3F3 - Ex. paper 2

Q1 $\{X_n\}_{n \geq 0}$ states $S = \{1, \dots, L\}$

Transition probabilities given by:

$$p(i_m | i_{m-1}, \dots, i_0) = P_{i_{m-1}, i_m}, \quad i_\ell \in S \quad \forall \ell \geq 1$$

a) Show $p(i_2 | i_1) = P_{i_1, i_2}$

Sol: In principle we don't know $p(i_2 | i_1)$

$$p(i_2 | i_1) = \frac{p(i_2, i_1)}{p(i_1)} = \frac{1}{p(i_1)} \sum_{i_0=1}^{L} p(i_0, i_1, i_2)$$

$$= \frac{1}{p(i_1)} \sum_{i_0 \in S} \underbrace{p(i_2 | i_1, i_0)}_{P_{i_1, i_2}} \cdot p(i_1, i_0)$$

$$= P_{i_1, i_2} \underbrace{\frac{1}{p(i_1)} \underbrace{\sum_{i_0 \in S} p(i_1, i_0)}_{p(i_1)}}_{=1}$$

b) Same reasoning:

$$p(i_m | i_{m-1}) = \frac{p(i_m, i_{m-1})}{p(i_{m-1})}$$

$$= \frac{1}{p(i_{m-1})} \sum_{\substack{i_0 \in S \\ \vdots \\ i_{m-2} \in S}} p(i_0 - i_{m-2}, i_{m-1}, i_m)$$

$$= \frac{1}{p(i_{m-1})} \sum_{i_0, \dots i_{m-2} \in S} \underbrace{p(i_m | i_0 - i_{m-1})}_{P_{i_{m-1}, i_m}} \cdot p(i_0 - i_{m-1})$$

$$= P_{i_{m-1}, i_m} \cdot \frac{1}{p(i_{m-1})} \cdot \Big(\underbrace{\sum_{i_0 - i_{m-2}} p(i_0 - i_{m-1})}_{p(i_{m-1})} \Big)$$

Q2 Bernoulli process with prob q
(i.e. $p(X_m = 1) = q$, $p(X_m = -1) = 1-q$)

a) • Prob. as a pmf:

$$(*) \quad p(X_0, \dots, X_m) = \prod_{i=0}^{m} p(x_i = X_i) = q^{\#X_m=1} \cdot (1-q)^{\#X_m=-1}$$

because the prob. of each state is independent

• Transition probabilities (i.e. probability matrix)

$$p(X_m = 1 \mid X_{m-1} = 1) = p(X_m = 1 \mid X_{m-1} = 0) = q$$
$$p(X_m = -1 \mid X_{m-1} = 1) = p(X_m = -1 \mid X_{m-1} = 0) = (1-q)$$

$$P = \begin{bmatrix} p(X_m = -1 \mid X_{m-1} = -1) & P(X_m = 1 \mid X_{m-1} = -1) \\ P(X_m = -1 \mid X_{m-1} = 1) & P(X_m = 1 \mid X_{m-1} = 1) \end{bmatrix} = \begin{bmatrix} 1-q & q \\ 1-q & q \end{bmatrix}$$

b) It can be argued from (*) that, since values at each time-step are independent and identically distributed, the process is unaffected by time shifts:

$$p(X_0 - X_m) = P(X_k - X_{k+m}) = \prod p(x_i = X'_i)$$

with X'_i either the value of X_i or X_{k+i}

then it is <u>strictly</u> stationary.

However, we can also stick to the characterisation of WSS:

1. Common mean for all X_m : $\mathbb{E}(X_m) = (1)q + (-1)\cdot(1-q) = 2q-1$
2. Finite variance : $\mathbb{E}(X_m^2) = 1^2 \cdot q + (-1)^2 \cdot (1-q) = 1$
3. Lag-dependent autocovariance function : $\mathbb{E}(X_m X_{m+k}) = \underbrace{\mathbb{E}(X_m)\,\mathbb{E}(X_{m+k})}_{\text{due to independence}}$ $K>0$
$= (2q-1)^2$

And the result holds regardless of lag k.

Q3 $S = \{1, \dots, L\}$, $\{X_n\}_{n \geq 0}$ taking values in S

$p(X_0 = X_s) = \lambda_s$ X_s any state $\in S$

$$\sum_{s \in S} \lambda_s = 1$$

$$P = \begin{bmatrix} \lambda_1 & \cdots & \lambda_L \\ \vdots & & \vdots \\ \lambda_1 & \cdots & \lambda_L \end{bmatrix}$$ is the transition prob. matrix.

