3F3 - Ex. paper 3

Q1 a. $R_{XX}[k] \overset{\text{Assuming WSS}}{=} \mathbb{E}\{X_m X_{m+k}\} = \mathbb{E}\{X_{m-k} X_m\} = R_{XX}[-k]$

$m = m+k$

b. $$S_x(\Omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-j\Omega m}$$

$$= r_{xx}[0] + \sum_{m=-\infty}^{-1} r_{xx}[m] e^{-j\Omega m} + \sum_{m=1}^{\infty} r_{xx}[m] e^{-j\Omega m}$$

$$= r_{xx}[0] + \sum_{m=1}^{\infty} \underbrace{r_{xx}[-m]}_{r_{xx}[m]} e^{j\Omega m} + \sum_{m=1}^{\infty} r_{xx}[m] e^{-j\Omega m}$$

$$= r_{xx}[0] + 2\sum_{m=1}^{\infty} r_{xx}[m] \cos \Omega m + j\left[\cancel{\sum r_{xx}[m] \sin \Omega m - \sum r_{xx}[m] \sin \Omega m}\right]$$

0

c. $$\mathbb{E}[(X_{m+k} - aX_m)^2] = \mathbb{E}X_{m+k}^2 - 2a\mathbb{E}\{X_{m+k}X_m\} + a^2\mathbb{E}X_m^2 \geq 0$$

$\Rightarrow R_{xx}[0](1+a^2) \geq 2a R_{xx}[K]$

- picking $a=1$ yields $\boxed{R_{xx}[0] \geq R_{xx}[k]}$
- But to prove $\max_k |R_{xx}[k]| = R_{xx}[0]$ we need $\boxed{R_{xx}[0] \geq -R_{xx}[k]}$. Picking $a=-1$ yields the desired result.

e. By definition:

$$S_x(e^{j(\Omega+2\pi)}) = \sum_{k=-\infty}^{\infty} R_{xx}[k] e^{-j(\Omega+2\pi)k}$$

$$= \sum_{k=-\infty}^{\infty} R_{xx}[k] e^{-j\Omega k} \cdot \underbrace{e^{-j2\pi k}}_{=1 \text{ for all } k}$$

$$= S_x(e^{j\Omega})$$

[Sorry, I skipped e. by mistake]

e. Cross-correlation function satisfies:

$$R_{xy}[k] = R_{yx}[-k]$$

$$R_{xy}[k] = \mathbb{E}\, X_m Y_{m+k} \underset{m = m+k}{=} \mathbb{E}\, X_{m-k} Y_m = R_{yx}[-k]$$

g. Cross power spectra satisfies: $S_{xy}(e^{j\Omega}) = S^*_{yx}(e^{j\Omega})$

$$S_{xy}(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} R_{xy}[m]\, e^{-j\Omega m}$$

$$= \sum_{m=-\infty}^{\infty} R_{yx}[-m]\, e^{-j\Omega m}$$

· "repairing" index m so that the argument of $R_{yx}[\cdot]$ is positive

$$= \sum_{m'=-\infty}^{\infty} R_{yx}[m']\, e^{j\Omega m'}$$

$$= S^*_{yx}(e^{j\Omega})$$

Q2 $\{X_m\}$ WSS $\mathbb{E}\, X_m < \infty$. Y_m is the output is the output of X_m passed through a LTI system with finite impulse response $\{h_m\}$.

$$X_m \longrightarrow \boxed{\text{LTI}} \longrightarrow Y_m = \sum_{K=-\infty}^{\infty} X_{m-p}\, h_p$$

Show

i. $$\boxed{\mathbb{E}\, Y_m = (\mathbb{E}\, X_m) \cdot \sum_{p=-\infty}^{\infty} h_p}$$

· from Y_m formula we have

$$\mathbb{E}\, Y_m = \mathbb{E}\left\{ \sum_{p=-\infty}^{\infty} X_{m-p}\, h_p \right\} \overset{\text{linearity of } \mathbb{E}(\cdot)}{=} \sum_{K=-\infty}^{\infty} (\mathbb{E}\, X_{m-p})\, h_p$$

$$= \mathbb{E}(\cancel{X_m}) \cdot \sum_{p=\infty}^{\infty} h_p$$

μ

Same mean μ for all X_m due to WSS.

ii. Derive formula for $R_{yy}[k]$:

$$R_{yy}[k] = \mathbb{E}\, Y_m\, Y_{m+k} = \mathbb{E}\left\{ \underbrace{\left(\sum_{p=-\infty}^{\infty} X_{m-p} h_p \right)}_{Y_m} \underbrace{\left(\sum_{q=-\infty}^{\infty} X_{m+k-q} \cdot h_q \right)}_{Y_{m+k}} \right\}$$

$$= \mathbb{E} \sum_{p=-\infty}^{\infty} h_p \cdot \left(X_{m-p} \sum_{q=-\infty}^{\infty} X_{m+k-q} \cdot h_q \right)$$

$$= \sum_{p=-\infty}^{\infty} h_p \underbrace{\left(\sum_{q=-\infty}^{\infty} \underbrace{\mathbb{E}(X_{m+k-q} \cdot X_{m-p})}_{R_{xx}[m+k-q-m+p] = R_{xx}[k+p-q]} \cdot h_q \right)}_{f[k+p] = \sum_{q=-\infty}^{\infty} h_q R_{xx}[k+p-q] = h_q * R_{xx}[k+p]}$$

$$= \sum_{p=-\infty}^{\infty} h_p \cdot f[k+p] = \sum_{p'=-\infty}^{\infty} \tilde{h}_{p'}\, f[k-p']$$

where a convolution with the reverse sequence $\tilde{h}$ was identified and $p' = -p$

