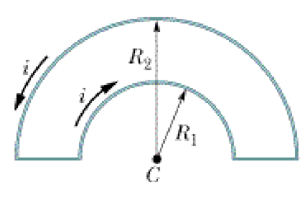
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02 **Fall 2007**

Problem Set 7 Solutions

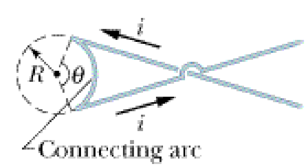


Problem 1: Quickies…

a) Two semicircular arcs have radii R_2 and R_1, carry current i, and share the same center of curvature C. What is the magnitude of the net magnetic field at C?

The inner semi-circle makes a field into the page, the outer one out of the page. The inner is closer and hence stronger, and hence the net field is into the page. In class you calculated the magnetic field from a semi-circle of radius R to be $B = \mu_0 i/4R$. So:

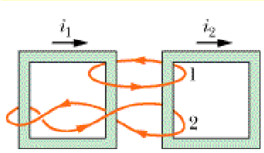
$$\vec{\mathbf{B}} = \frac{\mu_0 i}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \text{ into the page}$$



b) A wire with current i is shown at left. Two semi-infinite straight sections, both tangent to the same circle with radius R, are connected by a circular arc that has a central angle θ and runs along the circumference of the circle. The connecting arc and the two straight sections all lie in the same plane. If $B = 0$ at the center of the circle, what is θ?

The straight portions both make a field out of the page at the center of the circle while the arc makes one into the page. These must be equal so that the fields cancel. Two semi-infinite lines together make an infinite line, and we calculated (using Ampere's law) that the field from an infinite wire is $B = \mu_0 i/2\pi R$. The arc is just a fraction of a circle so it creates a fraction of the field that a whole circle does at its center: $B = (\mu_0 i/2R)(\theta/2\pi)$.

For these to be equal we must have $B = \mu_0 i/2\pi R = \mu_0 i\theta/4\pi R \quad \Rightarrow \quad \boxed{\theta = 2 \text{ radians}}$

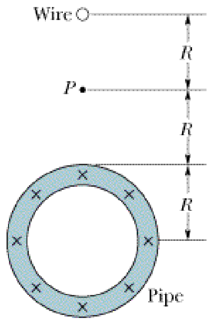


c) The figure at left shows two closed paths wrapped around two conducting loops carrying currents i_1 and i_2. What is the value of the integral for **(a)** path 1 and **(b)** path 2?

To do this you have to use the right hand rule to check whether the currents are positive or negative relative to the path. On path 1 i_1 penetrates in the negative direction while i_2 penetrates in the positive direction, so $\boxed{\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_o (i_2 - i_1)}$.

On path 2 i_1 penetrates twice in the negative direction and i_2 once in the negative direction so $\boxed{\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = -\mu_o (2i_1 + i_2)}$

Problem 2: Two Currents



In the figure at left a long circular pipe with outside radius R carries a (uniformly distributed) current i into the page. A wire runs parallel to the pipe at a distance of 3.00R from center to center. Find the current in the wire such that the ratio of the magnitude of the net magnetic field at point P to the magnitude of the net magnetic field at the center of the pipe is x, but it has the opposite direction.

The field at point P is due both to the pipe and the wire. The field at the center of the pipe is ONLY due to the wire. Since the direction of these two is opposite the current in the wire must create an opposite direction field from the pipe at point P and hence it must also be *into* the page.

The magnitude of the field at point P then is just the difference of the two, and realizing that we are outside of both, they both just look like long straight wires and hence:

$$\vec{\mathbf{B}}(P) = \frac{\mu_o i_{\text{pipe}}}{2\pi(2R)} - \frac{\mu_o i_{\text{Wire}}}{2\pi R} \text{ to the right;} \qquad \vec{\mathbf{B}}(\text{Pipe Ctr.}) = \frac{\mu_o i_{\text{Wire}}}{2\pi(3R)} \text{ to the left}$$

