MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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Problem Set 3 Solutions

Problem 1: Charges on a Square

Three identical charges $+Q$ are placed on the corners of a square of side a, as shown in the figure.

(a) What is the electric field at the fourth corner (the one missing a charge) due to the first three charges?

We'll just use superposition:

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{a\hat{\mathbf{i}}}{a^3} + \frac{a\hat{\mathbf{i}} + a\hat{\mathbf{j}}}{\left(\sqrt{2}a\right)^3} + \frac{a\hat{\mathbf{j}}}{a^3}\right) = \frac{Q}{4\pi\varepsilon_0 a^2}\left(1 + 2^{-3/2}\right)\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$

(b) What is the electric potential at that corner?

A common mistake in doing this kind of problem is to try to integrate the **E** field we just found to obtain the potential. Of course, we can't do that we only found the **E** field at a single point, not as a function of position. Instead, just sum the point charge potentials from the 3 points:

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{i \neq j}\frac{q_i}{r_{ij}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{a} + \frac{Q}{\sqrt{2}a} + \frac{Q}{a}\right) = \frac{Q}{4\pi\varepsilon_0 a}\left(2 + \frac{1}{\sqrt{2}}\right)$$

(c) How much work does it take to bring another charge, $+Q$, from infinity and place it at that corner?

The work required to bring a charge $+Q$ from infinity (where the potential is 0) to the corner is:

$$W = Q\Delta V = \frac{Q^2}{4\pi\varepsilon_0 a}\left(2 + \frac{1}{\sqrt{2}}\right)$$

(d) How much energy did it take to assemble the pictured configuration of three charges?

The work done to assemble three charges as pictured is the same as the potential energy of the three charges already in such an arrangement. Now, there are two pairs of charges situated at a distance of a, and one pair of charges situated at a distance of $\sqrt{2}a$, thus we have

Problem 1: Charges on a Square *continued…*

$$W = 2\left(\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a}\right) + 1\left(\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{\sqrt{2}a}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a}\left(2+\frac{\sqrt{2}}{2}\right)$$

Alternatively we could have started with empty space, brought in the first charge for free, the second charge in the potential of the first and so forth. We'll get the same answer.

Problem 2: Electric Potential

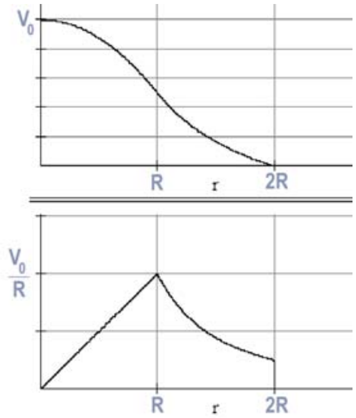
Suppose an electrostatic potential has a maximum at point P and a minimum at point M.

(a) Are either (or both) of these points equilibrium points for a negative charge? If so are they stable?

The electric field is the gradient of the potential, which is zero at both potential minima and maxima. So a negative charge is in equilibrium (feels no net force) at both P & M. However, only the maximum (P) is stable. If displaced slightly from P, a negative charge will roll back "up" hill, back to P. If displaced from M a negative charge will roll away from the potential minimum.

(b) Are either (or both) of these points equilibrium points for a positive charge? If so are they stable?

Similarly, both P & M are equilibria for positive charges, but only M is a stable equilibrium because positive charges seek low potential (this is probably the case that seems more logical since it is like balls on mountains).



Problem 3: A potential for finding charge
The electric potential *V(r)* for a distribution of charge with spherical symmetry is:

Region I: $V(r) = -\frac{V_0 r^2}{2R^2} + V_0$ for $0 < r < R$

Region II: $V(r) = \frac{V_0 R}{r} - \frac{V_0}{2}$ for $R < r < 2R$

Region III: $V(r) = 0$ for $r > 2R$

Where V_0 is a constant of appropriate units. This function is plotted above.
(a) What is the electric field $\vec{\mathbf{E}}(r)$ for this problem? Give your answer analytically for the three regions above and also plot the magnitude. Be sure to indicate the maximum value on the vertical axis for E.

$\vec{\mathbf{E}}(r) = E_r\hat{\mathbf{r}}$ where E_r is given below and plotted above at right.