Replacing $f[k-p']$ by the original expression in terms of $R_{xx}[\cdot]$:

$$R_{yy}[k] = \tilde{h} * \underbrace{h * R_{xx}}_{f} [k]$$

iii. WSS of $\{Y_n\}$

- Mean: calculated in part i: $\mathbb{E} Y_n = (\mathbb{E} X_n) \cdot \underbrace{\sum_p h_p}_{\text{finite if } \sum |h_p| < \infty}$ ✓
- Dependence on k: from ii, notice that $R_{yy}[k]$ depends on $R_{xx}[k]$ only. ✓
- Variance: $\mathbb{E}(Y_n^2) = \tilde{h} * h * \underbrace{R_{xx}[0]}_{\text{finite from WSS.}}$

since $\sum h_p \leq \sum |h_p|$

$$\mathbb{E}(Y_n^2) = \sum_{p'=-\infty}^{\infty} \tilde{h}_{p'} \left(\sum_{q=-\infty} h_q R[-q] \right)$$

$$\leq R_{xx}[0] \left(\sum_{p'=\infty}^{\infty} \tilde{h}_{p'} \sum_{q=-\infty}^{\infty} h_q \right) \quad \left(\text{where } \max_k |R[k]| = R[0] \text{ has been used} \right)$$

$$\leq R_{xx}[0] \underbrace{\left(\sum_{p'=\infty}^{\infty} |\tilde{h}_{p'}| \right)}_{<\infty} \underbrace{\left(\sum_{q=-\infty}^{\infty} |h_q| \right)}_{<\infty}$$

then $\mathbb{E} Y_n^2$ is finite

3. Def White noise

$$\boxed{C_{xx}[k] = \delta[k]\,\sigma_x^2}$$

$r[k]$ correlation

$c[k]$ covariance

a)

- $\mathbb{E}\,X_n = (1)p + (-1)(1-p) = 2p-1 = \mu$

- $C_{xx}[k] = \mathbb{E}\,X_n X_{n+k} - \mu^2$

$$= \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{E}(x_n^2) = 1 & \text{if } k = 0 \end{cases}$$

then it's white noise

$$\mathbb{E}\,X_n^2 = (1)^2 p + (-1)^2 (1-p) = 1$$

b. Sufficient condition for ergodicity:

$$\lim_{k \to \infty} C_{xx}[k] = 0$$

which trivially holds for white noise processes.

Q4 $R_{xx}[0] = 2.8$, $R_{xx}[1] = R_{xx}[-1] = 1$, $= 0$ otherwise

$X_m = d_m + v_m$ $\quad \mathbb{E}\, v_m = 0,\ \mathbb{E}\, v_m d_m = 0$

$R_{vv}[k] = \delta[k]\, 0.5$

The optimal causal linear filter must satisfy Wiener-Hopf equations:

$[R_{xx}]\,\underline{h} = \underline{R}_{xd}$ $\quad \underline{v}$ the lower bar indicates a vector.

- Remember that these relations come from the minimisation of the expected error: $J = \mathbb{E}(d_m - \hat{d}_m)^2$, where d_m is the variable we want to estimate and $\hat{d}_m$ is a linear estimator.
- In this case, Wiener-Hopf eqs. take the form:

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} R_{xd}[0] \\ R_{xd}[1] \end{bmatrix}.$$

- $R_{xx}[k]$ is known ✓✓

- $R_{xd}[k]$ is calculated from $R_{xd}[k] = \mathbb{E}(d_m + v_m)d_{m+k} = R_{dd}[k] + \underbrace{\cancel{\mathbb{E}\, v_m d_{m+k}}}_{0}$ uncorrelated noise.
- R_{dd} is in turn calculated from:

$$R_{xx}[k] = \mathbb{E}(d_m + v_m)(d_{m+k} + v_{m+k}) = R_{dd}[k] + R_{vv}[k] + \cancel{\mathbb{E}\, v_m d_{m+k}}^{\,0} + \cancel{\mathbb{E}\, d_m v_{m+k}}^{\,0}$$

$$R_{dd}[k] = R_{xx}[k] - R_{vv}[k]$$

- Evaluating in W-H eqs: $\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 2.8 & 1 \\ 1 & 2.8 \end{bmatrix}^{-1} \begin{bmatrix} 2.3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.795 \\ 0.073 \end{bmatrix}$

- Expected error:

$$\mathbb{E}(d_m - \underbrace{\hat{d}_m}_{\text{optimal}})^2 = \overbrace{R_{dd}[0]}^{2.3} - \sum R_{xd}[q]\, h_q \quad \text{(See Lecture handout)}$$

$$= 2.3 - [0.795 \quad 0.073] \begin{bmatrix} 2.3 \\ 1 \end{bmatrix} = 0.397$$

- Naive estimate $\hat{d}_m = X_m$ has MSE $\mathbb{E}(d_m - d_m - v_m)^2 = \mathbb{E}\, v_m^2 = 0.5 \Rightarrow$ optimal filter reduces MSE to ~70%

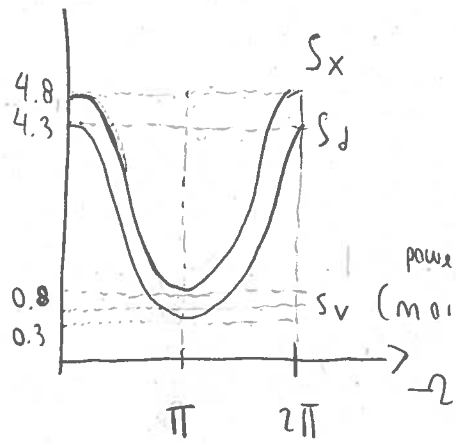
Q5 In the frequency domain:

$$H(e^{j\Omega}) = \underbrace{\frac{S_{xd}(e^{j\Omega})}{S_x(e^{j\Omega})}}_{\substack{\text{optimal filter} \\ \text{from Wiener-Hopf} \\ \text{eqs.}}} = \underbrace{\frac{S_d(e^{j\Omega})}{S_x(e^{j\Omega})}}_{\substack{\text{case of uncorrelated} \\ \text{noise}}}$$

