MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02 **Fall 2007**

Problem Set 4 Solutions

Problem 1: Parallel Plate Capacitor

A parallel-plate capacitor is charged to a potential V_0, charge Q_0 and then disconnected from the battery. The separation of the plates is then halved. What happens to

(a) the charge on the plates?
No Change. We aren't attached to a battery, so the charge is fixed.

(b) the electric field?
No Change. The charge is constant so, in the planar geometry, so is the field.

(c) the energy stored in the electric field?
Halves. The volume in which we have field halves, so the energy does too.

(d) the potential?
Halves. $V = E\, d$, so if d halves, so does V

(e) How much work did you do in halving the distance between the plates?
The work done is the change in energy. Energy, given the charge and potential, is:

$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV$$

The energy halves, so the change is half the initial energy: $\boxed{W = \Delta U = -\tfrac{1}{4}Q_0V_0}$

Notice the sign – you did negative work bringing the plates together because that is the way they naturally want to move; the field did positive work.

Problem 2: Capacitor

A parallel plate capacitor has capacitance C. It is connected to a battery of EMF ε until fully charged, and then disconnected. The plates are then pulled apart an extra distance d, during which the measured potential difference between them changed by a factor of 4. Below are a series of questions about how other quantities changed. Although they are related you do not need to rely on the answers to early questions in order to correctly answer the later ones.

a) Did the potential difference increase or decrease by a factor of 4?

$\boxed{\text{INCREASE}}$ DECREASE

Problem 2: Capacitor *continued…*

b) By what factor did the electric field change due to this increase in distance? Make sure that you indicate whether the field increased or decreased.

Since the charge cannot change (the battery is disconnected) the electric field cannot change either. $\boxed{\text{No Change!}}$

c) By what factor did the energy stored in the electric field change? Make sure that you indicate whether the energy increased or decreased.

The electric field is constant but the volume in which the field exists increased, so the energy must have increased. But by how much? The energy $U = \frac{1}{2}QV$. The charge doesn't change, the potential increased by a factor of 4, so the energy:

$$\boxed{\text{Increased by a factor of 4}}$$

d) A dielectric of dielectric constant κ is now inserted to completely fill the volume between the plates. Now by what factor does the energy stored in the electric field change? Does it increase or decrease?

Inserting a dielectric decreases the electric field by a factor of κ so it decreases the potential by a factor of κ as well. So now, by using the same energy formula $U = \frac{1}{2}QV$,

$$\boxed{\text{Energy decreases by a factor of } \kappa}$$

e) What is the volume of the dielectric necessary to fill the region between the plates? (Make sure that you give your answer only in terms of variables defined in the statement of this problem, fundamental constants and numbers)

How in the world do we know the volume? We must be able to figure out the cross-sectional area and the distance between the plates. The first relationship we have is from knowing the capacitance:

$$C = \frac{\varepsilon_0 A}{x}$$

where x is the original distance between the plates. Make sure you don't use the more typical variable d here because that is used for the distance the plates are pulled apart.

Next, the original voltage $V_0 = E\,x$, which increases by a factor of 4 when the plates are moved apart by a distance d, that is, $4\,V_0 = E\,(x+d)$. From these two equations we can solve for x:

$$4V_0 = 4Ex = E\left(x+d\right) \quad \Rightarrow \quad x = d/3$$

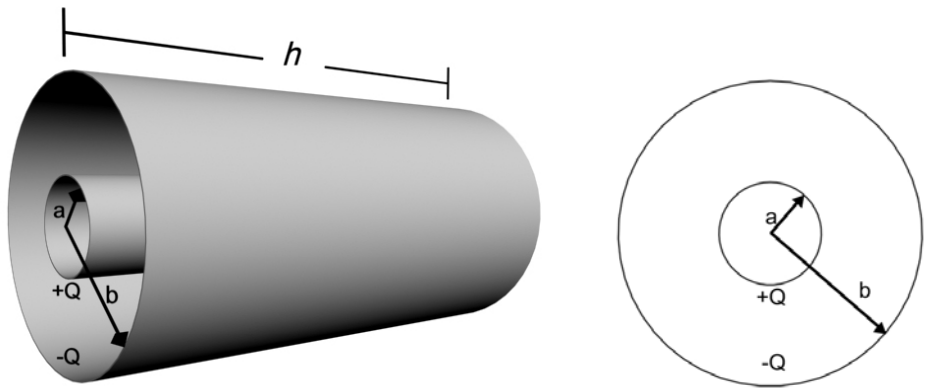
Now, we can use the capacitance to get the area, and multiply that by the distance between the plates (now $x + d$) to get the volume:

$$\text{Volume } = A\left(x+d\right) = \frac{xC}{\varepsilon_0}\left(x+d\right) = \frac{dC}{3\varepsilon_0}\left(\frac{d}{3}+d\right) = \boxed{\frac{4d^2C}{9\varepsilon_0}}$$

Problem 3: Capacitance of our Experimental Set-Up

In this experiment we will measure the potential difference between the pail and the shield, and make statements about the charge on the pail based on this. Here you will calculate the relationship between charge and potential.

