A Regression Analysis of Economic Performance

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Introduction

This presentation will analyze data from an economic study, rating the performances of select firms using six different independent variables and one response variable. All of the variables used in this study are common financial measurements found daily in the Financial world.

The data and some of the information in this presentation comes from a paper titled "Using Linear Regression In The Analysis of Financial-Economic Performances." [1]

Introduction

The independent variables used in this study include:

- Good Money (GM_i)
- Gross Operating Surplus (GOS_i)
- Operating Gross Margin Rate (R_{mb})
- Cashflow/Turnover (CFR_i)
- EBIT Margin ($\mathrm{M}_{\mathrm{EBIT}}$)
- Total Debt Turnover (R_{Dt})

and the response variable is:

- Economic Rate of Return (R_{ec})

Data

Firm	Economic Rate of Return	Good Money	Gross Operating Surplus	Operating Gross Margin Rate	Cash Flow/ Turnover	EBIT Margin	Total Debt Turnover
1	0.056	0.442	10383120	0.087	0.057	0.036	3.513
2	0.076	0.022	9869544	0.070	0.027	0.042	2.565
3	0.227	1.111	26685671	0.249	0.136	0.152	5.201
4	0.041	0.003	12138277	0.110	0.024	0.076	1.400
5	0.063	0.060	12032959	0.096	0.056	0.037	1.704
6	0.077	0.004	10798008	0.083	0.012	0.080	0.905
7	0.032	0.007	10696810	0.112	0.052	0.047	1.601
8	0.281	2.452	23165418	0.212	0.194	0.165	9.502
9	0.047	0.006	5421437	0.058	0.027	0.025	1.807
10	0.174	0.043	13324530	0.132	0.063	0.117	1.937
11	0.127	0.062	14044440	0.154	0.039	0.105	1.355
12	0.110	0.006	13390931	0.149	0.058	0.096	1.344
13	0.060	0.040	1492526	0.024	0.007	0.012	3.051
14	0.245	0.025	20313159	0.227	0.046	0.179	1.358

Data

Firm	Economic Rate of Return	Good Money	Gross Operating Surplus	Operating Gross Margin Rate	Cash Flow/ Turnover	EBIT Margin	Total Debt Turnover
15	0.047	0.044	3735283	0.053	0.023	0.017	3.479
16	0.122	0.087	13251207	0.153	0.083	0.094	3.600
17	0.004	0.095	3989975	0.046	0.057	0.004	2.723
18	0.135	0.177	14794181	0.222	0.034	0.136	0.703
19	0.163	0.165	11972850	0.144	0.087	0.115	2.385
20	0.060	0.044	3873616	0.082	0.023	0.030	1.400
21	0.066	0.009	9019110	0.108	0.059	0.062	1.338
22	0.116	0.164	3811847	0.056	0.003	0.043	1.878
23	0.090	0.004	14018015	0.207	0.110	0.073	1.116
24	0.150	0.074	11856849	0.159	0.061	0.099	1.243
25	0.136	0.261	8376274	0.194	0.064	0.078	1.706
26	0.137	0.025	15355836	0.279	0.092	0.132	0.795
27	0.229	0.058	19331352	0.287	0.180	0.139	2.343
28	0.120	0.346	4741264	0.073	0.039	0.046	3.295

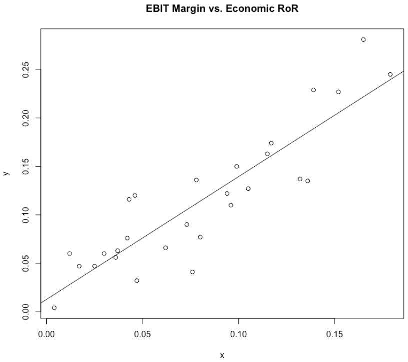
Regression Assumptions

Before we finish building a model we need to establish the regression assumptions. These assumptions include:

1. The random error is distributed normally with mean 0 and variance σ^2.
2. Each observation is independent of one another.
3. The variance is constant for any value of x.

Building Model

For our simple linear regression analysis I am going to choose the variable M_{EBIT} to be the regressor and the have the response variable be R_{ec}. Both of these measurement are dependent on the measurement EBIT. Here is a scatterplot of the two variables.