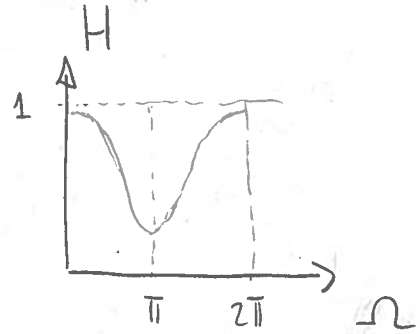
$$r_{dd}[k] = \begin{cases} 2.3 & k=0 \\ 1 & k=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$r_{xx}[k] = \begin{cases} 2.8 & k=0 \\ 1 & k=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Taking DTFT, $H(e^{j\Omega}) = \dfrac{2.3 + 2\cos\Omega}{2.8 + 2\cos\Omega}$



$$H(e^{j\Omega}) = \frac{1}{\underbrace{\dfrac{0.5}{2.3+2\cos\Omega}}_{SNR^{-1}} + 1}$$



Q 6

$$\epsilon_m = d_m - \sum_{p=-\infty}^{\infty} h_p x_{m-p}$$

a. $\mathbb{E}\,\epsilon_m \epsilon_{m+k} = \mathbb{E}\,(d_m - \sum h_p x_{m-p})(d_{m+k} - \sum h_q x_{m+k-q})$

$= \mathbb{E}\, d_m d_{m+k} \longrightarrow R_{dd}[k]$

$- \mathbb{E} \sum_{q=-\infty}^{\infty} h_q x_{m+k-q} d_m \longrightarrow \sum_q \underbrace{\mathbb{E}(x_{m+k-q} d_m)}_{R_{dx}[k-q]} \cdot h_q = h_q * R_{dx}[k]$

$- \mathbb{E} \sum_p h_p x_{m-p} d_{m+k} \longrightarrow \sum_p h_p \underbrace{\mathbb{E}\, x_{m-p} d_{m+k}}_{R_{xd}[k+p]} = h_{-q} * R_{xd}[k]$

$+ \underbrace{\sum h_p \sum_q h_q \underbrace{\mathbb{E}\, x_{m-p} x_{m+k-q}}_{R_{xx}[k+p-q]}} \longrightarrow \sum h_p f[k+p]$

$= h_{-p} * h_q * R_{xx}[k]$

$f[k+p] = h_q * R_{xx}[k+p]$ (See Q2)

$\mathbb{E}\,\epsilon_m \epsilon_{m+k} = R_{dd}[k] - h * R_{dx}[k] - h_{-q} R_{xd}[k]$
$+ h_{-p} * h_p * R_{xx}[k]$ Result from Q2

b. $S_\epsilon(e^{j\Omega}) = S_d(e^{j\Omega}) - H(e^{j\Omega}) S_{dx}(e^{j\Omega})$
$- H^*(e^{j\Omega}) S_{xd}(e^{j\Omega})$
$+ \underbrace{H^*(e^{j\Omega}) H(e^{j\Omega})}_{H^2(.)} S_x(e^{j\Omega})$

$\mathrm{DTFT}\{h_{-p}\}(e^{j\Omega})$
$= \sum_{p=-\infty}^{\infty} h_{-p} e^{-j\Omega p}$
$= \sum h_p e^{j\Omega p} = H^*(e^{j\Omega})$

(Just taking DTFT)

- Convolution in time domain is the product of spectra in frequency

c. If the filter $\{h_q\}$ is optimal, we know from Wiener-Hopf equations that

$$H(e^{j\Omega}) = \frac{S_{xd}(e^{j\Omega})}{S_x(e^{j\Omega})}$$

• Then, replacing in (b):

$$S_\epsilon(e^{j\Omega}) = S_d(e^{j\Omega}) + \overbrace{(H^{opt})^* \overbrace{\frac{S_{xd}}{\cancel{S_x}}}^{H^{opt}} \cdot \cancel{S_x(e^{j\Omega})}}^{|H|^2 S_x}$$

$$\cancel{- S_{xD} H^{opt*}(e^{j\Omega})}$$

$$- \underbrace{S_{DX}(e^{j\Omega})}_{S_{xd}^*(e^{j\Omega})} H^{opt}(e^{j\Omega}) \quad Q1.g$$

$$S_\epsilon(e^{j\Omega}) = S_d(e^{j\Omega}) - S_{xd}^*(e^{j\Omega}) H^{opt}(e^{j\Omega})$$

• Apply iDTFT to obtain $\mathbb{E}(\epsilon^2)$:

$$\mathbb{E}\, \epsilon_m^2 = R_{\epsilon\epsilon}[0]$$

And $R_{\epsilon\epsilon}[0] = \text{iDTFT}(S_\epsilon(e^{j\Omega}))$ evaluated in $k=0$, $e^{j\Omega k} = 1$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_\epsilon(e^{j\Omega})\, d\Omega$$

$$\mathbb{E}\, \epsilon_m^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(S_d(e^{j\Omega}) - S_{xd}^*(e^{j\Omega}) H^{opt}(\cdot)\right) d\Omega$$

d. With uncorrelated noise, $S_{xd}(e^{j\Omega}) = S_d(e^{j\Omega})$, and power spectrum is real and even: $S_d^* = S_d$.