Problem 3: A potential for finding charge *continued…*

Region I: $0 < r < R$

$$E_r = -\frac{\partial V}{\partial r} = \frac{V_0 r}{R^2}$$

Region II: $R < r < 2R$

$$E_r = -\frac{\partial V}{\partial r} = \frac{V_0 R}{r^2}$$

Region III: $r > 2R$

$$E_r = -\frac{\partial V}{\partial r} = 0$$

(b) Use Gauss's Law arguments to determine the distribution of charges that produces the above electric potential and electric field. When making a Gauss's Law argument, state the Gaussian surface you are using and what you conclude about the charge distribution by applying Gauss's Law to that surface.

First take a Gaussian sphere with radius *r* > *2R*. Since the electric field is zero on the surface of the sphere, the total charge enclosed by this sphere must be zero. Now take a Gaussian sphere with any radius *R* < *r* < *2R*. We have

$$\oiint_{\text{closed surface}} \vec{\mathbf{E}} \cdot \mathbf{d}\vec{\mathbf{A}} = \frac{Q_{inside}}{\varepsilon_o} = \frac{V_0 R}{r^2} 4\pi r^2 = 4\pi V_0 R$$

So for any *r* in the range *R* < *r* < *2R* $Q_{inside} = 4\pi\varepsilon_o V_0 R$, and for *r* > *2R* the charge enclosed is zero. The only way this can be is that we have a **surface charge at *r* = *2R*** with total charge $-4\pi\varepsilon_o V_0 R$ and therefore surface charge $\sigma = \frac{-\cancel{4\pi}\varepsilon_o V_0 R}{\cancel{4\pi}(2R)^2} = \boxed{\frac{-\varepsilon_o V_0}{4R} = \sigma}$

The fact that for *R* < *r* < *2R* the charge inside our Gaussian surface remains constant means that there is no charge in this region. We need only figure out the charge distribution for *r* < *R*. Suppose we have some charge density $\rho(r)$ in this region. Then for a sphere of radius r < R, Gauss's Law in this region, gives us

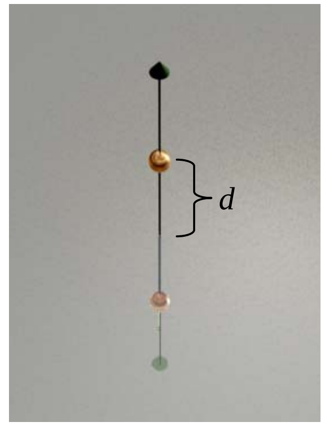
$$\oiint \mathbf{E} \cdot d\mathbf{A} = \frac{V_0 r}{R^2} 4\pi r^2 = 4\pi r^3 \frac{V_0}{R^2} = \frac{Q_{enc}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int_0^r \rho(r') 4\pi r'^2 dr' .$$

You can solve this by inspection or by taking the derivative of both sides with respect to r. We'll do the latter:

$$\frac{d}{dr}\left(4\pi r^3 \frac{V_0}{R^2}\right) = 12\pi r^2 \frac{V_0}{R^2} = \frac{d}{dr}\left(\frac{1}{\varepsilon_o}\int_0^r \rho(r') 4\pi r'^2 dr'\right) = \frac{\rho(r) 4\pi r^2}{\varepsilon_o}$$

$$\Rightarrow \rho(r) = \frac{\varepsilon_o}{4\cancel{\pi r^2}} 12\cancel{\pi r^2} \frac{V_0}{R^2} = \boxed{\frac{3\varepsilon_o V_0}{R^2} = \rho(r < R)}$$

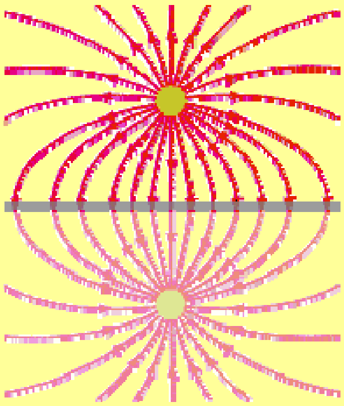
Problem 4: Reflecting on a Conducting Plane



A charge q lies a distance d above an infinitely large, uncharged, perfectly conducting, shiny piece of metal. (As you know from the first lab, conductors are equipotential surfaces).

(a) Draw this in a 2D view (as opposed to the perspective of the picture at right) and sketch some field lines. Pay close attention to (i.e. comment on) what they do when they hit the conductor.

Field lines must be perpendicular to a conductor (because a conductor is an equipotential). So the field lines must curve down from the point charge and land on the conductor:



Here I have shown not only the real field lines but also the mirror image of the field lines in the conductor. I do this to demonstrate that the field looks exactly like the field from a dipole. In other words, it looks like there is a –q charge a distance d below the surface of the conductor (just like in the mirror reflection of the charge in the metal).