- Show that $\{X_n\}_{n \geq 0}$ is strictly stationary

From matrix P, $p(X_n = X_s \mid X_{n-1} = X_{s'}) = \lambda_s$ regardless of s' (the previous state)

<u>pmf</u> : • $p(X_0, \dots, X_n) = p(X_0) \cdot p(X_1 \mid X_0) \cdot \dots \cdot p(X_n \mid X_{n-1})$
$\lambda_{X_0} \cdot \lambda_{X_1} \cdot \dots \cdot \lambda_{X_n}$

Strictly stationary: $p(X_0 - X_n) = p(X_K - X_{K+n})$

Just from the definition of a Markov chain

• $p(X_K, \dots, X_{K+n}) = p(X_K) \cdot p(X_{K+1} \mid X_K) \dots p(X_{n+K} \mid X_{n+k-1})$

Transition probs. remain invariant:

$p(X_{n+K} \mid X_{n+k-1}) = \lambda_{X_{n+k}}$ whatever value $X_{n+k} \in S$ may take. But we don't know $\boxed{p(X_K)}$

Let's calculate $p(X_1)$ [the marginal of X_1]

$$p(X_1) = \sum_{X_0 \in S} p(X_0, X_1) = \sum_{X_0 \in S} p(X_0) \cdot p(X_1 \mid X_0) = \lambda_{X_1} \cdot \cancel{\left(\sum_{X_0 = 1}^{S} \lambda_{X_0}\right)}^{1}$$

Advancing the same marginalisation $(k-1)$ steps, we conclude $p(X_K)$ and $p(X_0)$ are the same pmf, then $P(X_0 - X_n) = P(X_K - X_{n+k})$

$\boxed{Q4}$ AR(1)

$$X_n = aX_{n-1} + W_n$$

from independence: $\mathbb{E}(W_n W_m) = 0$ whenever $n \neq m$

$|a| < 1$ W_n i.i.d. and $\mathbb{E}(W_n) = 0$ $\mathbb{E}(W_n^2) = \sigma^2$

a) Assume X_n has constant mean μ, find μ.

$$\underbrace{\mathbb{E} X_n}_{\mu} = a \underbrace{\mathbb{E} X_{n-1}}_{\mu} + \underbrace{\mathbb{E} W_n}_{0}$$

$$\mu = a\mu$$

Since $|a| < 1$, the only value satisfying the Eq. is $\boxed{\mu = 0}$

b) Assuming $Var(X_n) = \sigma_X^2$, find σ_X

$$Var(X_n) = Var\{aX_{n-1} + W_n\}$$

$$\sigma_X^2 = \mathbb{E}\left\{\left(aX_{n-1} + W_n - \underbrace{\mathbb{E}\{aX_{n-1} + W_n\}}_{0}\right)^2\right\}$$

$$\sigma_X^2 = \mathbb{E}\left\{a^2 X_{n-1}^2 + 2aX_{n-1} \cdot W_n + W_n^2\right\}$$

$$\sigma_X^2 = a^2 \sigma_X^2 + 2a \underbrace{\mathbb{E}(X_{n-1} W_n)}_{0} + \underbrace{\mathbb{E} W_n^2}_{\sigma^2}$$

$X_{n-1} = aX_{n-2} + W_{n-1}$
only contains realizations of W
in times $< n \Rightarrow$ All expected values
of products are zero $\Rightarrow \mathbb{E} X_{n-1} W_n = 0$

$$\boxed{\sigma_X^2 = \frac{\sigma^2}{1 - a^2}}$$

Q5 Determine if the following processes are WSS.

a) $X_n = U$, U a random variable.

1. Mean: $\mathbb{E} X_n = \mathbb{E}(U)$. it depends on whether U has a defined mean

2. Finite variance $\mathbb{E} X_n^2 = \mathbb{E} U^2$

3. Auto covariance $\mathbb{E} X_n X_{n+k} = \mathbb{E} U^2 \Rightarrow$ not lag-dependent

The process is WSS provided both $\mathbb{E}(U)$ and $\mathbb{E}(U^2)$ are finite.

