EFFECTIVE ELASTIC PROPERTIES OF CRACKED SOLIDS

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ABSTRACT

The classical problem of effective elastic properties of cracked solids is critically reviewed. The predictions of the existing schemes are directly checked by computer experiments on a large number of sample arrays of interacting cracks. The main finding is that the approximation of non-interacting cracks (the simplest one) actually remains accurate at high crack densities and strong interactions. The underlying reason is that the competing interaction effects of shielding and amplification cancel each other (provided the mutual positions of cracks are random).

INTRODUCTION

This paper is a brief summary of the work presented in detail in [1,2].

The theory of effective elastic properties of cracked solids predicts degradation of stiffness, development of anisotropy, changes in wavespeeds caused by microcracking. Therefore, it is of obvious interest for solid mechanics, materials science, geophysics; it also constitutes a theoretical basis of various NDE techniques.

Most of the approaches to this problem have roots in the effective media theories of physics. For example, the approximation of *effective matrix* (a self-consistent scheme, in terminology of the mechanics of solids with inhomogeneities), in which a representative inhomogeneity is placed into the effective matrix, was first used in the problems of electrostatics in the XIX-th century. The method of *effective field*, and its simplest version - Mori-Tanaka's method - in which a representative inhomogeneity is embedded into the effective stress field, was first developed in 1940's - 1950's in connection with wave propagation problems.

At the same time, cracks constitute a distinctly special kind of inhomogeneities: they occupy no volume; stress fields generated by them are quite complex and have strong orientational dependence. As a result, cracked solids have many special features: the choice of crack density parameter is non-trivial; the effective properties are, generally, anisotropic; the approximation of non-interacting cracks has a wider than expected range of applicability. Many of the approximate methods, used in the mechanics of general two phase materials, degenerate and cannot be applied to a cracked solid. For example, bounds for the effective moduli, as a rule, cannot be established; Mori-Tanaka's method yields predictions coinciding with the approximation of non-interacting cracks.

Several approximate schemes for cracked solids have been suggested in literature. Their predictions are substantially different. A researcher or an engineer, trying to *use* the theory, may be confused by the choice of several models, yielding different results.

The present work attempts to introduce some clarity into this problem. We briefly review the simplest approach to the problem - the approximation of non-interacting cracks (developed as early as 1960, by Bristow[3]). We then present the results of extensive computer experiments, in which the problem was directly solved for a large number of sample arrays of cracks. The main finding is that the approximation of non-interacting cracks (the simplest one) actually remains accurate at high crack densities and strong interactions. The underlying reason is that the competing interaction effects of shielding and amplification cancel each other (provided the mutual positions of cracks are random).

THE APPROXIMATION OF NON-INTERACTING CRACKS

This simplest approximation, known since 1960, is briefly reviewed here.

In the assumption of non-interacting cracks (in the *isotropic* matrix material) the effective moduli can be found exactly, for an arbitrary crack orientation statistics, in both *2-D* and *3-D*. It is the simplest (and the only non-controversial) approximation to the problem; at the same time, it has a wider than expected range of applicability (see the results presented below). It is often called the approximation of small crack density (in fact, the two names are usually used as synonyms). We refrain, however, from using this term, since *these two assumptions are generally not equivalent* [1,2].

In the approximation of non-interacting cracks, each crack is regarded as an *isolated* one: it is embedded into the externally applied stress field σ and does not experience any influence of other cracks. Then the average crack opening displacement of each crack (COD) $\langle \mathbf{b} \rangle$ for each crack is expressed, in a simple way, in terms of $\mathbf{n}\cdot\sigma$ where $\mathbf{n}$ is a unit normal to the crack. Mutual positions of cracks do not matter in this approximation, hence averaging over a crack array is reduced to summation over orientations. Since the change in compliance $\Delta\mathbf{M}$ due to cracks is a sum of the isolated cracks' contributions, *the compliance is linear in crack density parameter*. Elastic *stiffnesses*, obtained by inversion of compliances, will, of course, be non-linear functions of the crack density, of the form $(1 + C\rho)^{-1}$. Linearization of this form corresponds to the assumption of small crack density. If, however, this form is left as it is, then, as seen below, the results remain accurate at high crack densities.

The problem of effective elastic moduli in the approximation of non-interacting cracks was first solved, for randomly oriented cracks, in both *2-D* and *3-D*, by Bristow (1960).

