quad \sigma_1 = -10 \text{ V/m} \cdot \varepsilon_0 \left[\text{C/m}^2\right]$$

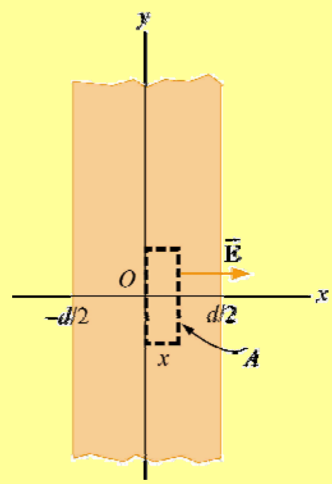
In a similar manner, $\sigma_2 = -10 \text{ V/m} \cdot \varepsilon_0 \left[\text{C/m}^2\right]$

Problem 4: Charge Slab

Consider a slab of insulating material which is infinitely large in two of its three dimensions, and has a thickness d in the third dimension. An edge view of the slab is shown in the figure. The slab has a uniform positive charge density ρ.

(a) Calculate the electric field everywhere, both inside and outside the slab.

There are two regions in which we need to calculate the electric field: inside and outside the slab. In both cases we will use a cylindrical shaped Gaussian "pillbox" (cross-sectional area A) perpendicular to the yz plane with one end in the yz plane and the other end containing the point $x > 0$, as shown in the figure. By symmetry, the electric field is zero in the yz plane and points in the $+x$ direction everywhere to the right of this plane. Thus, the only contribution to the surface integral is over the end cap containing the point x. Using Gauss's law:



$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0},$$

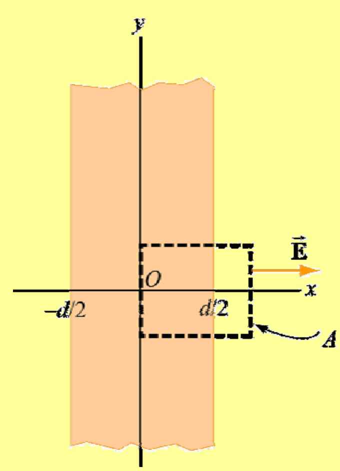
with $q_{\text{enc}} = \rho V_{\text{enc}} = \rho A x$, we obtain

$$EA = \frac{1}{\varepsilon_0}(\rho A x)$$

which implies that the electric field inside the slab is:

$$E = \frac{\rho x}{\varepsilon_0}$$

On the other hand, outside the slab ($|x| < d/2$), the charge enclosed by the Gaussian pillbox doesn't depend on x: $q_{\text{enc}} = \rho A d / 2$. Using Gauss's law, we have



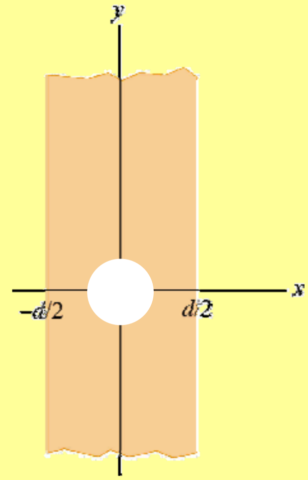
$$EA = \frac{1}{\varepsilon_0}(\rho A d / 2)$$

or $E = \rho d / 2\varepsilon_0$. In summary, the electric field due to the slab may be written as

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\rho d}{2\varepsilon_0}\hat{\mathbf{i}}, & x > d/2 \\ \dfrac{\rho x}{\varepsilon_0}\hat{\mathbf{i}}, & -d/2 < x < d/2 \\ -\dfrac{\rho d}{2\varepsilon_0}\hat{\mathbf{i}}, & x < -d/2 \end{cases}$$

Problem 4: Charge Slab *continued…*

(b) Now some material is removed around the origin, leaving a hole of radius R ($R < d$) there. What is the electric field on the x-axis for $|x| < d/2$?



