Q8

$$f(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2} \frac{x^2+y^2}{\sigma^2} \right\}$$

$$F(a \leq x \leq b,\ c < y \leq d)$$

$$\frac{1}{2\pi\sigma^2} \int_c^d \left( \int_a^b \exp -\frac{1}{2\sigma^2}(x^2+y^2)\, dx \right) dy = \frac{1}{2\pi\sigma^2} \int_c^d \exp -\frac{1y^2}{2\sigma^2}\, dy \cdot \int_a^b \left( \exp - \frac{1}{2}\frac{x^2}{\sigma^2} \right) dx$$

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$F(r)$?

<u>one way</u>: use polar coordinates

instead of $x, y$, use $r = \sqrt{x^2+y^2}$

$\theta = \tan\left(\frac{y}{x}\right)$

inverse transformation:

$$x(r,\theta) = r\cos\theta$$

$$y(r,\theta) = r\sin\theta$$

Using change of variables:

$$f_{r,\theta}(r,\theta) = f_X(\underbrace{x(r,\theta)}_{r\cos\theta}, \underbrace{y(r,\theta)}_{r\sin\theta}) \cdot |\det J(\theta, r)|$$

$$J = \begin{bmatrix} \frac{d\,x(r,\theta)}{dr} & \frac{d\,x(r,\theta)}{d\theta} \\ \frac{d\,y(r,\theta)}{dr} & \frac{d\,y(r,\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$|\det J| = |r\cos^2\theta + r\sin^2\theta| = r$$

$$N\left(x, y \,\middle|\, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\sigma^2 & 0\\0 & \sigma^2\end{bmatrix}\right)$$

$$f_X(r\cos\theta, r\sin\theta)\cdot r$$

$$\frac{1}{2\pi\sigma^2}\exp\left(\frac{-r^2\,[\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{\to 1}]}{2\sigma^2}\right) r$$

$$F_r(r) = \int_0^{2\pi} f_{\theta,r}(\theta, r)\,d\theta = \frac{1}{\sigma^2}\left(\exp\left\{\frac{-r^2}{2\sigma^2}\right\}\right) r$$

$$F_r(R) = \frac{1}{\sigma^2}\int_0^R r\,\exp\left(-\frac{1}{2\sigma^2}r^2\right) dr$$

primitive: of $f_r(r)$ $\quad -\exp\left(-\frac{1}{2\sigma^2}r^2\right) + C$

check: $\frac{d(\text{primitive})}{dr}$ yields exactly the function $f_r(r)$

$$\frac{d(P\cdot)}{dt} = -\exp\left(-\frac{1}{2\sigma^2}r^2\right)\cdot -\frac{1\,\cancel{2}\,r}{\cancel{2}\sigma^2}$$

$$= -\exp\left(-\frac{1}{2\sigma^2}r^2\right)\Bigg|_0^R = 0 - 1$$

$$\Rightarrow P.\ \boxed{CDF(r) = 1 - \exp\left(\frac{-1\,r^2}{2\sigma^2}\right)}$$

$C = 1$ must be added to ensure $CDF(\infty) = 1$

$CDF(0) = 0$