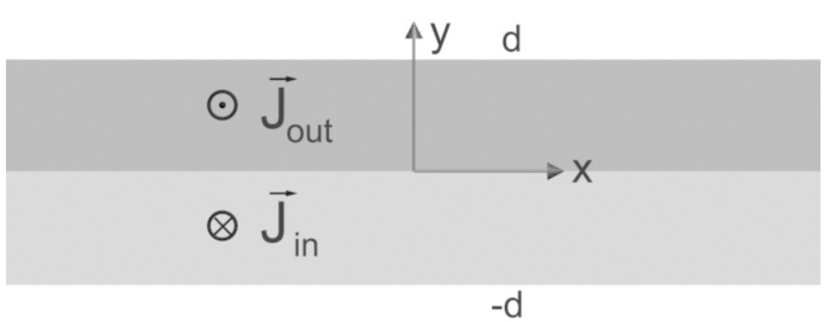
So the ratio

$$\frac{B(P)}{B(\text{Pipe Ctr.})} = x = \left(\frac{\mu_o i_{\text{pipe}}}{2\pi(2R)} - \frac{\mu_o i_{\text{Wire}}}{2\pi R}\right) \Big/ \frac{\mu_o i_{\text{Wire}}}{6\pi R} = 3\left(\frac{i_{\text{pipe}}}{2} - i_{\text{Wire}}\right) \Big/ i_{\text{Wire}} = 3\left(\frac{i_{\text{pipe}}}{2 i_{\text{Wire}}} - 1\right)$$

$$\Rightarrow \boxed{i_{\text{Wire}} = \frac{i_{\text{pipe}}}{2(x/3+1)} \text{ into the page}}$$

Problem 3: Current Slabs

The figure below shows two slabs of current. Both slabs of current are infinite in the x and z directions, and have thickness d in the y-direction. The top slab of current is located in the region $0 < y < d$ and has a constant current density $\vec{\mathbf{J}}_{out} = J\,\hat{\mathbf{z}}$ out of the page. The bottom slab of current is located in the region $-d < y < 0$ and has a constant current density $\vec{\mathbf{J}}_{in} = -J\,\hat{\mathbf{z}}$ into the page.

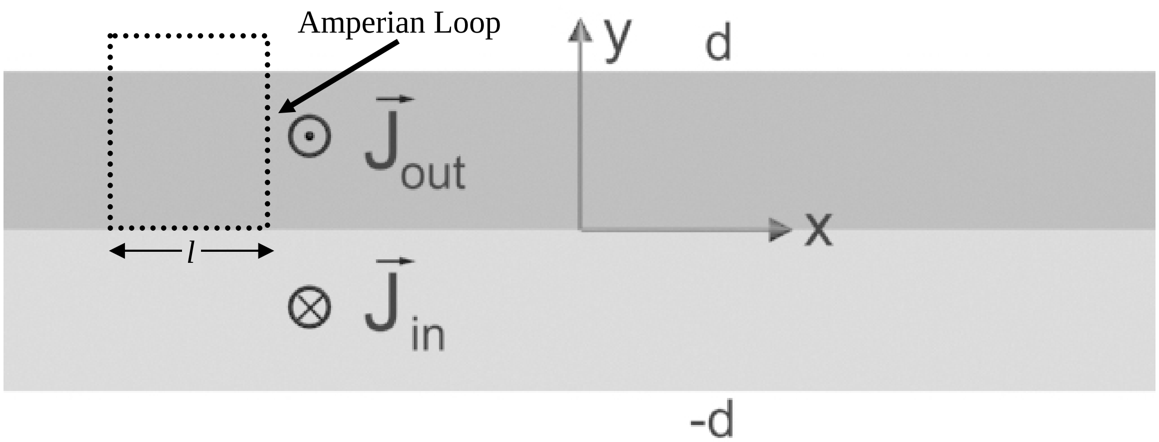


(a) What is the magnetic field for $|y| > d$? Justify your answer.

Zero. The two parts of the slab create equal and opposite fields for $|y| > d$.

