

18.02 Problem Set 5

Due Thursday 10/11/07, 12:45 pm in 2-106.

Part A (8 points)

Hand in the underlined problems only; the others are for more practice.

Lecture 13. Thu Oct. 4 Gradient, directional derivative, tangent plane.

Read: 13.8.

Work: 2D/ 1ab, 2b, 3ac, 8, 9; 2E/ 7.

Lecture 14. Fri Oct. 5 Lagrange multipliers.

Read: 13.9 through the top of p. 922 – one constraint only.

Work: 2I/ 1, 3.

Part B (19 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Write the names of all the people you consulted or with whom you collaborated and the resources you used.

Problem 1. (Thursday, 8 points: 2+1+1+4)

As in PS3, Problem 3, consider the function

$$f(x, y) = x^3 - xy^2 - 4x^2 + 3x + x^2y.$$

a) Find the maximum and minimum values of the directional derivative $\left. \frac{df}{ds} \right|_{\hat{\mathbf{u}}}$ at $(1/2, 1)$ as $\hat{\mathbf{u}}$ varies.

b) Say in which directions $\hat{\mathbf{u}}$ the maximum and minimum occur.

c) Find the direction(s) $\hat{\mathbf{u}}$ for which $\left. \frac{df}{ds} \right|_{\hat{\mathbf{u}}} = 0$.

d) Go to the applet (<http://math.mit.edu/18.02/applets/>) and select “directional derivatives” from the Show menu. Click to move the point in the contour plot as close as you can to $(1/2, 1)$. ((0.51, 1.01) is ok, but record the actual values of x and y that you use.) Rotate the direction $\hat{\mathbf{u}} = \langle \cos \theta, \sin \theta \rangle$ and find the numerical values the applet computes for df/ds and θ answering questions (a)-(c) above. In (b) and (c) state the geometric relationship between the yellow direction of the directional derivative $\hat{\mathbf{u}}$, the blue contour line, and the purple gradient vector on the contour plot. (The value of the directional derivative is on the lower right in red.)

Problem 2. (Thursday, 5 points: 2+3)

a) Find the direction from $(2, -1, 1)$ in which $g(x, y, z) = x^2 + yz$ decreases fastest.

b) Follow the line in the direction you found in part (a) to estimate, using linear approximation, the location of the point closest to $(2, -1, 1)$ at which $g = 2$. Do not use a calculator. Express your answer using fractions. Next, use a calculator to evaluate g at your point. (The value should be reasonably close to 2.)

Problem 3. (Friday, 6 points: 2+4)

a) Now we seek an exact answer to Problem 2b. Use the method of Lagrange multipliers to write down the system of equations satisfied by the point closest to $(2, -1, 1)$ at which $x^2 + yz = 2$. (Hint: it is easier to minimize the square of the distance).

b) Solve the equations to get a numerical approximation to four decimal places for the location of the point. Compare your answers with the approximation in 2(b). Was each coordinate of the approximate answer in 2(b) within 1/100 of the exact answer?

Suggestion: by considering the sum of the second and third equations, show that either $z = -y$ or $\lambda = 2$. Assuming $z = -y$, use the multiplier equations to express x, y, z in terms of λ , then substitute these formulas into the constraint equation. After clearing the denominator, you should obtain a degree 4 equation for λ . Use a calculator to solve it. Alternatively, in Matlab, `roots([a4, a3, a2, a1, a0])` finds the roots of the polynomial $a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. (There should be two real roots). Finally, check that the other case $\lambda = 2$ yields solutions (x, y, z) that are further away from $(2, -1, 1)$.