Building Model

From the scatterplot there seems to be evidence of an upward linear relationship between X and Y. Next, let's estimate the regression coefficients for our model above. To estimate β_0 we can use

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

and we can estimate β_1 using

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

where

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

Building Model

After performing these calculations we obtain $\hat{\beta}_0 = 0.01271747$ and $\hat{\beta}_1 = 1.267282$. So, the estimated regression equation is:

$$\hat{y} = 0.013 + 1.267\hat{x}$$

ANOVA Test

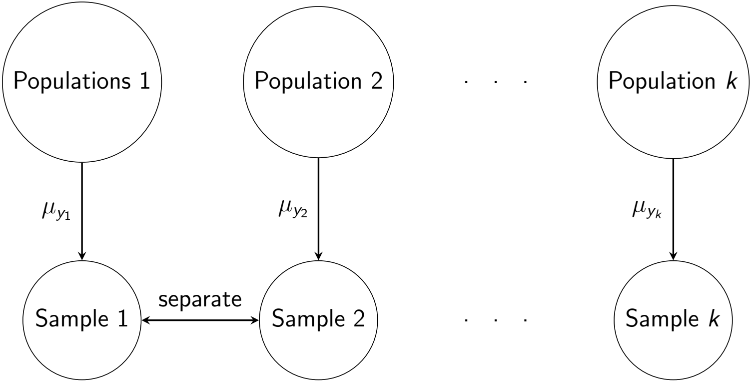
Let's test for significance of regression. There are a few ways to do this; let's start by constructing an ANOVA table. Each observation can be treated as its own population. For an analysis of variance test we are testing the following hypothesis:

$$H_0: \mu_{y_1} = \mu_{y_2} = \cdots = \mu_{y_k}$$
$$H_1: \text{at least one } \mu_{y_i} \text{ is different}$$

$$\iff$$

$$H_0: \sigma_B^2 = \sigma_W^2 = \sigma^2$$
$$H_1: \sigma_B^2 > \sigma_W^2 = \sigma^2$$

ANOVA Test



Populations 1
Population 2
Population k
μ_{y_1}
μ_{y_2}
μ_{y_k}
Sample 1
separate
Sample 2
Sample k

ANOVA Test

Now that we have established our hypothesis we can compute the ANOVA table in R. Below is the ANOVA table:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Regression	0.1039	1	0.1039	100.7644
Residual	0.0268	26	0.0010	
Total	0.1307	27		

Our test statistic for the ANOVA test is 100.7644. To compute a p-value, we can use a built-in R function to obtain 1.958e-10. This number is very small and we have a significant test at very small levels of significance. At the 1% level of significance we can conclude there is strong evidence for linear regression.

Test for Regression

There is another way to test for significance of linear regression under a simple linear regression model. We can calculate a confidence interval and check to see if 0 is contained in that interval. Let's find a 99% confidence interval to test for significance of regression.

Test for Regression

The estimated parameter $\hat{\beta}_1$ is normally distributed with mean β_1 and variance σ^2/S_{xx}. Since the variance is unknown we can use the mean squared error to estimate it. The confidence interval for the estimated β_1 parameters is:

$$\hat{\beta}_1 - t_{\alpha/2,n-2} \cdot s_e\left(\hat{\beta}_1\right) \leq \beta_1 \leq \hat{\beta}_1 + t_{\alpha/2,n-2} \cdot s_e\left(\hat{\beta}_1\right)$$

Using R, our 99% confidence interval for the regression parameter β_1 is:

$$0.9164791 \leq \beta_1 \leq 1.6180857$$

Test for Regression

After using these two different methods, I think it is safe to assume there is significant evidence for linear regression. In other words, we are 99% confident that the average slope of the regression model will be in the interval

$$(0.9164791, 1.6180857)$$

We could also perform a t-test to check for significance of regression, but this is already enough evidence.

Making Predictions

Let's try to predict the average Economic Rate of Return given that the EBIT Margin is 10%.

$$E[y|x = 0.1] = 0.013 + 1.267(0.1) = 0.139$$

In fact, we can make many predictions. Let's make many predicts for the average Economic Rate of Return and future values of Economic Rate of Return.

Making Predictions

Below let's define the confidence intervals for both the average Economic Rate of Return and future values of Economic Rate of Return. For the average Economic Rate of Return:

$$\hat{y}_0 - t_{\alpha/2,n-2} \cdot s_e\left(\hat{y}_0\right) \leq \mu_{y|x_0} \leq \hat{y}_0 + t_{\alpha/2,n-2} \cdot s_e\left(\hat{y}_0\right)$$

and the confidence interval for future values of Economic Rate of Return:

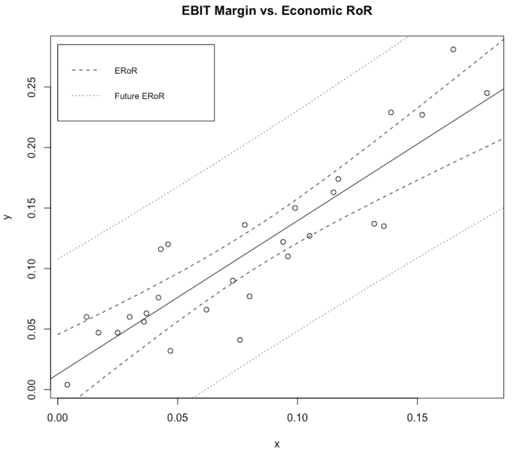
$$\hat{y}_0 - t_{\alpha/2,n-2} \cdot s_e\left(\Psi\right) \leq y_f \leq \hat{y}_0 + t_{\alpha/2,n-2} \cdot s_e\left(\Psi\right)$$