b) $X_n = A \cos(n f_0) + B \sin(n f_0)$

A, B satisfy $\mathbb{E}(A) = 0$, $\mathbb{E}(B) = 0$, $\mathbb{E}(AB) = 0$

$\mathbb{E}(A^2) = 2$ $\quad E(B^2) = 2$

1. $\mathbb{E}(X_n) = \underbrace{\mathbb{E}(A)}_{0} \cdot \cos n f_0 + \underbrace{\mathbb{E}(B)}_{0} \cdot \sin(n f_0) = 0$ for all n

(the sinusoids are deterministic)

2. $$\mathbb{E}(x_n^2) = \mathbb{E}\{A^2 \cos^2 n f_0 + B^2 \sin^2(n f_0) - 2AB \cos(n f_0) \sin(n f_0)\}$$

$$= \mathbb{E}(A^2) \cos^2 n f_0 + \mathbb{E}(B^2) \sin^2 n f_0 + 2 \underbrace{\mathbb{E}(AB)}_{0} \cdot \text{etc}$$

$$= 2 \cdot \underbrace{\{\cos^2 n f_0 + \sin^2 n f_0\}}_{1}$$

$= 2$ finite and same value for all n.

3. $$\mathbb{E} X_n X_{n+k} = \mathbb{E}\{(A \cos n f_0 + B \sin n f_0)(A \cos(n+k) f_0 + B \sin((n+k) f_0))\}$$

$$= \mathbb{E}\{A^2 \cos(n f_0) \cdot \cos(n+k) f_0 + B^2 \sin(n f_0) \sin(n+k) f_0 + AB \cdot \{\dots\}\}$$

$$= 2 \cdot [\cos n f_0 \cdot \cos(n+k) f_0 + \sin n f_0 \cdot \sin(n+k) f_0$$

From data book (standard trigonometric identity):

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

Replacing in $\mathbb{E}\, X_m X_{m+k}$

$$\mathbb{E}\, X_m X_{m+k} = 2 \cdot \Big\{ \underbrace{\frac{1}{2}[\cos(f_0 \cdot (-k)) + \cos(f_0 \cdot (2m+k))]}_{\cos(f_0 m)\cos(f_0 (m+k))}$$

$$+ \underbrace{\frac{1}{2}[\cos(f_0(-k)) + \cos f_0 (2m+k)]}_{\sin(f_0 m)\sin((m+k)f_0)} \Big\}$$

$$= \cos(f_0 k)$$

(cosine is an even function: $\cos(-a) = \cos(a)$)

$\Rightarrow$ WSS

Q5 c. $Y_n = X_n - X_{n-1}$

1. $\mathbb{E} Y_n = \mathbb{E} X_n - \mathbb{E} X_{n-1} = 2\,\mathbb{E} X_n = 2 \cdot (2q - 1)$

2. $\mathbb{E} Y_n^2 = \mathbb{E}\{X_n^2 + X_{n-1}^2 - 2 X_n X_{n-1}\}$

$= \mathbb{E}\underbrace{X_n^2}_{1} + \mathbb{E}\underbrace{X_{n-1}^2}_{1} - 2(2q-1)^2 = 2(1 - [2q-1]^2)$

$= 2 \cdot (4q - 4q^2)$

$= 8q(1-q)$

Both $\mathbb{E} Y_n$ and $\mathbb{E} Y_n^2$ are finite and equal for all n

3. $\mathbb{E} Y_n Y_{n+k} = \mathbb{E}\{(X_n - X_{n-1})(X_{n+k} - X_{n+k-1})\}$

$= \mathbb{E}\{X_n X_{n+k} - X_n X_{n+k-1} - X_{n-1} X_{n+k} + X_{n-1} X_{n+k-1}\}$

$|k| = 1 \Rightarrow \mathbb{E} Y_n Y_{n+k} = (2q-1)^2 - 1$

$|k| \geq 2 \Rightarrow \mathbb{E} Y_n Y_{n+k} = 0$

Then it depends on $|k|$ and not $(n, n+k)$

WSS ✓✓

Q6] AR(1): $X_n = a X_n + W_n$

$|a| < 1$, $\mathbb{E} W_n = 0$, $\mathbb{E} W_n^2 = \sigma^2$, $\mathbb{E} W_m W_n = 0$
$m \neq n$

a) $\mathbb{E}(X_n X_{n+k})$

Note that X_{n+k} can be expressed as a function of X_n and the "noises" $W_{n+1}, \cdots, W_{n+k}$ (for $k > 0$)

$$
\begin{aligned}
X_{n+k} &= a X_{n+k-1} + W_n \\
&= a \cdot [a X_{n+k-2} + W_{n+k-1}] + W_n \\
&= a[a[a X_{n+k-3} + W_{n+k-2}] + W_{n+k-1}] + W_n \\
&= \dots \\
&= a^k X_n + \underbrace{\sum_{l=0}^{k-1} a^l W_{n+k-l}}_{\text{contains terms from } W_{n+1} \text{ to } W_{n+k}}
\end{aligned}
$$

then $\mathbb{E} X_n X_{n+k} = \mathbb{E}\, a^k X_n^2 + \mathbb{E}\left(X_n \cdot \sum_{l=0}^{k-1} a^l W_{n+k-l}\right)$