(b) What is the electric field at the surface of the conductor directly beneath the point charge. HINTS: (1) From your field lines it should be clear that this is not just the field from the point charge. The conductor is doing something! (2) Your field lines should look familiar. What does your picture remind you of? If the field lines are the same then the field is the same!

To calculate the field we will just calculate the field at the center of a dipole:

$$\vec{\mathbf{E}}(r=0)=\underbrace{\frac{kq}{d^2}\left(-\hat{\mathbf{k}}\right)}_{\text{Real Charge}}+\underbrace{\frac{k(-q)}{d^2}\hat{\mathbf{k}}}_{\text{"Image" Charge}}=\boxed{-\frac{2kq}{d^2}\hat{\mathbf{k}}}$$

Here I have arbitrarily called the axis that the charge is sitting on the z-axis. You could choose to call it anything you like since the problem doesn't specify. But since you were asked the field you MUST specify the result as a vector (although the words "towards the plane" or "down" would also be ok, and are often better in an exam situation).

Note that the field is double what we would have if we just had the point charge without the plane – the reflected or "image" charge contributes an equal amount to the field.

Problem 4: Reflecting on a Conducting Plane *continued…*

(c) What is the magnitude of the electric field at the surface of the conductor as a function of distance *r* from the point just below the charge?

Here we just repeat what we did in (b), although now we really have to think about vectors.

$$\vec{\mathbf{E}}(r) = \frac{kq}{R^3}\vec{\mathbf{R}}_{\text{real}} + \frac{k(-q)}{R^3}\vec{\mathbf{R}}_{\text{image}} = \boxed{-\frac{2kq}{\left(d^2+r^2\right)^{3/2}}d\hat{\mathbf{k}}}$$

Again, the field is in the $-\hat{\mathbf{k}}$ direction, as it must be because it must be perpendicular to the conductor.

(d) What is the force (magnitude and direction) on the charge?

The charge feels a force not by the field that it itself generates (charges can never exert forces on themselves) but rather by the image charge.

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} = q\frac{k(-q)}{(2d)^3}2d\hat{\mathbf{k}} = \boxed{-\frac{kq^2}{4d^2}\hat{\mathbf{k}}}$$

Problem 5: Dipole

Consider two equal but opposite charges, both mass *m*, on the x-axis, +*Q* at (*a*, 0) and –*Q* at (-*a*, 0). They are connected by a rigid, massless, insulating rod whose center is fixed to a frictionless pivot at the origin. This is a dipole. A uniform field $\vec{\mathbf{E}} = E\hat{\mathbf{i}}$ is now applied.

(a) What is the force on the dipole due to this external field?

A dipole in a uniform field feels no force (the force on the positive charge is equal and opposite to the negative charge).

Now rotate the dipole to a small angle θ_0 (positive clockwise) from the x-axis.

(b) Now what is the force on the dipole?

The field is still uniform so the force is still zero.

Problem 5: Dipole *continued…*

(c) How much did the potential energy of the dipole change when it was rotated?

The change in potential energy of a charge *q* is given by:

$$\Delta U = q\Delta V = -q\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}$$

For example, our positive charge changes its potential energy by:

$$\Delta U = -Q\int_{\theta=0}^{\theta_0} E\hat{\mathbf{i}}\cdot a\,d\theta\,\hat{\boldsymbol{\theta}} = -QEa\int_{\theta=0}^{\theta_0} d\theta\,\hat{\mathbf{i}}\cdot\left(-\sin\theta\hat{\mathbf{i}}+\cos\theta\hat{\mathbf{j}}\right) = QEa\int_{\theta=0}^{\theta_0}\sin\theta\,d\theta$$
$$= -QEa\cos\theta\Big|_0^{\theta_0} = QEa\left(1-\cos\theta_0\right)$$

Note that if you don't know $\hat{\boldsymbol{\theta}}$ you could look it up or figure it out by just taking a couple of obvious locations ($\theta = 0°$, $\theta = 90°$).