Then $\mathbb{E}\, \epsilon_m^2$ simplifies to $\mathbb{E}\, \epsilon_m^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_d(e^{j\Omega})(1 - H^{opt}(e^{j\Omega}))\, d\Omega$

Q7 AR(1)

$$d_n = a_1 d_{n-1} + e_n \qquad \mathbb{E}(e_n) = 0, \ \mathbb{E}\, e_n^2 = \sigma_e^2$$

$$\mathbb{E}\, e_n e_m = 0 \quad \text{if } n \neq m$$

Hint: DTFT pair: $\displaystyle\sum_{k=-\infty}^{\infty} a^{|k|} e^{-j\Omega k} = \frac{1-a^2}{|1-ae^{-j\Omega}|^2}$

Taking z-transform of d_n:

$$D(z)(1-z^{-1}) = E(z)$$

$$D(z) = \frac{E(z)}{1-z^{-1}}$$

- d_n can be regarded as a noise process e_n passed through a system (LTI) given by $H(z) = \dfrac{1}{1-z^{-1}}$ (transfer function).
- We know that the power spectrum of the output of a linear system is $S_{out}(e^{j\Omega}) = S_{in}(e^{j\Omega})|H(e^{j\Omega})|^2$
- Then $S_d(e^{j\Omega}) = \dfrac{1}{|1-ae^{-j\Omega}|^2} \underbrace{S_e(e^{j\Omega})}_{\sigma_e^2}$
- Using the hint, $S_d(e^{j\Omega})$ can be rewritten as

$$S_d(e^{j\Omega}) = \sigma_e^2 \cdot \frac{1}{|1-ae^{-j\Omega}|^2}$$

$$= \sigma_e^2 \underbrace{\left(\sum_{k=-\infty}^{\infty} a^{|k|} e^{-j\Omega k}\right)}_{(1-a^2)/|1-ae^{-j\Omega}|^2} \frac{1}{1-a^2} = \sum_{k=-\infty}^{\infty} \left(\frac{a^{|k|}\sigma_e^2}{1-a^2}\right) e^{-j\Omega k}$$

If $S_d(e^{j\Omega})$ is the DTFT of $\left(\dfrac{\sigma_e^2 a^{|k|}}{1-a^2}\right)$, then

$$\boxed{R_{dd}[k] = \frac{\sigma_e^2 a^{|k|}}{1-a^2}}$$

- Alternatively, $R_{xx}[k]$ can be calculated as in Paper 2, and $S_d(e^{j\Omega})$ derived afterwards.

b. From W-H eqs. the optimal filter (in frequency) is

$$H(e^{j\Omega}) = \frac{S_d(e^{j\Omega})}{\underbrace{S_d(e^{j\Omega}) + S_v(e^{j\Omega})}_{S_x(e^{j\Omega})}}$$

S_{xd} in the general case, simplified now due to uncorrelated noise

$$H(e^{j\Omega}) = \frac{1}{1 + \frac{\sigma_v^2}{S_d(e^{j\Omega})}} = \frac{1}{1 + \frac{\sigma_v^2}{\sigma_e^2}|1 - a e^{-j\Omega}|^2}$$

$$= \frac{\sigma_e^2}{\sigma_e^2 + \sigma_v^2 [1 + a^2 - 2a\cos\Omega]}$$

Rearranging...

Q8

$$d_n = X_{n+m}$$

We want to estimate d_n with an infinite impulse response filter $\hat{d}_n = \sum_{p=-\infty}^{\infty} h_p X_{n-p}$

Error: $\epsilon_n = d_n - \sum_p h_p X_{n-p}$

From Wiener-Hopf equations we know h_p must satisfy

$$H(e^{j\Omega}) = \frac{S_{xd}(e^{j\Omega})}{S_x(e^{j\Omega})}$$ to be optimal.

Let's calculate $S_{xd}(e^{j\Omega})$

- $R_{xd}[k] = \mathbb{E}\, X_n \underbrace{X_{n+m+k}}_{d_{n+k}} = R_{xx}[m+k]$

$k' = m+k$
$k = k' - m$

- Power spectrum of $R_{xd}[k]$:

$$S_{xd}(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} R_{xx}[m+k]\, e^{-j\Omega k} = \sum_{k'=-\infty}^{\infty} R[k']\, e^{-j\Omega(k'-m)}$$

$$= e^{j\Omega m} \sum_{k'} R_{xx}[k]\, e^{j\Omega k} = e^{j\Omega m} S_x(e^{j\Omega})$$

$$H(e^{j\Omega}) = \frac{S_{xd}(e^{j\Omega})}{S_x(e^{j\Omega})} = e^{j\Omega m} \frac{\cancel{S_x(e^{j\Omega})}}{\cancel{S_x(e^{j\Omega})}} = e^{j\Omega m}$$

$$i\,\mathrm{DTFT}\{e^{j\Omega m}\}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega m} \cdot e^{j\Omega n}\, d\Omega = \begin{cases} 1 & n = -m \\ 0 & n \neq m \end{cases}$$

- $\frac{1}{2\pi}\int 1\, d\Omega$ if $n = -m$ $= 1$ $\quad = \delta[n+m]$
- Integrates 0 if $n \neq -m$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n+m)}\,d\Omega = \frac{1}{2\pi}\,\frac{e^{j\Omega(n+m)}}{j(n+m)}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\,\frac{e^{j\pi(n+m)} - e^{-j\pi(n+m)}}{j(n+m)} = 0$$

If $h[n] = \delta[n+m]$ the filter just picks the desired sample X_{n+m} ahead in the future (non-causality):

$$\hat{d}_n = \sum_{p=-\infty}^{\infty} h_p\, X[n-p]$$

$$= \dots + \underbrace{h_{p+1} X_{n-1}}_{0} + \underbrace{h_0 X_n}_{0} + \underbrace{h_{-1} X_{n+1}}_{0} + \dots + \underbrace{h_{-m} X_{n+m}}_{1\cdot X_{n+m}} + \dots \quad + 0 \dots,$$

Q 9

$$x_n = d_n + v_{1,n}$$

$v_{2,n}$ measured independently.

