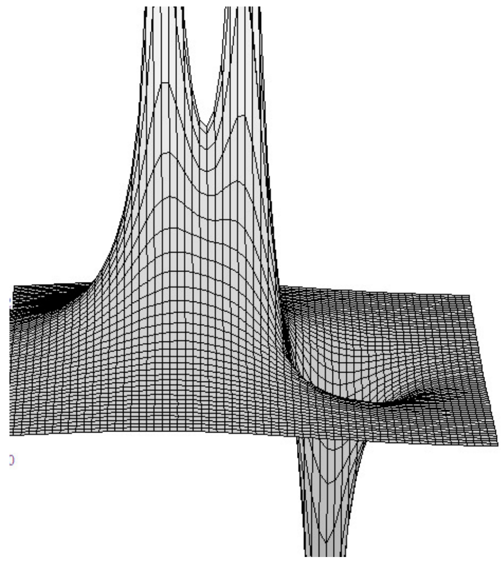
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02 **Fall 2007**

Problem Set 5 Solutions

Problem 1: Seven Short Questions. Circle your choice for the correct answer.

Question A (4 points out of 20 points):

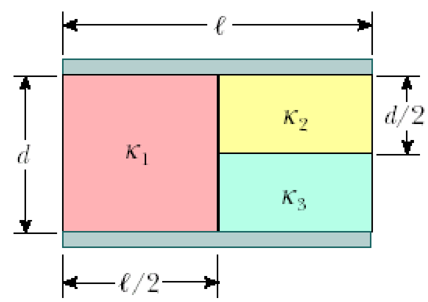


Three charges lie on the x-axis a distance *a* apart. The charges are numbered from 1 to 3 moving from left to right. A representation of the electric potential *V* of these three charges is shown above. Which one of the following statements is true?

The electric field is zero…

1. …at some point between charges 1 and 2 and also at some point between charges 2 and 3
2. …at some point between charges 1 and 2 but never between charges 2 and 3
3. …never between charges 1 and 2 but at some point between charges 2 and 3
4. …nowhere between charges 1 and 2 or between charges 2 and 3
5. I don't know (this answer is worth 1 point)

Question B (4 points out of 20 points):



Consider the capacitor at right, with $d << \ell$, filled with three different dielectrics, as pictured. What is the capacitance of this capacitor?

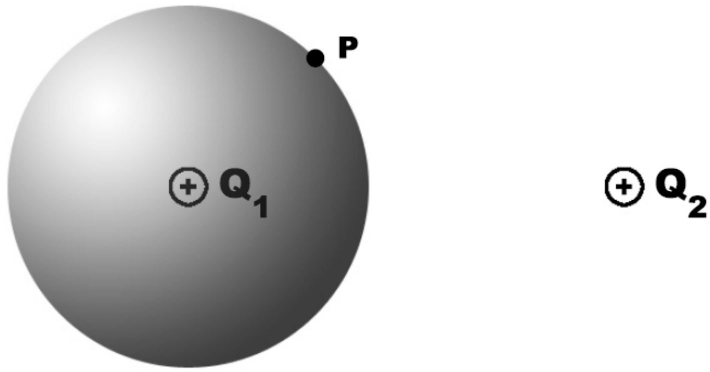
1) $C = \dfrac{d}{\varepsilon_0 A}\left(\dfrac{\kappa_1}{2} + \dfrac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)$ 2) $C = \dfrac{\varepsilon_0 A}{d}\left(\dfrac{\kappa_1}{2} + \dfrac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)$

3) $C = \dfrac{\varepsilon_0 A}{d}\left(\kappa_1 + \dfrac{\kappa_2\kappa_3}{\kappa_2+\kappa_3}\right)$ 4) $C = \dfrac{\varepsilon_0 A}{d}\left(\kappa_2 + \dfrac{\kappa_1\kappa_3}{\kappa_1+\kappa_3}\right)$

5) $C = \dfrac{3\varepsilon_0 A}{d}\left(\dfrac{\kappa_1\kappa_2\kappa_3}{\kappa_1+\kappa_2+\kappa_3}\right)$ 6) $C = \dfrac{3d}{\varepsilon_0 A}\left(\dfrac{\kappa_1\kappa_2\kappa_3}{\kappa_1+\kappa_2+\kappa_3}\right)$

7) I don't know (this answer is worth 1 point)

Question C (4 points out of 20 points):
In the figure below, a point charge $+Q_1$ is at the center of an imaginary spherical Gaussian surface and another point charge $+Q_2$ is outside of the Gaussian surface. Point P is on the surface of the sphere. Which one of the following statements is true?