Problem 3: Current Slabs *continued…*

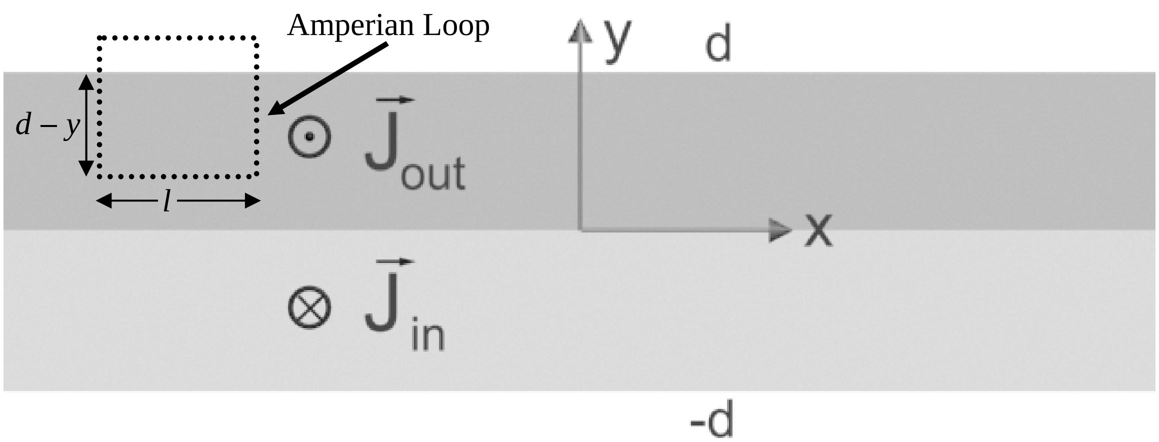
(b) Use Ampere's Law to find the magnetic field at $y = 0$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.



The field at $y = 0$ points to the right (both slabs make it point that way). So walk counter clockwise around the loop shown in the above figure and Ampere's Law gives:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl + 0 + 0 + 0 = \mu_o I_{enc} = \mu_o \left(Jld \right) \Rightarrow \boxed{\vec{\mathbf{B}} = \mu_o Jd\,\hat{\mathbf{i}}\ \left(\text{to the right}\right)}$$

c) Use Ampere's Law to find the magnetic field for $0 < y < d$. Show the Amperian Loop that you use and give the magnitude and direction of the magnetic field.

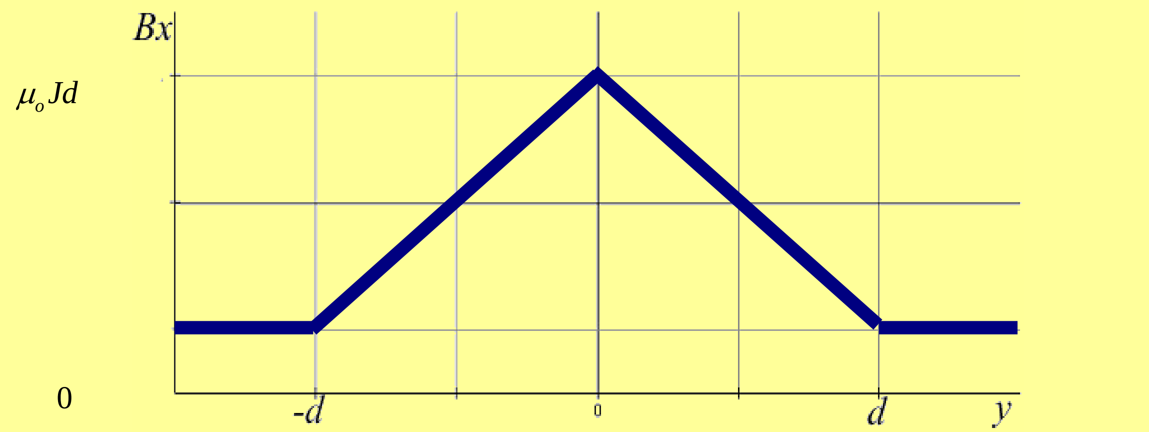


The field for $0 < y < d$ still points to the right. So walk counter clockwise around the loop shown in the above figure and Ampere's Law gives:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl + 0 + 0 + 0 = \mu_o I_{enc} = \mu_o Jl \left(d - y \right) \Rightarrow \boxed{\vec{\mathbf{B}} = \mu_o J \left(d - y \right) \hat{\mathbf{i}}\ \left(\text{to the right}\right)}$$

