18.02 Fall 2007 – Problem Set 4, Part B Solutions

1. a) LS (4) translates into $\begin{cases}(\mathbf{x}\cdot\mathbf{x})\,a+(\mathbf{x}\cdot\mathbf{u})\,b=(\mathbf{x}\cdot\mathbf{y})\\(\mathbf{x}\cdot\mathbf{u})\,a+(\mathbf{u}\cdot\mathbf{u})\,b=(\mathbf{y}\cdot\mathbf{u})\end{cases}$ or equivalently

$$A\,\mathbf{z}=\mathbf{r},\qquad \text{with}\qquad A=\begin{bmatrix}\mathbf{x}*\mathbf{x}' & \mathbf{x}*\mathbf{u}'\\ \mathbf{x}*\mathbf{u}' & \mathbf{u}*\mathbf{u}'\end{bmatrix},\qquad \mathbf{z}=\begin{bmatrix}a\\ b\end{bmatrix},\qquad \mathbf{r}=\begin{bmatrix}\mathbf{x}*\mathbf{y}'\\ \mathbf{y}*\mathbf{u}'\end{bmatrix}$$

In Matlab notation: `A=[x*x' x*u'; x*u' u*u']`, and `r=[x*y'; y*u']` (or even shorter: `A=[x;u]*[x' u']`, `r=[x;u]*y'`)

c) Matlab input and output (some of the output is removed for brevity)

```
>> x = [1:1:5]; y = [0.125 0.470 1.451 8.263 31.960]; u = [1 1 1 1 1];
>> A = [x*x' x*u'; x*u' u*u']; r = [x*y'; y*u']; z = inv(A)*r
z =   7.1463
    -12.9851                        (the best fit is: y = 7.146 x - 12.985)
>> v = z'*[x;u]
v =  -5.8388    1.3075    8.4538   15.6001   22.7464    (predicted values)
>> y - v
ans = 5.9638   -0.8375   -7.0028   -7.3371    9.2136     (the differences)
```

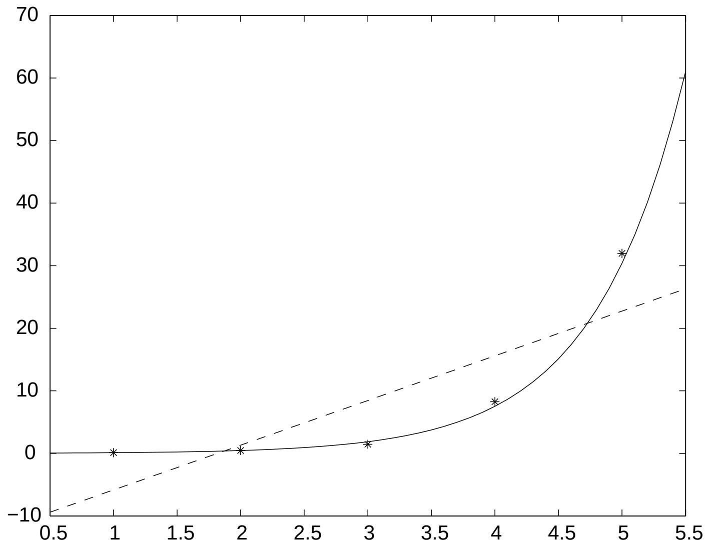
$a=7.146$, $b=-12.985$, and the largest error is about 9.2 (for the last data point).

d) Matlab input and output: (note A remains the same)

```
>> lny = log(y); r1 = [x*lny'; lny*u']; z1 = inv(A)*r1
z1 =  1.3955
     -3.5636                      (the best fit is: ln(y) = 1.395 x - 3.564)
>> v1 = exp(z1'*[x;u])
v1 =  0.1144    0.4618    1.8642    7.5254   30.3788     (predicted values)
>> y - v1
ans = 0.0106    0.0082   -0.4132    0.7376    1.5812      (the differences)
```

$a_1=1.395$, $b_1=-3.564$, and the largest error is about 1.6 (for the last data point).

e)



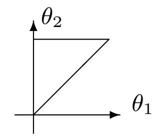
f) According to $y=e^{b_1}e^{a_1x}$, about 122.6 million iPods were sold in 2006, and about $3.49\cdot10^7$ million (34.9 trillion) iPods will be sold in 2015.

2. a) The three vertices are at $(1,0)$, $(\cos\theta_1, \sin\theta_1)$, and $(\cos\theta_2, \sin\theta_2)$.

When $\theta_1 < \theta_2$, the area of the triangle is given by

$$A(\theta_1,\theta_2) = \frac{1}{2}\begin{vmatrix} \cos\theta_1 - 1 & \sin\theta_1 \\ \cos\theta_2 - 1 & \sin\theta_2 \end{vmatrix} = \frac{1}{2}[(\cos\theta_1 - 1)\sin\theta_2 - \sin\theta_1(\cos\theta_2 - 1)]$$
$$= \frac{1}{2}[\sin\theta_1 + \sin(\theta_2 - \theta_1) - \sin\theta_2].$$

(This formula can also be obtained by splitting the triangle into three smaller triangles having O as a common vertex.)