$$\hat{d}_n = x_n - \sum_{p=-\infty}^{\infty} h_p v_{2,n-p}$$

$$e_n = d_n - \left(x_n - \sum_{p=-\infty}^{\infty} h_p v_{2,n-p} \right)$$

$$\frac{d\,\mathbb{E}\, e_n^2}{d\, h_q} = \mathbb{E}\left\{ 2\left(d_n - x_n + \sum h_p v_{2,n-p} \right) v_{2,n-q} \right\} = 0$$

$$\Rightarrow \underbrace{\mathbb{E}\, d_n v_{2,n-q}}_{0} - \underbrace{\mathbb{E}\, x_n v_{2,n-q}}_{} + \sum h_p \mathbb{E}(v_{2,n-p} v_{2,n-q}) = 0$$

$$- R_{xv_2}[-q] + \sum h_p R_{v_2v_2}[q-p]$$

or $-R_{v_2x}[q]$

Rearranging: all equations (for all q)

$$\begin{bmatrix} R_{v_2v_2}[0] & R_{v_2v_2}[-1] & \dots \\ R_{v_2v_2}[-1] & & \\ & & \\ & & R_{v_2v_2}[0] \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_p \end{bmatrix} = \underbrace{\begin{bmatrix} R_{v_2x}[0] \\ R_{v_2x}[1] \\ \vdots \\ R_{v_2x}[p] \end{bmatrix}}_{= R_{v_2v_1}[q]}$$

$$R_{v_2x}[q] = \mathbb{E}\, v_{2,n} \overbrace{x_{n+q}}^{d_{n+q} + v_{1,n+q}} = \underbrace{\mathbb{E}\, d_{n+q} v_{2,n}}_{0} + \mathbb{E}\, v_{2,n} v_{1,n+q} = R_{v_2v_1}[q]$$

b. If the noise v_2 is ergodic: $\hat{r}_{v_2v_2}[k] = \frac{1}{N} \sum_n v_{2,n} v_{2,n+k}$

c. $R_{v_2v_1}$ can be estimated as $\hat{R}_{v_2v_1}[k] = \frac{1}{N} \sum_n v_{2,n} x_{n+k}$

Since $R_{v_2v_1}[k] = \mathbb{E}(v_{2,n}[x_{n+k} - d_{n+k}]) = \mathbb{E}\, v_{2,n} x_{n+k} - \underbrace{\mathbb{E}\, v_{2,n} d_{n+k}}_{0} = R_{v_2x}[k]$

Q9.c (continuation): The assumption that d_n is stationary is not realistic in this application, since voice is a complex signal whose statistics change over time (although it can be modelled as locally stationary for most purposes). In the scenario studied in this question, whether the stationarity assumption holds or not does not impact the linear filter $\boldsymbol{h}$ calculated in part (a), since the filter acts as an estimator of noise v_1 given measurement v_2 rather than as a model of signal d_n itself: the correlations needed only involve v_1 and v_2, which are still assumed stationary. The relation

$$r_{v_2 x}[k] = \mathbb{E}v_{2,n}x_{2,n+k} = \mathbb{E}v_{2,n}(d_{n+k} + v_{1,n+k}) = \mathbb{E}v_{2,n}d_{n+k} + \mathbb{E}v_{2,n}v_{1,n+k} = r_{v_2 v_1}[k]$$

still applies since $\mathbb{E}v_{2,n}d_{n+k} = \mathbb{E}v_{2,n}\mathbb{E}d_{n+k} = (0) \cdot \mathbb{E}d_{n+k} = 0$ holds regardless of the stationarity of d_n (as long as both as d_n is uncorrelated with noise process $v_{2,n}$).

When modelling locally stationary signals like human voice, independent statistics (i.e., any relevant self- and cross-correlations) can be estimated for shorter frames of the time series and a Wiener-like scheme applied to each of them.

Q 10| $\{b_n\}_n$ Bernoulli $p = 0.5$

Measured signal: $x_n = b_n + 0.1\, b_{n-1}$

a. $r_{bx}[k]$

case $k=0$: $r_{bx}[0] = \mathbb{E}[b_n + 0.1\, b_{n-1}]b_n = \mathbb{E}\, b_n^2 + 0.1\, \mathbb{E}\, b_n b_{n-1}$

$= 1^2\, 0.5 + (-1)^2 0.5 + 0.1 \underbrace{(\mathbb{E}\, b_n)}\mathbb{E}(b_{n-1})$

$\mathbb{E}\, b_n = 1 \cdot 0.5 + (-1) \cdot 0.5$
$= 0$

$= 1|$

$k=1$ $r_{bx}[1] = \mathbb{E}\, b_n \cancel{\mathbb{E}\, b_{n+1}}^{\to 0} + 0.1\, \mathbb{E}\, b_n\, b_{n-1+1}$

$= 0.1\ \mathbb{E}\, b_n^2 = 0.1$

Note that for $k=1$ on any other value $r_{bx}[k] = 0$

$r_{xx}[k] = \mathbb{E}[b_n + 0.1\, b_{n-1}][b_{n+k} + 0.1\, b_{n+k-1}]$

$= \mathbb{E}\, b_n b_{n+k} + 0.1\, \mathbb{E}\, b_n\, b_{n+k-1} + 0.1\, \mathbb{E}\, b_{n-1}\, b_{n+k}$