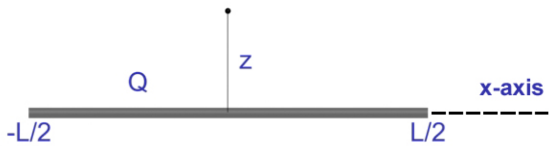


1. Both charges $+Q_1$ and $+Q_2$ contribute to the net electric flux through the sphere but only charge $+Q_1$ contributes to the electric field at point P on the sphere.
2. Both charges $+Q_1$ and $+Q_2$ contribute to the net electric flux through the sphere but only charge $+Q_2$ contributes to the electric field at point P on the sphere.
3. Only the charge $+Q_1$ contributes to the net electric flux through the sphere but both charges $+Q_1$ and $+Q_2$ contribute to the electric field at point P.
4. Only the charge $+Q_2$ contributes to the net electric flux through the sphere but both charges $+Q_1$ and $+Q_2$ contribute to the electric field at point P.
5. Only the charge $+Q_1$ contributes to the net electric flux through the sphere and to the electric field at point P on the sphere.
6. Only the charge $+Q_2$ contributes to the net electric flux through the sphere and to the electric field at point P on the sphere.
7. I don't know (this answer is worth 1 point)

Question D (4 points out of 20 points):
Consider an electric dipole immersed in a uniform electric field. Which of the following statements regarding the force $\vec{\mathbf{F}}$ and the torque $\vec{\boldsymbol{\tau}}$ is strictly true regardless of the orientation of the dipole:

1. $\vec{\mathbf{F}} \neq 0$ and $\vec{\boldsymbol{\tau}} \neq 0$.
2. $\vec{\mathbf{F}} = 0$ and $\vec{\boldsymbol{\tau}} \neq 0$.
3. $\vec{\mathbf{F}} \neq 0$ and $\vec{\boldsymbol{\tau}} = 0$.
4. $\vec{\mathbf{F}} = 0$ and $\vec{\boldsymbol{\tau}} = 0$.
5. $\vec{\mathbf{F}} = 0$ but $\vec{\boldsymbol{\tau}}$ may or may not be zero.
6. $\vec{\mathbf{F}}$ may or may not be zero but $\vec{\boldsymbol{\tau}} = 0$.
7. I don't know (this answer is worth 1 point)

Question E (4 points out of 20 points):



A rod of length L carries a charge per unit length λ uniformly distributed over its length. The magnitude of the electric field a distance z from the center of the rod along its perpendicular bisector (see sketch above) is given by the expression

1. $\frac{\lambda}{4\pi\varepsilon_o}\int_{-L/2}^{L/2}\frac{x\,dx}{\left[x^2+z^2\right]^{3/2}}$
2. $\frac{\lambda}{4\pi\varepsilon_o}\int_{-L/2}^{L/2}\frac{x\,dx}{\left[x^2+z^2\right]}$
3. $\boxed{\frac{\lambda}{4\pi\varepsilon_o}\int_{-L/2}^{L/2}\frac{z\,dx}{\left[x^2+z^2\right]^{3/2}}}$
4. $\frac{\lambda}{4\pi\varepsilon_o}\int_{-L/2}^{L/2}\frac{z\,dx}{\left[x^2+z^2\right]}$
5. $\frac{\lambda}{4\pi\varepsilon_o}\int_{-L/2}^{L/2}\frac{dx}{\left[x^2+z^2\right]}$
6. I don’t know (this answer is worth 1 point)

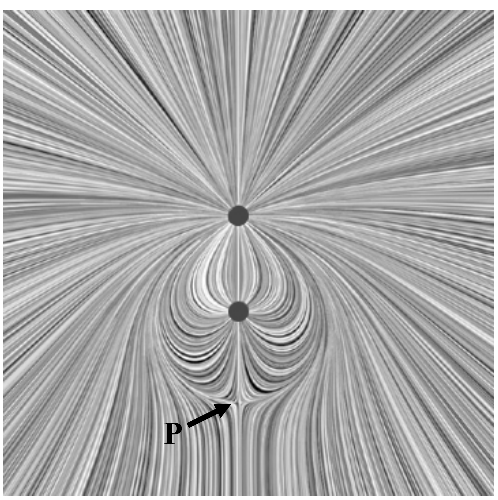
Problem 2: Back of the Envelope Calculation (5 Pts)
As always, you are not given enough information to exactly determine the answer to this question. Make your best estimates for unknowns, clearly indicating what your estimates are (e.g. Radius R ~ ….)