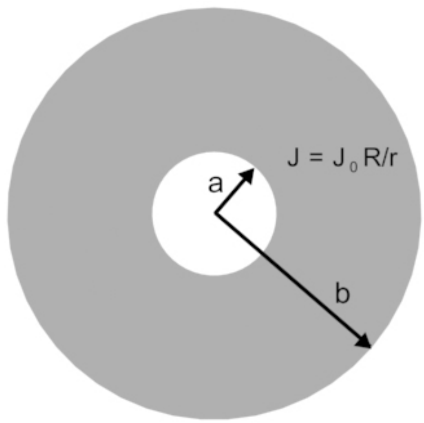
Problem 3: Current Slabs *continued…*

(d) Plot the x-component of the magnetic field as a function of the distance y on the graph below. Label your vertical axis.



Problem 4: Cable

A long cylindrical cable consists of a conducting cylindrical shell of inner radius a and outer radius b. The current density $\vec{\mathbf{J}}$ in the shell is out of the page (see sketch) and varies with radius as $J(r) = J_o \frac{R}{r}$ for $a < r < b$ and is zero outside of that range. Find the magnetic field in each of the following regions, indicating both magnitude and direction. Show your work and your Amperian loops.



Our Amperian loops are all circles with the appropriate radii

(a) $r < a$

For $r < a$, the enclosed current is $I_{\text{enc}} = 0$. Therefore, by Ampere's law, $\boxed{B = 0}$

(b) $a < r < b$

In the region $a < r < b$, the enclosed current is

$$I_{\text{enc}} = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = \int_a^r \left(\frac{J_0 R}{r} \right) 2\pi r \, dr = 2\pi J_0 R \int_a^r dr = 2\pi J_0 R \left(r - a \right)$$

Applying Ampere's law, we have

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \quad \Rightarrow \quad B(2\pi r) = \mu_0 2\pi J_0 R (r - a)$$

or

$$\vec{\mathbf{B}} = B\hat{\boldsymbol{\varphi}} = \frac{\mu_0 J_0 R (r-a)}{r} \hat{\boldsymbol{\varphi}} = \boxed{\mu_0 J_0 R \left(1 - \frac{a}{r} \right) \hat{\boldsymbol{\varphi}} = \vec{\mathbf{B}}}$$

The magnetic field points in the azimuthal direction.

Problem 4: Cable *continued…*

(c) $r > b$. Give also $B(r = b)$, the value of the magnetic field at $r = b$.

In the region $r > b$, the enclosed loop is

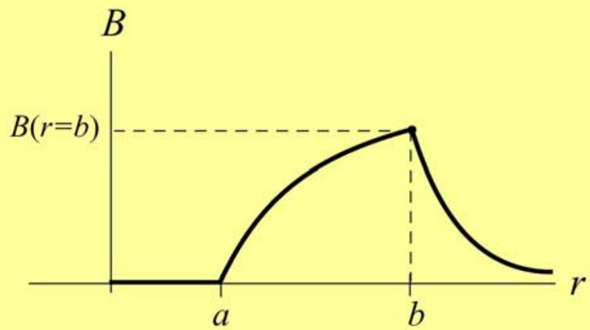
$$I_{\text{enc}} = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = \int_a^b \left(\frac{J_0 R}{r} \right) 2\pi r \, dr = 2\pi J_0 R \int_a^b dr = 2\pi J_0 R (b - a)$$

Applying Ampere's law, we have

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \quad \Rightarrow \quad B(2\pi r) = \mu_0 2\pi J_0 R(b - a) \Rightarrow \boxed{\vec{\mathbf{B}} = B\,\hat{\boldsymbol{\varphi}} = \frac{\mu_0 J_0 R(b-a)}{r}\hat{\boldsymbol{\varphi}}}$$

The magnitude of the B field at $r = b$ is $B(r = b) = \mu_0 J_0 R\left(1 - \dfrac{a}{b}\right)$

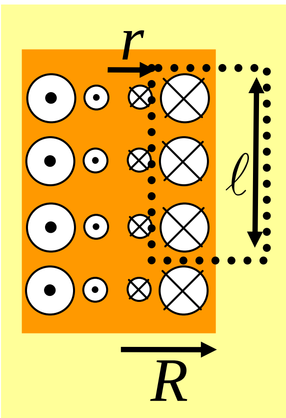
(d) Plot your answers for the magnitude of B above on the graph below.