Since averaging is reduced simply to integration over orientations, such calculations are easily repeated for *an arbitrary* (non-random) *orientational distribution*. Such calculations were done by a number of authors, see [1,2] for a review. For the arbitrary orientational distribution of cracks, the results can be written, in the most general form, in terms of the *crack density tensor* α (introduced by Vakulenko and Kachanov,1971, Kachanov,1972, and, in the corrected and most general form, by Kachanov,1980; see reviews [1,2] for references):

$$\alpha = \frac{1}{A}\sum l^{(i)2}\,\mathbf{n}^i\mathbf{n}^i \qquad \text{(rectilinear cracks of lengths } 2l^i, A \text{ is the representative area)} \qquad (1)$$

The use of the crack density tensor yields, in a unified way, results for any orientational distribution of cracks, without averaging over orientations. It also establishes the symmetry of the effective properties: since α is a symmetric second rank tensor and it enters the elastic potential through a simultaneous invariant with the stress tensor, the effective properties are always *orthotropic* (rectangular symmetry), with the axes of orthotropy coinciding with the principal axes of α. Moreover, the orthotropy is of a special, simplified type [1,2]. This fact is exact in 2-D and approximate (with good accuracy) in *3-D*.

In the simplest cases of *randomly oriented* cracks (isotropic effective properties) and *parallel* cracks (transversely isotropic effective properties) the results, for a *2-D* solid, are as follows.

$$E = E_0(1 + \pi\rho)^{-1}; \qquad \nu = \nu_0(1 + \pi\rho)^{-1} \quad \text{(random orientations)} \qquad (2)$$

$$E_1 = E_0(1+2\pi\rho)^{-1}; \qquad G = G_0[1+(2\pi G_0/E_0)\,\rho\,]^{-1} \quad \text{(parallel cracks)} \qquad (3)$$

where E_0, G_0 and ν_0 denote the moduli of the matrix without cracks (Young's modulus, shear modulus and Poisson's ratio, respectively), and ρ is the conventional scalar crack density parameter introduced by Bristow [3]:

$$\rho = \frac{1}{A}\sum l^{(i)2} \qquad (4)$$

Note that ρ is the linear invariant of the crack density tensor α.

In the *3-D* case, the analysis is similar; a complicating factor is that the orthotropy of the effective properties holds only approximately, although with good accuracy [1,2].

The fact of orthotropy allows one to establish a "natural" coordinate system - principal axes of α - in which the matrix of elastic moduli has its simplest form (orthotropic in *2-D* and approximately orthotropic in *3-D*).

The effect of dimensionality should be mentioned: comparison of the stiffness reduction due to cracks in*2-D* and and *3-D* shows that, at the same crack density, the reduction is *substantially weaker in 3-D*.

ANALYSIS OF CRACK INTERACTIONS

Solutions for *deterministic* arrays of arbitrary geometry, in both *2-D* and *3-D*, can be produced, by relatively simple means and with good accuracy, using the method of analysis of crack interactions developed by Kachanov in 1985, 1987 (see [1,2] for details). This method is briefly outlined below in the *2-D* version

The problem of a linear elastic solid with N cracks subjected to stress σ at infinity is equivalent to the problem with crack faces loaded by tractions $\mathbf{n}^i \cdot \sigma$ ($i = 1,...,N$) and stresses vanishing at infinity. The latter problem can be represented as a superposition of N sub-problems, each containing only *one* crack, but loaded by *unknown* tractions, comprising, in addition to the σ- induced traction, extra tractions accounting for interactions (to be found). Thus, traction $\mathbf{t}^i$ in the *i*-th sub-problem is a sum of the σ -induced traction $\mathbf{n}^i \cdot \sigma$ and tractions induced by the other cracks in the remaining sub-problems along the line l^i

$$\mathbf{t}^i(\xi^i) = \mathbf{n}^i \cdot [\sigma + \sum_{j \neq i} \sigma^j(\xi^i)] \quad \equiv \mathbf{t}^{i\,(0)} + \Delta\mathbf{t}^i(\xi^i) \tag{5}$$

where ξ^i is a current point on l^i and σ^j is the j-th crack-generated stress.

