18.02 Fall 2007 – Problem Set 8, Part B Solutions

1. $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2y + \frac{1}{3}y^3)\,dx = \int_{x_1}^{x_2}\Big(x^2 f(x) + \frac{1}{3}f(x)^3\Big)\,dx,$ and

$$\iint_R (x^2+y^2)\,dA = \int_{x_1}^{x_2}\int_0^{f(x)}(x^2+y^2)\,dy\,dx = \int_{x_1}^{x_2}\Big[x^2y+\frac{1}{3}y^3\Big]_0^{f(x)}\,dx$$
$$= \int_{x_1}^{x_2}\Big(x^2 f(x) + \frac{1}{3}f(x)^3\Big)\,dx.$$

These two integrals are therefore equal.

2. a) For $\theta(x,y) = \tan^{-1}(y/x)$:

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = -\frac{y}{x^2+y^2};\quad \frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}\quad \Rightarrow\ \nabla\theta = \vec{F}.$$

b) Because $\theta(x,y) = \tan^{-1}(y/x)$ is well-defined in the right half-plane ($x>0$) and $\vec{F} = \nabla\theta$, the fundamental theorem implies $\int_C \vec{F}\cdot d\vec{r} = \theta(x_2,y_2) - \theta(x_1,y_1) = \theta_2 - \theta_1$.

c) $\displaystyle\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_1}\frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^{\pi}\frac{(-\sin\theta)(-\sin\theta)+\cos\theta\cos\theta}{\cos^2\theta+\sin^2\theta}\,d\theta = \int_0^{\pi}d\theta = \pi.$

Similarly, $\displaystyle\int_{C_2}\vec{F}\cdot d\vec{r} = \int_0^{-\pi}d\theta = -\int_{-\pi}^{0}d\theta = -\pi.$

(Or geometrically: length(C_1) = length$(C_2) = \pi$, $\vec{F}\cdot\hat{\boldsymbol{T}} = 1$ on C_1; $\vec{F}\cdot\hat{\boldsymbol{T}} = -1$ on C_2)

d) $\displaystyle\operatorname{curl}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = 0.$

e) $\vec{F}$ is defined everywhere except at the origin, but is not conservative over its entire domain of definition. Indeed, the two line integrals computed in (c) both run from (1,0) to $(-1,0)$ but they are not equal, so path-independence fails. On the other hand, $\vec{F}$ is conservative over the half-plane $x>0$, where $\vec{F} = \nabla\theta$ and the fundamental theorem of calculus gives a formula for the line integral involving only the values of θ at the end points (as seen in part (b)).

Note: the fact that $\operatorname{curl}\vec{F} = 0$ does not imply that $\vec{F}$ has a well-defined potential function everywhere! This would only be true if $\vec{F}$ were defined over the entire plane (or more generally, a *simply connected region*). In fact, we can find a potential function for $\vec{F}$ over smaller regions such as the right half-plane $x>0$ (namely, the polar angle θ). However, if we consider the entire plane with just the origin removed, the polar angle coordinate θ is not well-defined as a single-valued differentiable function: its value "jumps" by 2π as we go around the origin. This is what causes conservativeness to fail.

3. a) $\vec{F} = r^n(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}})$: $\displaystyle\operatorname{curl}\vec{F} = \frac{\partial(yr^n)}{\partial x} - \frac{\partial(xr^n)}{\partial y} = nyr^{n-1}\frac{x}{r} - nxr^{n-1}\frac{y}{r} = 0.$

(Recall $r = \sqrt{x^2+y^2}$ gives $r_x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$, and similarly $r_y = \dfrac{y}{r}$.)

b) If $g = g(r)$, then $g_x = g'(r)\,\dfrac{x}{r}$ and $g_y = g'(r)\,\dfrac{y}{r}$ (by chain rule), so $\nabla g = \dfrac{g'(r)}{r}(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}})$. We must find g such that $g'(r)/r = r^n$, i.e. $g'(r) = r^{n+1}$. Two cases:

$n \neq -2$: $g(r) = \dfrac{1}{n+2}\,r^{n+2}$. $\qquad n = -2$: $g(r) = \ln(r)$.