Problem 5: Rotating Cylinder

A very long uniformly charged cylinder of charge $+Q$ and radius R and length L lies coaxial with the z-axis and is rotating counter clockwise about it with angular velocity ω. Ignoring end effects due to the finite length of the cylinder, what is the direction and magnitude of the magnetic field $\vec{\mathbf{B}}_{cylinder}$ inside and outside the cylinder?

HINT: What does this look like? Is there any place that the field is zero?



We will begin by considering the hint. What does this look like? When the cylinder rotates it carries charge around in circles, and hence like a series of current loops – a bunch of nested solenoids. Since these "solenoids" are long (we are instructed to ignore end effects due to the finite length) we can say that the B field is strictly zero outside of the cylinder. This means that we can use Ampere's law to calculate the field inside the cylinder. The symmetry, like that of solenoids, is "planar" – the magnetic field everywhere inside the cylinder will point up the axis of the cylinder (from the right hand rule). So we will use a rectangular Amperian loop, as pictured below:

Problem 5: Rotating Cylinder *continued…*

So, walking around the Amperian loop (dashed) in a clockwise fashion, we get:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B\ell + 0 + 0 + 0 = \mu_o I_{enc} \Rightarrow B = \mu_o I_{enc} / \ell$$

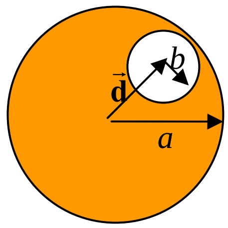
How much current is enclosed? The charge is uniformly distributed, so there is a charge density $\rho = Q/\pi R^2 L$. We can then consider the cylinder to be composed of a number of rings of radius r and area $dr\,dz$, which rotate with period $T = 2\pi/\omega$.

$$I_{\text{ring}} = \frac{Q_{\text{ring}}}{T} = 2\pi r dr dz \rho \frac{\omega}{2\pi} = \rho\omega r dr dz \Rightarrow I_{\text{enc}} = \int_{r=r}^{R} \int_{z=0}^{\ell} \rho\omega r dr dz = \frac{1}{2}\rho\omega\ell\left(R^2 - r^2\right)$$

$$\vec{\mathbf{B}}\left(r < R\right) = \frac{\mu_o}{2}\rho\omega\left(R^2 - r^2\right) = \frac{\mu_o Q\omega\left(R^2 - r^2\right)}{2\pi R^2 L} = \boxed{\frac{\mu_o Q\omega}{2\pi L}\left(1 - \left(\frac{r}{R}\right)^2\right) \text{ up}}$$

Problem 6: Cut up wire

A long solid copper wire of radius a has a cylindrical hole (radius b) bored out of it that is offset from the center by a vector $\vec{\mathbf{d}}$. A current I is flowing through the copper wire. Find the magnetic field everywhere inside the hole.



We will solve this problem using superposition, in this case of one wire of radius a with current running out of the page and of a "hole wire" of radius b, each with current density of magnitude $J = I/\pi\left(a^2 - b^2\right)$ thus making a hole, as pictured above.

To make our lives easier, let's rotate the coordinate system so that $\vec{\mathbf{d}}$ lies entirely along the x-axis, that is $\vec{\mathbf{d}} = d\hat{\mathbf{i}}$. We will then try to calculate the field at an arbitrary displacement $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ from the center of the wire, lying inside the hole.

Due to the wire we have a field pointing in the $\hat{\boldsymbol{\theta}}$ direction, whose magnitude we can obtain from Ampere's law (using an Amperian loop of radius $r = \sqrt{x^2 + y^2}$):

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B2\pi r = \mu_o I_{enc} = \mu_o J\pi r^2 \Rightarrow \vec{\mathbf{B}} = \tfrac{1}{2}\mu_o Jr\hat{\boldsymbol{\theta}} = \tfrac{1}{2}\mu_o Jr\left(-\tfrac{y}{r}\hat{\mathbf{i}} + \tfrac{x}{r}\hat{\mathbf{j}}\right) = -\tfrac{1}{2}\mu_o J\left(y\hat{\mathbf{i}} - x\hat{\mathbf{j}}\right)$$