Similarly, the negative charge moves from $\theta = \pi$ to $\theta = \pi + \theta_0$, changing its energy by:

$$\Delta U = (-Q)Ea\left(\cos\pi-\cos\left(\pi+\theta_0\right)\right) = -QEa\left(-1+\cos\left(\theta_0\right)\right) = QEa\left(1-\cos\theta_0\right)$$

So both charges change their potential energy by the same amount, and the total change in the dipole's potential energy is: $\boxed{\Delta U_{dipole} = 2QEa\left(1-\cos\theta_0\right)}$

Note that we could write this as follows:

$$\Delta U_{dipole} = -2QEa\left(\cos\theta_0-\cos 0\right) = -pE\left(\cos\theta_0-\cos 0\right) = U_{final}-U_{init}$$

and the potential energy of a dipole is $U_{dipole} = -\vec{\mathbf{p}}\cdot\vec{\mathbf{E}}$

(d) What is the torque on the dipole?

From class we had that the torque is given by

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}}\times\vec{\mathbf{E}} \qquad \Rightarrow \qquad \boxed{\tau = pE\sin\theta_0}$$

Note that one can actually derive this from the potential energy. Just as the force is the derivative of the potential energy with respect to position, the torque is its derivative with respect to angle: $\tau = \frac{\partial}{\partial\theta}\left(-pE\cos\theta\right) = pE\sin\theta$

Problem 5: Dipole *continued…*

(e) Now the dipole is released and allowed to rotate due to this torque. Describe the motion that it undergoes (i.e. what is its angle $\theta(t)$?)

Just thinking about it, the dipole wants to align with the field, so it will begin to rotate back towards $\theta = 0$. But since there is no friction it will overshoot but eventually turn around and come back. This is simple harmonic motion, just like you've done on the first couple problem sets, except now we are doing it for rotation. We need to show that the torque depends linearly on the displacement angle: $\tau = pE\sin\theta \approx pE\theta$ where we have used the small angle approximation to linearize the sine. So we can write out the rotational equivalent of $F = ma$: $\tau \approx pE\theta = I\alpha = I\ddot{\theta}$, where I is the moment of inertia, $I = 2ma^2$.

So we have:

$$\ddot{\theta} = -\frac{pE}{I}\theta = -\frac{2QaE}{2ma^2}\theta = -\frac{QE}{ma}\theta$$

where I have stuck in a minus sign because this is a restoring torque (it always causes rotation back towards $\theta = 0$). This is the same second order linear differential equation you wrote the answer to last week:

$$\boxed{\theta(t) = \theta_0 \cos(\omega t),\ \text{where } \omega = \sqrt{\frac{QE}{ma}}}$$

Problem 6: Line of Charge

Consider a very long conducting rod, radius R and charged to a linear charge density λ.

a) Calculate the electric field everywhere outside of this rod (i.e. find $\vec{\mathbf{E}}(\vec{\mathbf{r}})$).

This is easily calculated using Gauss's Law and a cylindrical Gaussian surface of radius r and length l. By symmetry, the electric field is completely radial (this is a "very long" rod), so all of the flux goes out the sides of the cylinder:

$$\oint\!\!\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi r l E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0} \Rightarrow \boxed{\vec{\mathbf{E}} = \frac{\lambda}{2\pi r \varepsilon_0}\hat{\mathbf{r}}}$$

b) Calculate the electric potential everywhere outside, where the potential is defined to be zero at a radius R_0 (i.e. $V(R_0) \equiv 0$)

To get the potential we simply integrate the electric field from R to wherever we want to know it (in this case r):

$$V(r) = V(r) - \underbrace{V(R_0)}_{0} = -\int_{R_0}^{r} \vec{\mathbf{E}}(\vec{\mathbf{r}}') \cdot d\vec{\mathbf{r}}' = -\int_{R_0}^{r} \frac{\lambda}{2\pi r' \varepsilon_0} dr' = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r')\Big|_{R_0}^{r} = \boxed{\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{R_0}{r}\right)}$$

Problem 7: Stupid Hobbies…

Some people like to do incredibly dangerous things. Like a guy I went to grad school with, Austin Richards (also known as Dr. Megavolt – see a movie of him at http://www.drmegavolt.com/index_gallery.html). Or Criss Angel, who performed a similar stunt on the "Tesla Coil" episode of his show mindfreak. Here are some pictures.

Pictures care of http://www.mindfreakconnection.com/

You'll note that while Dr. Megavolt takes strikes directly from the Tesla Coil (a device capable of making insanely high voltages), Criss Angel decides to get shocked from a small ball attached to the coil instead – convenient for the purposes of answering this question. At about what voltage was the Tesla coil for the strikes pictured above and about how much excess charge was on his hand (in the right picture) the instant before the strike was initiated?

