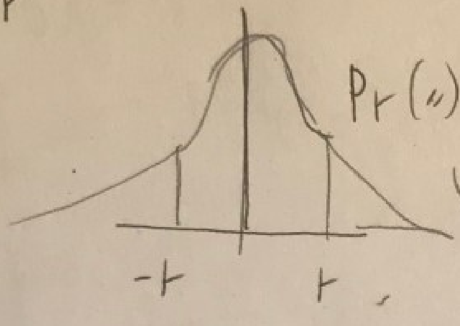
Q 6)

b) $\Pr(|v| < r)$

$$= \int_{-r}^{r} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2} \frac{t^2}{\sigma^2} \right\} dt \quad \text{by parity}$$



$$\Pr(\,) = \frac{1}{\sqrt{2\pi\sigma^2}} \, 2 \int_0^r \exp -\frac{1}{2} \frac{t^2}{\sigma^2} \, dt$$

$$\frac{d \Pr(\,)}{dr} = \sqrt{\frac{2}{\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2} \frac{t^2}{\sigma^2} \right)$$

c) $g(v) = v^2$

note that $g(v) = |v|^2 \Rightarrow$ we can use $p_{|v|}(x)$ directly

change of variables

$\Rightarrow$ $\cancel{f(x)}$ $y = g(x)$

$$\frac{d g^{-1}(y)}{dy} = \frac{d\sqrt{y}}{dy} =$$

$$p_y(y) = f(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right|$$

$$= \sqrt{\frac{2}{\pi\sigma}} \exp\left(-\frac{1}{2} \frac{(\sqrt{y})^2}{\sigma^2} \right) \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{y}} \exp\left(-\frac{1}{2} y/\sigma^2 \right)$$

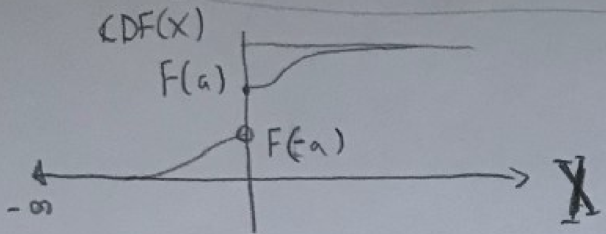
Q6

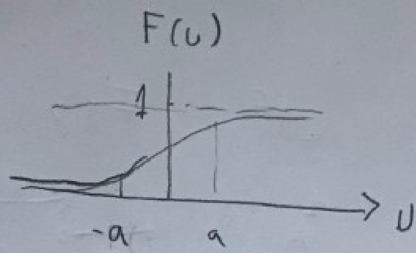
For $X < 0$:

$$\boxed{\begin{aligned} &\Pr(x < X) \\ &= \Pr(x = u + a < X) \\ &= F(X - a) \end{aligned}}$$

For $X \geq 0$:

$$\begin{aligned} &\Pr(x < X) \\ &= \Pr(u - a < X) \\ &= F(X + a) \end{aligned}$$





$$\frac{d\, CDF(X)}{dX} = \frac{d\left(F(x-a)\mathbb{1}_{[-\infty,0)} + F(x+a)\mathbb{1}_{[0,\infty)}\right)}{dx}$$

$$= N(x \mid a, \sigma^2)\mathbb{1}_{[-\infty,0]} + N(x \mid -a, \sigma^2)\mathbb{1}_{[0,\infty)}$$

$$+ \left[F(a) - F(-a)\right]\delta_0(x)$$

where:

- $$F(x-a) = \int_{-\infty}^{x-a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-t^2}{2\sigma^2}\right\} dt$$

$$\frac{dF(x-a)}{dt} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-t^2}{2\sigma^2}\right\}$$

- Same for $F(x+a)$

- Derivative of a vertical slope is a delta function. One can think of it the other way: the pdf needs a Dirac delta to make the CDF have a discontinuity.