At first glance this might seem very difficult. But we have already done the bulk of the work in doing the slab. Putting a hole in something is the same as adding equal and opposite charge to it, so what we need to do is superimpose a sphere of radius R and charge density $-\rho$. We calculated the field from a sphere in class, but it bears repeating. We break the problem into solving the field inside the sphere and outside, and use a spherical Gaussian surface of radius x. Inside:

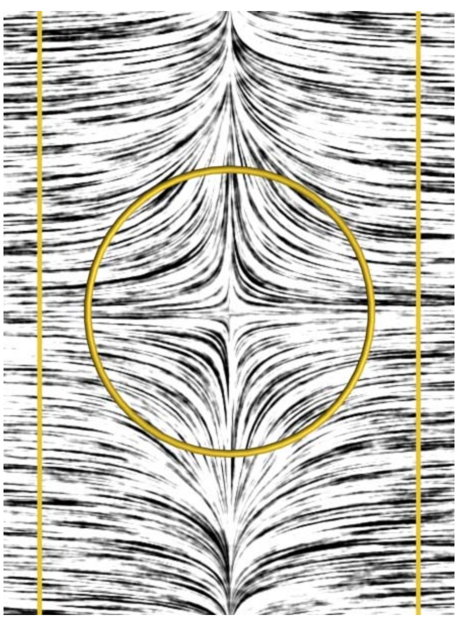
$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \cdot 4\pi x^2 = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{4}{3}\pi x^3(-\rho) \Rightarrow E = \frac{-\rho x}{3\varepsilon_0}$$

That is, the field strength grows linearly as we move towards the edge of the sphere. Here, the minus sign means that the field points inward, toward the origin. Outside we have:

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \cdot 4\pi x^2 = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{4}{3}\pi R^3(-\rho) \quad \Rightarrow \quad E = \frac{-\rho R^3}{3\varepsilon_0 x^2}$$

Now, adding this to the field we derived from the slab, we have:

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\rho x}{\varepsilon_0} - \dfrac{\rho x}{3\varepsilon_0}\hat{\mathbf{i}} = \dfrac{2\rho x}{3\varepsilon_0}\hat{\mathbf{i}}, & |x| < R \\[2ex] \dfrac{\rho x}{\varepsilon_0} - \dfrac{\rho R^3}{3\varepsilon_0 x^2}\hat{\mathbf{i}}, & \mathrm{R} \le |x| < d/2 \end{cases}$$



A couple of people asked me how you can have a field inside a hole – doesn't it somehow violate Gauss's law? The answer is no, as long as every field line going in also goes back out. That is exactly what happens here, with field lines going in along the y-axis and out on the x-axis:

Problem 4: Charge Slab *continued…*

(c) A small charge q with mass m is now placed on the x-axis at the right hand edge of the hole. How large and what sign must the charge q be so that it will undergo simple harmonic motion and have a velocity $v_{\max}$ at the origin?

The problem you've all been looking forward to – simple harmonic oscillations. As always, we need to find that the force is proportional to the displacement x of this charge.

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} = q\frac{2\rho x}{3\varepsilon_0}\hat{\mathbf{i}}$$

So the force is proportional to displacement x. The sign of q must be negative in order for this to be a restoring force. We can get the velocity at the origin by looking at the derivative of the position of the charge, which is sinusoidal in time:

$$x(t) = R\cos(\omega t) \;\Rightarrow\; v(t) = \frac{d}{dt}x(t) = -R\omega\sin\omega t \text{ where } \omega = \sqrt{\frac{2\rho|q|}{3\varepsilon_0 m}}$$

Here I used a cosine so that at t = 0 the charge is at the edge of the hole, $x = R$. Since all we want is the maximum velocity (at the origin) this isn't relevant, but its good to remember for problems when we do care about the explicit time dependence.

Anyway, using the given velocity, we can solve for q:

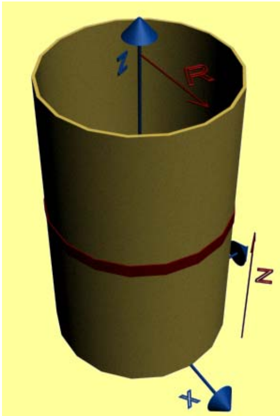
$$v = R\sqrt{\frac{2\rho|q|}{3\varepsilon_0 m}} = v_{\max} \quad\Rightarrow\quad \boxed{q = -\frac{3\varepsilon_0 m}{2\rho}\left(\frac{v_{\max}}{R}\right)^2}$$