Consider two nested cylindrical conductors of height h and radii a & b respectively. A charge $+Q$ is evenly distributed on the outer surface of the pail (the inner cylinder), $-Q$ on the inner surface of the shield (the outer cylinder).



(a) Calculate the electric field between the two cylinders ($a < r < b$).

For this we use Gauss's Law, with a Gaussian cylinder of radius r, height l

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi r l E = \frac{Q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{h} l \quad \Rightarrow \quad \boxed{E(r)_{a<r<b} = \frac{Q}{2\pi r \varepsilon_0 h}}$$

(b) Calculate the potential difference between the two cylinders:

The potential difference between the outer shell and the inner cylinder is

$$\Delta V = V(a) - V(b) = -\int_b^a \frac{Q}{2\pi r' \varepsilon_0 h} dr' = -\frac{Q}{2\pi\varepsilon_0 h} \ln r' \Big|_b^a = \boxed{\frac{Q}{2\pi\varepsilon_0 h}\ln\left(\frac{b}{a}\right) = \Delta V}$$

(c) Calculate the capacitance of this system, $C = Q/\Delta V$

$$C = \frac{|Q|}{|\Delta V|} = \frac{|Q|}{\frac{|Q|}{2\pi\varepsilon_0 h}\ln\left(\frac{b}{a}\right)} = \boxed{\frac{2\pi\varepsilon_o h}{\ln\left(\frac{b}{a}\right)} = C}$$

Problem 3: Capacitance of our Experimental Set-Up *continued*...

(d) Numerically evaluate the capacitance for your experimental setup, given:

$h \cong 15$ cm, $a \cong 4.75$ cm and $b \cong 7.25$ cm

$$C = \frac{2\pi\varepsilon_o h}{\ln\left(\frac{b}{a}\right)} = \frac{1}{2\cdot 9\times 10^9 \text{ m F}^{-1}} \frac{15 \text{ cm}}{\ln\left(\frac{7.25 \text{ cm}}{4.75 \text{ cm}}\right)} \cong \boxed{20 \text{ pF} \cong C}$$

Problem 4: What about charge on other surface?

In the previous problem we assumed that charge was located on the outer surface of the inner cylinder and the inner surface of the outer one, in other words, if both cylinders were charged. What if instead the cylinders were both neutral but exhibited charge separation due to the presence of a positive charge Q sitting in the interior of the inner cylinder? What now is the potential difference between the two cylinders?

The potential difference between the outer shell and the inner cylinder remains the same, since the electric field between them remains the same. That is,

$$\boxed{\Delta V = V(a) - V(b) = \frac{Q}{2\pi\varepsilon_0 h}\ln\left(\frac{b}{a}\right)}$$

Problem 5: Experimental Predictions

A. Prediction: Charging by Contact

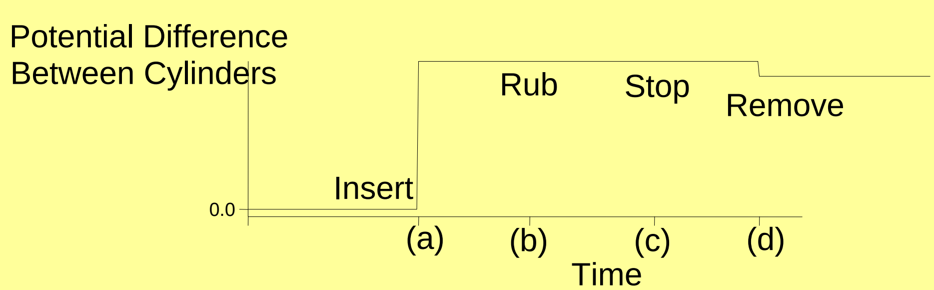
Sketch your prediction for the graph of potential difference vs. time for part 2 of this experiment. Indicate the following events on the time axis:

(a) Insert positive charge producer into pail

(b) Rub charge producer against inner surface of pail

(c) Stop rubbing (move away from surface, but leave inside pail)

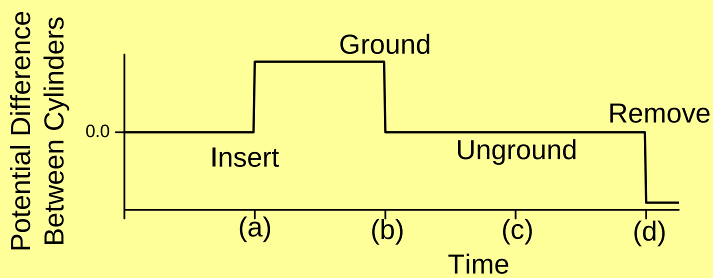
(d) Remove charge producer



Problem 5: Experimental Predictions *continued…*
B. Prediction: Charging by Induction