The key simplifying assumption of our method is that σ^j is taken as generated by a *uniform average* traction $<\mathbf{t}^j>$, yet unknown; the impact on the *i*-th crack of traction non-uniformities $\mathbf{t}^j - <\mathbf{t}^j>$ along the j-th crack is neglected. Decomposing the average $<\mathbf{t}^j>$ into a sum of normal and tangential averages, $<p^j>$ and $<\tau^j>$, we have

$$\mathbf{t}^i(\xi^i) = \mathbf{n}^i \cdot \sigma + \mathbf{n}^i \cdot \sum_{j \neq i} [<p^j> \sigma^j_n(\xi^i) + <\tau^j> \sigma^j_\tau(\xi^i)] \tag{6}$$

where σ^j_n and σ^j_τ are the "standard" stress fields, generated (in an infinite solid) by a single j-th crack loaded by uniform tractions of unit intensity, normal and shear, correspondingly. These fields (expressed in elementary functions) are available in fracture mechanics literature.

The unknown quantities in (6) are the average tractions $<\mathbf{t}^j>$ on cracks. They are found by averaging (6) along the *i*-th crack line and thus interrelating them by a system of N vectorial ($2N$ scalar) linear algebraic equations

$$<\mathbf{t}^i> = \mathbf{n}^i \cdot \sigma + \sum_{j=1}^{N} (\Lambda^{(ji)} - \delta^{ji}\, \mathbf{I}) \cdot <\mathbf{t}^j> \tag{7}$$

where δ^{ji} is Kronecker's delta and the (tensorial) *transmission* Λ *- factors* characterize transmission of the average tractions from one crack onto another ($\Lambda^{(ji)} \cdot <\mathbf{t}^j>$ is the average traction induced on a line l^i in a continuous material by the j-th crack loaded by a uniform traction $<\mathbf{t}^j>$). The Λ - factors are expressed in terms of line integrals of elementary functions and are easily calculated.

Solving (7) for average tractions, one obtains:

$$<\mathbf{t}^i> = \sum_{j=1}^{N} \Omega^{(ji)} \cdot <\mathbf{t}^j> = \sum_{j=1}^{N} \Omega^{(ji)} \mathbf{n}^j : \sigma \tag{8}$$

where $\Omega^{(ji)} = (2\delta^{ji}\mathbf{I} - \Lambda^{(ji)})^{-1}$. The case of non-interacting cracks is recovered by assuming $\Lambda^{(ji)} = \delta^{ji}\mathbf{I}$; then $\mathbf{t}^i = \mathbf{n}^i \cdot \sigma$.

Finding the average displacement discontinuity $<\mathbf{b}^i>$ as approximately proportional to the average traction $<\mathbf{t}^i>$ (Kachanov, 1987, see [1,2] for details) we obtain *an explicit expression for the effective compliancet ensor:*

$$\mathbf{M} = \mathbf{M}^o + (2\pi/A\,E_o') \sum_{i,j=1}^{N} l^{(i)2}\, \Omega^{(ij)}_{\{} \mathbf{n}^i \mathbf{n}^j{}_{\}} \tag{9}$$

where $\mathbf{M}^o$ is the compliance of the material without cracks and subscript braces denote all the appropriate symmetrizations (with respect to $p \leftrightarrow q$, $r \leftrightarrow s$, and $pq \leftrightarrow rs$ for M_{pqrs}). It incorporates information not only on the crack orientations and sizes, but, also, on their *mutual positions* (all these factors are reflected in Λ 's, and, thus, in Ω's).

The method yields results for any given arrangements of cracks. Such results constitute direct computer experiments on sample arrays and can be used to verify various approximate schemes. The method is accurate up to quite close spacings between cracks so that the results are accurate up to high crack densities (as confirmed by smallness of corrections provided by the alternating technique).

COMPUTER EXPERIMENTS

Sample *2-D* crack arrays contained 25 cracks and were generated with the help of random number generator, as realizations of certain crack statistics.

Two *orientation statistics* were assumed: randomly oriented cracks and parallel cracks; for each of them, six crack densities were assumed: $\rho = 0.10$; 0.15; 0.20; 0.25; 0.30 and 0.35 (in *2-D*, the densities of 0.25-0.35 can be considered as quite high).

Fifteen sample arrays were considered for each density.