The respective standard errors are:

$$s_e\left(\hat{y}_0\right) = \sqrt{\left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right) MS_{Res}} \qquad s_e\left(\Psi\right) = \sqrt{\left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right) MS_{Res}}$$

Making Predictions

Now we can construct bounds for a 99% confidence interval for both of these quantities. The graph is below:



R-Squared

Beyond the confidence intervals, there are other ways to check for linearity. Another statistic is called the R-squared value. This measure is the ratio between the sum of squares regression and the total sum of squares. Another way to think of this measure is like:

$$R^2 = \frac{SS_R}{SS_T} = \frac{SS_T - SS_{Res}}{SS_T}$$

In other words, the more error there is in the model (SS_{Res}), the lower the R-squared value will be [3]. The R-squared value for this model is 0.795. This is a relatively high value, but there can be room for improvement. Since, this is predicting the economic performance of firms, this is acceptable.

Building Model

Now let's build a multiple linear regression model.

For our multiple linear regression analysis, we will introduce five more regressors. The model will include all of the regressors from Table 1 with Economic Rate of Return as the response. We will use matrix formulation for the remainder of the regression analysis. The model is defined as:

$$\vec{y} = X\vec{\beta} + \vec{\varepsilon}$$

where,

Building Model

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{16} \\ 1 & x_{21} & x_{22} & \cdots & x_{26} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \cdots & X_{n2} & \cdots x_{n6} \end{bmatrix}$$

$$\vec{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_6 \end{bmatrix}, \qquad \vec{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Problems With Software

When building the model there was an issue because the inverse matrix was turning singular in R. Now the matrix was not actually singular, but was computationally singular, meaning it involved precision that R cannot handle. To avoid this issue I will scale the data, then unscale after I build the model. Using the coefficients γ_i and the scaled coefficients and β as the unscaled coefficients, we can establish the relationship:

$$\beta_0+\beta_1 x_1+\cdots+\beta_p x_p = \gamma_0+\gamma_1\left(\frac{x_1-\mu_1}{\sigma_1}\right)+\cdots+\gamma_p\left(\frac{x_p-\mu_p}{\sigma_p}\right)$$

Building Model

Using all of this, we are finally able to build our model:

$$y = -3.390 - 1.227x_1 - 4.845x_2 - 4.616x_3 + 2.256x_4 + 1.635x_5 + 1.364x_6$$

Multicollinearity

I am speculating the reasons why this complication happened was because of highly correlated data. This issue was not a surprise as it was mentioned in the paper the data came from [1]. This concept is called multicollinearity. This is a problem that happens when the data is highly correlated and can be fixed by analyzing the the Variance Inflation Factors (VIF). Below are the VIFs for this model, calculated in R.

```
       GM       GOS      OGMR       CFR     EBITM       TDT
 8.114270  6.843655 11.109788  6.427460  8.295903 10.179212
```

Multicollinearity

The rule of thumb for recognizing multicollinearity is VIF values above 10. Immediately we can see the regressors Operating Gross Margin Rate and Total Debt Turnover are potential risk for the model, so we will rid these of the model. Our new model now includes four regressors. Now let's rebuild our model without these:

$$y = 2.537 + 3.234x_1 - 5.002x_2 + 3.124x_4 + 1.505x_5$$

ANOVA Test

Next, let's compute an ANOVA table to see if our model is significant. We can compute these all manually in R.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Regression	0.1148	4	0.0.0287	41.3876
Residual	0.0159	23	0.0007	
Total	0.1307	27		

Our F statistic is 41.3876, which corresponds to a p-value of 3.4488e-10. At the 1% level of significance, we can conclude we have significant evidence of linear regression using the regressors listed above.

Adjusted R-Squared

The adjusted R-squared is a very helpful measure when more regressors are introduced to the model. The R-squared value increases as more regressors are introduced to the model, regardless of the importance of them. However, the adjusted R-squared addresses this issue by adjusting for this (dividing by degrees of freedom). The adjusted R-squared is defined as:

$$R^2_{\text{adj}} = 1 - \frac{SS_{Res}/n-p}{SS_T/n-1}$$

In this model we have both a high adjusted R-squared value and a high R-squared value. These can be interpreted in the following way. We have an adjusted R-squared value of 0.8568, so around 86% of the data is accounted for by the model.