Q11) bivariate Normal

$$f(x,y) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}[x-m_1, y-m_2]^T \Sigma^{-1} [x-m_1, y-m_2]\right\}$$

The covariance matrix is $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{bmatrix}$

$$\Sigma^{-1} = \frac{1}{\underbrace{\det \Sigma}_{\Delta}} \begin{bmatrix} \sigma_2^2 & -\rho \\ -\rho & \sigma_1^2 \end{bmatrix}, \quad \Delta = \sigma_1^2\sigma_2^2 - \rho^2$$

$$f(x,y) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2\Delta}[x-m_1, y-m_2] \overbrace{\begin{bmatrix} \sigma_2^2(x-m_1) - \rho(y-m_2) \\ \sigma_1^2(y-m_2) - \rho(x-m_1) \end{bmatrix}}^{\Sigma^{-1}\begin{bmatrix} x-m_1 \\ y-m_2 \end{bmatrix}}\right\}$$

$$= \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2\Delta}\left[\sigma_2^2(x-m_1)^2 - 2\rho(x-m_1)(y-m_2) + \sigma_1^2(y-m_2)^2\right]\right\}$$

$$= \frac{\exp\left\{\frac{-1}{2\Delta}\left[\sigma_2^2 x^2 - 2x(\sigma_2^2 m_1 + \rho[y-m_2])\right]\right\}}{2\pi|\Sigma|^{1/2}}$$

$$\cdot \overbrace{\exp\left\{\frac{-1}{2\Delta}\left[\sigma_2^2 m_1 + 2\rho[m_1 y - m_1 m_2] + \sigma_1^2(y-m_2)^2\right]\right\}}^{h(y)}$$

$$= \underbrace{\exp\left\{\frac{-1}{2\Delta/\sigma_2^2}\left[x^2 - 2x\left(m_1 + \frac{\rho}{\sigma_2^2}[y-m_2]\right)\right]\right\}}_{g(x,y)} \cdot \frac{1}{2\pi|\Sigma|^{1/2}} \cdot h(y)$$

$$f(x|y) = \frac{g(x,y) \cdot \cancel{h(y)} \cancel{\frac{1}{2\pi|\Sigma|^{1/2}}}}{\left[\int g(x,y)\,dx\right] \cancel{\frac{1}{2\pi|\Sigma|^{1/2}}} \cancel{h(y)}} = \frac{g(x,y)}{\int g(x,y)\,dx}$$

But by recalling $N(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2}(x^2 - 2x\mu + \mu^2)\right\}$

we recognize $f(x|y) = N\left(x \,\middle|\, m_1 + \frac{\rho}{\sigma_2^2}[y-m_2], \Delta/\sigma_2^2\right)$ easily b]. $f(y) = \dfrac{f(x,y)}{f(x|y)} =$

$$\frac{\dfrac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{\dfrac{-1}{2\Delta}\left[\sigma_2^2(x-m_1)^2 - 2\rho(x-m_1)(y-m_2) + \sigma_1^2(y-m_2)^2\right]\right\}}{\dfrac{1}{\sqrt{2\pi}\sqrt{\dfrac{|\Sigma|}{\sigma_2^2}}} \exp\left\{-\dfrac{1}{2\Delta/\sigma_2^2}\left[x^2 - 2x\left(m_1 + \dfrac{\rho}{\sigma_2^2}(y-m_2)\right) + \overbrace{\left(m_1 + \dfrac{\rho}{\sigma_2^2}(y-m_2)\right)^2}^{\text{expand}}\right]\right\}}$$

$\xrightarrow{\text{expand}} = m_1^2 + \dfrac{\rho^2}{\sigma_2^4}(y-m_2)^2 + \dfrac{2m_1\rho(y-m_2)}{\sigma_2^2}$

$|\Sigma| = \Delta$

$$\frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{-\frac{1}{2\Delta}\left[\cancel{\sigma_2^2(x-m_1)^2} - \cancel{\sigma_2^2[x^2 + 2x\,m_1 + m_1^2]}\right.\right.$$

$$-2\rho[\cancel{xy} - \cancel{m_1 y} - \cancel{m_2 x} + \cancel{m_1 m_2}] + 2\rho\cancel{(y-m_2)x}$$

$$-2m_1\rho(y-m_2)$$

$$\left.\left.+\sigma_1^2(y-m_2)^2 + \frac{\rho^2}{\sigma_2^2}(y-m_2)^2\right]\right\}$$

$$\Delta = \sigma_1^2\sigma_2^2 - \rho^2$$

$$= \frac{1}{\sqrt{2\pi\sigma_2^2}}\left\{\frac{-1}{2\Delta}\left[\sigma_1^2 + + \frac{\rho^2}{\sigma_2^2}\right](y-m_2)^2\right]\right\} = \frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2\sigma_2}(y-m_1)^2\right)$$

$f(y) =$

$\Rightarrow N(y \mid m_1, \sigma_2^2)$