The set of possible values for the angles is given by the inequalities $0 < \theta_1 < \theta_2 < 2\pi$. This is a triangular region in the (θ_1, θ_2) coordinates, with boundaries $\theta_1 = 0$, $\theta_2 = 2\pi$, and $\theta_1 = \theta_2$.

Note: if we ignored the information about the ordering of the vertices around the unit circle and allowed $\theta_1 > \theta_2$, then the determinant would be negative, and the correct formula for the area would be $|A(\theta_1, \theta_2)|$.

b) $\dfrac{\partial A}{\partial \theta_1} = \dfrac{1}{2}[\cos\theta_1 - \cos(\theta_2 - \theta_1)]$ and $\dfrac{\partial A}{\partial \theta_2} = \dfrac{1}{2}[\cos(\theta_2 - \theta_1) - \cos\theta_2]$, so critical points are solutions of $\cos\theta_1 = \cos(\theta_2 - \theta_1) = \cos\theta_2$. If $0 < \theta_1 < \theta_2 < 2\pi$, then $\cos\theta_1 = \cos\theta_2$ means that $\theta_2 = 2\pi - \theta_1$, and $\cos\theta_1 = \cos(\theta_2 - \theta_1)$ means that $\theta_1 = \theta_2 - \theta_1$, i.e. $\theta_2 = 2\theta_1$. We get the critical point $\theta_1 = 2\pi/3$, $\theta_2 = 4\pi/3$.

Note: if we allow two angles to be equal ($0 \le \theta_1 \le \theta_2 \le 2\pi$) then another solution of $\cos\theta_1 = \cos\theta_2$ is $\theta_1 = \theta_2$, which gives additional critical points $\theta_1 = \theta_2 = 0$ or 2π. These new critical points are on the boundary (in fact, at the corners) of the allowed set of values for (θ_1, θ_2), so we can safely ignore them.

c) At the critical point $\theta_1 = 2\pi/3$, $\theta_2 = 4\pi/3$, we have $A = 3\sqrt{3}/4$. The boundary of the (triangular) region where A is defined consists of three parts: $\theta_1 = 0$, $\theta_1 = \theta_2$, and $\theta_2 = 2\pi$. In each case, one of the terms in the formula for A becomes zero, and the two other terms cancel out, so that $A = 0$. (Geometrically: the boundary corresponds to the situation where two vertices of the inscribed triangle coincide with each other, in which case the area is zero).

By comparing the values of A at the critical point and near the boundary, the maximum of A is $3\sqrt{3}/4$, reached for $\theta_1 = 2\pi/3$ and $\theta_2 = 4\pi/3$, corresponding to an equilateral triangle. On the other hand, the minimum of A is zero, reached anywhere along the boundary, corresponding to a triangle with two equal vertices.

d) For $\theta_1 = 2\pi/3$ and $\theta_2 = 4\pi/3$, we have: $\dfrac{\partial^2 A}{\partial \theta_1^2} = \dfrac{1}{2}[-\sin\theta_1 - \sin(\theta_2 - \theta_1)] = -\dfrac{\sqrt{3}}{2}$, $\dfrac{\partial^2 A}{\partial \theta_1 \partial \theta_2} = \dfrac{1}{2}\sin(\theta_2 - \theta_1)] = \dfrac{\sqrt{3}}{4}$, and $\dfrac{\partial^2 A}{\partial \theta_2^2} = \dfrac{1}{2}[-\sin(\theta_2 - \theta_1) + \sin\theta_2] = -\dfrac{\sqrt{3}}{2}$.

Since $\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) - \left(\dfrac{\sqrt{3}}{4}\right)^2 > 0$ we have either a local maximum or a local minimum; since $\dfrac{\partial^2 A}{\partial \theta_1^2} < 0$ it is a local maximum.

3. a) $w_r = w_x x_r + w_y y_r = \cos\theta\, w_x + \sin\theta\, w_y$ and $w_\theta = w_x x_\theta + w_y y_\theta = -r\sin\theta\, w_x + r\cos\theta\, w_y$.

So $\begin{bmatrix} w_r \\ w_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}\begin{bmatrix} w_x \\ w_y \end{bmatrix}$, i.e. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$.

b) $w_x = w_r r_x + w_\theta \theta_x = \dfrac{x}{\sqrt{x^2+y^2}} w_r + \dfrac{-y/x^2}{1+(y/x)^2} w_\theta,$

$$w_y = w_r r_y + w_\theta \theta_y = \frac{y}{\sqrt{x^2+y^2}} w_r + \frac{1/x}{1+(y/x)^2} w_\theta \quad \Rightarrow B = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{bmatrix}.$$

c) In terms of r and θ, $B = \begin{bmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}$. So one checks $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as expected.

d) Using the expression of B for $(x,y) = (0,5)$: $\begin{bmatrix} w_x \\ w_y \end{bmatrix} = \begin{bmatrix} 0 & -1/5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_r \\ w_\theta \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.