In class I used a small Van de Graff generator that when fully charged was crackling. About how much charge was on it?

The crackling of the Van de Graff generator is due to corona discharge at its surface, which happens when the electric field at its surface reaches the breakdown field of the air, or E ~ 3 x 10^6 V/m. The radius is about R ~ 10 cm (probably less, but this is a nice round number to work with).

$$E = k_e \frac{q}{R^2} \quad \Rightarrow \quad q = \frac{ER^2}{k_e} \approx \frac{\left(3\times10^6 \text{ N/C}\right)\left(0.1\text{ m}\right)^2}{9\times10^9 \text{ N}\cdot\text{m}^2/\text{C}^2} \approx \frac{1}{3}\times10^{-5}\text{ C} \approx \boxed{3\ \mu\text{C}}$$

Problem 3: Charges (25 pts)



The grass seeds diagram at left represents the electric field created by two charges, each located on the y-axis at y = 0 and y = *d*. The charge at y = 0 is +*Q*.

(a) What is the sign of the charge at y = *d*?

POSITIVE $\quad$ $\boxed{\text{NEGATIVE}}$

They clearly are of opposite sign

(b) In which direction does the electric field point at the bottom of the figure on the y-axis?

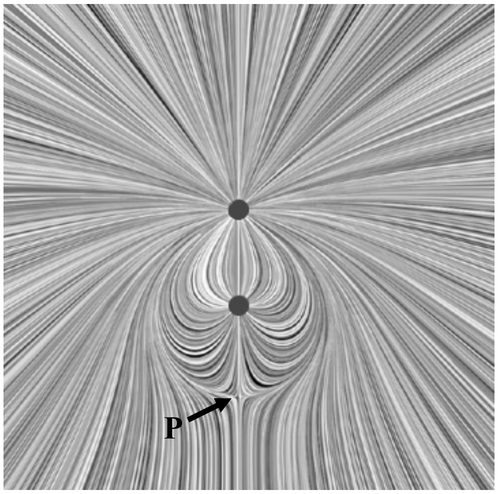
$\boxed{\text{UP}}$ $\quad$ DOWN

The large, negative charge acts as a sink

(c) Now consider the electric field at the point P (on the axis at y = -*d*). Is the electric field there

POSITIVE $\quad$ NEGATIVE $\quad$ or $\quad$ $\boxed{\text{ZERO}}$

Problem 3 *continued*: Charges



(d) Calculate the sign and magnitude Q_d of the charge located at y = *d*.

We know that the electric field at point P is zero, so we just need to find the charge that does this:

$$E_y = -k_e \frac{(+Q)}{d^2} - k_e \frac{(Q_d)}{(2d)^2} = 0$$

$$\boxed{Q_d = -4Q}$$

(e) Now consider a third point charge *q* presently located a very large distance away on the positive x-axis. How much work does it take to move this charge from its current location to point P? (NOTE: If you were unable to do part (d) but you need this value, simply refer to is as Q_d.)

To calculate the work we need to know the change in potential energy, which means we need to know the change in potential. We can calculate the potential at point P by superposition of point charge potentials:

$$V(P) = k_e \frac{(+Q)}{d} + k_e \frac{(Q_d)}{(2d)} = k_e \frac{Q}{d} + k_e \frac{(-4Q)}{(2d)} = -k_e \frac{Q}{d}$$

So the work done in bringing the charge from infinity (the fact that its on the x-axis is irrelevant, the potential at any infinity is 0!) is:

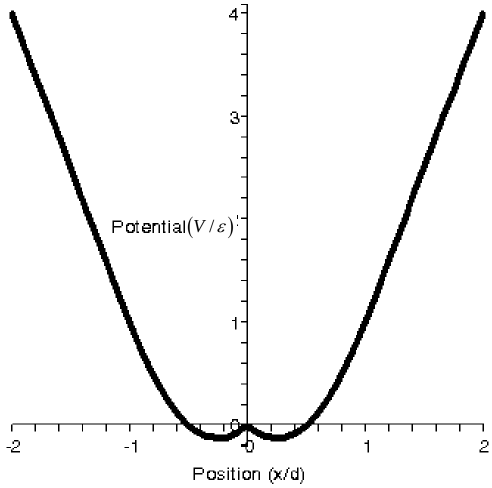
$$W = \Delta U = q\Delta V = qV(P) = \boxed{-k_e \frac{qQ}{d}}$$

Problem 4: Planar Potential (25 pts)

The electric potential *V(x,y,z)* for a planar charge distribution is given by:

$$V(x,y,z)=\begin{cases} -\varepsilon\left(3\frac{x}{d}+2\right) & \text{for } x<-d \\ \varepsilon\left(2\left(\frac{x}{d}\right)^2+\frac{x}{d}\right) & \text{for } -d\le x<0 \\ \varepsilon\left(2\left(\frac{x}{d}\right)^2-\frac{x}{d}\right) & \text{for } 0\le x<d \\ \varepsilon\left(3\frac{x}{d}-2\right) & \text{for } x>d \end{cases}$$

where d is the distance from the origin along the x axis where $V=\varepsilon$.



This function is plotted to the right, with the x-axis in units of d and the y-axis in units of ε.

(a) What is the electric field $\vec{\mathbf{E}}(x)$ for this problem?

Region I: -d > x

$$\vec{\mathbf{E}}=-\nabla V=-\frac{\partial V}{\partial x}\hat{\mathbf{i}}=\frac{3\varepsilon}{d}\hat{\mathbf{i}}$$

Region II: $-d\le x<0$

$$\vec{\mathbf{E}}=-\frac{\partial V}{\partial x}\hat{\mathbf{i}}=-\frac{\varepsilon}{d}\left(4\left(\frac{x}{d}\right)+1\right)\hat{\mathbf{i}}$$

Region III: $0\le x<d$

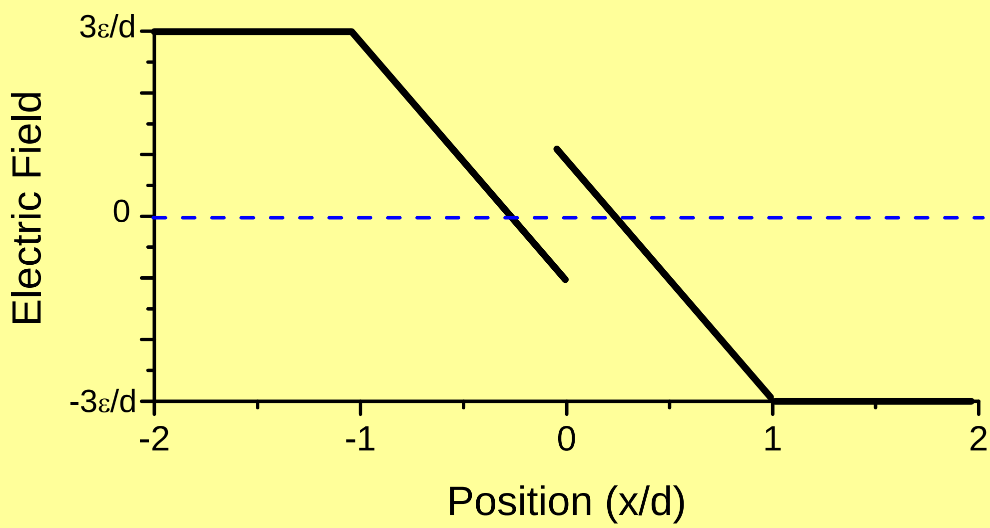
$$\vec{\mathbf{E}}=-\frac{\partial V}{\partial x}\hat{\mathbf{i}}=-\frac{\varepsilon}{d}\left(4\left(\frac{x}{d}\right)-1\right)\hat{\mathbf{i}}$$

Region IV: $x>d$

$$\vec{\mathbf{E}}=-\frac{\partial V}{\partial x}\hat{\mathbf{i}}=-\frac{3\varepsilon}{d}\hat{\mathbf{i}}$$

Problem 4 *continued*: Planar Potential

(b) Plot the electric field that you just calculated on the graph below. Be sure to properly label the y-axis (top and bottom) to indicate the limits of the magnitude of the E field!