Test for Regression

Now that we have established evidence of regression, let's test each coefficient using a t-test. Let's test whether the coefficient for Cash Flow contributes to the response. Let's test the following hypotheses:

$$H_0: \ \beta_4 = 0$$
$$H_a: \ \beta_4 \neq 0$$

The standard error can be calculated from the covariance matrix and the mean squared error. The standard error for the i^{th} regression coefficient is defined as $\sqrt{C_{ii} \cdot MS_{Res}}$.

Test for Regression

The test statistic can be defined as:

$$t = \frac{\hat{\beta}_4 - \beta_4}{s_e\left(\hat{\beta}_4\right)} = \frac{3.124 - 0}{0.1791} = 1.744$$

We can repeat this process for each coefficient. We can let software do the rest. We will test at the 5% level of significance.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	2.536287e-02	1.131812e-02	2.240909	3.498089e-02
XGM	3.233781e-02	1.327691e-02	2.435642	2.302719e-02
XGOS	-5.002444e-09	2.177756e-09	-2.297064	3.105353e-02
XCFR	3.124164e-01	1.790962e-01	1.744405	9.443960e-02
XEBITM	1.505211e+00	2.294826e-01	6.559149	1.081190e-06

Test for Regression

All coefficients are significant at the 5% level except for β_3, the one we tested earlier. We can conclude that β_0, β_1, β_2, and β_5 contribute to the response variable at the 95% confidence level.

Extra-Sum-of-Squares Method

Next, let's reduce our model one more time using the extra-sum-of-squares method. We will test the regressors x_1, x_2, and x_5. To do this method we will create two subsets for the regression coefficients, $\vec{\beta}_1 = (\beta_0\ \beta_3\ \beta_4\ \beta_6)^t$ and $\vec{\beta}_2 = (\beta_1\ \beta_2\ \beta_5)^t$. We want to test the following hypotheses:

$$H_0:\ \beta_1 = \beta_2 = \beta_5$$
$$H_a:\ \text{at least one } \beta_j \neq 0$$

Extra-Sum-of-Squares Method

Now we can define the test statistic and compute it in R.

$$F_0 = \frac{MS_R\left(\vec{\beta}_2|\vec{\beta}_1\right)}{MS_{Res}} = 0.0072/0.0006 = 11.0328$$

With the test statistic computed above, we can compute a p-value of 1.4716e-04. At the 1% level of significance, we can conclude that the regressors x_1, x_2, and x_5 contribute to the response variable given that the other regressors are already in the model.

Simultaneous Confidence Intervals

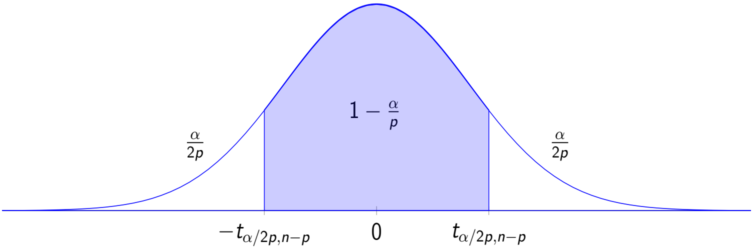
Next, let's compute simultaneous confidence intervals using the Bonferoni Method at the 5% confidence level using the optimal reduced model. For each β_j the confidence interval can be defined as:

$$\hat{\beta}_j - cv \cdot s_e\left(\hat{\beta}_j\right) \leq \beta_j \leq \hat{\beta}_j + cv \cdot s_e\left(\hat{\beta}_j\right)$$

where cv is the critical value for the simultaneous confidence intervals.

Simultaneous Confidence Intervals

Finding this critical value is a little different than normal. Since it is a simultaneous critical value, we must consider all of the regressors at once. Consider the graph below.



The critical value can be computed in R as 2.700.

Simultaneous Confidence Intervals

Next, we can construct our confidence interval by finding the standard error for each coefficient. Below are the confidence interval.

	SCIlower	SCIupper
(Intercept)	-5.319910e-03	5.825209e-02
XGM	8.475578e-03	7.599939e-02
XGOS	-8.817808e-09	2.171471e-09
XEBITM	8.212542e-01	2.105101e+00

There are a few observations we can make from this simultaneous confidence interval. Two of the confidence intervals contain 0, indicating they are not significant. We have already establish however that our model is significant. One thing to be careful about is the fact that these values are already very close to 0, so these confidence intervals may not be very helpful.

Checking Model Adequacy

The next topic is model adequacy. This idea will explore the initial assumptions we made at the beginning as whether they were correct assumptions or not. Statisticians may never know if their assumptions are correct, but there are many ways to sufficiently check. The main ways that we will check for our regression assumptions will be through:

1. Checking whether the current model is in-fact linear.
2. Checking if there is constant variance with respect to the random error.
3. Checking if the random error is normally distributed.