Locations of cracks within the representative area were random, i.e. positions of crack centers were uncorrelated. For *parallel* cracks, generation of such arrays was straightforward. For *randomly* oriented cracks, crack intersections had to be avoided; this was achieved by generating cracks successively and discarding the newly generated one if it intersects the already existing cracks and generating it again. Although such procedure, strictly speaking, violates the condition that crack centers are uncorrelated, we assume that it does not create errors of a *systematic* sign.

The sample arrays usually contained several cracks with spacings between them much smaller than the crack sizes. To ensure the accuracy of the results at such small spacings, the method outlined above was supplemented by the *alternating technique* (stress "feedbacks"), until the three digits accuracy for the effective moduli was reached. It actually produced only an insignificant correction (smaller than 3-4%) of the method's results, confirming the accuracy of the method.In order to limit the number of iterations in the alternating technique, the spacings between cracks were not allowed to be overly small. Namely, in the case of randomly oriented cracks, they were not smaller than 0.02 of the crack length; in the case of parallel cracks, they were not smaller than 0.15 of the crack length for those crack pairs that had a significant "overlap", and not smaller than 0.02 of the crack length otherwise.

In the case of randomly oriented cracks, each array was examined for *isotropy*: the effective Young's modulus was calculated in two perpendicular directions and only those arrays were kept for which the difference between these two moduli was smaller than 2%; the Young's modulus was then taken as the average over these two values. (Note that preferential orientations causing up to 5-7% variation of E with direction are usually not discernible by a "naked eye").

Fig.1 shows some of the generated arrays.

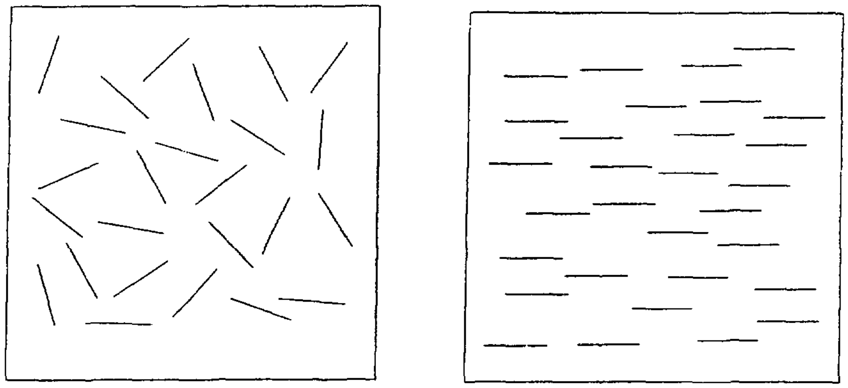

FIG.1. Two of the crack arrays examined (randomly oriented, $\rho = 0.25$ and parallel, $\rho = 0.30$).

The representative area A is assumed to constitute a part of a statistically homogeneous field of cracks (extending all the way to the external boundary) and stresses at the boundary Γ of A are assumed constant and equal to the remotely applied ones. This assumption is rigorously correct for non-interacting cracks, when cracks inside A do not experience any influence of those outside A. For *interacting* cracks, stresses fluctuate along Γ. The errors due to the mentioned assumption depend on the size of the sample. They cause *scatter* of results from one realization of the crack field statistics to another, but are not expected to produce errors of a *systematic* sign. Figs. 2 and 3 show that the scatter is relatively small.

RESULTS

Figs.2 and 3 show results for the Young's modulus for randomly oriented and parallel arrays. Vertical bars show scatter of the results from one sample to another.

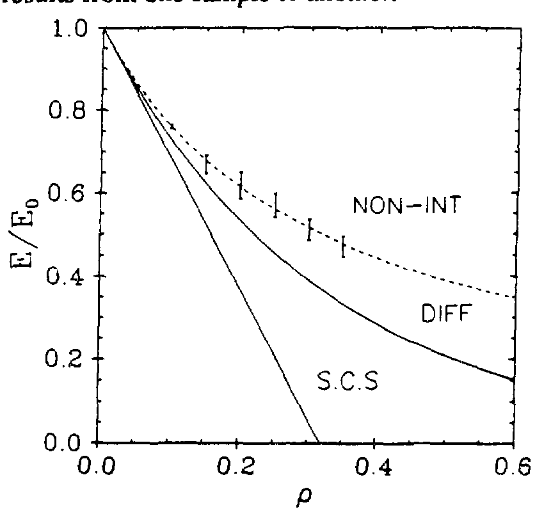

FIG.2. Effective Young's modulus, randomly oriented cracks: results for sample arrays vs predictions of the approximation of non-interacting cracks, self-consistent and differential schemes.