This is a pretty complicated equation, so lets check the units:

$$[q] \stackrel{?}{=} \frac{\left(\mathrm{C}^2/\mathrm{N}\times\mathrm{m}^2\right)\mathrm{kg}}{\mathrm{C}/\mathrm{m}^3}\left(\frac{\mathrm{m/s}}{\mathrm{m}}\right)^2 = (\mathrm{C/N})\,\mathrm{m}\cdot\mathrm{kg}\frac{1}{\mathrm{s}^2} = \mathrm{C}\ !$$

Problem 5: Uniformly Charged Cylinder

A cylindrical Plexiglas tube of length L, radius R carries a charge Q uniformly distributed over its surface. Find the electric field on the axis of the tube at one of its ends.



The charge is evenly distributed over the surface of the cylinder, so the charge density:

$$\sigma = \frac{Q}{A} = \frac{Q}{2\pi RL}$$

Now we need to calculate the electric field. I'm going to break the tube into rings, so first let me calculate the electric field on the axis of a ring of charge. By symmetry, only the component along the axis of the ring/cylinder (the z-axis as I have chosen it) will not disappear.

Problem 5: Uniformly Charged Cylinder *continued…*

Taking the thickness of the ring as *dz*, we have:

$$dE_z = -\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}z = -\frac{1}{4\pi\varepsilon_0}\frac{\sigma R d\varphi dz}{\left(z^2+R^2\right)^{3/2}}z$$

$$E_{\text{ring}} = -\int_{\varphi=0}^{2\pi}\frac{1}{4\pi\varepsilon_0}\frac{\sigma R d\varphi dz}{\left(z^2+R^2\right)^{3/2}}z = -\frac{1}{2\varepsilon_0}\frac{\sigma R dz}{\left(z^2+R^2\right)^{3/2}}z$$

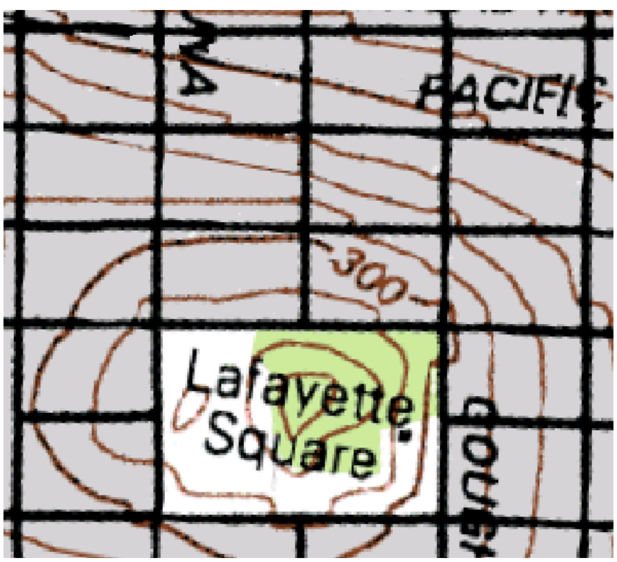
Now we just integrate up the rings to get the field

$$E_z = \int_{z=0}^{L} dE_{\text{ring}} = \int_{z=0}^{L} -\frac{1}{2\varepsilon_0}\frac{\sigma R}{\left(z^2+R^2\right)^{3/2}}zdz = \int_{u=R^2}^{L^2+R^2} -\frac{1}{2\varepsilon_0}\frac{\sigma R}{u^{3/2}}\frac{du}{2} = -\frac{\sigma R}{4\varepsilon_0}\left(\frac{u^{-1/2}}{-\frac{1}{2}}\right)\Bigg|_{R^2}^{L^2+R^2}$$

$$= \frac{\sigma R}{2\varepsilon_0}\left(\frac{1}{\sqrt{L^2+R^2}}-\frac{1}{R}\right) = -\frac{Q}{4\pi\varepsilon_0 L}\left(\frac{1}{R}-\frac{1}{\sqrt{L^2+R^2}}\right)$$

where I have used $\sigma = Q/2\pi RL$ and flipped the order of subtraction to highlight the fact that field points in the $-\hat{\mathbf{k}}$ direction (out the end of the cylinder).