But $\mathbb{E} X_n W_{n+k-l} = 0$ for all l because X_n contains noises up to W_n whereas W_{n+k-l} ranges from $n+1$ to $n+k$

$$\mathbb{E} X_n X_{n+k} = a^k\, \mathbb{E} X_n^2 = a^k \left(\frac{\sigma^2}{1-a^2}\right) \quad \text{from Q4.}$$

For negative lags the same reasoning can be applied:

$\mathbb{E} X_n X_{n-k} = \mathbb{E} X_m X_{m+k}$
$(m = n-k)$, concluding
$R(-k) = R(k)$

$$\boxed{\mathbb{E} X_n X_{n+k} = R_X(k) = \frac{a^{|k|}\sigma^2}{1-a^2}}$$

Q6.b)

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) \exp(-i 2\pi f k)$$

- if $R_X(k)$ is real and even:

$$S_X(f) = R_X(0) + 2 \sum_{k=1}^{\infty} R_X(k) \cos 2\pi f k$$

(real parts are equal and imaginary part is the additive inverse)

$$S_X(f) = \frac{\sigma^2}{1-a^2}\left[1 + 2\left(-1 + \underbrace{\sum_{k=0}^{\infty} a^k \cos 2\pi f k}_{(*)}\right)\right]$$

$(*)$ can be calculated as a special case of z-transform. However, we'll simply use convergent series arguments.

$$\sum_{k=0}^{\infty} \underbrace{a^k \exp(i 2\pi f k)}_{r = a\exp(i2\pi f)} = \left(\sum_{k=0}^{\infty} a^k \cos 2\pi f k\right) + i\left(\sum_{k=0}^{\infty} a^k \sin 2\pi f k\right)$$

$(*)$ is simply the real part of the series above.

$$(*) = \mathrm{Re}\left\{\sum_{k=0}^{\infty} r^k\right\} = \mathrm{Re}\left\{\frac{1}{1-r}\right\} = \mathrm{Re}\left[\frac{1}{1 - a\cos 2\pi f - i \sin 2\pi f}\right]$$

$$= \mathrm{Re}\left[\frac{1 - a\cos 2\pi f + i \sin 2\pi f}{(1 - a\cos 2\pi f)^2 + a^2 \sin^2 2\pi f}\right] = \frac{1 - a\cos 2\pi}{1 - 2a\cos 2\pi + a^2}$$

$$S_X(f) = \frac{\sigma^2}{1-a^2}\left[-1 + 2\,\frac{1 - a\cos 2\pi f}{1 - 2a\cos 2\pi f + a^2}\right]$$

$$= \frac{\sigma^2\left[\overset{+1}{-1} + \cancel{2a\cos 2\pi f} - a^2 + \cancel{2} - \cancel{2a\cos 2\pi f}\right]}{\left[1 - 2a\cos 2\pi f + a^2\right]\left[1 - a^2\right]}$$

$$= \frac{\sigma^2}{1 - 2a\cos 2\pi f + a^2}$$

Q6.c

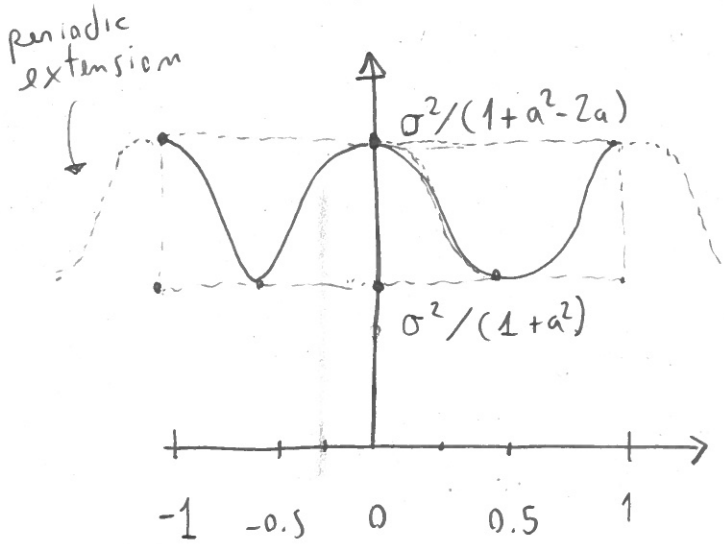
$$R_X(0) = \frac{\sigma^2}{1-a^2} = 5.26$$