For the hole, we are at $\vec{\mathbf{r}}_h = \left(x - d\right)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and use a similar calculation to find:

$$\vec{\mathbf{B}}_h = -\tfrac{1}{2}\mu_o Jr_h\hat{\boldsymbol{\theta}}_h = -\tfrac{1}{2}\mu_o Jr\left(-\tfrac{y}{r}\hat{\mathbf{i}} + \tfrac{x-d}{r}\hat{\mathbf{j}}\right) = \tfrac{1}{2}\mu_o J\left(y\hat{\mathbf{i}} - \left(x - d\right)\hat{\mathbf{j}}\right)$$

where the overall minus sign comes from the fact that the current is flowing the opposite direction in the "hole wire."

Now we can just sum to find the total field:

$$\vec{\mathbf{B}}_{total} = \vec{\mathbf{B}} + \vec{\mathbf{B}}_h = -\tfrac{1}{2}\mu_o J\left(y\hat{\mathbf{i}} - x\hat{\mathbf{j}}\right) + \tfrac{1}{2}\mu_o J\left(y\hat{\mathbf{i}} - \left(x - d\right)\hat{\mathbf{j}}\right) = \tfrac{1}{2}\mu_o Jd\hat{\mathbf{j}} = \boxed{\frac{\mu_o Id}{2\pi\left(a^2 - b^2\right)}\hat{\mathbf{j}}}$$

Note that it doesn't matter where you are inside the hole, just that you are inside.

Problem 7: Magnetic Field Strengths

A friend asks you to help with his problems with magnetic fields in his new house. Here are his comments:

> I had my inspector inspect the house and he concluded that the electromagnetic field levels inside the house were unacceptable. They ranged from a low of about 3 mG to a high of about 5.8 mG in the master bedroom, which is closest to the power lines running parallel to the back of the house at a distance of about 35 feet away. The inspector indicated to me that he considered levels above 2 mG as unacceptable and recommended that levels under 1 mG would be preferred. I could bury the lines (with dubious chances of reducing the EMF levels: a worker at the power company told me that it could make things worse because the lines would be closer to the house and the earth doesn't stop or mitigate EMFs but the inspector and some literature I've read postulated that with the wires in closer physical proximity the EMFs would cancel each other out to a greater extent) at a cost exceeding $20,000 just for the 50 feet or so in the back of my house, or live in the house and risking the health and possibly lives of myself and my family including any unborn children which may be especially susceptible to adverse health effects when they are in the womb).

Let's see what you should tell your friend.

a) Based on your real life experiences, how far apart do you think the wires are now (for the purposes of this problem assume that there are just two parallel wires carrying equal current in opposite directions – this is basically correct)?

From experience, the wires are probably about a meter apart and from the statement they are about 10 meters away (along the ground) from the house, and are probably up in the air about 10 meters. The field from these wires will clearly depend on the exact placement relative to each other, but clearly not too much, so I'll simply place them in a line from the house, with one wire $10\sqrt{2} \approx 14\text{ meters}$ away and the other, which is carrying current in the opposite direction, 15 meters away.

b) Based on (a) and the numbers he gave you how much current are they carrying?

The field from a long straight wire is $B = \mu_o I / 2\pi r$ (from Ampere's law). We have two wires a distance r_1 and r_2 away, where $r_2 = r_1 + \delta r$, so:

$$B = \frac{\mu_o I}{2\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\mu_o I}{2\pi}\left(\frac{r_2 - r_1}{r_1 r_2}\right) = \frac{\mu_o I}{2\pi}\left(\frac{\delta r}{r_1 r_2}\right)$$

Solving for I, we find:

$$I = \frac{2\pi B}{\mu_o}\frac{r_1 r_2}{\delta r} \approx \frac{2\pi\left(5.8\times10^{-7}\text{ T}\right)}{\left(4\pi\times10^{-7}\text{ T m A}^{-1}\right)}\frac{(14\,\text{m})(15\,\text{m})}{(1\,\text{m})} \approx (3\,\text{A/m})(200\text{ m}) = 600\text{ A}$$

Is this reasonable? He doesn't say what kind of wire it is, whether it is just the feed to his house (in which case this number would be too large), or if it is a high voltage feed for the neighborhood. If this later case, 600 A is quite reasonable. A neighborhood feed is typically at 2 kV, meaning that the wire is delivering a power of about 10^6 W, which would power about 100 houses (at 10 kW per house).

