18.02 Fall 2007 – Problem Set 7, Part B Solutions

1. a) $\displaystyle\int_1^a e^{-xy}\,dy = -\frac{1}{x}e^{-xy}\Big|_{y=1}^{y=a} = -\frac{1}{x}(e^{-ax}-e^{-x}) = \frac{e^{-x}-e^{-ax}}{x}.$

b)
$$\int_0^\infty \frac{e^{-x}-e^{-ax}}{x}\,dx = \int_0^\infty\int_1^a e^{-xy}\,dy\,dx = \int_1^a\int_0^\infty e^{-xy}\,dx\,dy$$
$$= \int_1^a -\frac{1}{y}e^{-xy}\Big|_{x=0}^{x=\infty}\,dy = \int_1^a \frac{dy}{y} = \ln a.$$

2. Average distance $\displaystyle = \frac{1}{\text{Area}}\iint r\,dA = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r\,r\,dr\,d\theta = \frac{1}{\pi a^2}\,2\pi\left[\frac{1}{3}r^3\right]_0^a = \frac{2}{3}a.$

3. The moment of inertia around the line $x=\bar{x}$ is

$$\bar{I} = \iint_R (x-\bar{x})^2\,\delta\,dA = \iint_R x^2\,\delta\,dA - 2\bar{x}\iint_R x\,\delta\,dA + \bar{x}^2\iint_R \delta\,dA$$
$$= I - 2\bar{x}(M\bar{x}) + \bar{x}^2 M = I - M\bar{x}^2.$$

(recall that $\bar{x} = \frac{1}{M}\iint_R x\,\delta\,dA$ is a constant.) So $I = \bar{I} + M\bar{x}^2$.

4. Rotate so that one vertex is at $(1,0)$. The other two vertices are at $(\cos\theta_1,\sin\theta_1)$ and $(\cos\theta_2,\sin\theta_2)$. The assumption that any point on the circle is as likely as any other and that the choice of θ_1 is independent of the choice of θ_2 means that average is the average with respect to $d\theta_1 d\theta_2$.

As in PS4, when $\theta_1<\theta_2$ the area of the triangle is given by

$$f(\theta_1,\theta_2) = \frac{1}{2}\begin{vmatrix}\cos\theta_1 - 1 & \sin\theta_1\\ \cos\theta_2 - 1 & \sin\theta_2\end{vmatrix} = \frac{1}{2}[(\cos\theta_1-1)\sin\theta_2 - \sin\theta_1(\cos\theta_2-1)]$$
$$= \frac{1}{2}[\sin\theta_1 + \sin(\theta_2-\theta_1) - \sin\theta_2].$$

If you integrate f over the whole square $0\le\theta_1\le 2\pi$, $0\le\theta_2\le 2\pi$, you get zero, the wrong answer. This is because the determinant is negative for $\theta_1>\theta_2$, and the correct formula for the area is $|f(\theta_1,\theta_2)|$.

One can deal with this problem in two equivalent ways. The first is to break the integral into two pieces, and integrate over $\theta_1>\theta_2$ and $\theta_1<\theta_2$. By symmetry these two integrals will give the same answer. The second way is to average over the half of the parameter space $0\le\theta_1\le\theta_2\le 2\pi$ for which the formula for the area is f. By symmetry the average is the same over each half, so it suffices to consider one half.

The area of the region $0\le\theta_1\le\theta_2\le 2\pi$ is $(2\pi)^2/2 = 2\pi^2$, and the average value is

$$\bar{f} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_{\theta_1}^{2\pi}\frac{1}{2}(\sin\theta_1 + \sin(\theta_2-\theta_1) - \sin\theta_2)\,d\theta_2 d\theta_1$$
$$= \frac{1}{4\pi^2}\int_0^{2\pi}(2\pi-\theta_1)\sin\theta_1 + \Big[-\cos(\theta_2-\theta_1)+\cos\theta_2\Big]_{\theta_1}^{2\pi}\,d\theta_1$$
$$= \frac{1}{4\pi^2}\int_0^{2\pi}\Big((2\pi-\theta_1)\sin\theta_1 + 2 - 2\cos\theta_1\Big)\,d\theta_1 = \frac{1}{\pi} - \frac{1}{4\pi^2}\int_0^{2\pi}\theta_1\sin\theta_1\,d\theta_1.$$

Integrating by parts, $\int_0^{2\pi} \theta \sin\theta d\theta = [-\theta\cos\theta]_0^{2\pi} - \int_0^{2\pi} (-\cos\theta) d\theta = -2\pi$, so the average area is $\bar{f} = \frac{1}{\pi} + \frac{1}{2\pi} = \frac{3}{2\pi}$. (The ratio of maximum to average is $\frac{3\sqrt{3}/4}{3/2\pi} = \sqrt{3}\pi/2 \approx 2.72$.)

5. $u = xy$, $v = y/x$: so $uv = y^2$ and $u/v = x^2$, which gives $x^2 + y^2 = uv + u/v$.
The Jacobian is $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v$.
Thus $du\,dv = \frac{2y}{x}\,dx\,dy$, and $dx\,dy = \frac{x}{2y}\,du\,dv = \frac{1}{2v}\,du\,dv$.

Limits of integration: $0 < xy < 1$, $1 < x < 2$. In uv-coordinates, the first inequality becomes $0 < u < 1$; and the second one becomes $1 < x^2 = u/v < 4$, or equivalently $v < u < 4v$, which means that $v < u$ and $v > \frac{1}{4}u$. So

$$\begin{aligned}
\iint_R (x^2+y^2)\,dx\,dy &= \int_0^1 \int_{u/4}^u \left(uv + \frac{u}{v}\right)\frac{1}{2v}\,dv\,du \\
&= \int_0^1 \int_{u/4}^u \left(\frac{u}{2} + \frac{u}{2v^2}\right) dv\,du \\
&= \int_0^1 \left[\frac{uv}{2} - \frac{u}{2v}\right]_{u/4}^u du \\
&= \int_0^1 \left(\left(\frac{1}{2}u^2 - \frac{1}{2}\right) - \left(\frac{1}{8}u^2 - 2\right)\right) du \\
&= \int_0^1 \left(\frac{3}{8}u^2 + \frac{3}{2}\right) du = \left[\frac{1}{8}u^3 + \frac{3}{2}u\right]_0^1 = \frac{1}{8} + \frac{3}{2} = \frac{13}{8}.
\end{aligned}$$

6. Let $u = 2x + 5y - 3$, $v = 3x - 7y + 8$. Then $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & -7 \end{vmatrix} = -29$.
Therefore $du\,dv = 29\,dx\,dy$, and so $dx\,dy = \frac{1}{29}\,du\,dv$.

$$\iint_{(2x+5y-3)^2+(3x-7y+8)^2<1} dx\,dy = \iint_{u^2+v^2<1} \frac{1}{29}\,du\,dv = \frac{\pi}{29}.$$

(The area of the unit disk is π.)