$+ 0.1^2\, \mathbb{E}\ b_{n-1}\, b_{n+k-1}$

$\underline{k=0|}$ $R_{xx}[0] = R_{bb}[0]\,[1 + (0.1)^2] = 1.01$

$\underline{k=1|}$ $R_{xx}[1] = 0.1$

$\underline{k=-1|}$ $R_{xx}[-1] = 0.1$

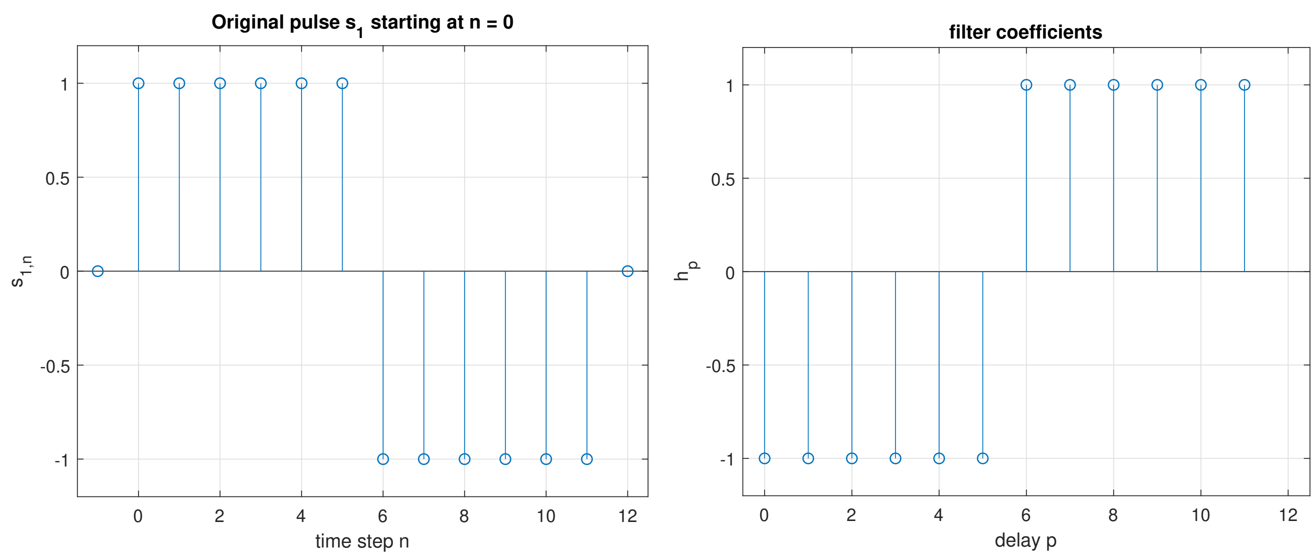
$|k| > 0$ $R_{xx}[k] = 0$

b. $\underbrace{\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix}}_{\underline{R}_{xx}} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{xb}[0] \\ R_{xb}[1] \end{bmatrix}}_{R_{xb}}$ $\begin{array}{l} \to R_{bx}[0] = 1 \\ \\ \to\!\!= R_{bx}[-1] = 0 \end{array}$

Q11 General explanation of the matched filter will be provided.

Unlike standard Wiener schemes where the objective is to generate a "clean" version of a signal (filtering, estimation) or a predict a future value (prediction, forecasting), the matched filter aims to **detect** the occurrence of a pulse with a known waveform that is buried under a layer of noise. From a mathematical point of view, it maximises the instant signal to noise ratio instead of minimising the mean squared error.

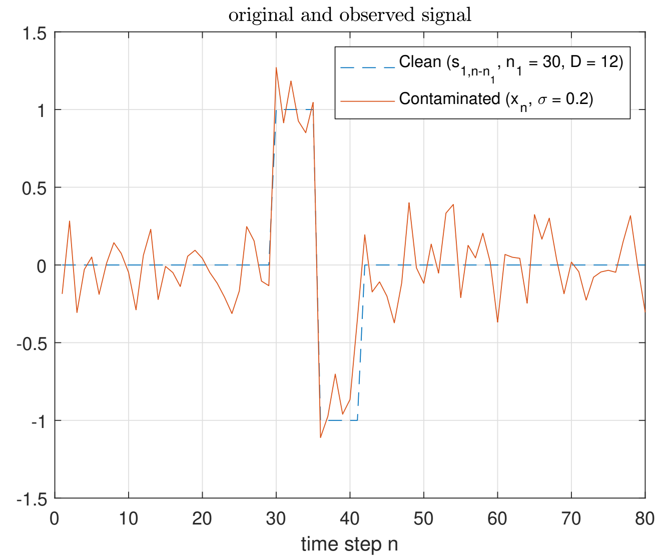
Q11.a In the matched filter, detection of pulse s_1 is carried out by setting coefficients h_p of linear filter $y_n = \sum_{n=0}^{P} h_p x_{n-p}$ to be the time-reversed version of s_1. Thus, at each time n the output of the filter is the dot product of the last D measurements $\bar{x}_n = [x_n, x_{n-1}, \cdots, x_{n-D+1}]^\top$ and the pulse in vector form $\bar{s}_1 = [s_{1,D}, s_{1,D-1}, \ldots, s_{1,1}]^\top$ $(P = D - 1)$: $y_n = \bar{s}^\top \bar{x}$.