Sketch your prediction for the graph of potential difference vs. time for part 3 of this experiment. Indicate the following events on the time axis:

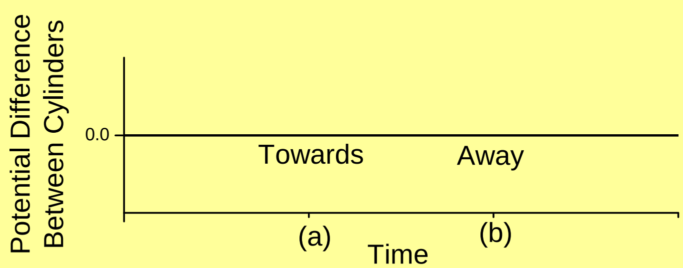
(a) Insert positive charge producer into pail
(b) Ground pail to shield
(c) Remove ground contact between pail and shield
(d) Remove charge producer



C. Prediction: Electrostatic Shielding

Sketch your prediction for the graph of potential difference vs. time for part 4 of this experiment. Indicate the following events on the time axis:

(a) Bring positive charge producer near (but outside of) shield
(b) Move charge producer away

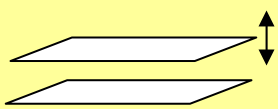


Problem 6: UROP Interview (AKA Shameless Plugging)

Our walker can move objects over about 5 millimeters with 0.1 to 1 nm resolution. We have a couple of ideas we'd like you to flesh out, both involving capacitive measuring devices. The first involves designing a parallel plate capacitor, with one plate fixed and the other moving towards or away from the first as the walker moves forward or backward. The second involves a cylindrical capacitor, where the inner plate of the capacitor is fixed and the outer plate of the capacitor moves forward or backward with the walker.

Below are the two methods with a summary of facts about each. We want to maximize d*C*/dz (the amount the capacitance will change while stepping) while keeping *C* of a measurable size.

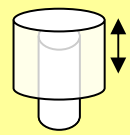
Parallel Plate Method



$$C = \frac{\varepsilon_o A}{d+z}$$

$$\frac{dC}{dz} = -\frac{\varepsilon_o A}{\left(d+z\right)^2}$$

Cylindrical Capacitor Method



$$C = \frac{2\pi\varepsilon_o\left(h-z\right)}{\ln\left(\frac{R+dR}{R}\right)} = \frac{2\pi\varepsilon_o\left(h-z\right)}{\ln\left(1+\frac{dR}{R}\right)} \approx \frac{2\pi\varepsilon_o\left(h-z\right)}{\frac{dR}{R}}$$

$$\frac{dC}{dz} = -\frac{2\pi\varepsilon_o R}{dR}$$

z = 0 .. 5 mm (distance of travel)

$\varepsilon_0 \sim 10^{-11}$ F/m

Parallel plate:

Maximize A = $\pi R^2 = \pi(3\text{ cm})^2 \sim 30\text{ cm}^2$

Minimize d ~ 0.1 mm

C ~ 300 pF to 6 pF

dC/dz ~ 3 μF/m to 1 nF/m

Cylindrical capacitor:

Maximize R ~ 3 cm

Minimize dR ~ 0.1 mm

Not so important: h = 6 mm

C ~ 100 pF to 20 pF

dC/dz ~ 20 nF/m

So they both end up being in about the same range. At dC/dz = 1 nF/m we get a spatial resolution of about 1 aF/(1 nF/m) = 1 nm which is pretty close to what we wanted. When the plates are very close the parallel plate capacitor does significantly better. But the cylindrical capacitor is linear (constant dC/dz) which is of huge benefit (easier to convert from C to position). It also turns out that it is easier to keep cylinders concentric than to keep planes parallel.

Using the cylindrical capacitor we get a measurement resolution of 50 pm.

Problem 7: Human Capacitor

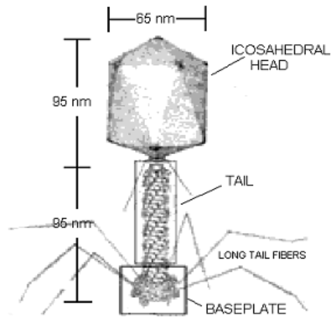
What, approximately, is the capacitance of a typical MIT student? Check out the exhibit in Strobe Alley (4^{th} floor of building 4) for a hint or just to check your answer.