Checking Model Adequacy

If these assumptions turn out to not be satisfied there may be a few actions taken. A transformation may be applied to change the model and perhaps fix the issue. We may also need to consider a different model, or in the worst case consider different sampling methods or re-sampling. We will only apply a transformation if necessary. Let's again state our model so we are clear what we are analyzing:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_5 x_5 + \varepsilon$$

Checking Model Adequacy

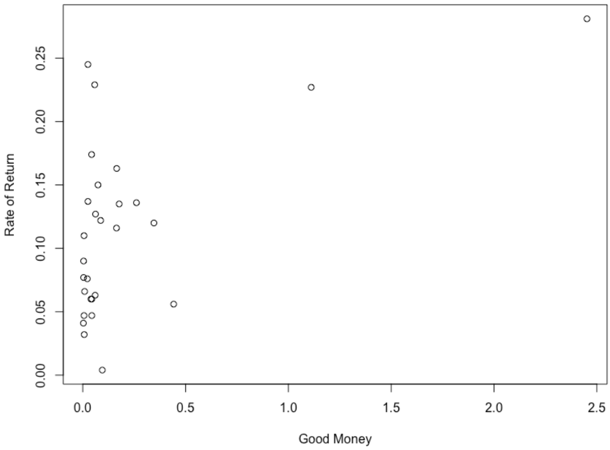
Our first method for checking model assumptions is to check for linearity. We have not calculated a complete regression analysis on the new model. We can do this in R.

```
Call:
lm(formula = Y ~ X)

Residuals:
      Min        1Q    Median        3Q       Max
-0.056457 -0.014874  0.001402  0.019397  0.060944

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.647e-02  1.177e-02   2.248  0.03401 *
XGM          4.224e-02  1.250e-02   3.378  0.00249 **
XGOS        -3.323e-09  2.035e-09  -1.633  0.11550
XEBITM       1.463e+00  2.377e-01   6.155 2.33e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.02743 on 24 degrees of freedom
Multiple R-squared:  0.8619,Adjusted R-squared:  0.8446
F-statistic: 49.92 on 3 and 24 DF,  p-value: 1.815e-10
```

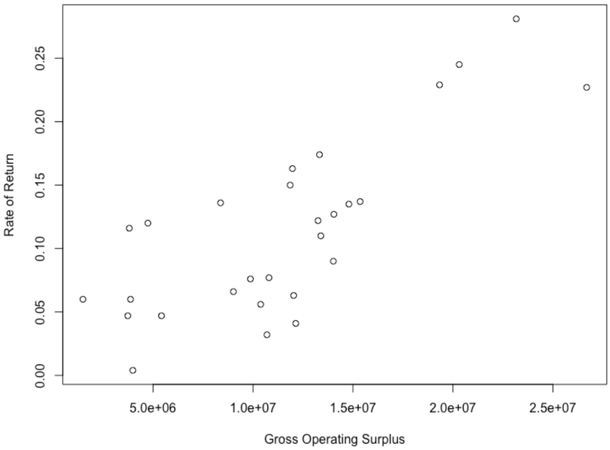
Checking Model Adequacy

There is an adjusted R-squared value of 0.8446 and an R-squared value of 0.8619. In my opinion, there is a clear linear relationship present. To further check this assumption, let's make plot of each regressor against the response variable.