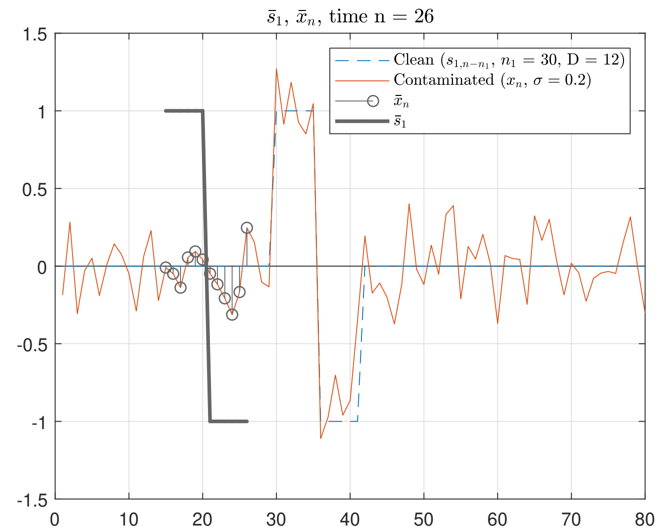
(a) Original pulse

(b) Values of optimal matched filter for pulse s_1

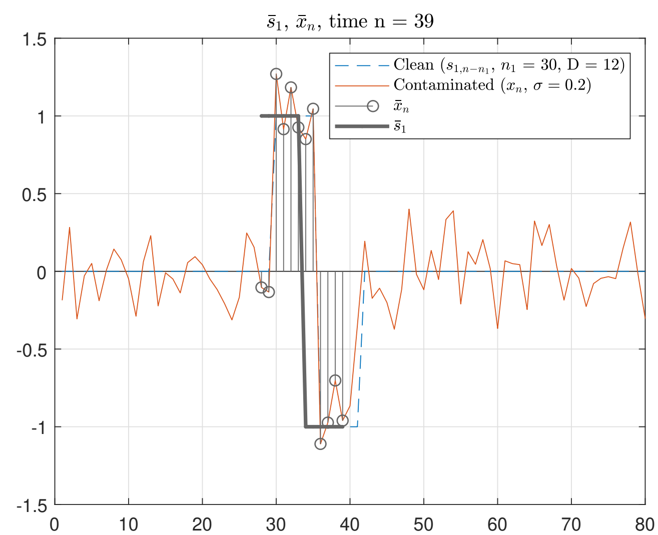
Informally, operation $y_n = \bar{s}^\top \bar{x}$ measures the degree of agreement between the two signals, and $\bar{s}$ could be thus regarded as a template signal. Consider $x_n = s_{1,n-n_1} + v_n$, $v_n \sim N(0, \sigma^2)$ with $n_1 = 30$ denoting the starting point of the pulse in the time axis. In the examples below, the last D (duration of the pulse) samples at time n are indicated with circles, and the corresponding values of h_p that accompany each sample are overlaid (grey line).



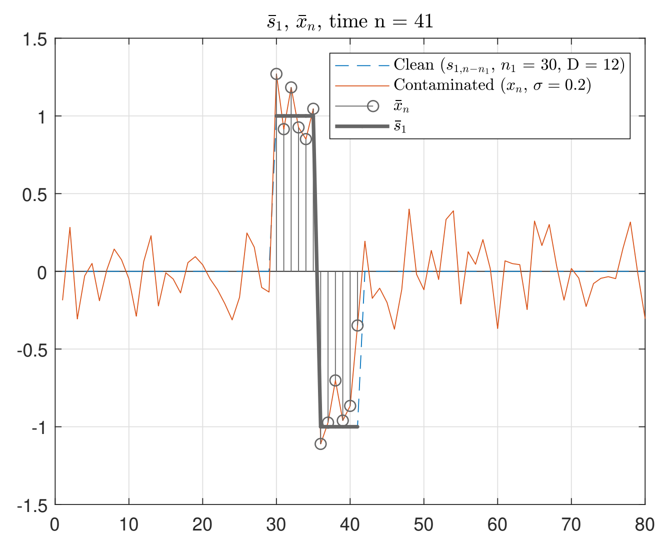
(a) Original pulse and contaminated measurement



(b) $\bar{x}_n$ and $\bar{s}_1$ when no pulse is present



(c) Partial agreement between $\bar{x}_n$ and $\bar{s}_1$ (filter output at a time close to the actual occurrence of the pulse)



(d) Highest degree of agreement between $\bar{x}_n$ and $\bar{s}_1$ (filter is computed exactly when the pulse occurs)

The figure below shows the output of the optimal matched filter at each time step. Clearly, the output reaches its maximum at time $n = 41$ (when the full pulse finishes). Since we observe the pulse through a noisy signal x_n, the output is noisy as well—other realisation of x_n would yield another y_n.

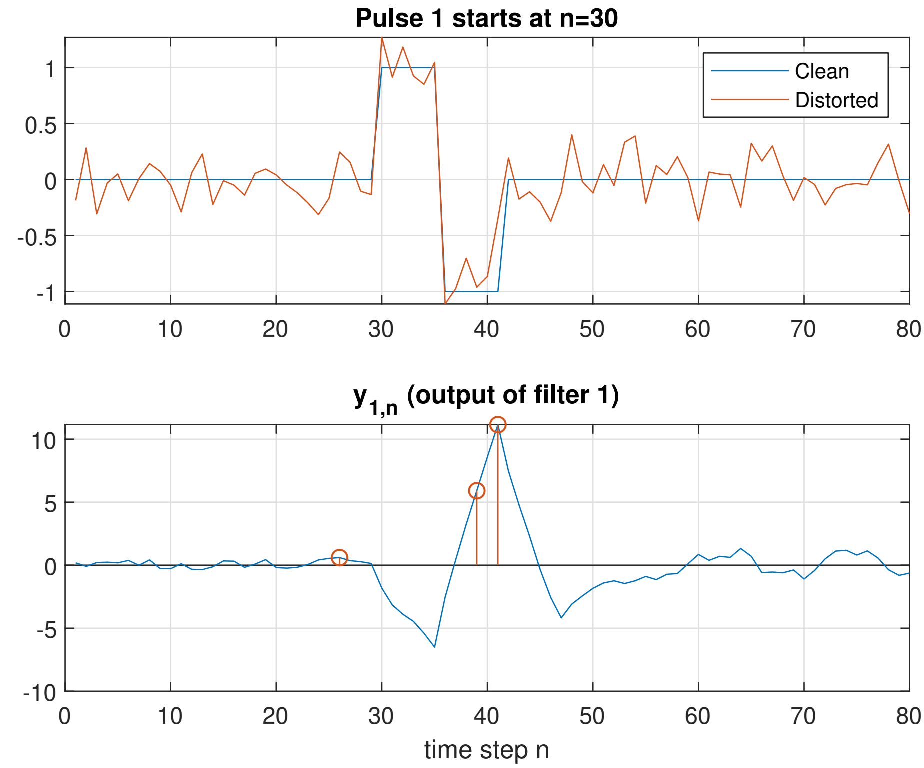


Figure 1: Realisation of x_n (up). Output of filter $y_{1,n}$, times considered in the previous example (n=26,39,41) indicated in red (down)

Q11.b At time of detection, the expected SNR of the optimal filter is $\|s\|^2/\sigma_v^2 = 12/\sigma_v^2$. Instead, using instant measurements would yield an expected SNR of $\|x_n\|^2/\sigma_v^2 = 1/\sigma_v^2$ ($x_n = s_{1,n-n_1} + v_n$). Although this can still be used as an indicator of the presence of the target (in the sense the expected SNR value when $s_{1,n-n_1} = 0$ would be just 0), it doesn't take into account the shape of the pulse, and would be ambiguous since exactly D positions in the time axis would yield the same value.