There are lots of ways to do this. The note in strobe alley tells us to use a cylinder of dimensions such that when filled with water it would be your mass. Personally I feel more like a sphere, of which we have already calculated the capacitance in class. All I need to know is my radius, a. As a first approximation, probably it's a meter (I'm certainly less than 10 m and more than 10 cm). So my capacitance should be about:

$$C \approx 4\pi\varepsilon_0 a \approx 1\text{m}/9\times10^9\ \text{F}^{-1}\ \text{m} \approx 100\ \text{pF}$$

Not a bad approximation – according to the measurement I'm really ~170 pF.
Note that I used the value for k_E instead of multiplying out $4\pi\varepsilon_0$. Always look for ways to recombine constants into things that you know (or that save you some steps).

Problem 8: DNA in T4 Phage



The T4 phage (pictured at right) has a strand of DNA about 50 μm long wound up inside of its head. DNA has a net charge of one electron every 1.7 Å.

(a) About how much energy does it take to put that charge into the head?

A strand of DNA that is 50 μm long will have about (50 μm)/(1.7 Å/e) ~ 3 x 10^5 e charge on it, or ~ 5 x 10^{-14} C. If we approximate the head as a sphere of radius a ~ 50 nm, it has a capacitance of $C_{\text{sphere}} = 4\pi\varepsilon_0 a$ ~ 5 x 10^{-18} F (tiny!).

(a) So the energy of the system is $\dfrac{Q^2}{2C} \sim \dfrac{\left(5\times10^{-14}\ \text{C}\right)^2}{2\left(5\times10^{-18}\ \text{F}\right)} \sim \boxed{3\times10^{-10}\ \text{J}}$

You probably don't have a good feeling for how much energy this is (you wouldn't notice if a bomb with this much energy – or a million times more – hit you). Hence the second question:

(b) About how much pressure is exerted on the head? (HINT: P = -dU/dV)

We need to take the derivative with respect to volume, so let's use the chain rule:

$$P = -\frac{dU}{dV} = -\frac{dU}{da}\frac{da}{dV} = -\frac{d}{da}\left(\frac{Q^2}{8\pi\varepsilon_0 a}\right)\left(\frac{d}{da}\frac{4}{3}\pi a^3\right)^{-1} = \frac{Q^2}{8\pi\varepsilon_0 a^2}\left(4\pi a^2\right)^{-1}$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 a^4} \sim \frac{\left(9\times10^9\ \text{Nm}^2\text{C}^{-2}\right)\left(5\times10^{-14}\ \text{C}\right)^2}{8\pi\left(50\times10^{-9}\ \text{m}\right)^4} \sim 10^{11}\ \text{Pascals} \sim \boxed{10^6\ \text{atm}}$$

This is HUGE. Which leads to question…

Problem 8: DNA in T4 Phage *continued…*

(c) Is this reasonable? Is there something that we are forgetting about?

This isn't reasonable – we have ignored something. There is no way that the DNA charge is left unbalanced. The phage is in some solution, and ions in that solution will naturally be attracted to the DNA and neutralize it. The charge is very important, however, as it (at least partially) leads to the twisting and folding of the DNA. Clearly if it weren't neutralized DNA would never be able to be squeezed into balls small enough to fit into the nuclei of cells.

Problem 9: More Reflections on a Conducting Plane

In the last problem set you considered a charge *q* a distance *d* above an infinitely large, uncharged, perfectly conducting, shiny piece of metal. Thinking about the answers to that problem,

(a) What is the charge density on the surface of the conductor as a function of position *r*?

To get the charge density we will just use the electric field we calculated last week, and recall the relationship between electric field and charge density outside a conductor:

$$E = \frac{\sigma}{\varepsilon_o} \quad \Rightarrow \quad \sigma(r) = \varepsilon_o E(r) = \boxed{-\frac{2qd}{4\pi\left(d^2+r^2\right)^{3/2}}}$$

(b) What is the total charge on the surface of the conductor (the part of the surface that is facing the charge, *not* the bottom side facing away)? Does this make sense?

The total charge is just the integral of the charge density over the entire surface. Since the density only depends on *r* I will integrate in rings outward:

$$Q = \iint\limits_{Plane} \sigma(r)\cdot dA = \int_{r=0}^{\infty} -\frac{2qd}{4\pi\left(d^2+r^2\right)^{3/2}}\cdot 2\pi r\, dr = -\frac{qd}{2}\int_{r=0}^{\infty}\frac{2r\, dr}{\left(d^2+r^2\right)^{3/2}}$$

$$= -\frac{qd}{2}\int_{u=d^2}^{\infty}\frac{du}{u^{3/2}} = -\frac{qd}{2}\left.\frac{u^{-1/2}}{-1/2}\right|_{d^2}^{\infty} = \boxed{-q}$$

This is one of those great problems where you know you must have done it right to get such a simple answer. This makes sense. We say we have an image charge –*q* and here we have calculated it.