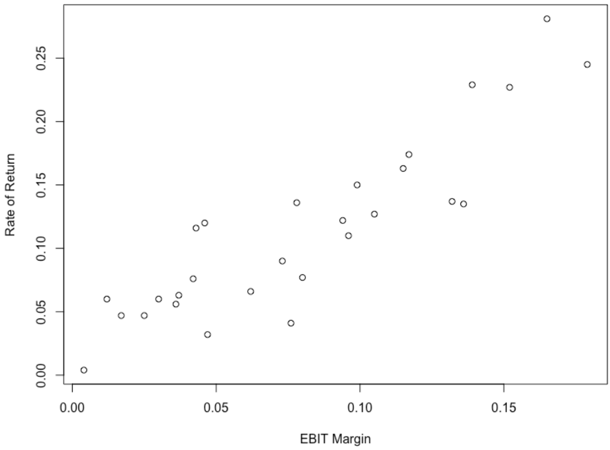
Checking Model Adequacy



Checking Model Adequacy



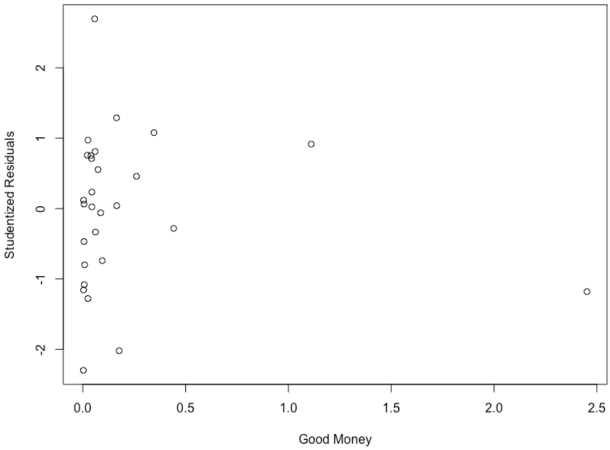
Checking Model Adequacy



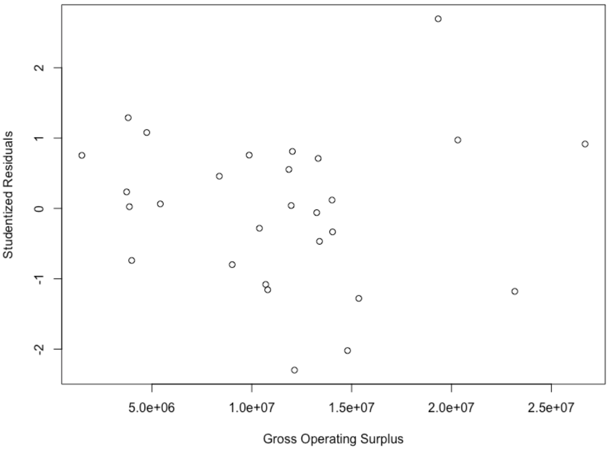
Checking Model Adequacy

There were very interesting results from these scatterplots. Even though our regression results from earlier pointed toward a strong linear relationship, we may need to reconsider our model. Let's move onto the assumption of constant variance. This can be checked by analyzing the residual plot against the fitted values. Also, also with the residuals plotted against the regressors as well. These plots will be below in that respective order.

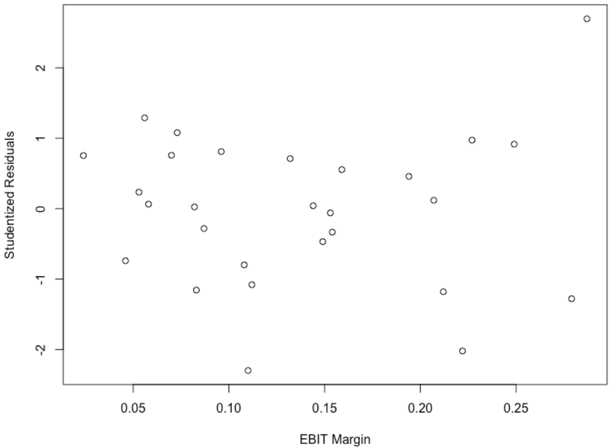
Checking Model Adequacy



Checking Model Adequacy



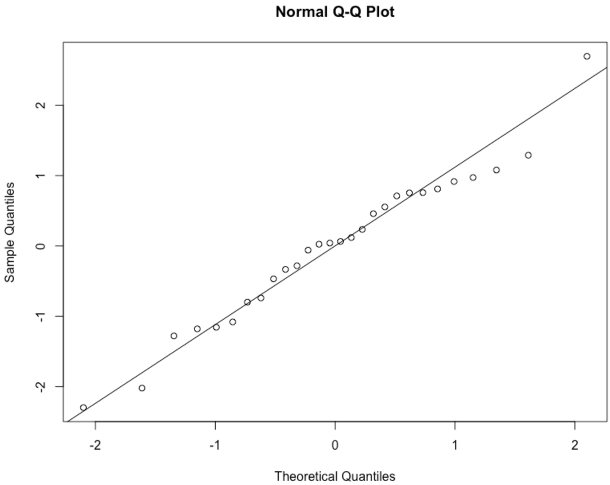
Checking Model Adequacy



Checking Model Adequacy

From these residual plot and the scatterplot above it is beginning to become obvious of some assumption violations. The first residual plot is not ideal as there are a few outliers. The last residual plot is good enough, but again the second plot is not ideal either. Let's continue with the last regression assumption of normality before we make a decision on how to proceed. Below is a normal probability plot of the studentized residuals.

Checking Model Adequacy



Transformation

The assumption of normality is satisfied. Consider the other assumption violations for regression we will try a transformation to try to fix these assumptions. The main violation that I have found is the violation of linearity. If we can fix this, I believe our model can become adequate. After trying many different transformations[2], consider the following model below:

$$\sin^{-1}\left(\sqrt{y}\right) = \beta_0 + \beta_1 \log\left(x_1\right) + \beta_2 x_2 + \beta_5 x_5 + \epsilon$$

With these transformations defined let's rerun these assumptions tests and let's also run another regression analysis below.