Problem 7: Magnetic Field Strengths *continued…*

c) If you bury them how much closer will they be to the house (how deep is a trench that power lines are buried in?) and how much closer to each other will the wires be?

Typically wires are buried 2-3 meters underground (ideally below the frost line, but not always). For these depths, we can essentially ignore the depth and say that the wires are approximately 10 m away. They will also typically be closer together, maybe 30 cm.

d) How much will this change the field in the master bedroom?

$$B = \frac{\mu_o I}{2\pi}\left(\frac{\delta r}{r_1 r_2}\right) \approx \frac{\left(4\pi\times10^{-7}\text{ T m A}^{-1}\right)\left(600\text{ A}\right)}{2\pi}\left(\frac{0.3\text{ m}}{100\text{ m}^2}\right) \sim 4\times10^{-7}\text{ T} = 4\text{ mG}$$

So burying the wires doesn't do much. We could make it do more by buying the wires very deeply or by putting them closer together. But it would be difficult to get an order of magnitude change.

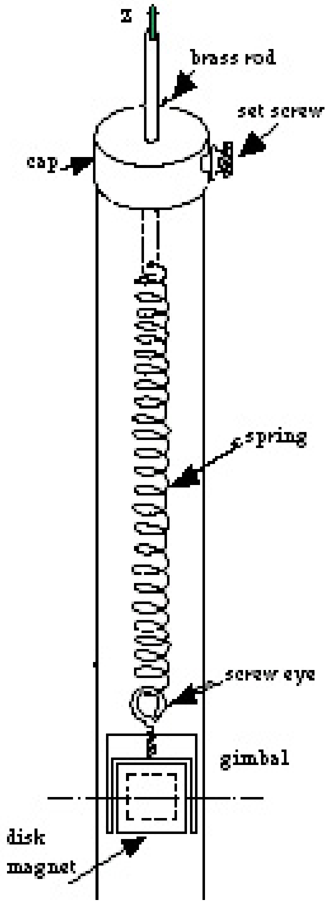
e) Most importantly, we need to know what to think about these field sizes. I'm sitting next to a desk lamp with an old fashioned power cord (the kind with the two wires running parallel in a molded plastic cover) less than a foot away from me. How does the field from this compare to the fields mentioned above?

Which leads us to the main question – how big is this field? You know that the Earth's field is half a Gauss, or 100 times larger than this. The Earth's field, however, is DC, while these fields are alternating at 60 Hz, and it is not unreasonable that, biologically, AC fields are different than DC fields. So let's think about my lamp. It has about 1 A (which at 120 V gives me 60 Watts – there is a factor of ½ for AC currents that you can ignore for the time being) flowing through it, the wires are about 1 cm apart, and I am sitting about 10 cm away from it. So the B field from that is:

$$B = \frac{\mu_o I}{2\pi}\left(\frac{\delta r}{r_1 r_2}\right) \approx \frac{\left(4\pi\times10^{-7}\text{ T m A}^{-1}\right)\left(1\text{ A}\right)}{2\pi}\left(\frac{0.01\text{ m}}{\left(0.1\text{ m}\right)^2}\right) \sim 2\times10^{-7}\text{ T} = 2\text{ mG}$$

Not very different! These fields are not particularly large. Certainly we could have made estimates that would push the values below the arbitrary threshold of 1 mG that the inspector claims he "prefers." But since there is no scientific evidence that these fields are dangerous I wouldn't recommend that this friend shell out $20k.

Problem 8: Force on a Dipole in the Helmholtz Apparatus



In class you calculated the magnetic field along the axis of a coil to be given by:

$$B_{axial} = \frac{N\,\mu_0\,I\,R^2}{2}\frac{1}{(z^2+R^2)^{3/2}}$$

where z is measured from the center of the coil.

