18.311 Pset 3

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Regular

TFP a22
a24

b14

<u>TFPa22</u>

$\psi = \psi(x, y, t)$. $\qquad \frac{dx}{dt} = u \quad \rightarrow x = ut + c$

$\psi_t + u\psi_x + v\psi_y = 0 \qquad \frac{dy}{dt} = v \quad y = vt + c.$

$$\frac{d\psi}{dr} = \underbrace{\left(\frac{dx}{dr}\right)}_{=u} \frac{\partial \psi}{\partial x} + \underbrace{\left(\frac{dy}{dr}\right)}_{=v} \frac{\partial \psi}{\partial y} + \underbrace{\left(\frac{dt}{dr}\right)}_{=1} \frac{\partial \psi}{\partial t} = 0.$$

Since $\frac{d\psi}{dr} = 0$, then ψ is const Along a characteristic curve

$$\psi = F(x_0, y_0)$$

- $\frac{dt}{dr} = 1 \quad \Rightarrow r = t.$

- $\frac{dx}{dr} = u. \quad \Rightarrow x = ur + x_0 = ut + x_0$

  $x_0 = x - ut \quad \Rightarrow x = ut + x_0. \Rightarrow \frac{dx}{dt} = u$

  Since ψ is constant along $\psi = F(x_0, y_0)$, just rearrange + differentiate.

- $\frac{dy}{dr} = v \quad \Rightarrow y = vr + y_0 = vt + y_0$

  $y_0 = y - vt \quad \Rightarrow y = vt + y_0 \quad \Rightarrow \frac{dy}{dt} = v.$

$\psi = F(x_0, y_0)$

Since we can say the characteristic is $\psi = F(x_0, y_0)$ then ψ is constant along this curve, You can find $x_0 = x - ut$, $y_0 = y - vt$.

Differentiate with respect to x + y: $\boxed{\frac{dx}{dt} = u \quad \frac{dy}{dt} = v}$

TFPa 24

$$x^2\psi_x + xy\psi_y = \psi^2$$

$\psi = 1$
$x = y^2$.

Characteristics $\frac{dx}{ds} = x^2$

$\frac{dy}{ds} = xy$

A) $x = \xi^2$
$y = \xi$
$-\infty < \xi < \infty$

initial condition

$x = \xi^2$ $\longrightarrow$ $x = \frac{1}{s_0 - s}$ $\Rightarrow$ $x = \frac{\xi^2}{1 - \xi^2 s}$
$y = \xi$ $\quad$ $y = C_0 x(s)$ $\quad$ $y = \frac{\xi}{1 - \xi^2 \cdot s}$
$s = 0$
$-\infty < \xi < \infty$

solve
If $x = 0$.
$\cancel{x^2\psi_x} + \cancel{xy\psi_y} = \psi^2$
$\Rightarrow$ $0 = \psi^2$.
$\Rightarrow$ $\psi = 0$.

If $x < 0$: $s > \frac{1}{\xi^2} \Rightarrow$ No!

$\Downarrow$
If $s = 0$:
$x = \xi^2$
$y = \xi$

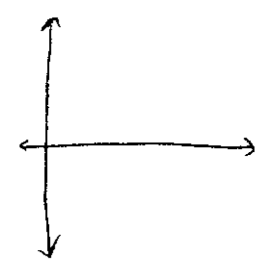
$$\frac{x}{y} = \frac{\xi^2}{\xi} = \xi$$

$s \to -\infty$, $x \to 0$
$y \to 0$.

$s \to \frac{1}{\xi^2}$, $x \to \infty$
$y \to \infty$ (if ξ positive)
$y \to -\infty$ (if ξ negative)

B) The characteristics are semi-infinite rays such that $x > 0$, and

$-\infty < \xi < 0$ OR $0 < \xi < \infty$



For $\xi = 0$, $0 = \frac{x}{y} \Rightarrow x = 0 \Rightarrow$ then the equation $x^2\psi_x + xy\psi_y = \psi^2$ becomes $0 = \psi^2$ and the constraint $\psi = 1$ is not satisfied when $s = 0$.

(C) $\frac{x}{y} = \xi$; $x = \frac{\xi^2}{1-\xi^2 \cdot s}$ $\Rightarrow x - x\xi^2 s = \xi^2$

$$\Rightarrow x\xi^2 s = x - \xi^2$$

$$s = \frac{x - \xi^2}{x\xi^2}$$

$$s = \frac{x - \frac{x^2}{y^2}}{x \cdot \frac{x^2}{y^2}} = \frac{\cancel{x}\left(1 - \frac{x}{y^2}\right)}{\cancel{x} \cdot \frac{x^2}{y^2}}$$

$$s = \frac{1 - \frac{x}{y^2}}{\frac{x^2}{y^2}} = \frac{\frac{y^2 - x}{y^2}}{x^2/y^2}$$

$$\boxed{s = \frac{y^2 - x}{x^2}}$$

This only applies for $x > 0$ and $y \neq 0$.

$$\frac{d\psi}{ds} = \psi^2$$

Since $\psi = 1$ at $s = 0$ $\Rightarrow$ $\boxed{\psi = \frac{1}{1-s}}$

Since $s = \frac{y^2 - x}{x^2}$ $\Rightarrow \psi = \frac{1}{1 - \left(\frac{y^2 - x}{x^2}\right)} = \frac{1}{\frac{x^2 - y^2 + x}{x^2}}$

$$\Rightarrow \boxed{\psi = \frac{x^2}{x^2 - y^2 + x}}$$

(D) Solution ψ not defined when $x > 0$ and $y \neq 0$.

not defined when $s \geq 1$ b/c when $s = 1$, $\psi = 0$ [singular]

Since $s < 1$ then $\frac{y^2 - x}{x^2} < 1$. so $y^2 - x < x^2$

$$\Rightarrow y^2 < x^2 + x.$$

Therefore the solution is defined when $\begin{cases} x > 0 \\ y \neq 0 \\ y^2 < x + x^2 \end{cases}$