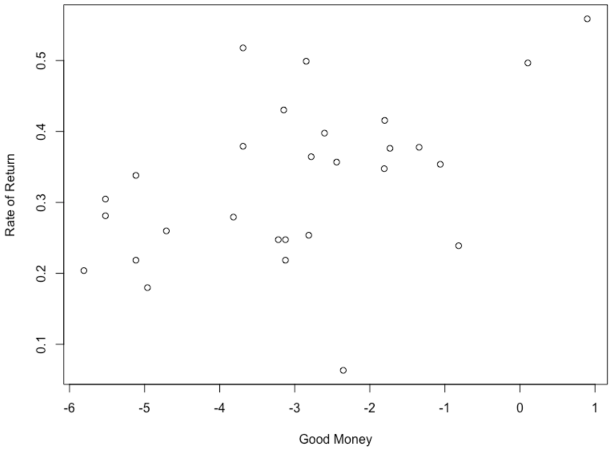
Transformation

```
Call:
lm(formula = nY ~ nX)

Residuals:
      Min        1Q    Median        3Q       Max
-0.133448 -0.017118  0.006608  0.030542  0.067791

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.229854   0.026772   8.586 6.33e-09 ***
nX1         0.017276   0.005481   3.152  0.00418 **
nX2         1.886722   0.192277   9.813 4.69e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.04723 on 25 degrees of freedom
Multiple R-squared:  0.8398,Adjusted R-squared:  0.827
F-statistic: 65.54 on 2 and 25 DF,  p-value: 1.142e-10
```

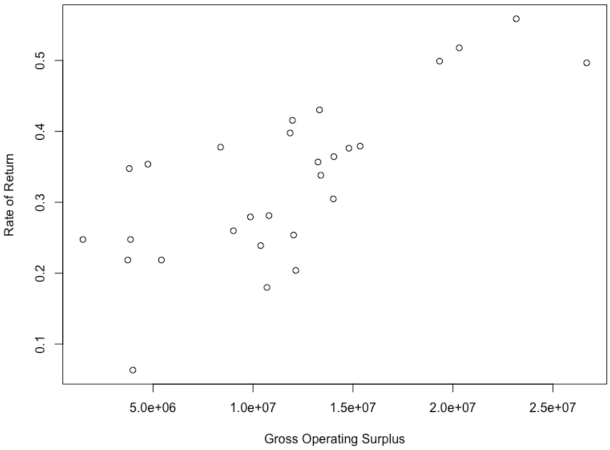
Transformation

	Residuals	Standardized	Studentized
1	-0.0447621509	-1.031108424	-1.032468405
2	0.0361384204	0.789548778	0.783425869
3	-0.0218475732	-0.516118109	-0.508406223
4	-0.0689927860	-1.574801933	-1.625726099
5	0.0026521641	0.058182850	0.057011178
6	-0.0042248116	-0.095431650	-0.093520575
7	-0.0529565651	-1.175148070	-1.184587678
8	0.0020534819	0.051197688	0.050165915
9	0.0298909170	0.673678123	0.666141215
10	0.0340472255	0.743561629	0.736730710
11	-0.0155419497	-0.336796855	-0.330743368
12	0.0154689202	0.346910008	0.340722086
13	0.0505809082	1.134566590	1.141417616
14	0.0139568321	0.334935855	0.328907557
15	0.0105636843	0.235692382	0.231187425
16	-0.0082132733	-0.177593218	-0.174114971
17	-0.1334481901	-3.062278990	-3.795559818
18	-0.0802963680	-1.784158908	-1.871301895
19	-0.0001064937	-0.002331898	-0.002284784
20	0.0149733506	0.329731617	0.323774487
21	-0.0056345835	-0.123901400	-0.121435374
22	0.0677913449	1.503029661	1.544078338
23	0.0324956878	0.732550851	0.725579932
24	0.0260405983	0.563289491	0.555444785
25	0.0238849308	0.525350251	0.517601033
26	-0.0360185631	-0.802021591	-0.796126189
27	0.0560716858	1.245647283	1.260214415
28	0.0554331565	1.252074702	1.267150850

Transformation



Transformation



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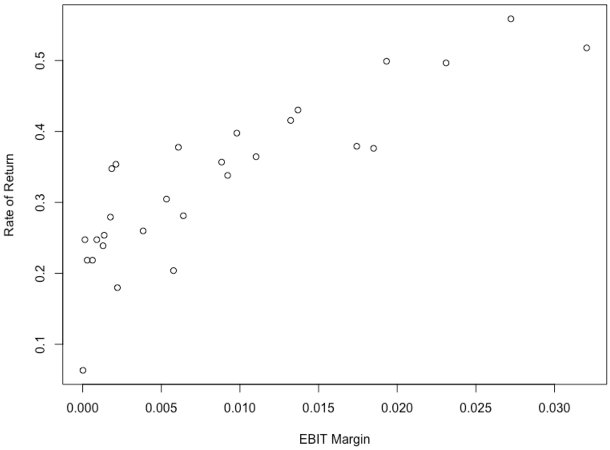