(HINT: Dry air breaks down at an electric field strength of about 3×10^6 V/m)

Judging from the picture, Criss is about a meter away from the ball when it arcs. Could be two meters, but it is easier to work with one meter, so I'll use that. If we make a simple minded assumption that V = Ed then the potential difference is given by:

$$3 \text{ x } 10^6 \text{ V/m x 1 m} \qquad \boxed{3\times10^6 \text{ V}}$$

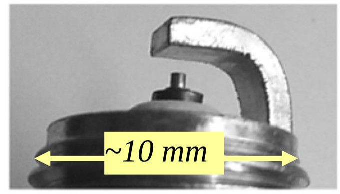
(hence Dr. Megavolt!). You may complain that clearly this is more like a ball of charge then a parallel plate capacitor so we should have used a point charge potential, kQ/r. But notice that even in this case V ~ Er, so the above is approximately correct. There is also a question of where the field equals the breakdown field. Fortunately, this is a back of the envelope question so the details don't matter so much.

We can determine a minimum charge by requiring the field to be at breakdown strength just outside his hand (or the ball). Let's make them spheres of radius 5 cm. Then:

$$E = kQ/r^2 \Rightarrow Q = r^2 E/k \sim (5 \text{ cm})^2 \, 3\times10^6 \text{ V m}^{-1} \left(9\times10^9 \text{ V m C}^{-1}\right)^{-1} \sim 8\times10^{-7} \text{ C} \sim 10^{12} e$$

I say that this is a minimum because the field is clearly breaking down a much further distance away (a meter) which would require a charge 400 (=20^2) times larger. The real charge has to be somewhere between these two extremes, although note that once breakdown begins to occur it is easier for it to propagate, so I'll estimate $\boxed{Q \sim 10^{-5} \text{ C} \sim 10^{13} e}$

Problem 8: Spark Plug



Pictured at right is a typical spark plug (for scale, the thread diameter is about 10 mm). About what voltage does your car ignition system need to generate to make a spark, if the breakdown field in a gas/air mixture is about 10 times higher than in air? For those of you unfamiliar with spark plugs, the spark is generated in the gap at the left of the top picture (top of the bottom picture). The white on the right is a ceramic which acts as an insulator between the high voltage center and the grounded outer (threaded) part.

From the picture, if the threads have a diameter of 10 mm, then the gap is about 0.5 mm. The breakdown is 10 times higher than in air, so its 3×10^7 V/m. So to get breakdown you need a voltage of

Potential difference 3×10^7 V/m x 0.5 mm $\boxed{15{,}000 \text{ V}}$

How do you get 15,000 V out of a car battery??? We'll talk about that later in the class.

Problem 9: High Voltage Power Lines

Estimate the largest voltage at which it's reasonable to hold high voltage power lines. Then check out this video, care of a Boulder City, Nevada power company.

In order to answer this question we have to think about what happens if we go to very high voltages. What breaks down? The previous couple questions were actually hints to how to answer this. The problem with high voltages is that they lead to high fields. And high fields mean breakdown.

You derived the voltage and field in problem 6:

$$E(r) = \lambda/2\pi\varepsilon_0 r;\ V(r) = (\lambda/2\pi\varepsilon_0)\ln(R_0/r) \ \Rightarrow \quad V(r) = E(r)\, r \ln(R_0/r)$$

The strongest field, and hence breakdown, appears at $r = R \sim 1$ cm, the radius of a power line (that makes the diameter just under 1 inch – it might be 3 or 4 times that big but probably not ten times). The voltage is defined relative to some ground, either another cable (probably $R_0 \sim 1$ m away) or at the most the real ground ($R_0 \sim 10$ m away).

So, $V_{\max} = E_{\max} R \ln(R_0/R) = \left(3\times10^6 \ {}^{\text{V}}\!/_{\text{m}}\right)(1\text{ cm})\ln(10\text{ m}/1\text{ cm}) \sim \boxed{2\times10^5 \text{ V}}$

As it turns out, a typical power-line voltage is about 250 kV, about as large as we estimate here. Some high voltage lines can even go up to 600 kV though (or double that for AC voltages). They must use larger diameter cables.

By the way, you can tell that breakdown is a real concern. In humid weather (during rainstorms for example) you will sometimes hear crackling coming from the power lines. This is corona discharge, a high voltage, low current breakdown, similar to the crackling you hear from the Van de Graff generator in class. The movie is of an arc discharge, a very high current phenomenon that can be very dangerous.