(c) Consider a cylinder of cross-sectional area A and length $3d$ lying along (coaxial with) the x-axis, and centered at the origin (i.e. lying between $x = -3d/2$ and $x = 3d/2$). How much net charge does this cylinder contain?

This is a Gauss's Law question. The only flux will be through the endcaps. In this case the left endcap has a negative flux (E is positive, entering the pillbox), as does the right endcap (E negative, entering the pillbox).

$$q_{\text{enclosed}} = \varepsilon_o \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = -\varepsilon_o \left(E_{\text{left}} A + E_{\text{right}} A \right) = -\frac{6\varepsilon_o A\varepsilon}{d}$$

(d) Now shift this cylinder to the right so that it lies between $x = d$ to $x = 3d$. Now how much charge does it contain?

The right endcap (at $x = 4d$) still has a negative flux, but the left endcap has E leaving the pillbox and hence a positive flux, equal and opposite to the flux through the right endcap. So, the net flux is zero and so is the charge enclosed.

Problem 4 *continued*: Planar Potential

(e) Finally, we think about the distribution of charges that produces the above electric potential and electric field – a single sheet of charge (charge density σ) and a single slab of charge (charge density ρ). Using Gauss's Law arguments determine where this slab and sheet are and what their charge densities are. Make sure to state the Gaussian surface(s) that you use and what you conclude about the charge distribution by applying Gauss's Law to those surfaces.

We can immediately see where the sheet of charge is by looking for a discontinuity in the electric field. This happens at $x = 0$, so that is where the sheet is. To find σ, we just put a very short Gaussian pillbox around that area (just to the left and right of $x = 0$)

$$q_{\text{enclosed}} = \varepsilon_o \iint \vec{\mathrm{E}} \cdot d\vec{\mathbf{A}} = \varepsilon_o \left(E_{\text{left}} A + E_{\text{right}} A \right) = 2\varepsilon_o \frac{\varepsilon}{d} A = \sigma A \quad \Rightarrow \quad \boxed{\sigma = 2\varepsilon_o \frac{\varepsilon}{d}}$$

Note that this is a positive charge, as it creates a positive field to its right and a negative field to its left.

Now we have to find the slab of charge. The field will be changing in the slab and constant outside of it, so the slab lies between $x = -d$ and $x = d$. Since the slope of the field is the same throughout, the charge density is uniform, and we can use a Gaussian pillbox that avoids the sheet of charge, say from $x = -d^-$ to $x = 0^-$. In this case both endcaps have a negative flux (E field entering)

$$q_{\text{enclosed}} = \varepsilon_o \iint \vec{\mathrm{E}} \cdot d\vec{\mathbf{A}} = \varepsilon_o A \left(E_{\text{left}} + E_{\text{right}} \right) = -\varepsilon_o A \left(\frac{3\varepsilon}{d} + \frac{\varepsilon}{d} \right) A = \rho A d \quad \Rightarrow \quad \boxed{\rho = -4\varepsilon_o \frac{\varepsilon}{d^2}}$$

Problem 5: Capacitor (25 pts)

Consider two nested spherical shell conductors, of radii a and $4a$. The inner conductor has a charge -Q placed on it. An equal and opposite charge +Q is placed on the outer conductor.

What is the capacitance of this configuration?

To calculate the capacitance we place a +Q on one plate and –Q on the other (this was already done for us). We then calculate the electric field, integrate to get the potential difference, and then use the definition of capacitance to calculate it.

In a spherical geometry the electric field looks like the field from a point charge, so

$$\vec{\mathbf{E}} = -k_e \frac{Q}{r^2}\hat{\mathbf{r}}$$

To get the potential difference we integrate the electric field:

$$\Delta V = V(r = 4a) - V(r = a) = -\int_a^{4a} -k_e \frac{Q}{r^2} dr = -k_e \frac{Q}{r}\bigg|_a^{4a} = -k_e Q\left(\frac{1}{4a} - \frac{1}{a}\right) = \frac{3k_e Q}{4a}$$

Note the this is positive, as it should be since the $4a$ plate is positive and hence at a higher potential.

All that remains is to calculate the capacitance:

$$C = \frac{Q}{\Delta V} = \frac{Q}{\frac{3k_e Q}{4a}} = \frac{4a}{3k_e} = \frac{16}{3}\pi\varepsilon_o a$$