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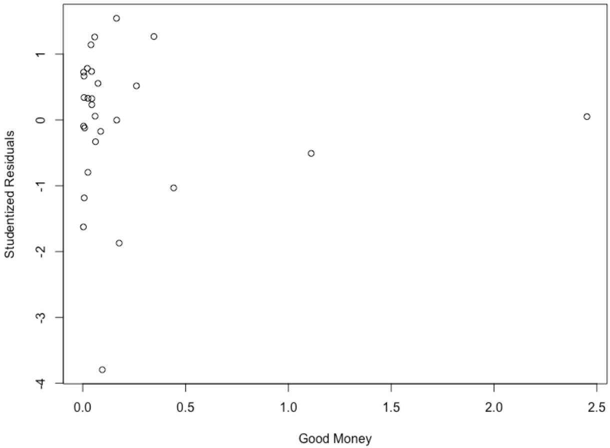

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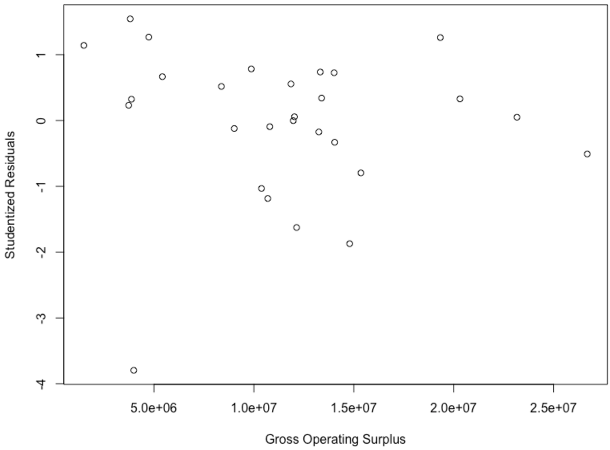
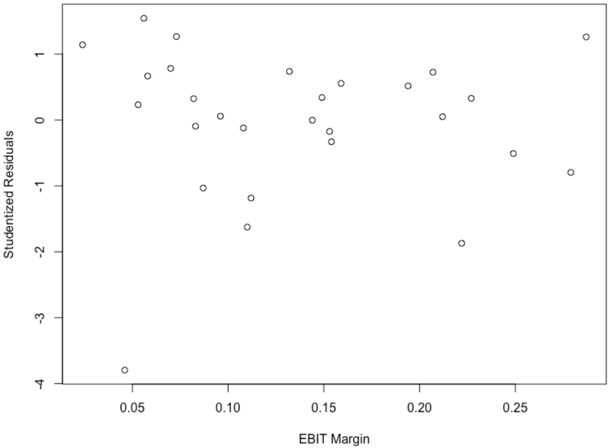
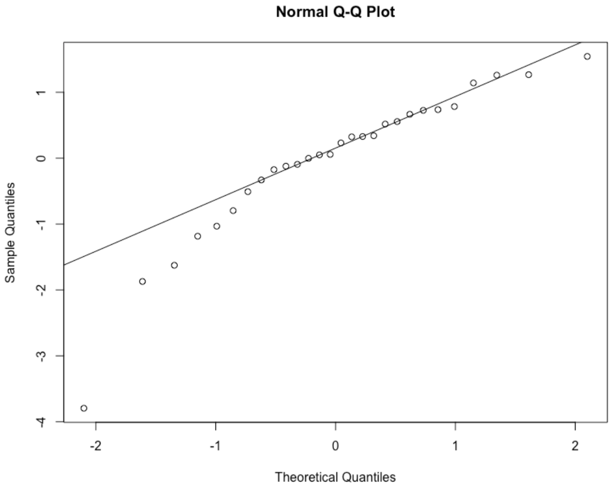


Transformation



Transformation



Transformation

The transformation looks like it helped our assumption of linearity. The scatterplots seemed to have improved some. However, the other assumptions mentioned earlier do not seem to have improved. To conclude, I think we should stick with the transformed model. The studentized residuals only produce one outlier, while the rest are below 2. This is good sign.

Conclusion

The regression analysis above revealed some interesting results, and unexpected results at that. I did not expect the issue arising regarding the computational singularity with R. This was an issue that was finally fixed with a lot of different attempts. Regardless, I believe that our initial simple linear regression analysis with the one regressor EBIT Margin was a superior model to the optimal model found in the multiple linear regression analysis. All of the other regression just seemed to add 'noise' to the data. Sometimes a best model is not always the more complicated model. This project just goes to show that making economic prediction can be very difficult and unreliable at times.

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