18.02 Fall 2007 – Problem Set 2, Part B Solutions

1. a) The two planes have normal vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = \langle 1,-3,1\rangle \times \langle -1,4,-2\rangle = \langle 2,1,1\rangle$. The plane containing A and B has equation $2x+y+z=8$, while the plane containing C and D has equation $2x+y+z=6$. To find the distance between these planes, we can compute the component along $\vec{n}$ of a vector joining the two planes, for example $\overrightarrow{AC}$:

$$\frac{\overrightarrow{AC}\cdot\vec{n}}{|\vec{n}|} = \frac{\langle -1,-3,3\rangle\cdot\langle 2,1,1\rangle}{\sqrt{2^2+1^2+1^2}} = -\frac{2}{\sqrt{6}} = -\sqrt{\frac{2}{3}}.$$

Hence the distance between the planes (which is the same as the distance between the lines (AB) and (CD)) is $\sqrt{2/3}$.

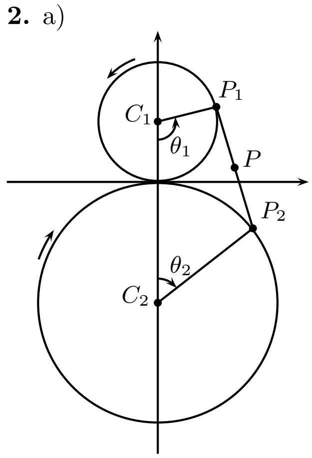
b) A parametric equation for (AB) is: $x = 2+t,\ y = 4-3t,\ z = t$. A parametric equation for (CD) is: $x = 1-t,\ y = 1+4t,\ z = 3-2t$. If $P_1 = (2+t_1, 4-3t_1, t_1)$ and $P_2 = (1-t_2, 1+4t_2, 3-2t_2)$ are points on the two lines, then

$$\overrightarrow{P_1P_2}\cdot\overrightarrow{AB} = \langle -1-t_1-t_2, -3+3t_1+4t_2, 3-t_1-2t_2\rangle\cdot\langle 1,-3,1\rangle = 11-11t_1-15t_2,$$

$$\overrightarrow{P_1P_2}\cdot\overrightarrow{CD} = \langle -1-t_1-t_2, -3+3t_1+4t_2, 3-t_1-2t_2\rangle\cdot\langle -1,4,-2\rangle = -17+15t_1+21t_2.$$

So $\overrightarrow{P_1P_2}$ is perpendicular to both lines when $11-11t_1-15t_2 = 0$ and $-17+15t_1+21t_2 = 0$, i.e. $t_1 = -4$ and $t_2 = 11/3$. For these values of the parameters, $P_1 = (-2,16,-4)$ and $P_2 = (-8/3, 47/3, -13/3)$, so $\overrightarrow{P_1P_2} = \langle -2/3, -1/3, -1/3\rangle$ and $|\overrightarrow{P_1P_2}| = \sqrt{6}/3 = \sqrt{2/3}$.

2. a)



At time t, the peg P_2 on the circle of radius 2 has rotated clockwise by $\theta_2 = t$, so the arc length from the origin to P_2 along the circumference is $2t$; since the circles roll without slipping, this arc length equals that from the origin to P_1 on the circle of radius 1. So P_1 has rotated counterclockwise by an angle $\theta_1 = 2t$.

Therefore $\overrightarrow{C_1P_1} = \langle \sin\theta_1, -\cos\theta_1\rangle = \langle \sin 2t, -\cos 2t\rangle$, while $\overrightarrow{C_2P_2} = \langle 2\sin\theta_2, 2\cos\theta_2\rangle = \langle 2\sin t, 2\cos t\rangle$. So $\overrightarrow{OP_1} = \overrightarrow{OC_1} + \overrightarrow{C_1P_1} = \langle 0,1\rangle + \langle \sin 2t, -\cos 2t\rangle = \langle \sin 2t, 1-\cos 2t\rangle$, and $\overrightarrow{OP_2} = \overrightarrow{OC_2} + \overrightarrow{C_2P_2} = \langle 0,-2\rangle + \langle 2\sin t, 2\cos t\rangle = \langle 2\sin t, -2+2\cos t\rangle$.

Since $\overrightarrow{OP} = \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2})$, we get

$$\overrightarrow{OP} = \langle \tfrac{1}{2}\sin 2t + \sin t, -\tfrac{1}{2} - \tfrac{1}{2}\cos 2t + \cos t\rangle.$$

b) $\vec{v} = \dfrac{d}{dt}\overrightarrow{OP} = \langle \cos 2t + \cos t, \sin 2t - \sin t\rangle$. At $t = 0$ we get $\vec{v} = \langle 2,0\rangle$ (// x-axis).

Next, $\vec{a} = \dfrac{d\vec{v}}{dt} = \langle -2\sin 2t - \sin t, 2\cos 2t - \cos t\rangle$; at $t=0$ we get $\langle 0,1\rangle$ (// y-axis).