Q11.c Addressed already in the solution of (a).

Q11.d False. Since we don't have access to the original values of pulse s_1 and instead observe it through a contaminated signal, the output reaches the maximum at the time of the occurrence of the pulse just *on average*. Typically, a detection is "announced" when the output surpasses a certain threshold or at the time when the filter output reaches its maximum value. Due to the randomness of the observation process, these conditions may happen in a time instant that does not coincide with the occurrence of the actual pulse (see the example below, where the maximum y_n occurs one time step after the actual pulse).

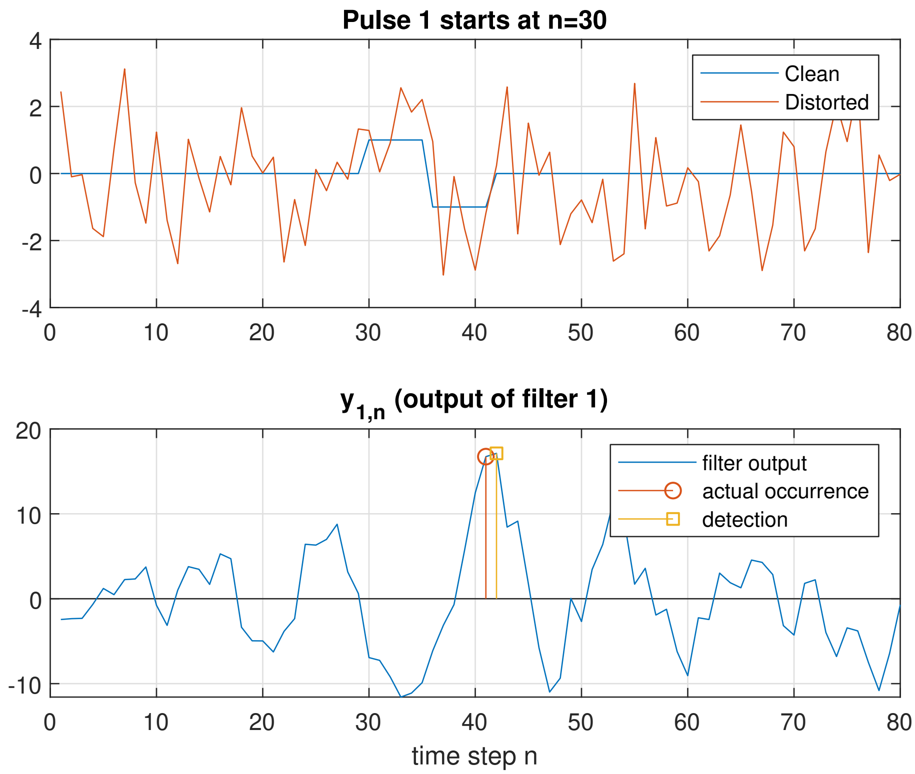
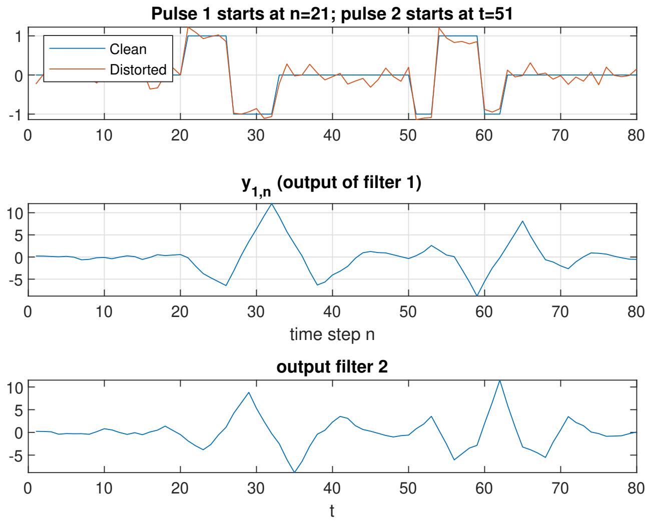
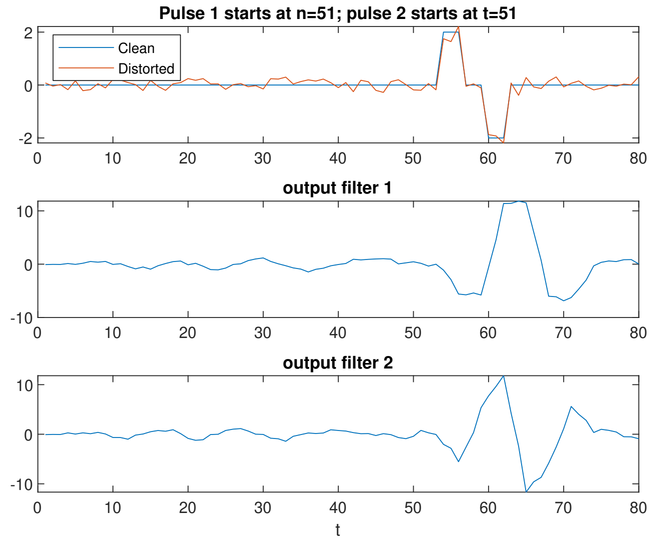


Figure 2: Example of filter output in a scenario of high observation noise

Q11.e Two matched filters, each of them tuned for a specific pulse s_1 or s_2, can be run in parallel to detect the presence of the pulse. Two examples are shown below.



(a) Pulses occurring at different times



(b) Pulses occuring at the same time