In this lab we will have a disk magnet (a dipole) suspended on a spring, which we will use to observe force s on dipoles due to different magnetic field configurations.

(a) Assuming we energize only the top coil (current running counter-clockwise in the coil, creating the field quoted above), and assuming that the dipole is always well aligned with the field and on axis, what is the force on the dipole as a function of position? (HINT: In this situation $F = \mu\ dB/dz$)

$$F = \mu\frac{dB_{axial}}{dz} = \mu\frac{N\,\mu_0\,I\,R^2}{2}\frac{d}{dz}(z^2+R^2)^{-3/2}$$

$$= \boxed{\mu\frac{N\,\mu_0\,I\,R^2}{2}\left(\frac{-3z}{(z^2+R^2)^{5/2}}\right)}$$

The force goes like –z, i.e. towards the coil center

(b) The disk magnet (together with its support) has mass *m*, the spring has spring constant *k* and the magnet has magnetic moment μ. With the current on, we lift the brass rod until the disk magnet is sitting a distance z_0 above the top of the coil. Now the current is turned off. How does the magnet move once the field is off (give both direction and distance)?

We must balance the magnetic force with the spring force:

$$F_B(z=z_0) = \mu\frac{N\,\mu_0\,I\,R^2}{2}\left(\frac{-3z_0}{(z_0^2+R^2)^{5/2}}\right) = F_{\text{spring}} = k\Delta z \Rightarrow \boxed{\Delta z = \mu\frac{N\,\mu_0\,I\,R^2}{2k}\left(\frac{-3z_0}{(z_0^2+R^2)^{5/2}}\right)}$$

When the magnetic field is turned off, the force from the magnetic field will disappear and the spring will relax upwards by this distance.

Problem 8: Force on a Dipole in the Helmholtz Apparatus *continued…*

(c) At what height(s) is the force on the dipole the largest?

To find this we just maximize the force function (find zeros of its derivative):

$$\frac{dF_B}{dz} = \frac{d}{dz}\left[\mu \frac{N\,\mu_0\,I\,R^2}{2}\left(\frac{-3z}{(z^2+R^2)^{5/2}}\right)\right] = 0$$

$$\Rightarrow \qquad 0 = \frac{d}{dz}\left(z\left(z^2+R^2\right)^{-5/2}\right) = \left(z^2+R^2\right)^{-5/2} - 5z^2\left(z^2+R^2\right)^{-7/2}$$

$$\Rightarrow \qquad 0 = z^2 + R^2 - 5z^2 \qquad \Rightarrow \qquad \boxed{z = \pm\frac{R}{2}}$$

(d) What is the force where the field is the largest?

Where the field is largest the force must be zero. You can either think "That's where an aligned dipole would like to be" or "Maximum field means derivative of field is zero means no force."

(e) Our coils have a radius R = 7 cm and N = 168 turns, and the experiment is done with I = 1 A in the coil. The spring constant k ~ 1 N/m, and μ ~ 0.5 A m^2. The mass m ~ 5 g is in the shape of a cylinder ~ 0.5 cm in diameter and ~ 1 cm long. If we place the magnet at the location where the spring is stretched the furthest when the field is on, at about what height will the magnet sit after the field is turned off?

To make the spring stretch the furthest we must be at the location of the largest force, a distance z_0=R/2 above the coil (from c). From above the spring will relax by:

$$-\Delta z = \mu \frac{N\,\mu_0\,I\,R^2}{2k}\left(\frac{3z_0}{(z_0^2+R^2)^{5/2}}\right) = \mu \frac{N\,\mu_0\,I}{2kR^2}\left(\frac{3\cdot\frac{1}{2}}{(\frac{1}{4}+1)^{5/2}}\right)$$

$$\approx \left(0.5\text{ A m}^2\right)\frac{(168)\left(4\pi\times10^{-7}\text{ T m/A}\right)(1\text{ A})}{2\left(1\text{ N/m}\right)\left(7\text{ cm}\right)^2}\left(\frac{3\cdot 32}{2\cdot 5^{5/2}}\right) = 9.2\text{ mm}$$

So the final position is a distance $3.5\text{ cm} + 9.2\text{ mm} \approx 4.4\text{ cm}$ above the center of the coil