Problem 6: Equipotentials Curves – Reading Topographic Maps



At left is a topographic map of a 0.4 mi square region of San Francisco. The contours shown are separated by heights of 25 feet (so from 375 feet to 175 feet above sea level for the region shown)

From left to right, the NS streets shown are Buchanan, Laguna, Octavia, Gough and Franklin. From top to bottom, the EW streets shown are Broadway, Pacific, Jackson, Washington, Clay (which stops on either side of the park) and Sacramento.

(a) In the part of town shown in the map above, which street(s) have the steepest runs? Which have the most level sections? How do you know?

You can tell how steep something is by looking at how quickly it passes through constant height contours (~ equipotentials). The steepest section is along Octavia between Pacific and Washington. The most level street is Jackson between Buchanan and Octavia, which runs parallel to the 275 foot contour and hence is very flat.

Problem 6: Equipotentials Curves – Reading Topographic Maps *continued*…

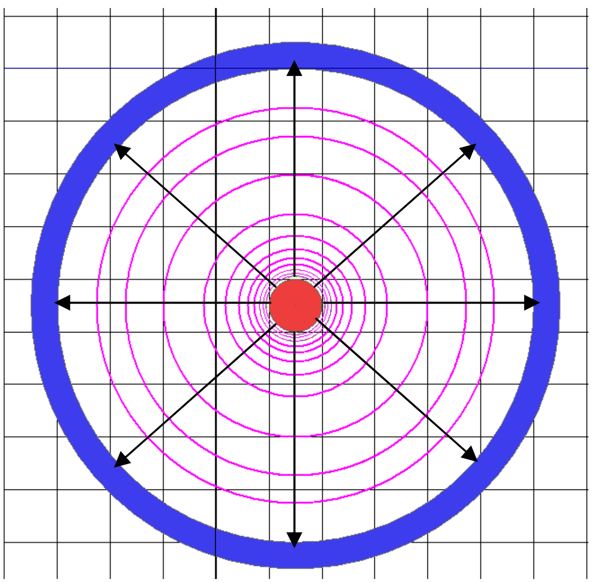
(b) How steep is the steepest street at its steepest (what is its slope in ft/mi)?

Looking at Octavia, it passes through 5 contours (125 feet) in two blocks (about 0.12 miles) so it has a slope of ~1000 ft/mi.

(c) Which would take more work (in the physics sense): walking 3 blocks south from Laguna and Jackson or 1 block west from Clay and Franklin?

Work is change in potential energy (and hence height). The change in height walking 3 blocks S on Laguna is almost nothing (you go up but come back down again). West on Clay from Franklin you rise 50 feet in the block, so that is more work.

Problem 7: Equipotentials, Electric Fields and Charge



One group did this lab and measured the equipotentials for a slightly different potential landscape then the ones you have been given (although still on a 1 cm grid).

Note that they went a little overboard and marked equipotential curves (the magenta circles) at V = 0.25 V, 0.5V and then from V = 1 V to V = 10 V in 1 V increments.

They followed the convention that red was their positive electrode (V = +10 V) and blue was ground (V = 0 V).

(a) Copy the above figure and sketch eight electric field lines on it (equally spaced around the inner conductor).

See black arrows – note that they are everywhere perpendicular to the equipotential surfaces

(b) What, approximately, is the magnitude of the electric field at r = 1 cm, 2 cm, and 3 cm, where r is measured from the center of the inner conductor? You should express the field in V/cm. (HINT: The field is the local slope (derivative) of the potential. Also, if you choose to use a ruler realize that the above reproduction of this group's results is not the same size as the original, where the grid size was 1 cm).

At r = 1 cm, V ~ 4 V and we move 1 V in about 1/5 cm.	E ~ 5 V/cm
At r = 2 cm, V ~ 1.5 V and we move about 1/2 V in 1/2 cm.	E ~ 1 V/cm
At r = 3 cm, V ~ 0.7 V and we move about 0.2 V in 1/2 cm.	E ~ 0.4 V/cm

(c) What is the relationship between the density of the equipotential lines, the density of the electric field lines, and the strength of the electric field?

The denser the equipotential lines and hence electric field lines, the stronger the field.