$$R_X(1) = \frac{\sigma^2}{1-a^2} \cdot a = 4.74$$

$$a = R_X(1)/R_X(0) \sim 0.9$$

$$\sigma^2 \sim 5.26 \cdot (1 - 0.9^2) = 0.999 \sim 1$$

$S_X(f)$



$S_X(f)$

$= \dfrac{\sigma^2}{1+a^2-2a}$ $f=0$

$= \dfrac{\sigma^2}{1+a^2}$ $f=1/4$

$= \dfrac{\sigma^2}{1+a^2+2a}$ $f=1/2$

$= \dfrac{\sigma^2}{1+a^2}$ $f=3/4$

$= \dfrac{\sigma^2}{1+a^2-2a}$ $f=1$

equivalently

Q7 $X_0 \sim N(\mu, \sigma_0^2)$ $\quad$ $X_0 \sim N(\mu, \sigma_0^2)$

$X_K \sim N(a X_{K-1}, \sigma^2)$ $\quad$ $W_K \sim N(0, \sigma^2)$

$X_K = a X_{K-1} + W_K$

find mean and covariance of the random vector $X_0 - X_K$

Sol 1

$$p_X(X_0, X_1 - X_K) = p(X_0) \cdot p(X_1 | X_0) \cdots p(X_K | X_{K-1}) \quad (*)$$

Is it clear?

$$p(X_0, X_1 - X_K) = p(\underbrace{X_K | X_0 - X_{K-1}}_{= p(X_K | X_{K-1})}) \cdot p(\underbrace{X_0 - X_{K-1}}_{= p(X_{K-1} | X_0 - X_{K-2}) \cdot p(X_0 - X_{K-2})})$$

$$p(X_0, X_1 - X_K) = N(X_K | a X_{K-1}, \sigma^2) \cdot N(X_{K-1} | a X_{K-2}, \sigma^2) \cdot (\ldots)$$

- Although valid, the representation provided by the pdf $(*)$ is not easily amenable for a characterization in terms of mean and covariance.
- Instead, note that X_0 and $W_1 - W_K$ are independent

Then $$p_{XW}(X_0, W_1 - W_K) = p(X_0) \prod_{l=1}^{K} p(W_K)$$

$$= N(X_0 | \mu, \sigma_0^2) \prod_{l=1}^{K} N(W_l | 0, \sigma^2)$$

It's easy to see that $X_0 - X_K$ is a linear transformation of $X_0, W_1 - W_K$

$$A \left\{ \begin{bmatrix} 1 & 0 & \text{---} & 0 \\ a & 1 & 0 \text{---} & 0 \\ a^2 & a & 1 & 0 \text{---} \\ a^K & \text{---} & \cdots & a \; 1 \end{bmatrix} \right. \begin{bmatrix} X_0 \\ W_1 \\ | \\ W_K \end{bmatrix} = \begin{bmatrix} X_0 \\ a X_0 + W_1 \\ \vdots \\ a^K X_0 + \sum_{l=0}^{K-1} a^l W_{K-l} \end{bmatrix}$$

Using change of variables:

Rows of A^{-1}

$$A\begin{bmatrix} x_0 \\ w_1 \\ | \\ w_K \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ | \\ x_K \end{bmatrix}}_{X} \Rightarrow \begin{bmatrix} x_0 \\ w_1 \\ | \\ w_K \end{bmatrix} = A^{-1}\begin{bmatrix} x_0 \\ | \\ x_K \end{bmatrix} = \begin{bmatrix} A_0^{-1}X \\ A_1^{-1}X \\ | \\ A_K^{-1}X \end{bmatrix}$$

$H(x) = A^{-1}x$ is the inverse function we need in the change of variables formula.

$$p(x_0 - x_K) = p_{xw}\left(A_0^{-1}X, A_1^{-1}X, \dots, A_K^{-1}X\right) \cdot \underbrace{|A^{-1}|}$$

the jacobian of a linear transformation is the matrix itself.