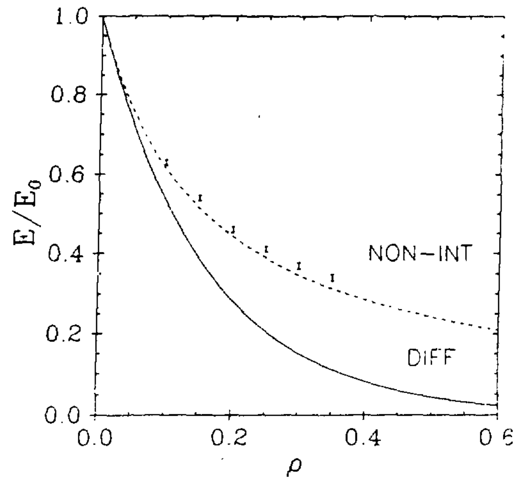

FIG.3. Effective Young's modulus, parallel cracks: results for sample arrays vs the approximation of non-interacting cracks and the differential scheme.

For *randomly oriented* cracks, the approximation of non-interacting cracks provides surprisingly good results, well into the domain of strong interactions where this approximation is usually considered inapplicable.

For *parallel* cracks, the approximation of non-interacting cracks still provides good results. However, there is a slight but distinguishable tendency for the stiffening overall effect of interactions, indicating a slight dominance of the shielding mode of interactions. The reason for this tendency is not fully clear. We suspect that the prohibition for spacings between substantially overlapping cracks to be smaller than 0.15 of the crack length (that made the sample arrays slightly non-random) may have created a slight "bias" in favour of shielding.

We also calculated the shear modulus for parallel crack arrays (for two arrays, $\rho = 0.15$ and 0.20). It differed from the predictions for non-interacting cracks by less than 1.5%.

Preservation of orthotropy for interacting cracks. As discussed above, for *non-interacting* cracks orthotropy (coaxial to α) is a rigorous result in *2-D*. Consistently with the fact that, due to cancellation of competing effects, the approximation of non-interacting cracks remains accurate at high crack densities, *orthotropy* can be expected to hold at high ρ.

We examined *deviation from orthotropy* in sample arrays of non-randomly oriented (but randomly *located*) cracks, by solving the interaction problem for two families of parallel cracks of equal density inclined at 30^0 to each other. At $\rho = 0.24$ (significant interactions) the deviation from orthotropy produced by interactions was very small: in the principal coordinate system of α the non-orthotropic compliances ΔM_{1112} and ΔM_{2212} were 10^{-3}- 10^{-4} of the orthotropic compliances, i.e. within the accuracy of the results.

This means that *characterization of crack arrays by the crack density tensor remains adequate even at high crack densities* when interactions are strong, and implies that the "natural" coordinate system (coaxial to α) remains the system of choice for calculation of the effective moduli.

DISCUSSION

The results show that, well into the interval of crack densities (ρ up to 0.35) where the interactions become significant, *the approximation of non-interacting cracks remains quite accurate*.

The underlying reason for this accuracy is *not* that the interactions can be neglected, but that the competing effects of stress shielding and stress amplification *cancel* each other.

This is a direct consequence of the fact that introduction of traction-free cracks does not change the average stress in the solid (provided *tractions* are prescribed on the boundary). Therefore, if a certain number of "new" cracks is introduced into the environment of the preexisting ones in a *random fashion*, these new cracks will, *on the average*, experience no effect of the preexisting cracks. In the language of stress superpositions, the additional tractions Δt^i will be of both amplifying and shielding nature on different cracks; on the average, their impacts will cancel each other.

We emphasize that this conclusion assumes the *absence of "bias"* in crack statistics towards either amplifying or shielding arrangements. Otherwise (in "ordered" arrays, for example) the impact of interactions can be very large, both in the direction of "softening" and "stiffening".

The reported results are for the *2-D* configurations. The approximation of non-interacting cracks will remain accurate in *3-D* as well, due to the same mechanism of cancellation of shielding and amplification effects. Moreover, we expect the scatter of the results to be smaller, since interactions are weaker in *3-D* (see [1,2]).

ACKNOWLEDGEMENT

This work was supported by the U.S. DOE, through grant to Tufts University.

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