$|A^{-1}| = |A|^{-1}$

$|L|$ = prod. of the diagonal terms if L is triangular

$$= \frac{N(A_0^{-1}X \mid \mu, \sigma^2)}{|A|} \cdot \prod_{l=1}^{K} N(A_l^{-1}X \mid 0, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi}^{K+1}\sqrt{\sigma^{-2}}^{K}\sqrt{\sigma_0^2}\,|A|} \exp\left\{-\frac{1}{2}\frac{(A_0^{-1}X - \mu)^2}{\sigma_0^2}\right\}$$

$$\cdot \prod_{l=1}^{K} \exp\left\{-\frac{1}{2}\frac{(A_l^{-1}X)^2}{\sigma^2}\right\}$$

Calculate the partial derivatives yourself

Rearranging

$$= \text{const} \cdot \exp\left\{-\frac{1}{2}\left[A^{-1}X - \begin{bmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right]^T \Sigma^{-1}\left[A^{-1}X - \begin{bmatrix} \mu \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right]\right\}$$

$$= \text{const} \cdot \exp\left\{-\frac{1}{2}\left[X - A\begin{bmatrix} \mu \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]^T A^{-T}\Sigma^{-1}A^{-1}\left[X - A\begin{bmatrix} \mu \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right\}$$

$$M_X = \begin{bmatrix} \mu \\ a\mu \\ a^2\mu \\ | \\ a^K\mu \end{bmatrix}$$

$$\boxed{\Sigma_X = A\,\Sigma\,A^T}$$

$$\Sigma = \begin{bmatrix} \sigma_0^2 & & 0 \\ & \sigma^2 & \\ 0 & & \sigma^2 \end{bmatrix}$$

Q8 MA process

$$\boxed{X_m = \sum_{i=1}^{q} b_i W_{m-i} + W_m}$$

MA (q)

$\{W_m\}_{m \in \mathbb{Z}}$

$\mathbb{E} W_m = 0$

$\mathbb{E} W_m^2 = \sigma^2$

$\mathbb{E} W_n W_m = 0$ if $n \neq m$

a) X_m as the output of a LTI filter with infinite response

$h_K = 1 \quad K = 0$

$h_K = b_K \quad K \in \{1, \dots, q\}$

$h_K = 0 \quad K \notin \{0, 1, \dots, q\}$

b) WSS 1. $\mathbb{E}\{X_m\} = \mathbb{E} \sum_{i=1}^{q} b_i W_{m-i} + \mathbb{E} W_m = 0$ ✓

2. $\mathbb{E}\{X_m^2\} = \mathbb{E}\left(\sum_{i=1}^{q} b_i W_{m-i} + W_m\right)^2 = \sigma^2\left[1 + \sum_{i=1}^{q} b_i^2\right]$ finite

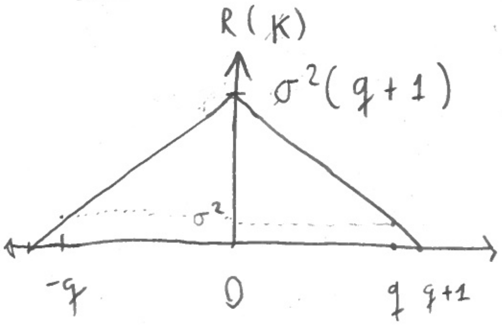
Because All cross-products satisfy $\mathbb{E}(\cdot) = 0$ ✓

3. $\mathbb{E} X_m X_{m+K} = \sigma^2 \left[\sum_{i=0}^{q-K} b_i b_{i+K}\right]$ $(b_0 = 1)$

$R(K) = \sigma^2 \sum_{i=-\infty}^{\infty} h_i h_{K+i}$

It is WSS ✓✓

c)



$R(K) = \sigma^2 (q + 1 - K)$

In that case all terms $b_i b_{i-K}$ are just 1 ($=1$, $=1$)

(d) $S_X(f) = \sum_{K=-2}^{2} R(K) \exp -i2\pi f K$

$$= R(0) + 2R(1)\cos 2\pi f + 2R(2)\cos 4\pi f$$

$$= \sigma^2 \left[3 + 4\cos 2\pi f + 2\cos 4\pi f \right]$$

for any b_1, b_2:

$$= \sigma^2 \left[(1 + b_1^2 + b_2^2) + 2\left\{ (b_0 b_1 + b_2 b_1)\cos 2\pi f + b_0 b_2 \cos 4\pi f \right\} \right]$$